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ON DISCRETIZATION OF C*-ALGEBRAS

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ABSTRACT. The C*-algebra of bounded operators on the separable infinite-dimensional Hilbert space cannot be mapped to an AW*-algebra in such a way that each unital commutative C*-subalgebra $C(X)$ factors normally through $\ell^\infty(X)$. Consequently, there is no faithful functor discretizing C*-algebras to AW*-algebras, including von Neumann algebras, in this way.

1. INTRODUCTION

In operator algebra it is common practice to think of a C*-algebra as representing a noncommutative analogue of a topological space, and to think of a W*-algebra as representing a noncommutative analogue of a measurable space. What would it mean to make precise the notion of a C*-algebra A as a 'noncommutative ring of continuous functions'? The present article explores the idea that one should first embed A in an appropriate noncommutative algebra of 'bounded functions on the underlying quantum set of the spectrum of A', just like any topological space embeds in a discrete one [1, 4]. It is tempting to demand that such a 'noncommutative function ring' be an atomic W*-algebra, but we work more generally under the mere assumption that they be AW*-algebras.

Write **Cstar** for the category of unital C*-algebras with unital $*$-homomorphisms, and **AWstar** for the category of AW*-algebras with unital $*$-homomorphisms whose restriction to the projection lattices preserve arbitrary least upper bounds.[1] The discussion above leads naturally to the following notion, in keeping with the programme of taking commutative subalgebras seriously [7, 14, 3, 15, 2], that has recently been successful [8, 5, 9, 6].

Definition. A *discretization* of a unital C*-algebra A is a unital $*$-homomorphism $\phi\colon A \to M$ to an AW*-algebra M whose restriction to each commutative unital C*-subalgebra $C \cong C(X)$ factors through the natural inclusion $C(X) \to \ell^\infty(X)$ via a morphism $\ell^\infty(X) \dashrightarrow M$ in **AWstar**, so that the following diagram commutes.

$$
\begin{array}{ccc}
A & \xrightarrow{\ \ \phi\ \ } & M \\
\big\uparrow & & \big\uparrow \\
C(X) & \hookrightarrow & \ell^\infty(X)
\end{array}
$$

This short note proves that this construction degenerates in prototypical cases.

[1]See [9, Lemma 2.2] for further characterizations of these morphisms.

Theorem. *If $\phi\colon B(H) \to M$ is a discretization for a separable infinite-dimensional Hilbert space H, then $M = 0$.*

For W*-algebras M, this obstruction concretely means that $B(H)$ has no nontrivial representation on a Hilbert space such that every (maximal) commutative $*$-subalgebra has a basis of simultaneous eigenvectors.

Consequently, discretization cannot be made into a faithful functor.

Corollary. *Let a functor $F\colon$ **Cstar** $\to$ **AWstar** have natural unital $*$-homomorphisms $\eta_A\colon A \to F(A)$. Suppose there are isomorphisms $F(C(X)) \cong \ell^\infty(X)$ for each compact Hausdorff space X that turn $\eta_{C(X)}$ into the inclusion $C(X) \to \ell^\infty(X)$. If a unital C*-algebra A has a unital $*$-homomorphism $\alpha\colon B(K) \to A$ for an infinite-dimensional Hilbert space K, then $F(A) = 0$.*

As the proof of the Theorem relies on the use of annihilating projections and on the Archimedean property of the partial ordering of positive elements in the discretizing AW*-algebra M, it is intriguing to note that this does not rule out faithful functors F as above from **Cstar** to the category **Cstar** or to the category of Baer $*$-rings with $*$-homomorphisms that restrict to complete orthomorphisms on projection lattices. A rather different approach to the problem of extending the embeddings $C(X) \hookrightarrow \ell^\infty(X)$ to noncommutative C*-algebras has recently appeared in [10]. We also remark that since the identity functor discretizes all finite-dimensional C*-algebras, this truly infinite-dimensional obstruction is independent of the Kochen–Specker theorem, a key ingredient in some previous spectral obstruction results [14, 3].

The rest of this note proves the Theorem and its Corollary.

2. Proof

Notation. Fix a separable infinite-dimensional Hilbert space $H = L^2[0,1]$, and consider its algebra $B(H)$ of bounded operators. Write D for the discrete maximal abelian $*$-subalgebra generated as a W*-algebra by the projections q_n onto the Fourier basis vectors $e_n = \exp(2\pi i n-)$ for $n \in \mathbb{Z}$. There is a canonical conditional expectation $E\colon B(H) \to D$ that sends $f \in B(H)$ to its diagonal part $\sum q_n f q_n$.

The main results rely upon the following mild strengthening of the recent solution of the Kadison–Singer problem [11].

Lemma 1. *Let A be any unital C*-algebra, and $\psi_0\colon D \to \mathbb{C}$ a pure state of D. The map $\psi_0 \cdot 1_A\colon D \to A$ given by $f \mapsto \psi_0(f) \cdot 1_A$ extends uniquely to a unital completely positive map $\psi\colon B(H) \to A$ given by $f \mapsto \psi_0(E(f)) \cdot 1_A$.*

Proof. We employ a standard reduction of the unique extension problem to Anderson's paving conjecture, as outlined, for instance, in [13].

The extension $\psi_0 \circ E$ is well known to be a pure state, proving existence. For uniqueness, let $\psi\colon B(H) \to A$ be any unital completely positive map extending $\psi_0 \cdot 1_A$. It suffices to show that $\psi = \psi \circ E$, as then $E(f) \in D$ for $f \in B(H)$ implies $\psi(f) = \psi(E(f)) = \psi_0(E(f)) \cdot 1_A$ as desired. As f is a linear combination of two self-adjoint elements, we may further assume that $f = f^* \in B(H)$. Replacing f with $f - E(f)$, we reduce to showing $\psi(f) = 0$ when $f = f^*$ and $E(f) = 0$. To this end, let $\varepsilon > 0$. By Anderson's paving conjecture, established in [11, 1.3], there exist projections $p_1, \ldots, p_n \in D$ with $\sum p_i = 1$ and $\|p_i f p_i\| \leq \varepsilon \|g\|$ for all i. As

$\psi|_D = \psi_0$ is a pure state, up to reordering indices we have $\psi(p_1) = 1$ and $\psi(p_i) = 0$ for $i > 1$.

By the Schwarz inequality for 2-positive maps [12, Exercise 3.4], for all $i > 1$ we have $\|\psi(p_i f)\|^2 \leq \|\psi(p_i p_i^*)\| \cdot \|\psi(f^* f)\| = 0$ since $\psi(p_i p_i^*) = \psi(p_i) = 0$. Thus $\psi(p_i f) = 0$ for all $i > 1$, making $\psi(f) = \sum_{i=1}^n \psi(p_i f) = \psi(p_1 f)$. A symmetric argument replacing f with $p_1 f$ yields $\psi(f) = \psi(p_1 f) = \psi(p_1 f p_1)$. Unitality of ψ furthermore gives $\|\psi\| = 1$ [12, Corollary 2.8], so that

$$\|\psi(f)\| = \|\psi(p_1 f p_1)\| \leq \|p_1 f p_1\| \leq \varepsilon \|f\|.$$

As ε was arbitrary, we deduce that $\psi(f) = 0$ as desired. $\qquad\square$

Note that Lemma 1 still holds with ψ merely 2-positive. Next we consider the continuous maximal abelian *-subalgebra $C = L^\infty[0,1]$ of $B(H)$.

Lemma 2. *Let $\psi\colon B(H) \to \mathbb{C}$ be the unique extension of a pure state of D. The restriction of ψ to C is the state given by integration (against the Lebesgue measure).*

Proof. Each $f \in C$ has diagonal part $E(f) = \int_0^1 f(x)\,\mathrm{d}x$ because

$$\langle f e_n, e_n \rangle = \langle f \cdot \exp(2\pi i n-), \exp(2\pi i n-) \rangle$$

$$= \int_0^1 f(x) \cdot e^{2\pi i n x} \cdot \overline{e^{2\pi i n x}}\,\mathrm{d}x$$

$$= \int_0^1 f(x)\,\mathrm{d}x.$$

Because we assumed that ψ is a pure state of D, we have $\psi = \psi \circ E$ as in Lemma 1. Hence $\psi(f) = \psi(E(f)) = \psi(\int_0^1 f(x)\,\mathrm{d}x) = \int_0^1 f(x)\,\mathrm{d}x$. $\qquad\square$

To prove the Theorem, recall that for an orthogonal set of projections $\{p_i\}$ in an AW*-algebra, $\sum p_i$ denotes their least upper bound in the lattice of projections.

Proof of Theorem. Write $C \cong C(X)$ and $D \cong C(Y)$ for compact Hausdorff spaces X and Y. The discretization $\phi\colon B(H) \to M$ is accompanied by the following commutative diagram, where α and β are morphisms in **AWstar**.

$$
\begin{array}{ccc}
C = L^\infty[0,1] \cong C(X) & \longrightarrow & \ell^\infty(X) \\
\big\updownarrow & & \big\downarrow \alpha \\
B(H) & \xrightarrow{\ \ \phi\ \ } & M \\
\big\updownarrow & & \big\uparrow \beta \\
D = \ell^\infty(\mathbb{Z}) \cong C(Y) & \longrightarrow & \ell^\infty(Y)
\end{array}
$$

The atomic projections $\delta_x \in \ell^\infty(X)$ for $x \in X$ and $\delta_y \in \ell^\infty(Y)$ for $y \in Y$ have respective images $p_x = \alpha(\delta_x) \in M$ and $q_y = \beta(\delta_y) \in M$. For each y, the map $\psi\colon B(H) \to q_y M q_y$ given by $\psi(f) = q_y \phi(f) q_y$ is completely positive and unital (where q_y is the unit of $q_y M q_y$). Its restriction to D is of the following form, where we consider $f \in D$ as an element of the function algebra $C(Y) \subseteq \ell^\infty(Y)$:

$$\psi(f) = q_y \phi(f) q_y = \beta(\delta_y f \delta_y) = \beta(f(y)\delta_y) = f(y) q_y.$$

Thus there is a pure state ψ_0 on D with $\psi|_D = \psi_0 \cdot q_y$. It follows from Lemma 1 that $\psi = (\psi_0 \circ E) \cdot q_y$. For $t \in [0,1]$, write $e_t = \phi(\chi_{[0,t]})$ for the image of the characteristic

function $\chi_{[0,t]} \in C$. Lemma 2 implies $\psi(\chi_{[0,t]}) = \left(\int_0^1 \chi_{[0,t]}(x)\,\mathrm{d}x \right) \cdot q_y = t q_y$, so

$$q_y e_t q_y = q_y \phi(\chi_{[0,t]}) q_y = \psi(\chi_{[0,t]}) = t q_y$$

for all $y \in Y$ and all $t \in [0,1]$.

Considering each projection $\chi_{[0,t]}$ as an element of $C(X)$, fix clopen sets $K_t \subseteq X$ such that $\chi_{[0,t]} = \sum_{x \in K_t} \delta_x$. Then $e_t = \phi(\chi_{[0,t]}) = \sum_{x \in K_t} p_x$ in M. Fix $n \in \mathbb{N}$, and set $J_i = K_{i/n} \setminus K_{(i-1)/n} \subseteq Y$. Note that $K_1 = X$, so that these J_i partition X into a disjoint union of n clopen sets. By construction,

$$\sum_{x \in J_i} p_x \;=\; \sum_{x \in K_{i/n}} p_x - \sum_{x \in K_{(i-1)/n}} p_x \;=\; e_{i/n} - e_{(i-1)/n}.$$

Now fix $x \in X$. Then $x \in J_i$ for some i, and $p_x \leq e_{i/n} - e_{(i-1)/n}$ as above. Thus

$$q_y p_x q_y \;\leq\; q_y(e_{i/n} - e_{(i-1)/n}) q_y \;=\; \tfrac{i}{n}\, q_y - \tfrac{i-1}{n}\, q_y \;=\; \tfrac{1}{n}\, q_y.$$

As n was arbitrary, we find that $q_y p_x q_y = 0$. Now $(p_x q_y)^*(p_x q_y) = q_y p_x q_y = 0$ gives $p_x q_y = 0$ for all $y \in Y$. Thus p_x is orthogonal to $\sum q_y = 1$ in M, whence $p_x = 0$ for all $x \in X$. It follows that $1 = \sum p_x = 0$ in M, and so $M = 0$. $\qquad\square$

Remark. We thank an anonymous referee for noticing that our arguments prevail without the full force of Kadison–Singer. This may be done as follows. Identifying the algebra $C(\mathbb{T})$ of continuous functions on the unit circle $\mathbb{T}$ with the subalgebra $\{f \mid f(0) = f(1)\} \subseteq C[0,1]$, it is known that $C(\mathbb{T})$ satisfies paving with respect to D. (Indeed, the algebra of Fourier polynomials—or more generally, the Wiener algebra $A(\mathbb{T})$—is a dense subalgebra of $C(\mathbb{T})$ and lies in the algebra $M_0 \subseteq B(H)$ of operators that are l_1-bounded in the sense of Tanbay [16] with respect to the Fourier basis $\{e_n \mid n \in \mathbb{Z}\}$. Thus $C(\mathbb{T})$ lies in the norm closure M of M_0, and [16] shows that all operators in M can be paved with respect to D.) An argument as in Lemma 1 shows that the completely positive map ψ in the proof of the Theorem is uniquely determined on $C(\mathbb{T})$, and a computation as in Lemma 2 shows that this extension is the state corresponding to the arclength measure on $\mathbb{T}$. The Theorem may now be proved in essentially the same manner, replacing C with $C(\mathbb{T})$.

The proof of the Corollary uses stability of discretizations in the following sense.

Lemma 3. *If $\phi\colon B \to M$ is a discretization, $\alpha\colon A \to B$ is a morphism in* **Cstar**, *and $\beta\colon M \to N$ is a morphism in* **AWstar**, *then $\beta \circ \phi \circ \alpha$ discretizes A.*

Proof. If $C(X) \subseteq A$ is a commutative C*-subalgebra, so is $C(Y) \cong \alpha[C(X)] \subseteq B$, making the top squares of the following diagram commute (where $\widehat{\alpha}\colon Y \to X$ is the continuous function corresponding to α via Gelfand duality).

$$
\begin{array}{ccccccc}
A & \xrightarrow{\;\alpha\;} & B & \xrightarrow{\;\phi\;} & M & \dashrightarrow{\;\beta\;} & N \\
\big\uparrow & & \big\uparrow & & \big\uparrow & & \\
C(X) & \xrightarrow{C(\widehat{\alpha})} & C(Y) & \hookleftarrow{\eta_{C(Y)}} & \ell^\infty(Y) & & \\
& \searrow_{\eta_{C(X)}} & \ell^\infty(X) & \dashrightarrow{\ell^\infty(\widehat{\alpha})} & & &
\end{array}
$$

The bottom triangle commutes by naturality of η. As all dashed arrows are morphisms in **AWstar**, so is their composite. $\qquad\square$

Proof of Corollary. Let $\gamma\colon C(X) \rightarrowtail A$ be the embedding of a commutative C*-subalgebra. The hypotheses ensure that the following diagram commutes, where $F(\gamma)$ is a morphism in **AWstar**, making $\eta_A\colon A \to F(A)$ a discretization.

$$
\begin{array}{ccc}
A & \xrightarrow{\quad\eta_A\quad} & F(A) \\
\gamma \uparrow & & \uparrow F(\gamma) \\
C(X) & \hookrightarrow \ell^\infty(X) & \cong F(C(X))
\end{array}
$$

Since K is infinite-dimensional, it is unitarily isomorphic to $H \otimes K$, so $a \mapsto a \otimes 1$ is a unital $*$-homomorphism $\iota\colon B(H) \to B(H) \otimes B(K) \cong B(K)$. Lemma 3 implies $\eta_A \circ \alpha \circ \iota\colon B(H) \to F(A)$ is a discretization, and the Theorem gives $F(A) = 0$. $\square$

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