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Sven Gösta Nilsson

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Abstract

The study of the effect of nuclear structure on internal conversion accompanying gamma-transitions of electric multipole type which is presented here is analogous to a treatment by Church and Weneser (Ref. 3) dealing with magnetic dipole transitions. The anomalous matrix elements found here, corresponding to the situation when the electron penetrates the nucleus, are of the form expected classically. The magnitude of these anomalous terms for the studied E1 case seems unfortunately to be somewhat dependent on more detailed assumptions about the Coulomb field inside the nucleus. Only terms up to the second order in a perturbation expansion in the electromagnetic field are considered here as in other published treatments of internal conversion. In view of an occurring cancellation, discussed in some detail in this work, this may not in all cases be a sufficiently accurate approximation.

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Introduction

The original calculations by Rose et al.[1] of conversion coefficients were based on the assumption of a point nucleus. Later Sliv et al.[2] merely calculated the corrections due to the change in the electron wave functions occurring when the assumption of a nuclear point charge is replaced by the assumption that nuclear charge and currents are distributed over a finite nuclear surface. Church and Weneser[3] have furthermore pointed out an additional effect of finite nuclear size that depends on the detailed intrinsic nuclear structure. This correction is associated with the fact that the electron may penetrate inside the nuclear surface. In this case there occur nuclear matrix elements for the process of internal conversion that are different from the nuclear matrix elements of gamma decay. This effect will on the average be small, as the electron has only a very small probability of being inside the nucleus.

However, Church and Weneser, who treated the case of M1 transitions, suggested that for nuclei where the M1 gamma transitions due to special nuclear selection rules may be particularly hindered, the anomalous nuclear matrix elements (electron inside nuclear surface) may become significant. The experimental evidence for this effect in M1 transitions so

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*On leave of absence from the University of Lund, Lund, Sweden.

[1]Cf., e.g., Rose, Goertzel, Spinrad, Harr, and Strong, Phys. Rev. 83, 79 (1951).

[2]L. A. Sliv and I. M. Band, Coefficients of Internal Conversion of Gamma Radiation, Acad. of Sciences, USSR, Moscow Leningrad 1956, issued in U.S.A. as Report 57 ICCKI, Phys. Dept., Univ. of Ill., Urbana, Ill.

[3]E. L. Church and J. Weneser, Phys. Rev. 104, 1382 (1956).

far seems meager.[4] The case seems to be that very few M1 transitions are hindered by more than a factor of 10^3 (except for so-called K hindrance, in which case usually also the anomalous element is small), and that the nuclear selection rules (e.g. corresponding to nonoverlap) that make the normal matrix elements small also weaken the anomalous matrix elements.

On the other hand, it has been found experimentally[5] that certain E1 transitions in the heavy-element region of strongly deformed nuclei, which particular transitions all are hindered by a factor 10^6 or more in comparison with the Moszkowski single-proton estimate,[6] also have conversion coefficients that deviate from those of Sliv's and Rose's by as much sometimes as a factor of 20. In all these experimental cases it is found that the E1 hindrance is due to selection rules in some particular quantum numbers appropriate in describing nucleonic states of strongly deformed nuclei.[7] It is furthermore found that, on the basis of the same quantum numbers, the anomalous matrix elements appear mainly unweakened.[4] At least qualitatively this effect thus seems rather well accounted for.

The purpose of this paper is mainly to study how these anomalous matrix elements occur in electric transitions.[8] The analysis of the experimental findings on the basis of the nucleonic quantum numbers appropriate to deformed nuclei is undertaken in a paper by S. G. Nilsson and J. O. Rasmussen (to appear in the Physical Review).[4]

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[4] Cf, however, S. G. Nilsson and J. O. Rasmussen, Phys. Rev. (forthcoming).

[5] Asaro, Hollander, Perlman, and Stephens, Phys. Rev. (forthcoming).

[6] S. A. Moszkowski, Theory of Multipole Radiation, in Beta- and Gamma-Ray Spectroscopy, ed. K. Siegbahn, (North Holland Publishing Co., Amsterdam, 1955), p. 273.

[7] S. G. Nilsson, Dan. Mat. Fys. Medd. 29, no 16 (1955).

[8] Calculations of a similar type have lately been reported in progress by T. Green and M. E. Rose in Bull. Am. Phys. Soc. Vol 2; 4 (1957).

General Formulation of the Conversion Problem for Electric Transition

Interaction of electrons and nucleons with the electromagnetic field

In the Coulomb gauge the interaction between the electrons and the nucleons via the electromagnetic field may be expressed as[9, 10]

$$H' = H^S + H^C, \quad \text{where} \tag{1}$$

$$H^C = \sum_n \frac{e_n e_E}{|\vec{r}_n - \vec{r}_E|} - v^{\text{av. Coul.}}(\vec{r}_E), \quad \text{and} \tag{1a}$$

$$H^S = \sum_n H_n(\vec{A}) + H_E(\vec{A}), \quad \text{where in turn} \tag{1b}$$

$$H_n(\vec{A}) = -\frac{e_n}{2M}[\vec{p}_n \cdot \vec{A}(\vec{r}_n) + \vec{A}(\vec{r}_n)\cdot \vec{p}_n] - C_n\, e_n[\vec{r}_n \times \vec{A}(\vec{r}_n)]\cdot \vec{s}_n, = \vec{J}_N \cdot A \tag{1c}$$

$$H_E(\vec{A}) = e_E\, \vec{a}_E \cdot \vec{A}(\vec{r}_E). = \vec{J}_e \cdot \vec{A} \tag{1d}$$

For this gauge, the photon field $\vec{A}$ is purely transversal. The scalar and longitudinal photons appearing in, e.g., the Lorentz gauge are here accounted for by the direct Coulomb interaction term H^C. (One may notice in Eq. (1a) that, as the unperturbed electrons are assumed to move in a static Coulomb field $v^{\text{av. Coul.}}$, this average Coulomb field is subtracted out of the perturbation term (1a).)

The expression (1c) accounting for the interaction of the nucleons with the transverse photon field, also includes a term arising from the spin-orbit coupling $C\,\vec{l}\cdot\vec{s}$ usually assumed as an important feature of the unperturbed nuclear Hamiltonian. It is necessary to take such a term into

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[9] The treatment given here is similar to that of G. Kramer, Z. Phys. 146, 187 (1956) and 147, 628 (1957); and to some extent, the notation of that reference is also adopted.

[10] Cf. also Alder, Bohr, Huus, Mottelson, and Winther, Revs. Modern Phys. 28, 432 (1956).

account to preserve the gauge invariance of the interaction. Such a term was first studied by J. H. D. Jensen and M. Goeppert Mayer.[11] An additional term of the type

$$\mu_n \cdot \frac{e_n}{2M} \vec{\sigma} \cdot [\vec{\nabla} \times \vec{A}(\vec{r}_n)],$$

representing the coupling of the magnetic moment of the nucleon with the electromagnetic field, may for electric transitions be neglected, as for such transitions this term is small of the order $\frac{W}{M}$ compared with the terms considered. [W is the transition energy and M is the nucleon mass.]

Finally, Eq. (1d) gives the relativistic interaction of electrons with the photon field on the basis of the Dirac equation. The velocity operator $\vec{\alpha}_E$ equals $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{\sigma}$ in the two-component representation. (For further discussion cf. p. 12).

The charge parameter e_n equals + e for a proton and zero for a neutron. Furthermore we have the electron charge $e_E = -e$. The nucleon mass is denoted M. The units used in Eqs. (1) - (1d), and also employed in the following, are such that $m = c = \hbar = 1$, where m is the electron mass.

The vector potential $\vec{A}$ of the transverse photon field is expanded in electric and magnetic multipole components.

$$\vec{A}(\vec{r}) = \sum_{L, M, \tau} \int_0^\infty dk [q^{\tau}_{LM}(k) \vec{A}^{\tau}_{LM}(kr) + q^{*\tau}_{LM}(k) \vec{A}^{*\tau}_{LM}(kr)] \quad (2)$$

In this expression τ represents the summation over the independent dual electric and magnetic multipole fields. As we limit ourselves to electric transitions the sum over τ is left out in the following.

In the normalization applied for the electromagnetic multipoles the creation and annihilation operators q and q^* have the following nonvanishing matrix elements:

$$\langle n | q | n+1 \rangle = \langle n+1 | q^* | n \rangle = \sqrt{\frac{2\pi}{k}} \sqrt{n+1}. \quad (3)$$

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[11] J. H. D. Jensen and M. Goeppert Mayer, Phys. Rev. 85, 1040 (1952).

The electric multipole operator is normalized as

$$\vec{A}^{(e)}_{LM} = \frac{1}{\sqrt{(\pi/2)L(L+1)}} \vec{\nabla} \times (\vec{r} \times \vec{\nabla}) j_L(kr) Y_{LM}. \tag{4}$$

It is easy to verify that $\vec{\nabla} \cdot \vec{A}^{(e)}_{LM} = 0$, as required of a transverse field.

The probability for the nucleus to emit gamma rays under transition from a state ϕ_i to a state ϕ_f with an energy W lower than the initial state is given as (in lowest order perturbation expansion, in which terms of order e^2 in T_γ are retained)

$$T_\gamma = 2\pi \sum_{n=1}^{A} \frac{2\pi}{W} \left| \int \phi_f^* \; H'_n \; \phi_i \; d\tau_n \right|^2, \tag{5}$$

where the factor $\frac{2\pi}{W}$ originates from the matrix elements of the photon field, Eq. (3).

The competing process of internal conversion now consists of removing an electron from the bound state ψ_i and ejecting it into the (free) state ψ_f, at the same time changing the nuclear state from ϕ_i to ϕ_f. The probability for this process may be written (when terms of lowest order e^4 are retained in T_e)

$$T_e = 2\pi \left| U_{fi}(W) \right|^2, \tag{6}$$

where

$$U_{fi}(W) = H^C_{fi}(W) + \sum_\nu (H^S_{f\nu} H^S_{\nu i})/(E^n_f + E^E_f - E_\nu). \tag{7}$$

The first term in (7) corresponds to a direct effect of the static Coulomb field. The second term, being a second-order term in this perturbation treatment, corresponds to two-step processes involving exchange of virtual photons between the electron and the nucleon. The energy of the initial state is written $E^n_i + E^E_i$, while the energy of the intermediate state including the energy of the emitted photon, is written summarily E_ν. The quantity W represents the energy difference between initial and final nuclear state $(E^n_i - E^n_f)$ or between final and initial electron state $(E^E_f - E^E_i)$.

Thus, in more detail, we have

$$H_{fi}^{C}(W) = \sum_{n=1}^{A} \int \psi_f^* \phi_f^* \frac{e_n e_E}{|\vec{r}_n - \vec{r}_E|} \psi_i \phi_i \, d\tau_E \, d\tau_n . \qquad (8)'$$

As $v^{\text{av. Coul}}\left(\vec{r}_E\right)$ depends only on the electron coordinate $\vec{r}_E$, it cannot connect different nuclear states, hence this term, though it appears in Eq. (1a), has been left out in (8).

The second term in Eq. (7) may be rewritten

$$H'' = \sum_{\nu} (H_{i\nu}^{S} H_{\nu f}^{S})/(E_f^n + E_f^E - E_\nu) = \sum_{LM} \sum_{n=1}^{A} \int_0^\infty dk \; e_n e_E \; \frac{2\pi}{k} \cdot$$

$$\cdot \left[\frac{\int \psi_f^* H_E(\vec{A}_{LM}) \psi_i \, d\tau_E \int \phi_f^* H_n(\vec{A}^*_{LM}) \phi_i \, d\tau_n}{W - k} + \right.$$

$$\left. + \frac{\int \psi_f^* H_E(\vec{A}^*_{LM}) \psi_i \, d\tau_E \int \phi_f H_n(\vec{A}_{LM}) \phi_i \, d\tau_n}{-W - k} \right] . \qquad (9)$$

Here $H_E(\vec{A}_{LM})$ is $e_E \vec{\alpha} \cdot \vec{A}_{LM}$, etc. The first term in parentheses in Eq. (9) corresponds to the situation that in the intermediate state a virtual photon of energy k has been emitted by the nucleus. The latter has in the process changed its state from ϕ_i to ϕ_f, corresponding to a loss of energy W. The second term corresponds to the alternative situation that the electron first emits the photon with energy k and in addition gains energy W.

Using the property $\vec{A}^*_{LM} = (-)^M \vec{A}_{L-M}$, one can easily contract the expression (9) to

$$H'' = -4\pi \int_0^\infty dk \sum_n e_n e_E \sum_{LM} \frac{(-)^M \int \psi_f^* H_E(\vec{A}_{LM}) \psi_i \, d\tau_E \int \phi_f^* H_n(\vec{A}_{L-M}) \phi_i \, d\tau_n}{W^2 - k^2} .$$

(10)

It turns out to be a major simplification of the problem to make a rearrangement of the matrix elements in Eq. (10) ahead of the k integration.

To this end it is convenient to rewrite A_{LM} as

$$\vec{A}^{(e)}_{LM} = \vec{A}'_{LM} + A''_{LM}, \tag{11}$$

where

$$\vec{A}'_{LM} = \frac{1}{\sqrt{(\pi/2)L(L+1)}} \vec{\nabla} \frac{\partial}{\partial r}(rj_L)Y_{LM} = \vec{\nabla} a'_{LM}, \tag{11a}$$

$$\vec{A}''_{LM} = \frac{1}{\sqrt{(\pi/2)L(L+1)}} k^2(\vec{r}\, j_L)Y_{LM}. \tag{11b}$$

(One may note that $\vec{\nabla}$ in Eq. (11a) operates only on $\vec{A}'_{LM}$ and not on the wave function.)

For the relativistic electron interaction Kramer[9] uses

$$\vec{\alpha} \cdot \vec{A}_{LM} = \vec{\alpha} \cdot \vec{\nabla} a'_{LM} = -i \left[\overset{o}{H}_E, a'_{LM} \right], \tag{12}$$

where $\overset{o}{H}_E$ is the unperturbed electron Hamiltonian and [,] denotes the commutator.

For the nuclear interaction, which we have here assumed nonrelativistic, we can derive a similar relation,

$$-\frac{1}{2M}\left\{(\vec{p}\cdot\vec{A}^{(e)}_{LM} + \vec{A}^{(e)}_{LM}\cdot\vec{p})_n\right\} - C_n(\vec{r}\times\vec{A}'_{LM})_n\cdot\vec{s}_n = -i\left[\overset{o}{H}_n, a'_{LM}\right], \tag{13}$$

where the unperturbed nuclear Hamiltonian is given as

$$\overset{o}{H}_n = \frac{1}{2M}\left(\vec{p}_n\right)^2 + C_n\vec{l}_n\cdot\vec{s}_n + V(\vec{r}_n), \tag{14}$$

if $V(\vec{r}_n)$ is assumed to be velocity independent.[12]

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[12] Indeed, if one assumes only that H_n is linear in $\vec{A}$, one can show generally that the interaction term in the "long wave length limit" may be written as the commutator: $-i\left[\overset{o}{H}_n, a'_{LM}\right]$.

Cf. R. G. Sachs and N. Austern, Phys. Rev. 81, 705 (1951).

In the nuclear matrix element of Eq. (10) one may leave out $\vec{A}''_{LM}$ altogether, as $kr \ll 1$ all over the nuclear volume. The essential contribution to the integral (10) comes from the pole $k = W$, where W in turn should be thought of as of the order of a few hundred kev.

Hence we have

$$\int \phi_f^* H_n(\vec{A}_{L-M})\phi_i \, d\tau_n = \frac{iW}{\sqrt{(\pi/2)L(L+1)}} \int \phi_f^* \frac{\partial}{\partial r}(rj_L) Y_{L-M} \phi_i \, d\tau_n. \tag{15}$$

In deriving the above relation Eq. (13) has been exploited.

Similarly, using Eq. (12), one can show

$$\int \psi_f^* H_E(\vec{A}_{LM})\psi_i \, d\tau_E = \frac{1}{\sqrt{(\pi/2)L(L+1)}} \left[\int \psi_f^* \left(-iW \frac{\partial}{\partial r}(rj_L) Y_{LM} + k^2 \vec{r} \cdot \vec{a} \; j_L Y_{LM}\right)\psi_i d\tau_E \right], \tag{16}$$

where j_L denotes $j_L(kr)$.

It is now convenient to perform the k integration in Eq. (10), using the relations[13]

$$\int_0^\infty \frac{j_L(kr_E) j_L(kr_n)}{W^2 - k^2} \, dk = -\frac{i\pi}{2W} \cdot j_L(Wr_<) h_L(Wr_>) + \frac{\pi r_<^L r_>^{-L-1}}{2W^2(2L+1)} \tag{17}$$

and

$$\int_0^\infty \frac{j_L(kr_E) j_L(kr_n) k^2}{W^2 - k^2} \, dk = -\frac{i\pi W}{2} j_L(Wr_<) \, h_L(Wr_>). \tag{18}$$

Here $r_<$ and $r_>$ denote respectively the smaller and the larger of the electron and nucleon coordinates. Furthermore, h_L is the spherical Hankel function of the first kind.

The total H'' may then be split up into two terms:

$$H'' = H''(1) + H''(2), \tag{19}$$

where the second part of Eq. (17) contributes to $H''(2)$, and where furthermore

$$H''(2) = -\sum_{n=1}^{A} \sum_{LM} (-)^M \frac{4\pi}{2L+1} \int \psi_f^* \phi_f^* \frac{r_<^L}{r_>^{L+1}} Y_{LM}(\vec{r}_E) Y_{L-M}(\vec{r}_n) \psi_i \phi_i \, d\tau_E \, d\tau_n. \tag{20}$$

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[13] G. N. Watson, Theory of Bessel Functions (Cambridge Univ. Press, New York, 1944) second ed., pp. 405, 429. For the prescriptions of perturbation theory concerning integration over the pole see P. A. M. Dirac, Quantum Mechanics (Oxford, 1947) third ed.

(For $H''(1)$ see Eq. (22).) On the other hand, expanding $H^C_{fi}(W)$ from Eq. (8), one obtains

$$H^C_{fi}(W) = \sum_{n=1}^{A} \sum_{L=0}^{\infty} \sum_{M} (-)^M \frac{4\pi}{2L+1} \int \psi_f^* \phi_f^* \frac{r_<^L}{r_>^{L+1}} Y_{LM}(\vec{r}_E) Y_{L-M}(\vec{r}_n) \psi_i \phi_i \, d\tau_E d\tau_n . \tag{21}$$

It is thus found that in the expression for $U_{fi}(W)$ given by Eq. (7) the multipole terms of $H''(2)$ cancel the multipole terms of H^C_{fi} one by one, apart from the monopole term in H^C_{fi}, which has no counterpart in the terms of H''. This corresponds physically to the fact that the transverse photons always carry angular momentum and that therefore a monopole transition can only take place with the help of the longitudinal and scalar photons, or equivalently expressed in terms of the gauge applied here, with the help of the static Coulomb field.

In the following we will leave out the monopole term. The remaining terms of $U_{fi}(W)$ are then all contained in $H^{II}(1)$:

$$H''(1) = -4\pi \sum_n e_n e_E \sum_{LM} \frac{(-)^M}{L(L+1)} \frac{1}{iW} \left[\int_0^\infty d\tau_E \psi_f^* O_E(h_L) \psi_i \int_0^{r_E} d\tau_n \phi_f^* O_n(j_L) \phi_i + \right.$$

$$\left. \int_0^\infty d\tau_n \phi_f^* O_n(h_L) \phi_i \int_0^{r_n} d\tau_E \psi_f^* O_E(j_L) \psi_i \right], \tag{22}$$

where

$$O_E(j_L) = \left\{ -iW \frac{\partial}{\partial r} \left[r j_L(Wr) \right] + \vec{\alpha} \cdot \vec{r} \, W^2 j_L(Wr) \right\} Y_{LM}, \tag{23}$$

$$O_n(j_L) = iW \frac{\partial}{\partial r} \cdot \left[r j_L(Wr) \right] Y_{L-M} . \tag{24}$$

Here the notation $\int_0^{r_n} d\tau_E$ implies a complete integration over all the electron angles, but an integration in r_E only from 0 to r_n, etc.

The expression (22) may be conveniently rewritten

$$H''(1) = -4\pi \sum_n e_n e_E \sum_{LM} \frac{(-)^M}{L(L+1)} \frac{1}{iW} \left\{ \int_0^\infty d\tau_E \, \psi_f^* O_E(h_L)\psi_i \cdot \int_0^\infty d\tau_n \, \phi_f^* O_n(j_L)\phi_i \right.$$

$$+ \int_0^\infty d\tau_n \phi_f^* O_n(h_L)\phi_i \int_0^{r_n} d\tau_E \, \psi_f^* O_E(j_L)\psi_i$$

$$\left. - \int_0^\infty d\tau_n \phi_f^* O_n(j_L)\phi_i \int_0^{r_n} d\tau_E \, \psi_f^* O_E(h_L)\psi_i \right\}. \qquad (25)$$

Evaluation of matrix elements involving electron wave functions

The next problem is to evaluate terms of the type

$$Q(r) = \int_0^r r_E^2 dr_E W \int_0^{2\pi} \int_0^\pi d\Omega_E \psi_f^* \left[\vec{a} \cdot \vec{r} \, W\zeta_L - i \frac{\partial}{\partial r}(r\zeta_L) \right] Y_{LM}\psi_i$$

Here ζ_L denotes alternatively j_L or h_L. The relativistic electron wave function corresponding to a central electrostatic field may be written

$$\psi = \begin{pmatrix} -if_\kappa \, \chi^\mu_{-\kappa} \\ g_\kappa \, \chi^\mu_\kappa \end{pmatrix} \qquad (26)$$

in the notation employed in, e.g., Rose's Multipole Fields.[14] The so-called small and large component radial functions are defined by

$$\frac{df_\kappa}{dr} = \frac{\kappa - 1}{r} f_\kappa - (E - v - 1) g_\kappa , \qquad (26a)$$

[14] M. E. Rose, Multipole Fields (New York 1955).

$$\frac{dg_\kappa}{dr} = (E - v + 1) f_\kappa - \frac{\kappa+1}{r} g_\kappa , \tag{26b}$$

where the electrostatic potential v(r) is later specified. For states with $\vec{\ell}$ and $\vec{s}$ "parallel" κ equals $-\ell-1$, and for states with $\vec{\ell}$ and $\vec{s}$ antiparallel κ equals ℓ. Furthermore, we have

$$\chi_\kappa^\mu = \sum_{m m_s} C^{\ell\ s\ j}_{m\ m_s \mu} F^s_{m_s} Y_{\ell m} \tag{27}$$

where $C^{\ell\ s\ j}_{m\ m_s \mu}$ are the vector coupling coefficients, adding $\vec{\ell}$ and $\vec{s}$ to a vector $\vec{j}$, and where $F^s_{m_s}$ is the Pauli spin function.

In this two-component representation we have

$$\vec{a} \cdot \vec{r} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{\sigma} \cdot \vec{r} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} r\sigma_r \tag{28}$$

where σ_r fulfills the relation

$$\sigma_r \chi_\kappa^\mu = -\chi_{-\kappa}^\mu \tag{29}$$

Thus σ_r changes, e.g., the angular eigenfunction $s_{1/2}$ into $p_{1/2}$, and in addition changes the sign.

Using Eqs. (28) and (29) one can show, employing the methods used by Rose in Ref. 14,

$$Q(r) = -iW \left\langle \kappa\mu \left| Y_{LM} \right| \kappa'\mu' \right\rangle \cdot S(r, \zeta_L) \tag{30}$$

where

$$S(r, \zeta_L) = \int_0^r r_E^2 dr_E \left\{ W r_E \zeta_L (-g_\kappa f_{\kappa'} + f_\kappa g_{\kappa'}) + \left[\frac{d}{dr_E} (r_E \zeta_L) \right] (g_\kappa g_{\kappa'} + f_\kappa f_{\kappa'}) \right\} , \tag{31}$$

where the primed coordinates denote the initial state and the unprimed the final state. For $x \to \infty$ this expression is equivalent, as shown by Kramer, to

$$S(\infty, \zeta_L) = \int r^2 dr \left[-i(\kappa - \kappa') \zeta_{L-1} (g_\kappa f_{\kappa'} + f_\kappa g_{\kappa'}) - iL \left\{ \zeta_{L-1} (g_\kappa f_{\kappa'} - f_\kappa g_{\kappa'}) + \zeta_L (g_\kappa g_{\kappa'} + f_\kappa f_{\kappa'}) \right\} \right] \tag{32}$$

which is the expression derived by Rose on the basis of point source fields.[1,14]

We now proceed to calculate N_γ, the number of photons emitted under change of nuclear state $\phi_i \rightarrow \phi_f$. From now on we simplify the notation by limiting ourselves to the case of single-particle transitions, i.e., we drop the sum over n. Using Eq. (5), one then obtains[15]

$$N_\gamma = \frac{4\pi^2 e_n^2}{W} \frac{1}{2I'+1} \cdot \sum_{mm'} \left| \sum_{LM} \frac{iW}{\sqrt{\frac{\pi}{2} L(L+1)}} \left\langle Im \left| \frac{\partial}{\partial r} (r j_L) Y_{L-M} \right| I'm' \right\rangle \right|^2$$

$$= \sum_L \frac{8\pi e_n^2}{L(L+1)} \frac{W}{2I'+1} \left| \left\langle I \left\| \frac{\partial}{\partial r} (r j_L) Y_L \right\| I' \right\rangle \right|^2 \qquad (33)$$

Here $\langle \| \ \| \rangle$ denotes the reduced matrix element in the usual definition.[16] In calculating N_γ we have summed over final nuclear substates m and averaged over initial nuclear substates m'.

The number of electrons emitted per second under change of the nuclear state from $\phi_i \rightarrow \phi_f$ is given from Eq. (6) as

$$N_e = \frac{2\pi}{2I'+1} \sum_{mm'\mu\mu'} \left| H''(1) \right|^2 \qquad (34)$$

where we sum over initial and final electron substates μ' and μ, average over initial nuclear m-states, and sum over final nuclear substates m'. (That we sum and not average over initial electron substates corresponds to the fact that all electron substates are occupied, and we do not care which of the substate electrons is emitted.)

Rearranging (25) we write:

$$H''(1) = 4\pi \cdot e_n e_E \sum_L \frac{1}{L(L+1)} \cdot \sum_M (-)^M \left\langle \kappa\mu \left| Y_{LM} \right| \kappa'\mu' \right\rangle \cdot \left\langle Im \left| X_{LM} \right| I'm' \right\rangle \qquad (35)$$

---

[15] This may easily be transformed into the familiar expression for the transition probability: (where M = m-m')

$$N_\gamma = \sum_L \frac{8\pi(L+1)}{L[(2L+1)!!]^2} W^{2L+1} \cdot \sum \left| \left\langle Im \left| e_n r^L Y_{L-M} \right| I'm' \right\rangle \right|^2$$

sum over final states, average over initial states

[16] G. Racah, Phys. Rev. 62, 438 (1942).

where

$$\left\langle Im \left| X_{LM} \right| I'm' \right\rangle = S(\infty, h_L) \left\langle Im \left| O_n(j_L) \right| I'm' \right\rangle + \left\langle Im \left| O_n(h_L)\, S(r, j_L) - O_n(j_L)\, S(r, h_L) \right| I'm' \right\rangle , \tag{35a}$$

where S and O_n are defined in Eqs. (31) and (24).

Thus

$$N_e = 32\pi^3 e_n^2 e_E^2 \sum_{LL'} \frac{1}{L(L+1)L'(L'+1)} \cdot \sum_{\substack{MM'mm' \\ \mu\mu'}} \frac{1}{(2I+1)(2j+1)} C^{L\,I'\,I}_{-Mm'\,m} C^{L'\,I'\,I}_{-M'm'm} C^{L\,j'j}_{M\mu'\mu} C^{L'j'j}_{M'\mu'\mu} \cdot \left\langle I \| X_L \| I' \right\rangle \cdot \left\langle \kappa \| Y_L \| \kappa' \right\rangle =$$

$$32\,\pi^3 e_n^2 e_E^2 \sum_L \frac{1}{L^2(L+1)^2} \cdot \frac{1}{2I'+1} \cdot \frac{1}{2L+1} \cdot \left\langle I \| X_L \| I' \right\rangle \left\langle \kappa \| Y_L \| \kappa' \right\rangle . \tag{36}$$

The "partial" conversion coefficient corresponding to a transition from electron state κ' to electron state κ can now be calculated from (33) and (36)

$$\alpha_{\kappa\kappa'}(EL) = \frac{N_e(L)}{N_\gamma(L)} = \overset{o}{\alpha}_{\kappa\kappa'} \left| 1 + \frac{\lambda}{S(\infty, h_L)} \right|^2 \tag{37}$$

where

$$\overset{o}{\alpha}_{\kappa\kappa'}(EL) = \frac{\pi e^2 W}{L(L+1)(2L+1)} (2j+1)(2\ell+1)(2j'+1)(2\ell'+1) \left(C^{\ell'\,\ell\,L}_{0\;\;0\;\;0} \right)^2 \cdot W^2(\ell j \ell' j'; \tfrac{1}{2} L) S^2(\infty, h_L) \tag{38}$$

$W(\ell j \ell' j'; \frac{1}{2} L)$ is the Racah coefficient in the conventional definition.[16] The quantity $S(\infty, h_L)$ equals $S(r, h_L)$ with $r = \infty$.

Equation (38) should be compared with Ref. 14. The sum $\sum_\kappa \overset{o}{\alpha}_{\kappa\kappa'}$ is equal to the conversion coefficient calculated by Rose on the basis of a nuclear point source, provided electron wave functions appropriate to a

central point charge are used in calculating $S(\infty, h_L)$. An improved value of $\overset{o}{a}_{\kappa\kappa'}$ is, however, obtained by the use of electron wave functions appropriate to a finite charge distribution. Finally

$$\lambda = \frac{\left\langle I \left|\left| O_n(h_L) \cdot S(r, j_L) - O_n(j_L) S(r, h_L) \right|\right| I' \right\rangle}{\left\langle I \left|\left| O_n(j_L) \right|\right| I' \right\rangle} \quad . \tag{39}$$

From now on we restrict ourselves to the case L = 1. To calculate λ we use the "long wave length" expansions:

$$j_1 \tilde{=} \frac{Wr}{3} , \tag{40a}$$

$$h_1 \tilde{=} -i \frac{1}{(Wr)^2} \quad . \tag{40b}$$

Thus

$$S(r, j_1) = \int_0^r r^2 dr \left[\frac{W^2 r^2}{3} (f_\kappa g_{\kappa'} - g_\kappa f_{\kappa'}) + \frac{2Wr}{3} (f_\kappa f_{\kappa'} + g_\kappa g_{\kappa'}) \right] \tag{41}$$

$$S(r, h_1) = \int_0^r r^2 dr \left[-i \frac{1}{Wr} (f_\kappa g_{\kappa'} - g_\kappa f_{\kappa'}) + i \frac{1}{(Wr)^2} (f_\kappa f_{\kappa'} + g_\kappa g_{\kappa'}) \right] \tag{42}$$

As the limit of integration corresponds to a distance less than or equal to the nuclear radius, the second term in Eqs (41) and (42) containing $f_\kappa f_{\kappa'} + g_\kappa g_{\kappa'}$ is expected to be highly dominant. However, its dominance is diminished for the case $\kappa = -\kappa'$ by a cancellation, and then the first term may be neglected only to the extent to which W can be treated as small compared with the electron rest mass. The first term is therefore retained for the present for the purpose of comparison.

To get an understanding of the mentioned cancellation, we now turn to the Dirac equations for an electron moving in a central field, Eq. (26a) and (26b). Corresponding to a homogeneous charge distribution inside the nucleus we assume an electrostatic potential inside the nuclear radius of the type[17]

$$v(r) = - \frac{e^2 Z}{2R} \left(3 - \frac{r^2}{R^2}\right) , \tag{43}$$

where R is the nuclear radius (in units used here ($R \cong 1.9 \times 10^{-2}$ for $A \cong 230$), and e the unit charge ($e^2 \cong \frac{1}{137}$ in these units). The depth of the Coulomb potential at the center of the nucleus is then $\cong 52\ mc^2$. It is useful to notice that $v(0) \cdot R = -\frac{3}{2} e^2 Z \cong -1$.

For such a potential one may find series expansions for the electron wave function inside the nucleus.[18] For $\kappa = + \kappa_o$ one has to lowest order in r

$$g_\kappa = {}^{o}g_\kappa \left(\frac{r}{R}\right)^{\kappa_o} + \ldots , \tag{44}$$

$$f_\kappa = {}^{o}f_\kappa \left(\frac{r}{R}\right)^{\kappa_o - 1} + \ldots , \tag{45}$$

with the additional relation

$${}^{o}g_\kappa = \frac{R}{2\kappa_o + 1} (E - v(0) + 1)\, {}^{o}f_\kappa . \tag{46}$$

For $\kappa = - \kappa_o$ the corresponding lowest-order terms are

$$g_\kappa = {}^{o}g_\kappa \left(\frac{r}{R}\right)^{\kappa_o - 1} , \tag{47}$$

$$f_\kappa = {}^{o}f_\kappa \left(\frac{r}{R}\right)^{\kappa_o} , \tag{48}$$

subject to the relation

$${}^{o}f_\kappa = - \frac{R}{2\kappa_o + 1} (E - v(0) - 1)\, {}^{o}g_\kappa . \tag{49}$$

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[17] Cf. L. A. Sliv, J. Exp. Theoret. Phys. 17, 1049 (1947).

[18] If one is interested only in the leading term, as we are here, the result depends only on the value of the potential at the origin v(0) and not on its detailed shape. Cf, however, Appendix B.

Thus for a transition from a state $\kappa' = -\kappa_o$ to a state $\kappa = \kappa_o$ we have *

$$f_\kappa f_{\kappa'} + g_\kappa g_{\kappa'} = \left(\frac{r}{R}\right)^{2\kappa_o - 1} {}^{o}_{f_\kappa} {}^{o}_{g_{\kappa'}} \frac{R}{2\kappa_o + 1} [2 + W + v'(0) - v(0)] \tag{50a}$$

Correspondingly, for $\kappa' = \kappa_o = -\kappa$,

$$f_\kappa f_{\kappa'} + g_\kappa g_{\kappa'} = \left(\frac{r}{R_o}\right)^{2\kappa_o - 1} {}^{o}_{g_\kappa} {}^{o}_{f_{\kappa'}} \frac{R}{2\kappa_o + 1} [2 - W - v'(0) + v(0)] \,. \tag{50b}$$

The term $f_\kappa g_{\kappa'} - g_\kappa f_{\kappa'}$ will be given for comparison here, even though the contribution to Eqs. (40)-(42) derived from this term will be neglected as small of order W compared with the electron rest mass.

For $\kappa' = -\kappa_o = -\kappa$ the following relation holds:

$$f_\kappa g_{\kappa'} - g_\kappa f_{\kappa'} = \left(\frac{r}{R}\right)^{2\kappa_o - 2} {}^{o}_{f_\kappa} {}^{o}_{g_{\kappa'}} \,. \tag{51a}$$

Correspondingly, for $\kappa' = \kappa_o = -\kappa$,

$$f_\kappa g_{\kappa'} - g_\kappa f_{\kappa'} = -\left(\frac{r}{R}\right)^{2\kappa_o - 2} {}^{o}_{g_\kappa} {}^{o}_{f_{\kappa'}} \,. \tag{51b}$$

We may now rewrite Eqs.(40)-(42) as:

for $\kappa = \kappa_o = -\kappa'$,

$$S(r, j_1) \cong \frac{2}{3} W \left(\frac{1}{R}\right)^{2\kappa_o - 2} \cdot \frac{1}{2\kappa_o + 1} \cdot \frac{1}{2\kappa_o + 3} {}^{o}_{f_\kappa} {}^{o}_{g_{\kappa'}} r^{2\kappa_o + 3} [2 + v'(0) - v(0)] , \tag{52}$$

$$S(r, h_1) \cong \frac{i}{W} 2 \left(\frac{1}{R}\right)^{2\kappa_o - 2} \cdot \frac{1}{2\kappa_o + 1} \cdot \frac{1}{2\kappa_o} {}^{o}_{f_\kappa} {}^{o}_{g_{\kappa'}} r^{2\kappa_o} [2 + v'(0) - v(0)] ; \tag{53}$$

and for $\kappa' = +\kappa_o = -\kappa$,

$$S(r, j_1) \cong \frac{2}{3} W \left(\frac{1}{R}\right)^{2\kappa_o - 2} \frac{1}{2\kappa_o + 1} \frac{1}{2\kappa_o + 3} {}^{o}_{g_\kappa} {}^{o}_{f_{\kappa'}} r^{2\kappa_o + 3} [2 - v'(0) + v(0)] , \tag{54}$$

*It is implicitly assumed in the conventional but not quite self consistent perturbation treatment presented here that the "unperturbed" Hamiltonian is the same in initial and final states. A generalization in line with the treatment of the problem of rearrangement collisions (see, L. I. Schiff, Quantum Mechanics, New York (1949)) would bear out the seeming conjecture behind Eqs. (50a, b). The author is indebted to Prof. G. Chew for a clarifying discussion on this point.

$$S(r, h_1) \cong \frac{i}{W^2}\left(\frac{1}{R}\right)^{2\kappa_o - 2} \frac{1}{2\kappa_o + 1} \cdot \frac{1}{2\kappa_o} \overset{o}{g}_{\kappa} \overset{o}{f}_{\kappa'} r^{2\kappa_o} [2 - v'(0) + v(0)] , \tag{55}$$

where Eqs. (54) and (55) might have been obtained from (52) and (53) by exchange of primed and unprimed quantities $\overset{o}{f}$, $\overset{o}{g}$ and $v(0)$.

The magnitude of $S(r, j_1)$ and $S(r, h_1)$ is thus greatly affected for the case $|\kappa'| = |\kappa|$ by the cancellation apparent in (50a) and (50b). It is thus found that – provided $|\kappa'| = |\kappa|$ – the sum $f_\kappa f_{\kappa'} + g_\kappa g_{\kappa'}$ is smaller by a factor $R(\approx 10^{-2})$ [19] then either of the two terms; i.e. the ratio

$$\frac{g_{-\kappa_o}}{f_{-\kappa_o}} \quad \text{is very nearly equal to} \quad -\frac{f_{\kappa_o}}{g_{\kappa_o}} \ .^{20}$$

A slight change in the effective electrostatic potential is, however, to be expected, as the nucleonic transition associated with the conversion process may cause a change in the nuclear charge distribution. Those terms of $S(r, \zeta_L)$ that are of lowest order in r will be affected mainly by a change in $v(0)$, the potential depth at the origin, as indicated in Eqs. (52) - (55). As shown in Appendix B, the terms of next higher order in r will in addition depend on changes in the "curvature" of the electrostatic potential assumed.

In summary, one may emphasize that the underlying assumptions in calculating the electron wave functions inside the nucleus imply an independent-particle description of the electrons, and imply a purely electrostatic potential of a simple shape, independent of electron and nuclear state, substituted into the Dirac equation.

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[19] As the cancellation corresponds to a weakening by an order of magnitude e^2 of the transition amplitude, it will probably be of interest to study higher-order perturbation terms in addition.

[20] Denoting $\frac{g_{-\kappa_o}}{f_{-\kappa_o}}$ by x and $\frac{f_{\kappa_o}}{g_{\kappa_o}}$ by y, one finds that x and y fulfill the equations

$$x' = x^2(E - v - 1) + x\frac{2\kappa_o}{r} + (E - v + 1),$$

$$y' = y^2(E - v + 1) + y\frac{2\kappa_o}{r} + (E - v - 1),$$

which thus are highly similar provided $|E - v| \gg 1$.

One might expect the cancellation effect to be relatively more sensitive to a relaxation of these conditions than the wave functions f_κ and g_κ themselves.

We rewrite the anomalous operator as[21]

$$O_n(h_1)S(r, j_1) - O_n(j_1)S(r, h_1) = r^{2\kappa_o+1} Y_{1-M} \frac{2}{3} F_{\kappa\kappa'} C_{corr} \quad . \qquad (56)$$

Values of $F_{\kappa\kappa'}$ and C_{corr} are listed in Table I. What is said above serves to emphasize the uncertainty of the factor C_{corr} for cases $|\kappa| = |\kappa'|$.

Table I

Values of $F_{\kappa\kappa'}$ and C_{corr} for some particular initial and final electron states.

Transition	κ'	κ	κ_o	$F_{\kappa\kappa'}$	C_{corr}
$s_{1/2} \rightarrow p_{1/2}$	-1	1	1	$\frac{1}{5} f_\kappa g_{\kappa'}$	$1 + \frac{v'(0)-v(0)}{2m}$
$p_{3/2} \rightarrow d_{3/2}$	-2	2	2	$\frac{3}{70} f_\kappa g_{\kappa'}$	
$p_{1/2} \rightarrow s_{1/2}$	1	-1	1	$\frac{1}{5} g_\kappa f_{\kappa'}$	$1 - \frac{v'(0)-v(0)}{2m}$
$s_{1/2} \rightarrow p_{3/2}$	-1	-2	1	$\frac{3}{10} \cdot \frac{1}{R} \cdot g_\kappa g_{\kappa'}$	1
$p_{3/2} \rightarrow s_{1/2}$	-2	-1	1		
$p_{1/2} \rightarrow d_{3/2}$	1	2	1	$\frac{3}{10} \cdot \frac{1}{R} f_\kappa f_{\kappa'}$	

[21] For $|\kappa| = |\kappa'|$, κ_o is defined as $|\kappa|$. For $|\kappa| \neq |\kappa'|$ we denote by κ_o the smallest of the two absolute values.

Evaluation of Nuclear Matrix Elements on the Basis of a Particular Nuclear Model

The last section dealt with an integration over the electron coordinates. Our next step consequently will be to integrate over the nucleon coordinates; in other words we will now study the nuclear matrix elements. This necessarily involves an assumption of a particular nuclear model.

As the El conversion anomalies are, so far, experimentally found in the regions of nuclei displaying features of a deformed nuclear shape, we will in the following limit ourselves to employing the nuclear wave functions of the unified model[22] that are appropriate for describing nuclei of large stable equilibrium deformations. It is found that these nuclei display a very simple coupling scheme. The large mass transport involved in the collective rotational motion (simple consequences of which are empirically encountered, e.g., in the occurrence of rotational energy bands in this region of nuclei) implies a low frequency for the latter mode of excitation. The rotational motion may thus occur with such a low frequency that its effect on the intrinsic nucleonic motion may to a good first approximation be neglected. The nuclear wave function may then be separated into two parts, one, D, describing the rotation of the system as a whole, one, X, describing the nucleonic motion with respect to a system fixed in the nucleus:[23]

$$\phi^{I}_{M} = D^{I}_{M K_o}(\alpha, \beta, \gamma)\ X_{K_o}(r''), \tag{57}$$

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[22] A. Bohr, Dan. Mat. Fys. Medd. 26, No. 14 (1952;)

A. Bohr and B. R. Mottelson, Dan. Mat. Fys. Medd. 27, No. 16 (1953). For the later modifications, see ref (10), where also a complete list of references is provided.

[23] To assure a definite parity the total wave function has to be symmetrized according to a prescription in Ref. 22. This symmetrization is of importance only for the case $K = 1/2$.

where α, β, γ are the eulerian angles defining the orientation of the nuclear symmetry axes with respect to the space-fixed system, and where I, M, and K are the total angular momentum, its projections on the space-fixed z-axis and on the nuclear symmetry axis, respectively. Furthermore, $\vec{r}''$ refers to the nucleonic coordinates defined with respect to a body fixed coordinate system. In this so-called adiabatic approximation the additional forces of the noninertial rotating system, Coriolis forces and centrifugal forces, have been neglected. The effect of Coriolis forces (which is the most important neglected effect) is to admix to the first order states of the same I, but with K different from K_o by one unit (and to higher order still more different K values). We may formally account for this effect by modifying Eq. (57):

$$\phi^{I}_{M} = \sum_{i} \epsilon_i \, D^{I}_{M K_i} \, X_{K_i} \qquad (57a)$$

Here $\epsilon_o \cong 1$, while all other ϵ_i are small. The smallness of ϵ_i provides a measure of the purity of the adiabatic approximation.[24]

The intrinsic wave function X_{K_o} is particularly simple in the coupling scheme appropriate to deformed nuclei.[o] The deformation of the nuclear field removes the degeneracy of the spherical shell-model states, apart from a twofold degeneracy corresponding to the retention of rotational symmetry. The nucleons now fill up the doubly degenerate levels pairwise. The angular momentum components along the nucleon axis (the relevant quantity) of a pair of nucleons cancel.

In an odd-A nucleus the nucleonic properties are essentially given by the wave function of the last odd nucleon.

In the cases of anomalous E1 conversion coefficients empirically encountered we have presumably to do with cases in which only the last odd nucleon is involved in the conversion process. The wave function X_{K_o} may then be thought of as the single-particle wave function of this particular unpaired nucleon. Calculations of intrinsic wave functions X_{K_o} by several authors are now available.[25, 7, 26]

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[24] For the effects of this K-impurity, cf. Alaga, Alder, Bohr, and Mottelson, Dan. Mat. Fys. Medd. 29, No. 9 (1955).

[25] S. A. Moszkowski, Phys. Rev. 99, 803 (1955).

[26] K. Gottfried, Phys. Rev. 103, 1017 (1956).

In order to be able to conveniently estimate the magnitude of the correction term and to apply the tabulated wave functions of Ref. 7 directly, it is convenient to introduce coordinates r' suitable to nuclear dimensions,

$$r'' = \sqrt{\frac{\hbar}{M\omega_o}} \cdot \frac{mc}{\hbar} \cdot r' \tag{58}$$

(which may be written $\frac{1}{\sqrt{M\omega_o}}$ r' in the units employed here). Here M is the proton mass and $\hbar\omega_o$, assumed equal to $\approx 80\, A^{-1/3}\, mc^2$, is the characteristic oscillator energy of the nuclear potential assumed in calculating the wave functions of Ref. 7.

We may now write considering the dominant anomalous conversion terms in K and L-conversion

$$\lambda = -\frac{i}{W^2} \cdot \frac{1}{M\omega_o} \cdot F_{\kappa\kappa'} \cdot x \tag{59}$$

where $F_{\kappa\kappa'}$ is given from Table I, and where

$$x = \frac{\left\langle X_{K_o} \left| C_{corr}\ r'^3\ Y_{1K_o-K'_o} \right| X_{K'_o} \right\rangle}{\left\langle X_{K_o} \left| r'\ Y_{1K_o-K'_o} \right| X_{K'_o} \right\rangle} . \tag{60}$$

provided the K impurity in the nuclear wave function (57a) is of negligible importance. However, one may generalize Eq. (60) to include the effect of K impurity. Such a modified expression for x reads

$$x = \frac{\sum_{ij} \epsilon_i \epsilon_j\ C^{I'}_{K'_j}\ {}^{1}_{K_i-K'_j}\ {}^{I}_{K_i} \left\langle X_{K_i}(r') \left| C_{corr}\ r'^3\ Y_{1K_i-K'_j} \right| X_{K'_j}(r') \right\rangle}{\sum_{ij} \epsilon_i \epsilon'_j\ C^{I'}_{K'_j}\ {}^{1}_{K_i-K'_j}\ {}^{I}_{K_i} \left\langle X_{K_i}(r') \left| r'\ Y_{1K_i-K'_j} \right| X_{K'_j}(r') \right\rangle} \tag{60a}$$

Using Eqs. (37) and (59) we now write the expression for the conversion coefficient in a more compact and final form

$$a_{\kappa'}(E1) = \sum_{\kappa} a_{\kappa\kappa'}(E1) = \sum_{\kappa} {}^{o}a_{\kappa\kappa'} \left| 1 - x \frac{i}{W^{3/2}} \cdot \frac{M_{\kappa\kappa'}}{e^{i\delta}\sqrt{{}^{o}a_{\kappa\kappa'}}} \right|^2 \tag{61}$$

The constant $M_{\kappa\kappa'}$, is given in Table II for the dominant terms of K and L conversion. It is calculated for the case Z = 91, A = 231 and should be sufficiently accurate for all the actinide nuclei. The numerical estimates of ${}^{o}f$ and ${}^{o}g$, on which the numerical values of $M_{\kappa\kappa'}$ are based, are discussed in Appendix A. In view of the discussion there it is apparent that the values of $M_{\kappa\kappa'}$ are only roughly approximate. Furthermore, ${}^{o}a_{\kappa\kappa'}$ is the partial conversion coefficient defined in Eq. (38) appropriate to the hypothetical case of negligible anomalous matrix elements. As has been pointed out earlier, the coefficients $\sum_{\kappa} {}^{o}a_{\kappa\kappa'}$ equal Rose's published conversion coefficients,[1] provided point-source electron wave functions are employed. Sliv's refined values[2] correspond to (a) using electron wave functions corrected for finite-size effects, (b) taking the nuclear correction term x into account in an average way on the basis of an assumption that nuclear currents are all confined to the nuclear surface. The results correspond formally to setting $x = R^2$, where the nuclear radius R is measured in units $\frac{\hbar}{\sqrt{M\omega_o}}$. No values of the partial conversion coefficients ${}^{o}a_{\kappa\kappa'}$ are at present available. The amplitude $\sqrt{{}^{o}a_{\kappa\kappa'}}$, furthermore, enters with a phase $e^{i\delta}$, the value of which is also unpublished. (This phase equals the phase of the quantity $S(\infty, h_L)$ defined in Eq. (32)).

The quantity x measures the strength of the anomalous matrix element r^3Y_1 compared with the $r\,Y_1$ matrix element. For a hypothetical completely unhindered case both have the order of magnitude one. In view of the smallness of the quantity $M_{\kappa\kappa'}$, x need to take on very large values in order for the effect of the anomalous terms to be pronounced. A large x requires a very small rY_1 matrix element; i.e., the gamma transition should be very hindered. As the angular dependence of both operators (rY_1 and r^3Y_1) is the same, it is apparent that a hindrance due to the K- selection rule (See Ref. 24) will in general not lead to a large x. However, the selection

rules due to nucleonic structure (and to a considerable extent borne out by the wave functions of e.g. Ref. 7 and understood largely in terms of the so-called asymptotic quantum numbers[7]) that effectively hinder most E1 transitions[27, 28, 29] in the region of odd-A elements in the rare earth region and beyond Ac are expected to be greatly relaxed for the r^3Y_1 operator. Cf. the scheme of selection rules displayed in Table III of Ref. 4.

A preliminary analysis of the experimentally anomalous conversion coefficients, reported by Asaro et al.[5] has been prepared for the Physical Review by S. G. Nilsson and J. O. Rasmussen.[4] In all the anomalous cases discussed there it appears that the rY_1 matrix element is empirically small by a factor 10^{-3} while correspondingly the r^3Y_1 matrix element appears unweakened or less weakened than the rY_1 element in terms of the asymptotic quantum numbers.[7]

The uncertainty in the correction factor C_{corr} above and the absence of published values of $\overset{o}{a}_{KK}$ makes a more detailed comparison difficult; except maybe in a particular case of exceedingly large experimental deviations from Rose's or Sliv's values (in which case the second term in Eq. (61) becomes dominant). This case seems to indicate either that the correction factors C_{corr} are unexpectedly large or that terms of higher order in the perturbation expansion, Eq. (7), are also of importance; Cf. Ref. 19.

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[27] S. G. Nilsson, Dissertation (Berlingska Boktr., Lund, 1955).

[28] D. Strominger and J. O. Rasmussen, Nuclear Phys. 3, 197 (1957).

[29] B. R. Mottelson and S. G. Nilsson, Dan. Mat. Fys. Medd. (forthcoming).

Table II

Values of $M_{\kappa\kappa'}$ for some particular initial and final electron states

Initial state		Final state	$10^6 M_{\kappa\kappa'}$
Shell	Orbital		
K	$1s_{1/2}$	$p_{1/2}$	-5.3
	"	$p_{3/2}$	3.9
L_I	$2s_{1/2}$	$p_{1/2}$	-2.1
	"	$p_{3/2}$	1.6
L_{II}	$2p_{1/2}$	$s_{1/2}$	-1.9
	"	$d_{3/2}$	0.1
L_{III}	$2p_{3/2}$	$s_{1/2}$	1.4
	"	$d_{3/2}$	*
	"	$d_{5/2}$	*

*The leading anomalous terms for these transitions is of the form $r^5 Y_1$; in addition the coefficient corresponding to $M_{\kappa\kappa'}$ is negligibly small.

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APPENDIX A

To get an approximate estimate of the coefficients $F_{\kappa\kappa'}$ [Cf. Eq. (56)] we need to know $\overset{o}{f}_{\kappa}$ and $\overset{o}{g}_{\kappa}$. Table III shows these quantities for the bound states $1s_{1/2}$, $2s_{1/2}$, $2p_{1/2}$, and $2p_{3/2}$, corresponding to the electron shells K, L_I, L_{II}, and L_{III} respectively. The third column gives values of $\overset{o}{f}$ and $\overset{o}{g}$ obtained from diagrams of Brysk and Rose,[30] with Z = 91, A = 231. Their values are corrected for "finite size" and in a somewhat approximate way for screening. [The values obtained under these assumptions are denoted (a); they are employed in calculating $F_{\kappa\kappa'}$.] The fourth column gives for comparison the corresponding values obtained by Reitz[31] for the K and L_I shell, Z = 92, $R = 1.91 \times 10^{-2}$, with screening taken into account [denoted (b)]. The constants of $\overset{o}{f}$ and $\overset{o}{g}$ in this column are estimated from the values of f and g at the nuclear surface, which values are given in Ref. 31, by matching solutions inside and outside the surface. For f and g inside, expansions of the type (B1) - (B4) of Appendix B have been used, retaining terms of the two lowest orders in r. Such expansions for the potential assumed in Eq. (43) have been studied by L. A. Sliv,[17] M. E. Rose[32] and others. The deviation between Columns 3 and 4 may provide a measure of the uncertainty involved in the values of $\overset{o}{f}$ and $\overset{o}{g}$.

The corresponding constants for the free solutions (which are assumed to be normalized, in the conventional manner, per unit energy mc^2) are exhibited in Table IV. The second and third columns [denoted by (c)] in this table list values of f and g at the nuclear surface corresponding to the simple free, regular, unscreened and zero-energy wave functions appropriate to a nuclear point charge. The effect of screening, and furthermore of the energy dependence, may be studied in Columns 4-7 [denoted by (b)]. These latter values are taken

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[30] H. Brysk and M. E. Rose, ORNL 1830 (1955).

[31] Reitz, Relativistic Wave Functions for a Fermi-Thomas-Dirac Statistical Atom (University of Chicago, 1949).

[32] M. E. Rose, Phys. Rev. 82, 389 (1951).

from Ref. 31. The last columns give the constants $\overset{o}{f}$ and $\overset{o}{g}$ (obtained by the matching procedure described in the previous paragraph) using only (a). This should be a satisfactory estimate for our purpose, as screening and energy dependence here appear to be minor corrections.

In review it might be stated that the second figure in the numbers listed often carries no significance.

Table III

The quantities $\overset{o}{g}$ and $\overset{o}{f}$ of bound electrons in K, L_I, L_{II}, L_{III} shells compiled from Refs. 30 and 31.

		(a)	(b)
$1s_{1/2}$	$\overset{o}{g}$	+3.0	+3.8
	$\overset{o}{f}$	-0.9	-1.7
$2s_{1/2}$	$\overset{o}{g}$	+1.2	+1.6
	$\overset{o}{f}$	-4.0(-1)	-6.2(-1)
$2p_{1/2}$	$\overset{o}{g}$	-1.3(-1)	--
	$\overset{o}{f}$	-4.1(-1)	--
$2p_{3/2}$	$\overset{o}{g}$	+2.6(-3)	--
	$\overset{o}{f}$	-4.3(-4)	--

Table IV

This table exhibits radial components g and f at the nuclear surface from free, regular, unscreened, zero-energy solutions, (c), and the corresponding quantities from Ref. 31; furthermore, values of $\overset{o}{g}$ and $\overset{o}{f}$ obtained on the basis of (c) are listed. Brackets [ ] denote values obtained by the additional use of Eqs. (46) and (49).

	(c)		(b) 0.05 mc^2		(b) 0.1 mc^2		(c)	
	g(R)	f(R)	g(R)	f(R)	g(R)	f(R)	$\overset{o}{g}$	$\overset{o}{f}$
$s_{1/2}$	5.4	--	5.5	-2.2	5.8	-2.3	6.5	[-2.1]
$p_{1/2}$	--	-2.0	-0.9	-2.3	-1.0	-2.5	[-0.8]	-2.4
$p_{3/2}$	1.4(-2)	--	1.6(-2)	-2.8(-3)	1.9(-2)	-3.3(-3)	1.6(-2)	[-3.1(-3)]
$d_{3/2}$	--	-2.4(-3)	-6.0(-4)	-3.3(-3)	-8.8(-4)	-4.9(-3)	[-5.5(-4)]	-2.6(-3)
$d_{5/2}$	4.6(-5)	--	2.2(-5)	-2.5(-6)	3.3(-5)	3.7(-6)	5.0(-5)	[-6.9(-6)]

APPENDIX B

As a cancellation takes place that greatly diminishes the magnitude of the usually leading anomalous term $r^3 Y_1$ in the conversion amplitude, it is of interest to study whether the terms of the next higher order are also subject to the same cancellation, or if they possibly may become dominant.

For $\kappa = -\kappa_o$ we assume the following expansions for f and g:

$$f_\kappa = f^o_\kappa \left(\frac{r}{R}\right)^{\kappa_o} \left[1 + a\left(\frac{r}{R}\right)^2 + \ldots\right], \tag{B1}$$

$$g_\kappa = g^o_\kappa \left(\frac{r}{R}\right)^{\kappa_o - 1} \left[1 + b\left(\frac{r}{R}\right)^2 + \ldots\right], \tag{B2}$$

and correspondingly for $\kappa = \kappa_o$

$$f_\kappa = f^o_\kappa \left(\frac{r}{R}\right)^{\kappa_o - 1} \left[1 + c\left(\frac{r}{R}\right)^2 + \ldots\right], \tag{B3}$$

$$g_\kappa = g^o_\kappa \left(\frac{r}{R}\right)^{\kappa_o} \left[1 + d\left(\frac{r}{R}\right)^2 + \ldots\right]. \tag{B4}$$

One then finds, from Eq. (26),

$$b = c = -\frac{\{[E - v(0)]^2 - 1\} R^2}{2(2\kappa_o + 1)} \tag{B5}$$

$$\left.\begin{matrix} a \\ d \end{matrix}\right\} = -\frac{\{[E - v(0)]^2 - 1\} R^2}{2(2\kappa_o + 3)} + \frac{1}{3} v(0) \frac{2\kappa_o + 1}{2\kappa_o + 3} \frac{1}{E - v(0) \mp 1} \tag{B6}$$

where the minus sign holds for a and the plus sign holds for d. Neglecting terms of order $\frac{W}{m}$ compared to terms of order 1 but retaining the terms of the lowest and next higher order in r, one finds, for $\kappa' = \kappa - \kappa_o$, $\kappa = \kappa_o$

$$f_\kappa f_{\kappa'} + g_\kappa g_{\kappa'} = \left(\frac{r^2}{RR'}\right)^{\kappa_o - 1} \cdot r \frac{f^o_\kappa g^o_{\kappa'}}{2\kappa_o + 1} \left[2 + v'(0) - v(0) + r^2 \frac{4 v(0)^2 (\kappa_o + 1)}{(2\kappa_o + 1)(2\kappa_o + 3)} \cdot D_{corr}\right], \tag{B7}$$

where

$$D_{corr} = 1 + \frac{\kappa_o}{2(\kappa_o+1)}[v'(0)-v(0)] + \frac{(2\kappa_o+1)^2}{12(\kappa_o+1)}\frac{1}{v(0)R^2}[-2\frac{R'-R}{R} + \frac{v'(0)-v(0)}{v(0)}]. \quad (B8)$$

The corresponding formulae for $\kappa' = \kappa_o$, $\kappa = -\kappa_o$ are obtained by exchanging primed and unprimed quantities f_κ, g_κ, $v(0)$, and R.

Thus, one has the same type of cancellation in the $r^5 Y_1$ terms as in the $r^3 Y_1$ terms. However, the cancellation is now sensitive not only to differences in $v(0)$ of the final and initial electron states, but also to differences in R, the "curvature" parameter of the potential. A change in R of 10% implies a D_{corr} of order 3-5 , neglecting the effects of changes in $v(0)$.

If terms of the leading order and of the next higher order in r are retained, Eq. (56) has to be modified:

$$O_n(h_1)\cdot S(r, j_1) - O_n(j_1)\cdot S(r, h_1) = \frac{2}{3} Y_{1-M} F_{\kappa\kappa'} r^{2\kappa_o+1}[C_{corr} + G_{\kappa\kappa'}(\frac{r}{R})^2 D_{corr}], \quad (B9)$$

where C_{corr} and $F_{\kappa\kappa'}$ are found in Table I, while $G_{\kappa\kappa'}$ and D_{corr} are listed in Table V. (In view of the discussion in Appendix A the uncertainty in $G_{\kappa\kappa'}$ might be of the order 50%.)

Table V shows that the terms of order $r^5 Y_1$ are on the average still small compared with the $r^3 Y_1$ terms. If, however, the single-particle transition is accompanied, e.g., by a great change in the nuclear deformation, there is a possibility that the $r^5 Y_1$ term due to D_{corr} may be of enough magnitude to affect the conversion coefficient significantly.

Table V

Approximate values of the constants $G_{\varkappa\varkappa'}$ and D_{corr} for some particular initial and final electron states, obtained under the assumption $Z \cong 90$, $A \cong 230$, and $W \ll mc^2$. Furthermore the relation $v(0)R \cong -1$ has been employed at some places.

Transition	$\varkappa'$	$\varkappa$	$\varkappa_o$	$-G_{\varkappa\varkappa'}$	D_{corr}
$s_{1/2} \rightarrow p_{1/2}$	-1	1	1	≈ 0.08	$1+\frac{1}{4}[v'(0)-v(0)] - \frac{3}{8R}[\frac{R-R'}{R} \cdot 2 + \frac{v'(0)-v(0)}{v(0)}]$
$p_{1/2} \rightarrow s_{1/2}$	1	-1	1	≈ 0.08	$1+\frac{1}{4}[v(0)-v'(0)] - \frac{3}{8R}[\frac{R'-R}{R} \cdot 2 + \frac{v(0)-v'(0)}{v(0)}]$
$s_{1/2} \rightarrow p_{3/2}$	-1	-2	1	≈ 0.08	1
$p_{3/2} \rightarrow s_{1/2}$	-2	-1	1	≈ 0.08	1
$p_{1/2} \rightarrow d_{3/2}$	1	2	1	≈ 0.08	1