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Metric Geometry in a Tame Setting

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Abstract of the Dissertation

Metric Geometry in a Tame Setting

by

Erik Walsberg

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2015

Professor Matthias J. Aschenbrenner, Chair

We prove basic results about the topology and metric geometry of metric spaces which are definable in o-minimal expansions of ordered fields.

The dissertation of Erik Walsberg is approved.

Yiannis N. Moschovakis

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Matthias J. Aschenbrenner, Committee Chair

University of California, Los Angeles

2015

To Sam.

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CHAPTER 1

Introduction

In this chapter we assume that $\mathcal{R}$ is an o-minimal expansion of the real field. By "definable" we mean "$\mathcal{R}$-definable, possibly with parameters from $\mathbb{R}$". Much of our work is done over general o-minimal expansions of ordered fields, we make this assumption now for the sake of simplicity. The aim of this thesis is to develop metric geometry in the o-minimal context. We study definable metric spaces, i.e. definable sets equipped with definable metrics. Let $X \subseteq \mathbb{R}^k$ be a definable set. It is natural to consider X as a metric space by equipping X with the restriction e of the euclidean metric on $\mathbb{R}^k$. The study of such metric spaces (X, e) is in essence the metric geometry of definable sets. There is now an interesting body of research around this topic, see for example [Val08] or [YC04].

The geometry of definable metric spaces is a generalization of the metric geometry of definable sets. It is in fact a substantial generalization as there are definable metric spaces which enjoy important topological and metric properties that definable sets equipped with euclidean metrics simply cannot have. There are metric spaces of great interest to metric geometers whose metric geometry is utterly unlike that of any definable set and which nevertheless are definable in a suitable o-minimal expansion of the real field. By studying the geometry of definable metric spaces we are able to study some geometric properties in the o-minimal context that have not previously been seen in this setting. For example, the behavior of Hausdorff dimension of definable metric spaces is more complicated than of definable sets. The Hausdorff dimension of (X, e) is equal to the o-minimal dimension of X, and is consequently a natural number. Indeed, something like this appears to hold over any first order expansion of the real field which is tame in any model-theoretic sense: Chris Miller and Phillip Hieronymi have shown, in as yet unpublished work, that if a first order expansion of the real field $\mathcal{M}$ does not define the integers then the Hausdorff dimension of any closed $\mathcal{M}$-definable set is a natural number. In contrast, there are even semialgebraic metric spaces with fractional Hausdorff dimension. If $r \in (0,1)$ is rational then $([0,1], e^r)$ is a semialgebraic metric space with Hausdorff dimension $\frac{1}{r}$, see Section 5.3. There are also important examples of definable metric spaces whose large-scale geometry is totally unlike that of any definable set. A metric space is said to be doubling if there is a natural number N such that any open ball of radius $2t$ contains at most N pairwise disjoint open balls of radius t. Euclidean space is doubling, and any subset of a doubling metric space equipped with the induced metric is doubling. In contrast there are important examples of definable metric spaces which are not doubling, such as the hyperbolic plane or symmetric spaces of noncompact type, see Section 5.4.

The central result in this thesis is Theorem 9.0.1:

Theorem. *Let (X, d) be a definable metric space. Exactly one of the following holds:*

i. There is an infinite definable $A \subseteq X$ such that (A, d) is discrete.

ii. There is a definable $Z \subseteq \mathbb{R}^k$ and a definable homeomorphism $(X, d) \to (Z, e)$.

This implies that any separable metric space which is definable in an o-minimal expansion of the real field is definably homeomorphic to a definable set equipped with its euclidean topology. In Section 5.8 we give examples of definable metric spaces which are not homeomorphic to any definable set. For instance, if G is a finitely generated subgroup of $Gl_n(\mathbb{R})$ there is a semilinear metric space which is homeomorphic to the disjoint union of continuum many copies of the geometric realization of the Cayley graph of G. At present we cannot classify definable metric spaces up to homeomorphism, see Problem 13.0.1.

In Chapter 10 we turn to the metric geometry of definable metric spaces. Theorem 10.2.9 states the following in slightly different language:

Theorem. *Suppose that $\mathcal{R}$ is polynomially bounded. Let (X, d) be a definable metric space. Exactly one of the following holds:*

i. There is an infinite definable $A \subseteq X$ such that (A, d) is discrete.

ii. There is a one-dimensional definable $A \subseteq X$ such that (A, d) is definably bilipschitz equivalent to a metric space of the form $([0,1], e^r)$ for an element $r \in (0,1)$ of the field of powers of $\mathcal{R}$.

iii. There is a partition $\{X_1, \ldots, X_n\}$ of X into definable sets such that id : $(X_i, d) \to (X_i, e)$ is locally bilipschitz for all $1 \leqslant i \leqslant n$.

Metric spaces of the form $([0,1], e^r)$ for $r \in (0,1)$ are viewed as generalized von Koch snowflakes, see Section 5.3. Our approach to this result and others is based on studying the metric geometry of (X, d) on a small neighborhood of a generic point of X, that is we investigate properties which hold locally around almost every point. In Chapter 10 we also study definable metric spaces on which the Hausdorff dimension agrees with the topological dimension. As a special case of Theorem 10.3.12 we have:

Theorem. *Suppose that $\mathcal{R}$ is polynomially bounded. Let (X, d) be a definable metric space such that the Hausdorff dimension of (X, d) agrees with the topological dimension of (X, d). Then almost every $p \in X$ has a d-open neighborhood U such that (U, d) is definably bilipschitz equivalent to an open subset of $(\mathbb{R}^l, e)$ where $l = \dim(X)$.*

Theorem 10.3.12 in its full generality is stated in terms of a refinement of the Hausdorff dimension which we introduce in Section 10.3.

In the last two chapters we prove results about $\bar{\mathbb{R}}_{\mathrm{an}}$- and $\bar{\mathbb{R}}$-definable metric spaces. In Chapter 11 we apply a theorem of Georges Comte, Jean-Marie Lion and Jean-Philippe Rolin [CR00]. This theorem allows us to bound the volume of a ball in terms of its radius, such bounds are very useful in metric geometry. For example, in Section 11.3 we show:

Proposition. *The Hausdorff dimension of an $\bar{\mathbb{R}}_{\mathrm{an}}$-definable metric space is either infinite or rational.*

In forthcoming joint work with Jana Maříková we generalize this proposition to polynomially bounded structures. Our proof of the following theorem does not use the Comte-Lion-Rolin theorem, it relies instead on the work of Jana Maříková and Masahiro Shiota on measure theory over o-minimal expansions of nonarchimedean ordered fields [MS].

Theorem. *Suppose that $\mathcal{R}$ is polynomially bounded. Then the Hausdorff dimension of a definable metric space is either infinite or an element of the field of powers of $\mathcal{R}$.*

In Chapter 12 we study Gromov-Hausdorff limits and ultralimits of sequences of elements of a semialgebraic family of proper metric spaces. This topic is naturally connected to the theory of T-convex expansions of o-minimal structures. This is because the Gromov-Hausdorff limits and ultralimits of a definable family of metric spaces are definable in suitable T-convex structures, see Proposition 8.0.1. In Proposition 12.1.1 we use the model theory of real closed valued fields to prove the following:

Proposition. *A proper metric space which is a Gromov-Hausdorff limit of a sequence of elements of a semialgebraic family of proper metric spaces is isometric to a semialgebraic metric space.*

Proposition 12.1.1, along with the other results in Chapter 12, is a consequence of a model-theoretic fact about definable sets of imaginaries in real closed valued fields. We give this fact in Proposition 4.3.1. In model-theoretic terminology Proposition 4.3.1 states that a definable set of imaginaries in a real closed valued field is either internal to the residue field or nonorthogonal to the value group. Proposition 4.3.1 follows from the elimination of imaginaries for real closed valued fields due to Timothy Mellor [Mel06]. All of this is based on the groundbreaking work of Deirdre Haskell, Ehud Hrushovski and Dugald Macpherson on imaginaries in algebraically closed valued fields [HHM06]. The material in Chapter 12 is inspired by earlier work on Hausdorff limits of sequences of elements of definable families of sets, which we explain below.

Let $\mathcal{A} = \{A_x : x \in \mathbb{R}^l\}$ be a semialgebraic family of bounded subsets of $\mathbb{R}^k$. We say that a closed subset A of $\mathbb{R}^k$ is a Hausdorff limit of $\mathcal{A}$ if there is a sequence of elements of $\mathcal{A}$ which converges in the Hausdorff metric to A. Let $(R, +, \times, \leqslant)$ be an ω-saturated real closed ordered field and let $\mathcal{O}$ be the convex hull of the integers in R. Then $(R, \mathcal{O})$ is a real closed valued field. The residue field of $(R, \mathcal{O})$ is canonically identified with $\mathbb{R}$. We take $(R, \mathcal{O})$ to be a two sorted structure with a sort for R and a sort for the residue field $\mathbb{R}$. Let $\mathrm{st} : \mathcal{O} \to \mathbb{R}$ be the residue map, i.e. the usual standard part map, and let $\mathrm{st} : \mathcal{O}^k \to \mathbb{R}^k$ be the map given by applying the standard part map coordinate-wise. Gregory Cherlin and Max Dickmann [CD83] showed that real closed valued fields admit quantifier elimination. Ludwig Bröcker [Bro94] used this quantifier elimination to prove that every $(R, \mathcal{O})$-definable subset of $\mathbb{R}^k$ is semialgebraic. This is also a consequence of a result of Françoise Delon, Corollary 2.23 of [Del82]. In model-theoretic terminology, $\mathbb{R}$ is *stably embedded* in $(R, \mathcal{O})$. In particular if $A \subseteq \mathcal{O}^k$ is $(R, \mathcal{O})$-definable then $\mathrm{st}(A)$ is semialgebraic. Bröcker [Bro92] used this model-theoretic fact to prove that a Hausdorff limit of a semialgebraic family of bounded sets is semialgebraic. One can show that the Hausdorff limits of $\mathcal{A}$ are exactly those sets of the form $\mathrm{st}(A^*_x)$ for elements $x \in R^l$, this is the o-minimal version of a nonstandard construction of Hausdorff limits which goes back to Abraham Robinson. Stable embeddedness of the residue field then implies that the Hausdorff limits of $\mathcal{A}$ are semialgebraic.

Our proof of Proposition 12.1.1 follows the same course: we use a nonstandard construction to show that the Gromov-Hausdorff limit is definable in $(R, \mathcal{O})$ and then apply a fact from the model theory of valued fields, Proposition 4.3.1, to show that the Gromov-Hausdorff limit is internal to the residue field. An application of stable embeddedness shows that the Gromov-Hausdorff limit is semialgebraic. It is possible to prove definability of Hausdorff limits without using model theory: Jean-Marie Lion and Patrick Speissegger gave a proof [LS04] and another proof has recently been given by Beata Kocel-Cynk, Wiesław Pawłucki, and Anna Valette [KPV14]. It is probably possible to give a purely o-minimal proof of the definability of Gromov-Hausdorff limits as well. Our aim is not to prove results in o-minimal geometry which cannot be proven by o-minimal means, but to find connections between o-minimal geometry over the real field and the model theory of valued fields.

CHAPTER 2

Conventions

Let $(R, +, \leqslant)$ be a totally ordered abelian group. We let $R^{>} = \{x \in R : x > 0\}$ and $R^{\geqslant} = \{x \in R : x \geqslant 0\}$, where 0 is the additive unit of R. Given $a, b \in R$ we say that a property holds for all $a < t \ll b$ if there is a $a < c < b$ such that the property holds for all $a < t < c$. We let $R_\infty = R \cup \{\infty\}$ where ∞ is a formal element which we declare to be larger then every element of R. We extend addition to R_∞ by declaring $a + \infty = \infty$ for any $a \in R_\infty$.

Let $(R, +, \times, \leqslant)$ be a real closed ordered field. By the **euclidean metric** on R^k we mean the function $e : R^k \times R^k \to R$ given by

$$e(\bar{x}, \bar{y}) = \sqrt{\sum_{i=1}^{k} [x_i - y_i]^2} \quad \text{for all } \bar{x}, \bar{y} \in R^k.$$

We use the notation e for the euclidean metric. We also use e to denote the restriction of the euclidean metric on R^k to a subset $A \subseteq R^k$. We write

$$\|\bar{x}\| = \sqrt{\sum_{i=1}^{k} x_i^2} \quad \text{for the norm of } \bar{x} \in R^k.$$

So $\|\bar{x} - \bar{y}\| = e(\bar{x}, \bar{y})$. The **euclidean topology** on a subset $A \subseteq R^k$ is the topology induced by e.

Given a set X and functions $f, g : X \to R$ we define:

$$\|f - g\|_\infty = \sup\{|f(x) - g(x)| : x \in X\}$$

if this supremum exists.

Throughout this thesis, $\mathcal{R}$ is an o-minimal expansion of a real closed ordered field $(R, +, \times, \leqslant)$. We assume that the reader is familiar with basic first-order model theory and o-minimality. We invoke the main results of [Dri98] freely. Throughout this thesis, by "definable" without modification we mean "$\mathcal{R}$-definable, possibly with parameters from R". When we write "definable", possibly for some structure other then $\mathcal{R}$, we always include the possibility of parameters.

Let $X \subseteq R^k$ be a definable set. A **path** in X is a definable function $\gamma : (0, \delta) \to X$ for some $\delta \in R^>_\infty$. We are almost always only interested in the behavior of a path γ for small values of $t \in (0, \delta)$. Any path $\gamma : (0, \delta) \to X$ can be extended to a path $R^> \to X$ by declaring $\gamma(s)$ equal to some fixed $q \in X$ for $s \geqslant \delta$, so we will sometimes assume that all paths have domain $R^>$.

Throughout, $\dim(X)$ is the o-minimal dimension of the definable set X. We say that a definable $A \subseteq X$ is **almost all** of X if $\dim(X \setminus A) < \dim(X)$. We say that a property holds for almost all points in X if the property holds on some definable subset of X which is almost all of X. A property holds **almost everywhere** on R^k if and only if it holds on an open dense subset of R^k.

We commonly have an elementary extension of $\mathcal{R}$, which we frequently call $\mathcal{R}^*$. Given an $\mathcal{R}$-definable $A \subseteq R^k$ we let A^* be the set defined over $\mathcal{R}^*$ by the same formula defining A. Given an $\mathcal{R}$-definable function f we let f^* be the $\mathcal{R}^*$-definable function defined by the same formula defining f. This does not depend on the choice of formulas defining A and f. We sometimes use this notation when our elementary extension is called something other then $\mathcal{R}^*$.

We put an equivalence relation on the set of definable functions $R^> \to R$ by declaring $f \sim g$ if and only if there is a $K \in R^>$ such that:

$$\frac{1}{K} f(t) \leqslant g(t) \leqslant K f(t) \quad \text{for all } 0 < t \ll 1.$$

We let $\operatorname{val}(f)$ be the equivalence class of a definable $f : R^> \to R$. We declare $\operatorname{val}(f) \leqslant \operatorname{val}(g)$ if and only if there is a $K \in R^>$ such that:

$$g(t) \leqslant K f(t) \quad \text{for all } 0 < t \ll 1.$$

This gives a total order on the set of equivalence classes. We will also sometime use val to denote a valuation on an ordered field. Throughout, the **archimedean valuation** on R is the valuation whose valuation ring is the convex hull of the integers in R.

Throughout this thesis, $\bar{\mathbb{R}}$ is the ordered field of real numbers $(\mathbb{R}, +, \times, \leqslant)$. We will sometimes call $\bar{\mathbb{R}}$ the **real field**. By **"semialgebraic"** we mean "$\bar{\mathbb{R}}$-definable". Let $\mathcal{V}$ be the real field considered as an ordered vector space over itself. Throughout, by **"semilinear"** we mean "$\mathcal{V}$-definable". We let $\bar{\mathbb{R}}_{\mathrm{exp}}$ be the real exponential field $(\mathbb{R}, +, \times, \leqslant, \exp)$. The real analytic field $\bar{\mathbb{R}}_{\mathrm{an}}$ is the expansion of $\bar{\mathbb{R}}$ by all functions of the form:

$$f(x) = \begin{cases} g(x) & \text{if } x \in [0,1]^k \\ 0 & \text{if } x \notin [0,1]^k \end{cases}$$

for real analytic functions $g : U \to \mathbb{R}$ where U is an open set containing $[0,1]^k$. Finally, $\bar{\mathbb{R}}_{\mathrm{an,exp}}$ is the expansion of $\bar{\mathbb{R}}_{\mathrm{an}}$ by the exponential function.

Let X be a topological space and let $A \subseteq X$. We denote the closure of A by $\mathrm{cl}(A)$ and the interior of A by $\mathrm{Int}(A)$. We denote the **frontier** of A by $\partial(A) = \mathrm{cl}(A) \setminus A$.

Let X be a set. Then $|X|$ is the cardinality of X and $|X|^+$ is the least cardinal strictly greater then $|X|$.

Given a function $f : A \to B$ between sets A, B we let $\mathrm{Graph}(f)$ be the **graph** of f, i.e. the set of $(a, b) \in A \times B$ such that $b = f(a)$.

CHAPTER 3

Metric Geometry

In this chapter we collect the metric geometry that we will need in this thesis. There are many references for this material, such as [BBI01]. *Throughout this chapter* $(\Lambda, +, \leqslant)$ *is a divisible ordered abelian group.*

3.1 Metric Spaces

A **Λ-valued metric space** is a set X together with a **Λ-valued metric**, that is, a function $d : X^2 \to \Lambda^{\geqslant}$ satisfying the following for all $x, y, z \in X$:

i. $d(x, y) = 0$ if and only if $x = y$.

ii. Symmetry: $d(x, y) = d(y, x)$.

iii. The triangle inequality: $d(x, z) \leqslant d(x, y) + d(y, z)$.

If (X, d) is a Λ-valued metric space and $A \subseteq X$ then the restriction of d to $A \times A$ gives a metric on A. We will also call this metric d and call the resulting metric space (A, d).

If d satisfies (ii) and (iii) but not necessarily (i) then d is a **Λ-valued pseudometric** on X and (X, d) is a **Λ-valued pseudometric space**. We canonically associate a metric space to a pseudometric (X, d) as follows: we put an equivalence relation $\sim$ on X by declaring $x \sim y$ if and only if $d(x, y) = 0$. Let $\pi : X \to X/\sim$ be the quotient map. We let

$$d'(\pi(x), \pi(y)) = d(x, y) \quad \text{for all } x, y \in X.$$

Then d' is easily seen to be a metric on $X/\sim$.

If d satisfies $d(x,z) \leqslant \max\{d(x,y), d(y,z)\}$ for all $x, y, z \in X$ then d is a **Λ-valued ultrametric.** We say that a Λ-valued metric space (X, d) is an **approximate ultrametric space** if there is a $\delta \in \Lambda^{>}$ such that $d(x,z) \leqslant \max\{d(x,y), d(y,z)\}+\delta$ holds for all $x, y, z \in X$.

Let (X, d) be a Λ-valued metric space and let $f : X \to Y$ be a bijection. The **push-forward** of d by f is the Λ-valued metric d' on Y satisfying $d'(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

A **pointed metric space** (X, d, p) is a Λ-valued metric space together with a distinguished basepoint $p \in X$.

If (X, d) is a Λ-valued metric space and $A \subseteq X$ is nonempty then we define the **diameter** of A to be

$$\mathrm{Diam}_d(A) = \sup\{d(x, y) : x, y \in A\}$$

when this supremum exists. We say that $A \subseteq X$ is **bounded** if there is a $t \in \Lambda^{>}$ such that $d(x, y) < t$ for all $x, y \in A$.

Given $p \in X$ and $r \in \Lambda^{>}$ we let $B_d(p, r) = \{y \in X : d(p, y) < r\}$ be the open d-ball with center p and radius r. We drop the d when it is clear from context. When we write "ball" without modification we mean "open ball". A **δ-ball** is an open ball with radius δ.

An **$\mathbb{R}$-valued metric space** is an $(\mathbb{R}, +, \leqslant)$-valued metric space. A metric space in the usual sense is an $\mathbb{R}$-valued metric space. We say that an $\mathbb{R}$-valued metric space (X, d) is **proper** if every closed and bounded subset of X is compact and we say that (X, d) is **preproper** if its completion is proper. We say that a $\mathbb{R}$-valued metric space (X, d) is **precompact** if its completion is compact, equivalently if for every $\delta \in \mathbb{R}^{>}$ there are finitely many δ-balls which cover X.

3.2 Maps Between Metric Spaces

Let (X, d) and (X', d') be pseudometric spaces. *Throughout this section x, y range over X and x', y' range over X'.* A subset $A \subseteq X$ is **ϵ-dense** if for every x there is an $a \in A$ such that $d(x, a) < \epsilon$. Given a subset $A \subseteq X$ a function $f : A \to X'$ is an **ϵ-approximate**

isometry from (X, d) to (X', d') if

i. A is ϵ-dense in X,

ii. $f(A)$ is ϵ-dense in X',

iii. $d(x, y) - \epsilon \leqslant d'(f(x), f(y)) \leqslant d(x, y) + \epsilon$ for all x, y.

We say that f as above is an **isometry** from (X, d) to (X', d') if it is a 0-approximate isometry. Note that we do not require an isometry of pseudometric spaces to be bijective or totally defined, but this follows when (X, d) and (X', d') are metric spaces. *For the remainder of this section (X, d) and (X', d') are Λ-valued metric spaces and $f : X \to X'$ is a map.*

We say that f is **uniformly continuous** if for some function $g : \Lambda^{>} \to \Lambda^{>}$ such that $g(t) \to 0$ as $t \to 0^{+}$ we have:

$$d'(f(x), f(y)) \leqslant g(d(x, y)) \qquad \text{for all } x, y.$$

The map f is a **uniform equivalence** if it is surjective and it satisfies

$$g_1(d(x, y)) \leqslant d'(f(x), f(y)) \leqslant g_1(d(x, y)) \quad \text{for all } x, y.$$

for two functions $g_1, g_2 : \Lambda^{>} \to \Lambda^{>}$ such that $g_1(t), g_2(t) \to 0$ as $t \to 0^{+}$. Uniform equivalences are homeomorphisms. We now suppose that Λ is an ordered $\mathbf{k}$-vector space for some ordered field $\mathbf{k}$. Given $\lambda \in \mathbf{k}^{>}$ we say that f is **λ-Lipschitz** if

$$d'(f(x), f(y)) \leqslant \lambda d(x, y) \qquad \text{for all } x, y.$$

We say that f is **Lipschitz** if it is λ-Lipschitz for some $\lambda \in \mathbf{k}^{>}$. Note that f is 1-Lipschitz if and only if $d(f(x), f(y)) \leqslant d(x, y)$ holds for all x, y. The map f is **λ-bilipschitz** if it satisfies

$$\frac{1}{\lambda} d(x, y) \leqslant d'(f(x), f(y)) \leqslant \lambda d(x, y) \qquad \text{for all } x, y$$

and f is **bilipschitz** if it is λ-bilipschitz for some $\lambda \in \mathbf{k}^{>}$. Note that bilipschitz maps are injective. We say that f is a **bilipschitz equivalence** if it is bilipschitz and surjective.

The metric spaces (X, d) and (X', d') are **bilipschitz equivalent** if there is a bilipschitz equivalence $(X, d) \to (X', d')$. We say that f is **locally Lipschitz** if every $p \in X$ has a neighborhood U such that the restriction of f to (U, d) is λ-Lipschitz for some $\lambda \in \mathbf{k}^>$, and we say that f is **locally bilipschitz** if every $p \in X$ has a neighborhood U such that the restriction of f is λ-bilipschitz for some $\lambda \in \mathbf{k}^>$.

For $\lambda \in \mathbf{k}^{>1}$ and $\beta \in \Lambda^>$, we say that f a (λ, β)-**quasi-isometry** if the image of f is β-dense in X' and

$$\frac{1}{\lambda} d(x, y) - \beta \leqslant d'(f(x), f(y)) \leqslant \lambda d(x, y) + \beta \qquad \text{for all } x, y.$$

We say that f is a **quasi-isometry** if it is a (λ, β)-quasi-isometry for some $\lambda \in \mathbf{k}^>, \beta \in \Lambda^>$. If there is a quasi-isometry $(X, d) \to (X', d')$ then we say that (X, d) and (X', d') are **quasi-isometric**.

3.3 Covers and Packing Inequalities

Let (X, d) be a Λ-valued metric space. For $\epsilon \in \Lambda^>$ we say that a set $B \subseteq X$ is ϵ-**separated** if $d(x, y) > \epsilon$ holds for every distinct $x, y \in B$. Given $A \subseteq X$ and $\epsilon \in \Lambda^>$ we define $N_d(A, \epsilon)$ to be the maximal cardinality of an ϵ-separated subset of A. For $t, \epsilon \in \Lambda^>$ we define $P_d(t, \epsilon)$ to be the maximal cardinality of a collection of pairwise disjoint balls of radius at least ϵ contained in a ball of radius t. We drop the d in the subscript when the metric is clear from context.

Lemma 3.3.1. *Let $p \in X$, $s, t \in \Lambda^>$. Then*

$$N(B(p, t), s) \leqslant P\left(t + \frac{s}{2}, \frac{s}{2}\right).$$

Proof. Let $\{x_i : i \in I\}$ be an s-separated subset of $B(p, t)$. The triangle inequality implies that $B(x_i, \frac{1}{2}s) \cap B(x_j, \frac{1}{2}s) = \emptyset$ when $i \neq j$. The triangle inequality also implies that $B(x_i, \frac{1}{2}s) \subseteq B(p, t + \frac{1}{2}s)$ holds for each $i \in I$. Thus $P(t + \frac{1}{2}s, \frac{1}{2}s) \geqslant |I|$. $\square$

3.3.1 The $5r$-covering Lemma

In this section we prove a basic proposition about covers of metric spaces which we will use to prove Fact 3.4.2. Our proof is taken from [Hei01].

Proposition 3.3.2. *Let (X, d) be a Λ-valued metric space and let $A \subseteq X$. Fix $\delta \in \Lambda^{>}$. There is a family $\{B(p_i, t_i) : i \in I\}$ of balls which covers A, satisfies $p_i \in A$ and $t_i \leqslant \delta$ for all $i \in I$ and satisfies*

$$B\left(p_i, \frac{1}{5}t_i\right) \cap B\left(p_j, \frac{1}{5}t_j\right) = \emptyset \qquad \text{when } i \neq j.$$

In the following proof we say that a family of sets is disjoint if and only if any two distinct elements of the family have empty intersection.

Proof. Let $\mathcal{B}$ be a family of balls of radius at most δ with centers in A which covers A. Let Ω be the collection of disjoint subfamilies $\mathcal{B}'$ of $\mathcal{B}$ such that if $B \in \mathcal{B}$ intersects a ball in $\mathcal{B}'$ then it intersects a ball in $\mathcal{B}'$ whose radius is at least half that of B. It is easy to see that Ω is closed under unions of chains. We apply Zorn to get a maximal element $\mathcal{C}$ of Ω. We show that every element of $\mathcal{B}$ intersects an element of $\mathcal{C}$. Towards a contradiction we suppose that $B \in \mathcal{B}$ does not intersect any element of $\mathcal{C}$ and that the radius of B is at least half the radius of any element of $\mathcal{B}$ which does not intersect any element of $\mathcal{C}$. Thus, if a ball $B' \in \mathcal{B}$ intersects a ball in $\mathcal{C} \cup \{B\}$ then it intersects a ball whose radius is at least half that of B'. Then $\mathcal{C} \cup \{B\} \in \Omega$, which contradicts maximality of Ω. Therefore every element of $\mathcal{B}$ intersects an element of $\mathcal{C}$. Let $\mathcal{C} = \{B(p_i, t_i) : i \in I\}$. We prove the proposition by showing that

$$A \subseteq \bigcup_{i \in I} B\left(p_i, 5t_i\right).$$

Fix $x \in A$. Let $B(q, r) \in \mathcal{B}$ be such that $x \in B(q, r)$. Let $B(p_i, t_i)$ be an element of $\mathcal{C}$ which intersects B and such that $t_i \geqslant \frac{1}{2}r$. Fix $z \in B(p_i, t_i) \cap B(q, r)$. Then

$$d(x, p_i) \leqslant d(x, q) + d(q, z) + d(z, p_i) < r + r + t_i \leqslant 5t_i$$

as required. $\square$

3.3.2 Doubling Metrics

In this subsection (X, d) *is an* $\mathbb{R}$*-valued metric space.* We say that (X, d) is K**-doubling** for $K \in \mathbb{N}$ if $P(2t, t) \leqslant K$ for all $t \in \mathbb{R}^{>}$. We say that (X, d) is **doubling** if it is K-doubling for some $K \in \mathbb{N}$. We omit the easy proof of the following fact:

Fact 3.3.3. *Suppose that* (X, d) *is doubling. There are* $C, r \in \mathbb{R}^{>}$ *such that*

$$P(t', t) \leqslant C\left(\frac{t'}{t}\right)^{r} \qquad \text{for all } t < t'.$$

Applying Lemma 3.3.1 we also have:

Fact 3.3.4. *If* (X, d) *is doubling then there are* $C, r \in \mathbb{R}^{>}$ *such that for any* $s, t \in \mathbb{R}^{>}$ *and* $p \in X$ *we have*

$$N(B(p, t), s) \leqslant C\left(\frac{t}{s}\right)^{r}.$$

Let μ be a finitely additive measure which is defined on the boolean algebra of subsets of X generated by the d-balls. We say that μ is K**-doubling** for $K \in \mathbb{N}$ if μ assigns every ball positive measure and if:

$$\mu[B(p, 2t)] \leqslant K\mu[B(p, t)] \qquad \text{for all } p \in X \text{ and } t \in \mathbb{R}^{>}.$$

We say that μ is **doubling** if it is K-doubling for some $K \in \mathbb{N}$. If μ is doubling then for every $n \in \mathbb{N}$ there is a $K \in \mathbb{N}$ such that $\mu[B(p, nt)] \leqslant K\mu[B(p, t)]$ holds for every p, t.

Lemma 3.3.5. *If* μ *is doubling then* (X, d) *is doubling.*

Proof. Suppose that μ is doubling and let $N \in \mathbb{N}$ be such that $\mu[B(p, 3t)] \leqslant N\mu[B(p, t)]$ for all p, t. Fix $p \in X$ and $t \in \mathbb{R}^{>}$. Suppose that $B(q_1, t), \dots, B(q_n, t)$ are pairwise disjoint balls such that $B(q_i, t) \subseteq B(p, 2t)$ for all $1 \leqslant i \leqslant n$. There is a $1 \leqslant j \leqslant n$ such that

$$\mu[B(q_j, t)] \leqslant \frac{1}{n}\mu[B(p, 2t)].$$

Fix such a j. The triangle inequality implies $B(p, 2t) \subseteq B(q_j, 3t)$. Thus $\mu[B(q_j, 3t)] \geqslant n\mu[B(q_j, t)]$ which implies that $n \leqslant N$. This proves the proposition. □

3.4 Hausdorff Measures and Dimension

Throughout this section (X, d) is an $\mathbb{R}$-valued metric space. In this section we only consider $\mathbb{R}$-valued metric spaces. Our proofs are adapted from [Hei01] and [Mat95].

3.4.1 Hausdorff Measures

A **dimension function** is a continuous, strictly increasing function $\mathbf{f} : \mathbb{R}^{\geqslant} \to \mathbb{R}^{\geqslant}$ such that $\mathbf{f}(0) = 0$. We define the **f-dimensional Hausdorff measure** $\mathcal{H}^{\mathbf{f}}$ on X. If $\mathbf{f}(t) = t^r$ then $\mathcal{H}^{\mathbf{f}}$ is the usual r-dimensional Hausdorff measure which we call $\mathcal{H}^r$. Given $\delta > 0$ and $A \subseteq X$ we let

$$\mathcal{H}^{\mathbf{f}}_{\delta}(A) = \inf \sum_{B \in \mathcal{B}} \mathbf{f}[\mathrm{Diam}_d(B)] \in \mathbb{R}_{\infty}.$$

where the infimum is taken over all countable collections of closed balls $\mathcal{B}$ which cover A and have diameter at most δ. We define

$$\mathcal{H}^{\mathbf{f}}(A) = \lim_{\delta \to 0^+} \mathcal{H}^{\mathbf{f}}_{\delta}(A).$$

The limit exists as $\mathcal{H}^{\mathbf{f}}_{\delta}(A)$ decreases with δ. It is a classical fact that $\mathcal{H}^{\mathbf{f}}$ gives a Borel measure on X.

Fact 3.4.1. *Let $\mathbf{f}, \mathbf{g}$ be dimension functions and $\lambda \in \mathbb{R}^{>}$.*

i. If $\mathbf{f}(t) \leqslant \lambda \mathbf{g}(t)$ when $0 < t \ll 1$ then $\mathcal{H}^{\mathbf{f}}(A) \leqslant \lambda \mathcal{H}^{\mathbf{g}}(A)$ holds for any $A \subseteq X$.

ii. Suppose that $\lim_{t \to 0^+} \frac{\mathbf{f}(t)}{\mathbf{g}(t)} = 0$. Then $\mathcal{H}^{\mathbf{f}}(A) = 0$ holds whenever $\mathcal{H}^{\mathbf{g}}(A) < \infty$ and $\mathcal{H}^{\mathbf{g}}(A) = \infty$ holds whenever $\mathcal{H}^{\mathbf{f}}(A) > 0$.

In particular if $r, s \in \mathbb{R}^{>}$ and $s < r$, then $\mathcal{H}^r(A) = 0$ whenever $\mathcal{H}^s(A) < \infty$ and $\mathcal{H}^s(A) = \infty$ whenever $\mathcal{H}^s(A) > 0$.

Proof. (i) follows immediately from the definition of $\mathcal{H}^{\mathbf{g}}$. (ii) follows immediately from (i). ☐

We say that a dimension function $\mathbf{g} : \mathbb{R}^{\geqslant} \to \mathbb{R}^{\geqslant}$ is **K-doubling** if there is a $K > 0$ such that $\mathbf{g}(2t) \leqslant K\mathbf{g}(t)$ when $0 < t \ll 1$. We say that $\mathbf{g}$ is **doubling** if it is K-doubling for some $K \in \mathbb{R}^{>}$. Note that if $\mathbf{g}$ is doubling then for any $k > 0$ there is a $K > 0$ such that $\mathbf{g}(kt) \leqslant K\mathbf{g}(t)$ when $0 < t \ll 1$. Any power function is doubling. The next fact is called the Mass Distribution Principle:

Fact 3.4.2. *Suppose (X, d) is separable and let μ be a finite Borel measure on X. Let $\mathbf{f}$ be a K-doubling dimension function. If there are $\lambda, \delta > 0$ such that*

$$\mu[B(p, t)] \geqslant \lambda\mathbf{f}(t) \quad \textit{holds for all } p \in A, t < \delta,$$

then $\mathcal{H}^{\mathbf{f}}(A) < \infty$ for all Borel $A \subseteq X$. If there are $\lambda, \delta > 0$ such that

$$\mu[B(p, t)] \leqslant \lambda\mathbf{f}(t) \quad \textit{holds for all } p \in A, t < \delta,$$

then $\mathcal{H}^{\mathbf{f}}(A) \geqslant \lambda^{-1}\mu(A)$ for all Borel $A \subseteq X$.

Proof. Fix $\delta, \lambda \in \mathbb{R}^{>}$ as in the hypothesis of the first claim. After decreasing δ if necessary we suppose that $\mathbf{f}(2t) \leqslant K\mathbf{f}(t)$ when $0 < t < \delta$. Applying Proposition 3.3.2 we let $\{B(p_i, t_i) : i \in \mathbb{N}\}$ be a family of open balls which covers A, and such that $p_i \in A$, $t_i < \delta$ and

$$B\left(p_i, \frac{1}{5}t_i\right) \cap B\left(p_j, \frac{1}{5}t_j\right) \qquad \text{when } i \neq j.$$

We have

$$\mathcal{H}^{\mathbf{f}}_{\delta}(A) \leqslant \sum_{i=0}^{\infty} \mathbf{f}[\operatorname{Diam} B(p, t_i)] \leqslant \sum_{i=0}^{\infty} \mathbf{f}[2t_i] \leqslant K^5 \sum_{i=0}^{\infty} \mathbf{f}\left(\frac{1}{5}t_i\right)$$

and

$$\sum_{i=0}^{\infty} \mathbf{f}\left(\frac{1}{5}t_i\right) \leqslant \frac{1}{\lambda}\sum_{i=0}^{\infty} \mu\left[B\left(p_i, \frac{1}{5}t_i\right)\right] \leqslant \frac{1}{\lambda}\mu(X).$$

As μ is finite we have $\mathcal{H}^{\mathbf{f}}(A) < \infty$.

We now prove the second claim. Let $\lambda, \delta \in \mathbb{R}^{>}$ be as in the hypothesis of the second claim. Fix $\epsilon > 0$. Let $\{B(p_i, t_i) : i \in \mathbb{N}\}$ be a family of closed balls which covers A with $t_i < \delta$ and satisfies

$$\sum_{i=0}^{\infty} \mathbf{f}[\operatorname{Diam} B(p_i, t_i)] \leqslant \mathcal{H}^{\mathbf{f}}(A) + \epsilon.$$

Without loss of generality we suppose that each element of the family has nonempty intersection with A. For each i let q_i be an element of $A \cap B(p_i, t_i)$ and let $r_i = \operatorname{Diam} B(p_i, t_i)$. The collection $\{B(p_i, r_i) : i \in \mathbb{N}\}$ covers A. We have

$$\mu(A) \leqslant \sum_{i=0}^{\infty} \mu[B(q_i, r_i)] \leqslant \lambda \sum_{i=0}^{\infty} \mathbf{f}(r_i) \leqslant \lambda[\mathcal{H}^{\mathbf{f}}(A) + \epsilon].$$

As this holds for every $\epsilon > 0$ we have $\lambda^{-1}\mu(A) \leqslant \mathcal{H}^{\mathbf{f}}(A)$. □

Fact 3.4.3. *Let* $\mathbf{f}$ *be a doubling dimension function. If there exists a Lipschitz surjection* $(X, d) \to (X', d')$ *then* $\mathcal{H}^{\mathbf{f}}(X, d) = 0$ *implies* $\mathcal{H}^{\mathbf{f}}(X', d') = 0$.

Proof. Let $\delta, K > 0$ be such that $\mathbf{f}(2t) \leqslant K\mathbf{f}(t)$ when $0 < t < \delta$. We suppose that $\mathcal{H}^{\mathbf{f}}(X, d) = 0$ and let $h : (X, d) \to (X', d')$ be a λ-Lipschitz surjection. Let $\epsilon \in \mathbb{R}^{>}$. Let $\{B(p, r_i) : i \in \mathbb{N}\}$ be a collection of balls with radius at most δ which covers X and satisfies

$$\sum_{i=0}^{\infty} \mathbf{f}[\operatorname{Diam} B(p_i, r_i)] < \epsilon.$$

For $i \in \mathbb{N}$ let $s_i = \operatorname{Diam} B(p_i, r_i)$. Consider the collection $\{B(f(p_i), \lambda s_i) : i \in \mathbb{N}\}$. It is easy to check that this collection covers X'. As $\operatorname{Diam} B(q_i, \lambda s_i) \leqslant 2\lambda s_i$,

$$\mathcal{H}^{\mathbf{f}}(X') \leqslant \sum_{i=0}^{\infty} \mathbf{f}(2\lambda s_i) \leqslant K \sum_{i=0}^{\infty} \mathbf{f}(s_i) \leqslant K\epsilon.$$

This inequality holds for every $\epsilon > 0$, so we have $\mathcal{H}^{\mathbf{f}}(X') = 0$. □

3.4.2 Hausdorff Dimension

The **Hausdorff dimension** of (X, d) is

$$\dim_{\mathcal{H}}(X, d) := \sup\{r \in \mathbb{R}^{>} : \mathcal{H}^{r}(X, d) = \infty\} \in \mathbb{R}_{\infty}.$$

Lemma 3.4.4. *If* $\dim_{\mathcal{H}}(X, d) < \infty$ *then* (X, d) *is separable.*

Proof. Suppose that $\dim_{\mathcal{H}}(X, d) < \infty$. Then for every $\delta > 0$, (X, d) admits a countable covering by balls of diameter at most δ. This implies that (X, d) is separable. □

We leave the proof of the following fact to the reader:

Fact 3.4.5. *Let $\{A_i : i \in \mathbb{N}\}$ be a countable collection of Borel sets which covers X. Then*

$$\dim_{\mathcal{H}}(X, d) = \sup\{\dim_{\mathcal{H}}(A_i, d) : i \in \mathbb{N}\}.$$

We now give a pointwise version of the Mass Distribution Principle.

Proposition 3.4.6. *Suppose (X, d) is separable. Let μ be a finite Borel measure on X. Let $A \subseteq X$ be Borel with $\mu(A) > 0$. Let $r > 0$. Suppose that for every $p \in A$*

$$\theta_r(p) := \lim_{t \to 0^+} \frac{\mu[B(p, t)]}{t^r} \in \mathbb{R}_\infty$$

exists. If $\theta_r(p) > 0$ at every $p \in A$ then $\dim_{\mathcal{H}}(A, d) \geqslant r$. If $\theta_r(p) < \infty$ at every $p \in A$ then $\dim_{\mathcal{H}}(A, d) \leqslant r$.

Proof. Suppose that $\theta_r > 0$. For $m, n \in \mathbb{N}$ let

$$C_{m,n} = \left\{ p \in A : \left[\forall 0 < t < \frac{1}{m} \right] \mu[B(p, t)] \geqslant \frac{1}{n} t^r \right\}.$$

Fact 3.4.2 implies that $\mathcal{H}^r(C_{m,n}) < \infty$, so $\dim_{\mathcal{H}}(C_{m,n}, d) \leqslant r$ holds for every m, n. As A is the union of the $C_{m,n}$, by Fact 3.4.5 we have $\dim_{\mathcal{H}}(X, d) \leqslant r$. Now we suppose that $\theta_r < \infty$. For $m, n \in \mathbb{N}$ let

$$B_{m,n} = \left\{ p \in A : \left[\forall 0 < t < \frac{1}{m} \right] \mu[B(p, t)] \leqslant n t^r \right\}.$$

Fact 3.4.2 implies that for each m, n we have $\mathcal{H}^r(B_{m,n}, d) \geqslant \frac{1}{n} \mu(B_{m,n})$. As A is the union of the $B_{m,n}$ we have $\mu(B_{m,n}) > 0$ for some m, n. For this m, n we have $\dim_{\mathcal{H}}(B_{m,n}) > s$ and so $\dim_{\mathcal{H}}(A, d) > s$. □

Proposition 3.4.7 below is an immediate consequence of Proposition 3.4.6. We use it to give an easy proof of a fact about Hausdorff dimensions of definable metric spaces, Proposition 11.3.2 below.

Proposition 3.4.7. *Suppose (X, d) is separable. Let μ be a finite Borel measure on X. Let $A \subseteq X$ be Borel and satisfy $\mu(A) > 0$. Suppose that for every $p \in A$*

$$\Theta(p) := \lim_{t \to 0^+} \frac{\log \mu[B(p, t)]}{\log(t)}$$

exists. If $\Theta(p) \geqslant r$ for all $p \in A$ then $\dim_{\mathcal{H}}(A, d) \geqslant r$. If $\Theta(p) \leqslant r$ for all $p \in A$ then $\dim_{\mathcal{H}}(A, d) \leqslant r$.

Fact 3.4.8. *Let (X, d) be separable and suppose that there is locally Lipschitz surjection $(X, d) \to (X', d')$. Then $\dim_{\mathcal{H}}(X, d) \geqslant \dim_{\mathcal{H}}(X, d')$.*

Proof. Let $f : (X, d) \to (X', d')$ be a locally Lipschitz surjection. As (X, d) is separable we let $\{U_i \subseteq X : i \in \mathbb{N}\}$ be a collection of open sets which covers X such that f is Lipschitz on each U_i. Let $U_i' = f(U_i)$. Fact 3.4.3 implies that $\dim_{\mathcal{H}}(U_i, d) \geqslant \dim_{\mathcal{H}}(U_i', d')$ holds for each i. Thus

$$\dim_{\mathcal{H}}(X', d') = \sup\{\dim_{\mathcal{H}}(U_i', d') : i \in \mathbb{N}\} \leqslant \sup\{\dim_{\mathcal{H}}(U_i, d) : i \in \mathbb{N}\} = \dim_{\mathcal{H}}(X, d).$$

□

3.5 Topological Dimension

In this section X is a topological space. We inductively define the topological dimension, $\dim_{\mathrm{top}}(X)$, of X:

i. $\dim_{\mathrm{top}}(X) = -1$ if and only if $X = \emptyset$.

ii. $\dim_{\mathrm{top}}(X) \leqslant n$ if and only if X has a neighborhood basis consisting of sets whose boundaries have topological dimension at most $n - 1$,

and $\dim_{\mathrm{top}}(X) \in \mathbb{N}_\infty$ is the least n such that $\dim_{\mathrm{top}}(X) \leqslant n$. We will need four facts about the topological dimension. The first can be proven with an easy inductive argument:

Fact 3.5.1. *If $A \subseteq X$ then $\dim_{\mathrm{top}}(A) \leqslant \dim_{\mathrm{top}}(X)$.*

The second is essentially trivial:

Fact 3.5.2. *Let $\{X_i : i \in I\}$ be a family of topological space and suppose X is the disjoint union of the X_i. Then*

$$\dim_{\mathrm{top}}(X) = \sup\{\dim_{\mathrm{top}}(X_i) : i \in I\}.$$

The last two require more effort, see [HW41] for a full account.

Fact 3.5.3. *The topological dimension of $\mathbb{R}^n$ equipped with the euclidean topology is n.*

Fact 3.5.4. *Let $\{F_i : i \in \mathbb{N}\}$ be a family of closed sets which cover X. Then*

$$\dim_{\text{top}}(X) = \sup\{\dim_{\text{top}}(F_i) : i \in \mathbb{N}\}.$$

We also define the Cantor rank of a point p of a topological space X, $rk(p)$. The rank is either an ordinal or ∞. The definition is inductive. The point p has rank zero if and only if it is isolated in X. Given an ordinal $\lambda > 0$, let $C_\lambda \subseteq X$ be the set of points with rank strictly less then λ. Then $rk(p) = \lambda$ if and only if $p \notin C_\lambda$ and p is isolated in $X \setminus C_\lambda$. And $rk(p) = \infty$ if and only if p does not have rank λ for any ordinal λ. The Cantor rank of X is the supremum of the Cantor ranks of points $p \in X$. If the $rk(X) < \infty$ then X has topological dimension zero.

3.6 Left-Invariant Metrics on Groups

Let G be a group. We say that a pseudometric d on G is **left-invariant** if $d(x, y) = d(zx, zy)$ holds for all $x, y, z \in G$. We say that a pseudometric d on a set X is a **coarse path pseudometric** if there is a $C \in \mathbb{R}^>$ such that for every $x, y \in X$ there is a sequence $x = z_0, \dots, z_m = y$ such that $d(z_i, z_{i-1}) \leqslant C$ for all $1 \leqslant i \leqslant m$ and

$$\sum_{i=1}^{m} d(x_{i-1}, x_i) \leqslant d(x, y) + C.$$

Suppose that G is a Lie group. We say that a left-invariant coarse path pseudometric d on G is a **nice** if in addition the following two conditions are satisfied:

i. Every compact subset of G is bounded with respect to d.

ii. Every d-ball is precompact.

Any left-invariant Riemannian metric on G is nice. We recall a fundamental result of Abels from [Abe04].

Proposition 3.6.1. *Any two nice left-invariant coarse path pseudometrics on a Lie group are quasi-isometric.*

3.7 Reductions, Ultralimits and Limits of Metric Spaces

In this section we describe ultralimits and GH-limits of sequences of $\mathbb{R}$-valued metric spaces, and the relationship between these two. The ideas in this section are due to Gromov [Gro81] and van den Dries - Wikie [DW84].

3.7.1 Reductions of Λ-valued Metric Spaces

Throughout this section Λ is an ordered vector space over the real field. We fix a distinguished positive element 1_Λ of Λ and embedd $\mathbb{R}$ into Λ by mapping t to $t1_\Lambda$. In this way we regard $\mathbb{R}$ as a subgroup of Λ. Let $\mathcal{O}$ be the convex hull of $\mathbb{R}$ in Λ. We say that a Λ-valued metric space (X, d) is **strongly bounded** if there is a $t \in \mathcal{O}$ such that $d(x, y) < t$ for all $x, y \in X$. Let $\mathrm{st} : \Lambda^{\geqslant} \to \mathbb{R}^{\geqslant}_{\infty}$ be given by $\mathrm{st}(x) = \sup\{t \in \mathbb{R} : t < x\}$. Let (X, d, p) be a Λ-valued pointed metric space. We associate an $\mathbb{R}$-valued pointed metric space to (X, d, p) which we call the **reduction** of (X, d, p). Let $\mathcal{O}(X, d, p) = \{q \in X : d(p, q) \in \mathcal{O}\}$. We will frequently drop the d or p when they are clear from context, as we do here. It is easy to check that $\mathrm{st}\, d$ is an $\mathbb{R}$-valued pseudometric on $\mathcal{O}(X)$. The reduction of (X, d, p) is the metric space associated to this pseudometric space. More explicitly, let $\sim$ be the equivalence relation on $\mathcal{O}(X)$ given by declaring $x \sim y$ when $\mathrm{st}\, d(x, y) = 0$. Let $\overline{\mathcal{O}(X)} = \mathcal{O}(X)/\sim$. It is easily seen that $\mathrm{st}\, d$ pushes forward to a metric on $\overline{\mathcal{O}(X)}$ which we also call $\mathrm{st}\, d$. We let $\overline{x}$ be the class of $x \in X$ in $\overline{\mathcal{O}(X)}$. The reduction of (X, d, p) is the pointed metric space $(\overline{\mathcal{O}(X)}, \mathrm{st}\, d, \overline{p})$. We define the reduction of a strongly bounded Λ-valued metric space without choosing a basepoint. Suppose (X, d) is strongly bounded. Then $\mathrm{st}\, d$ is an $\mathbb{R}$-valued pseudometric on X, the reduction of (X, d) is the associated metric space. That is, the reduction of (X, d) is the $\mathbb{R}$-valued metric space $(\overline{X}, \mathrm{st}\, d)$ where $\overline{X}$ is the quotient of X by the equivalence relation $\sim$, which is given by declaring $x \sim y$ if and only if $\mathrm{st}\, d(x, y) = 0$. For all $p \in X$, $\mathcal{O}(X, d, p) = X$, so we identify the reduction of (X, d) with that of (X, d, p) for any $p \in X$.

Lemma 3.7.1. *Let (X, d, p) be a pointed Λ-valued metric space. Suppose that every countable descending sequence of balls $\{B_i\}_{i\in\mathbb{N}}$ in (X, d) has nonempty intersection. Then $(\overline{X}, \mathrm{st}\, d, \overline{p})$ is complete.*

Proof. Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence of elements of X such that $\{\overline{x_i}\}_{i\in\mathbb{N}}$ is Cauchy. Without loss of generality we may suppose that for all i, $d(x_i, x_{i+1}) \leqslant 2^{-i}$. Let $B_i = B(x_i, 2^{-i})$. It is easy to see that the B_i form a decreasing sequence of balls. Let x_∞ be some element of $\bigcap_i B_i$. It is easy to see that the limit of the sequence $\{\overline{x_i}\}_{i\in\mathbb{N}}$ is $\overline{x_\infty}$. $\square$

The next lemma is trivial, but crucial.

Lemma 3.7.2. *Let (X, d) be a strongly bounded Λ-valued metric space and let $A \subseteq X$. For any $\epsilon \in \mathbb{R}^>$ we have $N(\overline{A}, \epsilon) = N(A, \epsilon)$.*

Proof. Suppose $x_1, \ldots, x_n \in A$. If $d(x_i, x_j) > \epsilon$ when $i \neq j$ then $\operatorname{st} d(\overline{x_i}, \overline{x_j}) > \epsilon$. This implies that $N(\overline{A}, \epsilon) \geqslant N(A, \epsilon)$. If $\operatorname{st} d(\overline{x_i}, \overline{x_j}) > \epsilon$ holds whenever $i \neq j$ we also have $d(x_i, x_j) > \epsilon$. This implies that $N(A, \epsilon) \geqslant N(\overline{A}, \epsilon)$. $\square$

Recall that a metric space is preproper if its completion is proper.

Lemma 3.7.3. *Let (X, d, p) be a pointed Λ-valued metric space. The reduction $(\overline{X}, \operatorname{st} d, \overline{p})$ is preproper if and only $N(B(p, t), \epsilon)$ is finite for all $p \in X$ and $t, \epsilon \in \mathbb{R}^>$. If (X, d) is strongly bounded then $(\overline{X}, \operatorname{st} d)$ is precompact if and only if $N(X, \epsilon)$ is finite for all $\epsilon \in \mathbb{R}^>$.*

Proof. If suffices to prove the second claim. Of course $(\overline{X}, \operatorname{st} d)$ is precompact if and only if $N(\overline{X}, \epsilon)$ is finite for every $\epsilon \in \mathbb{R}^>$. By Lemma 3.7.2 this holds if and only if $N(X, \epsilon)$ is finite for every $\epsilon \in \mathbb{R}^>$. $\square$

3.7.2 Ultralimits

For the remainder of this section $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$. Let $\mathbb{R}^{\mathcal{U}}$ be the corresponding ultrapower of $\mathbb{R}$. Let $\operatorname{st} : \mathbb{R}^{\mathcal{U}} \to \mathbb{R} \cup \{-\infty, \infty\}$ be the usual standard part map. Let $\{(X_i, d_i, p_i)\}_{i\in\mathbb{N}}$ be a sequence of $\mathbb{R}$-valued pointed metric spaces. We let $(X_{\mathcal{U}}, d_{\mathcal{U}}, p_{\mathcal{U}})$ be the ultraproduct of this family with respect to $\mathcal{U}$. Then $(X_{\mathcal{U}}, d_{\mathcal{U}}, p_{\mathcal{U}})$ is an $\mathbb{R}^{\mathcal{U}}$-valued metric space. The reduction of $(X_{\mathcal{U}}, d_{\mathcal{U}}, p_{\mathcal{U}})$ is the **metric ultraproduct** of the family with respect to $\mathcal{U}$, or the **metric ultralimit** of (X_i, d_i, p_i) as $i \to \mathcal{U}$.

Corollary 3.7.4. *Every metric ultraproduct is complete.*

Proof. As ultraproducts are ω-saturated every countable descending intersection of balls in $X_{\mathcal{U}}$ has nonempty intersection. Now apply Lemma 3.7.1. □

We let $B_i(p,t)$ be the open ball in X_i with center $p \in X_i$ and radius $t \in \mathbb{R}^>$.

Corollary 3.7.5. *The reduction of $(X_{\mathcal{U}}, d_{\mathcal{U}}, p_{\mathcal{U}})$ is proper if and only if $\lim_{i\to\mathcal{U}} N(B_i(p_i,n),\epsilon)$ is finite for every $\epsilon \in \mathbb{R}^>$. Suppose additionally that the diameters of the X_i are uniformly bounded. Then the reduction of $(X_{\mathcal{U}}, d_{\mathcal{U}})$ is compact if and only if $\lim_{i\to\mathcal{U}} N(X_i,\epsilon)$ is finite for every $\epsilon \in \mathbb{R}^>$.*

Proof. This follows immediately from basic properties of ultraproducts and from Lemma 3.7.3. □

3.7.3 GH-Convergence and GH-Ultralimits

In this section we define notions of convergence for bounded precompact $\mathbb{R}$-valued metric spaces and pointed preproper $\mathbb{R}$-valued metric spaces. Let (X,d) and (X',d') be bounded precompact $\mathbb{R}$-valued metric spaces. We define the **Gromov-Hausdorff distance** between (X,d) and (X',d') to be the infinimum of all $\delta \in \mathbb{R}^>$ for which there exists a δ-approximate isometry $(X,d) \to (X',d')$. We call this distance function d_{GH}. Note that the Gromov-Hausdorff distance between a bounded precompact metric space and its completion is zero. One can show that $d_{GH}((X,d),(X',d')) = 0$ holds if and only if the completions of (X,d) and (X',d') are isometric. We note that d_{GH} is not the Gromov-Hausdorff distance as usually defined, but d_{GH} is bounded from above and below by constant multiples of the usual Gromov-Hausdorff metric. In particular, our d_{GH} does not satisfy the triangle inequality. See [BBI01] for details. We say that a sequence of bounded precompact metric spaces $\{(X_i,d_i)\}_{i\in\mathbb{N}}$ **GH-converges**, or simply converges, to a bounded metric space (X_∞, d_∞) if

$$\lim_{i\to\infty} d_{GH}((X_i,d_i),(X_\infty,d_\infty)) = 0.$$

There is a natural way to extend this notion of convergence to sequences of pointed preproper metric spaces. We say that a sequence $\{(X_i,d_i,p_i)\}$ of pointed preproper metric spaces **GH-converges**, or simply converges, to a pointed metric space $(X_\infty,d_\infty,p_\infty)$, if for all $n \in \mathbb{N}$

the sequence $\{B_{d_i}(p_i, n)\}_{i\in\mathbb{N}}$ GH-converges to $B_{d_\infty}(p_\infty, n)$ as $i \to \infty$. If there is an $N \in \mathbb{N}$ such that each (X_i, d_i, p_i) has diameter at most N then, neglecting basepoints, the notions of GH-convergence that we have defined agree for $\{(X_i, d_i, p_i)\}_{i\in\mathbb{N}}$. We will also make use of the natural notion of convergence for a family of metric spaces $\{(X_i, d_i)\}_{i\in I}$ indexed by a directed set I.

Lemma 3.7.6. *Let $\{(X_i, d_i, p_i)\}_{i\in\mathbb{N}}$ be a sequence of preproper pointed metric spaces and let $(X_{\mathcal{U}}, d_{\mathcal{U}}, p_{\mathcal{U}})$ be the ultralimit of the sequence as $i \to \mathcal{U}$. Then $(\overline{\mathcal{O}(X_{\mathcal{U}})}, \operatorname{st} d)$ is proper if and only if (X_i, d_i, p_i) converges to $(\overline{\mathcal{O}(X_{\mathcal{U}})}, \operatorname{st} d, \overline{p_{\mathcal{U}}})$ as $i \to \mathcal{U}$.*

Proof. We only treat the case when the (X_i, d_i) have uniformly bounded diameters, so we neglect basepoints. Suppose first that $(\overline{X_{\mathcal{U}}}, \operatorname{st} d)$ is not compact. Let $\epsilon \in \mathbb{R}^{>}$ be such that $N(\overline{X_{\mathcal{U}}}, \epsilon)$ is infinite. Therefore $N(X_{\mathcal{U}}, \epsilon)$ is infinite. Then $N(X_i, \epsilon) \to \infty$ as $i \to \mathcal{U}$, therefore $\{(X_i, d_i)\}_{i\in\mathbb{N}}$ does not converge as $i \to \mathcal{U}$. Suppose now that $(\overline{X_{\mathcal{U}}}, \operatorname{st} d)$ is compact. Fix $\delta \in \mathbb{R}^{>}$. As $N(\overline{X_{\mathcal{U}}}, \delta)$ is finite, $N(X_{\mathcal{U}}, \delta)$ is finite. Let $\{x_1, \ldots, x_n\}$ be an ϵ-dense subset of $X_{\mathcal{U}}$. For each $1 \leqslant j \leqslant n$ let $\{y_j(i) \in X_i : i \in I\}$ be such that $y_j(i) \to x_j$ as $i \to \mathcal{U}$. For $\mathcal{U}$-many $i \in I$ the set $\{y_1(i), \ldots, y_n(i)\}$ is ϵ-dense in X_i. Furthermore for $\mathcal{U}$-many $i \in \mathbb{N}$ we have:

$$|d_i(y_j(i), y_k(i)) - \operatorname{st} d_{\mathcal{U}}(x_j, x_k)| < \epsilon \quad \text{for all } 1 \leqslant j, k \leqslant n.$$

For each $i \in I$ let $f_i : \{y_1(i), \ldots, y_n(i)\} \to \overline{X_{\mathcal{U}}}$ be given by $f_i(y_j(i)) = x_j$. Then f_i is an ϵ-isometry for $\mathcal{U}$-many i and so $d_{GH}((X_i, d_i), (\overline{X_{\mathcal{U}}}, \operatorname{st} d)) < \epsilon$ holds for $\mathcal{U}$-many i. Therefore (X_i, d_i) converges to $(\overline{X_{\mathcal{U}}}, \operatorname{st} d)$ as $i \to \mathcal{U}$. ☐

We say family of pointed proper metric spaces is **GH-precompact**, or simply precompact, if every sequence of elements of the family has a converging subsequence, likewise for a family of precompact metric spaces.

Proposition 3.7.7 (Gromov Precompactness)**.** *A family $\mathcal{X} = \{(X_i, d_i) : i \in I\}$ of compact metric spaces is GH-precompact if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $N(X_i, \epsilon) < N$ for all $i \in I$.*

Proof. Suppose $\epsilon \in \mathbb{R}^{>}$ is such that for all $N \in \mathbb{N}$ there is a $i \in I$ such that $N(X_i, \epsilon) \geqslant N$. For each $N \in \mathbb{N}$ let $i(N) \in I$ be such that $N(X_{i(N)}, \epsilon) \geqslant N$. No subsequence of $\{(X_{i(N)}, d_{i(N)})\}_{i \in \mathbb{N}}$ converges. We prove the other direction. Let $\{(X_{i(n)}, d_{i(n)})\}_{n \in \mathbb{N}}$ be a sequence of elements of $\mathcal{X}$. Let $(X_{\mathcal{U}}, d_{\mathcal{U}})$ be the ultraproduct of the sequence with respect to $\mathcal{U}$. For each $\epsilon \in \mathbb{R}^{>}$, $N(X_{\mathcal{U}}, \epsilon)$ is finite. Therefore $(\overline{X_{\mathcal{U}}}, \operatorname{st} d_{\mathcal{U}})$ is compact. It follows from Lemma 3.7.6 that $\{(X_{i(n)}, d_{i(n)})\}_{n \in \mathbb{N}}$ has a converging subsequence. $\square$

The corollary below is immediate.

Corollary 3.7.8. *Let $\mathcal{X}$ be as in the previous proposition. Suppose that for every $t, t' > 0$ there is a $N \in \mathbb{N}$ such that $P_{d_i}(t, t') < N$ holds for all $i \in I$. Then $\mathcal{X}$ is precompact.*

We use this corollary to prove the next lemma.

Lemma 3.7.9. *Let (X, d) be a proper doubling $\mathbb{R}$-valued metric space. The family*

$$\mathcal{B} = \{(B(q, \lambda), \lambda^{-1} d) : q \in X, \lambda \in \mathbb{R}^{>}\}$$

is precompact. Furthermore, if $p \in X$ then $\{(X, \lambda d, p) : \lambda \in \mathbb{R}^{>}\}$ is a precompact family of pointed proper metric spaces.

Proof. The second claim follows from the first. We prove the first claim. Fix $\epsilon \in \mathbb{R}^{>}$. As (X, d) is doubling there are $C, r \in \mathbb{R}^{>}$ such that we have

$$P_d(t', t) \leqslant C \left(\frac{t'}{t}\right)^r \qquad \text{for all } t, t' > 0.$$

Thus

$$P_{\lambda^{-1} d}(\lambda, \epsilon) = P_d(\lambda, \lambda\epsilon) \leqslant C \left(\frac{1}{\epsilon}\right)^r \text{ for all } \epsilon \in \mathbb{R}^{>}.$$

This implies that the family $\{(X, \lambda d, p) : \lambda \in \mathbb{R}^{>}\}$ is precompact. $\square$

3.7.4 Asymptotic Cones

In this subsection (X, d) is a proper $\mathbb{R}$-valued metric space. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Fix a basepoint $p \in X$. The **asymptotic cone** of (X, d) with respect to $\mathcal{U}$ is the ultralimit of $\{(X, \frac{1}{i}d, p)\}_{i\in\mathbb{N}}$ as $i \to \mathcal{U}$. Let ω be the class of the sequence $\{i\}_{i\in\mathbb{N}}$ in $\mathbb{R}^{\mathcal{U}}$. The asymptotic cone of (X, d) with respect to $\mathcal{U}$ is the reduction of

$$\left(\mathcal{O}\left(X_{\mathcal{U}}, \frac{1}{\omega}d_{\mathcal{U}}, p\right), \frac{1}{\omega}d\right).$$

For any $p, q \in X$ and $x \in X_{\mathcal{U}}$ the triangle inequality implies that $\mathrm{st}\,\frac{1}{\omega}d_{\mathcal{U}}(p, x) < \infty$ holds if and only if $\mathrm{st}\,\frac{1}{\omega}d_{\mathcal{U}}(q, x) < \infty$, so

$$\mathcal{O}\left(X_{\mathcal{U}}, \frac{1}{\omega}d_{\mathcal{U}}, p\right) = \mathcal{O}\left(X_{\mathcal{U}}, \frac{1}{\omega}d_{\mathcal{U}}, q\right) \quad \text{for all } p, q \in X.$$

So the asymptotic cone of (X, d) with respect to $\mathcal{U}$ does depend on the choice of p. In this section for ease of notation we let:

$$\mathcal{O}(X_{\mathcal{U}}) = \mathcal{O}\left(X_{\mathcal{U}}, \frac{1}{\omega}d_{\mathcal{U}}, p\right)$$

for a pointed proper metric space (X, d) and any $p \in X$. We let $\mathfrak{C}_{\mathcal{U}}(X, d)$ be the asymptotic cone of (X, d) with respect to $\mathcal{U}$. We leave the easy proof of the following lemma to the reader:

Lemma 3.7.10. *The following are equivalent:*

i. (X, d) *is bounded,*

ii. $\mathfrak{C}_{\mathcal{U}}(X, d)$ *is bounded,*

iii. $\mathfrak{C}_{\mathcal{U}}(X, d)$ *is a point.*

The following lemma is well-known to metric geometers.

Lemma 3.7.11. *Let* (X', d') *be a proper* $\mathbb{R}$*-valued metric. If* (X, d) *and* (X', d') *are quasi-isometric then* $\mathfrak{C}_{\mathcal{U}}(X, d)$ *and* $\mathfrak{C}_{\mathcal{U}}(X', d)$ *are bilipschitz equivalent.*

Proof. Let $f : (X, d) \to (X', d')$ be a (λ, β) quasi-isometry. Fix a basepoint $p \in X$ and let $p' = f(p)$. Then $\mathfrak{C}_{\mathcal{U}}(X, d)$ and $\mathfrak{C}_{\mathcal{U}}(X', d')$ are by definition the reductions of $(\mathcal{O}(X_{\mathcal{U}}), \frac{1}{\omega}d, p)$ and $(\mathcal{O}(X'_{\mathcal{U}}), \frac{1}{\omega}d', p')$ respectively. Let $f_{\mathcal{U}} : X_{\mathcal{U}} \to X'_{\mathcal{U}}$ be the map induced by f. As f is a (λ, β) quasi-isometry:

$$\frac{1}{\lambda}d_{\mathcal{U}}(x, y) - \beta \leqslant d'_{\mathcal{U}}(f_{\mathcal{U}}(x), f_{\mathcal{U}}(y)) \leqslant \lambda d_{\mathcal{U}}(x, y) + \beta \quad \text{for all } x, y \in X_{\mathcal{U}}.$$

dividing by z and taking standard parts we have:

$$\frac{1}{\lambda}\operatorname{st}\frac{1}{\omega}d_{\mathcal{U}}(x,y) \leqslant \operatorname{st}\frac{1}{\omega}d'_{\mathcal{U}}(f(x),f(y)) \leqslant \lambda\operatorname{st}\frac{1}{\omega}d_{\mathcal{U}}(x,y) \quad \text{for all } x,y \in X_{\mathcal{U}}.$$

Applying this inequality with $y = p$ we have $f_{\mathcal{U}}(x) \in \mathcal{O}(X'_{\mathcal{U}})$ if and only if $x \in \mathcal{O}(X_{\mathcal{U}})$ for all $x \in X_{\mathcal{U}}$. Then $f_{\mathcal{U}}$ induces a λ-bilipschitz map between the reductions

$$\overline{f_{\mathcal{U}}} : \left(\overline{\mathcal{O}(X_{\mathcal{U}})}, \operatorname{st}\frac{1}{\omega}d\right) \to \left(\overline{\mathcal{O}(X'_{\mathcal{U}})}, \operatorname{st}\frac{1}{\omega}d\right).$$

As $f(X_{\mathcal{U}})$ is β-dense in $X'_{\mathcal{U}}$; $f_{\mathcal{U}}[\mathcal{O}(X_{\mathcal{U}})]$ is $\frac{\beta}{\omega}$-dense in $(\mathcal{O}(X'_{\mathcal{U}}, \frac{1}{\omega}d)$. This implies that $\overline{f_{\mathcal{U}}}$ is surjective. □

Lemma 3.7.12. *If (X,d) is a approximate ultrametric space then $\mathfrak{C}_{\mathcal{U}}(X,d)$ is an ultrametric space.*

Proof. Suppose that $\delta \in \mathbb{R}^{>}$ is such that:

$$d(x,z) \leqslant \max\{d(x,y), d(y,z)\} + \delta \quad \text{for all } x,y,z \in X.$$

Dividing through by ω and taking standard parts we have:

$$\operatorname{st}\frac{1}{\omega}d_{\mathcal{U}}(x,z) \leqslant \max\left\{\operatorname{st}\frac{1}{\omega}d_{\mathcal{U}}(x,y), \operatorname{st}\frac{1}{\omega}d_{\mathcal{U}}(y,z)\right\} \quad \text{for all } x,y,z \in X_{\mathcal{U}}.$$

This implies that $\mathfrak{C}_{\mathcal{U}}(X,d)$ is an ultrametric space. □

We say that a Riemannian manifold has **polynomial volume growth** if the volume of a ball is bounded from above by a polynomial function of the radius. The proposition below combines the work of many people. See Breulliard's paper [Bre14] for a good overview of some of these ideas. We define Carnot metrics in Section 5.5 below.

Proposition 3.7.13. *Let $(G,\cdot)$ be a Lie group equipped with a left-invariant Riemannian metric with distance function d:*

i. If $\mathfrak{C}_{\mathcal{U}}(G,d)$ is locally compact then G has polynomial volume growth.

ii. If G has polynomial volume growth then (G,d) is quasi-isometric to a Carnot metric.

Now let $(G, \cdot)$ be a finitely generated group equipped with a word metric d:

i. If $\mathfrak{C}_{\mathcal{U}}(G, d)$ is locally compact then G has a nilpotent subgroup of finite index.

ii. If G has a nilpotent subgroup of finite index then (G, d) is quasi-isometric to a Carnot metric.

Proof (Sketch). We first let G be a Lie group equipped with a left-invariant Riemannian metric. Guivarc'h [Gui73] showed that G either has polynomial or exponential volume growth. It follows from the the work of Guivarc'h that if G has polynomial growth then G is quasi-isometric to a Carnot metric canonically associated to G. We assume that G has exponential growth and show that $\mathfrak{C}_{\mathcal{U}}(G)$ is not locally compact. Let μ and d be the measure and metric on G induced by the Riemannian metric, respectively. Let e be the identity element of G. Fix $r \in \mathbb{R}^{>}$. We show that $N(B_d(e, rt), t) \to \infty$ as $t \to \infty$. We let $\bar{e}$ be the element of $\mathfrak{C}_{\mathcal{U}}(G)$. corresponding to e. This implies that the ball in $\mathfrak{C}_{\mathcal{U}}(G, d)$ with center $\bar{e}$ and radius $2r$ in $\mathfrak{C}_{\mathcal{U}}(G)$ contains an infinite r-separated set. As this holds for every $r \in \mathbb{R}^{>}$, $\mathfrak{C}_{\mathcal{U}}(G, d)$ is not locally compact. Fixing $m \in \mathbb{N}$ we show that $N(B(e, rt), t) > m$ when t is sufficiently large. As G has exponential growth for some $K, \lambda \in \mathbb{R}^{>}$:

$$\mu[B(e, t)] \geqslant K \exp(\lambda t) \quad \text{for all } t \in \mathbb{R}^{>}.$$

Therefore, when t is sufficiently large we have $\mu[B(e, rt)] > m\mu[B(e, t)]$. As μ is left-invarient we have $\mu[B(e, rt)] > m\mu[B(p, t)]$ for all $p \in G$. Thus $B(e, rt)$ cannot be covered by any collection of t-balls of cardinality m when t is sufficiently large.

We now consider the case when G is a finitely generated group equipped with a word metric d. In [Sap15], Sapir showed, using Hrushovski's deep work on approximate groups [Hru12], that if $\mathfrak{C}_{\mathcal{U}}(G, d)$ is locally compact then G has a nilpotent subgroup of finite index. Suppose that G has a nilpotent subgroup of finite index, H. Then G is quasi-isometric to H. A theorem of Mal'cev, see [Har00] implies that H is a co-compact lattice in a nilpotent Lie group N, it follows that H is quasi-isometric to N. Then N is quasi-isometric to the nilshadow of N, so G is quasi-isometric to a Carnot metric. □

By Proposition 3.6.1 every nice left-invariant coarse path pseudometric on a Lie group is quasi-isometric to any left-invariant Riemannian metric. We also have:

Corollary 3.7.14. *Let G be a Lie group and d be a nice left-invariant coarse path pseudometric on G. If $\mathfrak{C}_{\mathcal{U}}(G,d)$ is locally compact then (G,d) is quasi-isometric to a Carnot metric.*

3.7.5 Tangent Cones

Let (X,d) be an $\mathbb{R}$-metric space and suppose that $p \in X$ has a precompact open neighborhood. A **tangent cone** of (X,d) at p is a GH-limit of a sequence of pointed metric spaces of the form:

$$\left\{\left(U, \frac{1}{t_i}d, p\right)\right\}_{i\in\mathbb{N}}$$

for some precompact neighborhood U of p and some sequence $\{t_i\}_{i\in\mathbb{N}}$ of elements of $\mathbb{R}^{>}$ such that $t_i \to 0$ as $i \to \infty$. It is easy to show that the limit, if it exists, will not depend on the choice of U. Lemma 3.7.9 implies that if (U,d) is doubling then any such sequence has a converging subsequence. If the limit exists for every such sequence $\{t_i\}_{i\in\mathbb{N}}$ and is unique up to isometry then we say that (X,d) has a unique tangent cone at p. We recall a theorem of le Donne, [Don11]. A Λ-valued metric space (Y,d) is **isometrically homogenous** if for every $p,q \in Y$ there is an isometry $\sigma : (Y,d) \to (Y,d)$ such that $\sigma(p) = q$.

Theorem 3.7.15. *Let (X,d) be a compact metric space and let μ be a doubling measure on (X,d). Suppose that (X,d) has a unique tangent cone at every point. Then the tangent cone of (X,d) at μ-almost every $p \in X$ is isometrically homogeneous.*

3.7.6 Conical Metric Spaces

A pointed metric space (X,d,p) is **conical** if for every $t > 0$ there is a map $h_t : X \to X$ which fixs p and satisfies $d(h_t(x),h_t(y)) = td(x,y)$ for all $x,y \in X$. The following question is a slightly weaker form of a question of Kinneberg and Le Donne, Question 1 of [KL14]:

Question 3.7.16. *Is every proper, connected, locally-connected, conical, isometrically ho-*

mogeneous metric space isometric to a Carnot group equipped with a left-invariant metric?

3.8 Normed Spaces

In this section we recall a well-known lemma about linear norms on finite dimensional vector spaces over ordered fields. In this section R is an ordered field. A **linear norm** on R^k is a map $\|,\|_N : R^k \to R^{\geqslant}$ such that for all $x, y \in R^k$ and $\lambda \in R$ we have:

i. $\|x\|_N = 0$ if and only if $x = 0$,

ii. $\|\lambda x\|_N = |\lambda|\|x\|_N$,

iii. $\|x + y\| \leqslant \|x\|_N + \|y\|_N$.

We associate a metric d to $\|,\|_N$ by declaring $d(x, y) = \|x - y\|_N$. We will make use of the following:

Fact 3.8.1. *Let d be a metric on R^k. Suppose that for all $x, y \in R^k$ and $\lambda > 0$ we have that $d(\lambda x, \lambda y) = \lambda d(x, y)$ and that $d(x, x + y) = d(0, y)$. Then there is a linear norm $\|,\|_N$ on R^k such that $d(x, y) = \|x - y\|_N$.*

Proof. Set $\|x\|_N = d(0, x)$. Clearly $\|x\|_N = 0$ if and only if $x = 0$. If $\lambda > 0$ then $\|\lambda x\|_N = d(0, \lambda x) = \lambda\|x\|_N$. As

$$d(0, x) = d(-x, x - x) = d(0, -x)$$

we have $\| - x\|_N = \|x\|_N$, so for any $\lambda \in R$ we have $\|\lambda x\|_N = |\lambda|\|x\|_N$. □

CHAPTER 4

T-Convexity

At several points in this thesis we use the theory T-convex structures begun by van den Dries and Lewenberg in [DL95] and [Dri97]. In this chapter we define T-convex structures, recall some basic results and prove a useful technical lemma. In Section 4.3 we record a fact about definable sets of imaginaries in real closed valued fields.

4.1 T-convex Structures

We assume that the language of $\mathcal{R}$ contains a symbol for every definable function $R \to R$. This ensures that $\mathcal{R}$ admits quantifier elimination. Let $\mathbb{K} = (K, +, \times, \dots)$ be an elementary extension of $\mathcal{R}$. If for every $b \in K$ the set $\{a \in R : a \leqslant b\}$ has a supremum in $R \cup \{-\infty, \infty\}$ then we say that $\mathbb{K}$ is **tame extension** of $\mathcal{R}$ and we say that $\mathcal{R}$ is a **tame substructure** of $\mathbb{K}$. We assume that all tame extension are nontrivial extensions, so as not to consider $\mathcal{R}$ to be a tame elementary extension of itself. If $\mathbb{K}$ is a tame elementary extension of $\mathcal{R}$ then we define a standard part map $\mathrm{st} : \mathbb{K} \cup \{-\infty, \infty\} \to \mathcal{R} \cup \{-\infty, \infty\}$ by

$$\mathrm{st}(b) = \sup\{a \in R : a < b\}.$$

The study of tame extensions began with the following theorem [MS94]:

Theorem 4.1.1 (Marker-Steinhorn)**.** *The following are equivalent:*

i. $\mathbb{K}$ *is a tame extension of* $\mathcal{R}$*,*

ii. Every type over $\mathcal{R}$ *realized in* $\mathbb{K}$ *is definable over* $\mathcal{R}$*.*

There are two particularly important kinds of tame extensions with which we are concerned. First, if $\mathcal{R}$ expands the ordered field of real numbers than every elementary extension of $\mathcal{R}$ is tame. Second, let ζ be a positive element of an elementary extension of $\mathcal{R}$ which is less then every positive element of $\mathcal{R}$. Then $\mathcal{R}(\zeta)$, the prime model over ζ, is a tame elementary extension of $\mathcal{R}$. Indeed, as $\mathcal{R}$ admits definable Skolem functions, every element of $\mathcal{R}(\zeta)$ is of the form $f^*(\zeta)$ for some $\mathcal{R}$-definable function $f : R^> \to R$ and in this case:

$$\operatorname{st} f^*(\zeta) = \lim_{t \to 0^+} f(t).$$

A **tame pair** is the expansion of $\mathbb{K}$ by a predicate defining R, denoted by $(\mathbb{K}, R)$. The theory of tame pairs is complete:

Theorem 4.1.2 (van den Dries - Lewenberg)**.** *Let $\mathbb{K}, \mathbb{K}'$ be tame extensions of $\mathcal{R}$. Then $(\mathbb{K}, R)$ and $(\mathbb{K}', R)$ are elementarily equivalent.*

Recall that the convex hull of a subfield of an ordered field is a subring, and that any convex subring of an ordered field is a valuation subring. We say that a convex subring $\mathcal{O} \subseteq K$ is **T-convex** if it is the convex hull of a tame substructure of $\mathbb{K}$. For the remainder of this section $\mathcal{O}$ is the convex hull of R in K. The expansion of $\mathbb{K}$ by a predicate defining $\mathcal{O}$ is called a **T-convex structure** and denoted $(\mathbb{K}, \mathcal{O})$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$ and let Γ be the value group of the associated valuation. It follows from the definition of a tame expansion that $b - \operatorname{st}(b) \in \mathfrak{m}$ for all $b \in \mathcal{O}$. We therefore identify the residue field of $(\mathbb{K}, \mathcal{O})$ with R and the residue map with st.

Theorem 4.1.3 (van den Dries - Lewenberg)**.** *The structure $(\mathbb{K}, \mathcal{O})$ admits elimination of quantifiers and is universally axiomatizable.*

Quantifier elimination implies that any $(\mathbb{K}, \mathcal{O})$-definable subset of K^n is a boolean combination of sets of the form $\{x : f(x) \in \mathcal{O}\}$ and $\{x : g(x) \notin \mathcal{O}\}$ for $\mathbb{K}$-definable functions $f, g : K^n \to K$. Every $\mathbb{K}$-definable set is of this form, as $A = \chi_A^{-1}(\mathfrak{m})$, where χ_A is the characteristic function of A. By replacing the functions g with their multiplicative inverses, we see in fact that every definable subset of K^n is a boolean combination of definable sets of the form $\{x : f(x) \in \mathcal{O}\}$ and $\{x : g(x) \in \mathfrak{m}\}$. The next three corollaries are nontrivial consequences of Theorem 4.1.3, they are due to van den Dries - Lewenberg:

Corollary 4.1.4. *Let $A \subseteq K^m$ and $f : A \to K^n$ be $(\mathbb{K}, \mathcal{O})$-definable. There is a finite partition $\mathcal{S}$ of A into $(\mathbb{K}, \mathcal{O})$-definable sets such that for each $S \in \mathcal{S}$ there is a $\mathbb{K}$-definable function $g : K^m \to K^n$ such that $f|_S = g|_S$.*

Corollary 4.1.5. *T-convex expansions of o-minimal structures admit definable Skolem functions.*

Corollary 4.1.6. *Every $(\mathbb{K}, \mathcal{O})$-definable subset of K is a finite union of convex sets, i.e. $(\mathbb{K}, \mathcal{O})$ is weakly o-minimal. It follows that Γ is also weakly o-minimal.*

The following corollary is proven in [Dri97]. The proof uses the Marker-Steinhorn theorem.

Corollary 4.1.7. *Every $(\mathbb{K}, \mathcal{O})$-definable subset of R^n is $\mathcal{R}$-definable.*

In model-theoretic terminology, $\mathcal{R}$ is **stably embedded** in $(\mathbb{K}, \mathcal{O})$. In particular, if $A \subseteq \mathcal{O}^n$ is $(\mathbb{K}, \mathcal{O})$-definable then $\mathrm{st}(A) \subseteq R^n$ is $\mathcal{R}$-definable. The last corollary of this section is a consequence of weak o-minimality of Γ, we will use it in the proof of proposition 12.3.3 below:

Corollary 4.1.8. *Let $\{A_x : x \in K^m\}$ be a $(\mathbb{K}, \mathcal{O})$-definable family of subsets of Γ. There is an $N \in \mathbb{N}$ such that if A_x is finite then $|A_x| \leqslant N$.*

4.2 The $(\mathbb{K}, \mathcal{O})$-approximation Lemma

In this section we prove a technical lemma which is very useful. It is an easy consequence of Corollary 4.1.4. We say that a family of sets $\{A_i : i \in I\}$ is **directed** if for every $i, j \in I$ there is a $k \in I$ such that

$$A_i \cup A_j \subseteq A_k.$$

Lemma 4.2.1. *Let $A \subseteq K^k$ be a $(\mathbb{K}, \mathcal{O})$-definable set. Let $f : A \to K^n$ be a $(\mathbb{K}, \mathcal{O})$-definable function.*

i. There is a $\mathbb{K}$-definable family of sets $\mathcal{A} = \{A_{s,t} : s, t \in K\}$ such that A is the union of a directed subfamily of $\mathcal{A}$.

ii. There is a $\mathbb{K}$*-definable family of sets* $\{A_{s,t} : s,t \in K\}$ *and a* $\mathbb{K}$*-definable family of functions*

$$\{g_{s,t} : A_{s,t} \to K^n : (s,t) \in K^2\}$$

such that $\mathrm{Graph}(f)$ *is the union of a directed subfamily of* $\{\mathrm{Graph}(g_{s,t}) : s,t \in K\}$.

In the proof we let $\mathcal{O}^{\geqslant 1} = \{b \in \mathcal{O} : b \geqslant 1\}$.

Proof. Applying the quantifier elimination to A we have $\mathbb{K}$-definable functions $s_{j,k}, t_{j,k} : K^k \to K$ for $1 \leqslant j,k, \leqslant M$ such that

$$A = \bigcup_{j=1}^{M} \bigcap_{k=1}^{M} [s_{j,k}^{-1}(\mathfrak{m}) \cap t_{j,k}^{-1}(\mathcal{O})].$$

For $s,t \in K$ we let:

$$A_{s,t} = \bigcup_{j=1}^{M} \bigcap_{k=1}^{M} \left[s_{j,k}^{-1}[-s,s] \cap t_{j,k}^{-1}[t^{-1},t]\right].$$

If $s \in \mathfrak{m}^{>}$ and $t \in \mathcal{O}^{>1}$ then $A_{s,t} \subseteq A$ and

$$A = \bigcup_{(s,t) \in \mathfrak{m}^{>} \times \mathcal{O}^{>1}} A_{s,t}.$$

This union is directed. Now let $f : A \to K^n$ be a $(\mathbb{K}, \mathcal{O})$-definable function. Let $\{S_1, \ldots, S_n\}$ be a partition of A into $(\mathbb{K}, \mathcal{O})$-definable sets and let $g_1, \ldots, g_n : A \to K^n$ be $\mathbb{K}$-definable functions such that $f|_{S_i} = g_i|_{S_i}$ for each i. Applying the quantifier elimination for each i there are finitely many $\mathbb{K}$-definable functions $s^i_{j,k}, t^i_{j,k}$ for $1 \leqslant j,k \leqslant M$ such that:

$$S_i = \bigcup_{j=1}^{M} \bigcap_{k=1}^{M} [(s^i_{j,k})^{-1}(\mathfrak{m}) \cap (t^i_{j,k})^{-1}(\mathcal{O})].$$

For $s,t \in K$ we let

$$T^i_{s,t} = \bigcup_{j=1}^{M} \bigcap_{k=1}^{M} (s^i_{j,k})^{-1}[-s,s] \cap (t^i_{j,k})^{-1}[t^{-1},t].$$

Let $B \subseteq K^2$ be the $\mathbb{K}$-definable set of (s,t) such that the $T^1_{s,t}, \ldots, T^n_{s,t}$ are pairwise disjoint. Note that $\mathfrak{m}^{\geqslant} \times \mathcal{O}^{\geqslant 1} \subseteq B$. We now define $f_{s,t}$. If $(s,t) \notin B$ then we set $A_{s,t} = \emptyset$. If $(s,t) \in B$ we let $A_{s,t} = T^1_{s,t} \cup \ldots, \cup T^n_{s,t}$ and let $f_{s,t} : A_{s,t} \to K^n$ agree with g_i on $T^i_{s,t}$. Thus

$$\mathrm{Graph}(f) = \bigcup_{(s,t) \in \mathfrak{m}^{\geqslant} \times \mathcal{O}^{\geqslant 1}} \mathrm{Graph}(f_{s,t})$$

and this union is directed. $\square$

The corollary below is a uniform version of part (i) of Lemma 4.2.1. We use this corollary in the proof of Proposition 12.3.2. We omit the proof.

Corollary 4.2.2. *Let $\mathcal{A} = \{A_r : r \in K\}$ be a $(\mathbb{K}, \mathcal{O})$-definable family of sets. There is a $\mathbb{K}$-definable family of sets $\{B_{r,s,t} : r, s, t \in K\}$ such that for every $r \in K$, A_r is union of directed a subfamily of $\{B_{r,s,t} : s, t \in K\}$.*

4.3 Imaginaries in Real Closed Valued Fields

In this section $\mathbb{K} = (K, +, \times, \leqslant)$ is a nonarchimedean real closed field, $\mathcal{O}$ is a convex subring of K, R is the residue field of the associated valuation and Γ is the value group. Then every convex subring of $\mathbb{K}$ is T-convex. In this section we record a fact about $(\mathbb{K}, \mathcal{O})$-definable sets of imaginaries, proposition 4.3.1. In chapter 12 we will apply this proposition to study Gromov-Hausdorff limit points of semialgebraic families of metric spaces. Proposition 4.3.1 can be proven using Mellor's elimination of imaginaries for real closed valued fields [Mel06] and the analysis of geometric sorts due to Haskell, Hrushovksi and Macpherson, see for example the proof of Lemma 2.6.2 in [HHM06]. The forthcoming [HW15] will contain another, more direct, proof of Proposition 4.3.1.

Proposition 4.3.1. *Let X be a $(\mathbb{K}, \mathcal{O})$-definable set of imaginaries. Exactly one of the following holds:*

i. There is a $(\mathbb{K}, \mathcal{O})$-definable map $X \to \Gamma$ with infinite image.

ii. For some k there is a $(\mathbb{K}, \mathcal{O})$-definable injection $X \to R^k$.

By applying model-theoretic compactness and the existence of $(\mathbb{K}, \mathcal{O})$-definable Skolem functions in the typical way one can prove a uniform version of Proposition 4.3.1. We leave the proof to the leader.

Proposition 4.3.2. *Let $\{X_a : a \in K^l\}$ be a $(\mathbb{K}, \mathcal{O})$-definable family of sets and let $\{\sim_a : a \in K^l\}$ be a $(\mathbb{K}, \mathcal{O})$-definable family of equivalence relations $\sim_a$ on X_a. There is a partition of K^l into $(\mathbb{K}, \mathcal{O})$-definable sets $\{A, B\}$, a $(\mathbb{K}, \mathcal{O})$-definable family $\{h_a : a \in A\}$ of functions*

$h_a : X_a \to \Gamma$, an R-definable family $\{Z_p : p \in R^n\}$ of subsets of R^k, a $(\mathbb{K}, \mathcal{O})$-definable function $j : B \to R^n$, and a $(\mathbb{K}, \mathcal{O})$-definable family $\{g_b : b \in B\}$ of functions $g_b : X_b \to Z_{j(b)}$ such that for all $a \in A$ and $x, y \in X_a$:

(1) If $x \sim_a y$ then $h_a(x) = h_a(y)$.

(2) h_a has infinite image,

and for all $b \in B$ and $x, y \in X_b$:

(3) $x \sim_b y$ holds if and only if $g_b(x) = g_b(y)$

(4) $Z_{j(b)} = g_b(X_b)$.

Proposition 4.3.3. *Suppose that $\mathbb{K}$ is $|\mathbb{R}|^+$-saturated and that $\mathcal{O}$ is the convex hull of the integers in K. Let X be a $(\mathbb{K}, \mathcal{O})$-definable set of imaginaries. If $|X| \leqslant |\mathbb{R}|$ then for some k there is a definable injection $X \to \mathbb{R}^k$.*

Proof. By Proposition 4.3.1 it suffices to show that if there is a $(\mathbb{K}, \mathcal{O})$-definable map $X \to \Gamma$ with infinite image then $|X| > |\mathbb{R}|$. It is enough to show that every infinite definable subset of Γ has cardinality at least $|\mathbb{R}|^+$. As Γ is weakly o-minimal it suffices to show that any open interval in Γ has cardinality at least $|\mathbb{R}|^+$. We fix $a, b \in K^>$ such that $\mathrm{val}(a) < \mathrm{val}(b)$ and show that $|(\mathrm{val}(a), \mathrm{val}(b))| \geqslant |\mathbb{R}|^+$. For each $N \in \mathbb{N}$ there are $c_1, \ldots, c_n \in (b, a)$ such that $\mathrm{val}(c_i) < \mathrm{val}(c_{i-1})$ for every i. As val is the archimedean valuation, it follows that for every $N \in \mathbb{N}$ and $K \in \mathbb{N}$ there are $c_1, \ldots, c_n \in (b, a)$ such that $Kc_{i-1} < c_i$. for all i. Applying $|\mathbb{R}|^+$-saturation of $\mathbb{K}$ there is an indexed set $\{c_\lambda : \lambda < |\mathbb{R}|^+\}$ of elements of (b, a) such that for every $K \in \mathbb{N}$ we have $Kc_\kappa < c_\lambda$ whenever $\kappa < \lambda$. Thus, if $\kappa, \lambda < |\mathbb{R}|^+$ and $\kappa \neq \lambda$ then $\mathrm{val}(c_\kappa) \neq \mathrm{val}(c_\lambda)$. □

4.4 Uniform Limits

Following [Dri05] we apply stable embeddedness to show definibility of pointwise and uniform limits of definable families of functions. We use this to construct the definable completion of a definable metric space in Chapter 7.

Lemma 4.4.1. *Let $A \subseteq R^k$ be definable and let $\mathcal{F} = \{f_x : x \in R^l\}$ be a definable family of functions $A \to R$. There is a definable family $\mathcal{G}$ of functions $A \to R$ such that a function $g : A \to R$ is an element of $\mathcal{G}$ if and only if there is a path γ in R^l such that g is the pointwise limit of $f_{\gamma(t)}$ as $t \to 0^+$.*

If $\mathcal{R}$ expands the real field then $\mathcal{G}$ is the set of pointwise limits of sequences of elements of $\mathcal{F}$.

This is essentially proven by van den Dries in [Dri05], however, he only worked over the reals. We only sketch the proof.

Proof (Sketch). Let $\mathcal{R}^* = (R^*, +, \times, \leqslant, \dots)$ be a tame elementary extension of $\mathcal{R}$. We consider the tame pair $(\mathcal{R}^*, R)$. Stable embeddedness implies that there is an $\mathcal{R}$-definable family $\mathcal{G} = \{g_x : x \in B\}$ of functions $A \to R$ such that a function $g : A \to R$ is an element of $\mathcal{G}$ if and only if for some $\alpha \in (R^*)^l$ we have:

$$g(a) = \operatorname{st} f^*_\alpha(a) \quad \text{for all } a \in A.$$

The completeness of the theory of tame pairs implies that $\mathcal{G}$ does not depend on the choice of $\mathcal{R}^*$. Suppose $\mathcal{R}^* = \mathcal{R}(\zeta)$ is the prime model over a positive infinitesimal element ζ of an elementary extension of $\mathcal{R}$. Then the elements of $\mathcal{G}$ are functions of the form $\operatorname{st} f^*_{\gamma^*(\zeta)} : A \to R$ for $\mathcal{R}$-definable paths γ in R^l. For any path γ in R^l:

$$\operatorname{st} f^*_{\gamma^*(\zeta)}(a) = \lim_{t \to 0^+} f_{\gamma(t)}(a) \quad \text{for all } a \in A.$$

Suppose that $\mathcal{R}$ expands the real field. Then every element of $\mathcal{G}$ is a pointwise limit of a sequence of the form

$$\{f_{\gamma(\frac{1}{i})}\}_{i \in \mathbb{N}} \quad \text{for a path } \gamma \text{ in } \mathbb{R}^l.$$

We show that every pointwise limit of a sequence of elements of $\mathcal{F}$ is an element of $\mathcal{G}$. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$ and let $\mathcal{R}^{\mathcal{U}} = (\mathbb{R}^{\mathcal{U}}, +, \times, \dots)$ be the ultrapower of $\mathcal{R}$ with respect to $\mathcal{U}$. Let $\{\alpha(i)\}_{i \in \mathbb{N}}$ be a sequence of elements of $\mathbb{R}^l$ such that $\{f_{\alpha(i)}\}_{i \in \mathbb{N}}$ pointwise converges as $i \to \infty$. We let α be the element of $(\mathbb{R}^{\mathcal{U}})^l$ corresponding to the sequence $\{\alpha_i\}_{i \in \mathbb{N}}$. Then

$$\operatorname{st} f^*_\alpha(a) = \lim_{i \to \infty} f_{\alpha(i)}(a) \quad \text{for all } a \in A.$$

□

In Proposition 7.2.2 we use the following corollary to construct the definable completion of a definable metric space.

Corollary 4.4.2. *There is a definable family $\mathcal{G}'$ of functions $A \to R$ such that a function g is an element of $\mathcal{G}'$ if and only if it is a uniform limit of a family $\{f_{\gamma(t)}\}_{t \in R^>}$ as $t \to 0^+$ for some definable $\gamma : R^> \to R^l$. If $\mathcal{R}$ expands the ordered field of reals then $\mathcal{G}'$ is the set of uniform limits of sequences of elements of $\mathcal{F}$.*

Proof (Sketch). We let $\mathcal{G}'$ be the set of $g \in \mathcal{G}$ such that for every $\epsilon \in R^>$ there is an $x \in R^l$ such that $\|g - f_x\|_\infty \leqslant \epsilon$. It is easy to check that this works. □

CHAPTER 5

Definition and Examples of Definable Metric Spaces

A **definable metric space** is a pair (X, d) consisting of a definable set $X \subseteq R^k$ for some $k \in \mathbb{N}$ and a definable $(R, + \leqslant)$-valued metric d on X. A **definable pseudometric space** (X, d) is a definable set X together with a definable $(R, +, \leqslant)$-valued pseudometric on X. Our first lemma is basic and used throughout:

Lemma 5.0.3. *Suppose that d' is a definable pseudometric on X. The associated metric space is definable.*

Proof. The metric space associated to (X, d') is the quotient $(X/{\sim}, d)$ where $x \sim y$ when $d'(x, y) = 0$ and d is the pushforward of d' to $X/{\sim}$ by the quotient map. As $\mathcal{R}$ admits elimination of imaginaries, there is a definable set Y and a definable function $\pi : X \to Y$ such that $\pi(x) = \pi(y)$ if and only if $x \sim y$. Let d'' be the pushforward of d' onto Y. Then (Y, d'') is isometric to $(X/{\sim}, d)$. □

5.1 Definable Sets with Euclidean Metrics

Let $X \subseteq R^k$ be a definable set. It is natural to equip X with the restriction e of the euclidean metric on R^k. There is now an interesting body of research around such metric spaces (X, e). We recall below only one result from this area, a beautiful theorem of Valette [Val08]. Valette in fact gave a classification of definable sets in a polynomially bounded o-minimal expansion of the real field up to bilipschitz equivalence, but we do not describe this. We will see that this theorem below fails for semialgebraic metric spaces.

Theorem 5.1.1. *Suppose that $\mathcal{R}$ is a polynomially bounded expansion of the real field. Let*

k *be the field of powers of* $\mathcal{R}$. *There are* $|\mathbf{k}|$*-many definable sets up to bilipschitz equivalence. Furthermore, a definable family of sets has only finitely many elements up to bilipschitz equivalence.*

In particular, as the field of powers of $\bar{\mathbb{R}}_{\mathrm{an}}$ is $\mathbb{Q}$, there are only countably many $\bar{\mathbb{R}}_{\mathrm{an}}$-definable sets up to bilipschitz equivalence.

5.2 Products, Sums and Cones

We describe, in the definable setting, some standard ways of constructing new metrics spaces from old.

5.2.1 Products

Given definable metrics (X, d) and (X', d') we put a metric $d_{\max}$ on $X \times Y$ by setting:

$$d_{\max}((x, x'), (y, y')) = \max\{d(x, x'), d'(y, y')\}.$$

Generally, if we refer to a product of definable metric spaces $(X, d) \times (X', d')$ we mean the product together with this maximum metric. The main exception to this is that we generally put the euclidean metric on R^k.

5.2.2 Disjoint Sums and Wedge Sums

Let $\mathcal{X} = \{(X_\alpha, d_\alpha) : \alpha \in R^l\}$ be a definable family of metric spaces. Let

$$X = \{(\alpha, x) : x \in X_\alpha\}$$

and let $\pi : X \to R^l$ be the coordinate projection. We put a metric d on X by declaring $d(x, y) = d_\alpha(x, y)$ when $\pi(x) = \pi(y) = \alpha$ and setting $d(x, y) = 1$ when $\pi(x) \neq \pi(y)$. The metric space (X, d) is homeomorphic to the disjoint union of the elements of $\mathcal{X}$.

We now let $\mathcal{Y} = \{(Y_\alpha, d_\alpha, p_\alpha) : \alpha \in R^l\}$ be a definable family of pointed metric spaces.

We construct the **wedge sum**:

$$\bigvee_{\alpha \in R^l} (Y_\alpha, d_\alpha, p_\alpha).$$

Let $Y = \{(\alpha, y) : y \in Y_\alpha\}$ and let $\pi : Y \to R^l$ be the natural projection. We put a pseudometric d on Y by declaring $d(x,y) = d_\alpha(x,y)$ when $\pi(x) = \pi(y) = \alpha$ and declaring

$$d(x,y) = d_{\pi(x)}(x, p_{\pi(x)}) + d_{\pi(y)}(y, p_{\pi(y)})$$

when $\pi(x) \neq \pi(y)$. Note that $d(p_\alpha, p_\beta) = 0$ for all $\alpha, \beta \in R^l$. The wedge sum is metric space associated to the pseudometric (Y, d).

5.2.3 The Euclidean Cone over a Metric Spaces

Suppose that $\mathcal{R}$ expands the ordered field of real numbers. Let $\mathcal{R}_{\cos}$ be the expansion of $\mathcal{R}$ by the restriction of cosine to $[0, \pi]$. As the restriction of cos to $[0, \frac{\pi}{2}]$ is the compositional inverse of the antiderivative of a semialgebraic function. A theorem of Karpinski and Macintyre on Pfaffian closure [KM97] thus implies that $\mathcal{R}_{\cos}$ is o-minimal, see also [Spe99]. Let (X, d) be a definable metric space. The **euclidean cone** over (X, d) is the $\mathcal{R}_{\cos}$-definable metric space with underlying set $X \times \mathbb{R}^{\geqslant}$ and metric

$$d((x,s),(y,t)) = \sqrt{s^2 + t^2 + 2st\cos[\min\{d(x,y), \pi\}]} :$$

See [BBI01] for more about Euclidean cones.

5.3 Snowflakes

A **gauge** is a a definable function $g : R^{\geqslant} \to R^{\geqslant}$ such that:

i. g is subadditive, i.e. $g(s+t) \leqslant g(s) + g(t)$ for all $s, t \in R^{>}$,

ii. $g(0) = 0$,

iii. g is strictly increasing and continuous.

Let (X, d) be a definable metric space. Then $g(d(x, y))$ is a definable metric on X. Such a metric space is called a **snowflake**, or a g-**snowflake**, of (X, d). The case when $g(t) = t^r$ for an element $r \in (0, 1)$ of the field of powers of $\mathcal{R}$ is of particular interest We call (X, d^r) the r-**snowflake** of (X, d). As $r \to 0^+$, d^r pointwise converges to the discrete $\{0, 1\}$-valued metric on X, so we will sometimes call such a discrete metric space a 0-**snowflake**.

For the remainder of this section we assume that $\mathcal{R}$ expands the real field. Why the name "snowflake"? The $\log(3)\log(4)^{-1}$-snowflake of $([0, 1], e)$ is bilipschitz equivalent to the classical von Koch snowflake, considered as a subset of $\mathbb{R}^2$ equipped with the induced euclidean metric. The fact below follows easily from the definition of Hausdorff dimension:

Fact 5.3.1. *Let (X, d) be a metric space. Then for any $r \in (0, 1]$ we have:*

$$\dim_{\mathcal{H}}(X, d^r) = \frac{1}{r} \dim_{\mathcal{H}}(X, d).$$

In particular the Hausdorff dimension of an r-snowflake of $([0, 1], e)$ is $\frac{1}{r}$.

Thus there are semialgebraic metric spaces with non-integral Hausdorff dimension. There are $\bar{\mathbb{R}}_{\exp}$-definable families of metric spaces such that the Hausdorff dimension of the elements of the family takes infinitely many distinct values, such as $\{([0, 1], e^r) : r \in (0, 1]\}$. The following fact is immediate from Fact 3.4.1.

Fact 5.3.2. *Let μ be the Lebesgue measure on $\mathbb{R}$. Let $(\mathbb{R}, d)$ be the g-snowflake of $(\mathbb{R}, e)$ for a gauge g. Let $\mathbf{h}$ be a definable dimension function.*

i. If $\lim_{t \to 0^+} \frac{g(t)}{\mathbf{h}(t)} = 0$ then $\mathcal{H}^{\mathbf{h}}(\mathbb{R}, d) = 0$.

ii. If $\lim_{t \to 0^+} \frac{g(t)}{\mathbf{h}(t)} = \infty$ then $\mathcal{H}^{\mathbf{h}}(A, d) = \infty$ for every Borel $A \subseteq X$ such that $\mu(A) > 0$.

5.3.1 Infinite Hausdorff Measure and Dimension

We construct a compact $\bar{\mathbb{R}}_{\exp}$-definable metric space with infinite Hausdorff dimension. Let $f(t) = -[\log(t)]^{-1}$. As $f'(0) = \infty$, Lemma 6.3.2 implies that there is a gauge g such that $g(t) = f(t)$ when $0 < t \ll 1$. Fact 5.3.2 implies that the g-snowflake of $([0, 1], e)$ has infinite

Hausdorff dimension. Proposition 10.3.6 below shows that if $\mathcal{R}$ is polynomially bounded then every $\mathcal{R}$-definable separable metric space has finite Hausdorff dimension.

Proposition **??** below shows that if $\mathcal{R}$ is polynomially bounded then a $\mathcal{R}$-definable metric space with Hausdorff dimension 1 contains a definable open set which is definably bilipschitz equivalenct to $([0,1], e)$. This does not hold in $\bar{\mathbb{R}}_{\text{exp}}$. Let $f(t) = t|\log(t)|$. As $f'(0) = \infty$, Lemma 6.3.2 implies that there is a gauge g such that $g(t) = f(t)$ when $0 < t \ll 1$. Fact 5.3.2 implies that $\dim_{\mathcal{H}}([0,1], f \circ e) = 1$ and that $\mathcal{H}^1(A) = \infty$ for any Borel $A \subseteq [0,1]$ with positive Lebesgue measure. Thus $\mathcal{H}^1$ is not σ-finite on an open subset of $([0,1], g \circ e)$. Corollary 10.3.11 shows that if the one-dimensional Hausdorff measure is σ-finite on a definable metric space with Hausdorff dimension one, then that definable metric space contains a definable open subset which is definably bilpschitz equivalence to $([0,1], e)$.

5.4 Metrics Defined using the Logarithm

A **logarithm** is an isomorphism of ordered abelian groups $(R, +, \leqslant) \to (R^>, \cdot, \leqslant)$. If $\mathcal{R}$ expands the real field then a theorem of Karpinski and Macintyre on Pfaffian closure [KM97] implies that the expansion of $\mathcal{R}$ by the usual logarithm is o-minimal, see also [Spe99]. *In this section we assume that $\mathcal{R}$ defines a logarithm.*

5.4.1 Hyperbolic Space

We define k-dimensional hyperbolic space. Let:

$$H = \{x \in R^k : \|x\| < 1\} \quad \text{and } S = \{x \in R^k : \|x\| = 1\}.$$

We describe the **Klein-Beltrami metric** d on H. Fix $x, y \in H$. Let L be the line in R^k passing through x and y. Let $\{p, q\} = L \cap S$. We declare:

$$d(x,y) = \frac{1}{2} \log \frac{\|y - p\| \|x - q\|}{\|x - p\| \|y - q\|}.$$

This metric is $\mathcal{R}$-definable.

Fact 5.4.1. *Suppose that $\mathcal{M}$ is a polynomially bounded expansion of the ordered field of* *reals. Then no $\mathcal{M}$-definable metric space is isometric to the hyperbolic plane.*

Proof. Suppose (H, d) is an $\mathcal{M}$-definable metric space which is isometric to the hyperbolic plane. Fix a basepoint $p \in H$. Let T be the boundary of $B(p, 1)$ and let $S = H \setminus B(p, 1)$. Let $\pi : S \setminus \{p\} \to T$ be the projection which takes each q to the unique element of T that lies on the hyperbolic line connecting p to q. That is, $\pi(q)$ is the unique element of T such that $d(\pi(p), p) + d(\pi(p), q) = d(p, q)$. Let

$$g(t) = \inf\{d(\pi(x), \pi(y)) : x, y \in S, d(p, x) = d(p, y) = t, d(x, y) = 1\}.$$

It follows from the hyperbolic law of sines that $g(t) \leqslant K \exp(-t)$ for some $K > 0$. Contradiction. □

5.4.2 The Hyperbolic Cone over a Definable Metric Space

Let (X, d) be a bounded definable metric space. The **hyperbolic cone** over (X, d) is the $\mathcal{R}$-definable metric space with underlying set $X \times (0, \mathrm{Diam}(X, d)]$ and metric:

$$d((x, s), (y, t)) = \log \left[\frac{d(x, y) + \max\{s, t\}}{\sqrt{st}} \right].$$

If $\mathcal{R}$ expands the real field and $|X| \geqslant 2$ then the hyperbolic cone over (X, d) is a negatively curved metric space in the sense of Gromov. See [BS00] for more information.

5.4.3 Noncompact Symmetric Spaces

In this subsection we let $|,|$ be the operator norm on $Gl_n(R)$:

$$|A| = \max\{\|A(x)\| : x \in R^n, \|x\| = 1\}.$$

Then

$$d(A, B) = \log \max\{|A^{-1}B|, |B^{-1}A|\}$$

is a left-invariant $\mathcal{R}$-definable pseudometric on $Gl_n(R)$. By restriction we obtain a left-invariant definable pseudometric on any definable subgroup of $Gl_n(R)$. Suppose for the remainder of the subsection that $\mathcal{R}$ expands the ordered field of real numbers. One can show, see [Abe04], that the restriction of d to any semisimple algebraic subgroup G of $Gl_n(\mathbb{R})$ is a nice left-invariant coarse path pseudometric on G. In particular, the restriction of d to a semisimple subgroup of $Gl_n(\mathbb{R})$ is quasi-isometric to any left-invariant Riemannian metric on G.

Let $G \subseteq Gl_n(\mathbb{R})$ be a semisimple group and let K be a semialgebraic maximal compact subgroup of G. Then the restriction of d to G pushes forward to a metric on the symmetric space G/K which is left-invariant under the action of G. See [Abe04] for more on this topic.

5.4.4 Hilbert Metrics

Hilbert metrics are analogues of hyperbolic space where the unit sphere is replaced by a definable, open, convex $U \subseteq R^k$. Let S be the boundary of U in R^k. Fix $x, y \in U$. Let L be the line in R^k connecting x and y, and let $\{p, q\} = L \cap S$. The **Hilbert metric** d on U is given by:

$$d(x, y) = \log \frac{\|y - p\| \|x - q\|}{\|x - p\| \|y - q\|}.$$

5.4.5 Approximate Ultrametrics

Any bounded metric space is trivially an approximate ultrametric space. We describe non-trivial definable approximate ultrametric spaces.

Fact 5.4.2. *Let (X, d) be a definable metric space. Then $(X, \log[1 + d])$ is an approximate ultrametric space.*

Proof. We first show that $\log[1+d]$ is a metric, it suffices to show that the triangle inequality holds. For all $x, y, z \in X$:

$$\log[1 + d(x, z)] \leqslant \log[1 + d(x, y) + d(y, z)] \leqslant \log[1 + d(x, y) + d(y, z) + d(x, y)d(y, z)]$$

$$= \log[(1 + d(x, y))(1 + d(y, z))] = \log[1 + d(x, y)] + \log[1 + d(y, z)].$$

Furthermore $\log[1 + d]$ satisfies an approximate ultrametric inequality as:

$$\log[1 + d(x, z)] \leqslant \log[1 + d(x, y) + d(y, z)] \leqslant \log(2) + \max\{\log[1 + d(x, y)], \log[1 + d(y, z)]\}$$

holds for all $x, y, z \in X$. $\square$

5.4.6 Multiplicative Metrics

In this subsection we no longer suppose that $\mathcal{R}$ defines a logarithm. An **additive** metric is an $(R, +, \leqslant)$-valued metric. We consider $(R^>, \cdot, \leqslant)$ as an ordered vector space over the field of powers $\mathbf{k}$ of $\mathcal{R}$ in the natural way. A definable **multiplicative metric space** (X, d) is a definable set X together with a definable $(R^>, \cdot, \leqslant)$-valued metric d on X. If $\mathcal{R}$ defines a logarithm then the logarithm of a multiplicative metric is an additive metric. All of the metric spaces described previously in this section can be considered as definable multiplicative metrics over arbitrary o-minimal expansions of ordered fields.

5.5 Carnot Metrics

In this section we describe (what we call) Carnot metrics. These are certain semialgebraic left-invariant metrics on certain nilpotent Lie groups, which we call Carnot Groups. There are many sources on Carnot groups and metrics such as [Mon02] and [Bel96]. Let $(G, \cdot)$ be a nilpotent Lie Group with underlying set $\mathbb{R}^k$. We identify the Lie algebra of G with $\mathbb{R}^k$. We say that G is a **Carnot group** if there are linear subspaces $V_1, \ldots, V_n \subseteq \mathbb{R}^k$ such that $\mathbb{R}^k = V_1 \oplus \ldots \oplus V_n$ and $[V_j, V_1] \subseteq V_{j+1}$ holds for all $1 \leqslant j \leqslant n$. For the remainder of this section G is a Carnot group. For $1 \leqslant i \leqslant n$ we set $h_i = \dim(V_1) + \ldots + \dim(V_i)$. Letting $e_1, \ldots, e_k$ be the standard basis vectors of $\mathbb{R}^k$ we can suppose without lose of generality that for each $1 \leqslant i \leqslant n$, $\{e_j : h_{i-1} < j \leqslant h_j\}$ is a basis for V_i. Then the group product on $\mathbb{R}^k$ is of the form

$$x \cdot y = x + y + Q(x, y)$$

where $Q(x, y) = (Q_1(x, y), \ldots, Q_n(x, y))$ for some polynomial functions $Q_i : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ of degree at most k.

For each $\lambda \in \mathbb{R}^>$ there is a dilation $\delta_\lambda : \mathbb{R}^k \to \mathbb{R}^k$ given by

$$\delta_\lambda(x_1, \ldots, x_k) = (\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_k} x_k)$$

where $\alpha_j = i$ when $h_{i-1} < j \leqslant h_i$. Each dilation δ_λ is an automorphism of $(G, \cdot)$. We define a norm $\|, \|_G$ on G by declaring

$$\|(x_1, \ldots, x_k)\|_G = \sum_{i=1}^{k} |x_i|^{\frac{1}{\alpha_i}}.$$

Then $\|, \|_G$ satisfies the following for all $x, y \in \mathbb{R}^k$ and $\lambda \in \mathbb{R}^>$:

i. $\|x\|_G = 0$ if and only if $x = 0$,

ii. $\|x^{-1}\|_G = \|x\|_G$,

iii. $\|\delta_\lambda(x)\|_G = \lambda\|x\|_G$,

iv. $\|x \cdot y\|_G \leqslant \|x\|_G + \|y\|_G$.

We put a left-invariant metric d on G by setting $d(x, y) = \|y^{-1}x\|_G$. We call such a metric space a **Carnot metric** on G. From what we have said the following is clear:

Fact 5.5.1. *The collection of Carnot groups with underlying set $\mathbb{R}^k$ forms a semialgebraic family of metric spaces.*

We recall a theorem of Pansu [Pan89], one of the marvels of metric geometry:

Theorem 5.5.2. *Let G_1 and G_2 be Carnot groups, equipped with Carnot metrics. If any open subset of G_1 is bilipschitz equivalent to any open subset of G_2 then G_1 and G_2 are isomorphic as groups. Furthermore if G_1 is nonabelian then G_1 is not bilipschitz equivalent to any subset of euclidean space equipped with the induced euclidean metric.*

In contrast, if X is an l-dimensional definable set then it follows from the C^1-cell decomposition that there is an open subset of (X, e) which is bilipschitz equivalent to an open subset of $(\mathbb{R}^l, e)$. Thus if X' is another l-dimensional definable set then there are definable open subsets $U \subseteq X, U' \subseteq X'$ which admit a definable bilipschitz equivalence $(U, e) \to (U', e)$.

There are many Carnot groups, see [Bel96].

Fact 5.5.3. *There are continuum many pairwise non-isomorphic Carnot groups with underlying set $\mathbb{R}^6$.*

Pansu's Theorem implies that there is a semialgebraic family of 6-dimensional metric spaces which contains continuum many elements up to bilipschitz equivalence and that there are semialgebraic metric spaces which do not admit bilipschitz embeddings into any euclidean space. As Carnot metrics are conical, every Carnot metric is isometric to any of its asymptotic cones. Lemma 3.7.11 shows that two quasi-isometric Carnot metrics are bilipschitz equivalent. Thus there are continuum many semialgebraic metric spaces up to quasi-isometry.

Finally, it is known that

$$\dim_{\mathcal{H}}(G, d) = \sum_{j=1}^{n} j \dim(V_j).$$

Thus $\dim_{\mathcal{H}}(G, d) > \dim(G)$ when $(G, \cdot)$ is not abelian. This gives examples of semialgebraic metric spaces on which the Hausdorff dimension is strictly greater than the topological dimension.

5.6 Euclidean Groups

We give another example of a definable group admitting a semialgebraic left-invariant metric. Let $E(n)$ be the isometry group of $(\mathbb{R}^n, e)$. Let $S \subseteq E(n)$ be a set of n linearly independent vectors in $\mathbb{R}^n$. If $\sigma, \tau \in E(n)$ are such that $\sigma(x) = \tau(x)$ for all $x \in S$ then $\sigma = \tau$. We put a semialgebraic left-invariant metric d on $E(n)$ by declaring:

$$d(\sigma, \tau) = \max\{\|\sigma(x) - \tau(x)\| : x \in S\} \quad \text{for all } \sigma, \tau \in E(n).$$

5.7 Metrics Associated To Definable Families

In this section we describe definable metric spaces associated to definable families of functions and sets.

5.7.1 Hausdorff Metrics

Let $A, B \subseteq R^k$ be definable. The **Hausdorff distance** d_H between A and B is the infimum of all $\delta \in R_\infty$ such that for every $a \in A$ there is a $b \in B$ satisfying $\|a - b\| \leqslant \delta$ and for all $b \in B$ there is a $a \in A$ such that $\|a - b\| \leqslant \delta$. This is an R_∞-valued pseudometric on the collection of definable subsets of R^k. Note that if A and B are bounded then $d_H(A, B) < \infty$ and that the Hausdorff distance between a definable set and its closure is zero.

Let $\mathcal{A} = \{A_x : x \in R^l\}$ be a definable family of subsets of R^k such that $d_H(A_x, A_y) < \infty$ for all $x, y \in R^l$. We put a definable pseudometric d on R^l by declaring

$$d(x, y) = d_H(A_x, A_y) \quad \text{for all } x, y \in R^l.$$

5.7.2 Metric Associated To Definable Families of Functions

Let $\mathcal{F} = \{f_x : x \in R^l\}$ be a definable family of functions $R^k \to R$. If each element of $\mathcal{F}$ is bounded then we put a uniform pseudometric d_∞ on R^l by declaring:

$$d_\infty(x, y) = \|f_x - f_y\|_\infty = \sup\{|f_x(a) - f_y(a)| : a \in R^k\} \quad \text{for all } x, y \in R^l.$$

If every element of $\mathcal{F}$ is C^r with bounded rth derivative then

$$d(x, y) = \max_{0 \leqslant i \leqslant r}\{\|f_x^{(i)} - f_y^{(i)}\|_\infty\}$$

is a definable pseudometric on R^l. Thomas considered definable metric spaces of this form in [Tho12].

5.7.3 Examples constructed using the CLR-volume Theorem

Let μ_m be the m-dimensional Lebesgue measure on $\mathbb{R}^k$. Comte, Lion and Rolin prove the following proposition in [CR00].

Proposition 5.7.1. *Let $\{A_x : x \in \mathbb{R}^l\}$ be an $\bar{\mathbb{R}}_{\mathrm{an}}$-definable family of subsets of $\mathbb{R}^k$. Then $\mu_m(A_x)$ is an $\bar{\mathbb{R}}_{\mathrm{an,exp}}$-definable function $\mathbb{R}^l \to \mathbb{R}_\infty$.*

This implies that if $f : \mathbb{R}^l \times \mathbb{R}^k \to \mathbb{R}$ is $\bar{\mathbb{R}}_{\text{an}}$-definable then

$$\int_{\mathbb{R}^k} f(a,x)\, dx$$

is an $\bar{\mathbb{R}}_{\text{an,exp}}$-definable, $\mathbb{R}_\infty$-valued function of $a \in \mathbb{R}^l$.

Let $\mathcal{F} = \{f_x : x \in \mathbb{R}^l\}$ be an $\bar{\mathbb{R}}_{\text{an}}$-definable family of functions $\mathbb{R}^k \to \mathbb{R}$. If every element of $\mathcal{F}$ is L^p integrable then

$$d_p(x,y) = \left[\int_{\mathbb{R}^k} |f_x(a) - f_y(a)|^p \, d\mu_k \right]^{\frac{1}{p}}$$

is an $\bar{\mathbb{R}}_{\text{an,exp}}$-definable pseudometric on $\mathbb{R}^l$. If $\{A_x : x \in \mathbb{R}^l\}$ is an $\bar{\mathbb{R}}_{\text{an}}$-definable family of bounded m-dimensional subsets of $\mathbb{R}^k$ then

$$d_\mu(x,y) = \mu[A_x \Delta A_y]$$

is an $\bar{\mathbb{R}}_{\text{an,exp}}$-definable pseudometric on $\mathbb{R}^l$.

If d, d' are metrics on X and $\lambda, \lambda' \in \mathbb{R}^>$ then $\lambda d + \lambda d'$ is a metric on X. Let $\{d_q : q \in \mathbb{R}^l\}$ be a uniformly bounded $\bar{\mathbb{R}}_{\text{an}}$-definable family of metrics on X and let $g : \mathbb{R}^l \to \mathbb{R}^>$ be an $\bar{\mathbb{R}}_{\text{an}}$-definable integrable function. It is easily checked, using the results quoted above, that

$$d(x,y) = \int_{\mathbb{R}^l} g(q) d_q(x,y)\, dq$$

is an $\bar{\mathbb{R}}_{\text{an,exp}}$-definable metric on X.

5.8 Definable Metrics with Noneuclidean Topology

In this section we give examples of definable metric spaces which are not homeomorphic to any definable set equipped with its euclidean topology.

5.8.1 Discrete Metric Spaces

Let X be a definable set. We put a discrete metric on X by declaring $d_{\text{disc}}(x,y) = 0$ when $x = y$ and $d_{\text{disc}}(x,y) = 1$ otherwise. There are many other discrete definable metric spaces:

Proposition 5.8.1. *Every definable metric space is definably quasi-isometric to a discrete definable metric space.*

Proof. Let (X, d) be a definable metric space. Let $d' : X^2 \to R^{\geqslant}$ be given by setting $d(x, y) = 0$ when $x = y$ and

$$d'(x, y) = \min\{1, d(x, y)\} \quad \text{when } x \neq y.$$

Then d' is a metric on X, (X, d') is discrete and $id : (X, d) \to (X, d')$ is a quasi-isometry. □

5.8.2 A Definable Metric Space with Cantor Rank 1

We give an example of a definable metric space with Cantor rank 1. Let d be the metric on $[0, 1]$ given by $d(x, y) = 0$ when $x = y$ and $d(x, y) = \max\{x, y\}$ otherwise. It is clear that d is symmetric and reflective. The ultrametric triangle inequality holds as:

$$\max\{x, y\} \leqslant \max\{\max\{x, z\}, \max\{y, z\}\} = \max\{d(x, z), d(y, z)\} \quad \text{for all } x, y, z \in X.$$

Every point in $[0, 1]$ is isolated except for 0, which is in the closure of $(0, 1)$. Therefore, $([0, 1], d)$ has Cantor rank 1. The metric d is $(R, \leqslant)$ definable.

Proposition 5.8.2. *Let (X, d) be a definable metric space which is $(R, \leqslant)$-definable. There is a $\delta \in R^{>}$ such that if $A \subseteq X$ has diameter less than δ then (A, d) is an ultrametric space.*

Proof. If $h : R^{>} \to R^{>}$ is an $(R, \leqslant)$-definable function then there is a $\delta \in R^{>}$ such that the restriction of h to $(0, \delta)$ is either constant or the identity, see [vdD]. Let $g : R^{>} \to R^{\geqslant}$ be the definable function given by:

$$g(t) = \sup\{d(x, z) : x, z \in X (\exists y \in X) d(x, y), d(y, z) \leqslant t\}.$$

The triangle inequality implies that $g(t) \leqslant 2t$ for all $t \in R^{>}$, so the definition of g makes sense. As $0 < g(t) \leqslant 2t$ for all $t \in R^{>}$ there is a $\delta > 0$ such that the restriction of g to $(0, \delta)$ is the identity or is the constant zero function. If $g|_{(0,\delta)} = 0$ then $x = y$ whenever $d(x, y) < \delta$, in this case (X, d) is discrete. Suppose that $g(t) = t$ when $0 < t < \delta$. Let $A \subseteq X$ have diameter less then δ, then:

$$d(x, z) \leqslant g(\max\{d(x, y), d(y, z)\}) = \max\{d(x, y), d(y, z)\} \quad \text{for all } x, y, z \in A.$$

Thus (A, d) is an ultrametric space. □

5.8.3 Metric Spaces Associated to Definable Simplicial Complexes

In this section $\mathcal{R}$ is an expansion of the reals considered as an ordered vector space over itself. We assume that the reader is familiar with simplicial complexes. Our source is the classic [Spa66]. An **abstract simplicial complex** $\mathcal{V} = (V, \mathcal{C})$ is a set V of vertices together with a collection $\mathcal{C}$ of finite subsets of V such that:

i. Every singleton subset of V is an element of $\mathcal{C}$,

ii. Every subset of an element of $\mathcal{C}$ is an element of $\mathcal{C}$.

We say that $\mathcal{V}$ is **finite-dimensional** if there is an $N \in \mathbb{N}$ such that every element of $\mathcal{C}$ has cardinality at most N. We only consider finite-dimensional simplicial complices. We say that $\mathcal{V}$ is **locally finite** if every element of $\mathcal{C}$ intersects only finitely many elements of $\mathcal{C}$. A **definable abstract simplicial complex** is a definable set V together with a definable family $\mathcal{C} = \{C_x : x \in \mathbb{R}^l\}$ of finite subsets of V such that $(V, \mathcal{C})$ is an abstract simplicial complex. We associate a topological space $|\mathcal{V}|$ to an abstract simplicial complex $\mathcal{V}$ called the **geometric realization** of $\mathcal{V}$. If $\mathcal{V}$ is locally finite then the construction we give is equivalent to any other construction of the geometric realization that the reader may have seen. We let $|\mathcal{V}|$ be the set of functions $\alpha : V \to [0, 1]$ such that:

$$\{v \in V : \alpha(v) \neq 0\} \in \mathcal{C} \quad \text{and} \quad \sum_{v \in V} \alpha(v) = 1.$$

We put a metric $d_{\mathcal{V}}$ on $|\mathcal{V}|$ by declaring:

$$d_{\mathcal{V}}(\alpha, \beta) = \sum_{v \in V} |\alpha(v) - \beta(v)|.$$

It is more customary to use an L^2 metric:

$$d_2(\alpha, \beta) = \sqrt{\sum_{v \in V} |\alpha(v) - \beta(v)|^2}.$$

If $\mathcal{V}$ is finite-dimensional then $id : (|\mathcal{V}|, d_{\mathcal{V}}) \to (|\mathcal{V}|, d_2)$ is bilipschitz. We use the L^1-metric as it ensures that:

Fact 5.8.3. *If $\mathcal{V}$ is a definable abstract simplicial complex then $(|\mathcal{V}|, d_{\mathcal{V}})$ is a definable metric space.*

This construction produces interesting semilinear metric spaces.

5.8.4 Cayley Graphs

Let G be a finitely generated group. We say that G is **definably representable** if there is a free action of G by definable functions on a definable set A. Any finitely generated subgroup of $Gl_n(\mathbb{R})$ is definably representable.

Question. Is every finitely generated group definably representable?

Suppose that G is a finitely generated group with a fixed symmetric set of generators $S = \{g_1, \dots, g_n\}$. Let $G \curvearrowright A$ be a free action of G on a definable set A. We define the **Cayley graph** of this action. This depends on the choice of S. We consider the elements of S as functions $A \to A$. The Cayley graph of $G \curvearrowright A$ is the graph $\mathcal{G}$ with vertex set A and edge set

$$E = \{(a, b) \in A : (\exists 1 \leqslant i \leqslant n) g_i(a) = b\}.$$

Then $\mathcal{G}$ is a definable graph, and the geometric realization of $\mathcal{G}$ is a definable metric space. As a graph, $\mathcal{G}$ is the disjoint union of continuum many copies of the usual Cayley graph of G. In this manner we construct definable metric spaces with unexpected properties.

Example (Sketch). We give an example of two homeomorphic semilinear metric spaces $(X, d), (X', d')$ such that if $\tau : (X, d) \to (X', d')$ is a homeomorphism then τ induces a non Lebesgue measurable map. We let:

$$(X, d) = (\mathbb{R}, e) \times (\mathbb{R}, d_{\text{disc}}).$$

Let $\lambda \in (0,1)$ be an irrational number. Let $\theta : [0,1] \to [0,1]$ be given by $\theta(t) = t + \lambda$ when $t + \lambda \in [0,1]$ and $g(t) = t + \lambda - 1$ otherwise. Consider the action of $(\mathbb{Z}, +)$ on $[0,1]$ generated by θ. Let $\mathcal{G}$ be the Cayley graph of this action and let (X', d') be the geometric realization of $\mathcal{G}$. As a topological space, (X', d') is the disjoint union of continuum many copies of $(\mathbb{R}, e)$, so (X', d') is homeomorphic to (X, d). We put an equivalence relation $\sim$ on X' by declaring $x \sim y$ when x and y lie in the same connected component of (X', d'). Recall that $[0,1]$ is the set of vertices of $\mathcal{G}$. Two elements of $[0,1]$ are $\sim$-equivalent if and only if they lie in the same orbit of the action of $(\mathbb{Z}, +)$. The restriction of $\sim$ is essentially the Vitali equivalence relation. Suppose that $h : (X', d') \to (X, d)$ is a homeomorphism. Two elements $x, y \in X$ are in the same connected component of (X, d) if and only if $h(x)$ and $h(y)$ lie in the same connected component of (X, d). Let $\tau : [0,1] \to \mathbb{R}$ be the restriction of $\pi \circ h$ to $[0,1]$. If $x, y \in [0,1]$ then $x \sim y$ holds if and only if $\tau(x) = \tau(y)$. It is well-known that such a map $\tau : [0,1] \to \mathbb{R}$ cannot be Lebesgue measurable.

Example. A definable metric space (Y, d) is **definably connected** if $\emptyset$ and Y are the only definable clopen subsets of Y. Let (X', d') be the definable metric space constructed in the previous example. We show that (X', d') is definably connected. This gives an example of a definably connected metric space which is not connected. Suppose that $A_1, A_2 \subseteq X'$ are disjoint nonempty clopen definable subsets of X. Fixing $p_1 \in A_1, p_2 \in A_2$ we let $B_1, B_2 \subseteq [0,1]$ be the set of vertices which lie in the connected component of p_1, p_2, respectively. As B_1 and B_2 are orbits of the action of $(\mathbb{Z}, +)$ on $[0,1]$, both B_1 and B_2 are dense in $([0,1], e)$. As B_1 and B_2 are definable this implies $B_1 \cap B_2 \neq \emptyset$, contradiction.

In the next example we use a simple topological fact. We leave the proof to the reader.

Fact 5.8.4. *Let $\mathcal{G}_1, \mathcal{G}_2$ be graphs where every vertex has degree at least 3. If the geometric realizations $|\mathcal{G}_1|$ and $|\mathcal{G}_2|$ are homeomorphic then $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic as graphs.*

Example 5.8.5 (Sketch). We show that the Trivialization Theorem does not hold for definable families of metric spaces. We construct a semialgebraic family of metric spaces which contains infinitely many elements up to homeomorphism. Let $S = \{x \in \mathbb{R}^2 : \|x\| = 1\}$, let $U(1)$ be the group of rotations of S and let τ be the reflection across the x-axis. If $\sigma \in U(1)$ then

$\{\sigma, \tau\}$ generate a definable action of a dihedral group D_σ on S. The group D_σ is finite if and only if σ is a rational rotation. Note that the Cayley graph of this action is 3-regular. Given $\sigma \in U(1)$ we let X_σ be the geometric realization of the Cayley graph of this action. Then $\{X_\sigma : \sigma \in U(1)\}$ is a semialgebraic family of metric spaces. Every connected component of X_σ is homeomorphic to the geometric realization of the Cayley graph of D_σ. Suppose that $\sigma, \eta \in U(1)$ and that $h : X_\sigma \to X_\eta$ is a homeomorphism. Then h maps connected components to connected components and thus induces a homeomorpism between the geometric realizations of the Cayley graphs of D_σ and D_η. Fact 5.8.4 implies that the Calyey graphs of D_σ and D_η are isomorphic as graphs. Thus, if σ and η are rational rotations such that $|D_\sigma| \neq |D_\eta|$ then X_σ is not homeomorphic to X_η. So the family $\{X_\sigma : \sigma \in U(1)\}$ contains infinitely many pairwise non-homeomorphic elements.

5.9 Subriemannian Metrics

We refer to [Mon02] for a general introduction to subriemannian geometry. Let $M \subseteq \mathbb{R}^k$ be a smooth submanifold of euclidean space and let D be a l-dimensional distribution on U for some $l < k$, i.e. D is a function which associates to each $p \in M$ an l-dimensional subspace of the tangent space of M at p. We say that a C^1 curve $\gamma : [0,1] \to M$ is **horizontal** if:

$$\gamma'(p) \in D_{\gamma(p)} \quad \text{for all } t \in \mathbb{R}.$$

We assume that any two points in M are connected by a horizontal curve. We put a metric d_D on M by declaring the distance between two points in M to be the infimum of the euclidean lengths of horizontal curves connecting them. Such a metric is said to be **subriemannian**. We now additionally assume that M is a real analytic submanifold of euclidean space. The following question has been of interest to subriemannian geometers:

Question 5.9.1. *For what real analytic distributions D on M is the associated subriemannian d_D metric subanalytic?*

There is now a body of work around this question. There are certain classes of real-analytic distributions D for which d_D is known to be subanalytic and other classes of real-analytic distributions for which the metric is known not to be subanalytic. See [AG01] and the references therein for more information. If d_D is subanalytic then the restriction of d_D to any compact subset of M is $\bar{\mathbb{R}}_{\mathrm{an}}$-definable. The Hausdorff dimension of a subriemannian metric space is strictly greater then its topological dimension. This gives examples of $\bar{\mathbb{R}}_{\mathrm{an}}$-definable metric spaces where the topological dimension is strictly less then the Hausdorff dimension.

CHAPTER 6

O-minimal Lemmas

In this chapter we prove a number of technical lemmas about o-minimal expansions of ordered fields to be used later. We do not claim that any of these lemmas are original.

6.1 Generic Properties

Throughout this thesis we study generic properties of definable metric spaces. To show that a property holds almost everywhere on a definable set X we will typically show that every definable open subset of X contains a definable open set on which the property holds. In this section we prove lemmas which we use to contruct open subsets on which generic properties hold.

Lemma 6.1.1. *Let X be a definable set and let $f : X \to R^{>}$ be a definable function. There is an open $V \subseteq X$ and a $\delta > 0$ such that $f(x) > \delta$ for all $x \in V$.*

Proof. Let $p \in X$ be a point at which f is continuous. Let V be a definable neighborhood of p such that $f(x) > \frac{1}{2}f(p)$ for all $x \in V$. □

Lemma 6.1.2 below is a generalization of Lemma 6.1.1. Lemma 6.1.1 follows by applying Lemma 6.1.2 to the family of sets of the form $A_t = \{x \in X : f(x) > t\}$. Recall that a family of sets $\{A_x : x \in I\}$ is directed if for every $x, y \in I$ there is a $z \in I$ such that $A_x \cup A_y \subseteq A_z$.

Lemma 6.1.2. *Let $\mathcal{A} = \{A_x : x \in R^l\}$ be a directed definable family of subsets of R^k such that*

$$\operatorname{Int}\left(\bigcup_{x \in R^l} A_x\right) \neq \emptyset.$$

There is an $x \in R^l$ such that A_x has nonempty interior.

Proof. We apply induction on k. Suppose $k = 1$. As $\mathcal{A}$ is directed and the union of $\mathcal{A}$ is infinite, for every $N \in \mathbb{N}$ there is an $x \in R^l$ such that $|A_x| > N$. Thus there is a $x \in R^l$ such that A_x is infinite and hence has nonempty interior in R. Suppose $k \geqslant 2$. Let B be a product of k open intervals which is contained in the union of $\mathcal{A}$. Let $\pi : B \to R^{k-1}$ be the projection onto the first coordinate. For each $x \in R^l$ let $B_x \subseteq R^{k-1}$ be the set of y such that $A_x \cap \pi^{-1}(y)$ has nonempty interior in $\pi^{-1}(y)$. The base case implies that for every $y \in R^{l-1}$ there is an $x \in R^l$ such that $A_x \cap \pi^{-1}(y)$ has nonempty interior in $\pi^{-1}(y)$. Thus

$$\bigcup_{x \in R^l} B_x = \pi(B).$$

As $\{B_x : x \in R^l\}$ is a directed family the inductive hypothesis implies that B_x has nonempty interior in R^{k-1}. If B_x has nonempty interior then the fiber lemma for o-minimal dimension implies that $\dim(A_x) = k$ and so A_x has nonempty interior. □

Lemma 6.1.3. *Let X be a definable set. Let $A \subseteq X \times R^{>}$ be a definable set such that $X \times \{0\} \subseteq \mathrm{cl}(A)$. There is an open $V \subseteq X$ and a $\delta > 0$ such that $V \times (0, \delta) \subseteq A$.*

Proof. Let $g : X \to R$ be given by

$$g(x) = \sup\{t \in (0,1) : \{x\} \times (0,t) \subseteq A\} \quad \text{for all } x \in X.$$

If $U \subseteq X$ is open and $g(x) = 0$ for all $x \in U$ then $U \times \{0\}$ does not lie in the closure of A. Thus there is an open $U \subseteq X$ such that $g(x) > 0$ for all $x \in U$. The lemma follows by applying Lemma 6.1.1. □

Lemma 6.1.4. *Let X be definable set and let $\{A_t : t \in R^{>}\}$ be a definable family of subsets of X such that $\dim(A_t) < \dim(X)$ for all $t \in R^{>}$. Then there is an open $V \subseteq X$ and a $\delta \in R^{>}$ such that $V \cap A_t = \emptyset$ when $0 < t \leqslant \delta$.*

Proof. Let $A = \{(x, t) \in X \times R^{>} : x \in A_t\}$. The fiber lemma for o-minimal dimension implies $\dim(A) < \dim(X) + 1$, it follows that the closure of A in $X \times R^{\geqslant}$ has dimension at most $\dim(X) - 1$. Thus $X \times \{0\}$ is contained in the closure of $[X \times R^{>}] \setminus A$. Apply Lemma 6.1.3. □

Lemma 6.1.5. *Let $U \subseteq R^k$ be open and definable. Let $\{f_t\}_{t \in R^>}$ be a definable family of functions $U \to R$ which pointwise converges to the constant zero function as $t \to 0^+$. There is an open $V \subseteq U$ such that $f_t|_V$ uniformly converges to zero as $t \to 0^+$.*

Proof. For each $x \in U$ we let $A_x \subseteq U$ be the set of y such that $f_t(y) \leqslant f_t(x)$ holds when $0 < t \ll 1$. For every $x, y \in U$ we either have $x \in A_y$ or $y \in A_x$, so the family $\{A_x : x \in U\}$ is directed. Applying Lemma 6.1.2 there is a $z \in U$ and open $V \subseteq U$ such that $V \subseteq A_z$. We declare $g(t) = f_t(z)$. For every $x \in V$ there is a $\delta > 0$ such that $f_t(x) \leqslant g(t)$ when $0 < t < \delta$. After applying Lemma 6.1.1 and replacing V with a smaller open set if necessary there is a $\delta > 0$ such that:

$$f_t(x) \leqslant g(t) \quad \text{for all } x \in V, 0 < t \leqslant \delta.$$

Thus $f_t|_V$ uniformly converges to the constant zero function as $t \to 0^+$. □

In the proof that follows we let $\nabla f(p)$ be the derivative of a definable function $f : R^k \to R$ at p if it exists. Then $\nabla f(p)$ is by definition a linear function $R^k \to R^k$. We take ∇f to be the function which maps each p to the derivative at p. We say that family of such derivatives converges to zero if it converges to zero with respect to the operator norm.

Lemma 6.1.6. *Let $U \subseteq R^k$ be open and definable. Let $\{f_t\}_{t \in R^>}$ be a definable family of functions $U \to R^>$ which converges uniformly to the constant zero function as $t \to 0^+$. There is an open $V \subseteq U$ such that f_t is C^1 on V when $0 < t \ll 1$ and such that $\nabla f_t|_U$ uniformly converges to the constant zero function as $t \to 0^+$.*

Proof. For each t let $W_t \subseteq U$ be the set of points at which f_t is C^1. It follows from Lemma 6.1.4 that there is an open $V' \subseteq V$ such that $V' \subseteq W_t$ when $0 < t \ll 1$. This proves the first claim. We now produce an open $V \subseteq V'$ such that $\nabla f_t|_U$ uniformly converges to the constant zero function as $t \to 0^+$. After replacing U with W if necessary we suppose that f_t is C^1 when t is sufficiently small. We let $\partial_i f_t$ be the the ith partial derivative of f_t. It suffices to find an open $V \subseteq U$ on which $\partial_i f_t$ uniformly converges to zero as $t \to 0^+$ for all i. We fix an i and produce an open $V \subseteq U$ such that $\partial_i f_t$ uniformly converges to zero as $t \to 0^+$, a repeated application of this argument proves the result for all i. Applying Lemma **??** it is enough to find an open $V \subseteq U$ such that $\partial_i f_t$ pointwise converges to zero as $t \to 0^+$. We suppose towards a contradiction that there is no such V. There is then an open $O \subseteq U$ such that $\lim_{t\to 0^+} \partial_i f_t(p) \neq 0$ holds at every $p \in O$. Then O contains an open subset O' such that either $\lim_{t\to 0^+} \partial_i f_t(p) > 0$ holds for all $p \in O'$ or $\lim_{t\to 0^+} \partial_i f_t(p) < 0$ holds for all $p \in O'$. We only treat the first case as the second follows in the same way. After replacing O with O' if necessary we suppose without lose of generality that $\lim_{t\to 0^+} \partial_i f_t(p) > 0$ holds for all $p \in O$. Thus for every $p \in O$ there are $\delta, \epsilon > 0$ such that $\partial_i f_t(p) > \delta$ when $0 < t \leqslant \delta$. After applying Lemma 6.1.1 and replacing O with a smaller open set if necessary we may suppose that for fixed $\delta, \epsilon > 0$ we have $\partial_i f_t(p) > \delta$ for all $0 < t \leqslant \epsilon$ and $p \in O$. Let e_i be the ith standard basis vector. Fix $x \in O$ and let $y \in O$ be such that $y = x + se_i$ for some $s > 0$. The Mean Value Theorem implies:

$$f_t(y) \geqslant s\delta + f_t(x) \geqslant s\delta \quad \text{for all } 0 < t \leqslant \epsilon.$$

This gives a contradiction as $f_t(y) \to 0$ as $t \to 0^+$. □

In the lemma below we let $\mathbb{G}(l, l-1)$ be the set of $(l-1)$-dimensional hyperplanes in R^l. We take $\mathbb{G}(l, l-1)$ to be a semialgebraic set, see 3.4 of [BCR98]. For a definable C^1-submanifold A of R^l and $p \in A$ we let T_pA be the tangent space of A at p, considered as an element of $\mathbb{G}(l, l-1)$.

Lemma 6.1.7. *Let $H \subseteq R^l$ be an $(l-1)$-dimensional hyperplane. Let $O \subseteq H$ be definable and open. Let $\{A_t : t \in R^>\}$ be a definable family of pairwise disjoint closed $(l-1)$-dimensional subsets of R^l such that A_t converges in the Hausdorff metric to O as $t \to 0^+$. Suppose also that $A_t \cap H = \emptyset$ for every $t \in R^>$. Let $\delta \in R^>$. There is a definable open $W \subseteq R^l$ such that:*

$$W \subseteq \bigcup_{t \in R^>} A_t,$$

and for every $t \in R^>$:

i. $A_t \cap W$ is a a C^1-submanifold of R^l.

ii. If $p \in A_t \cap W$ then $\|(H - T_p(A_t)\| < \delta$, *where we consider H and $T_p(A_t)$ to be elements of $\mathbb{G}(l, l-1)$.*

Proof. It suffices to consider the case $H = R^{l-1} \times \{0\}$. We consider O as a subset of R^{l-1}. Let

$$B = \bigcup_{t \in R^{>}} A_t.$$

As O is the Hausdorff limit of A_t as $t \to 0^+$, O is a subset of of the closure of B. Applying the proof of Lemma 6.1.3 there is a definable open $U \subseteq O$ and an $s \in R^{>}$ such that either $U \times (0, s] \subseteq B$ or $U \times [-s, 0) \subseteq B$. It is enough to consider the first case. Let $g : U \times (0, s] \to R^{>}$ be the function with takes each $p \in U \times (0, s]$ to the unique $t \in R^{>}$ such that $p \in A_t$. It follows that for each $p \in U$, $g_p(t) \to 0$ as $t \to 0^+$. For every $p \in U$ there is a $s' \in R^{>}$ such that the restriction of g_p to $(0, s')$ is continuous and strictly increasing. By Lemma 6.1.1 there is an open $U' \subseteq U$ and some $0 < s' < s$ such that for all $p \in U'$ the restriction of g_p to $(0, s')$ is continuous and strictly increasing. By replacing U and s with U' and s' if necessary we suppose with loss of generality that g_p is continuous and strictly increasing for all $p \in U$. After again applying Lemma 6.1.1 and replacing U with a smaller open set if necessary we may suppose that $r \in R^{>}$ is such that $g_p(s) > r$ holds for all $p \in U$. Thus for every $p \in U$ and $0 < t < r$ there is a unique $s' \in (0, s)$ such that $(p, s') \in A_t$. Let $f : (0, r) \times U \to R^{>}$ be the definable function such that $(p, f_t(p)) \in A_t$ holds for all $(p, t) \in (0, r) \times U$. Then,

$$\text{Graph}(f_t) = A_t \cap [U \times (0, s)] \quad \text{for all } t \in (0, r).$$

As f_t uniformly converges to the constant 0 function as $t \to 0^+$,we apply Lemma 6.1.6. Thus there is an open $V \subseteq U$ and an $0 < r' < r$ such that for each $t \in (0, r')$ the restriction of f_t to V is C^1 and the norm of the derivative of f_t uniformly converges to zero as $t \to 0^+$. It follows that $\text{Graph}(f_t|_V)$ is a C^1-submanifold and that for all $p \in V$ we have $\|H - T_p(A_t)\| \leqslant \delta$ when t is sufficiently small. As the union

$$\bigcup_{0<t<r'} \text{Graph}(f_t|_U)$$

is disjoint, the fiber lemma for o-minimal dimension gives an open $W \subseteq R^l$ contained in this union. W satisfies the conditions in the statement of the lemma. □

Lemma 6.1.8. *Let $F : R \times R^{\geqslant} \to R^{\geqslant}$ be a continuous function definable function such that $F(x, 0) = 0$ for all $x \in R$ and there is a $\lambda \in R^{>}$ such that $F(x, t) \leqslant \lambda t$ for all $(x, t) \in R \times R^{\geqslant}$. Then for all but finitely many $x \in R$:*

$$\lim_{t \to 0^+} \frac{F(x,t)}{t} = \lim_{t \to 0^+} \frac{F(x+t,t)}{t}.$$

Proof. We show that for every definable open $U \subseteq R$ this is an open interval $J \subseteq U$ such that

$$\lim_{t \to 0^+} \frac{F(x,t)}{t} = \lim_{t \to 0^+} \frac{F(x+t,t)}{t} \quad \text{for all } x \in J.$$

We only treat the case $U = R$. We let $F_t(x) = F(x, t)$. Then $F_t : R \to R^{\geqslant}$ uniformly converges to the constant zero function as $t \to 0$. By Lemma 6.1.6 there is an open interval J and a $\delta \in R^{>}$ such that the restriction of F_t to J is C^1 when $0 < t < \delta$ and such that the restriction of F'_t to J uniformly converges to the constant zero function as $t \to 0^+$. Fix $x \in J$ and $0 < t < \delta$. By the Mean Value Theorem:

$$\left| \frac{F(x+t,t)}{t} - \frac{F(x,t)}{t} \right| \leqslant \sup\{|F'_t(p)| : x < p < x+t\}$$

so

$$\lim_{t \to 0^+} \left| \frac{F(x+t,t)}{t} - \frac{F(x,t)}{t} \right| = 0.$$

□

Lemma 6.1.9. *Let $A \subseteq R^k$ be an l-dimensional definable set. There is a definable open $U \subseteq A$ and a coordinate projection $\pi : R^k \to R^l$ such that $\pi(U)$ is open and for some $\lambda \in R^{>}$ we have:*

$$\lambda_1 \|x - y\| \leqslant \|\pi(x) - \pi(y)\| \leqslant \|x - y\| \quad \textit{for all } x, y \in U.$$

Proof. Every coordinate projection is 1-Lipschitz, so we only consider the lower bound. Let $W \subseteq A$ be an l-dimensional open subset of A which is also a C^1-cell. Let $W' \subseteq R^l$ be open and definable and let $f : W' \to R^{k-l}$ be a definable C^1-function such that $\text{Graph}(f) = W$. Let $U' \subseteq W'$ be an open euclidean ball on which the partial derivatives of f are all bounded from above by a constant. It follows from the mean value theorem that there is a $\lambda \in R^{>}$ such that:

$$\|f(x') - f(y')\| \leqslant \lambda \|x' - y'\| \quad \text{for all } x', y' \in U'.$$

Let $\pi : W \to W'$ be the natural coordinate projection and let $U = \pi^{-1}(U')$. Fix $x, y \in U$ and let $x' = \pi(x)$ and $y' = \pi(y)$. Then

$$\|x - y\| = \|(x', f(x')) - (y', f(y'))\| \leqslant \|x' - y'\| + \|f(x') - f(y')\| \leqslant (1 + \lambda)\|x' - y'\|.$$

Thus

$$\frac{1}{1+\lambda}\|x - y\| \leqslant \|\pi(x) - \pi(y)\| \quad \text{for all } x, y \in U.$$

□

6.2 Hausdorff Limits

The lemma below allows us to construct the Hausdorff limit of a one-parameter family of definable subsets of the cube. It shows that any one-parameter family A_t of subset of $[0,1]^k$ converges in the Hausdorff metric as $t \to 0^+$.

Lemma 6.2.1. *Let $\{A_t : t \in R^>\}$ be a definable family of subsets of $[0,1]^k$, let A be the set of $(x,t) \in [0,1]^k \times R^{\geqslant}$ such that $x \in A_t$ and let*

$$A_0 = \{x \in [0,1]^k : (x, 0) \in \mathrm{cl}(A)\}.$$

Then A_t converges in the Hausdorff metric to A_0 as $t \to 0^+$.

Proof. We show that $d_H(A_t, A_0) \to 0$ as $t \to 0^+$. Suppose otherwise. Then there is a $\delta \in R^>$ such that $d_H(A_t, A_0) > \delta$ when t is sufficiently small. Then one of the following must hold:

(1) When t is sufficiently small there is a $x \in A_0$ such that $\|x - y\| > \delta$ for all $y \in A_t$.

(2) When t is sufficiently small there is a $y \in A_t$ such that $\|y - x\| > \delta$ for all $x \in A_0$.

Suppose that (1) holds. After replacing δ with a smaller element of $R^>$ if necessary and applying definable choice we let γ be a path in A_0 such that:

$$\|\gamma(t) - y\| > \delta \quad \text{for all } 0 < t < \delta, y \in A_t.$$

As A_0 is a closed subset of $[0,1]^k$, γ must have a limit γ_0 in A_0 as $t \to 0^+$. If $\|\gamma(t) - \gamma_0\| < \frac{\delta}{2}$ then the triangle inequality implies that if $0 < t < \delta$ then $\|\gamma_0 - y\| > \frac{\delta}{2}$ holds for all $y \in A_t$.

As γ_0 lies in the closure of A there is a $y \in A$ such that $\|y - \gamma_0\| < \frac{\delta}{2}$. Then $y \in A_t$ for $0 < t < \frac{\delta}{2}$. This gives a contradiction. A similar argument produces a contradiction from (2). $\square$

6.3 One-Variable Definable Functions

In this section we gather some results about one-variable definable functions. We say that a function $g : R^{\geqslant} \to R^{\geqslant}$ is a **gauge** if:

i. g is subadditive, i.e. $g(s+t) \leqslant g(s) + g(t)$ for all $s, t \in R^{>}$,

ii. $g(0) = 0$,

iii. g is strictly increasing and continuous.

Given a gauge g we say that $g'(0) = \infty$ if $g'(t) \to \infty$ as $t \to 0^+$.

Lemma 6.3.1. *Let $g : R^{\geqslant} \to R^{\geqslant}$ be a C^1-definable function such that $g(0) = 0$ and such that g' is decreasing. Then g is subadditive.*

Proof. We fix $0 < s < t$ and show that $g(s+t) \leqslant g(s) + g(t)$. By the Mean Value Theorem there are $a \in (0, s)$ and $b \in (t, t+s)$ such that

$$g'(a) = \frac{g(s) - g(0)}{s} \quad \text{and} \quad g'(b) = \frac{g(t+s) - g(t)}{s}.$$

As g' is decreasing we have $g'(b) \leqslant g'(a)$, so

$$g(t+s) - g(t) \leqslant g(s) - g(0)$$

and so $g(t+s) \leqslant g(t) + g(s)$. $\square$

Lemma 6.3.2. *Let $g : R^{\geqslant} \to R^{\geqslant}$ be a definable function which satisfies (ii) and (iii) above. If $g'(0) = \infty$ then there is $\delta \in R^{>}$ and a definable gauge f which agrees with g on $[0, \delta]$.*

Proof. Let $\delta \in R^{>}$ be such that g is C^1 on $(0, 2\delta]$. As $g'(0) = \infty$ and $\lim_{t\to 0^+} g'(t) = g'(0)$ we may suppose after decreasing δ if necessary that g' is decreasing on $(0, \delta]$ and that g is strictly increasing on $(0, \delta]$. Let $f : R^{\geqslant} \to R^{\geqslant}$ be given by $f(t) = g(t)$ when $0 < t < \delta$ and

$$f(t) = g(\delta) + g'(\delta)[t - \delta] \quad \text{for all } t \geqslant \delta.$$

It is easy to check that f is C^1 and that $f'(\delta) = g'(\delta)$. Then $f'(t) = f'(\delta)$ when $t > \delta$, so $f'(t)$ is decreasing. Lemma 6.3.1 implies that f is a gauge. $\square$

For the remainder of this section $\mathcal{R}$ is an expansion of the real field. Let $g : \mathbb{R}^{>} \to \mathbb{R}^{>}$ be definable. We say that g is **doubling** if there are $\delta, K \in \mathbb{R}^{>}$ such that $f(2t) \leqslant Kf(t)$ holds for all $0 < t \leqslant \delta$. If f is doubling then for any $k \in \mathbb{R}^{>}$ there is $K' \in \mathbb{R}^{>}$ such that $f(kt) \leqslant K'f(t)$ for all $0 < t \leqslant \delta$.

Lemma 6.3.3. *Let $f : \mathbb{R}^{>} \to \mathbb{R}^{>}$ be a definable function such that $\lim_{t\to 0^+} f(t) = 0$. The following are equivalent:*

i. f is doubling.

ii. There are $r \in (0, 1), \lambda \in \mathbb{R}^{>}$ such that $f(t) \geqslant \lambda t^r$ for all $0 < t \ll 1$.

Proof. Suppose that $f(2t) \leqslant Kf(t)$ holds when $0 < t \leqslant \delta$. Then $f(2^{-i}\delta) \geqslant K^{-i}f(\delta)$ for all $i \in \mathbb{N}$. Setting $r = \log_2(K)$ we have $f(2^{-i}\delta) \geqslant f(\delta)[2^{-i}]^r$ for all $i \in \mathbb{N}$. By o-minimality we have $f(t) \geqslant f(\delta)t^r$ when $0 < t \ll 1$. Now suppose that $f(t) \geqslant \lambda t^s$ holds for some $s \in (0, 1)$ when $0 < t \ll 1$. Let $r > 0$ be the infimum of all $s \in (0, 1)$ with this property. By 5.1 of [Mil94], r is an element of the field of powers of $\mathcal{R}$, so $g(t) = t^{-r}f(t)$ is a definable function. We show that g is doubling. This implies that f is doubling as a product of doubling functions is doubling. There are four cases to consider:

i. $\lim_{t\to 0^+} g(t) = \infty$,

ii. $0 < \lim_{t\to 0^+} g(t) < \infty$.

iii. $\lim_{t\to 0^+} g(t) = 0$ and $g'(0) = \infty$.

iv. $\lim_{t\to 0^+} g(t) = 0$ and $g'(0) < \infty$.

If (i) holds then $g(t) > g(2t)$ when $0 < t \ll 1$, so g is doubling. If (ii) holds then $\lim_{t\to 0^+} \frac{g(2t)}{g(t)} = 1$ so g is doubling. Suppose that (iii) holds. Lemma 6.3.2 implies that $g(2t) \leqslant 2g(t)$ for all $0 < t \ll 1$. We suppose that (iv) holds and derive a contradiction. As $g'(0) < \infty$ there is a $\lambda > 0$ such that $g(t) \leqslant \lambda t$ for all $0 < t \ll 1$. Then $f(t) \leqslant \lambda t^{r+1}$ when $0 < t \ll 1$. This contradicts the definition of r. $\square$

6.4 A Polynomial Volume Bound

In this section we assume that $\mathcal{R}$ expands the real field. We recall Proposition 6.4.1. This was proven by Yomdin and Comte, it follows from the proof of Corollary 5.2 in [YC04].

Proposition 6.4.1. *Let $\{A_x : x \in \mathbb{R}^l\}$ be a definable family of bounded subsets of $\mathbb{R}^k$ of dimension at most m. Let μ the m-dimensional Lebesgue measure. There is a $K \in \mathbb{R}^>$ such that:*

$$\mu(A_x) \leqslant K\,\mathrm{Diam}_e(A_x)^m \quad \textit{for all } x \in \mathbb{R}^l.$$

CHAPTER 7

Basics on Definable Metric Spaces

In this chapter we introduce definable analogues of completeness, compactness and properness. We construct the definable completion of a definable metric space and show that over expansions of the real field, definable completeness is equivalent to completeness and definable compactness is equivalent to compactness. The notion of definable completeness was first studied in the context of definable metrics on definable families of functions by Thomas [Tho12]. *Throughout this chapter* (X, d) *and* (X', d') *are definable metric spaces.*

7.1 Definable Completeness And Compactness

We make some definitions concerning paths in X. Note that a path in X need not be continuous with respect to the d-topology on X. For example, any injective $\gamma : R^{>} \to (R, d_{\text{disc}})$ is nowhere continuous. The path γ in X **converges** to $x \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < t < \delta$ then $d(\gamma(t), x) < \epsilon$. We also say that γ has limit x as $t \to 0^{+}$. The limit of γ is unique, provided that it exists. A path in X converges if it converges to some $x \in X$. A path γ in X is **Cauchy** if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < t, t' < \delta$ then $d(\gamma(t), \gamma(t')) < \epsilon$. An application of the triangle inequality proves that converging paths are Cauchy. A path γ in X is said to be bounded if the image of $R^{>}$ under γ is a bounded subset of (X, d). If the metric is not clear from context we will say that γ d-converges. A definable metric space is **definably complete** if every Cauchy path in it converges.

A definable metric space is **definably compact** if every path in it converges. A definable metric space is **definably proper** if every bounded path in it converges. A definably proper metric space is definably compact if and only if it is bounded. A closed subset of a definably proper definable metric space endowed with the induced metric is definably proper and a definably proper space is definably complete.

Lemma 7.1.1. *Let $h : (X, d) \to (X', d')$ be definable. The following are equivalent:*

i. h is continuous.

ii. If γ is a path in (X, d) which converges to $x \in X$ then $h \circ \gamma$ converges to $h(x)$.

Proof. It is quite obvious that (i) implies (ii). We prove the other implication. Suppose that h is not continuous at $x \in X$ and let $h(x) = x'$. There exists a $\delta > 0$ such that for every $\epsilon > 0$ there is a $y \in X$ satisfying $d(x, y) < \epsilon$ and $d'(h(y), x') > \delta$. Applying definable choice there is a path $\gamma : R^{>} \to (X, d)$ such that $d(\gamma(t), x) < t$ and $d'(h(\gamma(t)), x') > \delta$ holds for all $t > 0$. □

Lemma 7.1.2. *Let $A \subseteq X$ be definable.*

i. If (X, d) is definably complete then (A, d) is definably complete if and only if A is closed.

ii. If (X, d) is definably compact then (A, d) is definably compact if and only if A is closed.

iii. If (X, d) is definable compact then (A, d) is locally definably compact if and only if A is locally closed.

Proof. We only prove the first claim as the proofs of the other two claims are very similar and the idea is familiar from metric space topology. Suppose that (X, d) is definably complete and that A is closed. Every Cauchy path in A has a limit in X, and as A is closed the limit must be an element of A. Conversely, suppose that A is not closed. Let p be a frontier point of A. Applying definable choice we let γ be a path in A such that $d(\gamma(t), p) < t$ holds for all $t > 0$. Then γ is Cauchy with limit $p \notin A$. So (A, d) is not definably complete. □

Proposition 7.1.3. *Suppose that (X, d) is definably compact, and let $f : (X, d) \to (R, e)$ be definable and continuous. Then f has a maximum and a minimum.*

Proof. We prove that f has a maximum. Let $r = \sup\{(f(x) : x \in X\} \in R_\infty$. Let $\gamma : R^> \to X$ be a path in X such that $0 < r - (f \circ \gamma)(t) < r - t$ holds for all $t \in R^>$ if $r < \infty$ and $(f \circ \gamma)(t) > t$ holds for all $t \in R^>$ if $r = \infty$. As (X, d) is definably compact γ must converge to some point z. Continuity of f implies $f(z) = r$. This implies that $r < \infty$. □

Lemma 7.1.4. *A product of two definably compact metric spaces is definably compact.*

Proof. Suppose (X, d) and (X', d') are definably compact. Let $\pi : X \times X' \to X$ and $\pi' : X \times X' \to X'$ be the coordinate projections. Let γ be a path in $X \times X'$. Let $\xi = \pi \circ \gamma$ and $\xi' = \pi' \circ \gamma$. By definable compactness, ξ converges to some $x \in X$ and ξ' converges to some $x' \in X$. It follows immediately that $\gamma = (\xi, \xi')$ converges to (x, x'). □

Lemma 7.1.5. *Suppose that (X, d) is definably compact and infinite. If $0 < t \ll 1$ then there are $x, y \in X$ such that $d(x, y) = t$.*

Proof. One of the following holds:

i. If $0 < t \ll 1$ there there are $x, y \in X$ such that $d(x, y) = t$.

ii. If $0 < t \ll 1$ then $d(x, y) \neq t$ for all $x, y \in X$.

We assume that (ii) holds and derive a contradiction. Suppose $\epsilon > 0$ is such that if $0 < t < \epsilon$ then there do not exist $x, y \in X$ such that $d(x, y) = t$. Then if $x, y \in X$ and $d(x, y) < \epsilon$ then $x = y$, so (X, d) is discrete. As X is infinite there is an injective path $\gamma : R^> \to X$. As (X, d) is discrete, γ does not have a limit as $t \to 0^+$, contradiction. □

Proposition 7.1.6. *Let $f : (X, d) \to (X', d')$ be a definable continuous function. If (X, d) is definably compact then f is uniformly continuous. If (X, d) is definably compact and f is bijective then f is a uniform equivalence and hence a homeomorphism.*

Proof. Let $f : (X, d) \to (X', d')$ be a definable continuous map and suppose that (X, d) is definably compact. The proposition follows immediately in the case that X is finite, so we suppose that X is infinite. For $t \in R^>$ let

$$A_t = \{(x, y) \in X^2 : d(x, y) = t\}.$$

Applying Lemma 7.1.5 let $\epsilon > 0$ be such that if $t < \epsilon$ then $A_t \neq \emptyset$. Each A_t is closed, so it follows from the previous lemma that each A_t is a definably compact subset of X^2. Define $g : (0, \epsilon) \to R^{\geqslant}$ by

$$g(t) = \max\{d'(f(x), f(y)) : (x, y) \in A_t\}.$$

Proposition 7.1.3 shows that g is defined. We have $d'(f(x), f(y)) \leqslant g(d(x, y))$ so to show that f is uniformly continuous it suffices to show that $\lim_{t\to 0^+} g(t) = 0$. Suppose otherwise towards a contradiction. There are paths $\gamma_1, \gamma_2 : (0, \epsilon) \to X$ such that $d(\gamma_1(t), \gamma_2(t)) = t$ and

$$d((f(\gamma_1(t)), f(\gamma_2(t))) \geqslant \delta \quad \text{for some } \delta > 0.$$

Let x_1, x_2 be the limits of γ_1 and γ_2, respectively. Then $d(x_1, x_2) = 0$ so $x_1 = x_2$, but $d(f(x_1), f(x_2)) \geqslant \delta$, contradiction.

We now suppose in addition that f is a bijection and show that f is a uniform equivalence. Let $h : (0, \epsilon) \to R$ be the definable function given by

$$h(t) = \min\{d'(f(x), f(y)) : (x, y) \in A_t\}.$$

As A_t is definably compact $h(t)$ is defined. As f is injective for all $t > 0$ we have $h(t) > 0$. As $h(t) \leqslant g(t)$, $\lim_{t\to 0^+} h(t) = 0$. We have $h(d(x, y)) \leqslant d'(f(x), f(y))$, so f is a uniform equivalence. $\square$

We immediately have the following in the power bounded case:

Lemma 7.1.7. *Suppose that $\mathcal{R}$ is power bounded. Let $f : (X, d) \to (X', d')$ be a definable continuous map between definably compact spaces. Then for some $\lambda \in R^{>}$ and a positive element r of the field of powers of $\mathcal{R}$ we have:*

$$d'(f(x), f(y)) \leqslant \lambda d(x, y)^r \quad \textit{for all } x, y \in X.$$

If f is also a bijection then there are $\lambda_1, \lambda_2 \in R^{>}$ and positive elements r_0, r_1 of the field of powers of $\mathcal{R}$ such that:

$$\lambda_1 d(x, y)^{r_1} \leqslant d'(f(x), f(y)) \leqslant \lambda_2 d(x, y)^{r_2} \quad \textit{for all } x, y \in X.$$

The next lemma is used in a crucial way to prove Theorem 9.0.1, the main result of this thesis.

Lemma 7.1.8. *Let (Y, d) be a definably compact metric space. Let X be a definably compact subset of euclidean space which admits a definable continuous surjection $(X, e) \to (Y, d)$. Then there is a definable set X' and a definable homeomorphism $(X', e) \to (Y, d)$.*

Proof. Let $h : (X, e) \to (Y, d)$ be a definable continuous surjection. Let $E \subseteq X^2$ be the kernel of h, i.e., the equivalence relation

$$E = \{(x, y) \in X^2 : h(x) = h(y)\}.$$

We endow the set-theoretic quotient X/E with the quotient topology. Continuity of h implies that E is a closed, and hence definably compact, subset of X^2. We apply 2.15 of [Dri98] to obtain a definable set X' and a definable continuous map $p : X \to X'$ such that the kernel of p is E and the induced bijection $p_E : X/E \to (X', e)$ is a homeomorphism. As p is continuous, (X', e) is definably compact. Let $h_E : (X', e) \to (Y, d)$ be the map induced by h. Then h_E is a continuous bijection between definably compact metric spaces. Proposition 7.1.6 implies that h_E is a homeomorphism. □

We now prove two lemmas about extending definable functions:

Lemma 7.1.9. *Suppose (X', d') is definably complete. Let $A \subseteq X$ be definable and dense and let $f : (A, d) \to (X', d')$ be a uniformly continuous definable function. Then f admits a unique extension to a uniformly continuous definable function $(X, d) \to (X', d')$.*

Proof. We construct an extension; it is clear from that construction the the extension is unique. Let $g : R^{>} \to R^{>}$ be a definable continuous function such that $\lim_{t \to 0^+} g(t) = 0$ and $d'(f(x), f(y)) \leqslant g(d(x, y))$ whenever $x, y \in A$. Applying definable choice let $\psi : X \times R^{>} \to A$ be a definable function such for each $(x, t) \in X \times R^{>}$ we have $d(\psi_x(t), x) < t$. Fix $x \in X$ for the moment. For any $t, t' \in R^{>}$ we have

$$d'(f(\psi_x(t)), f(\psi_x(t'))) \leqslant g(d(\psi_x(t)), \psi_x(t'))).$$

As ψ_x is Cauchy this inequality implies that $f \circ \psi_x$ is Cauchy. Definable completeness of (X', d') implies that $f \circ \psi_x$ converges. Therefore, for any $x \in X$ we may let $\hat{f}(x)$ be the limit of $f \circ \psi_x$ in (X', d'). A computation shows that $\hat{f}$ is uniformly continuous:

$$d'(\hat{f}(x), \hat{f}(y)) = \lim_{t \to 0^+} d'(f(\psi_x(t)), f(\psi_y(t))) \leqslant \lim_{t \to 0^+} g(d(\psi_x(t), \psi_y(t))) = g(d(x, y)).$$

$\square$

The proof of Lemma 7.1.9 also shows the following:

Lemma 7.1.10. *Suppose (X, d) and (X', d') are definably complete. Let $A \subseteq X$ be definable and dense. Suppose that $f : A \to X'$ is a distance preserving function. Then f admits a unique extension to a distance preserving definable function $(X, d) \to (X', d')$.*

Immediately from this we have:

Lemma 7.1.11. *Let (X, d) and (X', d') be definably complete metric spaces. Let $A \subseteq X$ and $A' \subseteq X'$ be definable and dense. A definable isometry $(A, d) \to (A', d')$ admits a unique extension to a definable isometry $(X, d) \to (X', d')$.*

7.2 The Definable Completion

We construct the definable completion of a definable metric space. Our construction is an application of the following fact:

Fact 7.2.1. *Let $A \subseteq R^k$ be a definable set. Let $\mathcal{F} = \{f_x : x \in R^l\}$ be a definable family of functions $A \to R$ and $\mathcal{G} = \{g_x : x \in B\}$ be the definable family of functions $A \to R$ given by Corollary 4.4.2. Let d_∞ be the pseudometric on B given by $d_\infty(x, y) = \|f_x - f_y\|_\infty$. Then the metric space associated to (B, d_∞) is definably complete. If $\mathcal{R}$ expands the ordered field of reals then the metric space associated to (B, d_∞) is complete.*

Recall that $\mathcal{G}$ consists of those functions $A \to R$ which are uniform limits of families of the form $\{f_{\gamma(t)}\}_{t \in R^>}$ as $t \to 0^+$ for paths γ in R^l.

Proof. We show that the metric space associated to (B, d_∞) is definably complete. Fix a Cauchy path $\gamma : R^> \to (B, d_\infty)$. As γ is Cauchy, $f_{\gamma(t)}$ uniformly converges as $t \to 0^+$. Let $g_0 : A \to R$ be the uniform limit of $g_{\gamma(t)}$ as $t \to 0^+$. Applying definable choice we let $\psi : B \times R^> \to R^l$ be a definable function such that:

$$\|f_{\psi(x,t)} - g_x\|_\infty \leqslant t \quad \text{for all } x \in B, t \in R^>.$$

The triangle inequality implies:

$$\|f_{\psi(\gamma(t),t)} - g_0\|_\infty \leqslant \|f_{\psi(\gamma(t),t)} - g_{\gamma(t)}\|_\infty + \|g_{\gamma(t)} - g_0\|_\infty \leqslant t + \|g_{\gamma(t)} - g_0\|_\infty.$$

Thus $f_\psi(\gamma(t), t)$ uniformly converges to g_0 as $t \to 0^+$. Thus g_0 is an element of $\mathcal{G}$. If $\mathcal{R}$ is an expansion of the ordered field of real numbers then an similar argument with sequences in place of paths shows that the metric space associated to (B, d_∞) is complete. □

For our definable metric space (X, d) we construct a definably complete metric space $(\widetilde{X}, \widetilde{d})$ and a distance preserving definable map $(X, d) \to (\widetilde{X}, \widetilde{d})$ with dense image. It follows from Lemma 7.1.9 that the definable completion is left adjoint to the forgetful functor from the category of definably complete metric spaces and uniformly continuous definable maps to the category of all definable metric spaces and and uniformly continuous definable maps. It follows from Lemma 7.1.11 that the definable completion is unique up to definable isometry.

Proposition 7.2.2. *There is a definably complete metric space $(\widetilde{X}, \widetilde{d})$ and a definable isometry $(X, d) \to (\widetilde{X}, \widetilde{d})$ with dense image. Any uniformly continuous definable map $(X, d) \to (X, d')$ admits a unique extension to a uniformly continuous definable map between the definable completions $(\widetilde{X}, \widetilde{d}) \to (\widetilde{X'}, \widetilde{d'})$.*

Proof. We construct the definable completion. We use the Kuratowski Embedding. Fix a basepoint $p \in X$. For $x \in X$ let $d_x : X \to R$ be given by $d_x(y) = d(x, y)$. For each $x \in X$ consider the definable function $\tau_x : X \to R$ where $\tau_x(y) = d_x(y) - d_p(y)$. Each τ_x is bounded as we have

$$|d_x(y) - d_p(y)| = |d(x, y) - d(p, y)| \leqslant d(x, p) \quad \text{for all } y \in X.$$

We show that $\|\tau_x - \tau_y\|_\infty = d(x,y)$ for all $x, y \in X$. Fix $x, y \in X$. As

$$|\tau_x(z) - \tau_y(z)| = |d_x(z) - d_y(z)| \leqslant d(x,y) \quad \text{for all } z \in X,$$

we have $\|\tau_x - \tau_y\|_\infty \leqslant d(x,y)$. As $\tau_x(x) = -d(x,p)$ and $\tau_y(x) = d(x,y) - d(x,p)$ we have $\|\tau_x - \tau_y\|_\infty = d(x,y)$. Applying Corollary 4.4.2 let $\mathcal{G} = \{g_x : X \to R : x \in B\}$ be a definable family of functions whose elements are the uniform limit points of the family $\{\tau_x : x \in X\}$. For $a, b \in B$ let $\widetilde{d}(a,b) = \|g_a - g_b\|_\infty$. Let $(\widetilde{X}, \widetilde{d})$ be the metric space associated to the pseudometric space $(B, \widetilde{d})$. In Fact 7.2.1 we observed that $(\widetilde{X}, \widetilde{d})$ is definably complete. The natural map $X \to \widetilde{X}$ is clearly injective, distance-preserving and has dense image. The second claim follows from Lemma 7.1.9. □

Corollary 7.2.3. *Suppose that $\mathcal{R}$ expands the orderered field of reals. Then every definably complete metric space is complete.*

Proof. Fact 7.2.1 implies that the definable completion of any definable metric space is complete. Lemma 7.1.11 implies that a definable complete metric space is isometric to its definable completion. □

We now prove the analogue of Corollary 7.2.3 for definably compact spaces. We need two simple lemmas. We let $\mathcal{U}$ be a nonprincipal ultrafilter on the natural numbers and let $\mathcal{R}^* = (R^{\mathcal{U}}, +, \times, ...)$ be the corresponding ultrapower of $\mathcal{R}$. Given a sequence of elements $\{x_i\}_{i \in \mathbb{N}}$ of some $\mathcal{R}$-definable set A we let $[x_i]$ be the corresponding element of A^*.

Lemma 7.2.4. *Suppose that $\mathcal{R}$ expands the ordered field of reals. Let $\{x_i\}_{i \in \mathbb{N}}$ be a sequence of elements of X and $y \in X$ be such that $\operatorname{st} d^*([x_i], y) = 0$. Then there is a subsequence of $\{x_i\}_{i \in \mathbb{N}}$ which converges to y.*

Proof. Let $\epsilon \in \mathbb{R}^>$. As $d^*([x_i], y) < \epsilon$ we see that $d(x_i, y) < \epsilon$ for $\mathcal{U}$-many i. So for every such ϵ there is an $i \in \mathbb{N}$ such that $d(x_i, y) < \epsilon$. There is a subsequence of $\{x_i\}_{i \in \mathbb{N}}$ which converges to y. □

Lemma 7.2.5. *Let ζ be an infinitesimal positive element of an elementary extension of $\mathcal{R}$ and let $\mathcal{R}(\zeta)$ be the prime model over ζ. Let γ be a path in X. Then γ converges to $y \in X$ if and only if $\operatorname{st} d^*(\gamma(\zeta), y) = 0$.*

Proof. $\operatorname{st} d^*(\gamma(\zeta), y) = \lim_{t \to 0^+} d(\gamma(t), y)$. □

Proposition 7.2.6. *Suppose $\mathcal{R}$ expands the ordered field of reals. Then every definably compact metric space is compact.*

Proof. It is clear that a compact definable metric space is definably compact. Let (X, d) be a definably compact metric space. Consider the following sentence in the language of tame pairs:

$$\Theta = (\forall x \in X^*)(\exists y \in X)[\operatorname{st} d^*(x, y) = 0].$$

As (X, d) is definably compact, Lemma 7.2.5 implies that $(\mathcal{R}(\zeta), \mathcal{R}) \models \Theta$. As the theory of tame pairs is complete $(\mathcal{R}^{\mathcal{U}}, \mathcal{R}) \models \Theta$. By Lemma 7.2.4 every sequence in X admits a converging subsequence. Thus (X, d) is compact. □

We show how definability of Hausdorff limits follows directly from Proposition 7.2.2, Corollary 7.2.3 and Lemma 6.2.1.

Corollary 7.2.7. *Let $\mathcal{A} = \{A_x : x \in R^l\}$ be a definable family of bounded subsets of R^k. There is a definable family of $\{C_x : x \in B\}$ of closed and bounded subsets of R^k such that a closed and bounded $C \subseteq R^k$ is an element of $\mathcal{B}$ if and only if C is a Hausdorff limit of a family $\{A_{\gamma(t)}\}_{t \in R^>}$ as $t \to 0^+$ for some definable $\gamma : R^> \to R^l$. If $\mathcal{R}$ expands the ordered field of reals then $\mathcal{B}$ consists exactly of those compact subsets of $\mathbb{R}^k$ which are Hausdorff limits of sequences of elements of $\mathcal{A}$.*

Proof (Sketch). Let d_H be the Hausdorff metric. After replacing each element of $\mathcal{A}$ with its closure may assume that each element of $\mathcal{A}$ is closed. Let $\sim$ be the equivalence relation on R^l given by declaring $x \sim y$ if and only if $A_x = A_y$. Let $B \subseteq R^l$ be a definable set which contains one element of each $\sim$ class. It follows from these assumptions that the pseudometric d on B given by

$$d(x, y) = d_H(A_x, A_y)$$

is a metric. We let $(\widetilde{B}, \widetilde{d})$ be the definable completion of (B, d) and regard B as a subset of $\widetilde{B}$. Applying definable choice we let

$$h : \widetilde{B} \times R^> \to B$$

be a definable function such that $\widetilde{d}(h(b,t),b) \leqslant t$ for any $(b,t) \in \widetilde{B} \times R^{>}$. For $b \in B$ let $C_b \subseteq R^k$ be the set of p such that $(0,p)$ lies in the closure of

$$\{(t,q) : q \in B_{h(b,t)}\} \subseteq R^{>} \times R^k$$

The proof of Lemma 6.2.1 implies that $B_{h(b,t)}$ Hausdorff converges to C_b as $t \to 0^+$ for all $b \in B$. Suppose that $\mathcal{R}$ expands the real field. Then $(\widetilde{B}, \widetilde{d})$ is a complete metric space. From this one can easily show that $\{C_b : b \in \widetilde{B}\}$ is the Hausdorff closure of $\mathcal{A}$. $\square$

The next lemma will be used in the proof of Proposition 9.2.15 below to embed certain definable metric spaces in definably compact metric spaces.

Lemma 7.2.8. *Let $\{A_x : x \in R^l\}$ be a definable family of subsets of $[0,1]^k$. Let d be the pseudometric on R^l given by $d(x,y) = d_H(A_x, A_y)$ for all $x,y \in R^l$. Then the definable completion of the definable metric space associated to (R^l, d) is definably compact.*

Proof. Let (B,d) be the definable metric space associated to (R^l, d) and let $\pi : R^l \to B$ be the quotient map. Let $(\widetilde{B}, \widetilde{d})$ be the definable completion of (B,d). We show that an arbitrary path $\gamma : R^{>} \to \widetilde{B}$ converges. Applying definable choice let $\gamma' : R^{>} \to B$ be a path such that

$$\widetilde{d}(\gamma'(t), \gamma(t)) \leqslant t \quad \text{for all } t \in R^{>}.$$

It is enough to show that γ' converges. Let $\eta : R^{>} \to R^l$ be a path such that $\pi \circ \eta = \gamma'$. Let $D \subseteq R^k$ be the set of p such that $(0,p)$ is in the closure of

$$\{(t,q) : q \in A_{\eta(t)}\} \subseteq R^{>} \times R^k.$$

Lemma 6.2.1 implies that $A_{\eta(t)}$ converges to D in the Hausdorff metric as $t \to 0^+$. It follows that γ' has a limit in $\widetilde{B}$. $\square$

CHAPTER 8

Gromov-Hausdorff Limits of Definable Metric Spaces

In this chapter we establish some basic facts and prove some technical lemmas which we use in Chapter 12. We show that GH-limits of one-parameter definable families are unique and show as a corollary that asymptotic cones and tangent cones of definable metric spaces are conical.

In this chapter $\mathcal{R}$ expands the real field. We let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$ and let $\mathcal{R}^{\mathcal{U}}$ be the corresponding ultrapower of $\mathcal{R}$. Let $\mathcal{O}$ be the convex hull of the integers in $\mathbb{R}^{\mathcal{U}}$. Let $\mathrm{st} : \mathbb{R}^{\mathcal{U}} \to \mathbb{R} \cup \{\pm\infty\}$ be the standard part map. The proposition below follows immediately from the definitions in Section 3.7:

Proposition 8.0.1. *The reduction of an $\mathcal{R}^{\mathcal{U}}$-definable pointed metric space is $(\mathcal{R}^{\mathcal{U}}, \mathcal{O})$-definable. If $\mathcal{X}$ is a definable family of pointed proper metric spaces and $\{(X_i, d_i)\}_{i \in \mathbb{N}}$ is a sequence of elements of $\mathcal{X}$ then the ultralimit of (X_i, d_i) as $i \to \mathcal{U}$ is $(\mathcal{R}^{\mathcal{U}}, \mathcal{O})$-definable.*

Note that the reduction of an $\mathcal{R}^{\mathcal{U}}$-definable pointed metric space is an $(\mathcal{R}^{\mathcal{U}}, \mathcal{O})$-definable set of imaginaries.

8.1 Uniqueness of GH-Limits

A definable function $f : \mathbb{R}^{>} \to \mathbb{R}^{>}$ has a unique limit in $\mathbb{R}_{\infty}$ as $t \to 0^{+}$. In this section we explore the same phenomenon in the GH-setting.

Proposition 8.1.1. *Let $\mathcal{X} = \{(X_t, d_t, p_t) : t \in \mathbb{R}^{>}\}$ be a definable family of pointed proper metric spaces. If ζ, η are positive infinitesimal elements of $\mathbb{R}^{\mathcal{U}}$ then the reductions of $(\mathcal{O}(X^*_\zeta), \mathrm{st}\, d^*_\zeta, p^*_\zeta)$ and $(\mathcal{O}(X^*_\eta), \mathrm{st}\, d^*_\eta, p^*_\eta)$ are isometric. In particular if $\mathcal{X}$ is precompact then*

(X_t, d_t) *converges to a unique limit as* $t \to 0^+$.

Proof. In this proof we let $\mathcal{O}(X_t^*) = \mathcal{O}(X_t^*, d_t^*, p_t^*)$. Fix positive infinitesimals ζ, η. We construct a basepoint-preserving isometry between the reductions of $(\mathcal{O}(X_\zeta^*), d_\zeta^*, p_\zeta^*)$ and $(\mathcal{O}(X_\eta^*), d_\eta^*, p_\eta^*)$. It is enough to construct a basepoint-preserving isometry of pointed pseudometric spaces

$$(\mathcal{O}(X_\zeta^*), \operatorname{st} d_\zeta^*, p_\zeta^*) \to (\mathcal{O}(X_\eta^*), \operatorname{st} d_\eta^*, p_\eta^*).$$

As ζ and η have the same type over $\emptyset$ and $\mathcal{R}^{\mathcal{U}}$ is ω-saturated there is an automorphism σ of $\mathcal{R}^{\mathcal{U}}$ which maps ζ to η. As σ fixes $\mathbb{Q}$ pointwise and preserves the order $\operatorname{st}\sigma(r) = \operatorname{st}(r)$ holds for all $r \in \mathbb{R}^{\mathcal{U}}$. Then σ induces a bijection $\overline{\sigma} : X_\zeta^* \to X_\eta^*$ which maps p_ζ^* to p_η^*. For any $x \in X_\zeta^*$ we have $\operatorname{st} d_\zeta^*(p_\zeta^*, x) < \infty$ if and only if $\operatorname{st} d_\eta^*(p_\eta^*, \sigma(x)) < \infty$, so $\overline{\sigma}$ restricts to a bijection between $\mathcal{O}(X_\zeta^*)$ and $\mathcal{O}(X_\eta^*)$. Furthermore, $\operatorname{st} d_\zeta^*(x, y) = \operatorname{st} d_\eta^*(\sigma(x), \sigma(y))$ holds for all $x, y \in X_\zeta^*$. Therefore the restriction of $\overline{\sigma}$ to $\mathcal{O}(X_\zeta^*)$ gives the required isometry of pointed pseudometric spaces. $\square$

Proposition 8.1.2. *Let* (X, d) *be a definable metric spaces. The asymptotic cone of* (X, d) *is a conical metric space. If* (X, d) *has a tangent cone at* $p \in X$ *then the tangent cone is a conical metric space.*

Proof. We only prove the first claim as the proof the second is essentially the same. Fix a basepoint $p \in X$. Let ω be the element of $\mathbb{R}^{\mathcal{U}}$ corresponding to the sequence $\{i\}_{i \in \mathbb{N}}$. Let $\mathcal{O}(X^*) = \{q \in X^* : \operatorname{st} \frac{1}{\omega} d^*(p, q) < \infty\}$. Let $d_{\mathfrak{C}}$ be the metric induced by $\operatorname{st} \frac{1}{\omega} d^*$ on $\mathfrak{C}_{\mathcal{U}}(X, d)$. Fixing $\lambda \in \mathbb{R}^{>}$ we construct a map $h : \mathfrak{C}_{\mathcal{U}}(X, d) \to \mathfrak{C}_{\mathcal{U}}(X, d)$ which fix p and satisfies

$$d_{\mathfrak{C}}(h(x), h(y)) = \lambda d_{\mathfrak{C}}(x, y) \quad \text{for all } x, y \in \mathfrak{C}_{\mathcal{U}}(X, d).$$

It suffices to construct a basepoint preserving isometry of pseudometric spaces:

$$\left(\mathcal{O}(X^*), \operatorname{st} \frac{1}{\omega} d^*, p\right) \to \left(\mathcal{O}(X^*), \lambda \operatorname{st} \frac{1}{\omega} d^*, p\right).$$

As $\operatorname{st} \frac{1}{\omega} d^*(p, q) < \infty$ if and only if $\operatorname{st} \frac{\lambda}{\omega} d^*(p, q) < \infty$:

$$\mathcal{O}\left(X^*, \frac{\lambda}{\omega} d^*, p\right) = \mathcal{O}(X^*).$$

Thus as $\frac{1}{\omega}$ and $\frac{\lambda}{\omega}$ are both positive infinitesimals it follows from Proposition 8.1.1 that the reductions of $(\mathcal{O}(X^*), \operatorname{st}\frac{1}{\omega}d^*, p)$ and $(\mathcal{O}(X^*), \operatorname{st}\frac{\lambda}{\omega}d^*, p)$ are isometric. As $\operatorname{st}\frac{\lambda}{\omega}d^* = \lambda \operatorname{st}\frac{1}{\omega}d^*$, the reduction of $(O(X^*), \operatorname{st}\frac{\lambda}{\omega}d^*, p)$ is $(\mathfrak{C}_{\mathcal{U}}(X,d), \lambda d_{\mathfrak{C}}, p)$. □

Proposition 8.1.3. *Let (X,d) be a doubling definable metric space. Then (X,d) has a unique tangent cone at every point, and the tangent cone at μ-almost every point is conical and isometrically homogeneous.*

Proof. This follows directly from Lemma 3.7.9, the previous two propositions and the theorem of Le Donne, Theorem 3.7.15. □

The last proposition of this section will allow us to pass from $\mathcal{R}^{\mathcal{U}}$ to a larger elementary extension in Chapter 12. In the proposition below $\mathcal{M} = (M, +, \times, \leqslant, \ldots)$ is an elementary extension of $\mathcal{R}^{\mathcal{U}}$.

Proposition 8.1.4. *Let (X,d,p) be a $\mathcal{R}^{\mathcal{U}}$-definable pointed metric space and let $(X_{\mathcal{M}}, d_{\mathcal{M}}, p)$ be the pointed metric space defined by the same formula over $\mathcal{M}$. If the reduction of (X,d,p) is proper then $\overline{\mathcal{O}(X)} = \overline{\mathcal{O}(X_{\mathcal{M}})}$.*

Proposition 8.1.4 is a special case of the model-theoretic ideas in [LP01].

Proof. It is enough to show that the reductions of $(B(p,N),d)$ and $(B(p,N),d_{\mathcal{M}})$ are equal for all $N \in \mathbb{N}$. We therefore suppose without lose of generality that (X,d) is strongly bounded. We suppose that the reduction of (X,d) is compact.. The proposition follows by showing that for all $x \in X_{\mathcal{M}}$ there is a $y \in X$ such that $\operatorname{st} d_{\mathcal{M}}(x,y) = 0$. Fix $x \in X_{\mathcal{M}}$. Let $\epsilon \in \mathbb{R}^{>}$. As $N(X,\epsilon)$ is finite we let $\{y_1,\ldots,y_N\}$ be an ϵ-dense subset of X. Then $\{y_1,\ldots,y_n\}$ is ϵ-dense in $(X_{\mathcal{M}}, d_{\mathcal{M}})$, so there is an $1 \leqslant i \leqslant N$ such that $d_{\mathcal{M}}(x,y_i) \leqslant \epsilon$. Thus, for every $\epsilon \in \mathbb{R}^{>}$ there is a $y \in X$ such that $d_{\mathcal{M}}(x,y) \leqslant \epsilon$. As $\mathcal{R}^{\mathcal{U}}$ is ω-saturated there is a $y \in X$ such that $\operatorname{st} d_{\mathcal{M}}(x,y) = 0$. □

8.2 GH-Limits and Pointwise Limits

In this section we describe some situations in which the GH-limit of a family of metric spaces agrees with the pointwise limit. These technical results will be useful in later chapters. We say a family of metric spaces GH-converges to a pseudometric space if the family GH-converges to the associated metric space. Let $\{d_t : t \in \mathbb{R}^>\}$ be a definable family of metrics on the definable set X. We make the following assumptions:

i. For all $t \in \mathbb{R}^>$, the d_t-topology agrees with the euclidean topology on X,

ii. X is a definably compact subset of euclidean space,

iii. there is a $N \in \mathbb{N}$ such that the diameter of each (X, d_t) is bounded from above by N.

iv. (X, d_t) GH-converges as $t \to \infty$.

For $x, y \in X$ let

$$d_\infty(x, y) = \lim_{t\to\infty} d_t(x, y).$$

Uniform boundedness ensures that this limit exists. Then d_∞ is a pseudometric on X. Under all of these assumptions we have the following implication:

Lemma 8.2.1. *If every converging sequence $\{x_i\}_{i\in\mathbb{N}}$ of elements of X with limit x satisfies $\lim_{i\to\infty} d_i(x_i, x) = 0$ then (X, d_t) GH-converges to (X, d_∞) as $t \to \infty$.*

Proof. Let ω be the element of $\mathbb{R}^{\mathcal{U}}$ corresponding to the sequence $\{i\}_{i\in\mathbb{N}}$. We show that

$$\operatorname{st} : (X^*, \operatorname{st} d^*_\omega) \to (X, d_\infty)$$

is an isometry of pseudometric spaces. Fix $x, y \in X^*$. Let $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ be sequences of elements of X which represent x, y in the ultrapower, respectively Then

$$\operatorname{st} d^*_\omega(x, y) = \lim_{i\to\mathcal{U}} d_i(x_i, y_i).$$

In the euclidean topology on X, $x_i \to \operatorname{st}(x)$ and $y_i \to \operatorname{st}(y)$ as $i \to \mathcal{U}$. It follows from the assumption that $d_i(\operatorname{st}(x), x_i) \to 0$ and $d_i(\operatorname{st}(y), y_i) \to 0$ as $i \to \mathcal{U}$. As

$$|d_i(x_i, y_i) - d_i(\operatorname{st}(x), \operatorname{st}(y))| \leqslant d_i(\operatorname{st}(x), x_i) + d_i(\operatorname{st}(y), y_i),$$

we have

$$\lim_{i\to\mathcal{U}} d_i(x_i, y_i) = \lim_{i\to\mathcal{U}} d_i(\mathrm{st}(x), \mathrm{st}(y)) = d_\infty(\mathrm{st}(x), \mathrm{st}(y)).$$

Thus st is an isometry. □

The following is immediate.

Corollary 8.2.2. *If d_t uniformly converges to d_∞ as $t \to \infty$ then (X, d_t) GH-converges to the pseudometric space (X, d_∞) as $t \to \infty$.*

We apply the final technical lemma in the proof of Lemma 10.4.2:

Corollary 8.2.3. *Fix a basepoint $p \in \mathbb{R}^k$. Let $\{X_t : t \in \mathbb{R}^>\}$ be a monotone increasing definable family of definable open subsets of $\mathbb{R}^k$ such that for any n, $B_e(p, n) \subseteq X_t$ when t is sufficiently large. Let (X_t, d_t) be a definable family of metric spaces and $\lambda \in \mathbb{R}^{\geqslant}$ be such that each map $id : (X_t, d_t) \to (X_t, e)$ is λ-bilipschitz. Then (X_t, d_t, p) GH-converges to $(\mathbb{R}^k, d_\infty, p)$ as $t \to \infty$ where*

$$d_\infty(x, y) = \lim_{t\to\infty} d_t(x, y)$$

Proof. It is immediate that the limit defining d_∞ exists for all $x, x' \in \mathbb{R}^k$, that d_∞ is a pseudometric and that $id : (\mathbb{R}^k, d_\infty) \to (\mathbb{R}^k, e)$ is λ-bilipschitz. Fix $n \in \mathbb{N}$ and let B_t be the closed d_t-ball with center p and radius n. We let B_∞ be the closed d_∞ ball with center p and radius n. We prove the corollary by showing that (B_t, d_t, p) GH-converges to (B_∞, d_∞, p). For each t, $B_t \subseteq B_e(p, \lambda n) = C$. When t is sufficiently large $C \subseteq X_t$. Applying Lemma 8.2.1 we see that (C, d_t) GH-converges to (C, d_∞) as $t \to \infty$. As $B_\infty \subseteq C$ this implies that (B_t, d_t, p) GH-converges to (B_∞, d_∞) as $t \to \infty$. □

CHAPTER 9

Topology of Definable Metric Spaces

In this chapter (X, d) *is a definable metric space.* The main goal of this chapter is to prove the following:

Theorem 9.0.1. *Exactly one of the following holds:*

i. (X, d) *is not* ***definably separable****, i.e. there is an infinite definable* $A \subseteq X$ *such that* (A, d) *is discrete.*

ii. There is a definable set Z *and a definable homeomorphism*

$$(X, d) \to (Z, e).$$

The proof splits into two parts. First we show that every definably separable metric space is definably homeomorphic to a subspace of a definably compact metric space and then we show that every definably compact metric space is homeomorpic to a definable set equipped with its euclidean topology. We embedd definably separable spaces in definably compact metric spaces by studying the pseudometric d_H on X given by declaring $d_H(x, y)$ to be the Hausdorff distance between $\mathrm{Graph}(d_x)$ and $\mathrm{Graph}(d_y)$, where we set $d_x(z) = d(x, z)$. If X is a bounded subset of euclidean space and (X, d) is a bounded metric space then d_H is a psueodmetric on X and the completion of the metric space associated to (X, d_H) is definably compact. We show that every definably separable metric space is definably isometric to a definable metric space (X, d) for which d_H is a metric and $id : (X, d) \to (X, d_H)$ is a homeomorphism. This entails showing that any definably separable metric space (X, d) admits a partition into definable sets on which the d-topology agrees with the metric topology.

After constructing the embedding we use the aforementioned piecewise result and Aschenbrenner and Thamrongtanyalak's definable Micheal Selection Theorem to show that every definably compact metric space is a definable continuous image of a definably compact set. An application of Lemma 7.1.8 then shows that every definably compact metric space is homeomorphic to a definably compact set.

Along the way we prove some general facts about the topology of definable metric spaces. As a consequence we show that if $\mathcal{R}$ expands the real field then the topological dimension of (X, d) is the largest k for which there is a definable continuous injection $([0, 1]^k, e) \to (X, d)$.

9.1 Definably Separable Metric Spaces

We say that (X, d) is **definably separable** if every d-discrete definable subset of X is finite. This terminology is justified: If $\mathcal{R}$ is an expansion of the ordered field of reals then every infinite definable set has cardinality $|\mathbb{R}|$, so if (X, d) is not definably separable then X contains a discrete subset with cardinality $|\mathbb{R}|$, which implies that (X, d) is not separable. Theorem 9.0.1 thus implies that a metric space definable in an o-minimal expansion of the ordered real field is separable if and only if it is definably separable.

For the next lemma, we define the **local dimension** of (X, d) at $x \in X$ to be

$$\dim_x(X, d) = \min\{\dim B_d(x, t) : t > 0\}.$$

Lemma 9.1.1. *If (X, d) is definably separable then $\dim_x(X, d) = \dim(X)$ at almost every $x \in X$.*

Proof. We prove the contrapositive. To this effect we an e-open m-dimensional definable $U \subseteq X$ such that $\dim_x(X, d) < \dim(X)$ for all $x \in U$. After replacing U with a smaller open set with the same properties if necessary we may also suppose that $\dim_x(X, d) = m < n$ for all $x \in U$. Let $g : U \to R^>$ be a definable function such that $\dim[B_d(x, g(x))] = n$ for all $x \in U$. We set $B_x = B_d(x, g(x))$. We apply a uniform version of the good directions lemma see 4.3 in [Dri98]. We find an m-dimensional open $W \subseteq U$ and a linear projection $\pi : U \to R^n$ whose restriction to B_x is finite for each $x \in W$. As $\dim \pi(W) = n$ the fiber lemma for o-minimal dimension implies that there is a $q \in W$ such that $\dim \pi^{-1}(q)$ is $(m - n)$-dimensional. We fix such a q and let $J = \pi^{-1}(q)$. Then J is infinite. We show that

(J,d) is discrete. Fix $\beta \in J$. Let $\{\beta_1, \dots, \beta_N\} = B_\beta \cap J$. Then if $y \in J$ and $d(\beta, y) < d(\beta, \beta_i)$ holds for each i then $y = \beta$, so β is isolated in J. □

Lemma 6.1.1 now implies:

Lemma 9.1.2. *Suppose that (X,d) is not definably separable. There is a $t > 0$ and an infinite definable $A \subseteq X$ such that $d(x,x') > t$ for any distinct $x, x' \in A$.*

The next lemma shows that there is no definable compactifiction of a definable discrete metric space.

Lemma 9.1.3. *If (X,d) is definably compact and $B \subseteq X$ is definable then (B,d) is definably separable.*

Proof. We prove the contrapositive. Suppose that (B,d) is not definably separable. Suppose that $A \subseteq X$ and $t > 0$ satisfy the conditions in Lemma 9.1.2. A path in A is Cauchy if and only if it is eventually constant. Thus there is a path in X which is not Cauchy and so $(X.d)$ is not definably compact. □

9.2 A Definable Compactification

In this section we show that every definably separable metric space is definably homeomorphic to a definable subspace of a definably compact metric space. After this, it suffices to prove Theorem 9.0.1 for definably compact metric spaces. Along the way, we will prove several other results about arbitrary definable metric spaces.

We introduce some terminology. We say that a point $x \in X$ is **yellow** if there is a path in X which converges to x in the euclidean topology and converges in the d-topology to some $y \in X \setminus \{x\}$. We say that a point $x \in X$ is **blue** if it is not yellow. We say that definable metric space is **blue** if all of its points are blue. By a "blue metric space" we mean a "blue definable metric space". Our first goal is to show, in Proposition 9.2.3, that every definable metric space is definably isometric to a blue metric space. We first show that the set of blue points of (X,d) is definable. For $y \in X$ let $d_y : X \to R$ be given by $d_y(x) = d(y,x)$.

Lemma 9.2.1. *The following are equivalent for $x, y \in X$:*

i. There is a path in X which e-converges to x and d-converges to y.

ii. $(x, 0) \in \mathrm{cl}_e[\mathrm{Graph}(d_y)]$.

It follows that the set of yellow points is definable.

Proof. Suppose that γ is a path in X which e-converges to x and d-converges to y. Then $(\gamma(t), (d_y \circ \gamma)(t))$ is a path in $\mathrm{Graph}(d_y)$ which e-converges to $(x, 0)$. Conversely, suppose that $(x, 0) \in \mathrm{cl}_e[\mathrm{Graph}(d_y)]$. Let γ be a path in $\mathrm{Graph}(d_y)$ which e-converges to $(x, 0)$. Let $\pi_X : X \times R \to X$ and $\pi_R : X \times R \to R$ be the coordinate projections. Then $\pi_1 \circ \gamma$ converges to x and $\pi_R \circ \gamma$ converges to 0. As $\pi_X \circ \gamma = d_y \circ \pi_X \circ \gamma$, $\pi_X \circ \gamma$ is a path in X which e-converges to x and d-converges to y. □

As in Chapter 7, $(\widetilde{X}, d)$ is the definable completion of (X, d).

Lemma 9.2.2. *Let $x \in X$. Then x is a blue point of $(\widetilde{X}, d)$ if and only if any d-Cauchy path in X which e-converges to x must also d-converge to x.*

Proof. Suppose that x is a blue point of $\widetilde{X}$ and that γ is a d-Cauchy path which e-converges to x. It follows that γ d-converges in $\widetilde{X}$ and so γ must d-converge to x. We now prove the converse. Suppose that every d-Cauchy path in X which e-converges x also d-converges to x. Suppose that γ is a path in $\widetilde{X}$ which e-converges to x. Suppose that γ d-converges to $y \in \widetilde{X}$. Applying definable choice and the density of X in $\widetilde{X}$ there is a path η in X such that $d(\gamma(t), \eta(t)) < t$ for all $t \in R^{>}$. Thus η d-converges to y, so η is d-Cauchy. It follows from the assumption on x that η d-converges to x, so γ d-converges to x. □

Proposition 9.2.3. *Every definable metric space is definably isometric to a blue definable metric space.*

To prove Proposition 9.2.3 it suffices to prove the following claim:

Claim 9.2.4. *Almost every $x \in X$ is blue.*

Before proving this claim we suppose that it on holds every definable metric space and prove Proposition 9.2.3. We apply induction on $\dim(X)$. If $\dim(X) = 0$ then (X, e) and (X, d) are both discrete and so (X, d) is blue. Suppose $\dim(X) > 0$. We let A be the definable set of points of X which are yellow in $(\tilde{X}, d)$. Clearly $(X \setminus A, d)$ is blue and $\dim(A) < \dim(X)$. Applying the inductive assumption on (A, d) we produce a definable isometry $\tau : (A, d) \to (A', d')$ to a blue metric space (A', d'). Let X' be the disjoint union of $X \setminus A$ and A'. Let $\sigma : X \to X'$ be the natural bijection and let d' be the pushfoward of d by σ. Any path in X' is eventually contained in A' or $X \setminus A$. This implies that (X', d') is blue.

We prove a slight strengthening of Claim 9.2.4, where we use the full strength of the hypothesis in the inductive step.

Claim 9.2.5. *Almost every $x \in X$ is a blue point of $\widetilde{X}$.*

Proof. We apply induction on $\dim(X)$. If $\dim(X) = 0$ then both (X, e) and (X, d) are discrete and the claim trivially holds. We assume that $\dim(X) > 0$ and let $l = \dim(X)$. Let $\sigma : X \to R^l$ be an injective definable map. Let $X' = \sigma(X)$ and d' be the metric on X' given by $d'(\sigma(x), \sigma(y)) = d(x, y)$. Almost every $p \in X$ has a neighborhood U such that $\sigma|_U$ is a homeomorphism onto an open neighborhood of $\sigma(p)$, and if p has such a neighborhood then $\sigma(p)$ is blue in X' if and only if p is blue in X. It suffices to show that almost every point in (X', d') is blue.

We therefore suppose without loss of generality that $X \subseteq R^l$. We assume towards a contradiction that $\dim A = l$. We apply definable choice to produce definable functions $g : A \to \widetilde{X}$ and $\psi : A \times R^{>} \to X$ such that:

$$g(x) \neq x \quad \text{and} \quad e(\psi_x(t), x) < t, d(\psi_x(t), g(x)) < t \quad \text{for all } x \in A, t \in R^{>}.$$

Then ψ_x is a path in X which e-converges to x and d-converges to $g(x)$. Let $\pi : X \to R$ be the projection onto the last coordinate. Fix $y \in \pi(X)$. We have $\dim \pi^{-1}(y) \leqslant l - 1$ so we can apply the inductive assumption to $(\pi^{-1}(y), d)$. For almost every $x \in \pi^{-1}(y)$ we have the following: a d-Cauchy path in $\pi^{-1}(y)$ which e-converges to x also d-converges to x. This holds for every $y \in \pi(X)$ so we apply the fiber lemma for o-minimal dimension and conclude that for almost every $x \in X$ we have the following: a d-Cauchy path γ in X which e-converges to x and satisfies $(\pi \circ \gamma)(t) = \pi(x)$ when $0 < t \ll 1$ also d-converges to x. Let $A' \subseteq A$ be the definable set of points in A which satisfy this condition. Then A' is l-dimensional. If $x \in A'$ and $0 < t \ll 1$ then $(\pi \circ \psi_x)(t) \neq \pi(x)$. We let

$$A'_+ = \{x \in A' : [0 < t \ll 1] \longrightarrow [\pi(\psi_x(t)) > \pi(x)]\}$$

and

$$A'_- = \{x \in A' : [0 < t \ll 1] \longrightarrow [\pi(\psi_x(t)) < \pi(x)]\}.$$

As $A' = A'_+ \cup A'_-$, at least one of A'_+ or A'_- is l-dimensional. We suppose without loss of generality that $\dim A_+ = l$. Let $J \subseteq R$ be an interval and $B \subseteq R^{l-1}$ be a product of intervals such that $J \times B \subseteq A'_+$. For every $x \in J \times B$ we have $\pi(\psi_x(t)) > \pi(x)$ when $0 < t \ll 1$.

Claim 9.2.6. *For all $b \in J$ there is an open $V \subseteq B$ and $\delta, s > 0$ such that if $y \in (b, b+\delta) \times V$ then $d(\{b\} \times V, y) > s$.*

Fix $b \in J$. Suppose $x \in \{b\} \times B$. As $x \in A'$ there is a $s > 0$ such that if $z \in X$ satisfies $d(g(x), z) < s$ and $e(x, z) < s$ then $\pi(z) \neq b$. Applying Lemma 6.1.1 there is an open $V' \subseteq B$ and $s > 0$ such that if $x \in \{b\} \times V'$ and $z \in X$ satisfy $d(g(x), z) < s$ and $e(x, z) < s$ then $\pi(z) \neq b$. If $x, z \in \{b\} \times V'$ and $e(x, z) < s$ then $d(g(x), z) \geqslant s$. By shrinking V' if necessary we may assume that $\mathrm{Diam}_e(V') < s$. Now if $x, z \in \{b\} \times V'$ then $d(g(x), z) > s$. This implies that if $x \in \{b\} \times V'$ and $0 < t < s$ then $\pi(\psi_x(t)) > b$. Applying Lemma 6.1.3 to the image ψ we find an open $V \subseteq V'$ and a $\delta > 0$ such that

$$(b, b+\delta) \times V \subseteq \left\{ \psi_x(t) : x \in V', 0 < t < \frac{1}{2}s \right\}.$$

We show that V and δ satisfy the conditions of Claim 9.2.6. Let $y \in (b, b+\delta) \times V$ and $z \in \{b\} \times V$. Fix $(x, t) \in V' \times (0, \frac{s}{2})$ such that $\psi_x(t) = y$. We have $d(g(x), y) < \frac{s}{2}$ and $d(g(x), z) > s$ so the triangle inequality implies that $d(z, y) > \frac{1}{2}s$. This proves Claim 9.2.6.

Claim 9.2.7. *There is an interval $J' \subseteq J$, a product of intervals $B' \subseteq B$ and $s > 0$ such that if $y, z \in J' \times B'$ and $\pi(y) \neq \pi(z)$ then $d(x, y) > s$.*

Applying definable choice to Claim 9.2.6 we fix definable functions $f, g : J \to R$ and a definable set $V \subseteq J \times B$ such that each V_b is open and for all $b \in J$, if $y \in (b, b + f(b)) \times V_b$ then $d(\{b\} \times V_b, y) > g(b)$. There is an interval $J' \subseteq J$, a product of intervals $B' \subseteq B$ and $s, \delta > 0$ such that for all $b \in J'$:

i. $f(b) > s$ and $g(b) > \delta$,

ii. $B' \subseteq V_b$.

Here (i) is immediate. (ii) is immediate from the fiberwise openness theorem Theorem 2.2 of [Dri98].

By shrinking J' if necessary we suppose that $\text{Diam}_e(J') < \delta$. If $b \in J'$ then

$$\{x \in J' \times B' : \pi(x) > b\} \subseteq (b, b + \delta) \times V_b \subseteq (b, b + g(b)) \times V_b.$$

Thus if $x, y \in J' \times B'$ and $\pi(y) > \pi(x)$ then $d(x, y) > g(\pi(x)) > s$. This proves Claim 9.2.7.

Fix $x \in J' \times B'$. As we have $\pi(\psi_x(t)) \neq \pi(x)$ when $0 < t \ll 1$ and $\pi(\psi_x(t)) \to \pi(x)$ as $t \to 0^+$ we have $\pi(\psi_x(t)) \neq \pi(\psi_x(t'))$ when $0 < t < t' \ll 1$. Thus $d(\psi_x(t), \psi_x(t')) > s$ when $0 < t < t' \ll 1$. This contradicts the fact that ψ_x is d-Cauchy. This contradiction establishes Claim 9.2.5. $\square$

Every blue metric space admits a definable 1-Lipschitz injection into a definably compact metric space. We define the map ι_X in the case when X is a bounded subset of euclidean space. We indicate here how to extend this definition to the general case. Let X' be a definable bounded subset of euclidean space for which there is a definable homeomorphism $\tau : X \to X'$. Let d' be the pushforward of d onto X' by τ. We then take $\iota_X = \tau \circ \iota_{X'}$. Then ι_X depends on the choice of X' and τ. These choices will not matter, so we suppress them. Note that, in the situation described, (X', d') is blue if and only if (X, d) is blue. This ensures that ι_X is injective when (X, d) is blue.

Definition 9.2.8. Suppose that X is a bounded subset of euclidean space. Let $\mathbf{d}$ be the function on X^2 given by $\mathbf{d}(x, y) = \min\{d(x, y), 1\}$. It is easily checked that $\mathbf{d}$ is a metric and

$id : (X, d) \to (X, \mathbf{d})$ is 1-Lipschitz. For each $x \in X$ we let $\mathbf{d}_x : X \to R$ be $\mathbf{d}_x(y) = \mathbf{d}(x, y)$. We define a pseudometric d_H on X by setting $d_H(x, y)$ equal to the Hausdorff distance between $\mathrm{Graph}(\mathbf{d}_x)$ and $\mathrm{Graph}(\mathbf{d}_y)$. As X is a bounded subset of euclidean space and $\mathbf{d}$ is bounded from above by 1, each $\mathrm{Graph}(\mathbf{d}_x)$ is bounded and so $d_H(x, y) < \infty$ for every $x, y \in X$. For any bounded definable functions $f, g : X \to R$ we have:

$$d_H(\mathrm{Graph}(f), \mathrm{Graph}(g)) \leqslant \|f - g\|_\infty.$$

As $\mathbf{d}(x, y) = \|\mathbf{d}_x - \mathbf{d}_y\|_\infty$ the map $id : (X, d) \to (X, d_H)$ is 1-Lipschitz. Furthermore, as $\|\mathbf{d}_x - \mathbf{d}_y\|_\infty \leqslant 1$ for any $x, y \in X$, (X, d_H) is bounded. We let (B, d_H) be the metric space associated to the pseudometric (X, d_H) and let $(\widetilde{B}, d_H)$ be the definable completion of (B, d_H). As $(\widetilde{B}, \widetilde{\mathbf{d}}_H)$ is bounded, Lemma 7.2.8 implies that $(\widetilde{B}, d_H)$ is definably compact. We define ι_X to be the natural 1-Lipschitz map

$$\iota_X : (X, d) \to (\widetilde{B}, d_H).$$

Lemma 9.2.9. *If (X, d) is blue and X is a bounded of subset of euclidean space then ι is injective. Equivalently, if (X, d) is blue and X is a bounded subset of euclidean space then (X, d_H) is a metric space.*

Proof. We suppose that d_H is not a metric. Let x, y be distinct elements of X for which $d_H(x, y) = 0$. So

$$\mathrm{cl}_e[\mathrm{Graph}(\mathbf{d}_x)] = \mathrm{cl}_e[\mathrm{Graph}(\mathbf{d}_y)]$$

and in particular $(x, 0) \in \mathrm{cl}_e[\mathrm{Graph}(\mathbf{d}_y)]$. This contradicts Lemma 9.2.1. $\square$

By Proposition 9.2.3 we have:

Corollary 9.2.10. *Any definable metric space admits a continuous injection into a definably compact metric space.*

Lemma 9.2.11. *Let $A \subseteq X$ be definable. Then*

$$\dim[\partial_d(A)] < \dim(A).$$

If $\dim(A) = \dim(X)$ *then* $\dim[\mathrm{Int}_d(A)] = \dim(A)$.

Proof. It suffices to prove the lemma for any definable metric space definably isometric to (X, d). We therefore assume that (X, d) is blue. Thus d_H is a metric on X. The dimension inequality for Hausdorff limits (item (3) of Theorem 3.1 in [Dri05]) implies $\dim[\partial_{d_H}(A)] < \dim(A)$. Continuity of $id : (X, d) \to (X, d_H)$ implies that the d-frontier of A is contained in the d_H-frontier of A. So $\dim[\partial_d(A)] < \dim(A)$. Suppose that $\dim(A) = \dim(X)$. As $A \setminus \mathrm{Int}_d(A) = \partial_d(X \setminus A)$, we have

$$\dim[A \setminus \mathrm{Int}_d(A)] < \dim(X \setminus A) \leqslant \dim(A)$$

and so $\dim[\mathrm{Int}_d(A)] = \dim(A)$. $\square$

We use this dimension inequality to prove the following key technical lemma:

Lemma 9.2.12. *Let d' be another definable metric on X. Suppose that $\dim_x(X, d') = \dim(X)$ at almost every $x \in X$. Then $id : (X, d) \to (X, d')$ is continuous almost everywhere.*

Proof. Let $D \subseteq X$ be the set of points at which $id : (X, d) \to (X, d')$ is not continuous. We suppose towards a contradiction that $\dim(D) = \dim(X)$. Let $\pi_1 : X^2 \times R \to X$ be the projection onto the first coordinate. Let $Q \subseteq X^2 \times R$ be the set of (x, x', t) such that $x \in B_{d'}(x', t)$ and x is not in the d-interior of $B_{d'}(x', t)$. That is,

$$\{x \in X : (x, y, t) \in Q\} = \partial_d[B_{d'}(y, t)] \quad \text{for all } (y, t) \in X \times R^{>}.$$

By definition $D = \pi_1(Q)$. It follows that

$$\dim\{x \in X : (x, y, t) \in Q\} < \dim X \quad \text{for all } (y, t) \in X \times R^{>}.$$

By the fiber lemma for o-minimal dimension, $\dim(Q) < 2\dim(X) + 1$. Now we get a lower bound on $\dim Q$. Let $x \in D$. For some $t > 0$, x is not in the d-interior of $B_{d'}(x, t)$. If $(y, t') \in X \times R$ satisfies $x \in B_{d'}(y, t') \subseteq B_{d'}(x, t)$ then x is not in the d-interior of $B_{d'}(x, t)$ and $(x, y, t') \in Q$. Let $y \in B_{d'}\left(x, \frac{t}{3}\right)$ and $\frac{t}{3} < s < \frac{2t}{3}$. Then $x \in B_{d'}(x, \frac{t}{3})$ and by the triangle inequality, $B_{d'}(y, s) \subseteq B_{d'}(x, t)$, so $(y, s) \in Q_x$. We have shown:

$$B_{d'}\left(x, \frac{t}{3}\right) \times \left(\frac{t}{3}, \frac{2t}{3}\right) \subseteq Q_x.$$

So for each $x \in D$, $\dim(Q_x) \geqslant \dim_x(X, d') + 1$. We assumed that $\dim(D) = \dim(X)$ so we have $\dim(Q_x) = \dim(X) + 1$ at almost every $x \in D$. So $\dim(Q) = 2\dim(X) + 1$. Contradiction. $\square$

Corollary 9.2.13. *We have the following:*

i. $id : (X, d) \to (X, e)$ *is continuous almost everywhere.*

ii. There is a partition $\mathcal{X}$ *of* X *into cells such that if* $Y \in \mathcal{X}$ *then* $id : (Y, d) \to (Y, e)$ *is continuous.*

iii. If $\dim_x(X, d) = \dim(X)$ *at almost every* $x \in X$ *then both* $id : (X, e) \to (X, d)$ *and* $id : (X, d) \to (X, e)$ *are continuous almost everywhere on* X*.*

iv. If $\dim_x(X, d) = \dim(X)$ *at almost every* $x \in X$ *then almost every point* $p \in X$ *has an* e*- and* d*-open neighborhood* $V \subseteq X$ *such that* $id : (V, d) \to (V, d)$ *is a homeomorphism.*

v. If (X, d) *is definably separable then there is a partition* $\mathcal{X}$ *of* X *into cells such if* $Y \in \mathcal{X}$ *then* $id : (X, d) \to (Y, e)$ *is a homeomorphism.*

vi. If (X, d) *is definably separable then* (X, d) *is definably isometric to a definable metric space* (X', d') *such that* $id : (X', e) \to (X', d')$ *is continuous and* X' *is a locally closed subset of euclidean space.*

Proof. (i) follows immediately from Lemma 9.2.12. (iii) follows from (i) and Lemma 9.2.12. We prove (ii) by induction on $\dim(X)$. If $\dim(X) = 0$ we can take the $\mathcal{X}$ to be the singleton subsets of X. Suppose $\dim(X) > 0$. Let C be the set of points at which $id : (X, d) \to (X, e)$ is continuous. We can apply the inductive assumption to $(X \setminus C, d)$ and partition $X \setminus C$ into cells $X_1, \ldots, X_n$ on which $id_{d,e}$ is continuous. We then take some partition of C into cells $X_{n+1}, \ldots, X_m$. Let $\mathcal{X} = \{X_1, \ldots, X_n, X_{n+1}, \ldots, X_m\}$. We now prove (iv). Let $V \subseteq X$ be the set of points at which both $id : (X, e) \to (X, d)$ and $id : (X, d) \to (X, e)$ are continuous. Then $id : (V, d) \to (V, e)$ is a homeomorphism. (iii) implies $\dim(X \setminus V) < \dim(X)$. So $\dim(V) = \dim(X)$, so Lemma 9.2.11 implies that the d-interior of V is almost all of V. We let V' be the intersection of the d-interior of V with the e-interior of V in X. Then V' is almost all of V and $id : (V', d) \to (V', d)$ is a homeomorphism. (v) is proven in the same way as (ii). We prove (vi) using (v). Let $\mathcal{X}$ be the partition provided by (v) and let X' be the disjoint union of the elements of $\mathcal{X}$. Let d' be the natural pushforward of d onto X'. Then $id : (X', e) \to (X', d)$ is continuous. X' is locally closed as it is a disjoint union of cells. $\square$

Then next lemma is immediate from (iv) above and cell decomposition:

Lemma 9.2.14. *Let* $\dim(X) = l$. *Suppose that* $\dim_x(X, d) = l$ *at almost every* $x \in X$. *There is a definable injection* $((0,1)^l, e) \to (X, d)$ *which gives a homeomorphism between* $(0,1)^l$ *and a d-open subset of* X.

We now definably compactify definably separable metric spaces.

Proposition 9.2.15. *Every definably separable metric space is definably homeomorphic to a definable subspace of a definably compact metric space.*

To prove Propostion 9.2.15 it suffices to show that a definably separable (X, d) is definably isometric to a definable metric space for which the map ι is a homeomorphism onto its image; this shows that (X, d) is definably homeomorphic to a definable subspace of a definably compact metric space. This is established by the lemma below and Corollary 9.2.13.

Lemma 9.2.16. *Suppose that* $id : (X, e) \to (X, d)$ *is continuous. Then the topologies given by* d *and* d_H.

Proof. Continuity of $id : (X, e) \to (X, d)$ implies that (X, d) is blue, thus it is enough to show that $id : (X, d_H) \to (X, d)$ is continuous. Let $x \in X$ and $\delta > 0$. Let $\epsilon > 0$ be such that for all $y \in Y$, if $e(x, y) < \epsilon$ then $d(x, y) < \delta$. We assume that $\epsilon < \delta$. Suppose that $d_H(x, y) < \epsilon$. There is a point on $\mathrm{Graph}(d_y)$ whose euclidean distance from $(x, 0)$ is at most ϵ. Namely we have a x' such that the euclidean distance between $(x, 0)$ and $(x', d_y(x'))$ is at most ϵ. This implies that $e(x, x') < \epsilon$ and that $d(y, x') < \epsilon$. So $d(x, x') < \delta$ and the triangle inequality gives $d(x, y) < \epsilon + \delta < 2\delta$. So $id : (X, d_H) \to (X, d)$ is continuous. $\square$

To prove Theorem 9.0.1 it now suffices to show that every definably compact metric space is definably homeomorphic to a definable set equipped with its euclidean metric. This is done in the next section.

9.3 Covering Definably Compact Metric Spaces

In this section (X, d) *is a definably compact metric space.* We want to show that:

Claim 9.3.1. *There are definably compact sets* $Z_1, \ldots, Z_n$ *and definable continuous maps* $\rho_i : (Z_i, e) \to (X, d)$ *whose images cover* X*.*

Suppose that we have such Z_i and ρ_i. Let Z be the disjoint union of the Z_i and let $\rho : (Z, e) \to (X, d)$ be the map induced by the ρ_i. Then (Z, e) is definably compact and ρ is continuous, Lemma 7.1.8 gives a definable set Y and a definable homeomorphism $(X, d) \to (Y, e)$. Our construction of the Z_i is an application of the definable Michael's Selection Theorem, Theorem 4.1 of [AT13], which we now recall. Let $T \subseteq E \times R^m$. We say that T is **lower semi-continuous** if for every $(x, y) \in T$ and neighborhood $V \subseteq R^m$ of y there is a neighborhood U of x such that $T_y \cap V \neq \emptyset$ for every $y \in U$.

Theorem 9.3.2 (Aschenbrenner-Thamrongtanyalak)**.** *Let* E *be a definable locally closed set and let* $T \subseteq E \times R^m$ *be definable and lower semi-continuous such that each* T_x *is closed and convex. Then there is a definable continuous function* $\sigma : E \to R^m$ *such that* $\sigma(x) \in T_x$ *always.*

We will use the following lemma to apply Michael Selection:

Lemma 9.3.3. *Let* $E \subseteq R^l$ *and let* $S \subseteq E \times R^m$ *be lower semi-continuous. Let* $T \subseteq E \times R^m$ *be such that for every* $x \in E$*,* T_x *is the closure of the convex hull of* S_x *in* R^m*. Then* T *is lower semi-continuous.*

Proof. Fix $x \in E$ and let V be an open subset of R^m such that $T_x \cap V \neq \emptyset$. The intersection of V with the convex hull of S_x is nonempty, hence there are $p, p' \in S_x$ and $s \in (0, 1)$ such that $sp + (1 - s)p' \in V$. There are open $W, W' \subseteq R^m$ such that $p \in W$, $p' \in W'$ and if

$(q, q') \in W \times W'$ then $sq + (1-s)q' \in V$. Let U be a neighborhood of x such that if $y \in U$ then $S_y \cap W$ and $S_y \cap W'$ are both nonempty. Fix $q \in S_y \cap W$ and $q' \in S_y \cap W'$. Then $sq + (1-s)q'$ is an element of $T_y \cap V$. So T is lower semi-continuous. $\square$

We now prove Claim 9.3.1.

Proof. As it suffices to prove the claim for any definable metric space definably isometric to (X, d) we assume that X is a bounded subset of euclidean space. We apply induction on $\dim(X)$. The base case $\dim(X) = 0$ is trivial, we treat the case $\dim(X) > 0$. Applying Proposition 9.2.13 we partition X into cells $X_1, \ldots, X_n$ such that each $id : (X_i, d) \to (X_i, e)$ is a homeomorphism. For each i we construct a definably compact definable set Z_i and a definable continuous surjection

$$\rho_i : (Z_i, e) \to (\mathrm{cl}_d(X_i), d).$$

Fix i. To simplify notation we let $C = X_i$ and $A = \partial_d(C)$. As (C, e) is locally definably compact so (C, d) is locally definably compact. By Lemma 7.1.2 this implies that C is a locally closed subset of X. Then C is open in $\mathrm{cl}_d(C)$, so A is a d-closed subset of X. Thus (A, d) is definably compact. By Lemma 9.2.11 we have $\dim(A) < \dim(X)$, so we apply the inductive assumption to (A, d). We let $A' \subseteq R^m$ be a definably compact subset of euclidean space and let $\tau : (A, d) \to (A', e)$ be a homeomorphism. We let Y be the disjoint union of X_i and A'. We define $\tau' : \mathrm{cl}_d(X_i) \to Y$ to be identity on X_i and τ on A. We let d' be the pushforward of d by τ. Then (Y, d') is definably isometric to $(\mathrm{cl}_d(X_i), d)$, so it suffices to construct a definable continuous surjection $\rho : (Z, e) \to (Y, d')$ from a definably compact Z. We assume without loss of generality that the d- and e-topologies agree on A. Then A is a definably compact subset of euclidean space. We use the definable Michael's Selection Theorem to construct a bounded continuous definable function $\sigma : C \to R^m$ which satisfies the following for any path γ in C: if γ d-converges to $x \in A$ then $\lim_{t \to 0^+} (\sigma \circ \gamma)(t) = x$.

We first suppose we have such a σ. Let Z be the closure of $\mathrm{Graph}(\sigma)$ in the ambient euclidean space. Boundedness of C and σ ensures that Z is definably compact. Let $\pi_1 : Z \to C$ and $\pi_2 : Z \to R^m$ be the coordinate projections. We define a map $\rho : Z \to \mathrm{cl}_d(C)$

by setting $\rho(x) = \pi_1(x)$ when $x \in \text{Graph}(\sigma)$ and $\rho(x) = \pi_2(x)$ when $x \in \partial_e[\text{Graph}(\sigma)]$. First we must show that ρ does in fact take values in $\text{cl}_d(C)$. It suffices to show that $\pi_2(x) \in A$ when $x \in \partial_e[\text{Graph}(\sigma)]$, to this effect fix such an x. Let γ be a path in $\text{Graph}(\sigma)$ which e-converges to x. Then $\pi_1 \circ \gamma$ is a path in C which by definable compactness must d-converge to some $y \in \text{cl}_d(C)$. This y must be an element of A, otherwise $\pi_1 \circ \gamma$ would e-converge to an element of C and the continuity of σ would force $x \in \text{Graph}(\sigma)$. We have

$$\rho(x) = \pi_2(x) = \lim_{t \to 0^+} (\pi_2 \circ \gamma)(t) = \lim_{t \to 0^+} (\sigma \circ \pi_1 \circ \gamma)(t) = y \in \text{cl}_d(C).$$

We have shown that ρ takes values in $\text{cl}_d(C)$.

We now show that ρ is surjective. As C is obviously contained in the image of ρ it suffices to show that A is contained in the image of ρ. Let $y \in A$. Applying definable choice let γ be a path in C which d-converges to y. We have $\lim_{t \to 0^+}(\sigma \circ \gamma)(t) = y$. Letting z be the e-limit of $(f(t), (\sigma \circ \gamma)(t))$ as $t \to 0^+$ we have $z \in Z$ and $\rho(z) = y$.

We now show that ρ is continuous. We first show that ρ is continuous at every point in $\text{Graph}(\sigma)$. The restriction of ρ to $\text{Graph}(\sigma)$ is a continuous function, as projections are continuous and $id : (C, e) \to (C, d)$ is continuous. As $\text{Graph}(\sigma)$ is a cell it hence open in its closure, Z. Thus ρ is continuous at every point in $\text{Graph}(\sigma)$. We now show that ρ is continuous at every point in $\partial_e[\text{Graph}(\sigma)]$. Fix $x \in \partial_e[\text{Graph}(\sigma)]$. Let γ be a path in Z which e-converges to x. First suppose that $\gamma(t) \in [\text{Graph}(\sigma)]$ when t is sufficiently small. Then the d-limit of $(\rho \circ \gamma)(t)$ as $t \to 0^+$ equals the d-limit of $(\pi_1 \circ \gamma)(t)$ as $t \to 0^+$, which equals $\lim_{t \to 0^+}(\sigma \circ \pi_1 \circ \gamma)(t)$. As we showed above this equals $\rho(x)$. We now assume $\gamma(t) \in \partial_e[\text{Graph}(\sigma)]$ when $0 < t \ll 1$. As $\pi_2 : (\partial_e[\text{Graph}(\sigma)], e) \to (A, e)$ and $id : (A, e) \to (A, d)$ are continuous, the restriction of ρ to $\partial_e[\text{Graph}(\sigma)]$ is continuous. This implies that $\gamma(t) \to x$ as $t \to 0^+$.

We finally construct σ. Let $B \subseteq C \times R^m$ be the set of (x, y) such that $y \in A$ and $d(y, x) < 2d(A, x)$. As A is d-closed, $d(A, x)$ is always strictly positive, so B_x is always nonempty. We show that B is lower semi-continuous. Fix $z, y \in C$ satisfying $y \in B_z$. By the definition of B, $d(y, z) < 2d(A, z)$. We show that if $z' \in C$ satisfies

$$d(z, z') < \frac{1}{4}|2d(A, z) - d(y, z)|$$

then $y \in B_{z'}$. As the set of such z' is an e-open subset of C this gives lower semi-continuity. We have

$$|d(y, z') - d(y, z)| \leqslant d(z, z') < \frac{1}{4}|2d(A, z) - d(y, z)|$$

and

$$|2d(A, z) - 2d(A, z')| \leqslant 2d(z, z') < \frac{1}{2}|2d(A, z) - d(y, z)|.$$

These two inequalities give $d(y, z') < 2d(A, z')$ so $y \in B_{z'}$.

Let $D \subseteq X \times R^m$ be the definable set such that for each $x \in C$, D_x is the closure of the convex hull of B_x. Each D_x is closed and convex and D is lower semi-continuous. Applying the Michael's Selection Theorem let $\sigma : C \to R^m$ be a continuous definable map such that $\sigma(x) \in D_x$ holds for all $x \in C$. Fix a path γ in C which d-converges to $x \in A$. We show that $\lim_{t\to 0^+}(\sigma \circ \gamma)(t) = x$. Towards this we show that $B_{\gamma(t)}$ is contained in the d-ball with center x and radius $3d(x, \gamma(t))$. Fix $w \in B_{\gamma(t)}$. By definition of B we have $d(w, \gamma(t)) < 2d(A, \gamma(t))$. As $x \in A$ we have $d(A, \gamma(t)) \leqslant d(x, \gamma(t))$ and we compute:

$$d(x, w) \leqslant d(\gamma(t), x) + d(\gamma(t), w) \leqslant d(\gamma(t), x) + 2d(A, \gamma(t)) \leqslant 3d(\gamma(t), x).$$

As (A, d) is definably compact and the d-topology agrees with the e-topology on A, we apply Lemma 7.1.6 to produce a definable $h : R^{\geqslant} \to R^{\geqslant}$ satisfying

$$\|y - y'\| \leqslant h(d(y, y')) \quad \text{for all } y, y' \in A$$

and such that $\lim_{t\to 0^+} h(t) = 0$. We let $r(t) = h(3d(\gamma(t), x))$. Then $r(t) \to 0$ as $t \to 0^+$ and $B_{\gamma(t)} \subseteq B_e(x, r(t))$ for all $t \in R^>$. As euclidean balls are convex,

$$C_{\gamma(t)} \subseteq B_e(x, r(t)) \quad \text{for all } t \in R^>.$$

Thus $\|(\sigma \circ \gamma)(t) - x\| \leqslant r(t)$. The path $\sigma \circ \gamma$ e-converges to x. □

This completes the proof of Theorem 9.0.1. The uniform version of Theorem 9.0.1 follows by applying model-theoretic compactness and the existence of definable Skolem functions in the usual way:

Corollary 9.3.4. *Let $\{(X_\alpha, d_\alpha) : \alpha \in R^l\}$ be a definable family of metric spaces. The set of $\alpha \in R^l$ such that (X_α, d_α) is definably separable is definable. In fact, there is a definable family $\{A_\alpha : \alpha \in R^l\}$ of sets such that $A_\alpha \subseteq X_\alpha$, a definable family of sets $\{Z_\alpha : \alpha \in R^l\}$ and a definable family of functions $h_\alpha : X_\alpha \to Z_\alpha$ such that for every $\alpha \in R^l$ exactly one of the following holds:*

i. A_α is a d-discrete subset of X_α,

ii. h_α gives a homeomorphism $(X_\alpha, d_\alpha) \to (Z_\alpha, e)$.

The set of α for which h_α gives such a homeomorphism is definable.

From Lemma 7.1.7 and Theorem 9.0.1 we immediately have the following. Recall our standing assumption that (X, d) is definably compact.

Corollary 9.3.5. *Suppose that $\mathcal{R}$ is power bounded. Then (X, d) is definably isometric to a definable metric space (X', d') such that for some $\lambda_1, \lambda_2 > 0$ and positive elements r_1, r_2 of the field of powers of $\mathcal{R}$ such that:*

$$\lambda_1\|x' - y'\|^{r_1} \leqslant d'(x', y') \leqslant \lambda_2\|x' - y'\|^{r_2} \quad \text{for all } x', y' \in X'.$$

Combining Corollary 9.2.10 and Theorem 9.0.1 we have:

Corollary 9.3.6. *Every definable metric space (Y, d) is definably isometric to a definable metric space (Z, d') such that $id : (Z, d') \to (Z, e)$ is continuous.*

Proof. After replacing (Y, d) with a definably isometric space if necessary we can suppose that d_H is a metric on Y and that $id : (Y, d) \to (Y, d_H)$ is continuous. There is a definable set Z and a definable homeomorphism

$$\tau : (Y, d_H) \to (Z, e).$$

Let d' be the pushforward of d onto Z by τ. Then (Z, d') is definably isometric to (Y, d). We show that $id : (Z, d') \to (Z, e)$ is continuous by factoring it as a composition of continuous maps:

$$(Z, d') \xrightarrow{\tau^{-1}} (Y, d) \xrightarrow{id} (Y, d_H) \xrightarrow{\tau} (Z, e).$$

□

9.4 Product Structure of General Definable Metric Spaces

In this section we prove Proposition 9.4.1. As an application in the next section we prove Corollary 9.5.1, which characterizes the topological dimension of a metric space definable in an o-minimal expansion of the real field.

Proposition 9.4.1. *Almost every $x \in X$ is contained in a definable e- and d-open set $U \subseteq X$ which admits a definable homeomorphism*

$$(U, d) \to (I^n, e) \times (D, d_{\text{disc}})$$

where I is an open interval, $n \leqslant \dim(X)$, and D is a definable set. If $n = \dim(X)$ then we can take D to be a singleton.

Proof. Let $\dim(X) = m$. It suffices to prove the proposition for any definable metric space definably isometric to (X, d) as any definable isometry $(X, d) \to (X', d')$ locally gives a homeomorphism $(X, e) \to (X', e)$ at almost every point. Therefore after applying Corollary 9.3.6 we assume that $id : (X, d) \to (X, e)$ is continuous. Then every e-open subset of X is d-open. It suffices to fix an e-open, m-dimensional definable $W \subseteq X$ and find an e-open definable $U \subseteq W$ which satisfies the conditions of the proposition. We suppose, after shrinking W if necessary, that $\dim_x(X, d) = n$ for every $x \in W$. Then $\dim_x(W, d) = n$ at every $x \in W$. If $n = m$ we use Lemma 9.2.14 to find an e-open definable $U \subseteq W$ for which $id : (U, d) \to (U, e)$ is a homeomorphism and the proposition holds with D a singleton. We therefore assume that $n < m$. Let $f : W \to R^{>}$ be a definable function such that $\dim B(p, f(p)) = n$ for all $p \in W$. For $x \in W$ let $B_x = B(x, f(x))$. Applying a version of the good directions lemma in the same way as in the proof Lemma 9.1.1 we let $\pi : W \to R^n$ be a linear projection and $W' \subseteq W$ be an e-open m-dimensional definable set such that

$$|\pi^{-1}(w) \cap B_x| < \infty \quad \text{ for all } w \in R^n, x \in W.$$

After replacing W with W' if necessary we suppose that these properties hold for W. As in the proof of Lemma 9.1.1 each fiber of π is d-discrete. For all $x \in W$ there is a $t \in R^{>}$ such that $d(x, y) > t$ whenever $y \in W$ is distinct from x and satisfies $\pi(x) = \pi(y)$. After applying

Lemma 6.1.1 and replacing W with a smaller m-dimensional open set if necessary we fix a $t > 0$ such that if $x, y \in W$ and $\pi(x) = \pi(y)$ then $d(x, y) > t$. After replacing f with the function $\max\{f, \frac{1}{2}t\}$ if necessary we suppose that each B_x has radius at most $\frac{1}{2}t$. Note that we still have $\dim(B_x) = n$ and $\dim_y(B_x) = n$ at all $y \in B_x$. As $\mathrm{Diam}_d(B_x) < t$ the restriction of π to each B_x is injective. Furthermore if $B_x \cap B_y \neq \emptyset$ then the triangle inquality implies $d(x, y) < t$. Thus if $\pi(x) = \pi(y)$ then B_x and B_y have empty intersection. Injectivity of π on B_x implies $\dim \pi(B_x) = n$. As $\dim \pi(W) = n$ the fiber lemma for o-minimal dimension gives that some fiber of π is $(m-n)$-dimensional. We fix an $(m-n)$-dimensional fiber $F \subseteq W$ and let $G = \bigcup_{x \in F} B_x$. As the union is disjoint $\dim(G) = m$. As $\dim_y(B_x, d) = n$ at every $y \in B_x$ we apply Lemma 9.2.14 uniformly we take $\beta : F \times (0,1)^n \to G$ to be a definable function such that for each $x \in F$, $\beta_x[(0,1)^n]$ is an e-open subset of B_x and

$$\beta_x : ((0,1)^n, e) \to (\beta_x(I^n), d)$$

is a homeomorphism for each $x \in W$. We fix a definable $D \subseteq F$ with $\dim(D) = m - n$ and an interval $I \subseteq (0,1)$ such that the restriction of β to $D \times I^n$ gives a continuous map between euclidean topologies. We now replace β with the restriction of β to $D \times I^n$. By the o-minimal open mapping theorem, see [Woe96] or [Joh01], $\beta(D \times I^n)$ is an e-open subset of W. We let $U = \beta(D \times I^n)$. U is a d-open subset of X. Then (U, d) is, as a topological space, the disjoint union of the sets $\beta(\{b\} \times I^n)$. It follows that

$$\beta : (D, d_{\mathrm{disc}}) \times (I^n, e) \to (U, d)$$

is a homeomorphism. $\square$

The following two corollaries are weakenings of the previous proposition. We see in particular that every definable metric space is "almost locally definably compact".

Corollary 9.4.2. *Let (X, d) be an arbitrary definable metric space. Almost every $x \in X$ has a d-neighborhood U such that $id : (U, d) \to (U, e)$ is a homeomorphism.*

Corollary 9.4.3. *Almost every $x \in X$ has a d-neighborhood which is definably homeomorphic to an open subset of some R^k.*

Note that we could have $k = 0$.

9.5 Topological Dimension of Definable Metric Spaces

In this section we assume that $\mathcal{R}$ expands the ordered field of real numbers. We describe the topological dimension of a definable metric space (X, d). Recall the defination of topological dimension from Section 11.2.2.

Corollary 9.5.1. *The topological dimension of (X, d) is the maximal m for which there is a definable continuous injection $(I^m, e) \to (X, d)$ for an open interval $I \subseteq R$.*

Proof. Suppose $g : (I^m, e) \to (X, d)$ is a definable continuous injection. Then

$$\dim_{\text{top}}(X, d) \geqslant \dim_{\text{top}}(g(I^m), d) = \dim_{\text{top}}(I^m, e) = m.$$

We now suppose that there is no definable continuous injection $(I^m, e) \to (X, d)$ and show that $\dim_{\text{top}}(X, d) < m$. We apply induction to $\dim(X)$. Therefore we use Proposition 9.4.1 to partition (X, d) into definable sets U, A such that:

i. U is e-open and $\dim(A) < \dim(X)$.

ii. Every point in U has an e-neighborhood V which admits a definable homeomorphism

$$(V, d) \to (C, e) \times (D, d_{\text{disc}})$$

for definable sets C, D with $\dim(C) < m$.

There is no definable continuous injection $(I^m, e) \to (A, d)$, so the inductive assumption implies that $\dim_{\text{top}}(A, d) < m$. As (U, e) is separable we can cover U with countably many e-closed definable sets $\{F_i : i \in \mathbb{N}\}$ such that each F_i is contained in an e-open $V \subseteq X$ which satisfies the conditions on V in the statement. If V satisfies the conditions in the statement by Fact 3.5.2 then $\dim_{\text{top}}(V, d) = \dim(C) < m$, so $\dim_{\text{top}}(F_i, d) < m$. As

$$X = A \cup \bigcup_{i \in \mathbb{N}} F_i,$$

Fact 3.5.4 gives $\dim_{\text{top}}(X, d) \leqslant m$. $\square$

As an immediate consequence we have:

Corollary 9.5.2. $\dim_{\text{top}}(X, d) \leqslant \dim(X)$.

The proposition below is a complement to Corollary 9.5.1.

Proposition 9.5.3. *Let m be the maximal natural number for which there is a definable continuous surjection $(X, d) \to (I^m, e)$. Then $\dim(X) = m$.*

Proof. If $m > \dim(X)$ then there is no definable surjection $X \to I^m$. Corollary 9.3.6 and cell decomposition give the other inequality. □

Finally, we characterize zero-dimensional definable metric spaces.

Proposition 9.5.4. *The following are equivalent:*

i. *(X, d) is totally disconnected,*

ii. *(X, d) has finite Cantor rank, in fact $rk(X, d) \leqslant \dim(X)$,*

iii. *No definable subspace of X is definably homeomorphic to $([0, 1], e)$.*

Proof. It is clear that (ii) implies (i) and (iii). Every definable subspace of a totally disconnected metric space is totally disconnected, so (i) implies (iii). We show that (iii) implies (ii). We apply induction on $\dim(X)$. If $\dim(X) = 0$ then X is finite and thus has rank 0. Suppose that $\dim(X) > 0$. Let $D \subseteq X$ be the set of isolated points in X. We show that D is dense in X; to this effect we fix a nonempty open $U \subseteq X$. Applying Corollary 9.4.3 we find a definable open $V \subseteq U$ which is definably homeomorphic to an open subset of some $\mathbb{R}^k$. If $k \geqslant 1$ then V contains a definable set which is definably homeomorphic to $([0, 1], e)$. Thus we must have $k = 0$, so U contains an isolated point. Thus D is dense in X so Lemma 9.2.11 implies $\dim(X \setminus D) < \dim(X)$. No definable subspace of $(X \setminus D, d)$ is definably homeomorphic to $([0, 1], e)$, thus the inductive hypothesis implies $rk(X \setminus D, d) \leqslant \dim(X \setminus D)$. Then

$$rk(X, d) \leqslant rk(X \setminus D, d) + 1 \leqslant \dim(X).$$

□

CHAPTER 10

Metric Geometry of Definable Metric Spaces

Throughout this chapter (X, d) is an l-dimensional definable metric space. In this chapter we study bounds on d that can be obtained in terms of e. We say that (X, d) has **euclidean topology** if the d-topology agrees with the euclidean topology on X. In Section 10.1 we show that if (X, d) has euclidean topology then X contains an open set on which $\lambda e \leqslant d$ holds for some $\lambda > 0$. In Section 10.2 we consider upper bounds on d in terms of the euclidean metric. We show in Theorem 10.2.9 that if $\mathcal{R}$ is power bounded then exactly one of the following holds:

i. There is a definable $A \subseteq X$ such that (A, d) is definably bilipschitz equivalent to an r-snowflake of the unit interval for some $0 \leqslant r < 1$.

ii. There is a partition of X into definable sets $X_1, \ldots, X_n$ such that $id : (X_i, d) \to (X_i, e)$ is locally bilipschitz for every $1 \leqslant i \leqslant n$.

We call this the *Snowflake Dichotomy.* In Section 10.3 we give conditions on the Hausdorff dimension and measure on (X, d) which ensure that $id : (X, d) \to (X, e)$ is locally bilipschitz. Some of these results are phrased in terms of a refinement of the Hausdorff dimension which we introduce in Section 10.3. In Section 10.4 we prove that if $id : (X, d) \to (X, e)$ is biipschitz then the tangent cone of (X, d) at almost every point in X exists and is definably isometric to a normed linear space. Finally, in Section 10.5 we show as an application of the results in this chapter that every semilinear metric space contains an open set which is definably isometric to a definable subset of normed linear space.

10.1 A Lower Bound on d

Proposition 10.1.1. *Let (X, d) have euclidean topology. Almost every $x \in X$ has a neighborhood U such that $id : (U, d) \to (U, e)$ is Lipschitz.*

We gather some material for use in the proof. Let $\mathbb{G}(l, m)$ be the set of m-planes in R^l. We take $\mathbb{G}(l, m)$ to be a semialgebraic set, see 3.4 of [BCR98]. Two elements H, J of $\mathbb{G}(l, m)$ are **transverse** if the linear span of H, J in R^l is R^l. If $H, J \in \mathbb{G}(l, m)$ are transverse then there is a $\delta \in R^>$ such that if $P \in \mathbb{G}(l, m)$ satisfies $\|P - H\| < \delta$ then J, P are transverse. We say that two definable C^1-submanifolds $M, N \subseteq R^l$ are **transverse** if the tangent planes of M and N are transverse at every $p \in M \cap N$.

A **metric domain** is a definable metric space (X, d) such that $X \subseteq R^k$ is open and $id : (X, d) \to (X, e)$ is a homeomorphism.

Proof. To prove the proposition it is enough to fix a d-open $V \subseteq X$ and find a d-open $U \subseteq V$ such that $id : (U, d) \to (U, e)$ is Lipschitz. We only treat the case $V = X$. The general case follows in the same way. It is enough to construct a definable open $U \subseteq X$ and a Lipschitz injection $\Theta : (U, d) \to (R^l, e)$ for some l. Indeed, suppose we have such U and Θ. We factor $id : (U, d) \to (U, e)$ as the composition:

$$(U, d) \xrightarrow{\Theta} (\Theta(U), e) \xrightarrow{\Theta^{-1}} (U, e).$$

As $\Theta^{-1} : (\Theta(U), e) \to (U, e)$ is a definable bijection between euclidean metric spaces and it is locally bilipschitz almost everywhere. Then there is an open subset U' of U such that $id : (U', d) \to (U', e)$ is Lipschitz. We now construct Θ.

Claim 10.1.2. *For each $0 \leqslant i \leqslant \dim(X)$ there is an open $U_i \subseteq X$ and a definable Lipschitz function $\Theta_i : (U_i, d) \to (R^i, e)$ whose fibers have dimension at most $\dim(X) - i$.*

The claim implies that Θ_l has finite fibers and thus is generically locally injective, so U_l contains an open set U on which Θ_l is injective. We apply induction on i to prove the Claim. If $i = 0$ we take $U_0 = X$ and let Θ_0 be the unique map $X \to R^0$. When $i \geqslant 1$, Θ_i will be of the form

$$\Theta_i(x) = (d(A_1, x), \dots, d(A_i, x))$$

for certain definable closed sets $A_i \subseteq X$. This ensures that each Θ_i is Lipschitz as

$$|d(A_i, x) - d(A_i, y)| \leqslant d(x, y) \quad \text{for all } x, y \in U_i.$$

We proceed with the inductive step. Suppose that $U_i \subseteq X$ is open and that $\Theta_i : (U_i, d) \to (R^i, e)$ is a Lipschitz function whose fibers have dimension at most $l - i$. After replacing U_i with a smaller definable open set if necessary we may suppose that U_i is a cell with definably compact closure in X. We can replace (U_i, d) with any definably isometric definable metric space without loss of generality. Therefore we assume that (U_i, d) is a metric domain, and let $U_i \subseteq R^l$. After again replacing U_i with a smaller definable open set if necessary we suppose that Θ_i is C^1. Then the fibers of Θ_i are C^1-submanifolds of R^l. Let $g : U_i \to \mathbb{G}(l, l - i)$ be the function which takes x to the tangent plane of $\Theta_i^{-1}(y)$ at x where $y = \Theta_i(x)$. Fix $p \in U_i$. Let H be an $(l-1)$-dimensional hyperplane in R^l which is transverse to $g(p)$. Let $\delta \in R^>$ be such that if $J \in \mathbb{G}(l, l-1)$ satisfies $\|g(p) - J\|$ then J is transverse to H. There is a definable open $U' \subseteq U_i$ such that $\|g(x) - g(y)\| < \delta$ for all $x, y \in V$. After replacing U_i with such a U' if necessary we suppose that $\|g(x) - g(y)\| < \delta$ for all $x, y \in U_i$, so H is transverse to $g(x)$ for all $x \in U_i$. After possibly replacing H with an $(l-1)$-dimensional hyperplane parallel to H if necessary we also suppose that H that has nontrivial intersection with U_i. Let V be a definable open subset of X such that:

(1) $V \cap H \neq \emptyset$,

(2) $\mathrm{cl}(V) \subseteq U$,

(3) $\mathrm{cl}(V) \cap H = \mathrm{cl}(V \cap H)$.

We could take V to be small euclidean ball whose center is an element of H. Let $A = \mathrm{cl}(V \cap H)$, A is definably compact. We take $A_{i+1} = A$. For all $t \in R^>$ we define

$$B_t := \{x \in \mathrm{cl}(U_i) : d(A, x) = t\}.$$

We show that A is the Hausdorff limit of B_t as $t \to 0^+$. Statement (1) below implies that A is contained in the Hausdorff limit of B_t as $t \to 0^+$ and (2) below implies that A contains the Hausdorff limit of B_t as $t \to 0^+$.

(1) For all $x \in A$ there is a path γ in U_i which converges to x and satisfies $\gamma(t) \in B_t$ when $0 < t \ll 1$.

(2) If γ is a path in U_i such that $\gamma(t) \in B_t$ for all sufficiently small t then γ converges to some element of A.

We prove (1) for a given $x \in A$. As x is not isolated in U_i, for all $0 < t \ll 1$ there is a $y \in U_i$ such that $d(x, y) = t$. Applying definable choice let γ be a path in U_i such that $d(x, \gamma(t)) = t$ when $0 < t \ll 1$. We prove (2) for a path γ such that $\gamma(t) \in B_t$ for $t \in R^>$. As U_i has definably compact closure, γ must converge to some point p of $\mathrm{cl}(U_i)$ as $t \to 0^+$. As A is closed and $d(p, A) = 0$ we have $p \in A$.

Now apply Lemma 6.1.7. There is a definable open $W \subseteq U_i$ such that if $B_t \cap W$ is a C^1-submanifold of R^l for all $t \in R^>$ and

$$\|T_p B_t - H\| < \delta \quad \text{ for all } p \in B_t \cap W.$$

It follows that if $B_t \cap W \neq \emptyset$ then $B_t \cap W$ is transverse to each fiber of Θ_i. We set $U_{i+1} = W$. Let $\Theta_{i+1} : U_{i+1} \to R^{i+1}$ be given by

$$\Theta_{i+1}(x) := (\Theta_i(x), d(A, x)).$$

We finish the proof by showing that every fiber of Θ_{i+1} has dimension at most $l - i - 1$. Fix $\tilde{s} = (\bar{s}, s_{i+1}) \in R^{i+1}$ in the image of Θ_{i+1}. As

$$\Theta_{i+1}^{-1}(\tilde{s}) \subseteq \Theta_i^{-1}(\bar{s}) \cap [B_{s_{i+1}} \cap W]$$

and as $\Theta_i^{-1}(\bar{s})$ is transverse to $B_{s_{i+1}} \cap W$ we have:

$$\dim \Theta_{i+1}^{-1}(\tilde{s}) < \dim \Theta_i^{-1}(\bar{s}) \leqslant l - i.$$

$\square$

10.2 Upper Bounds on d

In this section we consider upper bounds on d in terms of e. For $1 \leqslant i \leqslant l$ let $\pi_i : X \to R^{l-1}$ be the projection from the ith coordinate. Let $b_1, \dots, b_l$ be the standard basis vectors of R^l. For each $1 \leqslant i \leqslant l$ we let $d^i : X \times R^{\geqslant} \to R$ be given by $d^i(x,t) = d(x, x + tb_i)$ when $x + tb_i \in X$ and $d^i(x,t) = 1$ otherwise. We write $d^i_x(t) = d^i(x,t)$.

Lemma 10.2.1. *Let (X,d) be a metric domain and $\mathbf{g}$ be a gauge. Suppose that $U \subseteq X$ is open and definable and fix $1 \leqslant i \leqslant l$.*

i. If $\mathrm{val}(d^i_x) \geqslant \mathrm{val}(\mathbf{g})$ for all $x \in U$ then there is a definable open $V \subseteq U$ and a $\lambda \in R^{>}$ such that

$$d(x,y) \leqslant \lambda \mathbf{g}(\|x-y\|)$$

holds for all $x, y \in V$ satisfying $\pi_i(x) = \pi_i(y)$.

ii. If $\mathrm{val}(d^i_x) \leqslant \mathrm{val}(\mathbf{g})$ for all $x \in U$ then there is a definable open $V \subseteq U$ and a $\lambda \in R^{>}$ such that

$$d(x,y) \geqslant \lambda \mathbf{g}(\|x-y\|)$$

holds for all $x, y \in V$ satisfying $\pi_i(x) = \pi_i(y)$.

Proof. We only prove the first claim as the second can be proven in the same way. Suppose $\mathrm{val}(d^i_x) \geqslant \mathrm{val}(\mathbf{g})$ for all $x \in U$. For all $x \in U$ there are $\delta, \lambda \in R^{>}$ such that $d(x, x+tb_i) \leqslant \lambda \mathbf{g}(t)$ when $0 < t < \delta$. Applying Lemma 6.1.1 there is an open $V \subseteq X$ and $\delta, \lambda \in R^{>}$ such that $d(x, x+tb_i) \leqslant \lambda \mathbf{g}(t)$ holds for all $x \in V$ and $0 < t < \delta$. After shrinking V if necessary we may suppose that $\mathrm{Diam}_e(V) < \delta$. Thus if $x, y \in V$ satisfy $\pi_i(x) = \pi_i(y)$ then $d(x,y) \leqslant \lambda \mathbf{g}(\|x-y\|)$. □

Lemma 10.2.2. *Let (X,d) be a metric domain and $\mathbf{g}$ be a gauge. Suppose that $U \subseteq X$ is open and definable and $1 \leqslant i \leqslant l$ is such that $\mathrm{val}(d^i_x) < \mathrm{val}(\mathbf{g})$ for all $x \in U$. There is a definable open $V \subseteq U$ a $\lambda \in R^{>}$ and a gauge $\tilde{\mathbf{g}}$ such that $\mathrm{val}(\tilde{\mathbf{g}}) < \mathrm{val}(\mathbf{g})$ and*

$$d(x,y) \geqslant \lambda \tilde{\mathbf{g}}(\|x-y\|)$$

for all $x, y \in V$ satisfying $\pi_i(x) = \pi_i(y)$.

Proof. For $x \in X$ we let $A_x \subseteq X$ be the set of y such that $\mathrm{val}(d^i_y) \leqslant \mathrm{val}(d^i_x)$. Then $\{A_x : x \in X\}$ is a directed definable family of sets and

$$\bigcup_{x \in X} A_x = X,$$

so we can apply Lemma 6.1.2 to produce a definable open $U' \subseteq U$ and a $y \in U$ such that $U' \subseteq A_y$. As $\mathrm{val}(d^i_y) < \mathrm{val}(\mathbf{g})$ it follows from Lemma 6.3.2 that there is a gauge $\tilde{\mathbf{g}}$ such that $\tilde{\mathbf{g}}(t) = d^i_y(t)$ when $0 < t \ll 1$. For each $x \in U'$, $\mathrm{val}(d^i_x) \leqslant \mathrm{val}(\tilde{\mathbf{g}})$. Now apply Lemma 10.2.1. $\square$

Lemma 10.2.3. *Let (X, d) be a metric domain and $\mathbf{g}$ be a gauge. Suppose $U \subseteq X$ is open such that $\mathrm{val}(d^i_x) \geqslant \mathrm{val}(\mathbf{g})$ for all $x \in U$ and $0 \leqslant i \leqslant l$. There is an open $V \subseteq U$ and a $\lambda \in R^>$ such that*

$$d(x, y) \leqslant \lambda \mathbf{g}(\|x - y\|) \quad \textit{holds for all } x, y \in V.$$

Proof. Applying Lemma 10.2.1 to each $1 \leqslant i \leqslant l$ we find an open $V \subseteq U$ and $\lambda \in R^>$ such that if $x, y \in V$ satisfy $\pi_i(x) = \pi_i(y)$ for some $1 \leqslant i \leqslant l$ then $d(x, y) \leqslant \lambda \mathbf{g}(\|x - y\|)$. After shrinking V if necessary we may suppose that V is a product of open intervals. Fixing

$$x = (x_1, \ldots, x_l) \in V \quad \text{and} \quad y = (y_1, \ldots, y_l) \in V,$$

we show that $d(x, y) \leqslant l\lambda \mathbf{g}(\|x - y\|)$. We define a sequence of elements $z_0, \ldots, z_l \in V$ by letting $z_0 = x$, letting

$$z_i = (y_1, \ldots, y_i, x_{i+1}, \ldots x_l) \quad \text{for all } 1 \leqslant i \leqslant l - 1,$$

and letting $z_l = y$. We have $\pi_i(z_{i-1}) = \pi_i(z_i)$ for every $1 \leqslant i \leqslant l$. As V is a product of open intervals, $z_i \in V$ for all $0 \leqslant i \leqslant l$. We compute:

$$d(x, y) \leqslant \sum_{i=1}^{l} d(z_{i-1}, z_i) \leqslant \sum_{i=1}^{l} \lambda \mathbf{g}(\|z_{i-1} - z_i\|).$$

For each i, $\|z_{i-1} - z_i\| = |y_i - x_i| \leqslant \|x - y\|$, so:

$$\sum_{i=1}^{l} \lambda \mathbf{g}(\|z_{i-1} - z_i\|) \leqslant l\lambda \mathbf{g}(\|x - y\|).$$

$\square$

Proposition 10.2.4. *Let (X, d) be a metric domain and $\mathbf{g}$ be a gauge. Almost every $x \in X$ has a definable neighborhood V on which one of the following holds for some $\lambda \in R^{>}$:*

i. $d(x, y) \leqslant \lambda \mathbf{g}(\|x - y\|)$ *for all $x, y \in U$.*

ii. For some coordinate projection $\pi : U \to R^l$ and gauge $\tilde{\mathbf{g}}$ such that $\mathrm{val}(\tilde{\mathbf{g}}) < \mathrm{val}(\mathbf{g})$ we have $d(x, y) \geqslant \lambda \tilde{\mathbf{g}}(\|x - y\|)$ for all $x, y \in U$ satisfying $\pi(x) = \pi(y)$.

Proof. If suffices to fix an open $U \subseteq X$ and find an open subset $V \subseteq U$ on which either (i) or (ii) holds. As usual we only treat the case $U = X$. For $1 \leqslant i \leqslant l$ we let $h^i : X \to R_\infty$ be given by

$$h^i(x) := \lim_{t \to 0^+} \frac{d(x, x + tb_i)}{\mathbf{g}(t)}$$

so that $h^i(x) < \infty$ if and only if $\mathrm{val}(d^i_x) \geqslant \mathrm{val}(\mathbf{g})$. There is an open $V' \subseteq X$ on which one of the following holds:

(1) For some i, $h^i(x) = \infty$ for all $x \in V'$,

(2) $h^i(x) < \infty$ for all $x \in V'$ and $1 \leqslant i \leqslant l$.

If (1) holds then Lemma 10.2.2 implies that there is an open $V \subseteq V'$ on which (i) holds. If (2) holds then Lemma 10.2.3 implies that there is an open $V \subseteq V'$ on which (ii) holds. $\square$

Corollary 10.2.5. *Let (X, d) be a metric domain where $X \subseteq R^l$. Almost every $x \in X$ has a definable neighborhood U on which exactly one of the following holds:*

i. $id : (U, d) \to (U, e)$ *is bilipschitz,*

ii. For some coordinate projection $\pi : X \to R^{l-1}$ and gauge $\mathbf{g}$ such that $\mathrm{val}(\mathbf{g}) < 1$ we have $d(x, y) \geqslant \mathbf{g}(\|x - y\|)$ for all $x, y \in U$ satisfying $\pi(x) = \pi(y)$.

Proof. Apply Proposition 10.2.4 to the gauge $\mathbf{g}(t) = t$ and then apply Proposition 10.1.1. $\square$

Proposition 10.2.6. *Suppose that $\mathcal{R}$ is power bounded. Let (X, d) have euclidean topology. Let $r \in (0, 1]$ be an element of the field of powers of $\mathcal{R}$. Almost every $x \in X$ has a neighborhood U on which one of the following holds for some $\lambda \in R^{>}$:*

i. *We have* $d(x,y) \leqslant \lambda\|x-y\|^r$ *for all* $x, y \in U$*,*

ii. *There is a linear projection* $\pi : U \to R^{l-1}$ *with* 1*-dimensional fibers, and an element* $s \in (0,r)$ *of the field of powers of* $\mathcal{R}$ *such that:*

$$\frac{1}{\lambda}\|x-y\|^s \leqslant d(x,y) \leqslant \lambda\|x-y\|^s \textit{ for all } x, y \in U \textit{ such that } \pi(x) = \pi(y).$$

Proof. We first assume that (X,d) is a metric domain and prove the proposition for X. As $\mathcal{R}$ is power bounded we take $\text{val}(d^i_x)$ to be an element of the field of powers of $\mathcal{R}$ for each $x \in X$. Then, for each $1 \leqslant i \leqslant l$, $\text{val}(d^i_x)$ is a definable function of $x \in X$ which takes only finitely many values. It suffices to fix an open $U' \subseteq X$ and find an open $U \subseteq U'$ on which either (i) or (ii) above holds. We only treat the case $U' = X$. By Lemma 10.2.3 there is an open $U \subseteq X$ and $1 \leqslant i \leqslant l$ such that either (i) holds on U or $\text{val}(d^i_x) > r$ holds for all $x \in U$. We suppose that $U \subseteq X$ is open and $\text{val}(d^i_x) > r$ for all $x \in U$. By replacing U by a smaller open set if necessary we suppose that $\text{val}(d^i_x)$ is constant and equal to some $s \in (0,r)$ on U. After applying Lemmas 10.2.1 and 10.2.2 and replacing U with a smaller open set if necessary there are $\lambda_1, \lambda_2 \in R^>$ such that:

$$\lambda_1\|x-y\|^s \leqslant d(x,y) \leqslant \lambda_2\|x-y\|^s \text{ for all } x, y \in U \text{ such that } \pi_i(x) = \pi_i(y).$$

We now prove the proposition for an arbitrary definable metric space (X,d) with euclidean topology. Let $X \subseteq R^k$. It suffices to fix an arbitrary l-dimensional open set $U' \subseteq X$ and find an l-dimensional open set $U \subseteq U'$ on which either (i) or (ii) above holds. After applying Lemma 6.1.9 if and replacing U' with a smaller l-dimensional open set if necessary there is a coordinate projection $\rho : R^k \to R^l$ such that $\rho(U')$ is open and for some $C \in R^>$ we have:

$$C\|x-y\| \leqslant \|\rho(x)-\rho(y)\| \leqslant \|x-y\| \quad \text{ for all } x, y \in U.$$

Let $W = \rho(U')$ and let d' be the pushforward of d along ρ. Then (W,d) is a metric domain. Suppose that $V \subseteq W$ is a definable open set satisfying (i) from the proposition for $\lambda \in R^>$. Let $U = \pi^{-1}(V)$. Then for all $x, y \in U$ we have:

$$d(x,y) = d'(\rho(x), \rho(y)) \leqslant \lambda\|\rho(x)-\rho(y)\|^r \leqslant \lambda\|x-y\|^r.$$

Thus $(V, d$ also satisfies (i). Suppose now that V satisfies (ii) for $\lambda \in R^{>}, s \in (0, r)$ and a coordinate projection π. Again let $U = \pi^{-1}(V)$. Let $\pi : U \to R^{l-1}$ be the composition of ρ and π. Then for all $x, y \in U$ such that $\pi'(x) = \pi(y)$:

$$d(x, y) = d'(\rho(x), \rho(y)) \leqslant \lambda \|\rho(x) - \rho(y)\|^s \leqslant \lambda \|x - y\|^s,$$

and

$$d(x, y) = d'(\rho(x), \rho(y)) \geqslant \frac{1}{\lambda} \|\rho(x) - \rho(y)\|^s \geqslant \frac{C^s}{\lambda} \|x - y\|^s.$$

□

Note that the proof may not work when our gauge is not a power function. For a general gauge $\mathbf{g}$, $\mathbf{g}(Ct)$ need not be bounded from above by a constant multiple of $\mathbf{g}(t)$. If $\mathcal{R}$ is not power bounded then a similar proof (which we omit) gives a weaker result.

Proposition 10.2.7. *Let (X, d) have euclidean topology. Let $r \in (0, 1]$ be an element of the field of powers of $\mathcal{R}$. Almost every $x \in X$ has a neighborhood U on which one of the following holds for some $\lambda \in R^{>}$:*

i. We have $d(x, y) \leqslant \lambda \|x - y\|^r$ for all $x, y \in U$.

ii. There is a coordinate projection $\pi : U \to R^{l-1}$ with 1-dimensional fibers and an element $s \in (0, r]$ of the field of powers such that $\lambda \|x - y\|^s \leqslant d(x, y)$ holds for all $x, y \in V$ such that $\pi(x) = \pi(y)$.

We now state a form of the Snowflake Dichotomy:

Proposition 10.2.8. *Suppose that $\mathcal{R}$ is power bounded and that (X, d) has euclidean topology. Almost every $x \in X$ has a neighborhood U on which exactly one of the following holds:*

i. $id : (U, d) \to (U, e)$ is bilipschitz.

ii. There is a coordinate projection $\pi : U \to R^{l-1}$ with 1-dimensional fibers and an element of the field of powers $s \in (0, 1)$ such that $id : (F, d) \to (F, e^s)$ is bilipschitz for every fiber F of π.

Proof. This follows by combining Proposition 10.1.1 with Proposition 10.2.6 with $r = 1$. $\square$

We can now prove the Snowflake Dichotomy.

Theorem 10.2.9. *Suppose that $\mathcal{R}$ is power bounded. Exactly one of the following holds:*

i. *There is a partition of X into definable sets $X_1, \ldots, X_n$ such that $id : (X_i, d) \to (X_i, e)$ is locally bilipschitz for every i.*

ii. *There is a definable 1-dimensional $A \subseteq X$ such that (A, d) is definably bilipschitz equivalent to an r-snowflake of the unit interval for some element $r \in [0, 1)$ of the field of powers of $\mathcal{R}$.*

Proof. We first suppose that (X, d) is not definably separable. Applying Lemma 9.1.2 let $A \subseteq X$ be an infinite definable set and $\epsilon \in R^{>}$ be such that $d(x, y) \geqslant \epsilon$ whenever $x, y \in A$ and $x \neq y$. Fix $p \in A$. There is a $t \in R^{>}$ such that $B(p, t) \cap A$ is infinite, fix such a t. After replacing A with $A \cap B(p, t)$ we suppose that A has diameter at most $2t$. Then the identity map gives a bilipschitz equivalence between (A, d) and the discrete $\{0, 1\}$-valued metric on A. After replacing A with a definable subset of A if necessary we assume there is a definable bijection $\tau : A \to [0, 1]$. Then τ induces a bilipschitz equivalence between (A, d) and the $\{0, 1\}$-valued discrete metric on $[0, 1]$. We now consider the case when (X, d) is definably separable. It is enough to prove the result for any definable metric space definably isometric to (X, d) so we suppose that (X, d) has euclidean topology. It should be clear that if (i) holds then (ii) cannot hold. We therefore assume that no definable subset of X is definably bilipschitz equivalent to a snowflake of the unit interval and prove that (i) holds. We apply induction on $\dim(X)$. If $\dim(X) = 0$ then the X_i are the singleton subsets of X. Suppose that $\dim(X) \geqslant 1$. Let $U \subseteq X$ be the set of points at which $id : (X, d) \to (X, e)$ is locally bilipschitz. Let $\{X_1, \ldots, X_m\}$ be a cell decomposition of U. Then $X \setminus U$ contains no snowflakes and $\dim(X \setminus U) < \dim(X)$, so we apply the inductive assumption to $(X \setminus U, d)$ and partition $X \setminus U$ into cells $X_{m+1}, \ldots, X_n$ on which $id : (X, d) \to (X, e)$ is locally bilipschitz. Then $\{X_1, \ldots, X_n\}$ is the required decomposition of X. $\square$

10.3 Fine Hausdorff Dimension and Hausdorff Dimension

Throughout this section $\mathcal{R}$ is an expansion of the ordered field of real numbers.

10.3.1 Fine Hausdorff Dimension

In this section we introduce a refinement of the Hausdorff dimension, the **fine Hausdorff dimension**, $\dim_{FH}$. This dimension depends on $\mathcal{R}$, but if $\mathcal{R}$ is polynomially bounded then $\dim_{FH}$ is canonically identified with $\dim_{\mathcal{H}}$. We let

$$\Omega = \{\mathrm{val}(f) : f : R^{>} \to R^{>} \text{ is definable and } t^q \leqslant f(t) \leqslant t^p \text{ for some } p, q \in \mathbb{Q}^{>}\},$$

and let $(\widetilde{\Omega}, \leqslant)$ be the completion (in the sense of linear orders) of $(\Omega, \leqslant)$. There is a canonical monotone map $\tau : \widetilde{\Omega} \to \mathbb{R}^{\geqslant}$ given by $\tau(p) = \sup\{r \in \mathbb{Q}^{>} : r < p\}$. If $\mathcal{R}$ is polynomially bounded then τ is a bijection. In this case we identify $\widetilde{\Omega}$ with $\mathbb{R}^{>}$.

Let (Y, d) be an arbitrary metric space and $\mathbf{f}, \mathbf{g}$ be definable dimension functions. We list two facts about the Borel measures $\mathcal{H}^{\mathbf{f}}, \mathcal{H}^{\mathbf{g}}$. They follow directly from Fact 3.4.1. Let $A \subseteq X$ be Borel.

(1) Suppose $\mathrm{val}(\mathbf{f}) = \mathrm{val}(\mathbf{g})$. Then $\mathcal{H}^{\mathbf{g}}(A) = 0$ if and only if $\mathcal{H}^{\mathbf{f}}(A) = 0$ and $\mathcal{H}^{\mathbf{g}}(A) = \infty$ if and only if $\mathcal{H}^{\mathbf{f}}(A) = \infty$.

(2) Suppose $\mathrm{val}(\mathbf{f}) < \mathrm{val}(\mathbf{g})$. If $\mathcal{H}^{\mathbf{f}}(A) = 0$ then $\mathcal{H}^{\mathbf{g}}(A) = 0$. If $\mathcal{H}^{\mathbf{g}}(A) > 0$ then $\mathcal{H}^{\mathbf{f}}(A) = \infty$.

The **fine Hausdorff dimension** of (Y, d) is

$$\dim_{FH}(Y, d) := \sup\{\gamma \in \Omega : [\mathrm{val}(\mathbf{g}) = \gamma] \longrightarrow [\mathcal{H}^{\mathbf{g}}(Y) = \infty]\} \in \widetilde{\Omega} \cup \{\infty\}.$$

The two facts above ensure that this definition makes sense. If $\mathcal{R}$ is polynomially bounded then $\dim_{FH}$ and $\dim_{\mathcal{H}}$ agree. Note also that $\dim_{FH}(Y, d) = \infty$ if and only if $\dim_{\mathcal{H}}(Y, d) = \infty$ and that we always have $\dim_{FH}(Y, d) \geqslant \dim_{\mathcal{H}}(Y, d)$. If $\mathbf{f}$ is a definable dimension function and $\mathrm{val}(\mathbf{f}) \in \Omega$ then by Lemma 6.3.3 $\mathbf{f}$ is a doubling function. Thus Fact 3.4.2 immediately implies the following form of the mass distribution principle for $\dim_{FH}$:

Lemma 10.3.1. *Let (Y, d) be a separable metric space and let μ be a finite Borel measure on Y. Let $\mathbf{f}$ be a definable doubling dimension function and $\delta \in \mathbb{R}^{>}$. If we have:*

$$\mu(E) \leqslant \mathbf{f}(\mathrm{Diam}_d(E)) \quad \textit{for all Borel } E \subseteq Y \textit{ such that } \mathrm{Diam}_d(E) \leqslant \delta$$

then $\dim_{FH}(Y, d) \geqslant \mathrm{val}(\mathbf{f})$.

Proof. As the diameter of a t-ball is at most $2t$ and as $\mathbf{f}$ is doubling, after shrinking δ if necessary, for some $K \in \mathbb{R}^{>}$ we have:

$$\mathbf{f}[\mathrm{Diam}_d\, B(p,t)] \leqslant \mathbf{f}(2t) \leqslant K\mathbf{f}(t) \quad \text{for all } p \in Y, 0 < t \leqslant \delta.$$

So for any $p \in X, 0 < t < \delta$ we have $\mu[B(p,t)] \leqslant K\mathbf{f}(t)$. Fact 3.4.2 implies:

$$\mathcal{H}^{\mathbf{f}}(Y) \geqslant K^{-1}\mu(X) > 0$$

so $\dim_{FH}(Y, d) \geqslant \mathrm{val}(\mathbf{f})$. □

We leave the proof of the following lemma to the reader:

Lemma 10.3.2. *Suppose that (Y, d) is a metric space and that $\{F_i\}_{i \in \mathbb{N}}$ is a collection of Borel sets which covers Y. Then*

$$\dim_{FH}(Y, d) = \sup\{\dim_{FH}(F_i, d) : i \in \mathbb{N}\}.$$

Lemma 10.3.3. *Suppose that $f : (Y, d) \to (Y', d')$ is surjective and locally Lipschitz. Then* $\dim_{FH}(Y, d) \geqslant \dim_{FH}(Y', d')$.

Proof. Apply the previous two lemmas is the same way as in the proof of Fact 3.4.8. □

10.3.2 Hausdorff Dimension and Fine Hausdorff Dimension of Definable Metric Spaces

Proposition 10.3.4. $\dim_{FH}(X, d) \geqslant \dim(X)$ *and* $\dim_{\mathcal{H}}(X, d) \geqslant \dim(X)$.

Proof. If (X,d) is not definably separable, then (X,d) is not separable and Lemma 3.4.4 implies $\dim_{\mathcal{H}}(X,d) = \dim_{FH}(X,d) = \infty$. We therefore assume that (X,d) has euclidean topology. By Proposition 10.1.1 there is an l-dimensional definable open $U \subseteq X$ such that $id : (U,d) \to (U,e)$ is Lipschitz. Thus Lemma 10.3.3 implies:

$$\dim_{FH}(U,d) \geqslant \dim_{FH}(U,e) \geqslant \dim_{\mathcal{H}}(U,e) = l,$$

and Lemma 3.4.8 implies that $\dim_{\mathcal{H}}(U,d) \geqslant \dim_{\mathcal{H}}(U,e) = l$. $\square$

In Section 5.3.1 we constructed an $\bar{\mathbb{R}}_{\exp}$-definable compact metric space with infinite Hausdorff dimension. In constrast to this, we have:

Proposition 10.3.5. *Suppose that $\mathcal{R}$ is polynomially bounded. Then (X,d) has finite Hausdorff dimension if and only if (X,d) is separable.*

Proof. Recall from Lemma 3.4.4 that if (X,d) is not separable then $\dim_{\mathcal{H}}(X,d) = \infty$. We show that (X,d) has finite Hausdorff dimension under the assumption that (X,d) has euclidean topology. Let $\{X_1,\ldots,X_n\}$ be a cell decomposition of X. Let

$$X_i(t) = \{x \in X_i : e(\partial(X_i),x) \geqslant t\}$$

and let

$$A_t = X_1(t) \cup \ldots \cup X_n(t).$$

Then X is the union of $\{A_i : i \in \mathbb{N}\}$. To show that $\dim_{\mathcal{H}}(X,d) < \infty$ it is enough to find an $r \in \mathbb{R}$ such that $\dim_{\mathcal{H}}(A_t,d) \leqslant r$ holds for all $t \in \mathbb{R}^{>}$. For all t, A_t is compact. For each $t \in \mathbb{R}^{>}$ let $h_t : (0,1) \to \mathbb{R}^{>}$ be the definable function given by

$$h_t(s) = \max\{d(x,y) : x,y \in A_t, \|x-y\| \leqslant s\}.$$

Then $d(x,y) \leqslant h_t(\|x-y\|)$ for all $x,y \in A_t$. By polynomial boundedness there is an $r \in \mathbb{R}^{>}$ such that for all $t \in \mathbb{R}^{>}$ there is a $\lambda \in \mathbb{R}^{>}$ such that $h_t(s) \leqslant \lambda s^r$ holds for all $0 \leqslant s \leqslant \mathrm{Diam}_e(A_t)$. Thus for all $t \in \mathbb{R}^{>}$ there is a $\lambda \in \mathbb{R}^{>}$ such that

$$d(x,y) \leqslant \lambda\|x-y\|^r \quad \text{for all } x,y \in A_t.$$

This inequality implies

$$\dim_{\mathcal{H}}(A_t, d) \leqslant \frac{1}{r}\dim(A_t) \leqslant \frac{l}{r} \quad \text{for all } t \in \mathbb{R}^{>}.$$

We apply Fact 3.4.5 to see that $\dim_{\mathcal{H}}(X, d) \leqslant \frac{l}{r}$. $\square$

Proposition 10.3.6. *Suppose that $\dim_{\mathcal{H}}(X, d) \leqslant n$. There is a partition $\mathcal{Y}$ of X into definable sets such that if $Y \in \mathcal{Y}$ and $p \in Y$ then there is a neighborhood $U \subseteq Y$ of p such that for some $\lambda_0, \lambda_1 \in \mathbb{R}^{\geqslant}$ we have:*

$$\lambda_0\|x-y\| \leqslant d(x,y) \leqslant \lambda_1\|x-y\|^{\frac{1}{n}} \quad \textit{for all } x, y \in U.$$

Proof. We apply induction on $\dim(X)$. If $\dim(X) = 0$ then $\dim_{\mathcal{H}}(X, d) = 0$ and the proposition holds. Suppose $\dim(X) \geqslant 1$. The proposition follows from the claim below:

Claim 10.3.7. *Almost every $p \in X$ has a neighborhood $U \subseteq X$ such that for some $\lambda \in \mathbb{R}^{>}$ we have:*

$$d(x,y) \leqslant \lambda\|x-y\|^{\frac{1}{n}} \quad \textit{for all } x, y \in U.$$

Suppose that the claim holds. Proposition 10.1.1 implies that almost every $p \in X$ has a neighborhood on which both the upper bound in the claim and the lower bound $\lambda_0\|x-y\| \leqslant d(x,y)$ holds for some $\lambda \in \mathbb{R}^{>}$. Let $W \subseteq X$ be the union of all definable euclidean open subsets U of X on which the bounds:

$$\lambda_0\|x-y\| \leqslant d(x,y) \leqslant \lambda_1\|x-y\|^{\frac{1}{n}} \quad \text{hold for some } \lambda_0, \lambda_1 \in \mathbb{R}^{>}.$$

As $\dim(X \setminus W) < \dim(X)$ the inductive hypothesis implies that $(X \setminus W, d)$ satisfies the proposition. Let $\mathcal{Y}'$ be a partition $X \setminus W$ into definable sets satisfying the conditions of the proposition. Then take $\mathcal{Y} = \mathcal{Y}' \cup \{W\}$.

The claim is a corollary to Proposition 10.2.7. This proposition shows that if the claim does not hold then there is a definable $A \subseteq X$ and an $0 < s < \frac{1}{n}$ such that $id : (A, d) \to (A, e^s)$ is Lipschitz, but this implies:

$$\dim_{\mathcal{H}}(A, d) \geqslant \frac{1}{s} > n.$$

Thus the claim holds. $\square$

The next technical lemma is used, together with Corollary 10.2.5, to characterize definable metric spaces on which $id : (X, d) \to (X, e)$ is generically locally bilipschitz.

Lemma 10.3.8. *Suppose that (X, d) is a metric domain with $X \subseteq \mathbb{R}^l$. Let $\pi : X \to \mathbb{R}^{l-1}$ be a coordinate projection and $\mathbf{g}$ be a gauge such that $\mathrm{val}(\mathbf{g}) < 1$ and $d(x, y) \geqslant \mathbf{g}(\|x - y\|)$ for all $x, y \in X$ satisfying $\pi(x) = \pi(y)$. Then $\dim_{FH}(X, d) > l$. Furthermore, $\mathcal{H}^l$ is not σ-finite on (X, d).*

Proof. Applying Proposition 10.1.1 we let $V \subseteq X$ be a definable open set such that $id : (V, d) \to (V, e)$ is K-Lipschitz on V for $K \in \mathbb{R}^>$. Let μ_i be the i-dimensional Lebesgue measure on $\mathbb{R}^l$. Let E be a Borel subset of V. Given $x \in \mathbb{R}^{l-1}$ we set $E(x) = \pi^{-1}(x) \cap E$. By Fubini:

$$\mu_l(E) = \int_{\pi(E)} \mu_1[E(x)]\, d\mu_{l-1},$$

and this integral is bounded from above by:

$$\mu_{l-1}[\pi(E)] \sup\{\mu_1[E(x)] : x \in \pi(E)\} \leqslant C\, \mathrm{Diam}_e[\pi(E)]^{l-1} \sup\{\mathrm{Diam}_e[E(x)] : x \in \pi(E)\}$$

where C is the volume of an $(l-1)$-dimensional euclidean ball with diameter 1. Furthermore:

$$\mathrm{Diam}_e[\pi(E)] \leqslant \mathrm{Diam}_e(E) \leqslant K\, \mathrm{Diam}_d(E).$$

The first inquality holds as π is 1-Lipschitz and the second inequality holds as $id : (V, d) \to (V, e)$ is K-Lipschitz. As $d(x, y) \geqslant \mathbf{g}(\|x - y\|)$ holds for all $x, y \in V$ satisfying $\pi(x) = \pi(y)$ we have:

$$\mathrm{Diam}_e[E(x)] \leqslant \mathbf{g}^{-1}(\mathrm{Diam}_d[E(x)]) \quad \text{for all } x \in \pi(E).$$

Trivially $\mathrm{Diam}_d[E(x)] \leqslant \mathrm{Diam}_d(E)$ for all $x \in \pi(E)$. This implies

$$\mathrm{Diam}_e[E(x)] \leqslant \mathbf{g}^{-1}(\mathrm{Diam}_d[E]) \quad \text{for all } x \in \pi(E).$$

Thus:

$$\sup\{\mathrm{Diam}_e[E(x)] : x \in \pi(E)\} \leqslant \mathbf{g}^{-1}(\mathrm{Diam}_d[E]).$$

Combining these bounds we have:

$$\mu_l(E) \leqslant CK^{l-1}\, \mathrm{Diam}_d(E)^{l-1} \mathbf{g}^{-1}(\mathrm{Diam}_d(E)).$$

We now wish to apply the mass distribution principal, Lemma 10.3.1. To do this we need to bound the measure of Borel subset of V in terms of a definable doubling function of its diameter. Let $\mathbf{h}(t) = \min\{\sqrt{t}, \mathbf{g}(t)\}$. Then $\mathrm{val}(\mathbf{h}) < 1$ and $\mathbf{h}$ is strictly increasing. As $t^2 \leqslant \mathbf{h}^{-1}(t)$ for all $t \in \mathbb{R}^>$, Lemma 6.3.3 implies that $\mathbf{h}^{-1}$ is doubling. As $\mathbf{g}^{-1}(t) \leqslant \mathbf{h}^{-1}(t)$ holds for all $t \in \mathbb{R}^>$ we have:

$$\mu_l(E) \leqslant CK^{l-1}\,\mathrm{Diam}_d(E)^{l-1}\mathbf{h}^{-1}(\mathrm{Diam}_d(E)) \quad \text{for all Borel } E \subseteq V.$$

Then $t^{l-1}\mathbf{h}^{-1}(t)$ is a doubling function of t, thus Lemma 10.3.1 implies:

$$\dim_{FH}(V, d) \geqslant l - 1 + \mathrm{val}(\mathbf{h}^{-1}) > l.$$

And also:

$$\mathcal{H}^l(E) = \infty \text{ for all Borel } E \subseteq V \text{ with } \mu_l(E) > 0.$$

This implies that V is not a countable union of Borel sets with finite l-dimensional Hausdorff measure, so $\mathcal{H}^l$ is not σ-finite on (X, d). □

Proposition 10.3.9. *Suppose that (X, d) has euclidean topology. If $\dim_{FH}(X, d) = \dim(X)$ then almost every $x \in X$ has a neighborhood U on which $id : (U, d) \to (U, e)$ is bilipschitz.*

Proof. Corollary 10.2.5 and Lemma 10.3.8 together prove the proposition when (X, d) is a metric domain. We reduce the more general case to the case of a metric domain. We fix an l-dimensional definable open $V \subseteq X$ and find an open $U \subseteq V$ on which $id : (V, d) \to (V, e)$ is bilipschitz. After applying Lemma 6.1.9 and replacing V with a smaller l-dimensional definable open set if necessary we suppose that V is a cell and that $\pi : V \to \mathbb{R}^l$ is a bilipschitz coordinate projection. Let $W = \pi(V)$. Let d' be the pushforward of d to W. Then (W, d') is a metric domain. As (W, d) is isometric to a subspace of (X, d), $\dim_{FH}(W, d) \leqslant \dim_{FH}(X, d) = l$. As W is l-dimensional, $\dim_{FH}(W, d) \geqslant l$. Thus $\dim_{FH}(W, d) = \dim(W)$ so there is an open $V' \subseteq X$ such that $id : (V', d') \to (V', e)$ is bilipschitz. Let $V = \pi^{-1}(V')$. Then $id : (V, d) \to (V, e)$ factors as a composition of bilipschitz maps:

$$(V, d) \xrightarrow{\pi} (V', d') \xrightarrow{id} (V', e) \xrightarrow{\pi^{-1}} (V, e).$$

□

In the same way we get a corollary whose statement does not mention the fine Hausdorff dimension; we omit the proof.

Corollary 10.3.10. *If $\mathcal{H}^l$ is σ-finite on (X,d) then $id : (X,d) \to (X,e)$ is locally a bilipschitz equivalence at almost every $x \in X$.*

The next corollary does not hold for Hausdorff dimension greater then one, r-snowflakes of the unit interval give counterexamples.

Corollary 10.3.11. *Suppose that (X,d) has Hausdorff dimension one and suppose that $\mathcal{H}^1$ is σ-finite on X. Then (X,d) contains a definable open set which is definably bilipschitz equivalent to $(0,1)$ equipped with the euclidean metric.*

Proof. If $\dim(X) = 0$ then $\dim_{\mathcal{H}}(X,d) = 0$, so we have $\dim(X) \geqslant 1$. Proposition 10.3.4 implies $\dim(X) = 1$. Corollary 10.3.10 implies that (X,d) contains a definable open set U such that $id : (U,d) \to (U,e)$ is bilipschitz. Lemma 6.1.9 implies that after replacing U with a smaller definable open set if necessary we have a definable bilipschitz equivalance $(U,e) \to (I,e)$ for some open interval $I \subseteq \mathbb{R}$. Rescaling and translating the interval suitably gives a definable bilipschitz equivalence $(I,e) \to ((0,1),e)$. Composing these maps gives the required definable bilipschitz equivalence. □

Applying the bound on topological dimension obtained in Section 9.5 we can now prove:

Theorem 10.3.12. *Suppose that the fine Hausdorff dimension of (X,d) agrees with the topological dimension of (X,d). Then almost every $p \in X$ has a d-open neighborhood U such that (U,d) is definably bilipschitz equivalent to an open subset of $(\mathbb{R}^l, e)$.*

Proof. Recall from Corollary 9.5.2 that the topological dimension of (X,d) is no greater then the o-minimal dimension of X and from Proposition 10.3.4 that the fine Hausdorff dimension of (X,d) is no greater then the o-minimal dimension of X. Thus, if the topological dimension of (X,d) agrees with the fine Hausdorff dimension then both must agree with the o-minimal dimension of X. Proposition 10.3.9 then gives an l-dimensional definable open $U \subseteq X$ such that $id : (U,d) \to (U,e)$ is bilipschitz. After applying Lemma 6.1.9 and replacing U

with a smaller l-dimensional definable open subset of X if necessary we let $\pi : U \to \mathbb{R}^l$ be a coordinate projection which gives a bilipschitz equivalence between (U, e) and an open subset of $\mathbb{R}^l$. $\square$

10.4 Tangent Cones

Let $U \subseteq R^k$ be definable and open and let $f : U \to R$ be a definable Lipschitz function. The **subdifferential** of f at $p \in U$ is the function $\nabla_p f : R^k \to R$ given by

$$\nabla_p f(x) = \lim_{t \to 0^+} \frac{f(p + tx) - f(p)}{t}.$$

The limit exists as f is Lipschitz. If f is differentiable at p then $\nabla_p f$ is the usual derivative of f at p. See [Roc70] for more about subdifferentials.

Lemma 10.4.1. *Let $f : U \to R$ be a definable Lipschitz function and let $p \in U$. Then $\nabla_p f(sx) = s\nabla_p f(x)$ for all $s \in R^>$.*

Proof. Suppose that $g : R^> \to R^>$ is a definable function which satisfies $g(t) \leqslant \lambda t$. Then g is differentiable at 0 and so by the chain rule:

$$\lim_{t \to 0^+} \frac{g(st)}{t} = s \lim_{t \to 0^+} \frac{g(t)}{t}.$$

The lemma follows by applying this fact to the function $g(t) = f(p + tx)$. $\square$

Suppose (W, d) is a metric domain such that $id : (W, d) \to (W, e)$ is λ-bilipschitz. Then the triangle inequality implies that for all $x, y, z, w \in W$:

$$|d(x, y) - d(z, w)| \leqslant d(x, z) + d(y, w) \leqslant \lambda\|x - z\| + \lambda\|y - w\| \leqslant \lambda\|(x, y) - (z, w)\|$$

so $d : (W^2, e) \to (R, e)$ is λ-Lipschitz. Thus the subdifferential of d exists at every $(p, p) \in W^2$. In the same way, if (X, d) is a metric domain such that $id : (X, d) \to (X, e)$ is locally bilipschitz then the subdifferential of d exists at every $(p, p) \in X^2$.

For the remainder of this section (X, d) is a metric domain such that $id : (X, d) \to (X, e)$ is locally bilipschitz. We study the subdifferential $\nabla_{(p,p)} d$ at generic $p \in X$. To simplify notation we let ∇_p be the subdifferential of d at $(p,p) \in X^2$. By definition:

$$\nabla_p(x,y) = \lim_{t \to 0^+} \frac{d(p+tx, p+ty)}{t}.$$

Lemma 10.4.2. *For all $p \in X$:*

i. ∇_p is a metric on R^l and $id : (R^l, \nabla_p) \to (R^l, e)$ is λ-bilipschitz.

ii. For all $t \in R^>$ and $x, y \in R^l$ we have $\nabla_p(tx, ty) = t\nabla_p(x,y)$.

iii. If $\mathcal{R}$ expands the real field then $(\nabla_p, \mathbb{R}^l)$ is isometric to the GH-tangent cone of (X,d) at p.

Note that (ii) implies that the pointed metric space $(R^l, \nabla_p, 0)$ is conical.

Proof. To simplify notation we suppose that p is the origin in R^l. For $t \in R^>$ let $d_t : (tX)^2 \to R$ be the metric given by $d_t(tx, ty) = td(x,y)$. For each t, $id : (tX, d_t) \to (tX, e)$ is λ-bilipschitz. Let $x, y \in R^l$. Then $x, y \in tX$ when t is sufficiently large, so the limit $\lim_{t\to\infty} d_t(x,y)$ makes sense. For any $x, y \in R^k$ we have by definition of ∇_p:

$$\nabla_p(x,y) = \lim_{t \to \infty} d_t(x,y).$$

This implies that ∇_p is a pseudometric on R^l. As each $id : (tX, d_t) \to (tX, e)$ is λ-bilipschitz, $id : (R^l, \nabla_p) \to (R^l, e)$ is also λ-bilipschitz. In particular, ∇_p is a metric. Item (ii) above is a special case of Lemma 10.4.1.

Suppose $\mathcal{R}$ expands the real field. Corollary 8.2.3 implies that the GH-limit of (tX, d_t) exists and is isometric to $(\mathbb{R}^l, \lim_{t\to\infty} d_t)$. Thus the GH-tangent cone of (X,d) at p is isometric to $(\mathbb{R}^l, \nabla_p)$. $\square$

With (iii) above as justification we say that (R^l, ∇_p) is the tangent cone of (X,d) at p. The remainder of this section is devoted to proving:

Proposition 10.4.3. *For almost every $p \in X$ the tangent cone (R^l, ∇_p) is a normed linear space. That is, ∇_p is a linear norm on R^l for almost every $p \in X$.*

The proof uses Lemma 6.1.8, which we recall below for the convenience of the reader:

Lemma 10.4.4. *Let $F : R \times R^{\geqslant} \to R^{\geqslant}$ be a definable function satisfying $F(x, 0) = 0$ and $F(x, t) \leqslant \lambda t$ for some $\lambda \in \mathbb{R}^{>}$. Then for almost every $x \in R$ we have*

$$\lim_{t \to 0^+} \frac{F(x, t)}{t} = \lim_{t \to 0^+} \frac{F(x + t, t)}{t}$$

Proof of Proposition 10.4.3. By part (iii) of Lemma 10.4.2 and Fact 3.8.1, ∇_p is a linear norm if and only if

$$\nabla_p(y, y + z) = \nabla_p(0, z) \quad \text{holds for all } y, z \in R^l.$$

If $\nabla_p(y + z, z) = \nabla_p(0, z)$ holds for almost every $y, z \in R^l$ then continuity of ∇_p implies that $\nabla_p(y, y + z) = \nabla_p(0, z)$ for all $y, z \in R^l$. Therefore it is enough to show:

$$\nabla_p(y, y + z) = \nabla_p(0, z) \quad \text{for almost all } p \in X, (y, z) \in R^l \times R^l.$$

By the fiber lemma for o-minimal dimension it suffices to fix $y, z \in R^l$ and show:

$$\nabla_p(y, y + z) = \nabla_p(0, z) \quad \text{for almost all } p \in X.$$

To this effect we fix $y, z \in R^l$. Let $\pi : X \to R^{l-1}$ be the orthogonal projection along y. That is, π is the orthogonal projection such that $\pi(x) = \pi(w)$ if and only if $w = x + ty$ for some $t \in R$. It suffices to fix a fiber of π and show that $\nabla_p(y, y + z) = \nabla_p(0, z)$ holds for almost all elements p of this fiber. Let L be a fiber of π. Fix $q \in L$, so $L = \{q + ty : t \in R\}$. Let $F : R \times R^{\geqslant} \to R$ be given by:

$$F(s, t) = d(q + sy, q + sy + tz).$$

Note that $F(s, 0) = 0$ and as d is λ-Lipschitz, $F(s, t) \leqslant \lambda t$. By Lemma 6.1.8 for almost every $s \in R$ we have:

$$\lim_{t \to 0^+} \frac{F(s, t)}{t} = \lim_{t \to 0^+} \frac{F(s + t, t)}{t}.$$

Suppose that s satisfies this equality. Letting $p = q + sy$ we have:

$$\lim_{t \to 0^+} \frac{F(s, t)}{t} = \lim_{t \to 0^+} \frac{d(q + sy, q + sy + tz)}{t} = \lim_{t \to 0^+} \frac{d(p, p + tz)}{t} = \nabla_p(0, z),$$

and

$$\lim_{t\to 0^+}\frac{F(s+t,t)}{t} = \lim_{t\to 0^+}\frac{d(q+sy+ty,q+sy+ty+tz)}{t}$$
$$= \lim_{t\to 0^+}\frac{d(p+ty,p+ty+tz)}{t} = \nabla_p(y,y+z).$$

Thus we have $\nabla_p(y,y+z)=\nabla_p(0,z)$ for almost every $p\in L$. $\square$

Corollary 10.4.5 is a kind of generalized C^1-cell decomposition.

Corollary 10.4.5. *Let (X,d) be a definable metric space such that $id:(X,d)\to(X,e)$ is locally bilipschitz. There is a partition $\mathcal{Y}$ of X into finitely many definable sets such that for each $Y\in\mathcal{Y}$ and $p\in Y$, the tangent cone of (Y,d) at p is isometric to a definable norm on R^k.*

It is easy to show via induction that if $C\subseteq R^k$ is an l-dimensional C^1-cell then there is a coordinate projection $\pi:\mathbb{R}^k\to\mathbb{R}^l$ whose restriction to C is locally bilipschitz. We apply this fact in the proof below.

Proof. We apply induction to $\dim(X)$. The base case $\dim(X)=0$ is trivial. Suppose $\dim(X)\geqslant 1$. Let $X_1,\ldots,X_n$ be a partition of X into C^1-cells and $\pi_i:X_i\to R^{\dim(X_i)}$ be locally bilipschitz coordinate projections. Let $X_i'=\pi_i(X_i)$ and let d_i be the pushforward of d onto X_i'. Each (X_i',d_i) is a metric domain and each $id:(X_i',d_i)\to(X_i',e)$ is locally bilipschitz. Applying Proposition 10.4.3, for each $1\leqslant i\leqslant n$ there is an open $U_i'\subseteq X_i'$ such that U_i' is almost all of X_i' and the tangent cone of (X_i',d_i) at each point in U_i' is a normed space. Let $U_i=\pi_i^{-1}(U_i')$. The tangent cone of (X_i,d) is a normed space at every $p\in U_i$. Let $U=U_1\cup\ldots\cup U_n$. As $\dim(X\setminus U)<\dim(X)$ we can apply the inductive assumption to construct a partition $\mathcal{Z}$ of $X\setminus U$ into definable sets such that for every $Z\in\mathcal{Z}$ and $p\in Z$ the tangent cone of (Z,d) at p is a normed space. We now take $\mathcal{Y}=\mathcal{Z}\cup\{U_1,\ldots,U_n\}$. $\square$

10.5 Semilinear Metric Spaces

Let $\mathcal{V}$ be R considered as an ordered vector space over itself, $\mathcal{V}$ is a definitional reduct of $\mathcal{R}$. In this section we consider $\mathcal{V}$-definable metric spaces. In Section 5.8 we saw that the topology of $\mathcal{V}$-definable metric spaces can be quite complex. The generic local structure of a semilinear metric is quite simple, almost every point has a semilinear neighborhood which is semilinearly isometric to an open subset of a euclidean space equipped with a linear norm. This fact is a consequence of the locally conical nature of semilinear sets. A set $A \subseteq R^k$ is **conical** at $p \in R^k$ if for all $q \in A$ and $0 < t < 1$ we have $p + tq \in A$. A set $A \subseteq R^k$ is **locally conical** at $p \in R^k$ if there is a $\delta \in R^>$ such that $B_e(p, \delta) \cap A$ is conical at p.

Fact 10.5.1. *A semilinear $A \subseteq R^k$ is locally conical at every $p \in R^k$.*

Proof. A convex subset of R^k is conical at every $p \in R^k$. A finite union of subset of R^k which are conical at every $p \in R^k$ is locally conical at every $p \in R^k$. If follows immediately from Proposition 1.7.6 of [Dri98] that every semilinear set is a finite union of convex sets. $\square$

We leave the verification of the following elementary fact to the reader:

Lemma 10.5.2. *Let $U \subseteq R^k$ be definable and open and let $f : R^k \to R$ be a definable continuous function. Suppose $q \in U$ is such that $f(q) = 0$. If* $\mathrm{Graph}(d)$ *is locally conical at $(q, 0)$ then q has a neighborhood W such that $f(x) = \nabla_p(x - p)$ for all $x \in W$.*

We can now prove that semilinear metric spaces are generically locally isometric to normed spaces.

Proposition 10.5.3. *Let (X, d) be semilinear. Almost every point $p \in X$ has a semilinear neighborhood U such that (U, d) is semilinearly isometric to an open subset R^l equipped with a linear norm.*

Sketch. It suffices to show that every d-open $W \subseteq X$ contains a d-open U on which the proposition holds. We only treat the case $W = X$. Corollary 9.4.2 implies that there is a d-open $V \subseteq X$ such V such that (V, d) has euclidean topology. Every open subset of X contains a semilinear open sets so after shrinking V if necessary we suppose that V is semilinear. After applying the semilinear cell decomposition, 1.7.6 of [Dri98], and shrinking V if necessary we suppose that V is an open subset of a hyperplane. Let $\pi : V \to R^n$ be a coordinate projection which bijectively maps V onto an open subset of R^n where $n = \dim(V)$.

It suffices to prove the result for $\pi(V)$ equipped with the pushforward of d by π. We therefore suppose without loss of generality that (X, d) is a semilinear metric domain. Then we have $X \subseteq R^l$. Every semilinear function $f : R^> \to R$ satisfying $\lim_{t \to 0^+} f(t) = 0$ is linear on a small interval $(0, \epsilon)$. Thus, the snowflake dichotomy implies that there is a definable open $V \subseteq X$ such that $id : (V, d) \to (V, e)$ is bilipschitz. The subdifferential of d exists at every $(p, p) \in V \times V$. After replacing V with a smaller open set if necessary we suppose that V is semilinear. Fix $p \in V$. Applying Lemma 10.5.2 to Graph(d), we see that p has a neighborhood W such that

$$d(x, y) = \nabla_p(x - p, y - p) \quad \text{for all } x, y \in W.$$

Applying Proposition 10.4.3 we fix a $p \in V$ such that ∇_p is a linear norm. Let U be a definable neighborhood of p such that $d(x, y) = \nabla_p(x-p, y-p)$ for all $x, y \in U$. After replacing U with a smaller definable open set if necessary we may suppose that U is semilinear, Then (U, d) is semilinearly isometric to an open subset of R^l equipped with the linear norm ∇_p. □

10.6 Doubling Metrics

In this section we assume that $\mathcal{R}$ expands the real field. In the Chapter 11 we use the following proposition to show that $\bar{\mathbb{R}}_{\text{an}}$-definable metric spaces are doubling almost everywhere.

Proposition 10.6.1. *Let (X, d) be a metric domain with finite Hausdorff dimension. Let μ be the l-dimensional Lebesgue measure on X. Suppose there is a definable function $f : X \times \mathbb{R}^{\geqslant} \to \mathbb{R}$ such that for some $\lambda_0, \lambda_1, \delta \in \mathbb{R}^>$ we have:*

$$\lambda_0 f(p, t) \leqslant \mu[B_d(p, t)] \leqslant \lambda_1 f(p, t) \quad \textit{for all } p \in X, 0 < \leqslant \delta.$$

Then almost every $p \in X$ has a neighborhood U such that (U, d) is doubling.

Proof. If suffices to show that every definable open $V \subseteq X$ contains a definable open $U \subseteq X$ such that (U, d) is doubling. Fix a definable open $V \subseteq X$. In this proof we let $B_d(p, t)$ be the d-ball in (V, d) with center p and radius t, likewise for $B_e(p, t)$. After applying

Proposition 10.3.6 and replacing V with a smaller open set if necessary we suppose that there are $K, r \in \mathbb{R}^{>}$ such that

$$d(x, y) \leqslant K\|x - y\|^r \quad \text{for all } x, y \in V.$$

This implies:

$$B_e\left(p, \frac{1}{K} t^{\frac{1}{r}}\right) \subseteq B_d(p, t) \quad \text{for all } p \in V, t \in \mathbb{R}^{>}.$$

There is a $K_0 \in \mathbb{R}^{>}$ such that

$$K_0 t^{\frac{l}{r}} \leqslant \mu[B_d(p, t)] \quad \text{for all } p \in V, t \in \mathbb{R}^{>}.$$

Applying Lemma 6.3.3 to $f(p, t)$ we see that for every $p \in V$ there is a $K_1 \in \mathbb{R}^{>}$ such that

$$f(p, 2t) \leqslant K_1 f(p, t) \quad \text{for all } 0 < t \ll 1.$$

After applying Lemma 6.1.1 and replacing V with a smaller definable open set if necessary there are $\delta, K_2 \in \mathbb{R}^{>}$ such that

$$f(p, 2t) \leqslant K_2 f(p, t) \quad \text{for all } p \in V, 0 < t \leqslant \delta.$$

Thus

$$\mu[B_d(p, 2t)] \leqslant \frac{\lambda_1 K_2}{\lambda_0} \mu[B_d(p, t)] \quad \text{for all } p \in V, 0 < t \leqslant \delta.$$

Lemma 3.3.5 implies that if $W \subseteq V$ has diameter at most δ then (W, d) is doubling. $\square$

CHAPTER 11

$\bar{\mathbb{R}}_{\mathrm{an}}$-definable Metric Spaces

In this chapter (X, d) is an $\bar{\mathbb{R}}_{\mathrm{an}}$-definable metric space. In this chapter we apply a theorem of Comte, Lion and Rolin to study $\bar{\mathbb{R}}_{\mathrm{an}}$-definable metric spaces. We use this theorem to bound the (Lebesgue) volume of balls in terms of their radius. The results in this chapter are proven using such bounds. In Section 11.3 we show that almost every point in an $\bar{\mathbb{R}}_{\mathrm{an}}$-definable metric space has a neighborhood on which the metric is doubling, in Section 11.3 we show that the Hausdorff dimension of an $\bar{\mathbb{R}}_{\mathrm{an}}$-definable metric space is an element of $\mathbb{Q}_{\infty}$. In Section 11.4 we show that an ultralimit of a sequence of elements of an $\bar{\mathbb{R}}_{\mathrm{an}}$-definable family of proper metric space is locally compact almost everywhere. As an application we obtain that every finitely generated group which is quasi-isometric to an $\bar{\mathbb{R}}_{\mathrm{an}}$-definable metric space is nilpotent by finite.

11.1 The CLR Theorem

We recall a beautiful theorem of Comte, Lion and Rolin [CR00]. Let $A \subseteq \mathbb{R}^n$ be $\bar{\mathbb{R}}_{\mathrm{an}}$-definable. A function $f : A \to \mathbb{R}$ is **constructible** if it is of the form

$$f(x) = p(G_1(x), \dots, G_m(x), \log H_1(x), \dots, \log H_n(x))$$

for $\bar{\mathbb{R}}_{\mathrm{an}}$-definable functions $G_i : A \to \mathbb{R}$, $H_i : A \to \mathbb{R}^{>}$ and a polynomial p. Every constructible function is $\bar{\mathbb{R}}_{\mathrm{an,exp}}$-definable.

Theorem 11.1.1 (Comte-Lion-Rolin). *Let $\{A_x : x \in \mathbb{R}^l\}$ be an $\bar{\mathbb{R}}_{\mathrm{an}}$-definable family of bounded m-dimensional subsets of $\mathbb{R}^k$ and let μ be the m-dimensional Lebesgue measure on $\mathbb{R}^k$. Then $\mu(A_x)$ is a constructible function of x.*

We use a preparation theorem for $\bar{\mathbb{R}}_{\mathrm{an,exp}}$-definable functions due to Lion and Rolin [LR97] to bound constructible functions. Van den Dries and Speissegger [DS02] prove similar results. The following proposition is an immediate special case of Theorem 4.11 in [DS02].

Proposition 11.1.2. *Let $f : \mathbb{R}^l \times \mathbb{R} \to \mathbb{R}$ be a constructible function. There is a partition $\mathcal{C}$ of $\mathbb{R}^l \times \mathbb{R}$ into finitely many $\bar{\mathbb{R}}_{\mathrm{an,exp}}$-definable cells such that for each $C \in \mathcal{C}$ there are $r_0, r_1 \in \mathbb{Q}$,$\epsilon_0, \epsilon_1 \in (0,1)$, $\bar{\mathbb{R}}_{\mathrm{an,exp}}$-definable functions $\theta_0, \theta_1, a : \mathbb{R}^l \to \mathbb{R}$ and an $\bar{\mathbb{R}}_{\mathrm{an,exp}}$-definable function $u : \mathbb{R}^l \times \mathbb{R} \to \mathbb{R}$ such that $|u(x,t) - 1| < \frac{1}{2}$ holds for all $(x,t) \in C$ and we have*

$$f(x,t) = |t - \theta_0(x)|^{r_0} |\log|t - \theta_0(x)| - \theta_1(x)|^{r_1} a(x) u(x,t) \quad \textit{for all } (x,t) \in C.$$

Furthermore, either $|t - \theta_0(x)| \leqslant \epsilon_0 |t|$ holds for all $(x,t) \in C$ or θ_0 is constant zero, and either

$$|\log|t - \theta_0(x)| - \theta_1(x)| \leqslant \epsilon_1 |\log|t - \theta_0(x)|| \quad \textit{holds for all } (x,t) \in C$$

or θ_1 is constant zero.

Corollary 11.1.3. *Let $f : \mathbb{R}^> \to \mathbb{R}^>$ be a constructible function which satisfies $f(t) < t^r$ for some $r \in \mathbb{R}^>$ when $0 < t \ll 1$. Then there are $\delta, \lambda_0, \lambda_1 \in \mathbb{R}^>$ and $r_0 \in \mathbb{Q}^>, r_1 \in \mathbb{Q}$ such that*

$$\lambda_0 t^{r_0} |\log(t)|^{r_1} \leqslant f(t) \leqslant \lambda_1 t^{r_0} |\log(t)|^{r_1} \quad \textit{for all } 0 < t \leqslant \delta.$$

Proof. Apply the previous proposition. Take $\delta, \lambda_0, \lambda_1 \in \mathbb{R}^>$ and $\theta_0, \theta_1 \in \mathbb{R}$ such that

$$\lambda_0 |t - \theta_0|^{r_0} |\log|t - \theta_0| - \theta|^{r_1} \leqslant f(t) \leqslant \lambda_1 |t - \theta_0|^{r_0} |\log|t - \theta_0| - \theta_1|^{r_1} \text{ for all } 0 < t \leqslant \delta.$$

If $\theta_0 \neq 0$ then we would have $|t - \theta_0| < |t|$ for all $0 < t < \delta$. This cannot hold when t is sufficiently small. We therefore have $\theta_0 = 0$. As $t \to 0^+$, $\log(t) \to -\infty$. Therefore $|\log(t) - \theta_1|$ is bounded from above and below by multiples of $\log(t)$ when t is sufficiently small. We finish the proof by showing $r_0 > 0$. Suppose $r_0 \leqslant 0$. Then $f(t) \geqslant \lambda_0 |\log(t)|^{r_1}$ when $0 < t \ll 1$. But when t is sufficiently small, $|\log(t)|^{r_1} \geqslant t^r$, contradiction. $\square$

We will use a uniform version of Corollary 11.1.3, we leave the proof to the reader.

Corollary 11.1.4. *Let $f : \mathbb{R}^l \times \mathbb{R}^> \to \mathbb{R}^>$ be a constructible function such that for every $p \in \mathbb{R}^l$ there is an $r \in \mathbb{R}^>$ such that $f(p,t) < t^r$ holds when $0 < t \ll 1$. Then there is a partition $\mathcal{C}$ of $\mathbb{R}^l$ into finitely many $\bar{\mathbb{R}}_{\mathrm{an,exp}}$-definable cells such that for each $C \in \mathcal{C}$ there are $r_0 \in \mathbb{Q}^>, r_1 \in \mathbb{Q}$ and $\bar{\mathbb{R}}_{\mathrm{an,exp}}$-definable functions $\lambda_0, \lambda_1, \delta : \mathbb{R}^l \to \mathbb{R}^>$ such that*

$$\lambda_0(p)t^{r_0}|\log(t)|^{r_1} \leqslant f(p,t) \leqslant \lambda_1(p)t^{r_0}|\log(t)|^{r_1} \quad \textit{for all } 0 < t \leqslant \delta(p).$$

11.2 Tangent Cones

In this section we prove the following proposition:

Proposition 11.2.1. *Almost every point in X has a neighborhood U such that (U,d) is doubling.*

Proof. It suffices to fix a d-open definable $U \subseteq X$ and find a d-open $V \subseteq U$ such that (V,d) is doubling. Let U be a definable d-open set. After applying Corollary 9.4.3 and replacing U with a definable d-open subset of U if necessary we may suppose that (U,d) is a cell with euclidean topology. We can replace (U,d) with any definably isometric definable metric space, so we may suppose that (U,d) is a metric domain. Now apply the CLR theorem and the previous proposition. □

Proposition 11.2.2. *Suppose that (X,d) has euclidean topology. Let μ be the* $\dim(X)$*-dimensional Lebesgue measure on X. Then (X,d) has a unique tangent cone at almost every $p \in X$ and the tangent cone at μ-almost every $p \in X$ is proper, isometrically homogeneous and conical.*

Proof. Immediate from Proposition 11.2.1 and Proposition 8.1.3. □

11.3 Hausdorff Dimension

In this section we show that the Hausdorff dimension of an $\bar{\mathbb{R}}_{\mathrm{an}}$-definable metric space is an element of $\mathbb{Q}_\infty$ and that the Hausdorff dimension of the elements of an $\bar{\mathbb{R}}_{\mathrm{an}}$-definable family of metric spaces takes only finitely many values. We first prove Lemma 11.3.1, which holds over any o-minimal expansion of the real field.

Lemma 11.3.1. *Suppose that (X, d) is precompact and has euclidean topology. Let μ be the $\dim(X)$-dimensional Lebesgue measure on X. There are $K, r \in \mathbb{R}^{>}$ such that:*

$$\mu[B(p,t)] \leqslant Kt^{r} \quad \textit{for all } p \in X, t \in \mathbb{R}^{>}.$$

Proof. Let $(\widetilde{X}, \widetilde{d})$ be the definable completion of (X, d), we regard X as a subset of $\widetilde{X}$. Then $(\widetilde{X}, \widetilde{d})$ is compact. We can therefore suppose that $(\widetilde{X}, \widetilde{d})$ has euclidean topology. Let μ be the $\dim(\widetilde{X})$-dimensional Lebesgue measure on $\widetilde{X}$. As $(\widetilde{X}, \widetilde{d})$ has euclidean topology, Lemma 7.1.7 implies that there exist $K, r \in \mathbb{R}^{>}$ such that:

$$\mathrm{Diam}_e[B_{\widetilde{d}}(p,t)] \leqslant Kt^{r} \quad \text{for all } p \in \widetilde{X}, t \in \mathbb{R}^{>}.$$

By Proposition 6.4.1 there are $K_0, s \in \mathbb{R}^{>}$ such that:

$$\mu[B_{\widetilde{d}}(p,t)] \leqslant K_0 t^{s} \quad \text{for all } p \in \widetilde{X}, t \in \mathbb{R}^{>}.$$

As $\dim(\widetilde{X}) = \dim(X)$ the restriction of μ to X is the $\dim(X)$-dimensional Lebesgue measure on X. So $\mu[B_d(p,t)] \leqslant K_0 t^s$ for all $p \in X$ and $t \in \mathbb{R}^{>}$. $\square$

Proposition 11.3.2. *The Hausdorff dimension of an $\bar{\mathbb{R}}_{\mathrm{an}}$-definable metric space is a rational number or infinite.*

Proof. If (X, d) is not separable then $\dim_{\mathcal{H}}(X, d) = \infty$. Applying Theorem 9.0.1 we suppose that (X, d) has euclidean topology. We apply induction to $\dim(X)$. If $\dim(X) = 0$ then $\dim_{\mathcal{H}}(X, d) = 0$. Suppose $\dim(X) \geqslant 1$. Let $l = \dim(X)$ and let μ be the l-dimensional Lebesgue measure on X. We may assume that X is a bounded subset of euclidean space. Let $\mathcal{X}$ be a partition of X into $\bar{\mathbb{R}}_{\mathrm{an}}$-definable cells. As $\dim_{\mathcal{H}}(X, d) = \max\{\dim_{\mathcal{H}}(Y, d) : Y \in \mathcal{X}\}$ it suffices to show that $\dim_{\mathcal{H}}(Y, d) \in \mathbb{Q}$ for each $Y \in \mathcal{X}$. By the inductive assumption it is enough to show that $\dim_{\mathcal{H}}(Y, d) \in \mathbb{Q}$ for each l-dimensional $Y \in \mathcal{X}$. We therefore reduce to the case that X is a cell and that (X, d) has euclidean topology. Applying the CLR-volume theorem and Corollary 11.1.3 there is a partition $\mathcal{C}$ of X into $\bar{\mathbb{R}}_{\text{an,exp}}$-definable cells such that for each $C \in \mathcal{C}$ there are $r \in \mathbb{Q}^{>}, s \in \mathbb{Q}$ with:

$$0 < \frac{\mu[B(p,t)]}{t^r|\log(t)|^s} < \infty \quad \text{for all } p \in C \text{ and } 0 < t \ll 1.$$

It suffices to show that $\dim_{\mathcal{H}}(C,d) \in \mathbb{Q}$ for every $C \in \mathcal{C}$. Applying the inductive assumption it is enough to show that $\dim_{\mathcal{H}}(C,d) \in \mathbb{Q}$ is a rational number for every l-dimensional $C \in \mathcal{C}$. Fix an l-dimensional $C \in \mathcal{C}$ and let r, s be as before. We show that $\dim_{\mathcal{H}}(C,d) = r$. As C is l-dimensional $\mu(C) > 0$. Therefore Proposition 3.4.7 implies that if

$$\lim_{t \to 0^+} \frac{\log \mu[B(p,t)]}{\log(t)} = r \quad \text{for all } p \in A,$$

then $\dim_{\mathcal{H}}(C,d) = r$. We prove this equality for a fixed $p \in A$. Let $\lambda_1, \lambda_2 \in \mathbb{R}^{>}$ be such that

$$\lambda_1 t^r|\log(t)|^s \leqslant \mu[B(p,t)] \leqslant \lambda_2 t^r|\log(t)|^s \quad \text{for all } 0 < t \ll 1.$$

Taking logs and dividing through by $\log(t)$ we have:

$$\frac{\log(\lambda_1)}{\log(t)} + r + \frac{s\log(|\log(t)|)}{\log(t)} \leqslant \frac{\log \mu[B(p,t)]}{\log(t)} \leqslant \frac{\log(\lambda_2)}{\log(t)} + r + \frac{s\log(|\log(t)|)}{\log(t)}$$

when $0 < t \ll 1$. Taking the limit as $t \to 0^+$ we have the desired equality. $\square$

The proposition below does not hold for $\bar{\mathbb{R}}_{\exp}$. The family of snowflakes,

$$\{([0,1], e^r) : 0 < r < 1\}$$

gives a counterexample.

Proposition 11.3.3. *Let* $\mathcal{X} = \{(X_\alpha, d_\alpha) : \alpha \in \mathbb{R}^l\}$ *be an* $\bar{\mathbb{R}}_{\text{an}}$*-definable family of metric spaces. Then* $\dim_{\mathcal{H}}(X_\alpha, d_\alpha)$ *takes only finitely many values.*

Our proof is in essence a uniform version of the proof of the previous proposition, so we omit some details.

Proof. We suppose that every element of $\mathcal{X}$ is a subset of $\mathbb{R}^k$. We let $B_\alpha(p,t)$ be the d_α-ball with center $p \in X_\alpha$ and radius t. Recall from Proposition 10.3.6 that $\dim_{\mathcal{H}}(X_\alpha, d_\alpha) = \infty$ if and only if (X_α, d_α) is not definably separable. We showed in Corollary 9.3.4 that the set of such α is definable. We therefore suppose that every element of $\mathcal{X}$ is definably separable. In the same way as in the proof of the previous proposition we are justified in assuming that every element of $\mathcal{X}$ is a cell. For each $i \in \mathbb{N}$ the set of α such that $\dim(X_\alpha) = i$ is definable. As it suffices to prove the result for each family $\{(X_\alpha, d_\alpha) : \alpha \in \mathbb{R}^l, \dim(X_\alpha) = i\}$ separately we can assume that $\dim(X_\alpha) = n$ for all $\alpha \in \mathbb{R}^l$. We apply induction on n. If $n = 0$ then every element of $\mathcal{X}$ has Hausdorff dimension zero, so we suppose $n \geqslant 1$. We let μ be the n-dimensional Lebesgue measure on $\mathbb{R}^k$ and let $f : \mathbb{R}^l \times \mathbb{R}^k \times \mathbb{R}^{>} \to \mathbb{R}$ be given by

$$f(\alpha, p, t) = \begin{cases} \mu[B_\alpha(p, t)] \quad \text{when } p \in X_\alpha. \\ 0 \quad \text{when } p \notin X_\alpha. \end{cases}$$

Applying Corollary 11.1.4 in the same way as in the proof of the previous proposition we let $\mathcal{C}$ be a partition of $\mathbb{R}^l \times \mathbb{R}^k$ into finitely many $\bar{\mathbb{R}}_{\text{an,exp}}$-definable cells such that for all $C \in \mathcal{C}$ there are $r \in \mathbb{Q}^{>}, s \in \mathbb{Q}$ such that:

$$0 < \frac{f(\alpha, p, t)}{t^r |\log(t)|^s} < \infty \qquad \text{for all } (\alpha, p, t) \in C \text{ and } 0 < t \ll 1.$$

For $C \in \mathcal{C}$ and $\alpha \in \mathbb{R}^l$ we let C_α be the set of $p \in X_\alpha$ such that $(\alpha, p) \in C$. As

$$\dim_{\mathcal{H}}(X_\alpha, d_\alpha) = \max\{\dim_{\mathcal{H}}(C_\alpha, d_\alpha) : C \in \mathcal{C}\} \quad \text{for all } \alpha \in \mathbb{R}^l$$

it suffices to fix a $C \in \mathcal{C}$ and prove the proposition for the family $\{(C_\alpha, d_\alpha) : \alpha \in \mathbb{R}^l\}$. Fix $C \in \mathcal{C}$. We let $r \in \mathbb{Q}^{>}, s \in \mathbb{Q}$ be such that $f(\alpha, p, t)$ is bounded from above and below by constant multiples of $t^r |\log(t)|^s$ when t is sufficiently small for all $(\alpha, p) \in C$. The proof of the previous proposition shows that if $\dim(C_\alpha) = n$ then $\dim_{\mathcal{H}}(C_\alpha, d_\alpha) = r$. The inductive assumption implies that the Hausdorff dimension of the elements of the family

$$\{(C_\alpha, d_\alpha) : \alpha \in \mathbb{R}^l, \dim(C_\alpha) < n\}$$

takes only finitely many values. $\square$

11.4 Weak Precompactness

Consider the semialgebraic family of metric spaces $\{([0,t], d_t) : t \in \mathbb{R}^{\geqslant}\}$ where d_t is given by

$$d_t(x,y) = \min\{|x-y|, 1\} \quad \text{when } x \neq y.$$

Each element of this family is compact and has diameter at most 1. For all $n \in \mathbb{N}$, $([0,n], d_n)$ contains $n-1$ pairwise disjoint 1-balls. Thus the sequence of $\{([0,n], d_n)\}_{n\in\mathbb{N}}$ has no converging subsequence. For any sequence $\{p_n\}_{n\in\mathbb{N}}$ such that $p_n \in [0,n]$ for all $n \in \mathbb{N}$, the sequence of balls $\{B_t(p_n, 1)\}_{n\in\mathbb{N}}$ converges to $([0,1], e)$ as $n \to \infty$. The ultralimit of $([0,t], d_t)$ as $t \to \infty$ is not compact, but any open ball in the ultralimit with radius 1 is precompact. The Weak Precompactness Theorem shows that this example is typical in $\bar{\mathbb{R}}_{\mathrm{an}}$:

Theorem 11.4.1. *Let $\mathcal{X} = \{(X_\alpha, d_\alpha, p_\alpha) : \alpha \in \mathbb{R}^l\}$ be an $\bar{\mathbb{R}}_{\mathrm{an}}$-definable family of pointed proper metric spaces. Let (Z, d) be an ultralimit of a sequence of elements of $\mathcal{X}$. Every open subset of Z contains a precompact open ball.*

This does not hold over $\bar{\mathbb{R}}_{\mathrm{exp}}$. Consider the family $\{(H, \frac{1}{t}d, p) : t \in \mathbb{R}^{>}\}$ of rescalings of the hyperbolic plane (H, d) with basepoint $p \in H$. The ultralimit of this family as $t \to \infty$ is the asymptotic cone of the hyperbolic plane, every compact subset of which has empty interior.

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. Let $\mathcal{R}^* = (R, +, \times, \dots)$ be the ultrapower of $\bar{\mathbb{R}}_{\mathrm{an}}$ with respect to $\mathcal{U}$. We let $\mathcal{O}$ be the convex hull of $\mathbb{Z}$ in R. In this section val is the archimedean valuation on R.

Lemma 11.4.2. *Let $h : \mathbb{R}^l \times \mathbb{R}^{>} \to \mathbb{R}^{>}$ be a constructible function. There is an $m \in \mathbb{N}$ such that*

$$|\{\operatorname{val} h^*(p,t) : t \in \mathbb{R}\}| < m \quad \textit{for all } p \in R^l.$$

Proof. Let $\mathcal{C} = \{C_1, \dots C_N\}$ be a partition of $\mathbb{R}^l \times \mathbb{R}^{>}$ into $\bar{\mathbb{R}}_{\mathrm{an,exp}}$-definable cells satisfying the conditions of Lemma 11.1.2 for h. We show that:

$$|\{h^*(p,t) : t \in \mathbb{R}^{>}\}| \leqslant 4N \quad \text{for all } p \in R^l.$$

Fix $p \in R^l$. We declare $g(t) = h^*(p,t)$. Let $I_n = \{t \in R^> : (p,t) \in C^*_n\}$ for all $1 \leqslant n \leqslant N$. Each I_n is an interval. We show that:

$$|\{\operatorname{val} g(t) : t \in \mathbb{R}^> \cap I_n\}| \leqslant 4 \quad \text{for all } 1 \leqslant n \leqslant N.$$

Fix $1 \leqslant n \leqslant N$ and let $I = I_n$. There are $r_0, r_1 \in \mathbb{Q}$ and $\lambda, \theta_0, \theta_1 \in R$ such that:

$$\frac{1}{2}\lambda|t-\theta_0|^r|\log|t-\theta_0|-\theta_1|^s \leqslant g(t) \leqslant \frac{3}{2}\lambda|t-\theta_0|^r|\log|t-\theta_0|-\theta_1|^s \text{ for all } t \in I.$$

Furthermore, either $\theta_0 = 0$ or there is an $0 < \epsilon_0 < 1$ such that $|t - \theta_0| \leqslant \epsilon_0|t|$ holds for all $t \in I$. Either $\theta_1 = 0$ or there is an $0 < \epsilon_1 < 1$ such that:

$$|\log|t-\theta_0|-\theta_1| \leqslant \epsilon_1|\log|t-\theta_0|| \quad \text{holds for all } t \in I.$$

As $I \cap \mathbb{R}$ is convex, if $|I \cap \mathbb{R}| \geqslant 3$ then $I \cap \mathbb{R}$ is infinite. If $|I \cap \mathbb{R}| \leqslant 2$ then $|\{\operatorname{val} g(t) : t \in I \cap \mathbb{R}\}| \leqslant 2$, so we suppose that $I \cap \mathbb{R}$ is infinite. We show that $\theta_0, \theta_1 \in \mathcal{O}$. Suppose that $\theta_0 \neq 0$. Then $|t - \theta_0| \leqslant |t|$ holds for some $t \in I \cap \mathbb{R}$, so $t \in \mathcal{O}$. As $I \cap \mathbb{R}$ is infinite there is a $t' \in I \cap \mathbb{R}$ such that $\operatorname{st}(\theta_0) \neq t'$, fix such a t'. Then $\log|t' - \theta_0|$ is an element of $\mathcal{O}$. If $\theta_1 \neq 0$ then,

$$|\log|t'-\theta_0|-\theta_1| \leqslant |\log|t'-\theta_0||$$

so θ_1 is also an element of $\mathcal{O}$. Taking valuations we have:

$$\operatorname{val} g(t) = \operatorname{val}(\lambda) + r\operatorname{val}|t-\theta_0| + s\operatorname{val}|\log|t-\theta_0|-\theta_1|$$

If $\operatorname{st}(\theta_0) \neq t$ and $\operatorname{st}\log|t-\theta_0| \neq \operatorname{st}(\theta_1)$ then

$$\operatorname{val}|t-\theta_0| = \operatorname{val}|\log|t-\theta_0|-\theta_1| = 0$$

and $\operatorname{val} g(x) = \operatorname{val}(\lambda)$. There is a unique $t \in \mathbb{R}$ such that $\operatorname{st}(\theta_0) = t$. There are at most two elements $t \in \mathbb{R}$ such that $\operatorname{st}\log|t-\theta_0| = \operatorname{st}(\theta_1)$. Thus $\operatorname{val} g(t)$ takes at most 4 values on $I \cap \mathbb{R}$. $\square$

We now prove Theorem 11.4.1 for $\mathcal{X}$. It suffices to, for all $n \in \mathbb{N}$, prove Theorem 11.4.1 for the family of precompact metric spaces $\{(B_\alpha(p_\alpha, n), d, p_\alpha) : \alpha \in \mathbb{R}^l\}$. We therefore suppose that there is an $N \in \mathbb{N}$ such that each element of $\mathcal{X}$ has diameter at most N. After applying

Corollary 9.3.4 we further suppose that each element of our family $\mathcal{X}$ has euclidean topology. After applying trivialization we suppose that $\mathcal{X}$ is a finite union of definable families of the form

$$\mathcal{X}_i = \{(X_i, d_\alpha) : \alpha \in \mathbb{R}^l\},$$

for definable sets $X_1, \ldots, X_n$. Any ultralimit of a sequence of elements of $\mathcal{X}$ is an ultralimit of sequence of elements of $\mathcal{X}_i$ for some $1 \leqslant i \leqslant n$, so it suffices to prove the theorem for each family $\mathcal{X}_i$ separately. We therefore can assume that

$$\mathcal{X} = \{(X, d_\alpha) : \alpha \in \mathbb{R}^l\}$$

for some definable set X and that each (X, d_α) has euclidean topology. Fix an $\alpha \in R^l$. We show that every d^*_α-ball with real radius contains a d^*_α-ball with real radius whose reduction is compact. We only show that there is a d^*_α-ball whose reduction is compact, the proof can be applied to any d^*_α-ball. Our proof will use a finitely additive R-valued measure μ on the boolean algebra of subsets of X^* generated by the d^*_α-balls. We first construct μ. Let $\{X_1, \ldots X_m\}$ be a cell decomposition of X. For each $1 \leqslant i \leqslant m$ let ν_i be the $\dim(X_i)$-dimensional Lebesgue measure on X. Let ν be the Borel measure on X given by:

$$\nu(A) = \sum_{i=1}^{m} \nu_i(A \cap X_i) \quad \text{for all Borel } A \subseteq X.$$

Let $h(\beta, p, t) = \nu[B_\beta(p, t)]$ for all $\beta \in \mathbb{R}^l$ and $(p, t) \in X \times \mathbb{R}^>$. Then h is a constructible function. We let $B_\alpha(p, t)$ be the d^*_α-ball with center $p \in X^*$ and radius $t \in R^>$. We define $\mu[B_\alpha(p, t)] = h^*(\alpha, p, t)$ for all $(p, t) \in X^* \times R^>$. Then μ naturally extends to a finitely additive R-valued measure on the boolean algebra of subsets of X^* generated by the d^*_α-balls. The following fact allows us to apply Lemma 3.3.5:

Fact 11.4.3. *We have $\mu[B_\alpha(p, t)] > 0$ for all $p \in X^*$ and $t \in R^>$.*

Proof. It suffices to show that $\nu[B_\beta(p, t)] > 0$ for all $\beta \in \mathbb{R}^l$ and $(p, t) \in X \times \mathbb{R}^>$. We fix an open $U \subseteq X$ and show that $\nu(U) > 0$. There is an $1 \leqslant i \leqslant m$ such that $U \cap X_i$ has nonempty interior in X_i, and $\nu_i(U \cap X_i) > 0$ for this i. Thus $\nu(U) > 0$. □

To prove Theorem 11.4.1 it suffices to verify the following claim:

Claim 11.4.4. *We show that there are $q \in X^*$ and $\delta \in \mathbb{R}^>$ such that if $0 < t < t' \leqslant 2\delta$ are real numbers then there is a $M \in \mathbb{N}$ such that:*

$$\mu[B_\alpha(p,t)] \leqslant M\mu[B_\alpha(p,t')] \quad \textit{for all } p \in B_\alpha(q,\delta).$$

It follows by an application Lemma 3.3.5 that $B_\alpha(q,\delta)$ has compact reduction when q, δ satisfy the conditions of the claim.

Proof (Claim 11.4.4). We let $d = d^*_\alpha$ and let $B(p,t)$ be the d-ball with center p and radius t. Applying Lemma 11.4.2 let $N \in \mathbb{N}$ be such that

$$|\{\operatorname{val}\mu[B(p,t)] : t \in \mathbb{R}^>\}| \leqslant N \quad \text{for all } p \in Z.$$

Let N be the least natural number for which this holds. Fix $q \in Z$ such that

$$|\{\operatorname{val}\mu[B(q,t)] : t \in \mathbb{R}^>\}| = N.$$

Then $\mu[B(q,t)]$ is an increasing function of t, so $\operatorname{val}\mu[B(q,t)]$ is a decreasing function of t. We can therefore partition $\mathbb{R}^>$ into N intervals on which $\mu[B(q,t)]$ is constant. Let $0 = s_0 < s_1 < s_2 < \ldots < s_{N-1}$ be real numbers such that for all $1 \leqslant i \leqslant N-1$, $\operatorname{val}\mu[B(q,t)]$ is constant on (s_{i-1}, s_i) and

$$\operatorname{val}\mu[B(q,t)] > \operatorname{val}\mu[B(q,t')] \quad \text{for all reals } 0 < t < s_i < t'.$$

Let $\epsilon \in \mathbb{R}^>$ be such that $\epsilon < \frac{1}{4}s_1$ and $0 < \epsilon < \frac{1}{4}[s_i - s_{i-1}]$ holds for all $1 \leqslant i \leqslant N-1$. Fix $p \in Z$ such that $d(q,p) < \epsilon$. Then the triangle inequality implies:

$$B(q,t) \subseteq B(p,t+\epsilon) \quad \text{and} \quad B(p,t-\epsilon) \subseteq B(q,t) \quad \text{for all } t \in R^>.$$

So for each $1 \leqslant i \leqslant N-1$ we have:

$$\operatorname{val}\mu[B(p,s_i+2\epsilon)] \leqslant \operatorname{val}\mu[B(q,s_i+\epsilon)] < \operatorname{val}\mu[B(q,s_i-\epsilon)] \leqslant \operatorname{val}\mu[B(p,s_i-2\epsilon)].$$

As

$$0 < s_1 - 2\epsilon < s_1 + 2\epsilon < s_2 - 2\epsilon < s_2 + 2\epsilon < \ldots s_N - 2\epsilon < s_N + 2\epsilon$$

we have $\operatorname{val}\mu[B(p, s_i + 2\epsilon) < \operatorname{val}\mu[B(p, s_{i-1} + 2\epsilon)]$ for all $1 \leqslant i \leqslant N-1$. This implies that that if $t, t' \in \mathbb{R}^>$ are such that $t, t' < s_1 - \epsilon$ then $\operatorname{val}\mu[B(p,t)] = \operatorname{val}\mu[B(p,t')..$ Let $\delta \in \mathbb{R}^>$ be such that $\delta < \epsilon, s_1 - \epsilon$. We have shown that if $p \in B(q, \delta)$ and $0 < t, t' < \delta$ are reals then $\operatorname{val}\mu[B(p,t)] = \operatorname{val}\mu[B(p,t')]$. Thus, if $q \in B(p,\delta)$ and $t \in \mathbb{R}$ is such that $0 < t < 2t < \delta$ then for some natural number M we have $\mu[B(p, 2t)] \leqslant M\mu[B(p,t)]$. $\square$

Theorem 11.4.1 can be used to give restrictions on the quasi-isometry type of an $\bar{\mathbb{R}}_{\mathrm{an}}$-definable metric space. See Section 5.5 for examples of semialgebraic metric spaces which are quasi-isometric to nilpotent groups with left-invariant Riemannian metrics. See Section 5.4.3 for examples of $\bar{\mathbb{R}}_{\exp}$-definable metric spaces which are quasi-isometric to semi-simple Lie groups with left-invariant Riemannian metrics, or to co-compact lattices in semi-simple Lie groups equipped with word metrics.

Corollary 11.4.5. *Suppose (X, d) is $\bar{\mathbb{R}}_{\mathrm{an}}$-definable and proper.*

i. Let G be a finitely generated group equipped with a word metric. If (X, d) is quasi-isometric to G then G is nilpotent-by-finite.

ii. Let G be a Lie group equipped with a left-invariant Riemannian metric. If (X, d) is quasi-isometric to G then (X, d) is quasi-isometric to a Carnot group equipped with a left-invariant Riemannian Metric.

In either case (X, d) is quasi-isometric to a Carnot metric.

Proof. Let G be a finitely generated group equipped with a word metric or a Lie group equipped with a left-invariant Riemannian metric. Suppose that (X, d) is quasi-isometric to G. Lemma 3.7.11 implies that the $\mathfrak{C}_{\mathcal{U}}(X, d)$ and $\mathfrak{C}_{\mathcal{U}}(G)$ are homeomorphic. Theorem 11.4.1 shows that $\mathfrak{C}_{\mathcal{U}}(G)$ contains a nonempty precompact open set. It is not hard to show that the asymptotic cone of a group equipped with a left-invariant metric is topologically homogeneous. As $\mathfrak{C}_{\mathcal{U}}(G)$ is topologically homogeneous, it is locally compact. The proposition is now a corollary of Proposition 3.7.13, which summarized deep and important work in group theory. $\square$

We use Corollary 11.4.5 to show that many definable groups do not admit "nice" $\bar{\mathbb{R}}_{\text{an}}$-definable metrics. Recall the notion of a nice coarse path psuedometric on a Lie group from Section 3.6 and recall that any nice coarse path psuedometric on a Lie group G is quasi-isometric to any left-invariant Riemannian metric on G. Let $(G, \cdot)$ be a $\bar{\mathbb{R}}_{\text{an}}$-definable group. After replacing $(G, \cdot)$ with a definable group which is definably isomorphic to $(G, \cdot)$ we suppose that $(G, \cdot)$ is a Lie group and that the Lie topology on G agrees with the euclidean topology on G, see [Pil88]. Then Proposition 11.4.5 implies the following:

Corollary 11.4.6. *Let d be a nice coarse path pseudometric on $(G, \cdot)$, considered as a Lie group. Then $(G, \cdot)$ has polynomial volume growth and (G, d) is quasi-isometric to a Carnot metric.*

CHAPTER 12

GH-limits of Semialgebraic Families of Metric Spaces

Throughout this chapter $\mathcal{X} = \{(X_\alpha, d_\alpha, p_\alpha) : \alpha \in \mathbb{R}^l\}$ *is a semialgebraic family of pointed proper metric spaces.* After applying Corollary 9.3.4 we assume without loss of generality that d_α-topology agrees with the euclidean topology on each X_α. When we say that a sequence of metric spaces converges we mean that it GH-converges. We say that a metric space (Y, d) is a limit point of $\mathcal{X}$ if and only if (Y, d) is proper and there is a sequence of elements of $\mathcal{X}$ which converges to (Y, d). We call (Y, d) a frontier point of $\mathcal{X}$ if (Y, d) is a limit point of $\mathcal{X}$ and (Y, d) is not isometric to any element of $\mathcal{X}$. In this chapter we study Gromov-Hausdorff limit points of $\mathcal{X}$. The main tool is a fact about definable sets of imaginaries in real closed valued fields, Proposition 4.3.1. In the Section 12.1 we show that the GH-limit points of a semialgebraic family of proper metric spaces are semialgebraic. In Section 12.2 we develop analogues of some of the results of [Dri05] for GH-limits. In Section 12.3 we prove some basic topological facts about reductions of $\mathcal{M}$-definable metric spaces. We use Theorem 11.4.1 to show that ultralimits of semialgebraic families of metric spaces are almost locally euclidean in the sense that every open set contains an open subset which is homeomorphic to an open subset of some euclidean space.

In this section $\mathcal{M} = (M, +, \times, \leqslant)$ is an $|\mathbb{R}|^+$-saturated real closed field and $\mathcal{O}$ is the convex hull of $\mathbb{Z}$ in M. We let $(\mathcal{M}, \mathcal{O})$ be the expansion of $\mathcal{M}$ by a predicate defining $\mathcal{O}$. As usual Γ is the value group of $(\mathcal{M}, \mathcal{O})$ and $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. The residue field is $\mathbb{R}$ and the residue map is the standard part st.

12.1 Definability of GH-limits of Sequences of Elements of $\mathcal{X}$

Let $\mathcal{X}^*$ be the $\mathcal{M}$-definable family $\{(X^*_\alpha, d^*_\alpha, p^*_\alpha) : \alpha \in M^l\}$. Lemma 3.7.6 and Proposition 8.1.4 show that every limit point of $\mathcal{X}$ is isometric to a reduction of an element of $\mathcal{X}^*$ and that every proper element of $\mathcal{X}^*$ is a limit point of $\mathcal{X}$. In this chapter we declare:

$$\mathcal{O}(X^*_\alpha) = \{y \in X^*_\alpha : \operatorname{st} d^*_\alpha(p^*_\alpha, y) < \infty\} \quad \text{for all } \alpha \in M^l.$$

Proposition 12.1.1. *Every limit point of $\mathcal{X}$ is isometric to a semialgebraic metric space.*

Proof. We fix an $\alpha \in M^l$ such that the reduction of $(X^*_\alpha, d^*_\alpha, p^*_\alpha)$ is proper and show that the reduction is isometric to a semialgebraic metric space. As $(\overline{\mathcal{O}(X^*_\alpha)}, \operatorname{st} d^*_\alpha)$ is proper, $|\overline{\mathcal{O}(X^*_\alpha)}| \leqslant |\mathbb{R}|$. Proposition 4.3.3 gives an $(\mathcal{M}, \mathcal{O})$-definable injection $i : \overline{\mathcal{O}(X^*_\alpha)} \to \mathbb{R}^k$ for some k. Let Z be the image of $\overline{\mathcal{O}(X^*_\alpha)}$ and let d be the pushforward of $\operatorname{st} d^*_\alpha$ onto Z by i. As $\mathbb{R}$ is stably embedded in $(\mathcal{M}, \mathcal{O})$ both Z and $d : Z^2 \to \mathbb{R}^{\geqslant}$ are semialgebraic, so (Z, d) is a semialgebraic metric space. $\square$

Lemma 12.1.2. *Suppose $\alpha \in M^l$. The reduction of $(X^*_\alpha, d^*_\alpha, p^*_\alpha)$ is proper if and only if there is an $(\mathcal{M}, \mathcal{O})$-definable injection $\overline{\mathcal{O}(X^*_\alpha)} \to \mathbb{R}^k$ for some k.*

Proof. It suffices to suppose that the reduction of $(X^*_\alpha, d^*_\alpha, p^*_\alpha)$ is not proper and show that $|\mathbb{R}| < \overline{\mathcal{O}(X^*_\alpha)}$. Fix $m \in \mathbb{N}$ such that the reduction of $(B(p^*_\alpha, m), d^*_\alpha)$ is not compact. Lemma 3.7.3 implies that there is a set $\{x_i : i \in \mathbb{N}\} \subseteq B(p^*_\alpha, m)$ and an $\epsilon \in \mathbb{R}^>$ such that $d^*_\alpha(x_i, x_j) \geqslant \epsilon$ when $i \neq j$. Applying $|\mathbb{R}|^+$-saturation of $\mathcal{M}$ we obtain a sequence $\{x_\lambda : \lambda < |\mathbb{R}|^+\}$ of elements $B(p^*_\alpha, m)$ such that $d(x_\lambda, x_\eta) \geqslant \epsilon$ for all $\lambda, \eta < |\mathbb{R}|^+$ such that $\lambda \neq \eta$. Thus $|\overline{B(p^*_\alpha, m)}| \geqslant |\mathbb{R}|^+$. $\square$

The proposition below follows directly from Proposition 12.1.1, Lemma 12.1.2, Proposition 4.3.2 and Corollary 9.3.4. We leave the details to the reader.

Proposition 12.1.3. *Let $H \subseteq M^l$ be the set of α such that the reduction of $(X^*_\alpha, d^*_\alpha, p^*_\alpha)$ is proper. Then H is $(\mathcal{M}, \mathcal{O})$-definable. There is a semialgebraic family $\mathcal{Y} = \{(Y_b, d_b) : b \in \mathbb{R}^n\}$ of proper metric spaces with euclidean topology, an $(\mathcal{M}, \mathcal{O})$-definable function $h : H \to \mathbb{R}^n$*

and an $(\mathcal{M},\mathcal{O})$-definable family $\{g_\alpha : \alpha \in H\}$ of functions $\mathcal{O}(X^*_\alpha) \to Y_{h(\alpha)}$ such that for all $\alpha \in H$ and all $x, y \in \mathcal{O}(X^*_\alpha)$:

i. $g_\alpha(\mathcal{O}(X^*_\alpha)) = Y_{h(\alpha)}$,

ii. For all $x, y \in \mathcal{O}(X^*_\alpha)$, $g_\alpha(x) = g_\alpha(y)$ if and only if $\operatorname{st} d^*_\alpha(x,y) = 0$,

iii. $d_{h(\alpha)}$ *is the pushforward of* $\operatorname{st} d^*_\alpha$ *onto* $Y_{h(\alpha)}$ *by* g_α.

Thus for all $\alpha \in H$, $(Y_{h(\alpha}, d_{h(\alpha)})$ is isometric to $(\overline{\mathcal{O}(X^*_\alpha)}, \operatorname{st} d^*_\alpha)$. Therefore element of $\mathcal{Y}$ is a limit point of $\mathcal{X}$ and every limit point of $\mathcal{X}$ is isometric to an element of $\mathcal{Y}$.

From Proposition 12.1.3 and Proposition 11.2.2 we have:

Corollary 12.1.4. *Let (X, d) be a compact semialgebraic metric space. Let μ be the* $\dim(X)$*-dimensional Lebesgue measure on X. The collection of all tangent cones of (X, d) forms a semialgebraic family of metric spaces. The tangent cone of (X, d) at μ-almost every $p \in X$ is proper, locally connected, isometrically homogeneous and conical.*

Proof. Proposition 11.2.2 shows that the tangent cone at μ-almost every point is proper, isometrically homogeneous and conical. Every tangent cone is isometric to a semialgebraic metric space with euclidean topology, so every tangent cone is locally connected. □

If Question 3.7.16 admits a positive answer then the tangent cone of (X, d) at μ-almost every point is isometric to a Carnot group equipped with a left-invariant metric.

12.2 Further Results on GH-limits

For the moment we let $\dim_{\mathbb{R}}(A)$ be the o-minimal dimension of a semialgebraic set A and $\dim_{\mathcal{M}}(B)$ be the weakly o-minimal dimension of an $(\mathcal{M},\mathcal{O})$-definable subset $B \subseteq M^k$. If $B \subseteq R^k$ is $\mathcal{M}$-definable then $\dim_{\mathcal{M}}(B)$ is the usual o-minimal dimension of B. In [Dri05] van den Dries showed that if $A \subseteq M^k$ and $\tau : A \to \mathbb{R}^l$ are $(\mathcal{M},\mathcal{O})$-definable then

$$\dim_{\mathbb{R}} \tau(A) \leqslant \dim_{\mathcal{M}}(A).$$

This inequality also follows from general model-theoretic ideas, $\dim_{\mathbb{R}}(A)$ and $\dim_{\mathcal{M}}(B)$ are the dp-ranks of $(\mathcal{M}, \mathcal{O})$-definable $A \subseteq \mathbb{R}^k$ and $B \subseteq M^k$, and it is known that definable functions cannot not raise dp-rank. See [Sim14] for more information. The inequality above implies that:

Proposition 12.2.1. *If (Z, d) is a limit point of $\mathcal{X}$ then:*

$$\dim_{\mathrm{top}}(Z, d) \leqslant \max\{\dim_{\mathrm{top}}(X_\alpha, d_\alpha) : \alpha \in \mathbb{R}^l\}.$$

Proof. Without lose of generality we take (Z, d) to be a semialgebraic metric space. As (Z, d) is proper, it is definably separable, so $\dim_{\mathrm{top}}(Z, d) = \dim_{\mathbb{R}}(Z)$. Let $\beta \in M^l$ be such that the reduction of $(X^*_\beta, d^*_\beta, p^*_\beta)$ is isometric to (Z, d). Then

$$\dim_{\mathbb{R}}(Z) \leqslant \dim_{\mathcal{M}}(X^*_\beta) \leqslant \max\{\dim_{\mathbb{R}}(X_\alpha) : \alpha \in \mathbb{R}^l\} = \max\{\dim_{\mathrm{top}}(X_\alpha, d_\alpha) : \alpha \in \mathbb{R}^l\}.$$

□

For the remainder of this section we assume that there is an $N \in \mathbb{N}$ such that every element of $\mathcal{X}$ has diameter at most N. The following proposition shows that a GH-limit point of $\mathcal{X}$ is a GH-limit of a family of the form $(X_{\gamma(t)}, d_{\gamma(t)})$ as $t \to 0^+$ for some definable path γ in $\mathbb{R}^l$.

Proposition 12.2.2. *Let $\alpha \in M^l$ be such that $(\overline{X^*_\alpha}, \operatorname{st} d^*_\alpha)$ is compact. Then there is a compact semialgebraic metric space (Y, d), a semialgebraic path $\gamma : \mathbb{R}^> \to \mathbb{R}^l$, and a semialgebraic family of partial functions $f_t : A_t \to Y$ such that $A_t \subseteq X_{\gamma(t)}$ for all $t \in \mathbb{R}^>$ and f_t gives a t-approximate isometry $(X_{\gamma(t)}, d_{\gamma(t)}) \to (Y, d)$ for all $0 < t \ll 1$. In particular $(X_{\gamma(t)}, d_{\gamma(t)})$ GH-converges to (Y, d) as $t \to 0^+$.*

Proof. Applying Proposition 12.1.3 we let (Y, d) be a compact semialgebraic metric space with euclidean topology and let $h : X^*_\alpha \to Y$ be a surjective $(\mathcal{M}, \mathcal{O})$ definable function such that:

$$d(h(x), h(y)) = \operatorname{st} d^*_\alpha(x, y) \quad \text{for all } x, y \in X^*_\alpha.$$

In particular $h(x) = h(y)$ if and only if $\operatorname{st} d^*_\alpha(x, y) = 0$. Applying the existence of $(\mathcal{M}, \mathcal{O})$-definable Skolem functions we let $g : X^*_\alpha \to Y^*$ be an $(\mathcal{M}, \mathcal{O})$-definable function such that

$\operatorname{st} g = h$. We show that g is a "$\mathfrak{m}$-isometry". As $\operatorname{st} g : X^*_\alpha \to Y$ is surjective, for all $y \in Y^*$ there is an $x \in X^*_\alpha$ such that $d^*(g(x), y) \in \mathfrak{m}$. Thus the image of g is "$\mathfrak{m}$-dense" in (Y^*, d^*). As (Y, d) has euclidean topology we have:

$$\operatorname{st} d^*(g(x), g(y)) = d(\operatorname{st} g(x), \operatorname{st} g(y)) = \operatorname{st} d^*_\alpha(x, y) \quad \text{for all } x, y \in X^*_\alpha.$$

So,

$$|d^*_\alpha(x, y) - d^*(g(x), g(y))| \in \mathfrak{m} \quad \text{for all } x, y \in X^*_\alpha.$$

Applying Lemma 4.2.1 let $\mathcal{G} = \{g_q : r \in R^2\}$ be a semialgebraic family of functions $A_q \to Y^*$ such that $A_q \subseteq X^*_\alpha$ for all $q \in M^2$ and such that $\operatorname{Graph}(g)$ is a union of a directed subset of $\{\operatorname{Graph}(g_q) : q \in M^2\}$.

Claim 12.2.3. *For every $\epsilon \in \mathbb{R}^>$ there is a $q \in M^2$ such that $g_q : (X^*_\alpha, d^*_\alpha) \to (Y^*, d^*)$ is an ϵ-approximate isometry.*

If $\operatorname{Graph}(g_q) \subseteq \operatorname{Graph}(g)$ and the image of g_q is ϵ-dense in Y^* then g_q is an ϵ-approximate isometry. Fix $\epsilon \in \mathbb{R}^>$. Let $\{y_1, \dots, y_m\}$ be an ϵ-dense subset of (Y, d). Then $\{y_1, \dots, y_m\}$ is also ϵ-dense in (Y^*, d^*). As $\{\operatorname{Graph}(g_q) : q \in M^2\}$ is directed there is a $q \in R^2$ such that $\operatorname{Graph}(g_q) \subseteq \operatorname{Graph}(g)$ and $\{y_1, \dots, y_m\}$ is contained in the image of g_q, then g_q is an ϵ-isometry for such a q. This proves the claim.

Now let $\{f_{\beta,q,z} : (\beta, q, z) \in \mathbb{R}^l \times \mathbb{R}^2 \times \mathbb{R}^m\}$ be a semialgebraic family of functions $A_{\beta,q,z} \to Y$ such that $A_{\beta,q,z} \subseteq X_\beta$ for all (β, q, z) and

$$g_q = f^*_{\alpha,q,z} \quad \text{holds for some } z \in M^m.$$

For every $\epsilon \in \mathbb{R}^>$ there is a $(\beta, q, z) \in \mathbb{R}^l \times \mathbb{R}^2 \times \mathbb{R}^m$ such that $f_{\beta,q,z}$ give an ϵ-approximate isometry $(X_\beta, d_\beta) \to (Y, d)$. The proposition follows by an application of definable choice. $\square$

The proof of Proposition 12.2.2 goes through uniformly and yields the following proposition. We use the notation of Proposition 12.1.3.

Proposition 12.2.4. *There is a semialgebraic function $\gamma : C \times \mathbb{R}^> \to \mathbb{R}^l$ and a semialgebraic family $\{f_{b,t} : (b, t) \in C \times \mathbb{R}^>\}$ of functions $A_{b,t} \to Y_b$ such that $A_{b,t} \subseteq X_{\gamma(b,t)}$ for all b, t and* *such that for each $b \in C$, $f_{b,t}$ give a t-isometry $(X_{\gamma(b,t)}, d_{\gamma(b,t)}) \to (Y_b, d_b)$ when $0 < t \ll 1$. In particular, $(X_{\gamma(b,t)}, d_{\gamma(b,t)})$ converges to (Y_b, d_b) as $t \to 0^+$ for all $b \in C$.*

The remainder of this section is devoted to proving Proposition 12.2.7, which states that the GH-frontier of $\mathcal{X}$ is in a certain sense smaller than $\mathcal{X}$. Our proof is an adaptation of Gabrielov's proof of the analogous result for Hausdorff limits, which is given in the last section of [Dri05]. Gabrielov's proof uses the Trivialization Theorem. In place of the Trivialization Theorem we use Lemma 12.2.5 and Corollary 12.2.6 below. We say that a definable family of compact metric spaces $\{(Z_b, d_b) : b \in D\}$ is **polished** if D is locally closed and whenever a sequence $\{b(i)\}_{i \in \mathbb{N}}$ of elements of A converges to an element $b \in A$ as $i \to \infty$ then $(Z_{b(i)}, d_{b(i)})$ converges to (Z_b, d_b) as $i \to \infty$.

Lemma 12.2.5. *Let $\mathcal{W} = \{(W_\alpha, d_\alpha) : \alpha \in Q\}$ be a semialgebraic family of compact metric spaces. There is a semialgebraic $Q' \subseteq Q$ such that every element of $\{(W_a, d_a) : a \in Q\}$ is semialgebraically isometric to an element of $\{(W_a, d_a) : a \in Q'\}$ and a partition of Q' into cells $Q'_1, \ldots, Q'_n$ such that $\{(W_a, d_a) : a \in Q'_i\}$ is polished for all $1 \leqslant i \leqslant n$.*

Proof. As before we may suppose that the d_α-topology agrees with the euclidean topology on each W_α. It suffices to partition Q into cells $P_1, \ldots, P_n$ and prove the lemma for each family $\{(W_a, d_a) : a \in P_i\}$. After applying trivialization we take $\mathcal{W}$ to be of the form $\{(W, d_\alpha) : \alpha \in Q\}$. We define a pseudometric $\widetilde{d}$ on Q by declaring:

$$\widetilde{d}(\alpha, \beta) = \sup\{|d_\alpha(x, y) - d_\beta(x, y)| : (x, y) \in W^2\} \quad \text{for all } \alpha, \beta \in Q.$$

The d_α are continuous functions on the compact space W^2, so this definition makes sense. If $\widetilde{d}(\alpha, \beta) = 0$ then $d_\alpha = d_\beta$ and so (W, d_α) and (W, d_β) are trivially definably isometric. We let Q' be a semialgebraic set which contains exactly one element from each $\widetilde{d}$-class. As W^2 is compact the space of bounded continuous functions $W^2 \to \mathbb{R}$ with the uniform topology is separable, thus $(Q', \widetilde{d})$ is separable. Applying item (iv) of Corollary 9.2.13 we partition Q' into cells $Q_1, \ldots, Q_m$ such that the $\widetilde{d}$-topology agrees with the euclidean topology on each Q_i. Suppose that $\{\beta(i)\}_{i \in \mathbb{N}}$ is a sequence of elements of Q_i with limit $\beta \in Q_i$. As $\widetilde{d}(\beta(i), \beta) \to 0$, $d_{\beta(i)}$ uniformly converges to d_β as $i \to \infty$. By Corollary 8.2.2, $(W, d_{\beta(i)})$ converges to (W, d_β) as $i \to \infty$. □

Corollary 12.2.6. *Let $\{(W_q, d_q) : q \in Q\}$ be a semialgebraic family of compact metric spaces. There are semialgebraic sets $Q' \subseteq Q, Q''$ and a definable bijection $\tau : Q'' \to Q'$ such that every element of $\mathcal{W}$ is isometric to an element of $\{(W_q, d_q) : q \in Q'\}$ and the family $\{(W_{\tau(q)}, d_{\tau(q)}) : q \in Q''\}$ is polished.*

Proof. We simply let Q'' be the disjoint union of the cells given by Lemma 12.2.5 and let $\tau : Q'' \to Q'$ be the natural map. $\square$

Proposition 12.2.7. *Let $\mathcal{X}$ and $\mathcal{Y}$ be as before. There is a definable subset $C' \subseteq C$ such that every frontier point of $\mathcal{X}$ is isometric to an element of $\{(Y_\beta, d_\beta) : \beta \in C'\}$ and $\dim(C') < \dim(A)$.*

In other words: "$\dim \partial_{GH}(\mathcal{X}) < \dim(\mathcal{X})$".

Proof. Applying Corollary 12.2.6 we assume without lose of generality that $\mathcal{X}$ is of form $\{(X_\alpha, d_\alpha) : \alpha \in A\}$ and is polished. Applying Proposition 12.2.4 we let $\gamma : C \times (0, 1) \to A$ be a semialgebraic function such the GH-distance between $(X_{\gamma(\beta,t)}, d_{\gamma(\beta,t)})$ and (Y_β, d_β) is at most t. We set $\gamma_b(t) = \gamma(b, t)$. For all $t \in \mathbb{R}^>$ we let $A_t = \{a \in A : e(\partial A, a) \geqslant t\}$. As A is bounded and locally closed, A is the union of the A_t and each A_t is compact.

Claim 12.2.8. *Let $b \in C$ and $t \in \mathbb{R}^>$. Suppose that for every $s' \in \mathbb{R}^>$ there is an $0 < s < s'$ such that $\gamma_b(s) \in A_t$. Then (Y_b, d_b) is isometric to an element of $\mathcal{X}$.*

Suppose that the conditions of the claim hold for $b \in C$ and $t \in \mathbb{R}^>$. Applying definable choice we let $q : \mathbb{R}^> \to A_t$ be a semialgebraic function such that $\gamma_{q(s)}(s) = b$ when $0 < s \ll 1$. As A_t is compact, $q(t)$ converges to some $p \in A_t$ as $t \to 0^+$. As $\mathcal{X}$ is polished, (X_p, d_p) is isometric to (Y_b, Y_b). Let B be the set of $b \in C$ for which there is a $t \in \mathbb{R}^>$ such that the condition in the claim holds for b, t. Then every frontier point of $\mathcal{X}$ is isometric to an element of $\{(Y_b, d_b) : b \in C \setminus B\}$. After replacing C with $C \setminus B$ we suppose that for all $b \in C$ and $t \in \mathbb{R}^>$ there is a $s' \in \mathbb{R}^>$ such that if $0 < s < s'$ then $\gamma_b(s) \notin A_t$.

Applying Corollary 12.2.6 we now assume without loss of generality that $\mathcal{Y}$ is polished.

We also suppose that C is a bounded subset of euclidean space. Let

$$C_t = \{b \in C : e(\partial C, b) \geqslant t\} \quad \text{for all } t \in \mathbb{R}^>.$$

Then C is the increasing union of the C_t and each C_t is compact. The following claim can be proven in the same way as the previous one:

Claim 12.2.9. *Let $a \in A$ and $t \in \mathbb{R}^>$. Suppose that for every $s' \in \mathbb{R}^>$ there is $0 < s < s'$ and $b \in C_t$ such that $\gamma_b(s) = a$. Then (Y_b, d_b) is isometric to (X_a, d_a).*

Thus we are justified in assuming that for any $a \in A$ and $t \in \mathbb{R}^>$ there is a $s' \in \mathbb{R}^>$ such that if $0 < s < s'$ and $b \in C_t$ then $\gamma_b(s) \neq a$

We put an equivalence relation $\sim$ on C. Given $b_1, b_2 \in C$ we declare $b_1 \sim b_2$ if and only if for every $t \in \mathbb{R}^>$ there is $a \in A$ and $c_1, c_2 \in C$ satisfying $\|b_1 - c_1\|, \|b_2 - c_2\| < t$ and $\gamma(c_1, s) = \gamma(c_2, s') = a$ for some $0 < s, s' < t$.

Claim 12.2.10. *Let $b, b' \in C$. If $b \sim b'$ then (Y_b, d_b) is isometric to $(Y_{b'}, d_{b'})$.*

Suppose that $b \sim b'$. Let $\eta_1, \eta_2 : \mathbb{R}^> \to C$ and $\alpha : \mathbb{R}^> \to A$ be semialgebraic functions such that for all $t \in \mathbb{R}^>$, $\|\eta_1(t) - b\|, \|\eta_2(t) - b'\| < t$ and

$$\gamma(\eta_1(t), s) = \gamma(\eta_2(t), s') = \alpha(t) \quad \text{for some } 0 < s, s' < t.$$

Then $(Y_{\eta_1(t)}, d_{\eta_1(t)})$ and $(Y_{\eta_2(t)}, d_{\eta_1(t)})$ converge to (Y_b, d_b) and $(Y_{b'}, d_{b'})$ as $t \to 0^+$, respectively. The definition of γ implies that the GH-distance between $(Y_{\eta_1(t)}, d_{\eta_2(t)})$ and $(Y_{\eta_2(t)}, d_{\eta_2(t)})$ goes to zero as $t \to 0^+$. So (Y_b, d_b) and $(Y_{b'}, d_{b'})$ are isometric. This proves Claim 12.2.10.

We prove the proposition by showing that $\dim(C/{\sim}) < \dim(A)$. We can then take C' to be any semialgebraic subset of C which contains exactly one element from each $\sim$-class. As $C/{\sim}$ is the increasing union of the family $\{C_t/{\sim} : t \in \mathbb{R}^>\}$, it is enough to show that $\dim(C_t/{\sim}) < \dim(A)$ for all $t \in \mathbb{R}^>$. To this effect fix $t \in \mathbb{R}^>$. Let $A' = \gamma(C_t \times (0, 1)) \subseteq A$. Applying definable choice we let $F : A' \to C_t$ be a semialgebraic function such that for all $a \in A$ there is an $s \in (0, 1)$ such that:

$$0 < s < 2 \inf\{s' > 0 : (\exists b \in C_t)[\gamma_b(s') = a]\}$$

and $\gamma(F(a), s) = a$. The remarks after Claim 12.2.9 imply that the infimum above is positive. Let P be the intersection of $\partial[\mathrm{Graph}(F)] \subseteq A \times C_t$ and $\partial(A) \times C_t$. Let $\pi : A \times C_t \to C_t$ be the coordinate projection. We show that every element of C_t is $\sim$-equivalent to an element of $\pi(P)$. As

$$\dim \pi(P) \leqslant \dim(P) \leqslant \dim \partial[\mathrm{Graph}(F)] < \dim[\mathrm{Graph}(F)] = \dim(A') \leqslant \dim(A)$$

this proves that $\dim(C_t/\sim) < \dim(A)$. Fix $b \in C_t$. As A is bounded $\gamma_b(t)$ converges to some point α in the closure of A as $t \to 0^+$. By Claim 12.2.8 $\alpha \notin A$, so $\alpha \in \partial(A)$. Let $\theta(t) = (F \circ \gamma_b)(t)$. As C_t is compact, θ converges to some element $b' \in C_t$ as $t \to 0^+$. As (α, b') is an element of P, so $b' \in \pi(P)$. It follows directly from the definition of $\sim$ that $b \sim b'$. □

12.3 GH-ultralimits of Sequences of Elements of $\mathcal{X}$

In this section we study reductions of $\mathcal{M}$-definable metric spaces. We make use of the following fact whose verification we leave to the reader:

Fact 12.3.1. *Suppose A is a subset of a metric space (Y, d) and $\epsilon > 0$ is such that if $d(x, y) \leqslant \epsilon$ and $x \in A$ then $y \in A$. Then A is clopen.*

Proposition 12.3.2 shows that Theorem 9.0.1 cannot hold for reductions of $\mathcal{M}$-definable metric spaces.

Proposition 12.3.2. *Let (X, d, p) be an $\mathcal{M}$-definable, definably separable pointed metric space. No infinite definable subset of $(\overline{\mathcal{O}(X)}, \mathrm{st}\, d)$ is discrete.*

We declare $\mathcal{O}(X) = \{y \in X : \mathrm{st}(p, y) < \infty\}$. If $A \subseteq \mathcal{O}(X)$ then we let $\overline{A}$ be the image of A in $\overline{\mathcal{O}(X)}$ under the quotient map.

Proof. We suppose can assume that the d-topology agrees with the euclidean topology on X. Suppose that $A \subseteq \mathcal{O}(X)$ is a $(\mathcal{M}, \mathcal{O})$-definable set such that $\overline{A}$ is infinite and discrete. For all $t \in M^{>}$ let $A_t \subseteq A$ be the set of $a \in A$ such that if $b \in A$ and $d(a, b) \leqslant t$ then $\overline{a} = \overline{b}$. Every element of A is an element of some A_t. Applying Lemma 4.2.1 let $\mathcal{B} = \{B_{q,t} : q \in M^2, t \in R\}$ be an $\mathcal{M}$-definable family of subsets of X such that A_t is the union of a directed subset of $\{B_{q,t} : q \in M^2\}$ for all $t \in M^>$. For each $(q,t) \in M^2 \times M$ we let $\sim_{q,t}$ be the 2-ary relation on $B_{q,t}$ given by declaring $a \sim_{q,t} b$ if and only if $d(a,b) \leqslant t$. If $B_{q,t} \subseteq A_t$ then $\sim_{q,t}$ is an equivalence relation on $B_{q,t}$. Fact 12.3.1 shows that if $B_{q,t} \subseteq A_t$ then each $\sim_{q,t}$-class is a clopen subset of $B_{q,t}$. It follows that for every $N \in \mathbb{N}$ there is are $(q,t) \in M^2 \times M$ such that $B_{q,t}$ has at least N definably connected components., This is a contradiction as the d-topology agrees with the euclidean topology on X. □

Proposition 12.3.3 is an analogue of Theorem 9.0.1 for reductions.

Proposition 12.3.3. *Let (X, d, p) be an $\mathcal{M}$-definable, definably separable pointed metric space. Exactly one of the following holds:*

i. There is an $\mathcal{M}$-definable $A \subseteq \mathcal{O}(X)$ and an $(\mathcal{M}, \mathcal{O})$-definable map $f : \overline{A} \to \Gamma$ with infinite image such that every fiber of f is clopen in $(\overline{A}, \operatorname{st} d)$.

ii. There is a semialgebraic $Z \subseteq \mathbb{R}^k$ and a definable homeomorphism

$$(\overline{\mathcal{O}(X)}, \operatorname{st} d) \to (Z, e).$$

Proof. It is enough to show that (i) above holds under the assumption that there is a $(\mathcal{M}, \mathcal{O})$-definable map $\overline{\mathcal{O}(X)} \to \Gamma$ with infinite image. We let $\overline{x} \in \overline{\mathcal{O}(X)}$ be the class of $x \in \mathcal{O}(X)$. Let $h : \overline{\mathcal{O}(X)} \to \Gamma$ be an $(\mathcal{M}, \mathcal{O})$-definable function with infinite image. Applying the existence of $(\mathcal{M}, \mathcal{O})$-definable Skolem functions we let $g : X \to M^>$ be a definable function such that:

$$\operatorname{val} g(a) = h(\overline{x}) \quad \text{for all } x \in \mathcal{O}(X).$$

Applying Lemma 4.2.1 let $\{A_q : q \in M^2\}$ be an $\mathcal{M}$-definable family of subsets of X and $\mathcal{F} = \{f_q : q \in M^2\}$ be an $\mathcal{M}$-definable family of functions $A_q \to M^>$ such that $\operatorname{Graph}(g)$ is the union of a directed subset of $\mathcal{F}$. As $\mathcal{F}$ is directed, for all $N \in \mathbb{N}$ there is $q \in M^2$ such that $\operatorname{Graph}(f_q) \subseteq \operatorname{Graph}(g)$ and $|\operatorname{val} f_q(A_q)| \geqslant N$. Thus by Corollary 4.1.8 there is a $q \in M^2$ such that $\operatorname{Graph}(f_q) \subseteq \operatorname{Graph}(g)$ and $\operatorname{val} f_q(A_q)$ is infinite. We fix such a $q \in M^2$, set $A = A_q$

and $f = f_q$. We show that every fiber of $\operatorname{val} f$ is clopen. If $x, y \in A$ and $\operatorname{st} d(x,y) = 0$ then $\operatorname{val} f(x) = \operatorname{val} f(y)$. As $\mathcal{M}$ is ω-saturated there is an $\epsilon \in \mathbb{R}^{>}$ and $N \in \mathbb{N}$ such that:

$$\frac{1}{N} f(x) \leqslant f(y) \leqslant N f(x) \quad \text{for all } x, y \in A \text{ such that } d(x,y) \leqslant \epsilon.$$

Fix $\gamma \in \Gamma$. If $x \in A$ is such that $\operatorname{val} f(x) = \gamma$ and $y \in A$ satisfies $d(x,y) \leqslant \epsilon$ then $\operatorname{val} f(y) = \gamma$. Fact 12.3.1 implies that $(\operatorname{val} f)^{-1}(\gamma)$ is clopen in $\overline{A}$. □

12.3.1 Applications of Weak Precompactness

Theorem 12.3.4. *Let (Z,d) be a reduction of an element of $\mathcal{X}^*$. Every open subset of Z contains an open ball which is definably homeomorphic to an open subset of some $\mathbb{R}^k$ where*

$$k \leqslant \max\{\dim(X_\alpha) : \alpha \in \mathbb{R}^l\}.$$

Proof. Theorem 11.4.1 implies that Z contains a precompact open ball B. Proposition 12.1.1 implies that B is definably isometric to a semialgebraic metric space (Y,d). Proposition 12.2.1 implies that $\dim(Y) \leqslant \max\{\dim(X_\alpha) : \alpha \in \mathbb{R}^l\}$. Corollary 9.5.1 shows that there is an open ball in (Y,d) which is definably homeomorphic to an open subset of $\mathbb{R}^{\dim(Y)}$. □

Corollary 12.3.5. *Let (Z,d) be as in the previous theorem. If Z is infinite then the topological dimension of (Z,d) is at least* 1.

Proof. Suppose that Z is infinite. Proposition 12.3.2 implies that (Z,d) has only finitely many isolated points. Thus there is an open $U \subseteq Z$ which does not contain any isolated point. Theorem 12.3.4 implies that U contains an open ball which is homeomorphic to an open subset of $\mathbb{R}^l$. If $l = 0$ then U would contain an isolated point, so $l \geqslant 1$. Thus $\dim_{\text{top}}(Z,d) \geqslant 1$. □

Recall from Fact 5.4.2 that if d is a metric then $\log[1+d]$ is an approximate ultrametric. Our final corollary shows in contrast that there are no nontrivial semialgebraic approximate ultrametrics.

Corollary 12.3.6. *Every semialgebraic proper approximate ultrametric space is bounded.*

Proof. Let (X, d) be a semialgebraic proper approximate ultrametric. By Lemma 3.7.12 the asymptotic cone of (X, d) is an ultrametric space, and so has topological dimension zero. This implies that the asymptotic cone of (X, d) is finite, and thus by Lemma 3.7.10 (X, d) is bounded. □

CHAPTER 13

Problems

We list a number of open problems which the present author finds interesting. *We assume that $\mathcal{R}$ expands the real field.*

Problem 13.0.1. *Is every definable metric space definably homeomorphic (or even homeomorphic) to a semilinear metric space?*

The Triangulation Theorem implies that every definable set is definably homeomorphic to a semilinear set. In Section 5.8 we construct some interesting examples of definable metric spaces which are not homeomorphic to any definable set. All of these examples are definably homeomorphic to semilinear metric spaces. We do not know if the disjoint union, as constructed in Subsection 5.8.5, of the semialgebraic family of metric spaces given in Example 5.8.5 is homeomorphic to a semilinear metric space. Proposition 10.5.3 shows that the metric geometry of semilinear metric spaces is very constrained, we know of no topological constraints on semilinear metric spaces other than those which hold for all definable metric spaces.

Problem 13.0.2. *Let (X, d) be a definable metric space with finite Hausdorff dimension. Prove that for almost every $p \in X$ the tangent cone of (X, d) at p exists and is isometric to a Carnot group equipped with a left-invariant metric.*

Let μ be the $\dim(X)$-dimensional Lebesgue measure on X. In Corollary 12.1.4 we showed that if (X, d) is semialgebraic then for μ-almost every $p \in X$ the tangent cone of (X, d) exists at p and is proper, locally connected, isometrically homogeneous and conical. If Question 3.7.16 admits a positive answer, at least for semialgebraic metric spaces, then the tangent cone of (X, d) at μ-almost every point is isometric to a Carnot group equipped with a left-invariant metric. We aim to generalize Corollary 12.1.4 to polynomially bounded o-minimal expansions of the real field in the forthcoming [HW15]. However, we do not know how to approach Problem 13.0.2 outside of the polynomially bounded setting.

Problem 13.0.3. *Let $\mathcal{X}$ be a semialgebraic family of proper metric spaces and let (Z, d) be a Gromov-Hausdorff ultralimit of a sequence of elements of $\mathcal{X}$. Is (Z, d) necessarily locally compact?*

Theorem 11.4.1 shows that every open subset of Z contains a precompact open subset set. We do not know of an example of such a (Z, d) which is not locally compact.

Problem 13.0.4. *For $n \leqslant 5$, how many n-dimensional semialgebraic metric spaces are there up to bilipschitz equivalence? How many are there up to quasi-isometry?*

Recall from Section 5.5 that there are continuum many six-dimensional semialgebraic metric spaces up to quasi-isometry. This is a consequence of Pansu's theorem and the fact that there are continuum many six-dimensional Carnot groups up to isomorphism. The only two-dimensional Carnot group up to isomorphism is $\mathbb{R}^2$ equipped with the usual vector addition. We do not know how may two-dimensional semialgebraic metric spaces there are up to quasi-isometry. There are other notions of equivalence of metric space which are of interest to metric geometers, see [Sem01] or [Hei01]. One can pose analogues of Problem 13.0.4 for these notions of equivalence.

References

[Abe04] Herbert Abels. "Reductive groups as metric spaces." In *Groups: topological, combinatorial and arithmetic aspects*, volume 311 of *London Math. Soc. Lecture Note Ser.*, pp. 1–20. Cambridge Univ. Press, Cambridge, 2004.

[AG01] Andrei Agrachev and Jean-Paul Gauthier. "On the subanalyticity of Carnot-Caratheodory distances." *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **18**(3):359–382, 2001.

[AT13] Matthias Aschenbrenner and Athipat Thamrongthanyalak. "Whitney's Extension Problem in o-minimal structures." *Preprint*, 2013.

[BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2001.

[BCR98] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Real algebraic geometry*, volume 36 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)].* Springer-Verlag, Berlin, 1998. Translated from the 1987 French original, Revised by the authors.

[Bel96] André Bellaïche. "The tangent space in sub-Riemannian geometry." In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pp. 1–78. Birkhäuser, Basel, 1996.

[Bre14] Emmanuel Breuillard. "Geometry of locally compact groups of polynomial growth and shape of large balls." *Groups Geom. Dyn.*, **8**(3):669–732, 2014.

[Bro92] Ludwig Bröcker. "Families of semialgebraic sets and limits." In *Real algebraic geometry (Rennes, 1991)*, volume 1524 of *Lecture Notes in Math.*, pp. 145–162. Springer, Berlin, 1992.

[Bro94] Ludwig Bröcker. "On the reduction of semialgebraic sets by real valuations." In *Recent advances in real algebraic geometry and quadratic forms (Berkeley, CA, 1990/1991; San Francisco, CA, 1991)*, volume 155 of *Contemp. Math.*, pp. 75–95. Amer. Math. Soc., Providence, RI, 1994.

[BS00] Mario Bonk and Oded Schramm. "Embeddings of Gromov hyperbolic spaces." *Geom. Funct. Anal.*, **10**(2):266–306, 2000.

[CD83] Gregory Cherlin and Max A. Dickmann. "Real closed rings. II. Model theory." *Ann. Pure Appl. Logic*, **25**(3):213–231, 1983.

[CR00] Jean-Marie Comte, Georges and and Jean-Philippe Rolin. "Nature log-analytique du volume des sous-analytiques." *Illinois J. Math.*, **44**(4):884–888, 2000.

[Del82] Françoise Delon. *Quelques propriétés des corps valués en théorie del molèles.* 1982. Thèse, Paris.

[DL95] Lou van den Dries and Adam H. Lewenberg. "T-convexity and tame extensions." *J. Symbolic Logic*, **60**(1):74–102, 1995.

[Don11] Enrico Le Donne. "Metric spaces with unique tangents." *Ann. Acad. Sci. Fenn. Math.*, **36**(2):683–694, 2011.

[Dri97] Lou van den Dries. "T-convexity and tame extensions. II." *J. Symbolic Logic*, **62**(1):14–34, 1997.

[Dri98] Lou van den Dries. *Tame topology and o-minimal structures*, volume 248 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1998.

[Dri05] Lou van den Dries. "Limit sets in o-minimal structures." In *Proceedings of the RAAG Summer School Lisbon 2003: O-minimal structures*, pp. 172 – 215. 2005.

[DS02] Lou van den Dries and Patrick Speissegger. "O-minimal preparation theorems." In *Model theory and applications*, volume 11 of *Quad. Mat.*, pp. 87–116. Aracne, Rome, 2002.

[DW84] Lou van den Dries and Alex Wilkie. "Gromov's theorem on groups of polynomial growth and elementary logic." *J. Algebra*, **89**(2):349–374, 1984.

[Gro81] Mikhael Gromov. *Structures métriques pour les variétés riemanniennes*, volume 1 of *Textes Mathématiques [Mathematical Texts].* CEDIC, Paris, 1981. Edited by J. Lafontaine and P. Pansu.

[Gui73] Yves Guivarc'h. "Croissance polynomiale et périodes des fonctions harmoniques." *Bull. Soc. Math. France*, **101**:333–379, 1973.

[Har00] Pierre de la Harpe. *Topics in geometric group theory.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.

[Hei01] Juha Heinonen. *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York, 2001.

[HHM06] Deirdre Haskell, Ehud Hrushovski, and Dugald Macpherson. "Definable sets in algebraically closed valued fields: elimination of imaginaries." *J. Reine Angew. Math.*, **597**:175–236, 2006.

[Hru12] Ehud Hrushovski. "Stable group theory and approximate subgroups." *J. Amer. Math. Soc.*, **25**(1):189–243, 2012.

[HW41] Witold Hurewicz and Henry Wallman. *Dimension Theory.* Princeton Mathematical Series, v. 4. Princeton University Press, Princeton, N. J., 1941.

[HW15] Ehud Hrushovski and Erik Walsberg. "Definable sets of imaginaries in T-convex Structures." *Preprint*, 2015.

[Joh01] Joseph Johns. "An open mapping theorem for o-minimal structures." *J. Symbolic Logic*, **66**(4):1817–1820, 2001.

[KL14] Kyle Kinneberg and Enrico Le Donne. "A Metric Characterization of Snowflakes of Euclidean Spaces." *Preprint*, 2014.

[KM97] Marek Karpinski and Angus Macintyre. "o-Minimal expansions of the real field: A characterization and an application to pfaffian closure." *Preprint*, 1997.

[KPV14] Beata Kocel-Cynk, Wiesław Pawłucki, and Anna Valette. "A short geometric proof that Hausdorff limits are definable in any o-minimal structure." *Adv. Geom.*, **14**(1):49–58, 2014.

[LP01] Daniel Lascar and Anand Pillay. "Hyperimaginaries and automorphism groups." *J. Symbolic Logic*, **66**(1):127–143, 2001.

[LR97] Jean-Marie Lion and Jean-Philippe Rolin. "Théorème de préparation pour les fonctions logarithmico-exponentielles." *Ann. Inst. Fourier (Grenoble)*, **47**(3):859–884, 1997.

[LS04] Jean-Marie Lion and Patrick Speissegger. "A geometric proof of the definability of Hausdorff limits." *Selecta Math. (N.S.)*, **10**(3):377–390, 2004.

[Mat95] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[Mel06] Timothy Mellor. "Imaginaries in real closed valued fields." *Ann. Pure Appl. Logic*, **139**(1-3):230–279, 2006.

[Mil94] Chris Miller. "Expansions of the real field with power functions." *Ann. Pure Appl. Logic*, **68**(1):79–94, 1994.

[Mon02] Richard Montgomery. *A tour of subriemannian geometries, their geodesics and applications*, volume 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[MS] Jana Maříková and Masahiro Shiota. "Measuring definable sets in o-minimal fields." *to appear in Israel J. Math.*

[MS94] David Marker and Charles I. Steinhorn. "Definable types in O-minimal theories." *J. Symbolic Logic*, **59**(1):185–198, 1994.

[Pan89] Pierre Pansu. "Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un." *Ann. of Math. (2)*, **129**(1):1–60, 1989.

[Pil88] Anand Pillay. "On groups and fields definable in *o*-minimal structures." *J. Pure Appl. Algebra*, **53**(3):239–255, 1988.

[Roc70] Ralph Tyrrell Rockafellar. *Convex analysis.* Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[Sap15] Mark Sapir. "On groups with locally compact asymptotic cones." *Internat. J. Algebra Comput.*, **25**(1-2):37–40, 2015.

[Sem01] Stephen Semmes. *Some novel types of fractal geometry.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2001.

[Sim14] Pierre Simon. *A Guide to NIP theories.* to be published in Lecture Notes in Logic, 2014.

[Spa66] Edwin H. Spanier. *Algebraic topology.* McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966.

[Spe99] Patrick Speissegger. "The Pfaffian closure of an o-minimal structure." *J. Reine Angew. Math.*, **508**:189–211, 1999.

[Tho12] Margaret E. M. Thomas. "Convergence results for function spaces over o-minimal structures." *J. Log. Anal.*, **4**:Paper 1, 14, 2012.

[Val08] Guillaume Valette. "On metric types that are definable in an o-minimal structure." *J. Symbolic Logic*, **73**(2):439–447, 2008.

[Woe96] Arthur Anderson Woerheide. *O-minimal homology.* ProQuest LLC, Ann Arbor, MI, 1996. Thesis (Ph.D.)–University of Illinois at Urbana-Champaign.

[YC04] Yosef Yomdin and Georges Comte. *Tame geometry with application in smooth analysis*, volume 1834 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2004.