Title
Crystal Approach to Affine Schubert Calculus

Journal
International Mathematics Research Notices, 2016(8)

ISSN
1687-3017

Authors
Morse, Jennifer
Schilling, Anne

Publication Date
2016

DOI
10.1093/imrn/rnv194

Peer reviewed

Crystal approach to affine Schubert calculus

Jennifer Morse* and Anne Schilling†

April 18, 2015

Abstract

We apply crystal theory to affine Schubert calculus, Gromov-Witten invariants for the complete flag manifold, and the positroid stratification of the positive Grassmannian. We introduce operators on decompositions of elements in the type-A affine Weyl group and produce a crystal reflecting the internal structure of the generalized Young modules whose Frobenius image is represented by stable Schubert polynomials. We apply the crystal framework to products of a Schur function with a k-Schur function, consequently proving that a subclass of 3-point Gromov–Witten invariants of complete flag varieties for $\mathbb{C}^n$ enumerate the highest weight elements under these operators. Included in this class are the Schubert structure constants in the (quantum) product of a Schubert polynomial with a Schur function s_λ for all $|\lambda^\vee| < n$. Another by-product gives a highest weight formulation for various fusion coefficients of the Verlinde algebra and for the Schubert decomposition of certain positroid classes.

1 Introduction

1.1 Background

The theory of crystal bases was introduced by Kashiwara [26, 27] in an investigation of quantized enveloping algebras $U_q(\mathfrak{g})$ associated to a symmetrizable Kac–Moody Lie algebra $\mathfrak{g}$. Integrable modules for quantum groups play a central role in two-dimensional solvable lattice models. When the absolute temperature is zero ($q = 0$), there is a distinguished *crystal basis* with many striking features. The most remarkable is that the internal structure of an integrable representation can be combinatorially realized by associating the basis to a colored oriented graph whose arrows are imposed by the Kashiwara (modified root) operators. From the crystal graph, characters can be computed by enumerating elements with a given weight and the tensor product decomposition into irreducible submodules is encoded by the disjoint union of connected components. Hence, progress in the field comes from having a natural combinatorial realization of crystal graphs.

Schubert calculus is a theory whose development also hinges on combinatorial methods, but its origin is in geometry. Initially motivated to determine the number of linear spaces of given dimension satisfying certain geometric conditions, the theory has grown to one that can address highly non-trivial curve counting including the calculation of Gromov–Witten invariants. The approach converts problems into computations with representatives for Schubert classes in the (quantum) cohomology ring of a flag variety. Hence, the basic problem is one of producing and working with explicit representatives.

*supported by NSF grants DMS–1001898, DMS–1301695 and a Simons Fellowship.
†supported by NSF grants DMS–1001256, OCI–1147247 and a Simons Fellowship.

Crystal theory and Schubert calculus convene naturally in a foundational example. From the geometric perspective, the problem is to count certain linear subspaces in projective space. Developments in algebraic geometry and topology convert the problem into the computation of intersection numbers of certain subvarieties in the Grassmannian $\mathrm{Gr}(a, n)$, which in turn are encoded by the structure constants of Schubert classes $\{\sigma_\lambda\}_{\lambda \subset (\mathrm{a}^{\mathrm{n-a}})}$ for the cohomology ring $H^*(\mathrm{Gr}(a, n))$. The computation is made concrete with a homomorphism ψ from the ring Λ of symmetric functions onto $H^*(\mathrm{Gr}(a, n))$. In particular,

$$\psi(s_\lambda) = \begin{cases} \sigma_\lambda & \text{if } \lambda \subset (a^{n-a}) \,, \\ 0 & \text{otherwise} \,, \end{cases}$$

where s_λ is a Schur function. Intersection numbers thus sit as coefficients in the Schur expansion of a product of two Schur functions.

The related representation theoretic example is $\mathfrak{g} = \mathfrak{gl}_n$. The heart of crystal theory realizes tensor multiplicities as highest weights (connected components) of a graph. In this case, the crystal is the graph whose vertices are Young tableaux and whose edges are imposed by coplactic operators introduced by Lascoux and Schützenberger [46, 47]. The number of connected components is the multiplicity $c^\nu_{\lambda,\mu}$ of the irreducible highest weight module V_ν in $V_\lambda \otimes V_\mu$:

$$V_\lambda \otimes V_\mu = \bigoplus_\nu c^\nu_{\lambda,\mu} V_\nu \,.$$

A look back to the early 1900s discovery that the Frobenius image of V_λ is the Schur function s_λ shows that the Grassmannian intersection numbers are $c^\nu_{\lambda,\mu}$ as well. An explicit rule to compute $c^\nu_{\lambda,\mu}$ by counting a subclass of Young tableaux was formulated by Littlewood and Richardson in 1934 [52], but the first proof only arrived 40 years later with Schützenberger [73].

This example provides a template within which the representation theory of different modules and the geometry of other varieties may be investigated. Even incremental variations inspire highly intricate combinatorics and leave unanswered questions. For example, Schubert calculus of the flag manifold Fl_n is highly developed — from a construction of Schubert classes $\{\sigma_w\}$ indexed by elements in the symmetric group S_n by Bernstein-Gelfand-Gelfand [8] and Demazure [16], to the explicit identification of the classes with polynomial representatives introduced by Lascoux and Schützenberger [48]. Nevertheless, an LR rule for the constants in

$$\sigma_u \cup \sigma_w = \sum_{v \in S_n} c^v_{u,w}\, \sigma_v \tag{1.1}$$

has yet to be discovered. Related efforts are summarized in Section 1.4 and confirm that it is not for lack of trying.

The main thrust of this article is to introduce crystals into a generalization of Schubert calculus centered around the affine Grassmannian $\mathrm{Gr} = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ for $G = SL(n, \mathbb{C})$, where $\mathbb{C}[[t]]$ is the ring of formal power series and $\mathbb{C}((t)) = C[[t]][t^{-1}]$ is the ring of formal Laurent series. Quillen (unpublished) and Garland and Raghunathan [22] showed that Gr is homotopy-equivalent to the group $\Omega SU(n, \mathbb{C})$ of based loops into $SU(n, \mathbb{C})$. Consequently, its homology and cohomology acquire algebra structures. In particular, it follows from Bott [12] that $H_*(\mathrm{Gr})$ and $H^*(\mathrm{Gr})$ can be identified with a subring $\Lambda_{(n)}$ and a quotient $\Lambda^{(n)}$ of the ring Λ of symmetric functions. On one hand, using the algebraic nil-Hecke ring construction, Kostant and Kumar [35] studied Schubert bases of $H^*(\mathrm{Gr})$ and Peterson [60] studied Schubert bases of $H_*(\mathrm{Gr})$,

$$\{\xi^w \in H^*(\mathrm{Gr}, \mathbb{Z}) \mid w \in \tilde{S}^0_n\} \quad \text{and} \quad \{\xi_w \in H_*(\mathrm{Gr}, \mathbb{Z}) \mid w \in \tilde{S}^0_n\} \,.$$

These are indexed by the subset of the affine symmetric group $\tilde{S}_n = \langle s_0, s_1, \ldots, s_{n-1}\rangle$ consisting of affine Grassmannian elements — representatives of minimal length in cosets of $\tilde{S}_n/S_n$.

On the other hand, a distinguished basis for $\Lambda_{(n)}$ comprised of elements called *k-Schur functions*, $\{s_w^{(k)} \mid w \in \tilde{S}_n^0\}$, came out of a study [42] of Macdonald polynomials ($k = n-1$). It was shown in [44] that the (affine-LR) coefficients in the products

$$s_u^{(k)}\, s_w^{(k)} = \sum_{v\in\tilde{S}_n^0} c_{u,w}^{v,k}\, s_v^{(k)} \tag{1.2}$$

contain all structure constants (Gromov-Witten invariants) for a quantum deformation of the cohomology of the Grassmannian (e.g. [36, 78]). A basis (of *dual k-Schur functions*) for $\Lambda^{(n)}$ was also introduced in [44] and therein generalized to a family $\{\mathcal{F}_{vw^{-1}}\}$ that alternately encodes the constants by

$$\mathcal{F}_{vw^{-1}} = \sum_{u\in\tilde{S}_n^0} c_{u,w}^{v,k}\, \mathcal{F}_u\,. \tag{1.3}$$

The two approaches converged when Lam proved [39] that the k-Schur basis is a set of representatives for the Schubert classes of $H_*(\mathrm{Gr})$ and the Schubert structure constants in homology exactly match the affine-LR coefficients (see also [40] for more details).

1.2 Crystals and the affine Grassmannian

In this article, we produce a crystal in the affine framework that has applications to affine-LR coefficients and to several other families of elusive constants. The k-Schur functions can be characterized using decreasing factorizations of elements in the type-A affine Weyl group. We introduce a set of operators (see Section 3.2) that act on a subclass of these factorizations and prove that the resulting graph is a $U_q(A_{\ell-1})$-crystal using the Stembridge local axioms [75] (see Theorem 3.5). At a basic level, we find that the crystals support generalized Young–Specht modules of S_n associated to permutation diagrams. These are the modules whose Frobenius images are stable Schubert polynomials F_w (also known as Stanley symmetric functions). In Theorem 4.11, we show that the Edelman–Greene decomposition into irreducible characters intertwines with our crystal operators.

We prove that the enumeration of highest weight elements in the crystal are k-Schur coefficients in the product of a Schur and a k-Schur function. These are in fact affine LR-coefficients (1.2) as it is known that there is an element $w \in \tilde{S}_n^0$ where $s_\mu = s_w^{(k)}$ anytime $\mu \subset (r^{n-r})$, for $1 \leq r < n$. A translation operator (defined in (5.12)) enables us to generalize the framework within which we can apply the crystal. We include two proofs detailing this application; one using crystal theory and another that extends the Remmel–Shimozono involution on tableaux [68].

(Theorem 5.10). *Let $v, w \in \tilde{S}_n^0$ and $\mu \subset (r^{n-r})$ for some $1 \leq r < n$. If $\ell(v) - \ell(w) \neq |\mu|$, then $c_{Rw_\mu,w}^{Rv,k} = 0$. Otherwise, if $vw^{-1} \in S_{\hat{x}}$ or $\ell(\mu) = 2$,*

$$c_{Rw_\mu,w}^{Rv,k} = \#\ \text{of highest weight factorizations of } vw^{-1} \text{ of weight } \mu,$$

where $S_{\hat{x}}$ denotes a finite subgroup of $\tilde{S}_n$ generated by a strict subset of $\{s_0, \ldots, s_{n-1}\}$ and R is a product of k-rectangle translation operators.

The crystal also connects to several families of intensely studied constants that arise as a subset of affine LR-coefficients. We discussed the genus 0, 3-point Gromov–Witten invariants of Grassmannians. In fact, our results apply more generally to the complete flag manifold. Quantum cohomology was defined for any Kähler algebraic manifold X. When $X = \mathrm{Fl}_n$, as a linear space, $\mathrm{QH}^*(\mathrm{Fl}_n) = H^*(\mathrm{Fl}_n) \otimes \mathbb{Z}[q_1, \ldots, q_{n-1}]$ for parameters $q_1, \ldots, q_{n-1}$. However, the multiplicative structure is defined by (where w_0 is the longest element in S_n)

$$\sigma_u *_q \sigma_w = \sum_v \sum_{\mathbf{d}} \langle u, w, v \rangle_{\mathbf{d}}\, q^{\mathbf{d}}\, \sigma_{w_0 v}\ , \tag{1.4}$$

where the structure constants are the 3-point Gromov–Witten invariants of genus 0, constants which count equivalence classes of certain rational curves in Fl_n. Peterson asserted that $\mathrm{QH}^*(G/P)$ of a flag variety is a quotient of the homology $H_*(\mathrm{Gr}_G)$ of the affine Grassmannian up to localization (details carried out in [41]). Consequently, $\langle u, w, v \rangle_{\mathbf{d}}$ arise as coefficients in (1.2) and in particular, when $\mathbf{d} = 0$, these include the Schubert structure constants of (1.1).

(Theorem 5.12) *For any $\mathbf{d} \in \mathbb{N}^{n-1}$ and $u, w, v \in S_n$ where u is Grassmannian with descent at position r, if $(Rv)w^{-1} \in S_{\hat{x}}$ for some $x \in [n]$, then*

$$\langle u, w, v \rangle_{\mathbf{d}} = \#\ \textit{of highest weight factorizations of } (Rv)w^{-1} \textit{ of weight } \mu,$$

where R is a translation defined in Theorem 5.12.

Subclasses of the affine LR-coefficients also include all Schubert structure constants (1.1) and all constants for the Verlinde (fusion) algebra of the Wess–Zumino–Witten model associated to $\widehat{su}(\ell)$ at level $n - \ell$. That is, it was shown in [44] that affine-LR coefficients contain the fusion coefficients $\mathcal{N}^{\nu}_{\lambda,\mu}$, defined for $\lambda, \mu, \nu \subseteq (n-\ell)^{\ell-1}$ by

$$L(\lambda) \otimes_{n-\ell} L(\mu) = \bigoplus_{\nu} \mathcal{N}^{\nu}_{\lambda,\mu} L(\nu)\,, \tag{1.5}$$

where the fusion product $\otimes_{n-\ell}$ is the reduction of the tensor product of integrable representations with highest weight λ and μ via the representation at level $n - \ell$ of $\widehat{su}(\ell)$.

A family of affine Stanley symmetric functions, indexed by affine elements $\widetilde{w} \in \tilde{S}_n$, was introduced in [38]. It was shown that these functions $\{F_{\widetilde{w}}\}_{\widetilde{w} \in \tilde{S}_n}$ reduce to dual k-Schur functions indexed by affine Grassmannian elements, to the stable Schubert polynomials when $\tilde{w}$ is a finite element of S_n, and to cylindric Schur functions of Postnikov [63] when $\tilde{w}$ has no braid relation. We discovered that there is a correspondence between the set $\tilde{S}_n$ of affine elements and certain skew shapes ν/λ (see Definition 5.5) and in fact prove that any affine Stanley $F_{\widetilde{w}}$ is a skew dual k-Schur function $\mathcal{F}_{w_\nu w_\lambda^{-1}}$.

This in hand, we connect our results to a finer subdivision of the Grassmannian than the usual Bruhat decomposition called the *positroid stratification* [62]. Its complexification was proven [30] to coincide with the projection of the Richardson decomposition of the flag manifold [54, 67, 11] and it was shown that each cohomology class is the image under ψ of an affine Stanley symmetric function. Because cylindric Schur functions give access to Gromov–Witten invariants for the Grassmannian and they are contained in the set $\{F_{\widetilde{w}}\}$, it is implied in [30] that a subset of the positroid varieties relates to quantum cohomology of Grassmannians. We extend their result, proving that Gromov–Witten invariants for the complete flag manifold arise in the Schubert decomposition of the cohomology class of any positroid variety. We then apply the crystal on affine factorizations to this study.

1.3 Outline

Basic notation is reviewed in Section 2 regarding the affine Weyl group, crystals, and the Stembridge local axioms [75]. The crystal operators on affine factorizations for certain $\widetilde{w} \in \tilde{S}_n$ are introduced in Section 3. We discuss their properties and the theorem that the resulting graph

$B(\widetilde{w})$ is a crystal in the category of integrable highest weight crystals for type A. A proof that the Stembridge axioms hold is relegated to Appendix A. Section 4 shows that $B(\widetilde{w})$ supports the generalized Young–Specht module of a permutation diagram associated to $\widetilde{w}$. In Section 5, we connect affine Stanley symmetric functions to dual k-Schur functions and use the crystal to describe various k-Schur structure constants in the product of a Schur function and a k-Schur function. The subsections relate these coefficients to families of constants that arise from quantum flag varieties, WZW fusion, Schur times Schubert polynomials, and positroid varieties. In Section 6, we produce a sign-reversing involution on $B(\widetilde{w})$ that refines the Remmel and Shimozono [68] proof of the classical LR-rule. A second proof of Theorem 5.10 arises consequently.

1.4 Related work

Manifestly positive combinatorial formulas for structure coefficients for the Verlinde fusion algebra, or the quantum cohomology of the Grassmannian, and for the full flag have been actively sought for some time now. In the fusion case, Tudose [76] gave a combinatorial interpretation when μ has at most two columns in her thesis. For $n = 2, 3$, positive formulas are known [4, 2] as well as when λ and μ are rectangles [69]. Korff and Stroppel [34] give a formula using the plactic algebra, however their formula involves signs. Subsequently, Korff [32, 33] gave a new algorithm for the calculation of the fusion coefficients using relations to integrable models. In [44], it was shown that the fusion and three point Gromov–Witten invariants for Grassmannians form a special case of the k-Schur function structure coefficients. The fusion case of Theorem 5.10 was treated in [58].

Knutson formulated a conjecture for the quantum Grassmannian Littlewood–Richardson coefficients in terms of puzzles [31] as presented in [13]. Coskun [15] gave a positive geometric rule to compute the structure constants of the cohomology ring of two-step flag varieties in terms of Mondrian tableaux. A proof of the puzzle conjecture was recently given by Buch et al. [14]. In the flag case, Fomin, Gelfand and Postnikov [19] computed the quantum Monk rule which was extended in [61] to the quantum Pieri rule. Berg, Saliola and Serrano [7] computed the Littlewood–Richardson coefficients for k-Schur functions for the case which is equivalent to the quantum Monk rule. Denton [17] proved a special k-Littlewood–Richardson rule when there is a single term without multiplicity.

The Schur times (quantum) Schubert polynomial coefficients fall within the realm of Theorem 5.12 and have received much attention during the last years. Lenart [50, 51] used growth diagrams and plactic relations to approach this problem. Benedetti and Bergeron [5, 6] relate the Schur times Schubert problem to k-Schur function structure coefficients using strong order. The Schur times quantum Schubert coefficients are addressed by Mészáros, Panova and Postnikov [57] using the Fomin–Kirillov algebra in the hook and two-row case. As we will see in Section 5 this is the opposite extreme from the cases treated in Theorem 5.12.

Acknowledgments

We would like to thank the ICERM program "Automorphic Forms, Combinatorial Representation Theory and Multiple Dirichlet Series" during the spring 2013 and IHES in Orsay for hospitality, where part of this work was done. Both authors would like to thank the Simons Foundation for sabbatical support.

Many thanks to Avi Dalal, Nate Gallup and Mike Zabrocki for their help with the implementation of weak tableaux and k-charge during Sage Days 49 in Paris. This work benefitted from computations with Sage [71, 72]. Finally, we would like to thank Sara Billey, Dan Bump, Patrick Clarke, Adriano Garsia, Thomas Lam, Luc Lapointe, Mark Shimozono, and Josh Swanson for

enlightening discussions.

2 Preliminaries

Here we review background on affine permutations and crystals that will be used throughout the article. Otherwise, definitions and notation will be introduced as needed. In particular, our convention for partitions and tableaux is summarized in Section 4.

2.1 Extended affine symmetric group

Fix $n \in \mathbb{Z}_{>0}$. The *affine symmetric group* $\tilde{S}_n$ is the Coxeter group generated by $\langle s_0, s_1, \ldots, s_{n-1}\rangle$ satisfying the relations

$$\begin{aligned} s_i^2 &= 1 && \text{for all } i, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for all } i, \\ s_i s_j &= s_j s_i && \text{if } |i-j| > 1\,, \end{aligned} \tag{2.1}$$

where indices are taken modulo n (we will work mod n without further comment). The subgroup generated by $s_1, \ldots, s_{n-1}$ is isomorphic to the symmetric group S_n. A word $i_1 i_2 \cdots i_m$ in the alphabet $[n] = \{0, 1, \ldots, n-1\}$ corresponds to the affine permutation $w = s_{i_1} \cdots s_{i_m} \in \tilde{S}_n$. The length $\ell(w)$ of $w \in \tilde{S}_n$ is defined by the length of its shortest word. Any word of this length is said to be *reduced*.

There is a concrete realization of $\tilde{S}_n$ as the affine Weyl group $\tilde{A}_{n-1}$ [53]. Affine permutations are bijections w from $\mathbb{Z} \to \mathbb{Z}$ where $w(i+n) = w(i) + n$ for all i and where

$$\sum_{i=1}^{n} (w(i) - i) = 0\,. \tag{2.2}$$

Since an affine permutation w is determined by its tuple of values $[w(1), w(2), \ldots, w(n)]$, we often use only this *window* to represent it. The length of w can be determined by counting the appropriate notion of inversions. In particular, the *left inversion vector* $\operatorname{linv}(w) = (\alpha_1, \ldots, \alpha_n)$ is the composition where α_i records the number of positions $-\infty < j < i$ such that $w(j) > w(i)$. It was proven [70, 10] that

$$\ell(w) = |\operatorname{linv}(w)|\,. \tag{2.3}$$

We shall also have the need to work in a larger setting with *extended* affine permutations, bijections as before but without requiring condition (2.2). The set of these elements forms the *extended affine symmetric group* which can be realized by adding a generator τ to $\tilde{S}_n$ where $\tau(i) = (i+1)$. It is subject to the relation $\tau s_i = s_{i+1}\tau$. For any extended affine permutation w, there is a unique non-negative integer r where $w = \tau^r v$ and $v \in \tilde{S}_n$. Note then that

$$v = [w(1) - r, w(2) - r, \ldots, w(n) - r]\,. \tag{2.4}$$

The extended affine symmetric group contains $\tilde{S}_n$ as a normal subgroup. Its coset decomposition is given by subsets $\tilde{S}_{n,r}$ made up of elements w with the property that $\sum_{i=1}^{n}(w(i) - i) = rn$.

The set $\tilde{S}_n^0$ of *affine Grassmannian elements* is the set of minimal length coset representatives of $\tilde{S}_n / S_n$. Representatives are given by those $w \in \tilde{S}_n$ for which all reduced words end in 0. An element w is affine Grassmannian if and only if its window is increasing. We shall also define an extended affine permutation w to be affine Grassmannian when $w(1) < w(2) < \cdots < w(n)$.

2.2 Kashiwara crystals and Stembridge local axioms

Kashiwara [26, 28] introduced a *crystal* as an edge-colored directed graph satisfying a simple set of axioms. Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra with associated root, coroot and weight lattices $Q, Q^\vee, P$. Let I be the index set of the Dynkin diagram and denote the simple roots, simple coroots and fundamental weights by α_i, $\alpha_i^\vee$ and Λ_i ($i \in I$), respectively. There is a natural pairing $\langle \cdot, \cdot \rangle \colon Q^\vee \otimes P \to \mathbb{Z}$ defined by $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}$.

Definition 2.1. An *abstract* $U_q(\mathfrak{g})$*-crystal* is a nonempty set B together with maps

$$\begin{aligned} \mathrm{wt} \colon B &\to P \\ \tilde{e}_i, \tilde{f}_i \colon B &\to B \cup \{\mathbf{0}\} \qquad \text{for all } i \in I \end{aligned}$$

satisfying

1. $\tilde{f}_i(b) = b'$ is equivalent to $\tilde{e}_i(b') = b$ for $b, b' \in B$, $i \in I$.

2. For $i \in I$ and $b \in B$

$$\begin{aligned} \mathrm{wt}(\tilde{e}_i b) &= \mathrm{wt}(b) + \alpha_i \qquad \text{if } \tilde{e}_i b \in B, \\ \mathrm{wt}(\tilde{f}_i b) &= \mathrm{wt}(b) - \alpha_i \qquad \text{if } \tilde{f}_i b \in B. \end{aligned}$$

3. For all $i \in I$ and $b \in B$, we have $\varphi_i(b) = \varepsilon_i(b) + \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle$, where

$$\begin{aligned} \varepsilon_i(b) &= \max\{d \geq 0 \mid \tilde{e}_i^d(b) \neq \mathbf{0}\}, \\ \varphi_i(b) &= \max\{d \geq 0 \mid \tilde{f}_i^d(b) \neq \mathbf{0}\}. \end{aligned} \tag{2.5}$$

Remark 2.2. Although the above axioms are sometimes used to define only semi-normal crystals, this suffices here since we consider crystals coming from $U_q(\mathfrak{g})$-representations, all of which are semi-normal.

Remark 2.3. The axioms of Definition 2.1 define an edge-colored directed graph with vertex set B by drawing an edge $b \xrightarrow{i} b'$ when $\tilde{f}_i(b) = b'$.

Abstract crystals do not necessarily correspond to crystals coming from $U_q(\mathfrak{g})$-representations. Stembridge [75] provided a simple set of local axioms that uniquely characterize the crystals corresponding to representations of simply-laced algebras. We briefly review his axioms here.

Let $A = [a_{ij}]_{i,j \in I}$ be the Cartan matrix of a simply-laced Kac–Moody algebra $\mathfrak{g}$ (off-diagonal entries are either 0 or -1). In this paper we mainly consider the Cartan matrix of type $A_{\ell-1}$. An edge-colored graph X is called A-regular if it satisfies the following conditions (P1)-(P6), (P5'), and (P6'):

(P1) All monochromatic directed paths in X have finite length. In particular X has no monochromatic circuits.

(P2) For every $i \in I$ and every vertex x, there is at most one edge $y \xrightarrow{i} x$ and at most one edge $x \xrightarrow{i} z$.

We introduce the notation

$$\Delta_i \varepsilon_j(x) = \varepsilon_j(x) - \varepsilon_j(\tilde{e}_i x), \qquad \Delta_i \varphi_j(x) = \varphi_j(\tilde{e}_i x) - \varphi_j(x),$$

whenever $\tilde{e}_i x$ is defined, and

$$\nabla_i \varepsilon_j(x) = \varepsilon_j(\tilde{f}_i x) - \varepsilon_j(x), \qquad \nabla_i \varphi_j(x) = \varphi_j(x) - \varphi_j(\tilde{f}_i x),$$

whenever $\tilde{f}_i x$ is defined, where ε_i and φ_i are defined as in (2.5).

For fixed $x \in X$ and a distinct pair $i, j \in I$, assuming that $\tilde{e}_i x$ is defined, require

(P3) $\Delta_i\varepsilon_j(x) + \Delta_i\varphi_j(x) = a_{ij}$, and

(P4) $\Delta_i\varepsilon_j(x) \leq 0$, $\Delta_i\varphi_j(x) \leq 0$.

Note that for simply-laced algebras $a_{ij} \in \{0, -1\}$ for $i, j \in I$ distinct. Hence (P3) and (P4) allow for only three possibilities:

$$(a_{ij}, \Delta_i\varepsilon_j(x), \Delta_i\varphi_j(x)) = (0, 0, 0), (-1, -1, 0), (-1, 0, -1).$$

Assuming that $\tilde{e}_i x$ and $\tilde{e}_j x$ both exist, we require

(P5) $\Delta_i\varepsilon_j(x) = 0$ implies $y := \tilde{e}_i\tilde{e}_j x = \tilde{e}_j\tilde{e}_i x$ and $\nabla_j\varphi_i(y) = 0$.

(P6) $\Delta_i\varepsilon_j(x) = \Delta_j\varepsilon_i(x) = -1$ implies $y := \tilde{e}_i\tilde{e}_j^2\tilde{e}_i x = \tilde{e}_j\tilde{e}_i^2\tilde{e}_j x$ and $\nabla_i\varphi_j(y) = \nabla_j\varphi_i(y) = -1$.

Dually, assuming that $\tilde{f}_i x$ and $\tilde{f}_j x$ both exist, we require

(P5') $\nabla_i\varphi_j(x) = 0$ implies $y := \tilde{f}_i\tilde{f}_j x = \tilde{f}_j\tilde{f}_i x$ and $\Delta_j\varepsilon_i(y) = 0$.

(P6') $\nabla_i\varphi_j(x) = \nabla_j\varphi_i(x) = -1$ implies $y := \tilde{f}_i\tilde{f}_j^2\tilde{f}_i x = \tilde{f}_j\tilde{f}_i^2\tilde{f}_j x$ and $\Delta_i\varepsilon_j(y) = \Delta_j\varepsilon_i(y) = -1$.

Stembridge proved [75, Proposition 1.4] that any two A-regular posets P, P' with maximal elements x, x' are isomorphic if and only if $\varphi_i(x) = \varphi_i(x')$ for all $i \in I$. Moreover, this isomorphism is unique. Let $\lambda = \sum_{i\in I}\mu_i\Lambda_i$. Denote by $B(\lambda)$ the unique A-regular poset with maximal element b such that $\varphi_i(b) = \mu_i$ for all $i \in I$.

Theorem 2.4. *[75, Theorem 3.3] If $\mathfrak{g}$ is a simply-laced Kac–Moody Lie algebra with Cartan matrix A, then the crystal graph of the irreducible $U_q(\mathfrak{g})$-module of highest weight λ is $B(\lambda)$.*

From now on we call crystals corresponding to $U_q(\mathfrak{g})$-modules simply $U_q(\mathfrak{g})$-crystals. As Theorem 2.4 shows, for simply-laced types, it can be checked whether a crystal is a $U_q(\mathfrak{g})$-crystal by checking axioms (P1)-(P6').

An element $u \in B$ is called *highest weight* if $\tilde{e}_i u = \mathbf{0}$ for all $i \in I$. A crystal B is in the category of highest weight integrable crystals if for every $b \in B$, there exists a sequence $i_1, \ldots, i_h \in I$ such that $\tilde{e}_{i_1}\cdots\tilde{e}_{i_h} b$ is highest weight. One of the most important applications of crystal theory is that crystals are well-behaved with respect to taking tensor products.

Theorem 2.5. *[27, 59, 49] If B is a $U_q(\mathfrak{g})$-crystal in the category of integrable highest-weight crystals, then the connected components of B correspond to the irreducible components and the irreducible components are in bijection with the highest weight vectors.*

3 Crystal on affine factorizations

We start this section by defining affine factorizations, elements of $\tilde{S}_n$ with a decreasing feature that appear prominently in the geometry and combinatorics of the affine Grassmannian Gr. We introduce operators on a distinguished subset of these elements and prove properties needed to show that they are crystal operators for quantum algebra representations of type A. In Section 3.4, we show how the crystal operators can be extended to act on a different subset of affine factorizations.

In subsequent sections, we shall see that the operators support certain Young–Specht modules and that they specialize to the reflection, raising and lowering crystal operators of [29] on semi-standard Young tableaux. We will also give applications of the resulting crystal graph to the affine and positive Grassmannian, Gromov–Witten invariants, and fusion rules.

3.1 Affine factorizations

Let $i_1 \cdots i_\ell$ be a sequence with each $i_r \in [n]$. The word $i_1 \cdots i_\ell$ is *cyclically decreasing* if no number is repeated and $j-1\,j$ does not occur as a subword for any $j \in [n]$ (recall that we take all indices mod n). If $i_1 \cdots i_\ell$ is cyclically decreasing, then we say the permutation $w = s_{i_1} \cdots s_{i_\ell}$ is cyclically decreasing. Define the *content* of a permutation $w \in \tilde{S}_n$ as

$$\mathrm{con}(w) = \{i \in [n] \mid i \text{ appears in a reduced word for } w\}.$$

Note that this set can be obtained from a single reduced word for w and is independent of the reduced word chosen. Moreover, a cyclically decreasing permutation is uniquely determined by its content. Hence, we often abuse notation and write the cyclically decreasing words for the actual permutation.

Foremost, cyclically decreasing elements describe the structure in homology $H_*(\mathrm{Gr})$. For any $u \in \tilde{S}_n^0$, the Pieri rule is

$$\xi_{s_{r-1}\cdots s_1 s_0}\xi_u = \sum_{v:\ell(v)=r} \xi_{vu}, \tag{3.1}$$

over all cyclically decreasing permutations $v \in \tilde{S}_n$ where $\ell(vu) = r + \ell(u)$ and $vu \in \tilde{S}_n^0$. The focal point of our study is a set of distinguished products of cyclically decreasing elements. For any composition $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{N}^\ell$ and $w \in \tilde{S}_n$ of length $|\alpha| := \alpha_1 + \cdots + \alpha_\ell$, an *affine factorization* of w of weight α is a decomposition of the form $w = w^\ell \cdots w^1$, where w^i is a cyclically decreasing permutation of length α_i for each $1 \leq i \leq \ell$. We denote the set of affine factorizations of w by $\mathcal{W}_w$, and the subset of these having weight α is $\mathcal{W}_{w,\alpha}$. Their enumeration describes more general homology products; for $u \in \tilde{S}_n^0$,

$$\xi_{s_{\alpha_\ell-1}\cdots s_1 s_0} \cdots \xi_{s_{\alpha_1-1}\cdots s_1 s_0}\xi_u = \sum_{v\in\tilde{S}_n^0} \mathcal{K}_{vu^{-1},\alpha}\, \xi_v, \tag{3.2}$$

where $\mathcal{K}_{w,\alpha} = |\mathcal{W}_{w,\alpha}|$ for any $w \in \tilde{S}_n$.

The generating functions of affine factorizations were considered by Lam in [38] as *affine Stanley symmetric functions.* Defined for any $\tilde{w} \in \tilde{S}_n$ by

$$F_{\tilde{w}}(x) = F_{\tilde{w}} = \sum_{w^\ell\cdots w^1\in\mathcal{W}_{\tilde{w}}} x_1^{\ell(w^1)} \cdots x_\ell^{\ell(w^\ell)}, \tag{3.3}$$

the functions connect to several notable families. At the fundamental level, when $w \in S_n$, these are precisely the functions constructed by Stanley in [74] with the specific intention of realizing the number of reduced words for w as the coefficient of $x_1 x_2 \cdots x_\ell$. These "Stanley symmetric functions" had in fact been studied earlier by Lascoux and Schützenberger [48] as the stable limit of Schubert polynomials $\mathfrak{S}_w(x)$. More generally, we will prove that affine Stanley symmetric functions are none other than the dual k-Schur functions (1.3) of [44] and we will discuss the tie between $F_{\tilde{w}}$ and cohomology classes of positroid varieties [62, 30].

3.2 The crystal operators

We define operators $\tilde{e}_r$, $\tilde{f}_r$, and $\tilde{s}_r$ that act on the r-th and $(r+1)$-st factors in an affine factorization by altering the contents of these consecutive factors. The alteration is determined by a process of pairing reflections in their respective contents. The process is independent of r and it can thus be defined on a product of two cyclically decreasing factors.

Given a cyclically decreasing permutation $u \in \tilde{S}_n$, since $\mathrm{con}(u)$ is strictly contained in $[n]$, there exists some $x \in [n]$ such that $u \in S_{\hat{x}}$, where we have defined

$$S_{\hat{x}} = \langle s_0, s_1, \ldots, \hat{s}_x, \ldots, s_{n-1} \rangle \subseteq \tilde{S}_n \, .$$

Therefore, for such a fixed x, there is a unique reduced word for u given by the decreasing arrangement of entries in $\mathrm{con}(u)$ taken with respect to the order

$$x-1 > x-2 > \cdots > 0 > n-1 > \cdots > x+1 \, . \tag{3.4}$$

Consider cyclically decreasing permutations $u, v \in S_{\hat{x}}$. The *uv-pairing with respect to x* is defined by pairing the largest $b \in \mathrm{con}(u)$ with the smallest $a > b$ in $\mathrm{con}(v)$ using the ordering in (3.4). If there is no such a in $\mathrm{con}(v)$ then b is unpaired. The pairing proceeds in decreasing order on elements of $\mathrm{con}(u)$, and with each iteration previously paired letters of $\mathrm{con}(v)$ are ignored.

Example 3.1. Let $n = 14, u = s_{12}s_5s_9s_8s_2$, and $v = s_7s_6s_4s_1s_0s_{13}s_{11}$. The uv-pairing with respect to $x = 10$ proceeds from left to right on the words for u and v given by writing $\mathrm{con}(u)$ and $\mathrm{con}(v)$ in decreasing order with respect to x:

$$(9, 8, 5_1, 2_2, 12_3)(7, 6_1, 4_2, 1, 0, 13_3, 11) \, .$$

Here the pairs are denoted by matching subscripts. The uv-pairing with respect to $x = 3$ is

$$(2, 12_1, 9_2, 8_3, 5_4)(1, 0_3, 13_1, 11_2, 7, 6_4, 4) \, .$$

For $n = 5, u = s_1s_0$, and $v = s_4s_3s_1$, the uv-pairing with respect to 2 is

$$(1, 0_1)(1_1, 4, 3) \, .$$

Given $w \in S_{\hat{x}}$ for some $x \in [n]$, the crystal operators are defined to act by changing unpaired entries in adjacent factors of a factorization $w^\ell \cdots w^1$ of w. Since all the factors in an affine factorization of w lie in $S_{\hat{x}}$, we can pair any two adjacent factors with respect to x and set

$$L_r(w^\ell \cdots w^1) = \{b \in \mathrm{con}(w^{r+1}) \mid b \text{ is unpaired in the } w^{r+1}w^r\text{-pairing}\} \, ,$$

$$R_r(w^\ell \cdots w^1) = \{b \in \mathrm{con}(w^{r}) \mid b \text{ is unpaired in the } w^{r+1}w^r\text{-pairing}\} \, .$$

Definition 3.2. Fix $x \in [n]$. We define operators on cyclically decreasing $u, v \in S_{\hat{x}}$ as follows:

(i) $\tilde{e}_1(uv) = \tilde{u}\tilde{v}$ where $\tilde{u}$ and $\tilde{v}$ are the unique cyclically decreasing elements with

$$\mathrm{con}(\tilde{u}) = \mathrm{con}(u) \backslash \{b\} \quad \text{and} \quad \mathrm{con}(\tilde{v}) = \mathrm{con}(v) \cup \{b - t\}$$

for $b = \min(L_1(uv))$ and $t = \min\{i \geq 0 \mid b - i - 1 \notin \mathrm{con}(u)\}$. If $L_1(uv) = \emptyset$, $\tilde{e}_1(uv) = \mathbf{0}$.

(ii) $\tilde{f}_1(uv) = \tilde{u}\tilde{v}$ where $\tilde{u}$ and $\tilde{v}$ are the unique cyclically decreasing elements with

$$\mathrm{con}(\tilde{u}) = \mathrm{con}(u) \cup \{a + s\} \quad \text{and} \quad \mathrm{con}(\tilde{v}) = \mathrm{con}(v) \backslash \{a\}$$

for $a = \max(R_1(uv))$ and $s = \min\{i \geq 0 \mid a + i + 1 \notin \mathrm{con}(v)\}$. If $R_1(uv) = \emptyset$, $\tilde{f}_1(uv) = \mathbf{0}$.

(iii) $\tilde{s}_1 = \tilde{f}_1^{q-p}$ if $q > p$ and $\tilde{s}_1 = \tilde{e}_1^{p-q}$ if $p > q$ where $p = |L_1(uv)|$ and $q = |R_1(uv)|$. When $p = q$, $\tilde{s}_1$ is the identity map.

Given an affine factorization $w^\ell \cdots w^1$ for $w \in S_{\hat{x}}$, $\tilde{e}_r, \tilde{f}_r, \tilde{s}_r$ are defined for $1 \leq r < \ell$ by

$$\begin{aligned}
\tilde{e}_r(w^\ell \cdots w^1) &= w^\ell \cdots \tilde{e}_1(w^{r+1}w^r) \cdots w^1 \, , \\
\tilde{f}_r(w^\ell \cdots w^1) &= w^\ell \cdots \tilde{f}_1(w^{r+1}w^r) \cdots w^1 \, , \\
\tilde{s}_r(w^\ell \cdots w^1) &= w^\ell \cdots \tilde{s}_1(w^{r+1}w^r) \cdots w^1 \, .
\end{aligned} \tag{3.5}$$

Remark 3.3. Definition 3.2 is well-defined since the cyclically decreasing permutations $\tilde{u}$ and $\tilde{v}$ are uniquely defined by a strict subset of $[n]$ giving their contents. In particular, $\mathrm{con}(v)\cup\{b-t\} \subseteq [n]\backslash\{x\}$ since $x \notin \mathrm{con}(u)\cup\mathrm{con}(v)$ and $b-t \in \mathrm{con}(u)$ and similarly, $\mathrm{con}(u)\cup\{a+s\} \subseteq [n]\backslash\{x\}$.

Example 3.4. We appeal to the pairings computed in Example 3.1 to compute the images of the following affine factorizations. With $n = 14, u = s_{12}s_5s_9s_8s_2$, and $v = s_7s_6s_4s_1s_0s_{13}s_{11}$, and $x = 10$, we have

$$\tilde{e}_1(uv) = (9, 5_1, 2_2, 12_3)(8, 7, 6_1, 4_2, 1, 0, 13_3, 11)\,,$$
$$\tilde{f}_1(uv) = (9, 8, 7, 5_1, 2_2, 12_3)(6_1, 4_2, 1, 0, 13_3, 11)\,,$$
$$\tilde{s}_1(uv) = (9, 8, 7, 5_1, 2_2, 1, 12_3)(6_1, 4_2, 0, 13_3, 11)\,.$$

For the same u and v, but now with $x = 3$, we have

$$\tilde{e}_1(uv) = (12_1, 9_2, 8_3, 5_4)(2, 1, 0_3, 13_1, 11_2, 7, 6_4, 4)\,.$$

Pairing $uv = (s_1s_0)(s_1s_4s_3) \in \tilde{S}_5$ with respect to $x = 2$ yields $\tilde{e}_1(uv) = (0_1)(1_1, 0, 4, 3)$.

For $x \in [n]$ and any $w \in S_{\hat{x}}$, consider the graph $B(w)$ whose vertices are the affine factorizations $\mathcal{W}_w$ with ℓ factors (some of which might be trivial) and whose I-colored edges $x \xrightarrow{i} y$ for $x, y \in B(w)$ are determined by

$$\tilde{f}_i\, x = y\,.$$

In Appendix A, we show that $B(w)$ is a $U_q(A_{\ell-1})$-crystal graph by proving that the Stembridge axioms spelled out in Section 2.2 are satisfied.

Theorem 3.5. *For $x \in [n]$ and any $w \in S_{\hat{x}}$, $B(w)$ is a $U_q(A_{\ell-1})$-crystal.*

Consequently, by Theorem 2.5, the connected components of $B(w)$ are in bijection with highest weight vectors as defined below.

Definition 3.6. Fix $x \in [n]$, $w \in S_{\hat{x}}$, and a composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ with $|\alpha| = \ell(w)$. The factorization $w^\alpha \in \mathcal{W}_{w,\alpha}$ is *highest weight* when $\tilde{e}_r w^\alpha = \mathbf{0}$ for all $1 \le r < \ell$. That is, w^α is highest weight if there is no unpaired residue in w^{r+1} in the $w^{r+1}w^r$-pairing with respect to x for every $r = 1, \ldots, \ell-1$.

Example 3.7. The crystal $B(s_3s_4s_1s_2)$ of type A_2 is displayed in Figure 1. It has two highest weight elements $(1)(s_3s_1)(s_4s_2)$ and $(s_1)(s_3)(s_4s_2)$ of weights $(2, 2)$ and $(2, 1, 1)$, respectively. In Sage, this crystal can be generated by

```
sage: W = WeylGroup(['A',4], prefix='s')
sage: w = W.from_reduced_word([3,4,1,2])
sage: B = crystals.AffineFactorization(w,3)
sage: view(B)
```

3.3 Properties of the crystal operators

Here we establish that the crystal operators of Definition 3.2 map factorizations of w to factorizations of the same element w so that indeed $\tilde{e}_i, \tilde{f}_i \colon \mathcal{W}_w \to \mathcal{W}_w \cup \{\mathbf{0}\}$. To this end, we carefully study properties of our pairing process.

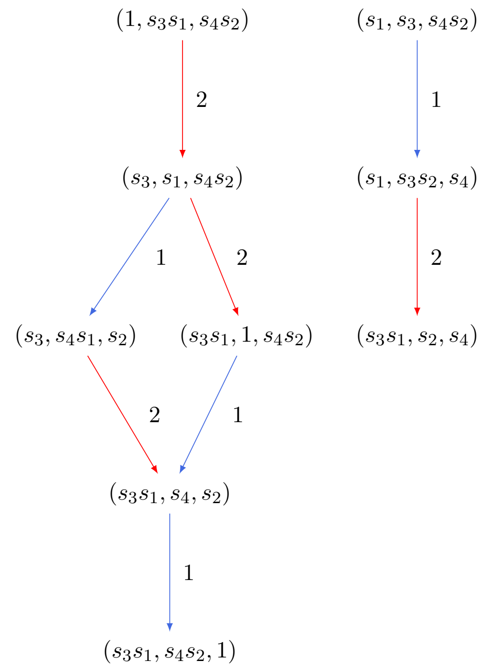

Figure 1: The crystal $B(w)$ for $w = s_3s_4s_1s_2 \in S_5$ with 3 factors (of type A_2)

Lemma 3.8. *Consider $x \in [n]$ and cyclically decreasing permutations $u, v \in S_{\hat{x}}$ where there exists $b = \min(L_1(uv))$. For the cyclically decreasing elements $u_1, u_2, v_1,$ and v_2 defined by* $\mathrm{con}(u_1) = \{z \in \mathrm{con}(u) \mid b < z < x\}$, $\mathrm{con}(u_2) = \{z \in \mathrm{con}(u) \mid x + 1 \leq z < b - t - 1\}$ $\mathrm{con}(v_1) = \{z \in \mathrm{con}(v) \mid b + 1 < z < x\}$, $\mathrm{con}(v_2) = \{z \in \mathrm{con}(v) \mid x + 1 \leq z < b - t\}$, *where $t = \min\{i \geq 0 \mid b - i - 1 \notin \mathrm{con}(u)\}$, we have the decompositions*

$$u = u_1\,(s_b\,s_{b-1}\cdots s_{b-t})\,u_2 \quad \text{and} \quad v = v_1\,(s_b \cdots s_{b-t+1})v_2\,. \tag{3.6}$$

If $\ell(uv) = \ell(u) + \ell(v)$, then uv is the product of the cyclically decreasing elements

$$\tilde{u} = u_1(s_{b-1}\,\cdots s_{b-t})u_2 \quad \text{and} \quad \tilde{v} = v_1(s_b \cdots s_{b-t+1})s_{b-t}\,v_2 \tag{3.7}$$

and, under the uv-pairing, every element of $\mathrm{con}(v_1)$ is paired with something in $\mathrm{con}(u_1)$ and every element of $\mathrm{con}(u_2)$ is paired with something in $\mathrm{con}(v_2)$.

Proof. Given that $b = \min(L_1(uv))$ exists, $b \in \mathrm{con}(u)$ and we have $b \neq x$. Therefore, $u = u_1\,(s_b\,s_{b-1}\cdots s_{b-t}\,\hat{s}_{b-t-1})\,u_2$ as claimed. Since the pairing process proceeds from largest to smallest on entries of $\mathrm{con}(u)$, and $b \in \mathrm{con}(u)$ is unpaired, $b+1 \notin \mathrm{con}(v)$ and every element of $\mathrm{con}(v_1)$ is paired with something in $\mathrm{con}(u_1)$. Further, since b is the *smallest* unpaired element, $b-1, \ldots, b-t$ are paired with $b, \ldots, b-t+1 \in \mathrm{con}(v)$ and every element in $\mathrm{con}(u_2)$ is paired with something in $\mathrm{con}(v_2)$.

Note since $b+1 \notin \mathrm{con}(v)$ by definition of b, $v = v_1\,(s_b \cdots s_{b-t+1})v_2$ for $\mathrm{con}(v_2) = \{z \in \mathrm{con}(v) \mid x+1 \leq z \leq b-t\}$. Equipped also with $u = u_1\,(s_b\,s_{b-1}\cdots s_{b-t}\,\hat{s}_{b-t-1})\,u_2$, we can commute to obtain

$$u = u_1\,u_2\,(s_b\,s_{b-1}\cdots s_{b-t}) \quad \text{and} \quad v = (s_b \cdots s_{b-t+1})v_1v_2\,.$$

A succession of Coxeter relations (2.1) implies that

$$(s_b\,s_{b-1}\cdots s_{b-t})(s_b \cdots s_{b-t+1}) = (s_{b-1}\,\cdots s_{b-t})(s_b \cdots s_{b-t}).$$

Therefore, $uv = UV$ for $U = u_1u_2(s_{b-1}\cdots s_{b-t})$ and $V = (s_b\cdots s_{b-t+1})s_{b-t}\,v_1\,v_2$. The conditions on the content of u_2 in Lemma 3.8 allow us again to commute to find that $U = u_1(s_{b-1}\cdots s_{b-t})u_2 = \tilde{u}$, where we see that $\tilde{u}$ is cyclically decreasing, $\mathrm{con}(\tilde{u}) = \mathrm{con}(u)\backslash\{b\}$, and $\ell(\tilde{u}) = \ell(u) - 1$.

On the other hand, the length of V is at most $\ell(v)+1$ since it only differs from v by the additional generator s_{b-t}. If we assume that $\ell(uv) = \ell(\tilde{u}V) = \ell(u) - 1 + \ell(v) + 1$, then equality $\ell(V) = \ell(v)+1$ must hold. This given, $b-t \notin \mathrm{con}(v_1)\cup\mathrm{con}(v_2)$. Since $x+1 \leq b-t \leq b$ and $\mathrm{con}(v_1) = \{z \in \mathrm{con}(v) \mid b+1 < z < x\}$, we can commute to find that $V = v_1(s_b\cdots s_{b-t+1})s_{b-t}\,v_2 = \tilde{v}$ is cyclically decreasing. □

The next lemma follows in the same fashion.

Lemma 3.9. *Consider $x \in [n]$ and cyclically decreasing permutations $u, v \in S_{\hat{x}}$ where there exists $a = \max(R_1(uv))$. For the cyclically decreasing elements $u_1, u_2, v_1,$ and v_2 defined by*
$\mathrm{con}(u_1) = \{z \in \mathrm{con}(u) \mid a+s < z < x\}$, $\mathrm{con}(u_2) = \{z \in \mathrm{con}(u) \mid x+1 \leq z < a-1\}$
$\mathrm{con}(v_1) = \{z \in \mathrm{con}(v) \mid a+s+1 < z < x\}$, $\mathrm{con}(v_2) = \{z \in \mathrm{con}(v) \mid x+1 \leq z < a\}$
where $s = \min\{i \geq 0 \mid a+i+1 \notin \mathrm{con}(v)\}$, we have the decompositions

$$u = u_1\,(s_{a+s-1}\,s_{a+s-2}\cdots s_a)\,u_2 \quad \textit{and} \quad v = v_1\,(s_{a+s}\cdots s_{a+1}s_a)v_2\,. \tag{3.8}$$

If $\ell(uv) = \ell(u)+\ell(v)$, then uv is the product of the cyclically decreasing elements

$$\tilde{u} = u_1(s_{a+s}s_{a+s-1}\,\cdots s_a)u_2 \quad \textit{and} \quad \tilde{v} = v_1(s_{a+s}\cdots s_{a+1})\,v_2 \tag{3.9}$$

and, under the uv-pairing, every element of $\mathrm{con}(v_1)$ is paired with something in $\mathrm{con}(u_1)$ and every element of $\mathrm{con}(u_2)$ is paired with something in $\mathrm{con}(v_2)$.

The fundamental task of crystal operators is to send a factorization of w to another factorization of w, with a carefully incremented weight change. From now on we fix $\ell = \ell(\beta)$ to be the length of all weights, where if necessary some parts of β might be zero. Let α_r be the r-th simple root of type $A_{\ell-1}$. This given, we can specify the weight change under the crystal operators of Definition 3.2 and show they are inverses of each other.

Proposition 3.10. *Fix $x \in [n]$ and $w \in S_{\hat{x}}$. If $w^\beta := w^\ell\cdots w^1 \in \mathcal{W}_{w,\beta}$, then for any $1 \leq r < \ell$,*

1. *$\tilde{e}_r(w^\beta) \in \mathcal{W}_{w,\beta+\alpha_r}$ and $\tilde{f}_r(w^\beta) \in \mathcal{W}_{w,\beta-\alpha_r}$, or w^β is annihilated,*
2. *$\tilde{s}_r(w^\beta) \in \mathcal{W}_{w,s_r\beta}$ where s_r acts on β by interchanging β_r and β_{r+1},*
3. *$\varepsilon_r(w^\beta) = |L_r(w^\beta)|$ and $\varphi_r(w^\beta) = |R_r(w^\beta)|$,*
4. *$\tilde{f}_r\,\tilde{e}_r(w^\beta) = w^\beta$, or w^β is annihilated. The same is true for $\tilde{e}_r\,\tilde{f}_r(w^\beta)$.*

Proof. Fix $x \in [n]$ and $w \in S_{\hat{x}}$. By the definition of $\tilde{e}_r, \tilde{f}_r, \tilde{s}_r$ for any r given in (3.5), it suffices to consider $uv \in \mathcal{W}_{w,(\beta_1,\beta_2)}$ where $\ell(v) = \beta_1$, $\ell(u) = \beta_2$ and $\ell(uv) = \beta_1+\beta_2$ and to prove

$$\tilde{s}_1(uv) \in \mathcal{W}_{w,(\beta_2,\beta_1)}\,, \quad \tilde{e}_1(uv) \in \mathcal{W}_{w,(\beta_1+1,\beta_2-1)}\,, \quad \tilde{f}_1(uv) \in \mathcal{W}_{w,(\beta_1-1,\beta_2+1)}\,,$$

or uv is annihilated.

To this end, if $L_1(uv) = \emptyset$, $\tilde{e}_1$ annihilates uv. Otherwise, $b = \min(L_1(uv))$ exists and by Lemma 3.8, $w = uv = \tilde{u}\tilde{v}$ where $\tilde{u}$ and $\tilde{v}$ are cyclically decreasing permutations with $\mathrm{con}(\tilde{u}) = \mathrm{con}(u)\backslash\{b\}$ and $\mathrm{con}(\tilde{v}) = \mathrm{con}(v)\cup\{b-t\}$. In fact, $\tilde{e}_1(uv) = \tilde{u}\tilde{v}$ by the definition of $\tilde{e}_1$. Note that $\ell(\tilde{u}) = \beta_2 - 1$ since it is obtained by deleting one generator from the cyclically decreasing permutation u. On the other hand, $\tilde{v}$ is obtained by adding one generator to v and therefore $\ell(\tilde{v}) \leq \beta_1+1$. By assumption $\ell(uv) = \beta_1+\beta_2 = \ell(\tilde{u}\tilde{v}) \leq \ell(\tilde{u})+\ell(\tilde{v}) \leq \beta_2-1+\beta_1+1$. Therefore, $\ell(\tilde{v}) = \beta_1+1$ and we have proven $\tilde{e}_1(uv) = \tilde{u}\tilde{v} \in \mathcal{W}_{uv,(\beta_1+1,\beta_2-1)}$.

It is also clear from the above discussion that all unbracketed letters in u_1 in uv remain unbracketed in $\tilde{u}\tilde{v}$ implying that $\varepsilon_1(uv) = |L_1(uv)|$. Other cases in (2) and (3) follow in a similar manner.

To prove (4), again consider $uv \in \mathcal{W}_{w,(\beta_1,\beta_2)}$. If $L_1(uv) = \emptyset$, then $\tilde{e}_1$ annihilates uv. Otherwise let $b = \min(L_1(uv))$. For $\tilde{u}\tilde{v} = \tilde{e}_1(uv)$, recall the decompositions of (3.7):

$$\tilde{u} = u_1(s_{b-1}\cdots s_{b-t})\,u_2 \quad \text{and} \quad \tilde{v} = v_1\,(s_b\cdots s_{b-t+1}s_{b-t})\,v_2\,. \tag{3.10}$$

Proceed with the $\tilde{u}\tilde{v}$-pairing on the largest to smallest entries of $\mathrm{con}(\tilde{u})$. Every entry in $\mathrm{con}(v_1)$ is paired to something in $\mathrm{con}(u_1)$ by Lemma 3.8. Next, $b-1,\ldots,b-t \in \mathrm{con}(\tilde{u})$ are paired with $b,\ldots,b-t+1 \in \mathrm{con}(\tilde{v})$ and we find that $b-t \in \mathrm{con}(\tilde{v})$ is unpaired. Therefore, $\max(R_1(\tilde{u}\tilde{v})) = b-t$. The conditions on $\mathrm{con}(v_1)$ from Lemma 3.8 tell us that $b+1 \notin \mathrm{con}(\tilde{v})$ implying by the definition of $\tilde{f}_1$ that $\tilde{f}_1\tilde{e}_1(uv) = uv$.

The proof for $\tilde{e}_1\tilde{f}_1$ follows in a similar manner. $\square$

Example 3.11. Let $n = 8$, $u = s_4s_3s_2s_1s_0s_7$ and $v = s_5s_2s_1s_0$. Since the uv-pairing with respect to $x = 6$ is $uv = (4_1, 3, 2, 1_2, 0_3, 7_4)(5_1, 2_2, 1_3, 0_4)$, we find that

$$\tilde{e}_1(uv) = (4_1, 3, 1_2, 0_3, 7_4)(5_1, 2_2, 1_3, 0_4, 7)\,.$$

It is not hard to check that $\tilde{f}_1$ acts on this by deleting the 7 from the right factor and adding a 2 to the left with braid relations, and indeed $\tilde{f}_1\tilde{e}_1(uv) = uv$.

3.4 Two factor case

In this section, we show that when w has only two factors, we can drop the assumption that $w \in S_{\hat{x}}$ for some $x \in [n]$. We define crystal operators in the two factor case by reducing to the $w \in S_{\hat{x}}$ case and then proceeding as in Section 3.2.

Let $w \in \tilde{S}_n$ with $uv \in \mathcal{W}_{w,(\beta_1,\beta_2)}$. Do the following initial bracket algorithm: Whenever i is in $\mathrm{con}(u)$ and $i+1$ is in $\mathrm{con}(v)$, bracket them. Now either:

1. i is in a block of the form $(\ldots[b\cdots b-t]\ldots)(\ldots[b\cdots b-t+1]\ldots)$ where $b \in \mathrm{con}(u)$ is unbracketed under the initial bracketing; we assume t to be maximal; or

2. i is in a block of the form $(\ldots[b-1\cdots b-t]\ldots)(\ldots[b\cdots b-t]\ldots)$ where $b-t \in \mathrm{con}(v)$ is unbracketed under the initial bracketing; we assume t to be maximal; or

3. i is in a block of the form $(\ldots[b-1\cdots b-t]\ldots)(\ldots[b\cdots b-t+1]\ldots)$ with $b \notin \mathrm{con}(u)$ and $b-t \notin \mathrm{con}(v)$; again assume that t is maximal.

Remark 3.12. Note that in Case (1) above $b+1 \notin \mathrm{con}(v)$ since otherwise b in $\mathrm{con}(u)$ would be bracketed. Similarly, in Case (2) $b-t-1 \notin \mathrm{con}(u)$ since otherwise $b-t$ in $\mathrm{con}(v)$ would be bracketed.

Lemma 3.13. *Let $w \in \tilde{S}_n$ and $uv \in \mathcal{W}_{w,(\beta_1,\beta_2)}$. Then either $w \in S_{\hat{x}}$ for some $x \in [n]$ or there exists an $i \in \mathrm{con}(uv) = [n]$ in Case (3) above.*

Proof. If $w \in S_{\hat{x}}$, we are done. So assume that $w \notin S_{\hat{x}}$ for any x. Note that since u is cyclically decreasing, there exists at least one letter $j \in [n]$ such that $j \notin \mathrm{con}(u)$. The same holds for v. Since all letters in $[n]$ appear in $\mathrm{con}(uv)$ neither $\mathrm{con}(u)$ nor $\mathrm{con}(v)$ can be empty. Hence there must be a letter $a \in \mathrm{con}(u)$ such that $a+1 \notin \mathrm{con}(u)$. Since all letters appear in $\mathrm{con}(uv)$, we must have $a+1 \in \mathrm{con}(v)$. This implies that we have at least one initially bracketed letter in $\mathrm{con}(uv)$. Now assume by contradiction that all initially bracketed letters are in Cases (1) or (2) above.

If $\mathrm{con}(uv)$ contains a block $(\dots[b \cdots b-t]\dots)(\dots[b \cdots b-t+1]\dots)$, then $b+1 \notin \mathrm{con}(v)$ (else we are in Case (3) or t is not maximal). Hence $b+1 \in \mathrm{con}(u)$. Let s be maximal such that $b+j \in \mathrm{con}(u)$ for $1 \le j \le s$, but $b+j \notin \mathrm{con}(v)$. If $b+j = b-t$ (where recall that we take all letters $\mod n$), then all letters in $[n]$ occur in $\mathrm{con}(u)$, which is not possible. Hence another block $(\dots[b' \cdots b'-t']\dots)(\dots[b' \cdots b'-t'+1]\dots)$ or $(\dots[b'-1 \cdots b'-t']\dots)(\dots[b' \cdots b'-t']\dots)$ must occur. If only blocks of the first form occur, then as in the previous argument all letters occur in $\mathrm{con}(u)$, which is a contradiction. But note since $s>0$ and $b+s+1 = b'-t'$, we have that $b'-t'-1 \in \mathrm{con}(u)$, which by Remark 3.12 means that we are not in Case (2), so we must be in Case (3), contradicting our assumptions.

If we had started with a block of Case (2) initially, we would have arrived at a contradiction in similar fashion.

This proves that Case (3) must occur. $\square$

By Lemma 3.13 and its proof, there exists a b such that $b \notin \mathrm{con}(u)$, $b \in \mathrm{con}(v)$ and $b-1$ is of Case (3). Remove the initially bracketed $(b-1, b)$-pair in $\mathrm{con}(uv)$. Now it is not hard to check that all definitions and properties of the crystal operators on affine factorizations of Sections 3.2 and 3.3 still go through with $x = b$ (and any braid or commutation relations still hold even with the $(b-1,b)$-pair present). Hence we have crystal operators in the two factor case as well, even if $uv \notin S_{\hat{x}}$ for any $x \in [n]$.

Theorem 3.14. *For any $w \in \tilde{S}_n$ for which there is an affine factorization into two factors, $B(w)$ carries the structure of an $U_q(\mathfrak{sl}_2)$-crystal.*

A common generalization of Theorems 3.5 and 3.14 (a "crystal theorem") for more general $w \in \tilde{S}_n$ would be extremely interesting. Since a generic affine Stanley symmetric function does not have a nonnegative Schur expansion, such a theorem will not exist without generalizing the notion of crystal. However, there are large classes of affine permutations for which the expansion is Schur positive (modulo a natural ideal). These cases would encode as highest weights invariants tied to the WZW Verlinde fusion algebra and positroid decompositions (discussed further in Section 5).

4 Young–Specht modules

The crystal $B(w)$ for $w \in S_n$ corresponds to representations carrying an action of the symmetric group called *Young–Specht modules* (also called Specht modules in [66]). These modules $\mathcal{S}^D$ are associated to finite subsets D of $\mathbb{N} \times \mathbb{N}$ called diagrams. Their origin was in Young's work [79] to explicitly produce the irreducible representations of the symmetric group. He required only Ferrers diagrams, the graphical depiction of a partition $\lambda = (\lambda_1, \dots, \lambda_m)$ with non-increasing positive integer entries obtained by stacking rows of λ_i boxes in the left corner (with its smallest row at the top). Here $\ell(\lambda) := m$ is called the length of the partition λ. The set of Young–Specht modules indexed by Ferrers diagrams λ, where $|\lambda| = \sum_{i=1}^m \lambda_i = \ell$, is a complete set of irreducible S_ℓ-modules.

It has since been established that other subclasses of Young–Specht modules are fundamental as well. For example, Young–Specht modules indexed by skew-shaped diagrams give $\mathfrak{sl}_\ell$-representations, and their decomposition as a direct sum of irreducible submodules

$$\mathcal{S}^{\nu/\lambda} = \bigoplus_{\mu} c^{\nu}_{\lambda,\mu}\, \mathcal{S}^{\mu} \tag{4.1}$$

yields multiplicities $c^{\nu}_{\lambda,\mu}$ that are given by the acclaimed *Littlewood–Richardson (LR) rule* (details to follow).

Another notable family consists of the Young–Specht modules indexed by Rothe diagrams of permutations, defined uniquely for each $w \in S_n$ to be

$$D(w) = \{(i, w(j)) \mid 1 \leq i < j \leq n, w(i) > w(j)\}\,.$$

Our primary goal in this section is to provide the crystal for these. We also discuss how a subcase of our construction yields a new characterization of the $\mathfrak{sl}_\ell$-crystal [29], and we give a number of results concerning the decomposition of Young–Specht modules into their irreducible components.

Before we begin, recall that the definition of $\mathcal{S}^D$ requires *fillings* f of D, which are bijections $f\colon D \mapsto \{1, \ldots, \ell\}$ where $\ell = |D|$. The Young–Specht module carries a natural left action of S_ℓ on fillings by the permutation of entries. The row group $R(f)$ of a filling f is the subgroup of permutations $\sigma \in S_\ell$ which act on f by permuting entries within their row and similarly, the column group $C(f)$ is the subgroup that permutes entries within their columns. The Young symmetrizer of a filling f is given by

$$y_f = \sum_{p \in R(f)} \sum_{q \in C(f)} \operatorname{sign}(q) qp \in \mathbb{C}[S_\ell]\,.$$

Definition 4.1. For each diagram D and filling f, the Young–Specht module $\mathcal{S}^D$ is the S_ℓ-module $\mathbb{C}[S_\ell]y_f$, where $\ell = |D|$.

4.1 Young–Specht modules and crystals for skew shapes

A foundational example in crystal theory is the $\mathfrak{sl}_\ell$-crystal [46, 47, 29] on skew tableaux which, by Schur–Weyl duality, can be associated to the Young–Specht modules $\mathcal{S}^D$ for skew shapes D. Our point of departure is to recall the crystal on tableaux and to show that it is a special case of $B(w)$ on affine factorizations.

The vertices of the $\mathfrak{sl}_\ell$-crystal crystal $B(\nu/\lambda)$ consist of the *semi-standard skew tableaux* $\mathrm{SSYT}(\nu/\lambda)$ over the alphabet $\{1, 2, \ldots, \ell\}$. Here $t \in \mathrm{SSYT}(\nu/\lambda)$ when it is a filling of the diagram $D = \nu/\lambda$ with letters placed non-decreasing across rows and increasing up columns. Its *weight* is defined by the composition $\mu = (\mu_1, \ldots, \mu_\ell)$ where μ_i records the number of times i occurs in t.

Crystal operators $\tilde{e}_i$ and $\tilde{f}_i$ for $1 \leq i < \ell$ are defined on $t \in \mathrm{SSYT}(\nu/\lambda)$ using a bracketing of the letters i and $i+1$ in t. Scan the columns of t from right to left, bottom to top. When a letter $i+1$ appears, pair it with the closest previously scanned i in this scanning order that has not yet been paired (if possible). Then $\tilde{f}_i(t)$ is the skew tableau obtained from t by changing the rightmost unpaired i into an $i+1$. If none exists, $\tilde{f}_i(t) = \mathbf{0}$. Similarly, $\tilde{e}_i(t)$ is obtained from t by changing the leftmost unpaired $i+1$ into an i and if none exists, $\tilde{e}_i(t) = \mathbf{0}$.

Example 4.2. In the following skew tableau, bracketed letters 2 and 3 are indicated in red and the crystal operators $\tilde{e}_2$ and $\tilde{f}_2$ act on the letter in the bold box.

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \color{red}{3} & & & & & & \\ \hline & 1 & \color{red}{2} & 2 & \color{red}{3} & & \\ \hline & & 1 & 1 & \color{red}{2} & \boxed{3} & 3 \\ \hline \end{array} \quad \begin{array}{c} \rightarrow \tilde{e}_2 \rightarrow \\ \leftarrow \tilde{f}_2 \leftarrow \end{array} \quad \begin{array}{|c|c|c|c|c|c|c|}\hline \color{red}{3} & & & & & & \\ \hline & 1 & \color{red}{2} & 2 & \color{red}{3} & & \\ \hline & & 1 & 1 & \color{red}{2} & \boxed{2} & 3 \\ \hline \end{array}$$

Recall that Theorem 2.5 indicates that highest weights correspond to irreducible components. In this setting, $t \in B(\nu/\lambda)$ is highest weight if $\tilde{e}_i t = \mathbf{0}$ for all $1 \le i < \ell$ and thus the multiplicities in (4.1) are given by the following combinatorial objects.

Crystal version of Littlewood–Richardson Rule.
$c^{\nu}_{\lambda,\mu}$ is the number of all semi-standard skew tableaux $t \in \mathrm{SSYT}(\nu/\lambda, \mu)$ of shape ν/λ and weight μ such that t is highest weight.

Although the first rigorous proof did not appear until 1977 [73], this rule was originally formulated in 1934 [52] by counting objects called *Yamanouchi tableaux*. In particular, the Littlewood–Richardson rule specifies that $c^{\nu}_{\lambda,\mu}$ counts the number of semi-standard tableaux of shape ν/λ and weight μ with a Yamanouchi row word (that is, reading right to left and bottom to top, there are never more $i+1$'s than i's, for all i). It is not hard to see that the condition of being Yamanouchi is equivalent to being highest weight in the crystal.

The crystal $B(w)$ introduced in Section 3 reduces to the crystal $B(\nu/\lambda)$ when $w \in S_n \subset \tilde{S}_n$ is 321-*avoiding* – that is, when none of its reduced words contain a braid $s_i s_{i+1} s_i$. This subclass of permutations is in bijection [9] (see also [23, Theorem 2.3.1(i)]) with skew shapes fitting inside a rectangle by removing all rows and columns without cells from the Rothe diagram $D(w)$.

Proposition 4.3. *If $w \in S_n$ is 321-avoiding, let $D(w) = \nu/\lambda$ be the corresponding skew shape. As $U_q(\mathfrak{sl}_\ell)$-crystals, $B(\nu/\lambda)$ over the alphabet $\{1, 2, \ldots, \ell\}$ is isomorphic to $B(w)$ with ℓ factors.*

Proof. Identify each cell (i,j) in a skew semi-standard tableau $t \in \mathrm{SSYT}(\nu/\lambda, \mu)$ with a label $j - i + \ell(\nu)$. From t we are going to produce an affine factorization w^μ of w of weight μ. For each letter $1 \le r \le \ell$ in t, record the labels of all letters r from right to left in t. This yields a decreasing word w^r which is the r-th factor from the end in $w^\mu = w^\ell \cdots w^1$. It is not hard to see that the bracketing rules for letters r and $r+1$ in t are equivalent to the bracketing rules in factors r and $r+1$ in w^μ. Then $\tilde{e}_r$ transforms the leftmost unbracketed letter $r+1$ in t to an r. This corresponds precisely to moving the rightmost unbracketed label from the $(r+1)$-th factor to the r-th factor in w^μ. Since w does not contain any braids, the label moves unchanged between the factors. $\square$

Example 4.4. We label each cell (i,j) in the skew tableau of Example 4.2 by $j - i + \ell(\nu)$:

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \color{red}{3_1} & & & & & & \\ \hline & 1_3 & \color{red}{2_4} & 2_5 & \color{red}{3_6} & & \\ \hline & & 1_5 & 1_6 & \color{red}{2_7} & \boxed{3_8} & 3_9 \\ \hline \end{array}$$

so that the factorization is $w^\mu = (9\mathbf{8}\color{red}{61}\color{black})(\color{red}{754}\color{black})(653)$. The colored letters are the bracketed ones and the bold entry is the label moved by $\tilde{e}_2$.

4.2 Young–Specht modules and crystals for Rothe diagrams

For any affine permutation $\tilde{w} \in S_{\hat{x}} \subset \tilde{S}_n$, the crystal $B(\tilde{w})$ on affine factorizations of $\tilde{w}$ gives the structure of more general Young–Specht modules. Since $S_{\hat{x}}$ for any $x \in [n]$ is isomorphic to

S_n, we can associate a permutation $\tau_x(\tilde{w})$ in S_n to each $\tilde{w} \in S_{\hat{x}}$ by shifting the generators of $\tilde{w}$ by $-x \mod n$. We thus define the diagram of $\tilde{w}$ to be

$$\tilde{D}(\tilde{w}) := D(\tau_x(\tilde{w}))\,.$$

The crystal $B(\tilde{w})$ on affine factorizations of $\tilde{w} \in S_{\hat{x}}$ gives the structure of the modules $\mathcal{S}^{\tilde{D}(\tilde{w})}$.

Theorem 4.5. *For any $\tilde{w} \in S_{\hat{x}} \subset \tilde{S}_n$ with $x \in [n]$, the decomposition of the Young–Specht module $\mathcal{S}^{\tilde{D}(\tilde{w})}$ into irreducible submodules is*

$$\mathcal{S}^{\tilde{D}(\tilde{w})} = \bigoplus_{\lambda} a_{\tilde{w},\lambda}\, \mathcal{S}^{\lambda}\,, \tag{4.2}$$

where the multiplicity $a_{\tilde{w},\lambda}$ is the number of highest weight factorizations in $\mathcal{W}_{\tilde{w},\lambda}$.

Proof. For $v \in S_n$, it can be deduced from results in [37, 65, 66] that the Frobenius characteristic of $\mathcal{S}^{D(v)}$ is the Stanley symmetric function $F_v(x)$, where we recall these can be viewed as a special case of the functions in (3.3). Therefore, if $\tilde{w} \in S_{\hat{x}} \subset \tilde{S}_n$ for some $x \in [n]$, we have that

$$\mathrm{char}(\mathcal{S}^{\tilde{D}(\tilde{w})}) = F_{\tau_x(\tilde{w})} = F_{\tilde{w}}\,. \tag{4.3}$$

By Theorem 3.5, $B(\tilde{w})$ is a $U_q(\mathfrak{sl}_\ell)$-crystal in the category of integrable highest weight crystals. Hence by Theorem 2.5, the irreducible components are in one-to-one correspondence with highest weight vectors. Selecting the highest weight vectors of weight λ yields the result. $\square$

From the previous result and Theorem 3.14, the statement can instead be interpreted on the level of symmetric functions by recalling that the Schur functions s_λ are characters of the irreducible Young modules $\mathcal{S}^\lambda$.

Corollary 4.6. *For any $\tilde{w} \in S_{\hat{x}} \subset \tilde{S}_n$ with $x \in [n]$, or for any $\tilde{w} \in \tilde{S}_n$ and $\ell(\lambda) \leq 2$, the coefficient $a_{\tilde{w},\lambda}$ in*

$$F_{\tilde{w}}(x) = \sum_{\lambda} a_{\tilde{w},\lambda}\, s_\lambda \tag{4.4}$$

enumerates the highest weight factorizations in $\mathcal{W}_{\tilde{w},\lambda}$.

Example 4.7. Example 3.7 shows that the crystal $B(w)$ of type A_2 for $w = s_3s_4s_1s_2 \in S_5$ has two highest weight vectors, one of weight $(2,1,1)$ and one of weight $(2,2)$, matching the Schur expansion of the Stanley symmetric function $F_{s_3s_4s_1s_2} = s_{(2,2)} + s_{(2,1,1)}$:

```
sage: W = WeylGroup(['A',4],prefix='s')
sage: w = W.from_reduced_word([3,4,1,2])
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.schur()
sage: s(w.stanley_symmetric_function())
s[2, 1, 1] + s[2, 2]
```

For $w \in S_n$, the coefficient $a_{w,\lambda}$ of a Schur function s_λ in $F_w(x)$ was previously characterized by Fomin and Greene; they proved [20, Theorem 1.2] that $a_{w,\lambda}$ counts the number of semi-standard tableaux of shape λ' (the transpose of λ) whose column-reading word is a reduced word of w. Corollary 4.6 thus implies such tableaux are in bijection with highest weight factorizations.

Corollary 4.8. *For any permutation $\tilde{w} \in S_{\hat{x}} \subset \tilde{S}_n$ and partition λ with $\ell(\lambda) \leq \ell$, the cardinality of the set*

$$\{v^\ell \cdots v^1 \in \mathcal{W}_{\tilde{w},\lambda} \mid \tilde{e}_i(v^\ell \cdots v^1) = \mathbf{0} \text{ for all } 1 \leq i < \ell\}$$

of highest weight factorizations equals the number of semi-standard tableaux of shape λ' whose column-reading word is a reduced word of $\tilde{w}$.

As we will show in Theorem 4.11, this result can be proved bijectively by extending the Edelman–Greene (EG) correspondence [18] between reduced words for $w \in S_n$ and pairs of certain same-shaped row and column increasing tableaux (P, Q). In fact, the bijection applies to the full crystal rather than just highest weight elements.

The basic operation needed for the EG-correspondence is a variant of RSK-insertion. Namely, the EG-insertion of letter a into row r of a tableau is defined by picking out the smallest letter $b > a$ in row r. If no such b exists, the letter a is placed at the end of row r. If $b = a+1$ and a is also contained in row r, then $a+1$ is inserted into row $r+1$. Otherwise, b is replaced by a, and b is inserted into row $r+1$. In the last two cases, we say that b has been bumped. This given, an insertion tableau P and recording tableau Q are constructed from a reduced word $w_\ell \cdots w_2 w_1$ starting from $P^0 = Q^0 = \emptyset$ and iteratively defining P^i by inserting w_i into the bottom row of P^{i-1}. Letters are bumped until a letter a is to be inserted into a row r containing no letter larger than a, at which point a is put at the end of row r. Q^i is then defined by adding i to the end of row r in Q^{i-1}. Finally, $P = P^\ell$ and $Q = Q^\ell$.

Theorem 4.9. *[18] Each reduced word for $w \in S_n$ corresponds to a unique pair of tableaux (P, Q) of the same shape, where the column reading of the transpose of P is a reduced expression for w and Q is standard.*

For $\tilde{w} \in S_{\hat{x}}$, we more generally define a map on $\mathcal{W}_{\tilde{w},\alpha}$ where

$$\varphi_{\mathrm{EG}} \colon v^\ell \cdots v^1 \mapsto (P, Q)\,,$$

for an appropriate pair of tableaux (P, Q) with Q now semi-standard of weight $\alpha = (\alpha_1, \ldots, \alpha_\ell)$. In particular, let $P^0 = Q^0 = \emptyset$ and define P^i, for $i = 1, \ldots, \ell$, by EG-inserting into P^{i-1} the word $w_{\alpha_i} \cdots w_1$, where $\mathrm{con}(v^i) = \{w_1, \ldots, w_{\alpha_i}\}$ and $w_{\alpha_i} > \cdots > w_1$ under the order of (3.4). Q^i is defined by adding letter i to Q^{i-1} in cells given by $\mathrm{shape}(P^i)/\mathrm{shape}(P^{i-1})$. Set $\varphi^Q_{\mathrm{EG}}(v^\ell \cdots v^1) = Q$.

Remark 4.10. EG-insertion enjoys many of the same properties as RSK-insertion. For example, given that cell c_y is added to a tableau when y is EG-inserted, and cell c_x is added when x is then EG-inserted into the result, c_x lies strictly east of c_y when $x > y$, and c_x lies strictly higher than c_y when $x < y$.

Theorem 4.11. *For any $\tilde{w} \in S_{\hat{x}} \subset \tilde{S}_n$, the map φ^Q_{EG} is a crystal isomorphism*

$$B(\tilde{w}) \cong \bigoplus_\lambda B(\lambda)^{\oplus a_{\tilde{w},\lambda}}\,.$$

In particular,

$$\varphi^Q_{\mathrm{EG}} \circ \tilde{e}_i = \tilde{e}_i \circ \varphi^Q_{\mathrm{EG}} \qquad \text{and} \qquad \varphi^Q_{\mathrm{EG}} \circ \tilde{f}_i = \tilde{f}_i \circ \varphi^Q_{\mathrm{EG}}.$$

Proof. Fix $\tilde{w} \in S_{\hat{x}}$ for some $x \in [n]$. We first note that φ_{EG} is a bijection between $\mathcal{W}_{\tilde{w},\alpha}$ and the set of pairs of same-shaped tableaux (P, Q) where the column-reading word of the transpose of P is a reduced expression for $\tilde{w}$ and Q is semi-standard of weight α. That is, given $v^\ell \cdots v^1 \in \mathcal{W}_{\tilde{w},\alpha}$, let $(P, Q) = \varphi_{\mathrm{EG}}(v^\ell \cdots v^1)$ and recall that $P = P^\ell$ where P^ℓ is defined by inserting the (distinct) letters of $\mathrm{con}(v^\ell)$ from smallest to largest into $P^{\ell-1}$. By Remark 4.10, $Q^\ell/Q^{\ell-1}$ is a horizontal $\ell(v^\ell)$-strip and we iteratively find Q to be semi-standard of weight α. The column reading word of the transpose of P is a reduced expression for $\tilde{w}$ by Theorem 4.9. It is not difficult to see that the process is invertible by reverse EG-bumping letters from P^i that lie in the positions determined by cells of $\mathrm{shape}(Q^i)/\mathrm{shape}(Q^{i-1})$ taken from right to left.

Let us now denote the letters in $\mathrm{con}(v^{i+1})$ by $y_{\alpha_{i+1}} \cdots y_1$ and the letters in $\mathrm{con}(v^i)$ by $x_{\alpha_i} \cdots x_1$ in the order in (3.4). Let $a = x_j$ be the leftmost unbracketed letter in the pairing in Section 3.2. Inserting the letters $x_1, \ldots, x_{\alpha_i}$ under the EG-insertion yields α_i insertion paths that move strictly to the right in the tableaux P^i by Remark 4.10. Since $a = x_j$ is the leftmost unbracketed letter in $\mathrm{con}(v^i)$, by Lemma 3.9 there exists an index $1 \leq m \leq \alpha_{i+1}$ such that $x_j < y_m < y_{m+1} < \cdots < y_{\alpha_{i+1}}$ and $y_1 < y_2 < \cdots < y_{m-1} < x_{j-1}$ in the order (3.4). In addition, all letters $y_1, \ldots, y_{m-1}$ are bracketed under the crystal bracketing which means that the insertion paths of these letters are weakly to the left of the insertion path of x_{j-1} and no letter can bump x_j. Also, the letters $x_{j+1}, \ldots, x_{\alpha_i}$ are bracketed under the crystal bracketing so that of the letters i in Q^i corresponding to the insertion paths of $x_j, \ldots, x_{\alpha_i}$ precisely one is not bracketed with an $i+1$ in Q^{i+1}.

Now under $\tilde{f}_i$ the letter $a = x_j$ moves from $\mathrm{con}(v^i)$ to the letter $a+s$ in $\mathrm{con}(v^{i+1})$. As a result, the insertion paths of $x_{j+1}, \ldots, x_{\alpha_i}$ either stay (partially) in their old track or move left (partially) to the insertion of the previously inserted letter. Similarly, the insertion paths of the corresponding y_h move (partially) left. The new letter $a+s$ in v^{i+1} after the application of $\tilde{f}_i$, then causes the previously unpaired letter i in Q^{i+1} in the insertion to become an $i+1$, possibly by shifting the insertion paths of the subsequent y_h to the right. This proves the claim for $\tilde{f}_i$.

The proof for $\tilde{e}_i$ is similar. $\square$

A by-product of Theorem 4.11 is a bijective proof of Corollary 4.8, where the tableau associated to a highest weight element $v^\ell \cdots v^1$ is the transpose of the insertion tableau $\varphi^P_{\mathrm{EG}}(v^\ell \cdots v^1) = P$. It further gives a crystal theoretic analogue of the relation between the Edelman–Greene insertion of a reduced word of $w \in S_n$ and the RSK insertion of its peelable word given in [65].

Example 4.12. Given the highest weight factorization $v^3v^2v^1 = (1)(2)(32)$, with weight $\lambda = (2,1,1)$ of the permutation $s_1s_2s_3s_2 \in S_4$, the successive insertion of v^i, for $i = 1,2,3$ yields

$$\left(\begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}\, , \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} \right) \qquad \left(\begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 2 & 3 \\ \hline \end{array}\, , \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array} \right) \qquad \left(\begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array}\, , \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array} \right) = (P, Q)\,.$$

The column-reading word of the transpose of P is 3123, indeed a reduced word for $s_1s_2s_3s_2$, demonstrating the bijective correspondence of Corollary 4.8.

Another immediate outcome of our crystal $B(w)$ is Stanley's famous result [74] that the number of reduced expressions for the longest element $w_0 \in S_n$ is equal to the number of standard tableaux of staircase shape $\rho = (n-1, n-2, \ldots, 1)$. Namely, in $B(w_0)$ there is only one highest weight element given by the factorization $(s_1)(s_2s_1)(s_3s_2s_1)\cdots(s_{n-1}s_{n-2}\cdots s_1)$. Hence $B(w_0)$ is isomorphic to the highest weight crystal $B(\rho)$. The reduced words of w_0 are precisely given by the factorizations of weight $(1,1,\ldots,1)$. In $B(\rho)$ they are the standard tableaux of shape ρ. The bijection between the reduced words of w_0 and standard tableaux of shape ρ induced by the crystal isomorphism is precisely φ^Q_{EG} (which due to our conventions of reading the factorization from right to left gives the transpose of the standard tableau from the straight EG-insertion).

5 Highest weights and geometric invariants

Here we study families of constants including Gromov–Witten invariants for flag varieties (and in particular, Schubert polynomial structure constants (1.1)), the structure constants for the Verlinde (fusion) algebra of the Wess–Zumino–Witten model, and the decomposition of positroid classes into Schubert classes. Our approach is to apply the $B(\tilde{w})$-crystal introduced in Section 3 to affine Schubert calculus.

To be precise, as discussed in the introduction, $H_*(\mathrm{Gr})$ is isomorphic to the subring

$$\Lambda_{(n)} = \mathbb{Z}[h_1, \ldots, h_{n-1}]$$

of the ring of symmetric functions Λ, where $h_r = \sum_{i_1 \leq \cdots \leq i_r} x_{i_1} \cdots x_{i_r}$. Representatives for the Schubert homology classes are given by a basis for $\Lambda_{(n)}$ made up of symmetric functions called *k-Schur functions* (hereafter, $k = n-1$). Denoted by $s_{\tilde{u}}^{(k)}$, these are indexed by $\tilde{u} \in \tilde{S}_n^0$. The importance of this basis to our study is that the Schubert structure constants for $H_*(\mathrm{Gr})$ match the coefficients in

$$s_{\tilde{u}}^{(k)} \, s_{\tilde{w}}^{(k)} = \sum_{\tilde{v} \in \tilde{S}_n^0} c_{\tilde{u},\tilde{w}}^{\tilde{v},k} \, s_{\tilde{v}}^{(k)}\,, \tag{5.1}$$

and the families of constants under study here arise as subsets of these *affine Littlewood–Richardson coefficients*.

After recalling the definition of k-Schur functions, we begin by proving that the affine Stanley symmetric functions are none other than functions that arose by studying duals of the k-Schur functions. In doing so, we can apply the crystal $B(\tilde{w})$ to the study of affine LR numbers and subsequently, to the study of the aforementioned constants. In this section, to avoid confusion, we use the convention that affine permutations are denoted by $\tilde{w}$ and usual permutations of S_n appear as w – without the tilde.

5.1 The affine Stanley/dual k-Schur correspondence

There are many equivalent formulations for the k-Schur basis. In the spirit of our presentation, we use its characterization in terms of the homogeneous basis $\{h_\lambda\}_{\lambda \in \mathcal{P}^n}$, where $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$ and $\mathcal{P}^n = \{\lambda \mid \lambda_1 < n\}$ is the set of partitions with all parts shorter than n. This expansion relies on the matrix $\mathcal{K}$ whose entries,

$$\mathcal{K}_{\tilde{w},\mu} = |\mathcal{W}_{\tilde{w},\mu}|\,,$$

enumerate affine factorizations of $\tilde{w} \in \tilde{S}_n^0$ with fixed weight $\mu \in \mathcal{P}^n$.

The matrix $\mathcal{K}$ is square since the set of affine Grassmannian elements $\tilde{S}_n^0$ is in bijection with $\mathcal{P}^n$. Namely, since the window of an affine Grassmannian $\tilde{w} \in \tilde{S}_n^0$ is increasing, $\lambda = \mathrm{linv}(\tilde{w})$ is weakly decreasing and its last entry is zero. Thus, taking the transpose partition λ', the map

$$\mathcal{LC}\colon \tilde{w} \mapsto \mathrm{linv}(\tilde{w})' \tag{5.2}$$

sends $\tilde{S}_n^0 \to \mathcal{P}^n$ and in fact, it is bijective [10]. We use $\tilde{w}_\lambda$ to denote the inverse image of $\lambda \in \mathcal{P}^n$.

Remark 5.1. The map $\mathcal{LC}$ is well-defined on the set of all extended affine Grassmannian permutations and, for fixed r, gives a bijection between the affine Grassmannian elements of $\tilde{S}_{n,r}$ and $\mathcal{P}^n$. In particular, $\mathcal{LC}(\tilde{v}) = \lambda$ for $\tilde{v} = \tau^r \tilde{w}_\lambda$.

Example 5.2. For $\tilde{w} = [-2, 0, 1, 4, 12] \in \tilde{S}_5^0$, the left inversion vector is $\mathrm{linv}(\tilde{w}) = (3, 2, 2, 1, 0)$ and its conjugate is $\mathcal{LC}(\tilde{w}) = (4, 3, 1) \in \mathcal{P}^5$. Thus, in our notation, $\tilde{w} = \tilde{w}_{(4,3,1)}$.

The matrix $\mathcal{K}$ is also unitriangular and thus characterizes functions, for each $w \in \tilde{S}_n^0$, by

$$s_{\tilde{w}}^{(k)} = \sum_{\mu \in \mathcal{P}^n} \overline{\mathcal{K}}_{\mu,\tilde{w}}\, h_\mu\,, \tag{5.3}$$

where $\overline{\mathcal{K}} = \mathcal{K}^{-1}$. The set of these functions defines the *k-Schur basis* for $\Lambda_{(n)}$.

The ring $\Lambda_{(n)}$ is naturally Hopf dual to the quotient $\Lambda^{(n)} = \Lambda/\langle m_\lambda \mid \lambda_1 \geq n\rangle$, where $m_\lambda = \sum x_{\alpha_1}^{\lambda_1} x_{\alpha_2}^{\lambda_2} \cdots x_{\alpha_\ell}^{\lambda_\ell}$ over tuples $(\alpha_1, \ldots, \alpha_\ell) \in \mathbb{N}^\ell$ with distinct entries. The elements $\{m_\lambda\}_{\lambda \in \mathcal{P}^n}$ may be chosen as representatives of the dual algebra. Duality can be used to produce a second basis, now for the algebra $\Lambda^{(n)}$. Namely, a basis $\{\mathcal{F}_\lambda\}_{\lambda \in \mathcal{P}^n}$ can be characterized as the unique set of elements in the subspace $\Lambda^{(n)}$ that are dual to $\{s_{\tilde{u}}^{(k)}\}_{\tilde{u} \in \tilde{S}_n^0}$ by appealing to the pairing

$$\langle\ \cdot\ ,\ \cdot\ \rangle\ :\ \Lambda_{(n)} \times \Lambda^{(n)} \to \mathbb{Q}\,, \tag{5.4}$$

where $h_\mu \in \Lambda_{(n)}$ and $m_\lambda \in \Lambda^{(n)}$ are dual elements. That is, $\langle h_\mu, m_\lambda\rangle = \delta_{\lambda\mu}$.

These elements were first studied as a special case of the *dual k-Schur functions*, a family defined for any skew diagram of $\mathcal{D} = \{\nu/\lambda \mid \ell(\tilde{w}_\nu \tilde{w}_\lambda^{-1}) = \ell(\tilde{w}_\nu) - \ell(\tilde{w}_\lambda)\}$ by

$$\mathcal{F}_{\nu/\lambda} = \sum_{\mu \in \mathcal{P}^n} \mathcal{K}_{\tilde{w}_\nu \tilde{w}_\lambda^{-1},\mu}\, m_\mu\,. \tag{5.5}$$

The original motivation for their study was to produce affine LR (and WZW-fusion) coefficients.

Theorem 5.3. *[44][Theorem 28] For any $\lambda \subset \nu \in \mathcal{P}^n$,*

$$\mathcal{F}_{\nu/\lambda} = \sum_{\mu \in \mathcal{P}^n} c_{\tilde{w}_\lambda,\tilde{w}_\mu}^{\tilde{w}_\nu,k}\, \mathcal{F}_\mu\,. \tag{5.6}$$

Lam then introduced affine Stanley functions $F_{\tilde{w}}$ for $\tilde{w} \in \tilde{S}_n$ in [38] and proved that

$$\mathcal{F}_{\nu/\lambda} = F_{\tilde{w}_\nu \tilde{w}_\lambda^{-1}}\,. \tag{5.7}$$

The converse was not readily apparent at the time; it was believed that affine Stanley symmetric functions were more general than dual k-Schur functions. As it turns out, we discovered that every affine element of $\tilde{S}_n$ can be associated to a skew diagram in such a way that the corresponding affine Stanley symmetric function is a skew dual k-Schur function.[1]

For this, it is convenient to work not only with $\tilde{S}_n = \tilde{S}_{n,0}$, but with the extended affine symmetric group as defined in Section 2.1. Our correspondence hinges on an injection from permutations of S_n into $\tilde{S}_{n,\binom{n}{2}}$ defined by

$$\mathfrak{af} : w \mapsto [w(1), w(2)+n, w(3)+2n, \ldots, w(n)+(n-1)n]\,.$$

This map was introduced in [45] as the crux of an association between Gromov–Witten invariants for flag manifolds and affine LR-coefficients (further discussed in Section 5.3). We shall need two properties of the interplay between $\mathfrak{af}$ and length that come out of a close examination of linv (e.g. [45]): For $\tilde{v} \in \tilde{S}_n^0$ and $u \in S_n$,

$$\ell(\tilde{v}\mathfrak{af}(\mathrm{id})) = \ell(\tilde{v}) + \ell(\mathfrak{af}(\mathrm{id})) \qquad \text{and} \qquad \ell(\mathfrak{af}(u)) = \ell(\mathfrak{af}(\mathrm{id})) - \ell(u)\,. \tag{5.8}$$

In fact, the left inversion vector of $\tilde{v}\mathfrak{af}(\mathrm{id})$ and of $\mathfrak{af}(u)$ is a partition. Precisely, affine permutations in the image of $\mathfrak{af}$ always have increasing windows. Moreover, when $\tilde{v} = [\tilde{v}(1) < \cdots < \tilde{v}(n)]$ is affine Grassmannian, $\tilde{v}\mathfrak{af}(\mathrm{id}) = [\tilde{v}(1), \tilde{v}(2)+n, \ldots, \tilde{v}(n)+(n-1)n]$ has an increasing window as well. This allows for a well-defined action of $\mathcal{LC}$ on the range of $\mathfrak{af}$ as well as on the product of elements of the form $\tilde{v}\mathfrak{af}(\mathrm{id})$ for $\tilde{v} \in \tilde{S}_n^0$.

[1] Thomas Lam mentioned that he knows a different (unpublished) proof of this fact.

Lemma 5.4. *[45] The composition $\mathcal{LC} \circ \mathfrak{af}$ is a bijection from S_n onto the set of partitions $\{\lambda \mid \square_{k-1} \subseteq \lambda \subseteq \square_k\}$, where $\square_k = (k, k-1, k-1, \ldots, 1^k)$ is the concatenation of all k-rectangles. Further, for any $\nu \in \mathcal{P}^n$ containing $\square_k$, there is a unique $\tilde{v} \in \tilde{S}_n^0$ such that $\mathcal{LC}(\tilde{v}\mathfrak{af}(\mathrm{id})) = \nu$.*

Definition 5.5. For $\tilde{w} \in \tilde{S}_n$, consider the decomposition $\tilde{w} = \tilde{v}u$ where $u \in S_n$ is the permutation that rearranges the window of $\tilde{w}$ into increasing order and $\tilde{v} \in \tilde{S}_n^0$ is the resulting affine Grassmannian element. Define

$$\kappa \colon \tilde{w} \mapsto \nu/\lambda\,, \tag{5.9}$$

for $\nu = \mathcal{LC}(\tilde{v}\mathfrak{af}(\mathrm{id}))$ and $\lambda = \mathcal{LC}(\mathfrak{af}(u^{-1}))$.

Proposition 5.6. *The map κ is a bijection*

$$\kappa \colon \tilde{S}_n \to \{\nu/\lambda \mid \nu, \lambda \in \mathcal{P}^n \text{ and } \square_{k-1} \subseteq \lambda \subseteq \square_k \subseteq \nu\}\,.$$

Moreover, if $\kappa(\tilde{w}) = \nu/\lambda$, then $\ell(\tilde{w}) = |\nu| - |\lambda|$ and $\tilde{w} = \tau^{\binom{n}{2}}\tilde{w}_\nu\tilde{w}_\lambda^{-1}\tau^{-\binom{n}{2}}$.

Proof. Given $\tilde{\tau}, \tilde{w} \in \tilde{S}_n$, consider their decompositions $\tilde{\tau} = \tilde{b}\sigma$ and $\tilde{w} = \tilde{v}u$ where $\sigma, u \in S_n$ and $\tilde{b}, \tilde{v} \in \tilde{S}_n^0$ according to Definition 5.5. If $\kappa(\tilde{w}) = \kappa(\tilde{\tau})$, then $\nu = \mathcal{LC}(\tilde{b}\mathfrak{af}(\mathrm{id})) = \mathcal{LC}(\tilde{v}\mathfrak{af}(\mathrm{id}))$ and $\lambda = \mathcal{LC}(\mathfrak{af}(\sigma^{-1})) = \mathcal{LC}(\mathfrak{af}(u^{-1}))$. Since $\mathcal{LC}$ is a bijection between $\tilde{S}_{n,\binom{n}{2}}$ and $\mathcal{P}^n$, we have that $\tilde{b}\mathfrak{af}(\mathrm{id}) = \tilde{v}\mathfrak{af}(\mathrm{id})$ and $\mathfrak{af}(\sigma^{-1}) = \mathfrak{af}(u^{-1})$ implying that κ is injective. Moreover, for a generic $\tilde{w} = \tilde{v}u$, Lemma 5.4 indicates that $\square_{k-1} \subseteq \mathcal{LC}(\mathfrak{af}(u^{-1})) \subseteq \square_k \subseteq \mathcal{LC}(\tilde{v}\mathfrak{af}(\mathrm{id}))$ and thus the correct codomain has been specified.

To complete the proof that κ is bijective, we construct the preimage of ν/λ assuming that $\square_{k-1} \subseteq \lambda \subseteq \square_k \subseteq \nu$ and $\nu, \lambda \in \mathcal{P}^n$. In particular, we take $\tilde{w} = \tilde{v}u$ for the unique $u \in S_n$ and $\tilde{v} \in \tilde{S}_n^0$ such that $\nu = \mathcal{LC}(\tilde{v}\mathfrak{af}(\mathrm{id}))$ and $\lambda = \mathcal{LC}(\mathfrak{af}(u^{-1}))$ that is guaranteed to exist by Lemma 5.4.

Now consider $\kappa(\tilde{w}) = \nu/\lambda$ and the usual decomposition $\tilde{w} = \tilde{v}u$. Note that $\ell(\tilde{w}) = \ell(\tilde{v}) + \ell(u)$. On the other hand, $\nu = \mathcal{LC}(\tilde{v}\mathfrak{af}(\mathrm{id}))$ and $\lambda = \mathcal{LC}(\mathfrak{af}(u^{-1}))$ imply that $\ell(\tilde{v}\mathfrak{af}(\mathrm{id})) = |\nu|$ and $\ell(\mathfrak{af}(u^{-1})) = |\lambda|$. From (5.8), we have that $\ell(\tilde{v}\mathfrak{af}(\mathrm{id})) = \ell(\tilde{v}) + \ell(\mathfrak{af}(\mathrm{id}))$ and $\ell(\mathfrak{af}(u^{-1})) = \ell(\mathfrak{af}(\mathrm{id})) - \ell(u)$. Therefore, $|\nu| - |\lambda| = \ell(\tilde{v}) + \ell(u) = \ell(\tilde{w})$. Lastly,

$$\tilde{w} = \tilde{v}u = (\tilde{v}\mathfrak{af}(\mathrm{id}))\left(\mathfrak{af}(u^{-1})\right)^{-1} = \left(\tau^{\binom{n}{2}}\tilde{w}_\nu\right)\left(\tau^{\binom{n}{2}}\tilde{w}_\lambda\right)^{-1}$$

since $\tilde{v}\mathfrak{af}(\mathrm{id}), \mathfrak{af}(u^{-1}) \in \tilde{S}_{n,\binom{n}{2}}$ and $\nu = \mathcal{LC}(\tilde{v}\mathfrak{af}(\mathrm{id}))$ and $\lambda = \mathcal{LC}(\mathfrak{af}(u^{-1}))$. ☐

Corollary 5.7. *For every $\tilde{w} \in \tilde{S}_n$, the affine Stanley symmetric function $F_{\tilde{w}}$ is the dual k-Schur function $\mathcal{F}_{\kappa(\tilde{w})}$.*

Proof. For any $\tilde{w} \in \tilde{S}_n$, if $\nu/\lambda = \kappa(\tilde{w})$ then $\tilde{w} = \tau^{\binom{n}{2}}\tilde{w}_\nu\tilde{w}_\lambda^{-1}\tau^{-\binom{n}{2}}$ by Proposition 5.6. Since τ^r acts on affine elements by a cyclic shift of the generators, the number of affine factorizations of $\tilde{w}$ with weight μ is the same as the number of affine factorizations of $\tilde{w}_\nu\tilde{w}_\lambda^{-1}$ with weight μ. By the definition of dual k-Schur and affine Stanley functions, $F_{\tilde{w}} = \mathcal{F}_{\nu/\lambda}$. ☐

Example 5.8. For $\tilde{w} = s_3s_0s_2s_3 = [-1, 4, 5, 2] \in \tilde{S}_4$, we find $\tilde{v} = [-1, 2, 4, 5]$ and $u = s_2s_3 = [1, 3, 4, 2]$. From this, $\mathfrak{af}(u^{-1}) = [1, 8, 10, 15]$ and since $\mathfrak{af}(\mathrm{id}) = [1, 6, 11, 16]$, we have $\tilde{v}\mathfrak{af}(\mathrm{id}) = [-1, 6, 12, 17]$. Thus, $\nu = \mathcal{LC}([-1, 6, 12, 17]) = (3, 2, 2, 1^5)$ and $\lambda = \mathcal{LC}([1, 8, 10, 15]) = (3, 1^5)$ implying $\tilde{w}_\nu\tilde{w}_\lambda^{-1} = s_1s_2s_0s_1$. This indeed is $\tilde{w} = s_3s_0s_2s_3$ up to a cyclic shift by 2 in the generators.

5.2 Structure constants for $H_*(\mathrm{Gr})$

We are now in a position to connect the crystal $B(\tilde{w})$ with homology of the affine Grassmannian Gr. We prove that the highest weights of $B(\tilde{w})$ are affine LR coefficients which in turn are the Schubert structure constants of $H_*(\mathrm{Gr})$.

The crystal applies to a subclass of the k-Schur structure constants that appear in products of a k-Schur function with a Schur function indexed by $\mu \subset (r^{n-r})$, for any $1 \leq r < n$. These indeed are affine LR-coefficients of (5.1) by the k-Schur property that, for any $\mu \subset (r^{n-r})$,

$$s^{(k)}_{\tilde{w}_\mu} = s_\mu \,. \tag{5.10}$$

We first revisit the affine Stanley symmetric functions with this restriction in hand. While Theorem 5.3 explains that the dual k-Schur expansion of $F_{\tilde{w}}$ yields positive coefficients, it does not suggest the same about the Schur expansion coefficients. In fact, many of the Schur coefficients are not positive. However, when $\mu \subset (r^{n-r})$, we shall prove that they are positive and in particular include Gromov–Witten invariants for full flags.

Proposition 5.9. *For any $\tilde{w} \in \tilde{S}_n$, consider the Schur expansion*

$$F_{\tilde{w}} = \sum_\mu a_{\tilde{w},\mu}\, s_\mu \,. \tag{5.11}$$

For $\mu \subset (r^{n-r})$ with $1 \leq r < n$, the coefficient $a_{\tilde{w},\mu}$ is the (non-negative) affine LR-coefficient $c^{\tilde{w}_\nu,k}_{\tilde{w}_\lambda,\tilde{w}_\mu}$, where $\kappa(\tilde{w}) = \nu/\lambda$.

Proof. Let $\nu/\lambda = \kappa(\tilde{w})$. By Theorem 5.3 and Corollary 5.7, we have

$$F_{\tilde{w}} = \mathcal{F}_{\nu/\lambda} = \sum_{\mu \in \mathcal{P}^n} c^{\tilde{w}_\nu,k}_{\tilde{w}_\lambda,\tilde{w}_\mu}\, \mathcal{F}_\mu \,.$$

Since $\mathcal{F}_\mu$ and $s^{(k)}_\mu$ are dual under the Hall inner product, when $\mu \subset (r^{n-r})$ we use (5.10) to find

$$c^{\tilde{w}_\nu,k}_{\tilde{w}_\lambda,\tilde{w}_\mu} = \langle F_{\tilde{w}}, s^{(k)}_\mu \rangle = \langle F_{\tilde{w}}, s_\mu \rangle = a_{\tilde{w},\mu} \,,$$

and the claim follows. □

Having identified particular Schur coefficients in the affine Stanley (dual k-Schur) function as affine LR coefficients, we use highest weights in the crystal to describe them. For the greatest generality of results, we appeal to another property of k-Schur functions. It relies on a family of operators $\{R_i\}_{1\leq i<n}$ that act by changing the window $[\tilde{u}_1, \ldots, \tilde{u}_n]$ of an element $\tilde{u} \in \tilde{S}_{n,r}$ to

$$R_i\tilde{u} = [\tilde{u}_1 - (n-i), \tilde{u}_2 - (n-i), \ldots, \tilde{u}_i - (n-i), \tilde{u}_{i+1} + i, \ldots, \tilde{u}_n + i] \,. \tag{5.12}$$

Note that the application of various R_i commutes. It was shown in [45] that when $\tilde{u}$ is affine Grassmannian,

$$\mathrm{linv}(R_r\tilde{u}) = \mathrm{linv}(\tilde{u}) + \sum_{i=1}^{r} (n-r)e_i \,, \tag{5.13}$$

where the e_i denote standard basis vectors. Note then that $R_i(\mathrm{id}) = \tilde{w}_{(i^{n-i})}$ and hence the $R_i := R_i(\mathrm{id})$ are called *k-rectangles*. In this case, the k-Schur function $s^{(k)}_{R_i}$ is simply $s_{(i^{n-i})}$ by (5.10). We shall extensively use that the multiplication of a k-Schur function by such a term is trivial. By [43, Theorem 40],

$$s^{(k)}_{R_i}\, s^{(k)}_{\tilde{u}} = s^{(k)}_{R_i\tilde{u}} \,. \tag{5.14}$$

Theorem 5.10. *Consider $\tilde{v}, \tilde{w} \in \tilde{S}_n^0$ and $\mu \subset (r^{n-r})$ where $1 \le r < n$. Let $R = \prod R_i$ be any product of k-rectangles. If $\ell(\tilde{v}) - \ell(\tilde{w}) \neq |\mu|$, then $c_{R\tilde{w}_\mu, \tilde{w}}^{R\tilde{v},k} = 0$. Otherwise, if either (i) $\tilde{v}\tilde{w}^{-1} \in S_{\hat{x}}$ for some $x \in [n]$ or (ii) $\ell(\mu) = 2$, then*

$$c_{R\tilde{w}_\mu, \tilde{w}}^{R\tilde{v},k} = \#\ \textit{of highest weight factorizations of}\ \tilde{v}\tilde{w}^{-1}\ \textit{of weight}\ \mu\,.$$

Proof. Given $\mu \subset (r^{n-r})$ for any $1 \le r < n$, first consider the case when R is the empty product. For this, we examine the coefficients in

$$s_{\tilde{w}_\mu}^{(k)}\, s_{\tilde{w}}^{(k)} = \sum_{\tilde{v}} c_{\tilde{w}_\mu, \tilde{w}}^{\tilde{v},k}\, s_{\tilde{v}}^{(k)}\,. \tag{5.15}$$

Degree conditions on polynomials imply that $c_{\tilde{w}_\mu, \tilde{w}}^{\tilde{v},k} = 0$ unless $\ell(\tilde{v}) = \ell(\tilde{w}) + |\mu|$. Otherwise, Proposition 5.9 indicates that the coefficients are equivalent to $a_{\tilde{z},\mu}$, where $\tilde{z} = \tilde{v}\tilde{w}^{-1}$. By Corollary 4.6, this is the number of highest weight factorizations of $\tilde{v}\tilde{w}^{-1}$ of weight μ.

Now multiply both sides of (5.15) by $s_R^{(k)}$ for any product $R = \prod R_i$ and use Equation (5.14) to find that $s_{R\tilde{w}_\mu}^{(k)}\, s_{\tilde{w}}^{(k)} = \sum_{\tilde{v}} c_{\tilde{w}_\mu, \tilde{w}}^{\tilde{v},k}\, s_{R\tilde{v}}^{(k)}$. Hence,

$$c_{R\tilde{w}_\mu, \tilde{w}}^{R\tilde{v},k} = c_{\tilde{w}_\mu, \tilde{w}}^{\tilde{v},k}\,, \tag{5.16}$$

and the result thus holds in the generality stated. □

5.3 Flag Gromov–Witten invariants

Let Fl_n be the complete flag manifold (chains of vector spaces) in $\mathbb{C}^n$. Fl_n admits a cell decomposition into *Schubert cells*, indexed by permutations of S_n. Details on their construction can be found in [56]. The set of Schubert classes $\{\sigma_w\}_{w \in S_n}$ forms a basis for the cohomology ring and counting points in the intersection of Schubert varieties (closures of Schubert cells) amounts to taking the cup product (1.1) of Schubert classes in $H^*(\mathrm{Fl}_n)$ [8, 16]. In turn, Lascoux and Schützenberger introduced an explicit set of polynomial representatives $\mathfrak{S}_w$ called Schubert polynomials whose structure coefficients,

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w c_{u,v}^w\, \mathfrak{S}_w\,, \tag{5.17}$$

give these intersection numbers.

As a linear space, the *quantum cohomology* of Fl_n is $\mathrm{QH}^*(\mathrm{Fl}_n) = H^*(\mathrm{Fl}_n) \otimes \mathbb{Z}[q_1, \ldots, q_{n-1}]$ for parameters $q_1, \ldots, q_{n-1}$. Its intricacy is in the multiplicative structure, defined by

$$\sigma_u * \sigma_w = \sum_v \sum_{\mathbf{d}} q_1^{d_1} \cdots q_{n-1}^{d_{n-1}} \langle u, w, v \rangle_{\mathbf{d}}\, \sigma_{w_0 v}\,, \tag{5.18}$$

where the structure constants $\langle u, w, v \rangle_d$ are 3-point Gromov–Witten invariants of genus 0 which count equivalence classes of certain rational curves in Fl_n. The combinatorial study of these invariants for flag manifolds is wide open. In fact, a manifestly positive formula in the case that $q_1 = \cdots = q_{n-1} = 0$, when the invariants reduce to coefficients in (5.17), has been an open problem for 40 years.

Our approach is to use an identification of the Gromov–Witten invariants with affine LR–coefficients that was made in [45]. It requires the map $\mathfrak{af} : S_n \to \tilde{S}_{n,\binom{n}{2}}$, and we thus allow for k-Schur functions indexed by extended affine permutations with increasing window. That is, using (5.2), we set $s_{\tilde{w}}^{(k)} = s_{\tilde{w}_{\mathcal{LC}(\tilde{w})}}^{(k)}$. Note that then affine LR coefficients may be indexed by extended affine Grassmannian elements in $\tilde{S}_{n,r}$ as well.

Theorem 5.11. *(proven in [45]) For any $u, v, w \in S_n$, the degree d, 3-point Gromov–Witten invariants of genus zero for complete flags are the affine LR–coefficients given by*

$$\langle u, w, w_0 v\rangle_{\mathbf{d}} = c_{\mathfrak{af}(u),\mathfrak{af}(w)}^{\mathfrak{af}_{\mathbf{d}}(v),k}\,, \tag{5.19}$$

where

$$\mathfrak{af}_{\mathbf{d}}(v) = \prod_{i=1}^{n-1} R_i^{d_{i-1}+d_{i+1}-2d_i+1}\,\mathfrak{af}(v)\,. \tag{5.20}$$

If $\mathfrak{af}_{\mathbf{d}}(v)$ is not affine Grassmannian or $\ell(u)+\ell(w)=\ell(v)+2\sum_{i=1}^{n-1} d_i$, then $\langle u, w, w_0 v\rangle_{\mathbf{d}} = 0$.

Theorem 5.10 thus applies to the study of Gromov–Witten invariants. The imposed conditions translate to the study of the natural subclass of $\langle u, w, w_0 v\rangle_{\mathbf{d}}$ where u is a coset representative of $S_n/S_r \times S_{n-r}$ for some r. A set of representatives is given by the *Grassmannian* permutations of S_n, characterized by having exactly one descent in position r. Each Grassmannian permutation u can be identified with a partition $\lambda(u) = (\lambda_1, \ldots, \lambda_r)$ in the $r \times (n-r)$ rectangle by setting $\lambda_i = |\{u_j \mid u_{r+1-i} > u_j \text{ and } j > r\}|$. We also use the complement partition $\lambda^\vee$ to λ in this rectangle; that is, $\lambda^\vee = (n-r-\lambda_r, \ldots, n-r-\lambda_1)$.

Theorem 5.12. *For any $\mathbf{d} \in \mathbb{N}^{n-1}$ and $u, w, v \in S_n$ where u is Grassmannian with descent at position r, let $\mu' = \lambda(u)^\vee$. If $\ell(v) \neq \ell(u)+\ell(w)-2\sum_i d_i$, then $\langle u, w, w_0 v\rangle_d = 0$. Otherwise, if (i) $(R\,v)w^{-1} \in S_{\hat{x}}$ for some $x \in [n]$ or (ii) $\ell(\mu) = 2$, then*

$$\langle u, w, w_0 v\rangle_{\mathbf{d}} = \#\text{ of highest weight factorizations of } (R\,v)w^{-1} \text{ with weight } \mu$$

for $R = R_r \prod_{i=1}^{n-1} R_i^{d_{i-1}+d_{i+1}-2d_i}$.

Proof. We shall use the correspondence $\langle u, w, w_0 v\rangle_{\mathbf{d}} = c_{\mathfrak{af}(u),\mathfrak{af}(w)}^{\mathfrak{af}_{\mathbf{d}}(v),k}$ of (5.19) as our guide. We first examine $\mathfrak{af}(u)$ in the case that $u \in S_n$ has exactly one descent at position r. It was shown in [45] that

$$\mathrm{linv}(\tilde{u}) = \lambda(u)^\vee\,, \tag{5.21}$$

for $\tilde{u} = \prod_{i\neq r} R_i^{-1}\mathfrak{af}(u) \in \tilde{S}_{n,p}$, where $p = \binom{n}{2}$. In particular, $\tilde{u}$ is affine Grassmannian and therefore has a unique decomposition $\tilde{u} = \tau^p \tilde{w}_\mu$ for some $\mu \in \mathcal{P}^n$. In fact, $\mu' = \lambda(u)^\vee$ since $\mathrm{linv}(\tilde{u}) = \mathrm{linv}(\tilde{w}_\mu) = \mu'$. Therefore, $\mathfrak{af}(u) = \prod_{i\neq r} R_i \tau^p \tilde{w}_\mu$ for $\mu' = \lambda(u)^\vee$.

Next consider $\mathfrak{af}_{\mathbf{d}}(v) = \left(\prod_{i\neq r} R_i\right) R\,\mathfrak{af}(v)$ where $R = R_r \prod_{i=1}^{n-1} R_i^{d_{i-1}+d_{i+1}-2d_i}$. From (5.14), it can be deduced that

$$\langle u, w, w_0 v\rangle_{\mathbf{d}} = c_{\prod_{i\neq r} R_i \tau^p \tilde{w}_\mu, \mathfrak{af}(w)}^{\prod_{i\neq r} R_i\, R\,\mathfrak{af}(v),k} = \begin{cases} c_{\tilde{w}_\mu, \tau^{-p}\mathfrak{af}(w)}^{\tilde{w}_\nu,k} & \text{if } R\,\mathfrak{af}(v) = \tau^p \tilde{w}_\nu\,, \\ 0 & \text{otherwise.} \end{cases} \tag{5.22}$$

We thus assume that $R\,\mathfrak{af}(v) = \tau^p \tilde{w}_\nu$ and consider the quantity $q(u,w,v) = \ell(u)+\ell(w)-2\sum_i d_i - \ell(v) = \ell(\mathfrak{af}(\mathrm{id})) - \ell(v) - 2\sum_i d_i - \ell(\mathfrak{af}(\mathrm{id})) + \ell(w) + \ell(u)$. Using that $\mu' = \lambda(u)^\vee$ and the length conditions of (5.8), we have that $q(u,w,v) = \ell(\mathfrak{af}(v)) - 2\sum_i d_i + (n-r)r - |\mu| - \ell(\mathfrak{af}(w))$. By (2.3) and Property (5.13), we then have

$$\ell(u)+\ell(w)-\ell(v)-2\sum_i d_i = \ell(R\mathfrak{af}(v)) - \ell(\mathfrak{af}(w)) - \ell(\tilde{w}_\mu)\,.$$

If $q(u,w,v) \neq 0$, then degree conditions on polynomial multiplication imply that $c_{\tilde{w}_\mu, \tau^{-p}\mathfrak{af}(w)}^{\tilde{w}_\nu,k} = 0$. Otherwise, note that $\tilde{w}_\nu(\tau^{-p}\mathfrak{af}(w))^{-1} = \tau^{-p}R\mathfrak{af}(v)\mathfrak{af}(\mathrm{id})^{-1}w^{-1}\tau^p = \tau^{-p}(Rv)w^{-1}\tau^p$. Since τ acts on reduced expressions by adding a constant to each letter, the number of affine factorizations of $\tau^{-p}(Rv)w^{-1}\tau^p$ and of $(Rv)w^{-1}$ are the same. Moreover, if $(Rv)w^{-1} \in S_{\hat{x}}$ for some $x \in [n]$, then $\tilde{w}_\nu(\tau^{-p}\mathfrak{af}(w))^{-1} \in S_{\widehat{\tau^p x}}$. The result thus follows from Theorem 5.10. $\square$

Corollary 5.13. *For* $\mathbf{d} \in \mathbb{N}^{n-1}$ *and* $u, w, v \in S_n$ *where* u *has exactly one descent at position* r*, let* $\mu' = \lambda(u)^\vee$*. If* $\ell(v) \neq \ell(u)+\ell(w)-2\sum_i d_i$ *then* $\langle u, w, w_0 v\rangle_{\mathbf{d}} = 0$*. Otherwise, if* $\ell(u) \geq n(n-r-1)$*, then*

$$\langle u, w, w_0 v\rangle_{\mathbf{d}} = \#\text{ of highest weight factorizations of } (Rv)w^{-1} \text{ of weight } \mu,$$

where $R = R_r \prod_{i=1}^{n-1} R_i^{d_{i-1}+d_{i+1}-2d_i}$.

Proof. We again use identification (5.22) of the invariants $\langle u, w, w_0 v\rangle_{\mathbf{d}}$ with $c^{\tilde{w}_\nu,k}_{\tilde{w}_\mu,\tau^{-p}\mathfrak{af}(w)}$, where $\tau^p\tilde{w}_\nu = R\mathfrak{af}(v)$. The substitution of monomials in terms of Schur functions (via the inverse Kostka matrix $\bar{K}$) into the monomial expansion (5.5) of skew dual k-Schur functions gives an alternating expression,

$$c^{\tilde{w}_\nu,k}_{\tilde{w}_\mu,\tilde{w}_\lambda} = \sum_{\alpha:|\alpha|=\ell(\tilde{w}_\nu\tilde{w}_\lambda^{-1})} \mathcal{K}_{\tilde{w}_\nu\tilde{w}_\lambda^{-1},\alpha}\,\overline{K}_{\alpha,\mu}\,, \tag{5.23}$$

for the affine LR coefficients. Thus, they count a subset of affine factorizations of $\tilde{v} = \tilde{w}_\nu\mathfrak{af}(w)^{-1}\tau^p$ of weight α, where $|\alpha| = |\mu|$. Recall that an affine factorization of $\tilde{v}$ has weight α only if $\ell(\tilde{v}) = |\alpha|$. Since $\ell(u) = |\lambda(u)| \geq n(n-r-1)$, we have that $|\mu| < n$ and hence $\ell(\tilde{v}) < n$. But $\tilde{v} \in \tilde{S}_n$ and therefore $\tilde{v} \in S_{\hat{x}}$ for some $x \in [n]$. The result then follows by Theorem 5.12. $\square$

Theorem 5.12 and its corollary apply to the problem of describing structure constants in the product of a Schur polynomial by a (quantum) Schubert polynomial [48, 19]. Recall (e.g. [21, Section 10.6, Proposition 8]) that when $u \in S_n$ is Grassmannian with a descent at position r, the Schubert polynomial $\mathfrak{S}_u$ is simply the Schur polynomial $s_{\lambda(u)}(x_1, \ldots, x_r)$. Thus, we can address the coefficients in

$$s_\lambda(x_1, \ldots, x_r)\,\mathfrak{S}_w = \sum_{v \in S_n} \langle u, w, w_0 v\rangle_0\,\mathfrak{S}_v\,,$$

and in the quantum ($\mathbf{d} \neq 0$) analog.

In [57], the Fomin-Kirillov algebra is used to study the Gromov–Witten invariants $\langle u, w, w_0 v\rangle_{\mathbf{d}}$ for $u, v, w \in S_n$ where u is Grassmannian and $\lambda(u)$ is a hook shape. Their conditions on u imply $|\lambda(u)| < n$ and thus suggest a relation to a subset of the cases treated in Corollary 5.13. However, despite satisfying a number of symmetry properties (e.g. [64]), the Gromov–Witten invariants are not symmetric under the complementing of $\lambda(u)$ and there is no apparent relation.

Corollary 5.13 can be used to give new results for the classical case by setting $\mathbf{d} = 0$. For these, there is an even simpler highest weight formulation.

Corollary 5.14. *Let* $u, v, w \in S_n$ *where* u *has exactly one descent at position* r*. If* $\ell(v) \neq \ell(u) + \ell(w)$*, then* $\langle u, w, w_0 v\rangle_0 = 0$*. Otherwise, if either (i)* $(R_r v)w^{-1} \in S_{\hat{x}}$ *for some* $x \in [n]$ *or (ii)* $\ell(\mu) = 2$ *for* $\mu' = \lambda(u)^\vee$*, then*

$$\langle u, w, w_0 v\rangle_0 = \#\text{ of highest weight factorizations of } (R_r v)w^{-1} \text{ of weight } \mu.$$

Example 5.15. Let $u = (1, 2, 4, 7, 3, 5, 6)$, $w = (3, 1, 5, 4, 2, 6, 7)$, and $v = (4, 2, 5, 7, 1, 3, 6)$ be permutations of S_7 in one-line notation. Since u is Grassmannian with its descent at position $r = 4$, we find $\mu' = (3, 3, 2)$ by taking the complement of $\lambda(u) = (3, 1)$. Note that $\ell(w) = 5$ and $\ell(v) = 9$ and indeed $\ell(u) = \ell(v) - \ell(w)$. By Corollary 5.14, we compute the coefficient of $\mathfrak{S}_v$ in

$s_{\lambda(u)}(x_1,\ldots,x_4)\mathfrak{S}_w$ by counting the highest weight factorizations of $\sigma = (R_4 v)w^{-1}$ with weight $(3,3,2)$.

We note that $w^{-1} = (2,5,1,4,3,6,7)$ and use (5.12) to find $R_4 v = [1,-1,2,4,5,7,10]$. This gives $\sigma = s_6 s_2 s_3 s_4 s_3 s_1 s_2 s_0$ whose affine factorizations $\sigma = v^3 v^2 v^1$ of weight $\mu = (3,3,2)$ satisfy $\ell(v^3) = 2, \ell(v^2) = 3, \ell(v^1) = 3$ and each has a word that is decreasing with respect to $4 > 3 > 2 > 1 > 0 > 6$ (since $\sigma \in S_{\tilde{5}}$). Possible affine factorizations are:

$$\mathrm{con}(v^3)\mathrm{con}(v^2)\mathrm{con}(v^1) = (26)(431)(420) \quad (26)(310)(432) \quad (42)(316)(420) \quad (21)(436)(420)\ ,$$

and valid highest weights satisfy the extra condition that all elements in factor $\mathrm{con}(v^i)$ are paired with an element of $\mathrm{con}(v^{i-1})$, for each $i > 1$. Since $(26)(310)(432)$ is the only such factorization, we find $\langle u, w, w_0 v\rangle_0 = 1$.

5.4 Quantum cohomology of the Grassmannian and fusion coefficients

As with the quantum cohomology of full flags, the small quantum cohomology ring of the Grassmannian $\mathrm{QH}^*(\mathrm{Gr}(r,n))$ is a deformation of the usual cohomology. As a linear space, this is the tensor product $H^*(\mathrm{Gr}(r,n)) \otimes \mathbb{Z}[q]$ and $\{\sigma_\lambda\}_{\lambda\subset(r^{n-r})}$ forms a $\mathbb{Z}[q]$-linear basis of $\mathrm{QH}^*(\mathrm{Gr}(r,n))$. Multiplication is a q-deformation of the product in $H^*(\mathrm{Gr}(r,n))$, defined by

$$\sigma_\lambda * \sigma_\mu = \sum_{\substack{\nu\subset(r^{n-r})\\ |\nu|=|\lambda|+|\mu|-dn}} q^d\, \langle\lambda,\mu,\nu\rangle_d\, \sigma_{\nu^\vee}\ .$$

where the $\langle\lambda,\mu,\nu\rangle_d$ are the 3-point Gromov–Witten invariants of genus 0 for the Grassmannian. These constants count the number of maps $f\colon \mathbb{P}_1 \to \mathrm{Gr}(r,n)$ whose image has degree d and meets generic translates of Schubert varieties associated to λ, μ, and ν.

Theorem 5.16. *[44, Theorem 5.6] For $\lambda,\mu,\nu \subset (r^{n-r})$, $\langle\lambda,\mu,\nu\rangle_d = c_{w_\lambda,w_\mu}^{w_{\hat\nu},n-1}$, where $\hat\nu$ is constructed from ν by adding d n-rim hooks, each starting in column r and ending in the first column.*

This theorem combined with Theorem 5.10 shows that the crystal on affine factorizations applies directly the quantum cohomology of the Grassmannian. Furthermore, Postnikov [63] defined cylindric Schur functions $s_{\nu/d/\lambda}$ indexed by skew cylindric shapes $\nu/d/\lambda$ and proved that the 3-point genus 0 Gromov–Witten invariants for the Grassmannian $\langle\lambda,\mu,\nu\rangle_d$ appear in the Schur expansion of the toric Schur functions [63, Theorem 5.3] (which are restrictions to finitely many variables)

$$s_{\nu/d/\lambda}(x_1,\ldots,x_r) = \sum_{\mu\subseteq((n-r)^r)} \langle\lambda,\mu,\nu\rangle_d\, s_\mu(x_1,\ldots,x_r)\ . \tag{5.24}$$

Lam proved in [38, Theorem 36] that the cylindric Schur functions are precisely the subset of affine Stanley symmetric functions (or skew dual k-Schur functions) whose index set is the affine permutations containing no braid relation $s_i s_{i+1} s_i$, also called 321-*avoiding affine permutations.* This suggests that there is a crystal structure on 321-avoiding affine factorizations that would describe the 3-point, genus 0, Gromov–Witten invariants of the Grassmannian in general without the conditions imposed by Theorem 5.10. Buch et al. [14] recently proved Knutson's puzzle rule for the Schubert structure constants on two-step flag varieties, which by [13, Corollary 1] yield Gromov–Witten invariants defining the (small) quantum cohomology ring $\mathrm{QH}^*(\mathrm{Gr}(r,n))$ of a Grassmann variety in type A.

It was proven [1, 3] that the structure constants of the quantum cohomology of the Grassmannian are related to the fusion coefficients. We now reformulate our results in the fusion setting. For $n > \ell \geq 1$, consider the quotient $R^{\ell n} = \Lambda_{(\ell)}/I^{\ell n}$, where $I^{\ell n}$ is the ideal generated by Schur functions that have exactly $n - \ell + 1$ rows of length smaller than ℓ:

$$I^{\ell n} = \left\langle s_\lambda \,\middle|\, \#\{j \mid \lambda_j < \ell\} = n - \ell + 1 \right\rangle .$$

The Verlinde (fusion) algebra of the WZW model associated to $\widehat{su}(\ell)$ at level $n-\ell$ is isomorphic to the quotient of $R^{\ell n}$ modulo the single relation $s_\ell \equiv 1$ [24, 25, 77]. The fusion coefficient $\mathcal{N}_{\lambda,\mu}^{\nu}$ is defined for $\lambda, \mu, \nu \subseteq ((n-\ell)^{\ell-1})$ as in (1.5). It was shown in [44] that

$$\mathcal{N}_{\lambda,\mu}^{\nu} = c_{\tilde{w}_{\lambda'}\tilde{w}_{\mu'}}^{\tilde{w}_{\hat{\nu}},k} ,$$

where $\hat{\nu} = (\ell^{(|\lambda|+|\mu|-|\nu|)/\ell}, \nu')$.

Corollary 5.17. *[58] Let $\lambda, \mu, \nu \subseteq ((n-\ell)^{\ell-1})$ and $\hat{\nu} = (\ell^{(|\lambda|+|\mu|-|\nu|)/\ell}, \nu')$. Whenever $\tilde{w}_{\hat{\nu}}\tilde{w}_{\lambda'}^{-1} \in S_{\hat{x}}$ for some $x \in [n]$ or $\ell(\mu') = 2$, the fusion coefficient $\mathcal{N}_{\lambda,\mu}^{\nu}$ is the number of highest weight factorizations of $\tilde{w}_{\hat{\nu}}\tilde{w}_{\lambda'}^{-1}$ of weight μ'.*

5.5 Positroid stratification

The real Grassmannian variety $\mathrm{Gr}(r,n)$ can be represented by the quotient space of $r \times n$ matrices (with at least one non-zero maximal minor) modulo the left action of the group GL_r of real $r \times r$ matrices with non-zero determinant. The best understood cellular decomposition of the Grassmannian is given by the disjoint union of Schubert cells Ω_λ indexed by partitions $\lambda \subset ((n-r)^r)$. The totally nonnegative Grassmannian $\mathrm{Gr}(r,n)_{\geq 0}$ is a subdivision of the Grassmannian $\mathrm{Gr}(r,n)$, presented in [62] by the $r \times n$ matrices X with all nonnegative Plücker coordinates $\Delta_I(X)$ (maximal $r \times r$ minors).

For each point $X \in \mathrm{Gr}(r,n)_{\geq 0}$, the *positroid* of X is defined by

$$\mathcal{M}_X = \left\{ I \in \binom{[n^+]}{r} \mid \Delta_I(X) > 0 \right\} ,$$

where $\binom{[n^+]}{r}$ is the set of r-element subsets of $\{1, \dots, n\}$. When the collection of points

$$\mathring{\Pi}_{\mathcal{M}} = \{X \in \mathrm{Gr}(r,n)_{\geq 0} \mid \mathcal{M}_X = \mathcal{M}\}$$

is non-empty, it is called a *positroid cell*. Postnikov established that the indexing sets for positroid cells are given by various combinatorial objects such as Grassmann necklaces, Le diagrams, and plabic graphs.

It turns out that the complexification of positroid cells relates to the projected Richardson stratification for the full flag manifold. In particular, the positroid cells are *open positroid varieties*, defined in [30] to be an intersection of n Schubert cells, taken with respect to the n cyclic rotations of the standard flag. The closure of the open positroid varieties are called the *positroid varieties*. Knutson, Lam, and Speyer proved that the positroid varieties are Richardson varieties projected to $\mathrm{Gr}(r,n)$ and they introduced yet another equinumerous indexing set; the set of *bounded affine permutations*,

$$\mathrm{Bound}(r,n) = \{\tilde{w} \in \tilde{S}_{n,r} \mid i \leq \tilde{w}(i) \leq i+n\} .$$

We shall denote positroid varieties by $\Pi_{\tilde{w}}$, for $\tilde{w} \in \mathrm{Bound}(r,n)$, and refer the reader to [30] for details on the map from $\tilde{w}$ to $\mathcal{M}$.

Remarkably, the cohomology classes of positroid varieties can be represented by a projection of affine Stanley symmetric functions. Consider the map $\psi : \Lambda \to H^*(\mathrm{Gr}(r,n))$ where

$$\psi(s_\lambda) = \begin{cases} [\Omega_\lambda] & \text{when } \lambda \subset (r^{n-r}) \\ 0 & \text{otherwise.} \end{cases}$$

It was proven ([30], Theorem 7.1) that for each $\tilde{w} \in \mathrm{Bound}(r,n)$,

$$\psi(F_{\tilde{w}}) = [\Pi_{\tilde{w}}] \in H^*(\mathrm{Gr}(r,n))\,.$$

It was also shown there that as long as $\tilde{w}$ is 321-avoiding, the decomposition of positroid classes into Schubert classes is given by Gromov–Witten invariants for the Grassmannian. More generally, we use Proposition 5.9 to show that for any bounded $\tilde{w}$, the decomposition is given by certain k-Schur structure constants where now Gromov–Witten invariants for flags appear.

Theorem 5.18. *Let $\tilde{w} \in \mathrm{Bound}(r,n)$. The cohomology class of a positroid decomposes over Schubert classes as*

$$[\Pi_{\tilde{w}}] = \sum_{\lambda \subset ((n-r)^r)} a_{\tilde{w},\lambda}\,[\Omega_\lambda]\,,$$

where the set of $a_{\tilde{w},\lambda}$ are affine LR-coefficients, which include 3-point Gromov–Witten invariants for the complete flag Fl_n.

Proof. Recall that the cohomology $H^*(\mathrm{Gr}(r,n))$ can be presented as the quotient

$$\phi\colon H^*(\mathrm{Gr}(r,n)) \cong \Lambda/\mathcal{I}\,,$$

where the ideal $\mathcal{I} = \langle e_i, h_j \mid i > r, j > n-r\rangle$. Since $\mathcal{I}$ is spanned by the Schur functions s_λ whose shapes do not fit inside the $r \times (n-r)$-rectangle, isomorphism ϕ is given by identifying the Schubert class $[\Omega_\lambda]$ to (the coset of) the Schur function s_λ. The map ψ thus decomposes into the composition of ϕ and $\bar{\psi}$, where $\bar{\psi}$ is the quotient map from Λ to $\Lambda/\mathcal{I}$ defined by annihilating each s_λ for $\lambda \not\subset ((n-r)^r)$. Thus we have

$$\bar{\psi}(F_{\tilde{w}}) = \sum_{\mu \subset ((n-r)^r)} a_{\tilde{w},\mu}\, s_\mu$$

and by Proposition 5.9 the coefficients $a_{\tilde{w},\mu}$ are affine LR-coefficients. □

Applying Theorem 5.12 (or Corollary 4.6) to this result, we can explicitly describe the decomposition in terms of the crystal in many cases.

Corollary 5.19. *When $\tilde{w} \in S_{\hat{x}} \subseteq \mathrm{Bound}(r,n)$ for some $x \in [n]$,*

$$[\Pi_{\tilde{w}}] = \sum_{\lambda \subset ((n-r)^r)} a_{\tilde{w},\lambda}\,[\Omega_\lambda]\,,$$

where $a_{\tilde{w},\lambda} = |\{v \in \mathcal{W}_{\tilde{w},\lambda} \mid \tilde{e}_i(v) = \mathbf{0}\ \forall i\}|$ counts highest weight factorizations of the crystal $B(\tilde{w})$.

6 Crystal operator involution

The broad goal of this article is to introduce the powerful theory of crystals into the combinatorial study of affine Schubert calculus and Gromov–Witten invariants. However, Proposition 5.9 suggests that existing crystal theory is not enough to address invariants $\langle u, w, v\rangle_d$ without the limitation that u is Grassmannian. It is with this in mind that we give here an alternate proof of Theorem 5.10 that circumvents the need for Theorem 2.5.

An elegant proof of the formulation of the Littlewood–Richardson rule as highest weights in the $\mathfrak{sl}_\ell$-crystal of Section 4 was given by Remmel and Shimozono [68]. They substitute a Schur function, in the product of Schur functions, by its expansion in terms of homogeneous symmetric functions to obtain

$$s_\lambda s_\mu = s_\lambda \sum_\alpha \overline{K}_{\alpha,\mu}\, h_\alpha \;= \sum_{\nu,\alpha} K_{\nu/\lambda,\alpha}\overline{K}_{\alpha,\mu}\, s_\nu\,.$$

Here, the Kostka numbers $K_{\nu/\lambda,\alpha}$ enumerate the set $\mathrm{SSYT}(\nu/\lambda,\alpha)$ of skew tableaux of shape ν/λ and weight α, whereas $\overline{K}_{\alpha,\mu}$ is an alternating sum. To be precise, let m be a positive integer which weakly exceeds the length of the partitions α and μ, let S_m be the symmetric group of permutations in m letters, and define $\rho = (m-1, m-2, \ldots, 1, 0)$. Then in [68, Eq. (1.11)] the inverse Kostka matrix was expressed as

$$\overline{K}_{\alpha,\mu} = \sum_{\sigma\in S_m} (-1)^{\mathrm{sign}(\sigma)}\,, \tag{6.1}$$

over permutations σ where $\sigma(\mu+\rho)-\rho \in S_m\alpha$. From this,

$$s_\lambda s_\mu = \sum_{\nu,\alpha}\sum_{(T,\sigma)} (-1)^{\mathrm{sign}(\sigma)} s_\nu\,,$$

where $\sigma \in S_m$ such that $\sigma(\mu+\rho)-\rho \in S_m\alpha$ and $T \in \mathrm{SSYT}(\nu/\lambda,\alpha)$. The trick to canceling the negative terms lies in an involution that is defined using $\tilde{s}_i\tilde{e}_i$ for a suitable i, where $\tilde{e}_i$ are the crystal raising operators on tableaux and $\tilde{s}_i(T) = \tilde{e}_i^p(T)$ if $p := \varepsilon_i(T) - \varphi_i(T) \geq 0$ and $\tilde{s}_i(T) = \tilde{f}_i^{-p}(T)$ if $p < 0$ are the reflections within an i-string. The Littlewood–Richardson rule follows since the Yamanouchi tableaux are fixed points under the involution.

In the same spirit, we produce a sign-reversing involution using the crystal operators on affine factorizations.

Definition 6.1. For any $\mu \subset (a^{n-a})$ and $w \in S_{\hat{x}}$ for some fixed $x \in [n]$, define θ to act on the set of pairs $\bigcup_\beta \{\sigma \in S_m \mid \sigma(\mu+\rho)-\rho = \beta\} \times \mathcal{W}_{w,\beta}$ for $m > \ell(w)$ by

$$\theta(\sigma, w^\beta) = \begin{cases} (\sigma, w^\beta) & \text{when} \quad X = \emptyset,\\ (s_r\sigma, \tilde{s}_r\tilde{e}_r(w^\beta)) & \text{otherwise}\,,\end{cases}$$

where $X = \bigcup_i L_i(w^\beta)$ and $r = \max\{i \mid \max(X) \in L_i(w^\beta)\}$.

Proposition 6.2. *For any $\mu \subset (a^{n-a})$ and $w \in S_{\hat{x}}$ for some fixed $x \in [n]$, θ is an involution whose fixed points are $\{(\mathrm{id}, w^\mu) \mid w^\mu$ is a highest weight factorization of w with weight $\mu\}$.*

Proof. Consider a factorization w^β of $w \in S_{\hat{x}}$ with $r = \max\{i \mid \max(X) \in L_i(w^\beta)\}$ and let $w^\gamma = \tilde{s}_r\tilde{e}_r(w^\beta)$. It is known for crystal operators (see for example [68]) that $(\tilde{s}_r\tilde{e}_r)^2$ either acts as the identity or annihilates an element. Hence our main task in verifying that θ is an involution is to prove

$$\theta^2(\sigma, w^\beta) = (s_r^2\sigma, (\tilde{s}_r\tilde{e}_r)^2(w^\beta))$$

when $X \neq \emptyset$ since $(s_r^2\sigma, (\tilde{s}_r\tilde{e}_r)^2(w^\beta)) = (\sigma, w^\beta)$ by the above argument. To do so, we must show that $r = \max\{i \mid \max(Y) \in L_i(w^\gamma)\}$ for $Y = \bigcup_i L_i(w^\gamma)$ when $X \neq \emptyset$.

Let $uv = w^{\beta_{r+1}}w^{\beta_r}$ and $UV = \tilde{s}_1\tilde{e}_1(uv)$ and set $z = \max(L_1(uv))$. Consider

$$u = (u_1 s_z u_2) \quad \text{and} \quad v = (v_1 v_2) \tag{6.2}$$

for $\operatorname{con}(u_1) = \{c \in \operatorname{con}(u) \mid c > z\}$ and $\operatorname{con}(v_1) = \{c \in \operatorname{con}(v) \mid c \geq z\}$. We start by showing that $\max(L_1(UV)) = z$ and that

$$U = (u_1 s_z U_2) \quad \text{and} \quad V = (v_1 V_2)\,, \tag{6.3}$$

where the elements in $\operatorname{con}(U_2)$ and $\operatorname{con}(V_2)$ are smaller than z. To this end, let $\tilde{u}\tilde{v} = \tilde{e}_1(uv)$ and set $p = |L_1(\tilde{u}\tilde{v})| > 0$ and $q = |R_1(\tilde{u}\tilde{v})|$. Then

$$UV = \begin{cases} \tilde{e}_1^{p-q}(\tilde{u}\tilde{v}) & \text{if } p > q, \\ \tilde{u}\tilde{v} & \text{if } p = q, \\ \tilde{f}_1^{q-p}(\tilde{u}\tilde{v}) & \text{if } p < q\,. \end{cases}$$

Let $b = \min(L_1(uv))$ and $t = \max\{i \geq 0 \mid b - i - 1 \notin \operatorname{con}(u)\}$. (3.7) tells us that $\tilde{u}$ differs from u by the deletion of b and $\tilde{v}$ differs from v by the addition of $b - t$. We deduce from Lemma 3.8 that $\max(R_1(\tilde{u}\tilde{v}))) = b - t$ (implying that $q > 0$) and $\max(L_1(\tilde{u}\tilde{v})) = z$ unless $L_1(\tilde{u}\tilde{v}) = \emptyset$.

When $p = q$, $q > 0$ implies that $\max(L_1(UV)) = \max(L_1(\tilde{u}\tilde{v})) = z$ and (6.3) holds. When $p > q$, we can iterate our deduction to find that $\max(L_1(UV)) = z$ (since $p - q < p$) and that U differs from u by the deletion of letters smaller than z and V differs from v by the addition of letters smaller than z implying (6.3). If $p < q$, we have that $UV = \tilde{f}_1^{q-p-1}(uv)$ since $\tilde{f}_1\tilde{e}_1 = \mathrm{id}$ by Proposition 3.10. Lemma 3.9 can then be used to prove our claim in this last case.

Now given that $\max(L_1(UV)) = z$, since $\tilde{s}_r$ and $\tilde{e}_r$ act on w^β by changing only factors uv into UV, it suffices to verify

$$\max\{i \mid \max(Y) \in L_i(\tau UV\sigma)\} = 2 \;\text{ for } Y = \bigcup_i L_i(\tau UV\sigma)\,,$$

where $\tau = w^{\beta_{r+2}}$ and $\sigma = w^{\beta_{r-1}}$. Consider the decomposition

$$\tau\, u = (\tau_1\tau_2)(u_1 s_z u_2)\,,$$

where $\operatorname{con}(\tau_1) = \{c \in \operatorname{con}(\tau) \mid c \geq z\}$. If $\tilde{z} = \max(L_1(\tau u))$, then our choice of r implies $\tilde{z} < z$. Therefore, every element in τ_1 is paired with something in u_1 in the $\tau\, u$ pairing. Since $\tau\, U = (\tau_1\tau_2)(u_1 s_z U_2)$, we find that $\max(L_1(\tau U)) < z$.

Next consider

$$v\sigma = (v_1 v_2)(\sigma_1\sigma_2)\,,$$

where $\operatorname{con}(v_1) = \{c \in \operatorname{con}(v) \mid c \geq z\}$ and $\operatorname{con}(\sigma_1) = \{c \in \operatorname{con}(\sigma) \mid c > z\}$. Since our choice of r implies here that $\max(L_1(v\sigma)) \geq z$, we have that every element in $\operatorname{con}(v_1)$ is paired with something in $\operatorname{con}(\sigma_1)$. Since $V = (v_1 V_2)$, we have that $\max(L_1(V\sigma)) \leq z$.

Let (σ, w^β) be a fixed point of θ. If $\bigcup_i L_i(w^\beta) = \emptyset$, then every element of $\operatorname{con}(w^{\beta_i})$ is paired with something in $\operatorname{con}(w^{\beta_{i-1}})$ and in particular, $\beta_{i-1} \leq \beta_i$ for all i. Given $\sigma(\mu + \rho) - \rho = \beta$, we have that $\beta_j = \mu_i + j - i$ where $\sigma(i) = j$. Since β and μ are partitions, $\beta_1 = \max\{\mu_i + 1 - i\} = \mu_1$ implying that $\sigma(1) = 1$. By iteration, $\sigma = \mathrm{id}$ and $\beta = \mu$. By Definition 3.6, w^β is a highest weight factorization of weight μ. $\square$

Remark 6.3. Note that the proof of Proposition 6.2 goes through in almost the identical manner if we defined $r = \min\{i \mid \max(X) \in L_i(w^\beta)\}$ instead of $r = \max\{i \mid \max(X) \in L_i(w^\beta)\}$. In the $k \to \infty$ limit, these choices correspond to choosing the rightmost violation of the Yamanouchi word condition in the row and column reading of the tableau, respectively. In [68], row reading was chosen, so $r = \min\{i \mid \max(X) \in L_i(w^\beta)\}$ in our formulation.

The previous proposition immediately implies that

$$\sum_{\beta} \sum_{\substack{(\sigma, w^\beta) \\ \theta(\sigma,w^\beta)\neq(\sigma,w^\beta)}} (-1)^{\mathrm{sign}(\sigma)} = 0\,,$$

leaving only the θ-fixed points (id, w^μ), where w^μ is a highest weight factorization.

Corollary 6.4. *For any $\mu \subset (a^{n-a})$ and $w \in S_{\hat{x}}$ for some fixed $x \in [n]$,*

$$\sum_{\beta} \sum_{(\sigma, w^\beta)} (-1)^{\mathrm{sign}(\sigma)} = \#\{w^\mu \in \mathcal{W}_{w,\mu} \mid w^\mu \text{ is highest weight }\}\,,$$

where the sum is over pairs (σ, w^β) with $\sigma(\mu+\rho)-\rho = \beta$ and w^β is an affine factorization of w with weight β.

The involution θ allows us to give an alternate proof of Theorem 5.10.

Alternative proof of Theorem 5.10. Since (5.14) implies that $c^{Rv,k}_{Ru,w} = c^{v,k}_{u,w}$, we are only concerned with products $s_\mu s^{(k)}_w$ for some $\mu \subset (a^{n-a})$. Substitution of formula (6.1) for inverse Kostka numbers into the h-expansion of a Schur function $s_\mu = \sum_\alpha \overline{K}_{\alpha,\mu}\, h_\alpha$ gives

$$s_\mu = \sum_{\alpha \in \mathcal{P}^n} \sum_{\substack{\sigma \in S_m \\ \sigma(\mu+\rho)-\rho \in S_m\alpha}} (-1)^{\mathrm{sign}(\sigma)}\, h_\alpha\,,$$

where $\rho = (m-1, m-2, \ldots, 1, 0)$ and $m \leq n$ is weakly bigger than the number of parts in μ. This given, we use (3.2) to k-Schur expand the product of h_α with a k-Schur function $s^{(k)}_w$ in

$$s_\mu\, s^{(k)}_w = \sum_{\alpha \in \mathcal{P}^n} \sum_{\substack{\sigma \in S_m \\ \sigma(\mu+\rho)-\rho \in S_m\alpha}} (-1)^{\mathrm{sign}(\sigma)}\, h_\alpha\, s^{(k)}_w = \sum_{v \in \tilde{S}^0_n} \sum_{\alpha \in \mathcal{P}^n} \mathcal{K}_{vw^{-1},\alpha} \sum_{\substack{\sigma \in S_m \\ \sigma(\mu+\rho)-\rho \in S_m\alpha}} (-1)^{\mathrm{sign}(\sigma)}\, s^{(k)}_v\,.$$

In fact, $\mathcal{K}_{vw^{-1},\alpha} = \mathcal{K}_{vw^{-1},\beta}$ for any rearrangement β of α leading us to the expression

$$s_\mu\, s^{(k)}_w = \sum_{v \in \tilde{S}^0_n} \sum_{\beta} \sum_{(\sigma, w^\beta)} (-1)^{\mathrm{sign}(\sigma)}\, s^{(k)}_v\,,$$

where the sum is over pairs (σ, w^β) with $\sigma(\mu+\rho)-\rho = \beta$ and w^β is an affine factorization of vw^{-1} with weight β. Corollary 6.4 yields the desired result.

Note that if μ has only two parts, we can choose $m = 2$ and in this case θ is still defined as in Definition 6.1 with $r = 1$ by Section 3.4. □

A Appendix

Proof of Theorem 3.5. The proof proceeds by checking Stembridge's local axioms of Section 2.2. We freely use the properties of the operators $\tilde{e}_i$ and $\tilde{f}_i$ established in Section 3.3. The lengths of the monochromatic directed paths are given by $\varepsilon_r(w^\beta)$ and $\varphi_r(w^\beta)$ for every $w^\beta \in \mathcal{W}_w$. By Proposition 3.10 (3) these are given by the number of unbracketed letters, which is finite. This shows (P1) of Stembridge's local axioms. (P2) is also ensured by the definition of the crystal operators on affine factorizations.

Proof of (P3) and (P4)

Next we consider axioms (P3) and (P4) by proving that

$$(a_{ij}, \Delta_i\varepsilon_j(w^\beta), \Delta_i\varphi_j(w^\beta)) \in \{(0,0,0), (-1,-1,0), (-1,0,-1)\}\,.$$

If $a_{ij} = 0$, then $\tilde{e}_i$ and $\tilde{e}_j$ (resp. $\tilde{f}_j$) commute, so that indeed $(a_{ij}, \Delta_i\varepsilon_j(w^\beta), \Delta_i\varphi_j(w^\beta)) = (0,0,0)$ in this case. Next consider $a_{ij} = -1$, so that $j = i-1$ or $j = i+1$. First assume $j = i-1$. Without loss of generality we may assume that $w^\beta = yuv$ is a product of three factors and $i = 2$. Since by assumption $\tilde{e}_2(yuv)$ is defined, one letter moves from factor y to factor u. Call s_c the new generator in u under $\tilde{e}_2$. Recall that as in Lemma 3.8, we may write

$$uv = (u_1 s_b \cdots s_{b-t} u_2)(v_1 s_b \cdots s_{b-t+1} v_2), \tag{A.1}$$

where $b = \min(L_1(uv))$, all letters in $\mathrm{con}(v_1)$ are paired with something in $\mathrm{con}(u_1)$, and every element in $\mathrm{con}(u_2)$ is with with something in $\mathrm{con}(v_2)$. Note that

1. If $c > b$ we have $\varepsilon_1(\tilde{e}_2 yuv) = \varepsilon_1(yuv) + 1$ and $\varphi_1(\tilde{e}_2 yuv) = \varphi_1(yuv)$ since still all letters in $\mathrm{con}(v_1)$ are paired and there is one extra unpaired letter in $\mathrm{con}(u_1) \cup \{c\}$ after the application of $\tilde{e}_1$. Hence $(a_{21}, \Delta_2\varepsilon_1(yuv), \Delta_2\varphi_1(yuv)) = (-1,-1,0)$.

2. If $c < b-t$, we have two cases:

   (a) If c does not pair with a letter in $\mathrm{con}(v_2)$, then as before $\varepsilon_1(\tilde{e}_2 yuv) = \varepsilon_1(yuv)+1$ and $\varphi_1(\tilde{e}_2 yuv) = \varphi_1(yuv)$, so that again $(a_{21}, \Delta_2\varepsilon_1(yuv), \Delta_2\varphi_1(yuv)) = (-1,-1,0)$.

   (b) If c does pair with a letter in $\mathrm{con}(v_2)$, then $\varepsilon_1(\tilde{e}_2 yuv) = \varepsilon_1(yuv)$ and $\varphi_1(\tilde{e}_2 yuv) = \varphi_1(yuv) - 1$, so $(a_{21}, \Delta_2\varepsilon_1(yuv), \Delta_2\varphi_1(yuv)) = (-1,0,-1)$.

This proves (P3) and (P4) for $j = i-1$.

Now assume that $j = i+1$. Without loss of generality we may assume that $w^\beta = uvy$ is a product of three factors and $i = 1$. Since by assumption $\tilde{e}_1(uvy)$ is defined, one letter moves from factor v to factor y under $\tilde{e}_1$. Call s_c the generator in v that disappears. We can write uv again as in (A.1). Note that

1. If $c \in \mathrm{con}(v_1)$, then $\varepsilon_2(\tilde{e}_1 uvy) = \varepsilon_2(uvy)+1$ and $\varphi_2(\tilde{e}_1 uvy) = \varphi_2(uvy)$ since one less letter is bracketed in $\mathrm{con}(u_1)$. Hence $(a_{12}, \Delta_1\varepsilon_2(uvy), \Delta_1\varphi_2(uvy)) = (-1,-1,0)$.

2. If $c \in \{b-t+1, b-t+2, \ldots, b\}$, then the letter $c-1$ becomes unbracketed in $\mathrm{con}(u)$, so that $\varepsilon_2(\tilde{e}_1 uvy) = \varepsilon_2(uvy)+1$ and $\varphi_2(\tilde{e}_1 uvy) = \varphi_2(uvy)$ and hence $(a_{12}, \Delta_1\varepsilon_2(uvy), \Delta_1\varphi_2(uvy)) = (-1,-1,0)$.

3. If $c \in \mathrm{con}(v_2)$, we have two cases

   (a) If c is paired with a letter $c' \in \mathrm{con}(u_2)$ and c' does not find a new bracketing partner in $\mathrm{con}(v_2)$ after c is removed, then $\varepsilon_2(\tilde{e}_1 uvy) = \varepsilon_2(uvy)+1$ and $\varphi_2(\tilde{e}_1 uvy) = \varphi_2(uvy)$ and hence $(a_{12}, \Delta_1\varepsilon_2(uvy), \Delta_1\varphi_2(uvy)) = (-1,-1,0)$.

   (b) If c is paired with a letter $c' \in \mathrm{con}(u_2)$, but c' finds a new bracketing partner in $\mathrm{con}(v_2)\backslash\{c\}$, or if c is not bracketed with a letter in $\mathrm{con}(u_2)$, then $\varepsilon_2(\tilde{e}_1 uvy) = \varepsilon_2(uvy)$ and $\varphi_2(\tilde{e}_1 uvy) = \varphi_2(uvy) - 1$, so that $(a_{12}, \Delta_1\varepsilon_2(uvy), \Delta_1\varphi_2(uvy)) = (-1,0,-1)$.

This completes the proof of (P3) and (P4) for $j = i+1$.

Proof of (P5)

Next we prove (P5). When $a_{ij} = 0$, then $\tilde{e}_i$ commutes with $\tilde{e}_j$ and $\tilde{f}_j$, so that the conditions of (P5) follow easily. Next assume that $j = i - 1$. It suffices again to consider $w^\beta = yuv$ with $i = 2$. Then by the analysis in (P3) and (P4) above, we have that $\Delta_2\varepsilon_1(yuv) = 0$ only when $c < b - t$ and c pairs with a letter in $\mathrm{con}(v_2)$. In this case $\tilde{e}_1$ moves the letter b from u to the letter $b - t$ in v before and after $\tilde{e}_2$, so that $\tilde{e}_1\tilde{e}_2(yuv) = \tilde{e}_2\tilde{e}_1(yuv)$. In addition $\nabla_1\varphi_2(\tilde{e}_1\tilde{e}_2yuv) = \varphi_2(\tilde{e}_2\tilde{e}_1yuv) - \varphi_2(\tilde{f}_1\tilde{e}_1\tilde{e}_2yuv) = \varphi_2(\tilde{e}_1yuv) + 1 - \varphi_2(yuv) - 1 = 0$ since by Lemma 3.8 all letters in $\mathrm{con}(u_1s_b \cdots s_{b-t})$ are bracketed with letters in y and hence do contribute to neither $\varphi_2(yuv)$ nor $\varphi_2(\tilde{e}_1yuv)$ (and $\tilde{e}_1$ moves b). This proves (P5) when $j = i - 1$.

Now assume that $j = i + 1$. It suffices to consider $w^\beta = uvy$ with $i = 1$, and by the above analysis of cases (P3) and (P4) we have $\Delta_1\varepsilon_2(uvy) = 0$ only when $c \in \mathrm{con}(v_2)$ and either c is paired with a letter $c' \in \mathrm{con}(u_2)$, but c' finds a new bracketing partner in $\mathrm{con}(v_2) \setminus \{c\}$ after the application of $\tilde{e}_1$, or if c is not bracketed with a letter in $\mathrm{con}(u_2)$. Since the letters in $\mathrm{con}(v_2)$ remain bracketed and b from $\mathrm{con}(u)$ is moved to $b - t$ in $\mathrm{con}(v)$, we have $\tilde{e}_1\tilde{e}_2(uvy) = \tilde{e}_2\tilde{e}_1(uvy)$. A similar computation as in the case $j = i - 1$ shows that $\nabla_2\varphi_1(\tilde{e}_1\tilde{e}_2uvy) = 0$. This concludes the proof of (P5).

Proof of (P6)

For the proof of (P6) assume that $\Delta_i\varepsilon_j(w^\beta) = \Delta_j\varepsilon_i(w^\beta) = -1$. Without loss of generality we may assume that $w^\beta = yuv$, $i = 2$, and $j = 1$. Then let us write

$$\begin{aligned} yuv &= y(u_1s_{b_1} \cdots s_{b_1-t_1}u_2)(v_1s_{b_1} \cdots s_{b_1-t_1+1}v_2) \\ yuv &= (y_1s_{b_2} \cdots s_{b_2-t_2}y_2)(u'_1s_{b_2} \cdots s_{b_2-t_2+1}u'_2)v \end{aligned}$$

in the decomposition according to Lemma 3.8 with respect to $\tilde{e}_1$ and $\tilde{e}_2$, respectively. All letters in $\mathrm{con}(v_1)$ pair with letters in $\mathrm{con}(u_1)$ and all letters in $\mathrm{con}(u_2)$ pair with letters in $\mathrm{con}(v_2)$. In addition, all letters in $\mathrm{con}(u'_1)$ pair with letters in $\mathrm{con}(y_1)$ and all letters in $\mathrm{con}(y_2)$ pair with letters in $\mathrm{con}(u'_2)$.

By the analysis of (P3) and (P4), $\Delta_2\varepsilon_1(yuv) = -1$ unless the new letter $b_2 - t_2$ in $\mathrm{con}(u)$ under $\tilde{e}_2$ satisfies $b_2 - t_2 < b_1 - t_1$ and pairs with a letter in $\mathrm{con}(v_2)$. This implies in particular that $\mathrm{con}(u_2) \subseteq \mathrm{con}(u'_2)$.

By the analysis of (P3) and (P4), $\Delta_1\varepsilon_2(yuv) = -1$ if $b_1 \notin \mathrm{con}(u'_2)$ or $b_1 \in \mathrm{con}(u'_2)$ but b_1 is paired with $c' \in y_2$ for the $\tilde{e}_2$ bracketing and c' cannot find a new bracketing partner in u when b_1 is removed by $\tilde{e}_1$.

First assume that $b_2 - t_2 \geq b_1 - t_1$. Since $b_2 - t_2$ is the new letter in u under $\tilde{e}_2$ this means in particular that $b_2 - t_2 > b_1$ (since all letters $b_1 - t_1, \ldots, b_1 - 1, b_1$ already appear in $\mathrm{con}(u)$). Hence $b_1 \in \mathrm{con}(u'_2)$ and we can write yuv as

$$(y_1s_{b_2} \cdots s_{b_2-t_2}\overbrace{y_2}^{s_{c'}\in})(u'_1s_{b_2} \cdots s_{b_2-t_2+1}\overbrace{\tilde{u}_2s_{b_1} \cdots s_{b_1-t_1}u_2}^{u'_2})(v_1s_{b_1} \cdots s_{b_1-t_1+1}v_2),$$

where possibly $\tilde{u}_2 = 1$. The letter b_1 is paired with $c' \in \mathrm{con}(y_2)$. Since c' cannot find a new bracketing partner when b_1 is removed by $\tilde{e}_1$, all letters in $\mathrm{con}(\tilde{u}_2)$ must be paired with letters in $\mathrm{con}(y_2)$. Now computing $\tilde{e}_1\tilde{e}_2^2\tilde{e}_1(yuv)$ we obtain

- under $\tilde{e}_1$ the letter b_1 moves from u to $b_1 - t_1$ in v;
- under $\tilde{e}_2$ the letter c' moves from y to some letter c'' in u; since it is smaller than $b_1 - t_1$ it must bracket with $b_1 - t_1$ in v (or some other letter in v);

- under $\tilde{e}_2$ the letter b_2 moves from y to $b_2 - t_2$ in u;
- under $\tilde{e}_1$ the rightmost unbracketed letter $b_2 - t_2 \geq i > b_1$ moves from u to v.

Next computing $\tilde{e}_2\tilde{e}_1^2\tilde{e}_2(yuv)$ we obtain

- under $\tilde{e}_2$ the letter b_2 moves from y to $b_2 - t_2$ in u;
- under $\tilde{e}_1$ the letter b_1 moves from u to $b_1 - t_1$ in v;
- under $\tilde{e}_1$ the rightmost unbracketed letter $b_2 - t_2 \geq i > b_1$ moves from u to v;
- under $\tilde{e}_2$ the letter c' moves from y to c'' in u; again it is bracketed with a letter in v.

This shows that $z := \tilde{e}_1\tilde{e}_2^2\tilde{e}_1(yuv) = \tilde{e}_2\tilde{e}_1^2\tilde{e}_2(yuv)$. It remains to verify that $\nabla_1\varphi_2(z) = \nabla_2\varphi_1(z) = -1$. By the above explicit description of the action of $\tilde{e}_1$, we find that $\varphi_2(\tilde{e}_2^2\tilde{e}_1 yuv) = \varphi_2(\tilde{e}_1 yuv) + 2 = \varphi_2(yuv) + 2$. When we act with $\tilde{e}_1$ on $\tilde{e}_2^2\tilde{e}_1 yuv$, the rightmost unbracketed letter $b_2 - t_2 \geq i > b_1$ moves from u to v. If $i = b_2 - t_2$, then certainly $\varphi_2(z) = \varphi_2(yuv) + 1$. Otherwise the letter in $\mathrm{con}(y)$ which was before bracketed with i, brackets with $b_2 - t_2$ after the application of $\tilde{e}_1$. Hence again $\varphi_2(z) = \varphi_2(yuv) + 1$. Altogether $\nabla_1\varphi_2(z) = -1$. Similarly, $\varphi_1(\tilde{e}_1^2\tilde{e}_2 yuv) = \varphi_1(\tilde{e}_2 yuv) + 2 = \varphi_1(yuv) + 2$. When we act with $\tilde{e}_2$ on $\tilde{e}_1^2\tilde{e}_2 yuv$, the new letter c'' brackets with $b_1 - t_1$ or another letter in v_2. Hence $\varphi_1(\tilde{e}_2\tilde{e}_1^2\tilde{e}_2 yuv) = \varphi_1(yuv) + 1$. This implies $\nabla_2\varphi_1(z) = -1$. This concludes the proof of (P6) when $b_2 - t_2 \geq b_1 - t_1$.

Next assume that $b_1 - t_1 = b_2 - t_2 + 1$. In this case $s_{b_2} \cdots s_{b_2-t_2+1}$ and $s_{b_1} \cdots s_{b_1-t_1}$ in u overlap. In particular $b_1 \leq b_2$, so that yuv can be written as

$$(y_1 s_{b_2} \cdots s_{b_2-t_2} y_2)(\overbrace{u_1' s_{b_2} \cdots}^{u_1} s_{b_1} \cdots \underbrace{s_{b_2-t_2+1}}_{=s_{b_1-t_1}} \overbrace{u_2}^{=u_2'})(v_1 s_{b_1} \cdots s_{b_1-t_1+1} v_2)\,.$$

Computing $\tilde{e}_1\tilde{e}_2^2\tilde{e}_1(yuv)$ we obtain

- under $\tilde{e}_1$ the letter b_1 moves from u to $b_1 - t_1$ in v;
- under $\tilde{e}_2$ the letter $b_1 - 1$ moves from y to $b_2 - t_2$ in u;
- under $\tilde{e}_2$ the letter b_2 moves from y to b_1 in u;
- under $\tilde{e}_1$ the letter b_1 moves from u to a letter $i \leq b_1 - t_1 - 1$ in v.

Similarly, computing $\tilde{e}_2\tilde{e}_1^2\tilde{e}_2(yuv)$ we have

- under $\tilde{e}_2$ the letter b_2 moves from y to $b_2 - t_2$ in u;
- under $\tilde{e}_1$ the letter $b_2 - t_2 = b_1 - t_1 - 1$ moves from u to a letter $i \leq b_1 - t_1 - 1$ in v;
- under $\tilde{e}_1$ the letter b_1 moves from u to a letter $b_1 - t_1$ in v;
- under $\tilde{e}_2$ the letter $b_1 - 1$ moves from y to $b_2 - t_2$ in u.

This implies that $z := \tilde{e}_1\tilde{e}_2^2\tilde{e}_1(yuv) = \tilde{e}_2\tilde{e}_1^2\tilde{e}_2(yuv)$. By very similar arguments to the previous case we also have $\nabla_1\varphi_2(z) = \nabla_2\varphi_1(z) = -1$. This concludes the proof of (P6) when $b_1 - t_1 = b_2 - t_2 + 1$.

Finally assume that $b_2 - t_2 < b_1 - t_1 - 1$. Then we have

$$(y_1 s_{b_2} \cdots s_{b_2-t_2} y_2)(\overbrace{u_1 s_{b_1} \cdots s_{b_1-t_1} \tilde{u}_1}^{=u_1'} s_{b_2} \cdots s_{b_2-t_2+1} u_2')(v_1 s_{b_1} \cdots s_{b_1-t_1+1} v_2).$$

Computing $\tilde{e}_1\tilde{e}_2^2\tilde{e}_1(yuv)$ we obtain

- under $\tilde{e}_1$ the letter b_1 moves from u to the letter $b_1 - t_1$ in v;
- under $\tilde{e}_2$ the letter b_2 moves from y to $b_2 - t_2$ in u. Since by assumption $\Delta_2\varepsilon_1(yuv) = -1$, the new letter $b_2 - t_2$ in u does not pair with a letter in $\mathrm{con}(v_2)$;
- since b_1 was moved from u by $\tilde{e}_1$, there is at least one free letter in $\mathrm{con}(y_1)$ which is not bracketed with a letter in u. Let c' be the smallest such letter. Under $\tilde{e}_2$ the letter c' moves from y to a letter $c'' < b_1 - t_1$ in u;
- under $\tilde{e}_1$ the letter $b_2 - t_2$ moves from u to a letter $i \leq b_2 - t_2$ in v.

Next computing $\tilde{e}_2\tilde{e}_1^2\tilde{e}_2(yuv)$ yields

- under $\tilde{e}_2$ the letter b_2 moves from y to $b_2 - t_2$ in u; again since $\Delta_2\varepsilon_1(yuv) = -1$, the new letter $b_2 - t_2$ in u does not pair with a letter in $\mathrm{con}(v_2)$;
- under $\tilde{e}_1$ the letter $b_2 - t_2$ in u moves to a letter $i \leq b_2 - t_2$ in v;
- under $\tilde{e}_1$ the letter b_1 in u moves to $b_1 - t_1$ in v;
- under $\tilde{e}_2$ the same letter c' from the previous case moves from y to a letter $c'' < b_1 - t_1$ in u.

Again, this show that $z := \tilde{e}_1\tilde{e}_2^2\tilde{e}_1(yuv) = \tilde{e}_2\tilde{e}_1^2\tilde{e}_2(yuv)$. By very similar arguments to the previous case we also have $\nabla_1\varphi_2(z) = \nabla_2\varphi_1(z) = -1$. This concludes the proof of (P6).

Proof of (P5') and (P6')

(P5') and (P6') follow from duality. On $x+1 < \cdots < 0 < \cdots < x-2 < x-1$ define the order reversing map $* : i \mapsto 2x - i$. We extend this map to words $a = a_1 \cdots a_h \mapsto a_h^* \cdots a_1^*$ and affine factorizations $* : w^\beta = w^\ell \cdots w^1 \mapsto w^{1*} \cdots w^{\ell*}$, where w^{i*} is the element in $S_{\hat{x}}$ corresponding to $\mathrm{con}(w^i)^*$. Note that the bracketing for the letters in the factors w^{r+1} and w^r used for the crystal operators is equivalent to bracketing all pairs $i \in w^{r+1}$ and $i+1 \in w^r$, removing these mentally, then bracketing all pairs $i \in w^{r+1}$ and $i+2 \in w^r$ from the remaining letters etc.. This shows that $* \circ \tilde{e}_r = \tilde{f}_{\ell-r} \circ *$ and $* \circ \tilde{f}_r = \tilde{e}_{\ell-r} \circ *$. Hence (P5') and (P6') follow from (P5) and (P6). $\square$

References

[1] S. Agnihotri, *Quantum cohomology and the Verlinde algebra*, Ph.D. thesis, University of Oxford, 1995. 5.4

[2] A. Barker, D. Swinarski, L. Vogelstein, J. Wu, *A new proof of a formula for the type A_2 fusion rules*, preprint August 2014. 1.4

[3] A. Bertram, I. Ciocan-Fontanine, and W. Fulton, *Quantum multiplication of Schur polynomials*, J. Algebra **219** (1999), no. 2, 728–746. 5.4

[4] L. Bégin, P. Mathieu, M. A. Walton, *$su(3)_k$ fusion coefficients*, Modern Phys. Lett. A **7** (1992), no. 35, 3255–3265. 1.4

[5] C. Benedetti, N. Bergeron, *Schubert polynomials and k-Schur functions*, preprint arXiv:1209.4956 1.4

[6] C. Benedetti, N. Bergeron, *Fomin-Greene monoids and Pieri operations*, Fields Institute Communications **71** (2014) 305–338. 1.4

[7] C. Berg, F. Saliola, L. Serrano, *The down operator and expansions of near rectangular k-Schur functions*, J. Combin. Theory Ser. A **120** (2013), no. 3, 623–636. 1.4

[8] I.N. Bernstein, I.M. Gel'fand, S.I. Gel'fand, *Schubert cells, and the cohomology of the spaces* G/P, Uspehi Mat. Nauk **28** (1973), no. 3(171), 3–26. 1.1, 5.3

[9] S. Billey, W. Jockusch, R. Stanley, *Some combinatorial properties of Schubert polynomials*, J. of Alg. Comb. **2** (1993), 345–374. 4.1

[10] A. Björner, F. Brenti, *Affine permutations of type A*, Electron. J. Combin. **3** (1996), no. 2, Research Paper 18, 35 pp. 2.1, 5.1

[11] K. Brown, K. Goodearl, M. Yakimov, *Poisson structures on affine spaces and flag varieties. I. Matrix affine Poisson space*, Adv. Math. **206** (2006), 567–629. 1.2

[12] R. Bott, *The space of loops on a Lie group*, Michigan Math. J. **5** (1958), 35–61. 1.1

[13] A. S. Buch, A. Kresch, H. Tamvakis, *Gromov–Witten invariants on Grassmannians*, J. Amer. Math. Soc. **16** (2003), no. 4, 901–915. 1.4, 5.4

[14] A. S. Buch, A. Kresch, K. Purbhoo, H. Tamvakis, *The puzzle conjecture for the cohomology of two-step flag manifolds*, preprint arXiv:1401.1725. 1.4, 5.4

[15] I. Coskun, *A Littlewood–Richardson rule for two-step flag varieties*, Invent. Math. **176** (2009), no. 2, 325–395. 1.4

[16] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, I. Ann. Sci. École Norm. Sup. (4) **7** (1974) 53–88. 1.1, 5.3

[17] T. Denton, *Canonical decompositions of affine permutations, affine codes, and split k-Schur functions*, Electron. J. Combin. **19** (2012), no. 4, Paper 19, 41 pp. 1.4

[18] P. Edelman, C. Greene, *Balanced tableaux*, Adv. in Math. **63** (1987), no. 1, 42–99. 4.2, 4.9

[19] S. Fomin, S. Gelfand, A. Postnikov, *Quantum Schubert polynomials*, J. Amer. Math. Soc. **10** (1997), no. 3, 565–596. 1.4, 5.3

[20] S. Fomin, C. Greene, *Noncommutative Schur functions and their applications*, Discrete Math. **193** (1998), no. 1-3, 179–200. 4.2

[21] W. Fulton, *Young tableaux*, With applications to representation theory and geometry. London Mathematical Society Student Texts, **35**, Cambridge University Press, Cambridge, 1997. 5.3

[22] H. Garland, M. .S. Raghunathan, *A Bruhat decomposition for the loop space of a compact group: a new approach to results of Bott*, Proc. Nat. Acad. Sci. U.S.A. **72** (1975), no. 12, 4716–4717. 1.1

[23] A. Garsia, *The Saga of reduced factorizations of elements of the symmetric group*, Publications du Laboratoire de Combinatoire et d'Informatique Mathématique **29**, 2002 (`lacim.uqam.ca/publications_pdf/29.pdf`). 4.1

[24] F. M. Goodman, H. Wenzl, *Littlewood–Richardson coefficients for Hecke algebras at roots of unity*, Adv. Math. **82** (1990), no. 2, 244–265. 5.4

[25] V.G. Kac, *Infinite-dimensional Lie algebras*, Third edition. Cambridge University Press, Cambridge, 1990. xxii+400 pp. ISBN: 0-521-37215-1. 5.4

[26] M. Kashiwara, *Crystalizing the q-analogue of universal enveloping algebras*, Comm. Math. Phys. **133** (1990), no. 2, 249–260. 1.1, 2.2

[27] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), no. 2, 465–516. 1.1, 2.5

[28] M. Kashiwara, *On crystal bases*, Representations of groups (Banff, AB, 1994), 155–197, CMS Conf. Proc., 16, Amer. Math. Soc., Providence, RI, 1995. 2.2

[29] M. Kashiwara, T. Nakashima, *Crystal graphs for representations of the q-analogue of classical Lie algebras*, J. Algebra **165** (1994), no. 2, 295–345. 3, 4, 4.1

[30] A. Knutson, T. Lam, D. E. Speyer, *Positroid varieties: juggling and geometry*, Compos. Math. **149** (2013), no. 10, 1710–1752. 1.2, 3.1, 5.5

[31] A. Knutson, T. Tao, *Puzzles and (equivariant) cohomology of Grassmannians*, Duke Math. J. **119** (2003), no. 2, 221–260. 1.4

[32] C. Korff, *The* $su(n)$ *WZNW fusion ring as integrable model: a new algorithm to compute fusion coefficients*, Infinite analysis 2010 – Developments in quantum integrable systems, 121–153, RIMS Kokyuroku Bessatsu, **B28**, Res. Inst. Math. Sci. (RIMS), Kyoto, 2011. 1.4

[33] C. Korff, *Cylindric versions of specialised Macdonald functions and a deformed Verlinde algebra*, Comm. Math. Phys. **318** (2013), no. 1, 173–246. 1.4

[34] C. Korff, C. Stroppel, *The* $\widehat{\mathrm{sl}}(n)_k$*-WZNW fusion ring: a combinatorial construction and a realisation as quotient of quantum cohomology*, Adv. Math. **225** (2010), no. 1, 20–268. 1.4

[35] B. Kostant, S. Kumar, *The nil Hecke ring and cohomology of* G/P *for a Kac-Moody group* G, Adv. in Math. **62** (1986), no. 3, 187–237. 1.1

[36] M. Kontsevich, *Enumeration of rational curves via torus actions*, in: The moduli space of curves, Progr. Math. **129** (1995) Birkhäuser Boston, Boston, MA, 335–368. 1.1

[37] W. Kraśkiewicz, P. Pragacz, *Schubert functors and Schubert polynomials*, European J. Combin. **25** (2004), no. 8, 1327–1344. 4.2

[38] T. Lam, *Affine Stanley symmetric functions*, Amer. J. Math. **128** (2006), no. 6, 1553–1586. 1.2, 3.1, 5.1, 5.4

[39] T. Lam, *Schubert polynomials for the affine Grassmannian*, J. Amer. Math. Soc. 21 (2008), no. 1, 259–281. 1.1

[40] T. Lam. L. Lapointe, J. Morse, A. Schilling, M. Shimozono, M. Zabrocki, *k-Schur functions and affine Schubert calculus*, Fields Institute Monographs **33**, 2014 (Springer), DOI 10.1007/978-1-4939-0682-6 (arXiv:1301.3569). 1.1

[41] T. Lam, M. Shimozono, *Quantum cohomology of* G/P *and homology of affine Grassmannian*, Acta Math. **204** (2010), no. 1, 49–90. 1.2

[42] L. Lapointe, A. Lascoux, J. Morse, *Tableau atoms and a new Macdonald positivity conjecture*, Duke Math. J. 116 (2003), no. 1, 103–146. 1.1

[43] L. Lapointe, J. Morse, *A k-tableau characterization of k-Schur functions*, Adv. Math. 213 (2007), no. 1, 183–204. 5.2

[44] L. Lapointe, J. Morse, *Quantum cohomology and the k-Schur basis*, Trans. Amer. Math. Soc. 360 (2008), pp. 2021–2040. 1.1, 1.1, 1.2, 1.4, 3.1, 5.3, 5.16, 5.4

[45] L. Lapointe, J. Morse, *Flag Gromov–Witten invariants and k-Littlewood Richardson coefficients*, in preparation. 5.1, 5.4, 5.2, 5.3, 5.11, 5.3

[46] A. Lascoux, M.-P. Schützenberger, *Sur une conjecture de H. O. Foulkes*, C. R. Acad. Sci. Paris Sér. A-B **286** (1978), no. 7, A323–A324. 1.1, 4.1

[47] A. Lascoux, M.-P. Schützenberger, *Croissance des polynômes de Foulkes-Green*, C. R. Acad. Sci. Paris Sér. A-B **288** (1979), no. 2, A95–A98. 1.1, 4.1

[48] A. Lascoux, M.-P. Schützenberger, *Polynômes de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), no. 13, 447–450. 1.1, 3.1, 5.3

[49] P. Littelmann, *A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras*, Invent. Math. **116** (1994), no. 1-3, 329–346. 2.5

[50] C. Lenart, *A unified approach to combinatorial formulas for Schubert polynomials*, J. Algebraic Combin. **20** (2004), no. 3, 263–299. 1.4

[51] C. Lenart, *Growth diagrams for the Schubert multiplication*, J. Combin. Theory Ser. A **117** (2010), no. 7, 842–856. 1.4

[52] D.E. Littlewood, A. R Richardson, *Group characters and algebra*, Philos. Trans. R. Soc. Lond. Ser. A **233** (1934) 99–141. 1.1, 4.1

[53] G. Lusztig, *Some examples of square integrable representations of semisimple p-adic groups*, Trans. Amer. Math. Soc. **277** (1983), no. 2, 623–653. 2.1

[54] G. Lusztig, *Total positivity for partial flag varieties*, Represent. Theory **2** (1998), 70–78. 1.2

[55] I. G. Macdonald, Symmetric functions and Hall polynomials. Second edition. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[56] L. Manivel, Symmetric functions, Schubert polynomials and degeneracy loci. SMF/AMS Texts and Monographs, 3. American Mathematical Society, Providence, RI, 2001. viii+167 pp. ISBN: 0-8218-2154-7. 5.3

[57] K. Mészáros, G. Panova, A. Postnikov, *Schur times Schubert via the Fomin–Kirillov algebra*, The Electronic Journal of Combinatorics **21** (1) (2014), P1.39. 1.4, 5.3

[58] J. Morse, A. Schilling, *A combinatorial formula for fusion coefficients*, 24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012), 735–744, Discrete Math. Theor. Comput. Sci. Proc., AR, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2012. 1.4, 5.17

[59] T. Nakashima, *Crystal base and a generalization of the Littlewood-Richardson rule for the classical Lie algebras*, Comm. Math. Phys. **154** (1993), no. 2, 215–243. 2.5

[60] D. Peterson, *Quantum cohomology of G/P*, Lecture notes, M.I.T., Spring 1997. 1.1

[61] A. Postnikov, *On a quantum version of Pieri's formula*, Advances in geometry, 371–383, Progr. Math., **172**, Birkhäuser Boston, Boston, MA, 1999. 1.4

[62] A. Postnikov, *Total positivity, Grassmannians, and networks*, preprint 2005, `http://www-math.mit.edu/~apost/papers/tpgrass.pdf`. 1.2, 3.1, 5.5

[63] A. Postnikov, *Affine approach to quantum Schubert calculus*, Duke Mathematical Journal **128** (2005), no. 3, 473–509. 1.2, 5.4

[64] A. Postnikov, *Symmetries of Gromov–Witten invariants*, Advances in Algebraic Geometry Motivated by Physics, Contemporary Mathematics **276** (2001), 251–258. 5.3

[65] V. Reiner, M. Shimozono, *Plactification*, Journal of Algebraic Combinatorics **4** (1995), 331–351. 4.2, 4.2

[66] V. Reiner, M. Shimozono, *Percentage-avoiding, northwest shapes, and peelable tableaux*, Journal of Combinatorial Theory, Series A **82** (1998), 1–73. 4, 4.2

[67] K. Rietsch, *Closure relations for totally nonnegative cells in G/P*, Mathematical Research Letters **13** (2006), 775–786. 1.2

[68] J. Remmel, M. Shimozono, *A simple proof of the Littlewood-Richardson rule and applications*, Discrete Math. **193** (1998), no. 1-3, 257–266. 1.2, 1.3, 6, 6, 6.3

[69] A. Schilling, M. Shimozono, *Fermionic formulas for level-restricted generalized Kostka polynomials and coset branching functions*, Commun. Math. Phys. **220** (2001) 105–164. 1.4

[70] J. Shi, *The Kazhdan-Lusztig cells in certain affine Weyl groups*, Lecture Notes in Math., Vol **1179**, Springer, Berlin, 1986. 2.1

[71] W. A. Stein et al. *Sage Mathematics Software (Version 5.12).* The Sage Development Team, 2013. `http://www.sagemath.org`. 1.4

[72] The Sage-Combinat community. Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, 2008. `http://combinat.sagemath.org`. 1.4

[73] M.-P. Schützenberger, *La correspondance de Robinson*, Combinatoire et représentation du groupe symétrique (Actes Table Ronde CNRS, Univ. Louis-Pasteur Strasbourg, Strasbourg, 1976), pp. 59–113. Lecture Notes in Math., Vol. 579, Springer, Berlin, 1977. 1.1, 4.1

[74] R.P. Stanley, *On the number of reduced decompositions of elements of Coxeter groups*, European J. Combin. **5** (1984), no. 4, 359–372. 3.1, 4.2

[75] J. R. Stembridge, *A local characterization of simply-laced crystals*, Transactions of the AMS **355** (2003) 4807–4823. 1.2, 1.3, 2.2, 2.4

[76] G. Tudose, *On the combinatorics of $sl(n)$-fusion algebra*, Ph.D., York University (Canada), 2002 , 99 pages (see also arXiv:0008.5034). 1.4

[77] M. A. Walton, *Fusion rules in Wess-Zumino-Witten models*, Nuclear Phys. B **340** (1990), no. 2-3, 777–790. 5.4

[78] E. Witten, *Quantum cohomology in string theory*, Comm. Math. Phys. **118** (1988), no. 3, 411–449. 1.1

[79] A. Young, *The collected papers of Alfred Young (1873–1940)*, With a foreword by G. de B. Robinson and a biography by H. W. Turnbull. Mathematical Expositions, No. 21. University of Toronto Press, Toronto, Ont., Buffalo, N. Y., 1977. xxvii+684 pp. 4

(Jennifer Morse) Department of Mathematics, Drexel University, Philadelphia, PA 19104

Email address: `morsej@math.drexel.edu`

(Anne Schilling) Department of Mathematics, UC Davis, One Shields Ave., Davis, CA 95616-8633

Email address: `anne@math.ucdavis.edu`