Title
Superflatness

Author
Boocher, Adam Lee

Publication Date
2013

Superflatness

by

Adam Lee Boocher

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor David Eisenbud, Chair
Professor Bernd Sturmfels
Professor Katherine O'Brien O'Keeffe

Fall 2013

Superflatness

Abstract

Superflatness

by

Adam Lee Boocher

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor David Eisenbud, Chair

One way to obtain geometric information about a homogeneous ideal is to pass to a monomial ideal via a flat degeneration. Flatness is strong enough to ensure this degeneration preserves the Hilbert function, which allows us to make geometric statements about the original ideal. Although it is by no means trivial, full analysis of monomial ideals is aided by a wealth of interactions with combinatorics, topology, and commutative algebra. However, since flatness only goes so far, finer invariants than the Hilbert function cannot typically be detected via this technique.

One finer invariant is the minimal free resolution. Originally introduced by Hilbert, free resolutions encode algebraic relations among the generators of an ideal. Numerically, the data of a free resolution are the graded Betti numbers which detect surprising geometric information. In recent years there has been much study devoted to the relationship between the modules occurring in a free resolution (collectively called *syzygies*) and geometric invariants.

Flatness is not strong enough to guarantee that the free resolution will be preserved upon degeneration. In fact, in some sense, the expected behavior is that the resolution will become more poorly behaved. This dissertation studies situations in which flat degenerations preserve more than they ought, and how these *superflat* degenerations allow us to better understand the resolution of our original ideal. It contains a brief introduction followed by three self-contained chapters.

In Chapter 2 we study ideals associated to sparse-generic matrices, those whose entries are distinct variables and zeros. Such matrices were studied by Giusti and Merle in [GM82] where they computed some invariants of their ideals of maximal minors. Here we extend these results by computing a minimal free resolution for all such sparse determinantal ideals. We do so by introducing a technique for pruning minimal free resolutions when a subset of the variables is set to zero. Our technique correctly computes a minimal free resolution in two cases of interest: resolutions of monomial ideals, and ideals resolved by the Eagon-Northcott Complex. As a consequence we can show that sparse determinantal ideals have a linear resolution over $\mathbb{Z}$, and that the projective dimension depends only on the number of

columns of the matrix that are identically zero. Finally, we show that all such ideals have the property that regardless of the term order chosen, the Betti numbers of the ideal and its initial ideal are the same. In particular the nonzero generators of these ideals form a universal Gröbner basis.

Chapter 3 presents joint work with Elina Robeva and initiates a systematic study of ideals minimally generated by a universal Gröbner basis. We call such an ideal robust. We show that robust toric ideals generated by quadrics are essentially determinantal. We then discuss two possible generalizations to higher degree, providing a tight classification for determinantal ideals, and a counterexample to a natural extension for Lawrence ideals. We close with a discussion of robustness of higher Betti numbers.

Chapter 4 is joint work with Federico Ardila concerning the closure of linear spaces in a product of projective lines. Let L be an linear space in $\mathbb{A}^n$. We study the closure $\widetilde{L}$ in $(\mathbb{P}^1)^n$ and show that the degree, defining equations, graded Betti numbers, and universal Gröbner basis of its defining ideal $I(\widetilde{L})$ are all combinatorially determined by the linear matroid associated to L. We explicitly compute these invariants. In so doing, we study the set of monomial initial ideals of $I(\widetilde{L})$.

Contents

Acknowledgments

I am grateful to my advisor, David Eisenbud, for his dedicated mentorship and support throughout my graduate career. I am also thankful for all of the teachers and friends who helped teach me mathematics. It is impossible to thank everyone, but in particular I would like to thank Andy Kustin for sharing his love of free resolutions with me. By luck I had the chance to see Andy give a seminar talk about the Eagon-Northcott Complex which inspired my first main project of graduate school. I thank Claudia Polini for teaching me Commutative Algebra when I was an undergraduate, and Juan Migliore, for first showing me the relationship between algebra and geometry. At Berkeley, I thank Bernd Sturmfels for introducing me to combinatorial questions close to my research, and for encouraging collaborations with Elina Robeva and Federico Ardila who graciously agreed to have our joint work appear in this dissertation. I am grateful to Katherine O'Brien O'Keeffe for agreeing to serve on my dissertation committee and for her support from outside the math department. I thank my friends and fellow graduate students for everything great and small they did to help me throughout my graduate school career - Stuart Ambler, David Berlekamp, Melody Chan, Charley Crissman, Michael Daub, George Emblom, Daniel Erman, Andrew Fanoe, Claudiu Raicu, James Tener, Carol Wallace, Emma Whitten, and countless others. Finally, on a personal note, I'd like to thank my family for all of their support and also the members of St. Mark's Berkeley Choir for years of wonderful music and friendship.

Chapter 1

Introduction

Let $S = k[x_0, \ldots, x_n]$ be a polynomial ring over a field k. If M is a graded S-module, then one important invariant of M is its minimal free resolution, which is an exact sequence

$$0 \longrightarrow F_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where the F_i are free modules chosen to have rank as small as possible. Such a resolution is unique up to isomorphism and many geometric invariants can be obtained only from the ranks and generating degrees of the F_i. We define the *graded Betti numbers* to be these ranks

$$\beta_{i,j}(M) := \dim_k(\operatorname{Tor}_i^S(M, k))_j.$$

In what follows we assume that $M = S/I$, for a homogeneous ideal I and interpret M as the coordinate ring of a projective variety.

An effective way to compute Betti numbers is to pass to a monomial ideal via a flat degeneration. Typically this arises in the context of a Gröbner degeneration coming from a monomial term order $<$, and the resulting monomial ideal is called the initial ideal, denoted $\operatorname{in}_< I$. We say such a degeneration is flat because there exists an ideal $\tilde{I} \subset S[t]$ such that $S[t]/\tilde{I}$ is a flat $k[t]$-module with the property that

$$S[t]/\tilde{I} \otimes_{k[t]} k[t, t^{-1}] \cong S/I[t, t^{-1}], \quad S[t]/\tilde{I} \otimes_{k[t]} k[t]/(t) \cong S/\operatorname{in}_< I.$$

Flatness ensures that the Hilbert function of S/I is equal to that of $S/\operatorname{in}_< I$, but once we pass to Betti numbers we obtain only an inequality:

$$\beta_{i,j}(S/I) \leq \beta_{i,j}(S/\operatorname{in}_< I) \tag{1.1}$$

(See [Pee11]).

This inequality is typically strict. Indeed, equality for $i = 1$ is equivalent to the fact that I is minimally generated by a Gröbner basis with respect to $<$, which is hardly typical behavior. Various authors have considered cases where equality holds in 1.1 for a particular term order (see [CHT06, BR07, JW07]), and for the case $i = 1$, papers supplying examples and non-examples alike about Gröbner bases abound. In this dissertation, we take the approach of analyzing the much stronger condition that equality holds for *all* term orders $<$, and for all i and j. In other words, we seek to understand those ideals I whose initial ideals are not only flat degenerations, but superflat!

The three chapters that follow all share a component of our superflat theme. The goal of this thesis is to show that not only is superflatness an interesting property, but also one that occurs in many classical settings. Each chapter is meant to be self-contained, although there is a natural progression. It is worth noting, that this dissertation began by studying matrices of zeros and variables in an attempt to compute the minimal free resolution of their ideal of maximal minors. In completing that project, the importance of ideals with superflat degenerations became apparent.

In order to make precise statements, we have adopted the word *robust* to refer to ideals minimally generated by a universal Gröbner basis, and say an ideal has *robust Betti numbers* if equality holds for all i, j, and $<$ in Equation 1.1.

Sparse Determinantal Ideals

In Chapter 2 we study ideals associated to sparse-generic matrices, those whose entries are distinct variables and zeros. For example consider the ideals of maximal minors of the following two matrices:

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \end{pmatrix} \quad X' = \begin{pmatrix} 0 & 0 & x_3 & 0 & x_5 \\ 0 & 0 & y_3 & y_4 & y_5 \\ z_1 & z_2 & 0 & 0 & z_5 \end{pmatrix}.$$

Figure 1.1: A Generic Matrix and a Specialization

Such ideals were studied by Giusti and Merle in [GM82] where they computed some invariants of their ideals of maximal minors. In particular they proved that the codimension, primeness and Cohen-Macaulayness of such ideals depend only on perimeter of the largest block of zeros. For example, their result says that the codimension of the ideals in the figure are given by

$$\operatorname{codim} I_3(X) = 3, \quad \operatorname{codim} I_3(X') = 2.$$

It is natural to ask how the minimal free resolutions of such ideals change as we add or remove zeros. For example, the two matrices in Figure 1 yield ideals whose free resolutions have the following Betti tables:

	0	1	2	3
total:	1	10	15	6
0:	1	.	.	.
1:	.	.	.	.
2:	.	10	15	6

	0	1	2	3
total:	1	7	9	3
0:	1	.	.	.
1:	.	.	.	.
2:	.	7	9	3

In Chapter 2 we prove that the minimal free resolution of a sparse determinantal ideal is always given by a direct summand of the Eagon-Northcott complex, and as such the resolution is always linear. We also show that except in degenerate cases when an entire column consists of zeros the projective dimension is always equal to that of the generic case.

We prove this by introducing a technique for pruning minimal free resolutions when a subset of the variables is set to zero. Our technique correctly computes a minimal free resolution in two cases of interest: resolutions of monomial ideals, and ideals resolved by the Eagon-Northcott Complex. Finally, we show that all such ideals have the property that, regardless of the term order chosen, the Betti numbers of the ideal and its initial ideal are the same. In particular the nonzero generators of these ideals form a universal Gröbner basis. In other words, the ideals are robust and have robust Betti numbers.

Robust Toric Ideals

Chapter 3 presents joint work with Elina Robeva and initiates a systematic study of ideals minimally generated by a universal Gröbner basis, which we call robust. Robust ideals were essential in the proof of the results in Chapter 2, but few large classes of robust ideals are known. We begin by studying prime ideals generated by binomials. Such ideals are called toric ideals and enjoy many connections with combinatorics. In this chapter we show that robust toric ideals generated by quadrics are essentially determinantal. We then discuss two possible generalizations to higher degree, providing a tight classification for determinantal ideals, and a counterexample to a natural extension for Lawrence ideals. We close with a discussion of robustness of higher Betti numbers.

Closures of Linear Spaces

Chapter 4 presents joint work with Federico Ardila concerning the closure of linear spaces in a product of projective lines. Such closures are a natural extension of the construction of Lawrence ideals considered in chapter 3. Let L be an linear space in $\mathbb{A}^n$. We study the closure $\widetilde{L}$ in $(\mathbb{P}^1)^n$ and show that the degree, defining equations, graded Betti numbers, and universal Gröbner basis of its defining ideal $I(\widetilde{L})$ are all combinatorially determined by the linear matroid associated to L.

Example 1.2. *To the linear ideal*

$$I = \langle x_1 + x_2 + x_6,\ x_2 - x_3 + x_5,\ x_3 + x_4 \rangle\,.$$

given by $r = 3$ independent equations in $n = 6$ variables, and the corresponding linear subspace $L \subset k^6$ of dimension $d = n - r = 3$, we associate the $r \times n$ matrix whose rows correspond to our 3 equations:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

We regard the columns of A as a point configuration in $\mathbb{P}^{r-1} = \mathbb{P}^2$, respectively, as shown in Figure 1.2. The affine dependence relations among the points correspond to the linear dependence relations among the columns of the matrix. A different generating set for I would give a different point configuration with the same affine dependence relations.

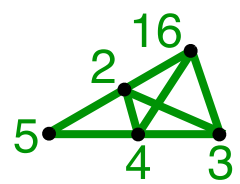


Figure 1.2: A point configuration $A \subset \mathbb{P}^2$ corresponding to the linear ideal I.

It is known that the minimal universal Gröbner basis of I is given by the cocircuits *of I: the linear forms in L using an inclusion-minimal set of variables.*

$$I = \langle x_1 + x_2 + x_6,\ x_1 + x_3 - x_5 + x_6,\ x_1 - x_4 - x_5 + x_6,\ x_2 - x_3 + x_5,\ x_2 + x_4 + x_5,\ x_3 + x_4 \rangle$$

We identify the cocircuits with their support sets $126, 1356, 1456, 235, 245,$ *and* 34. *They are the complements of the* hyperplanes $345, 24, 23, 146, 136,$ *and* 1256 *spanned by A. Our main result claims that the homogenized cocircuits minimally generate $\widetilde{I}$, and give a universal Gröbner basis:*

$$\widetilde{I} = \langle x_1y_2y_6 + y_1x_2y_6 + y_1y_2x_6,\ x_1y_3y_5y_6 + y_1x_3y_5y_6 - y_1y_3x_5y_6 + y_1y_3y_5x_6, \ldots, x_3y_4 + y_3x_4 \rangle$$

Similarly, other invariants of the matroid yield information about the multi-degree, initial ideals, and Betti numbers. Again it turns out that the ideal $\widetilde{I}$ is robust and has robust Betti numbers.

Chapter 2

Resolutions of Sparse Determinantal Ideals

This chapter presents the paper "Free Resolutions and Sparse Determinantal Ideals" [Boo12] which has been published in *Math Research Letters*, with only minor changes.

2.1 Introduction

Let S be a polynomial ring over K, where K is any field or $\mathbb{Z}$. By a *sparse generic matrix*, we mean a $k \times n$ matrix X' (with $k \leq n$) whose entries are distinct variables and zeros, and will denote by $I_k(X')$ its ideal of maximal minors, which we call a *sparse determinantal ideal*. For example, the two matrices below are both sparse generic matrices.

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix} \quad X' = \begin{pmatrix} 0 & 0 & x_3 & 0 \\ 0 & 0 & y_3 & y_4 \\ z_1 & z_2 & 0 & 0 \end{pmatrix}.$$

Figure 2.1: A Generic Matrix and a Specialization

Sparse generic matrices and determinantal ideals were studied by Giusti and Merle in [GM82] where they showed that the codimension, primeness, and Cohen-Macaulayness of $I_k(X')$ depend only on the perimeter of the largest subrectangle of zeros in X'. In this paper we continue the story by studying the homological invariants of these ideals and describe explicitly how to compute their minimal free resolution in terms of the arrangement of zeros. In particular we prove that except in trivial cases, the projective dimension and regularity of such ideals is the same as in the generic case:

Theorem 2.1. *Let X' be a $k \times n$ sparse generic matrix with no column identically zero, and $I = I_k(X')$ its ideal of maximal minors. If $I \neq 0$ then* $\operatorname{reg} S/I = k$ *and* $\operatorname{pdim} S/I = n-k+1$.

Finally, if X is a generic $k \times n$ matrix, then the Betti numbers of S/I satisfy

$$\beta_{ij}(S/I_k(X')) \leq \beta_{ij}(S/I_k(X)), \quad \textit{for all } i, j.$$

Sparse generic matrices can be thought of as generic matrices after setting some variables equal to zero. For an arbitrary ideal, it is difficult to describe how the minimal free resolution changes after setting some linear forms equal to zero. Indeed, the Betti numbers, projective dimension, and regularity can be wildly different before and after specialization. However, in the case of determinantal ideals, which are resolved by Eagon-Northcott complex, there is a simple greedy algorithm that can be used to compute the minimal free resolution of any sparse determinantal ideal. This is the basis for our proof of Theorem 2.1. The following example illustrates our method:

Example 2.2. *Consider the matrices X and X' in Figure 2.1. We begin with the Eagon-Northcott complex that resolves $S/I_3(X)$:*

$$0 \longrightarrow S^3 \xrightarrow{\begin{pmatrix} x_4 & y_4 & z_4 \\ x_3 & y_3 & z_3 \\ x_2 & y_2 & z_2 \\ x_1 & y_1 & z_1 \end{pmatrix}} S^4 \xrightarrow{\begin{pmatrix} \Delta_{123} & -\Delta_{124} & \Delta_{134} & -\Delta_{234} \end{pmatrix}} S$$

where Δ_J denotes the minor indexed by the columns in J. Now suppose we want to resolve $S/I_3(X')$. Naively we might just set $x_1, x_2, x_4, y_1, y_2, z_3$ and z_4 equal to zero - i.e. tensor with $T = S/(x_1, x_2, x_4, y_1, y_2, z_3)$. The result is:

$$0 \longrightarrow T^3 \xrightarrow{\begin{pmatrix} 0 & y_4 & 0 \\ x_3 & y_3 & 0 \\ 0 & 0 & z_2 \\ 0 & 0 & z_1 \end{pmatrix}} T^4 \xrightarrow{\begin{pmatrix} 0 & 0 & x_3y_4z_1 & -x_3y_4z_2 \end{pmatrix}} T$$

Notice that the first two columns of the rightmost matrix are redundant, and hence, so are the first two rows of the leftmost matrix. Deleting the corresponding summand of T^4 we obtain:

$$0 \longrightarrow T^3 \xrightarrow{\begin{pmatrix} 0 & 0 & z_2 \\ 0 & 0 & z_1 \end{pmatrix}} T^2 \xrightarrow{\begin{pmatrix} x_3y_4z_1 & x_3y_4z_2 \end{pmatrix}} T$$

And now similarly prune the first matrix:

$$0 \longrightarrow T^1 \xrightarrow{\begin{pmatrix} z_2 \\ z_1 \end{pmatrix}} T^2 \xrightarrow{\begin{pmatrix} x_3y_4z_1 & -x_3y_4z_2 \end{pmatrix}} T$$

In this case the resulting sequence of maps is a minimal free resolution of $T/I_3(X')$. This exemplifies what we call the Pruning Technique.

We will define and study the pruning technique in Section 2.2. Our main result on pruning is the following:

Theorem 2.3. *Suppose that $I \subset S$ is an ideal in a polynomial ring and Z is a subset of the variables. If $T = S/(Z)$ then the pruning technique computes a minimal free resolution of $S/I \otimes T$ as a T-module in the following two cases:*

- *I is a monomial ideal.*
- *I is a determinantal ideal resolved by the Eagon-Nortcott Complex*

In Section 2 we also discuss a homological interpretation of pruning. One feature of this interpretation is that it can be used (see Corollaries 2.10 and 2.22) to describe the shape of the Betti table of $\mathrm{Tor}_1(S/I, S/(x))$ where x is a variable and I is either a monomial ideal or a sparse determinantal ideal.

Our proof of Theorem 2.3 proceeds in two cases. For monomial resolutions, we study an $\mathbb{N}^n$ grading. For determinantal ideals, we use the result of Sturmfels, Zelevinsky, and Bernstein [SZ93, BZ93] that shows that the maximal minors of a generic matrix are a universal Gröbner basis for the ideal I that they generate. Since setting variables equal to zero is *almost* like taking them last in a term order, it is natural to study the free resolution of initial ideals of $I_k(X)$ when X is generic. For example, the aforementioned Gröbner basis result says that for any term order "$<$",

$$\beta_1(S/I) = \beta_1(S/\operatorname{in}_< I).$$

We extend this to show that in fact the maximal minors are a universal Gröbner resolution in the following sense:

Theorem 2.4. *Let X be a (sparse) generic matrix and let I denote its ideal of maximal minors. Then for any term order $<$, we have*

$$\beta_{ij}(S/I) = \beta_{ij}(S/\operatorname{in}_< I) \quad \textit{for all } i, j.$$

In particular, every initial ideal of I has a linear resolution.

We note that the analagous result does not hold for lower order minors. In fact, even the 2×2 minors of a 3×3 matrix are not a universal Gröbner basis. [SZ93]

Theorem 2.4 provides a new class of squarefree Cohen-Macaulay monomial ideals generated in degree k that have a linear resolution. Combining the techniques of pruning and taking initial ideals, we can obtain a class of squarefree monomial ideals with linear resolutions that sit inside of the Eagon-Northcott complex. Finally, although the proofs rely on the Gröbner basis property, the pruning algorithm itself is algebra free - it only involves an eraser.

Acknowledgments

Throughout the course of this project, many calculations were performed using the software Macaulay2 [Sti]. The author is grateful to David Eisenbud, Daniel Erman, Andy Kustin, and Claudiu Raicu for many helpful conversations.

2.2 The Pruning Technique

In this section we define and study the pruning technique. Throughout, S will denote a polynomial ring over K, where K is any field or $\mathbb{Z}$. The variable names may change for convenience, but should always be clear from the context. By $Z \subset S$ we will always mean a subset of the variables or as an an abuse of notation, the ideal that they generate in S. We set $T := S/Z$.

The pruning technique is a way of approximating a T-resolution of $M \otimes T$ starting from an S-resolution of M. To do so, we essentially tensor the given resolution with T and erase any obvious excess. The definition here - which makes precise the method outlined in Example 2.2 - requires a choice of basis, but as we will discuss later, this is mostly for convenience.

Definition 1. *Let $C_\bullet$ be a complex of free S-modules with choice of bases (so we have a matrix for each map)*

$$F_t \xrightarrow{A_t} F_{t-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{A_1} F_0 \ .$$

Let Z be a subset of the variables. We define the pruning of $C_\bullet$ with respect to Z *to be the complex of $T := S/Z$-modules obtained from $C_\bullet$ by the following algorithm:*

Let $i = 1$

For $i \leq t$ do:

In the matrix A_i, set all variables in Z equal to zero. Set A_i equal to this new matrix, and set U equal to the set indexing which columns of A_i are identically zero.

Replace, $\{A_{i+1}, F_i, A_i\}$ with news maps, and modules obtained by simply deleting the rows, basis elements, and columns, respectively, corresponding to U.

Let $i = i + 1$.

The resulting sequence of maps with bases is naturally a sequence of T-modules, which we will denote $P(C_\bullet, Z)$.

Proposition 2.5. *If $C_\bullet$ is a complex, then so is $P(C_\bullet, Z)$. In addition, if the entries of the matrices of $C_\bullet$ are in the homogeneous maximal ideal, then the same is true for those of $P(C_\bullet, Z)$.*

Proof. It is clear that if $A_i \cdot A_{i-1} = 0$ then the same is true once we set variables in Z equal to zero. Further, any column that is identically 0 in A_{i-1} essentially makes the corresponding row in A_i irrelevant for the product to be zero. Indeed the non-identically-zero columns of A_{i-1} must now necessarily pair to zero with the corresponding rows of A_i. This is exactly what the pruning process does. Finally, since pruning only erases entries, the second claim of the Proposition is clear. □

In some cases, the pruning technique preserves exactness:

Theorem 2.6. *Let I be a monomial ideal in a polynomial ring S with n variables, and let $C_\bullet$ be a minimal free resolution of S/I with $\mathbb{N}^n$ homogeneous bases. If Z is an ideal generated by a subset of the variables then $P(C_\bullet, Z)$ is a minimal free resolution of $S/I \otimes S/Z$ as an S/Z module.*

The proof follows from a careful study of the
mathbbN^n grading. We will use a similar technique below to study the case of the Eagon-Northcott complex.

Proof. We may assume that $Z = (x_1, \ldots, x_r)$. By the grading of $C_\bullet$, the maps will be of the form

$$\begin{array}{ccc} \bigoplus_{\substack{\text{not all} \\ b_{i1},\ldots,b_{ir}=0}} S(-b_{i1}, \ldots - b_{ir}, \ldots, -b_{in}) & & \bigoplus_{\substack{\text{not all} \\ a_{j1},\ldots,a_{jr}=0}} S(-a_{j1}, -a_{j2}, \ldots, -a_{jn}) \\ \bigoplus & \xrightarrow{M_i} & \bigoplus \\ \bigoplus S(0, \ldots, 0, -c_{i\ r+1}, \ldots, -c_{in}) & & \bigoplus S(0, \ldots, 0, -d_{j\ r+1}, \ldots, -d_{jn}) \end{array}$$

where the matrix M_i has the form

$$\left(\begin{array}{c|c} A_i & 0 \\ \hline C_i & D_i \end{array} \right).$$

By the grading, it is clear that every nonzero entry in the submatrix C_i is divisible by some $x_i \in Z$. In this notation, the beginning of the resolution of S/I is:

$$F_2 \xrightarrow{\left(\begin{array}{c|c} A_2 & 0 \\ \hline C_2 & D_2 \end{array} \right)} F_1 \xrightarrow{\left(\begin{array}{c|c} C_1 & D_1 \end{array} \right)} S$$

where the first matrix is a row matrix consisting of the generators of I. Thus the pruning algorithm, will commence by deleting the columns in C_1, the rows of A_2, obtaining

$$F_2 \xrightarrow{\left(\begin{array}{c|c} C_2 & D_2 \end{array} \right)} F_1' \xrightarrow{\left(\begin{array}{c} D_1 \end{array} \right)} S\ .$$

Now, inductively we can see that the pruning algorithm will successively prune each matrix M_i down to the matrix D_i. Hence $P(C_\bullet, Z)$ is the complex of T of modules whose ith map is given by D_i.

To see that $P(C_\bullet, Z)$ is a resolution, notice that any element $v = (v_1, \ldots, v_k)$ in the kernel of D_i trivially extends to the element $w = (0, \ldots, 0, v_1, \ldots, v_k)$ which is in the kernel of M_i. By the exactness of the original complex, we deduce that w is in the image of M_{i+1}, say $w = M_{i+1}(u)$. Finally, since every entry of C_{i+1} is zero mod Z, we have the following equality over S/Z:

$$v = \pi(w) = \pi(M_{i+1}(u)) = (C_{i+1}|D_{i+1})(u) = D_{i+1}(\overline{u})$$

where π is the obvious projection sending w to v and $\overline{u}$ consists of the last entries of u. Hence mod Z, v is in the image of D. □

The pruning technique does not preserve exactness in general, as the following example shows:

Example 2.7. *Consider the Buchsbaum-Rim resolution of the generic 2 by 3 matrix M:*

$$0 \longrightarrow S^1 \xrightarrow{\begin{pmatrix} \Delta_{23} \\ -\Delta_{13} \\ \Delta_{12} \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x & y & z \\ a & b & c \end{pmatrix}} S^2 \; .$$

This is a minimal free resolution of coker M*. Here Δ_{ij} denotes the ij minor of the presentation matrix. Pruning by setting x and y to zero yields*

$$0 \longrightarrow T^1 \xrightarrow{\begin{pmatrix} bz \\ -az \\ 0 \end{pmatrix}} T^3 \xrightarrow{\begin{pmatrix} 0 & 0 & z \\ a & b & c \end{pmatrix}} T^2$$

which is not exact since the kernel of the second map contains the element $(b, -a, 0)^T$, which is not in the image of the first.

We note that the pruning process has only been defined for complexes with a choice of bases. We have chosen this definition because it is all we need for the main results in this paper, and we feel that it highlights the important aspects of monomial resolutions, and the Eagon-Northcott complex. However, we could easily modify our definition to allow row and column operations over K. In fact, pruning can be defined without referring to matrices at all, simply by tensoring the given resolution with T and then taking successive quotients by the free module of degree zero syzygies at each stage. A further generalization might be to also include saturating by dividing through by common factors, which would remedy the problem with Example 2.7. We plan to study this generalization in the future.

Another interpretation of pruning is as follows: If $F_\bullet \to M$ is a minimal free resolution and x is a variable, then a general pruning technique should "work" exactly when the minimal free resolution of $M \otimes S/(x)$, is a direct summand of $F_\bullet \otimes S/(x)$. The following general result gives a necessary and sufficient condition for this to occur.

Proposition 2.8. *Let $F_\bullet$ be a minimal free resolution of a graded S-module M and let $x \in S$ be any homogeneous polynomial. By $F'_\bullet$ we will denote the complex of $S/(x)$-modules obtained by tensoring $F_\bullet$ with $S/(x)$. If H denotes $H_1^{S/(x)}(F'_\bullet)$, then the following are equivalent:*

1. *The minimal free resolution of* $M' := M \otimes S/(x)$ *is a direct summand of* $F'_\bullet$.
2. *There is a split inclusion of the minimal free resolution of* H *as an* $S/(x)$*-module into* $F'_\bullet[1]$.

Proof. We being by noting that since $\operatorname{Tor}_i^S(M, S/(x)) = 0$ for $i > 1$ we have $H_j(F'_\bullet) = 0$ for all $j > 1$.

$(i) \implies (ii)$: Let $G_\bullet$ be a minimal free resolution of M'. Then (i) says that there are projection maps π such that the following diagram commutes:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & G_n & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & G_0 \\ & & \uparrow{\scriptstyle \pi} & & & & \uparrow{\scriptstyle \pi} & & \uparrow{\scriptstyle \pi} \\ \cdots & \longrightarrow & F'_n & \longrightarrow & \cdots & \longrightarrow & F'_1 & \longrightarrow & F'_0 \end{array}.$$

Letting $K_\bullet$ denote $(\ker \pi)_\bullet$, we see that $K_\bullet$ split injects into $F'_\bullet$. To see that $K_\bullet$ is a resolution of H, notice that the long exact sequence of homology implies that

$$\cdots \to H_{i+1}(G_\bullet) \to H_i(K_\bullet) \to H_i(F'_\bullet) \to H_i(G_\bullet) \to \cdots$$

is exact. Since $H_j(F'_\bullet) = 0$ for all $j > 1$, and $G_\bullet$ is exact, we conclude that $H_j(K_\bullet) = 0$ for $j \geq 2$. Finally, we obtain the exact sequence:

$$0 \to H_1(K_\bullet) \to H_1(F'_\bullet) \to 0 \to H_0(K_\bullet) \to M' \xrightarrow{=} M' \to 0$$

and we see that $H_1(K_\bullet) \cong H$, so that $K_\bullet[-1]$ is a minimal free resolution of H, and hence $K_\bullet$ split-injects into $F'_\bullet[1]$.

$(ii) \implies (i)$: Suppose that we have a minimal free resolution $K_\bullet \to H$ which split injects into $F'_\bullet[1]$. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & F'_n & \longrightarrow & \cdots & \longrightarrow & F'_1 & \longrightarrow & F'_0 \\ & & \uparrow{\scriptstyle \phi} & & & & \uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle \phi} \\ \cdots & \longrightarrow & K_{n-1} & \longrightarrow & \cdots & \longrightarrow & K_0 & \longrightarrow & 0 \end{array}.$$

Taking cokernels of each map, and applying the long exact sequence of homology as in the first part of the proof, we see that $(\operatorname{coker} \phi)_\bullet$ is a minimal free resolution of M'. □

Remark 2.9. *Notice that in general, if* $K_\bullet \to H$ *is a resolution, then there is always a (non-canonical) map of complexes:* $\phi : K_\bullet \to F'_\bullet[1]$. *The mapping cone of* ϕ *will be a (typically non-minimal) free resolution of* M'. *In cases where pruning works,* ϕ *can be taken to be an inclusion.*

Corollary 2.10. *If* I *is a monomial ideal, and* x *is a variable, then*

$$\beta_{ij}\left(\frac{I : x}{I}\right) \leq \beta_{ij}(I) \text{ for all } i, j$$

Proof. Let $F_\bullet \to S/I$ be a minimal free resolution. By Theorem 2.6, the minimal free resolution of $S/I \otimes S/(x)$ is a direct summand of $F'_\bullet = F_\bullet \otimes S/(x)$. Hence by Proposition 2.8, the resolution of $H_1(F'_\bullet) \cong (I : x)/I$ is a direct summand of $F'_\bullet[1]$. In particular, the degrees and ranks of the free modules appearing in a minimal free resolution of $(I : x)/I$ can be no larger than those appearing in $F'_\bullet[1]$. Since $F[1]$ is a minimal free resolution of I we see the desired inequality. □

2.3 Initial Ideals of $I_k(X)$

In order to prove Theorems 2.1 and 2.3, it is useful to study the various initial ideals of $I_k(X)$ when X is a generic $k \times n$ matrix. By term order, we will always mean a monomial term order $<$, so that the initial ideal will be monomial.

In general, when passing to an initial ideal, we expect homological invariants to change. Indeed, since passing to the initial ideal is a flat deformation, we have

$$\beta_{ij}(S/\operatorname{in}_< I) \geq \beta_{ij}(S/I) \text{ for all } i, j$$

and typically these inequalities are strict. (For a great exposition, see [HH11]). For instance, the first Betti numbers are equal if and only if the ideal is minimally generated by a Gröbner basis with respect to the term order. In this vein, Sturmfels, Zelevinsky, and Bernstein have shown in [SZ93, BZ93] that the maximal minors form a universal Grobner basis for $I := I_k(X)$. This proves, for instance, that $\beta_1(S/\operatorname{in}_< I) = \beta_1(S/I) = \binom{n}{k}$ for any term order. In this section we prove

Theorem 2.11. *If $I := I_m(X)$ is the ideal of maximal minors of a generic matrix X and $<$ is any term order, then*

$$\beta_{ij}(S/\operatorname{in}_< I) = \beta_{ij}(S/I) \quad \textit{for all } i, j.$$

In particular, every initial ideal is a Cohen-Macaulay, squarefree monomial ideal with a linear free resolution. Further, the resolution can be obtained from the Eagon-Northcott complex by taking appropriate lead terms of each syzygy.

For certain orders, analyzing the initial ideal explicitly is manageable. For example, diagonal term orders were viewed in the context of basic double links in [GMN13] where they proved such initial ideals are Cohen-Macaulay. In general, however, not all term orders have “nice” descriptions. Instead we use the following fact:

Lemma 2.12. *[Sturmfels-Zelevinsky [SZ93]] For any monomial term order $<$, the initial ideal $\operatorname{in}_< I$ is squarefree and has a primary decomposition of the form*

$$\operatorname{in}_< I = \bigcap_\alpha I_\alpha$$

where α ranges over all subsets $\{j_1, j_2, \ldots, j_c\}$ of $\{1, \ldots, n\}$with $c = n - k + 1$, and $I_\alpha = (x_{i_1 j_1}, \ldots, x_{i_c j_c})$.

Remark 2.13. *[SZ93] gives an explicit description of these components in terms of the monomial order $<$, but we will not need that much detail in what follows.*

Proof of Theorem 2.11. Let $<$ be any term order, and write $\operatorname{in} I = \operatorname{in}_< I$. We will show that

$$\{x_{11} - x_{21}, \ldots, x_{11} - x_{k1}\} \cup \{x_{12} - x_{22}, \ldots, x_{12} - x_{k2}\} \cup \cdots$$

$$\cdots \cup \{x_{1n} - x_{2n}, \ldots, x_{1n} - x_{kn}\}$$

is a regular sequence on $S/\operatorname{in} I$. Indeed, once this is shown, we know that the Betti numbers of $\operatorname{in} I$ are the same as those of the ideal obtained by substituting the relations induced by the regular sequence above. These are precisely the substitutions $x_{ij} = x_{1j}$ for all i, j. Since $\operatorname{in} I$ is the ideal generated by the leading term of each minor, these substitutions deform $\operatorname{in} I$ into the ideal J consisting of all squarefree degree k monomials in $K[x_{11}, \ldots, x_{1n}]$. The resolution of this ideal is well known. In particular, its Betti numbers are equal to those in the Eagon-Northcott complex, and $\beta_{ij}(S/\operatorname{in} I) = \beta_{ij}(S/J) = \beta_{ij}(S/I)$ as required.

To prove that the sequence defined above is a regular sequence, we successively modify the primary decomposition described in Lemma 2.12 after each substitution. Since in the end, we will only compute with the ideal formed by substituting $x_{ij} = x_{1j}$, we study these *substitution ideals*.

Set $K = \operatorname{in} I$ and suppose $K = \bigcap P_i$ as in the Lemma. Since we will inductively apply the following argument, we first highlight the following properties that we will use about K:

- K has no minimal generators that contain a product of two elements from the same column of X.
- The ideals $P_i = (x_{i_1}, \ldots, x_{i_c})$ are generated by variables in different columns of X.

Let x_{ij} be any variable with $i \neq 1$. For the ease of notation, we will write *sub* to denote the substitution $x_{ij} \to x_{1j}$. We claim that the following two monomial ideals are equal:

$$(K)_{sub} = \bigcap (P_i)_{sub}.$$

Indeed, since substitution is just a ring map, $K \subset \cap P_i$ implies that $K_{sub} \subset \cap (P_i)_{sub}$.

Conversely, suppose that f is a minimal generator of $\cap (P_i)_{sub}$. Notice that f does not involve x_{ij}. We have two cases:

Case 1: x_{1j} does not divide f. In this case, the membership of f in $(P_i)_{sub}$ guarantees membership in (P_i) since the factors of f relevant to ideal membership do not change under our substitution.

Case 2: x_{1j} divides f, say $f = x_{1j} g$. Consider the element $h = x_{ij} f$. Since h is divisible by both x_{ij} and x_{1j}, and since f is in $\cap (P_i)_{sub}$, we know h is in fact in each ideal P_i. Thus $h = x_{ij} f = x_{ij} x_{1j} g \in K$. But since K has no minimal generators divisible by $x_{ij} x_{1j}$ we know that either $x_{ij} g$ or $x_{1j} g$ must be in K. Under the substitution, both of these elements will be sent to f, so that $f \in K_{sub}$.

Notice that if we next replace K and P_i with $(K)_{sub}$ and $(P_i)_{sub}$, then K and P_i still satisfy the bulleted properties above. Therefore, we may inductively apply our argument to the next substitution $x_{ij} \to x_{1j}$ to complete the proof. $\square$

Remark 2.14. *It is a very rare property for an ideal be minimally generated by a Gröbner basis, and it is an even rarer property for $\beta_{ij}(S/I) = \beta_{ij}(S/\operatorname{in}_< I)$ for $i > 0$, for every term order. Indeed, there are ideals that are minimally generated by a Gröbner basis, but whose initial ideals have still have larger Betti numbers than the ideal itself. It would be interesting to study whether such behavior is possible for ideals minimally generated by a universal Gröbner basis. But apart from determinantal ideals, monomial ideals, and trivial examples of ideals whose generators have all coprime terms, the author does not know of any other (large) classes of ideals with this property. It may be that the symmetry inherent in being a universal Gröbner basis is enough to ensure that all Betti numbers remain constant upon passing to the initial ideal.*

Question 2.15. *Do there exist ideals minimally generated by a universal Gröbner basis whose initial ideals have distinct Betti tables?*

Having shown the Betti numbers of S/I and $S/\operatorname{in}_< I$ are equal, a natural question is how to obtain a minimal free resolution for $S/\operatorname{in} I$. We next show that this can easily be obtained from the Eagon-Northcott complex.

Since our pruning technique is defined only for complexes where the maps are represented by matrices, we need to specify what we mean by "Eagon-Northcott complex". By this, we will always mean the complex whose first map consists of the minors Δ_J and whose later maps are of the form

$$D_a(S^k) \otimes \wedge^{a+k}(S^n) \to D_{a-1}(S^k) \otimes \wedge^{a+k-1}(S^n)$$

where D_i is the divided power algebra and the matrices are chosen with respect to the natural basis $e_1^{(n_1)} \cdots e_k^{(n_k)} \otimes f_{j_1} \wedge \cdots \wedge f_{j_\ell}$, where $e_1, \ldots, e_k$ and $f_1, \ldots, f_n$ are bases for the rows and columns of X.

Remark 2.16. *Notice that with this choice of basis, the first matrix in the complex consists of the minors Δ_J, and all syzygy matrices are essentially multiplication tables between the rows and columns. For this reason we notice that each entry is simply a variable $\pm x_{ij}$ and that no variable appears twice in the same row or column.*

Now let w be any set of weights on the variables x_{ij}. Then since we can always choose a monomial order $<_w$ which refines that of w, we have

$$\beta_{ij}(S/I) \leq \beta_{ij}(S/\operatorname{in}_w I) \leq \beta_{ij}(S/\operatorname{in}_{<_w} I).$$

By Theorem 2.11, we have equality.

For a weight w, we can homogenize any $f \in S$ by taking the leading term to be the one of highest weight, and multiplying smaller order terms by appropriate powers of a parameter t. We denote the homogenization f^h and will write I^h for the ideal by I^h the ideal

$$I^h = \{f^h \mid f \in I\} \subset S[t].$$

Similarly, we can homogenize any map between free S-modules.

Example 2.17. *If we consider the Eagon-Northcott complex on the matrix with weights*

$$X = \begin{pmatrix} x & y & z \\ a & b & c \end{pmatrix}, \quad w = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

we could homogenize the maps to obtain

$$0 \longrightarrow \begin{matrix} S[t](-5) \\ \oplus \\ S[t](-6) \end{matrix} \xrightarrow{\begin{pmatrix} z & ct \\ -y & -b \\ x & a \end{pmatrix}} \begin{matrix} S[t](-3) \\ \oplus \\ S[t](-4) \\ \oplus \\ S[t](-4) \end{matrix} \xrightarrow{(\Delta^h_{12}, \Delta^h_{13}, \Delta^h_{23})} S[t]$$

where $\Delta^h_{12} = xb - ay,\ \Delta^h_{13} = xct - az,\ \Delta^h_{23} = cyt - bz.$

In this example the above is a minimal free resolution of I^h. This is always true, which we prove now.

Proposition 2.18. *Let w be an integral weight order on the variables and let $E_\bullet$ denote the Eagon-Northcott complex. Then $E^h_\bullet$ is a minimal free resolution of $S[t]/I^h$.*

Proof. We notice that $I^h = (\Delta^h_J)$ since the Δ_J form a universal Gröbner basis, so we just need to show that $E^h_\bullet$ is exact. To show this, it suffices to show that $E^h_\bullet$ is exact after tensoring with $S[t]/(t)$ - in other words, after erasing each entry divisible by t. By Remark 2.16 the surviving columns of each matrix will be linearly independent over K. But since

$$\beta_{ij}(S/I) = \beta_{ij}(S/\operatorname{in}_w I) = \beta_{ij}(S/I^h) \text{ for all } i, j.$$

we see that these columns in fact span the full space of syzygies. □

Corollary 2.19. *To obtain the minimal free resolution of $S/\operatorname{in}_< I$ simply set $t = 0$ in the resolution $E^h_\bullet$ defined above.*

2.4 Minimal Free Resolution of Determinantal Ideals

In this section we compute the minimal free resolution of the ideal $I_k(X')$ where X' is a sparse generic matrix. This section was inspired by the work of Giusti and Merle in [GM82]. Throughout this section, X and X' will denote generic and sparse generic matrices respectively.

Since a matrix with a column identically equal to zero is essentially a $k \times (n-1)$ matrix, we will assume X' has no column identically zero. We also assume that $I_k(X')$ is not the zero ideal. This is equivalent to the fact that there is no rectangle of zeros in X' whose perimeter is greater than $2n+1$. (See [GM82])

Theorem 2.20. *Let $X = (x_{ij})$ be a generic $k \times n$ matrix and Z be a subset of the variables. Let X' be the sparse generic matrix with variables in Z set to zero. If $E_\bullet$ is the Eagon-Northcott Complex with standard bases that resolves $S/(I_k(X))$, then the result of pruning - $P(E_\bullet, Z)$ is a minimal free resolution of $S/I_k(X')$ as an S/Z module.*

Proof. Let $I = I_k(X)$. To simplify notation, we will use z_{ij} to denote the variables in Z, and use x_{ij} to denote the other variables. Assign a grading on S by assigning weights

$$w(z_{ij}) = 1, \quad w(x_{ij}) = 2.$$

Under this grading, the ideal I is no longer homogenous.

By Proposition 2.18 $E^h_\bullet$ is a resolution of $S[t]/I^h$. In particular,

$$I^h = (\Delta^h_J)$$

where J runs over all the $k \times k$ minors.

Further, there is a dichotomy

$$w(\Delta^h_J) = 2k \iff \Delta_J \neq 0 \mod Z,$$

$$w(\Delta^h_J) < 2k \iff \Delta_J = 0 \mod Z.$$

By virtue of the simplicity of the maps in the Eagon Northcott complex, every matrix after the first contains entries that are simply variables of S. Hence, with respect to our grading every element in these matrices is either of degree one or two before homogenization. After homogenizing we can split our resolution into pieces: One corresponding to the strand that resolves the "surviving" minors of weight $2k$, and the other consisting of everything else. Explicitly, the ith map of $E^h_\bullet$ will look like:

$$\begin{matrix} \bigoplus_{a_j < 2k+2i+2} S[t](-a_j) \\ \bigoplus \\ \bigoplus S[t](-2k-2i-2) \end{matrix} \xrightarrow{M_i} \begin{matrix} \bigoplus_{b_j < 2k+2i} S[t](-b_j) \\ \bigoplus \\ \bigoplus S[t](-2k-2i) \end{matrix}$$

where the matrix M_i has the form

$$\left(\begin{array}{c|c} A_i & T_i \\ \hline C_i & D_i \end{array}\right).$$

From the grading alone we can deduce three things:

- The nonzero entries of T_i are divisible by t since the degree shift is more than two.
- The nonzero entries of C_i have degree at most one. (i.e. they are z_{ij})
- The nonzero entries of D_i have degree two (i.e. they are $z_{ij}t$ or x_{ij}.)

Note that this implies that if we take the matrix D_i modulo Z or modulo t we get the same result. Denote this matrix F_i:

$$F_i := D_i \bmod t = D_i \bmod Z.$$

Therefore when we set t equal to zero in $E^h_\bullet$, we obtain a complex $E'_\bullet$ where all matrices take the following form

$$\left(\begin{array}{c|c} A_i & 0 \\ \hline C_i & F_i \end{array}\right).$$

This is analogous to the decomposition we had in the monomial case. By the same argument in the proof of Theorem 2.6 we conclude that modulo the variables in Z, the complex $F_\bullet$ is equal to $P(E_\bullet, Z)$ and is a minimal free resolution of $S/I \otimes S/(Z) \cong S/I_k(X')$. □

Corollary 2.21. *If X and X' are as above, then*

- *$S/I_k(X')$ has regularity k*
- *$\beta_{ij}(S/I_k(X')) \leq \beta_{ij}(S/I_k(X))$ for all i, j.*
- *$S/I_k(X')$ has projective dimension $n - k + 1$.*

Proof. Let $I' = I_k(X')$. By Theorem 2.20, the minimal free resolution of S/I' is given by pruning the Eagon-Northcott complex, and as such, the degrees of syzygies do not change. Hence the regularity is equal to k, the generating degree of the ideal, which proves the first statement.

Notice that each time we add a zero to our matrix, we can compute a minimal free resolution by pruning, and as such the Betti numbers can only possibly decrease. This shows the second statement.

We compute the projective by using induction on k and n. Since the only $1 \times n$ matrices with no columns identically equal to zero are generic matrices, the base case is trivial. Similarly, $k \times k$ matrices give rise to a principal ideal of minors, which have projective dimension 1.

Since I' is nonzero, we can assume without a loss of generality that $D = \Delta_{1\cdots k} \neq 0$, and that a nonzero term of D is a multiple of x_{k1}. Notice that by pruning, the projective dimension can only decrease by adding more zeros, so it is sufficient to compute the projective dimension in the case when the first column has $k-1$ zeros. Thus we may assume X' has the form

$$X' = \begin{pmatrix} 0 & \star & \dots & \star \\ \vdots & \vdots & \cdots & \vdots \\ 0 & \star & \cdots & \star \\ x_{k1} & \star & \cdots & \star \end{pmatrix} = \left(\begin{array}{cccc} 0 & & & \\ \vdots & & M' & \\ 0 & & & \\ \hline x_{k1} & \star & \cdots & \star \end{array}\right)$$

Let Y denote the matrix of the rightmost $n-1$ columns of X'. Then

$$I' : x_{k1} = I_{k-1}(M') \;\text{ and }\; (I', x_{k1}) = (x_{k1}) + I_k(Y).$$

M' is a sparse generic matrix and since the minor $\Delta_{2\cdots n}$ (indices refer to those of X') of M' is nonzero by assumption, $I_{k-1}(M')$ is nonzero. We have two cases:

- Case 1: Suppose that some column j of M' is identically zero. Then since $D \neq 0$ we know that $j > n$, and since X' had no column identically zero, the kj entry of X' must be nonzero. Hence $\Delta_{\{2\cdots n\}\cup\{j\}} \neq 0$, so that $I_k(Y)$ is nonzero. In this case, by induction, $\operatorname{pdim} S/I_k(Y) = n-k$.

- Case 2: If no column of M' is identically zero, then by induction,

$$\operatorname{pdim} S/(I' : x_{k1}) = \operatorname{pdim} S/I_{k-1}(M) = n-k+1, \quad \operatorname{pdim} S/I_k(Y) \leq n-k$$

  the last inequality is strict if and only if $I_k(Y)$ is the zero ideal.

In either case, we have

$$\max\left(\operatorname{pdim} S/I_k(Y)+1, \operatorname{pdim} S/I_{k-1}(M')\right) = n-k+1.$$

Since the resolution of $S/(I', x_{k1})$ can be obtained by tensoring the resolution of $S/I_k(Y)$ with the Koszul complex on x_{k1} we see that

$$\operatorname{pdim} S/(I', x_{k1}) = \operatorname{pdim} S/I_k(Y) + 1$$

and that the minimal free resolution of $S/(I', x_{k1})$ is linear after the first map. Applying the Horseshoe Lemma to the exact sequence

$$0 \longrightarrow S/(I' : x_{k1})(-1) \longrightarrow S/I' \longrightarrow S/(I', x_{k1}) \longrightarrow 0\,,$$

we see that a free resolution of S/I' can be computed as the direct sum of the minimal free resolutions of $S/(I' : x_{k1})$ and $S/(I', x_{k1})$. Finally, since $S/(I' : x_{k1}) \cong S/I_{k-1}(M')$

has a linear resolution by Theorem 2.20, this implies that except for the extra generator in homological degree 0, the direct sum of the resolutions of the outside two modules is in fact a *minimal* free resolution of S/I'. Hence:

$$\begin{aligned} \operatorname{pdim} S/I' &= \max\left(\operatorname{pdim} S/(I', x_{k1}), \operatorname{pdim} S/(I' : x_{k1})\right) \\ &= \max\left(\operatorname{pdim} S/I_k(Y) + 1, \operatorname{pdim} S/I_{k-1}(M')\right) \\ &= n - k + 1. \end{aligned}$$

□

We close this section by proving a result analogous to Corollary 2.10.

Corollary 2.22. *Let X' be a spare $k \times n$ generic matrix and $I = I_k(X')$. If x is any variable appearing in X' then $(I : x)/I$ has a linear resolution as an $S/(x)$-module. Furthermore, its Betti numbers are precisely the difference between those of $I_k(X')$ and $I_k(X'')$ where X'' is the matrix X' with x substituted for zero.*

Proof. Let $F_\bullet$ be a minimal free resolution of S/I. By inductively applying Theorem 2.20, we see that the minimal free resolution of $S/I_k(X'')$ can be obtained by pruning $F_\bullet$. This precisely says that the minimal free resolution of $S/I_k(X'')$ is a direct summand of $F_\bullet \otimes S/(x)$. Since $H_1(F_\bullet \otimes S/(x)) = \operatorname{Tor}_1^S(S/I, S/(x)) \cong (I : x)/I$, Proposition 2.8 shows that the resolution of $(I : x)/I$ injects into $(F_\bullet \otimes S/(x))[1]$ and hence has a linear resolution. The statement about Betti numbers follows since from the short exact sequence of complexes used in the proof of Proposition 2.8 □

2.5 Applications and Examples

Merle and Giusti's result in [GM82] was particularly beautiful because it showed that several invariants of $I_k(X')$ depended only on one number - the length of the perimeter of the largest subrectangle of zeros in the sparse generic matrix X'. In this vein, Corollary 2.21 can be interpreted as saying that the projective dimension depends only on the number of columns that are identically zero. The next natural question seems to be how the Betti numbers depend on the placement of zeros in the matrix. Notice that if $\operatorname{codim} I_k(X') = n - k + 1$ then $I_k(X')$ is a perfect ideal, and hence the Eagon-Northcott complex itself is a resolution.

In smaller codimension, however, it is easy to produce matrices with the same perimeter of zeros, but yet whose ideals have a different number of minimal generators. One might hope that the perimeter and number of generators are sufficient to compute all the Betti numbers. However, the following example shows two matrices that give rise to ideals with the same codimension and number of generators, but have different Betti numbers.

Example 2.23.

X'	$\operatorname{codim} I_3(X')$	*perimeter of zeros*	*Betti Table of* $(S/I_3(X'))$
$\begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & x & y & z \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & a & b & c \\ d & e & f & g & h & w \end{pmatrix}$	2	10	$\begin{array}{\|ccccc} 1 & - & - & - & - \\ - & - & - & - & - \\ - & 10 & 18 & 12 & 3 \end{array}$
$\begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & a & b \\ \boldsymbol{0} & \boldsymbol{0} & c & d & \boldsymbol{0} & \boldsymbol{0} \\ e & f & g & h & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}$	2	10	$\begin{array}{\|ccccc} 1 & - & - & - & - \\ - & - & - & - & - \\ - & 10 & 17 & 10 & 2 \end{array}$

This suggests that whatever dependence the Betti numbers have on the arrangement of zeros is subtle. However, in the case of codimension $n-k$, we have the following:

Theorem 2.24. *Let X' be a sparse $k \times n$ generic matrix and let $I' = I_k(X')$. If $\operatorname{codim} I' = n-k$ then the Betti numbers of S/I' depend only on the number of identically vanishing minors of X'.*

The proof follows from the more general lemma from Boij-Soderberg Theory ([BS12]):

Lemma 2.25. *If I is an ideal generated in degree d with a linear resolution such that $\operatorname{codim} I = \operatorname{pdim} S/I - 1$ then the Betti table of S/I is determined by the minimal number of generators $\mu(I)$.*

Proof. Let $\operatorname{pdim} S/I = r$. By Boij-Soderberg Theory, the Betti table of S/I is a linear combination over $\mathbb{Q}$ of two pure diagrams B_1 and B_2 corresponding to the sequences

$$(0, d, d+1, \ldots, d+r), \text{ and } (0, d, d+1, \ldots, d+r-1)$$

respectively. If $\beta(S/I)$ denotes the Betti table of S/I then we have

$$\beta(S/I) = a_1 B_1 + a_2 B_2.$$

By equating the zeroth and first Betti numbers on each side, we obtain the following equations

$$\begin{aligned} a_1 + a_2 &= 1 \\ \binom{d+r}{d} a_1 + \binom{d+r-1}{d} a_2 &= \mu(I) \end{aligned}$$

from which we can determine a_1, a_2 and hence $\beta(S/I)$. □

Next, we answer a question of Giusti and Merle concerning when the ideals $I_k(X')$ are radical.

Proposition 2.26. *If X' is any sparse generic matrix, then the nonzero minors are a universal Gröbner basis for the ideal they generate. In particular, for each term order, the initial ideal is squarefree, and thus $I_k(X')$ is a radical ideal.*

Proof. Let Y be a generic $k \times n$ matrix with entries z_{ij} and x_{ij} corresponding to the zero and nonzero entries of X' respectively. Let $<$ be any term order on the variables supporting $I_k(X')$. Then extend this to an order $<_2$ on the z_{ij} where the z_{ij} are weighted last. Let $f \in I_k(Y)$. Then if $f = \sum c_J \Delta_J(X')$ is nonzero, consider the element

$$\overline{f} = \sum c_J \Delta_J(Y).$$

Then since the z_{ij} are weighted last, $\operatorname{in} \overline{f} = \operatorname{in} f$. And thus $\operatorname{in} f$ is divisible by some $m_0 = \operatorname{in} \Delta_J(Y) = \operatorname{in} \Delta_J(X')$. □

Monomial Ideals with Linear Resolutions

A corollary of our work is that we can produce many monomial ideals in any degree that have linear resolutions. For example, by Theorem 2.11, we know that if we choose any monomial term order $<$ and any generic matrix X, then the initial ideal $I_k(X)$ with respect to $<$ has a linear resolution. The proof of this fact carries through to work for generic matrices with zeros as well. Also, in the spirit of the proof of Theorem 2.11 we can also set any entries in the same column equal to each other, and obtain yet another ideal with a linear resolution. Hence we have the following:

Theorem 2.27. *Let X' be a generic $k \times n$ matrix with zeros and let $<$ be any monomial term order. Then the initial ideal $J = \operatorname{in}_< I_k(X')$ is an ideal with a linear resolution. Furthermore, if $\{(x_i, y_i)\}$ is any collection of variables such that for each i, x_i and y_i are in the same column of X' then the ideal $J_{x \to y}$ where we substitute y_i for x_i still has a linear resolution.*

If we apply this theorem by setting each variable in each column to the same variable (say y_i) then we will obtain a squarefree monomial ideal in $K[y_1, \dots, y_n]$ which has a linear resolution. This proves

Corollary 2.28. *Let X' be a generic $k \times n$ matrix with zeros. Let J denote the ideal generated by all such $\prod y_{i_1} \cdots y_{i_k}$ such that the $\det X'_{i_1,\dots,i_k} \neq 0$. Then J has a linear resolution.*

Questions and Future Work

It is interesting to ask to what extent the pruning technique works in general. There are two directions in which one could attempt to answer this question:

Question 2.29.

1. *For what other classes of ideals does the pruning technique compute a minimal resolution after setting variables equal to zero? For example, what can be said for determinantal ideals of lower order minors of sparse determinantal matrices.*

2. *How does pruning work when we prune by setting arbitrary linear forms equal to zero? For example, when can we use a pruning technique to compute the minimal free resolution of determinantal ideals of (non-generic) matrices of linear forms?*

One interesting case for Question (ii) is the resolution of the ideal of 2×2 minors of an arbitrary $2 \times n$ matrix of linear forms. In [ZNZN00], the authors computed Gröbner bases and a free resolution of all such ideals. In the cases where the matrix is sparse generic, our resolution agrees with theirs, but they show that in general the regularity can be as large as $n - 1$. It is not clear how a pruning technique could be used to prune the linear Eagon-Northcott complex to a nonlinear resolution. However, there may be an interpretation via mapping cones as in Remark 2.9.

Another special case of Question 2.29 is the case when the linear forms are the difference of two variables. In other words, how does the minimal free resolution of an ideal change as variables are set equal to one another? This question must necessarily be difficult, since any ideal can be obtained from a generic complete intersection (in many variables) by successively setting variables equal to one another. However, in some cases it may be possible to give an effective answer.

Chapter 3

Robust Toric Ideals

The content of this chapter is the paper [BR13] and is joint work with Elina Robeva. It has been submitted for publication with only minor changes.

3.1 Introduction

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k. We call an ideal *robust* if it can be minimally generated by a universal Gröbner basis, that is, a collection of polynomials which form a Gröbner basis with respect to all possible monomial term orders. Robustness is a very strong condition. For instance, if I is robust then the number of minimal generators of each initial ideal is the same:

$$\mu(I) = \mu(\mathrm{in}_< I) \quad \text{for all term orders } <.$$

In general, we can only expect an inequality.

The study of ideals minimally generated by a Gröbner basis (for some term order) is ubiquitous. In [CHT06], Conca et al studied certain classical ideals and determined when they are minimally generated by some Gröbner basis. In the study of Koszul algebras, one of the most fruitful approaches has been via G-quadratic ideals - those generated by a quadratic Gröbner basis. We are not aware, however, of any systematic study of ideals minimally generated by a universal Gröbner basis; *robust ideals*.

For trivial reasons, all monomial and principal ideals are robust. Simple considerations show that robustness is preserved upon taking coordinate projections and joins (see Section 2). However, nontrivial examples of robust ideals are rare. A difficult result of [BZ93, SZ93] (recently extended by [Boo12] and [CNG13]) shows that the ideal of maximal minors of a generic matrix of indeterminates is robust. In the toric case, the broadest known class of examples is the set of Lawrence ideals. In this paper we study robustness and provide a classification of robust toric ideals generated in degree two. It turns out that in this setting, robust toric ideals are essentially determinantal, (and thus Lawrence as well). On the other hand, we show that robustness does not in general classify Lawrence ideals.

The paper is organized as follows: In section 2, we prove our main result, Theorem 3.1 characterizing robust toric ideals generated in degree two. The methods are mainly combinatorial. In Sections 3 and 4 we pose two questions concerning extensions of Theorem 3.1 using Lawrence ideals. We provide negative and positive answers respectively. Section 5 closes with a discussion of "robustness of higher Betti numbers," our original motivation for this project.

3.2 Quadratic Robust Toric Ideals are Determinantal

In the sequel, by *toric ideal* we will always mean a prime ideal generated minimally by homogeneous binomials with nonzero coefficients in k. By the support of a polynomial we mean the set of variables appearing in its terms.

Definition 2. *A set F of polynomials in S is called robust if F is a universal Gröbner basis and the elements of F minimally generate their ideal.*

If a set of polynomials F can be written as a union $F = G \cup H$ of polynomials in disjoint sets of variables, then we say that G is a *robust component* of F. If F admits no such decomposition, then we say the set F is *irreducibly robust*. Notice that robustness is preserved under these disjoint unions, so to classify robust ideals, it suffices to study the irreducible ones. We remark that the ideal of F corresponds to the join of the varieties corresponding to the G and H.

The goal of this section is to prove the following theorem:

Theorem 3.1. *Let F be an irreducibly robust set consisting of irreducible quadratic binomials. Then F is robust if and only if $|F| = 1$ or F consists of the 2×2 minors of a generic $2 \times n$ matrix*

$$\begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix}$$

up to a rescaling of the variables.

Remark 3.2. *In the statement of the Theorem, we only assume that the generators are irreducible. It turns out that this is sufficient to show that the ideal they generate is prime.*

Notice that one direction follows immediately from the results of [SZ93] which show that the 2×2 minors are a universal Gröbner basis. To prove the converse, our technique is essentially to eliminate certain combinations of monomials from appearing in F. To simplify notation, we will omit writing coefficients in the proofs when it is clear that they do not affect the argument. In particular, we treat the issue of coefficients only in tackling the proof of Theorem 3.1 itself and not in earlier lemmas.

Lemma 3.3. *Let F be a robust set of prime quadratic binomials. Then no monomial appears as a term in two different polynomials in F.*

Proof. Suppose that the monomial m appears in the polynomials $f, g \in F$. Let $<$ be a Lex term order taking the support of m to be first. Since f and g are prime, $<$ will select m as the lead term of both f and g, and applying Buchberger's algorithm, we would obtain a degree zero syzygy of the elements in F, contradicting the minimality of F. □

Proposition 3.4. *If F is robust, and $0 \leq k \leq n$, then so is $F \cap k[x_1, \ldots, x_k]$.*

Proof. Write $F_k = F \cap k[x_1, \ldots, x_k]$. It is clear that F_k minimally generates the ideal (F_k). Let $<$ be any term order on $k[x_1, \ldots, x_k]$. Extend $<$ to a term order $<_S$ on S, taking $x_1, \ldots, x_k$ last. Then since F is a Gröbner basis with respect to $<_S$, by basic properties of Gröbner bases, we know that F_k will be a Gröbner basis with respect to $<$. □

The above proposition is extremely useful, because in our analysis it will be helpful to assume we are working in a ring with few variables. We will use this reduction extensively in the following main technical lemma. We use the letters $a, \ldots, z$ when convenient for ease of reading.

Lemma 3.5. *Let F be a robust set of prime quadratic binomials:*

(a) F cannot contain two polynomials of the form

$$f = x^2 + yz, \;\; g = xy + m$$

or

$$f = x^2 + y^2, \;\; g = xy + m$$

or

$$f = x^2 + y^2, \;\; g = xz + m.$$

where m is any monomial.

(b) F cannot contain two polynomials of the form

$$f = x_i x_j + x_k x_l, \;\; g = x_i x_k + x_p x_q$$

(here we do not assume i, j, k, l, p, q are distinct.)

(c) F cannot contain two polynomials of the form

$$f = x^2 + yz, \;\; g = xw + m$$

or

$$f = x^2 + yz, \;\; g = yw + m$$

where m is any monomial.

(d) If $f, g \in F$ are two polynomials whose supports share a variable, then all terms of f and g are squarefree.

(e) *If F contains two polynomials whose supports share a variable, then (up to coefficients) F must contain the 2×2 minors of a generic matrix.*

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

Proof. a) We prove the first statement. The proofs of the others are similar. Suppose that $f, g \in F$. Notice that by primality, m cannot contain a factor of y. Let $<$ be the lex term order with ($y > x > z >$ all other variables). Then the S-pair of f and g is $x^3 - mz$, whose lead term is x^3. Since F is a Gröbner basis with respect to $<$ we must have some polynomial h whose lead term divides x^3. Since x^2 is not the lead term of f, $h \neq f$ and we must have two distinct polynomials in F with x^2 appearing. This contradicts Lemma 3.3.

b) Suppose that $f, g \in F$. By restricting to the subring $k[x_i, x_j, x_k, x_p, x_q]$, Proposition 3.4 tells us we can assume F involves only these variables. Notice by primality (and part (a)) we know that k and i are distinct from j, l, p, q. Let $<$ be the lex term order with ($x_k > x_i >$ all other variables.) Then the S-pair of f and g is $x_i^2 x_j - x_l x_p x_q$, whose lead term is $x_i^2 x_j$. As in part a) we must have some polynomial $h \in F$ whose lead term divides $x_i^2 x_j$. The only possible monomials are x_i^2 and $x_i x_j$. And since $x_i x_j$ appears in f (and is not a lead term) we must have a polynomial $h = x_i^2 + x_a x_b \in F$ for some $a, b \in \{i, j, k, l, p, q\}$. So F contains

$$f = x_i x_j + x_k x_l, \; g = x_i x_k + x_p x_q, \; h = x_i^2 + x_a x_b.$$

Applying part a), and primality, we know that $a, b \in \{l, p, q\}$. By part a), we know that $x_a x_b$ must be squarefree, and since $x_p x_q$ already appears, we can say (renaming p and q if necessary,) that $x_a x_b = x_l x_p$. But now choosing $<$ to be the lex term order with ($x_l > x_p > x_i > x_k >$ all other variables,) we see that the S-pair of f and h is $x_i^2 x_k - x_i x_j x_k$ whose lead term is $x_i^2 x_k$ which is only divisible by the monomials x_i^2 and $x_i x_k$, neither of which can be a lead term of a polynomial in F by Lemma 3.3.

c) We will prove the first statement. The second proof is similar. Suppose that $f, g \in F$. First restrict, using Proposition 3.4 to assume we are working only with the variables x, y, z, w and the factors of m. Let $<$ be the lex term order with ($w > x >$ all other variables. Taking the S-pair of f and g, we obtain $wyz - mx$, whose lead term is wyz. Since this must be divisible by the lead term of some polynomial $h \in F$, without loss of generality, we assume $h = wy + n$. Consider now the possibilities for n. By primeness n cannot contain a factor of w or y. By part b) it cannot contain a factor of x or z. Hence, the only possible options left are that the factors of n are contained in the factors of m. But this means that m, n are ab, a^2 for some (new, distinct) variables a, b. As in (b), we can conclude this is impossible.

d) This follows immediately from parts a) - c).

e) We assume that F contains two polynomials whose supports intersect. By d) we can assume that these polynomials are squarefree, and we write them as $p = ae - bd, q = af - m_1 m_2$ where m_i represents some variable. Notice that by primality and part b), neither m_1 nor m_2 can be a, e or f. Nor can $m_1 m_2 = bd$ (since it would be a repetition). Hence we may as well assume m_1 is different from the other variables, and call it c. There are now two cases: Either m_2 is also a new variable g, or it isn't, in which case we can see that without loss of generality, $m_2 = d$. Rewriting: If p, q are two polynomials whose supports intersect, then they must contain either 6 or 7 distinct variables.

In the case of 6 variables, restrict F to the subring $k[a, b, c, d, e, f]$. Now

$$p = ae - bd, \ q = af - cd.$$

Computing an S-pair with the lex order ($a > b > d >$ all other variables) we obtain $fbd - ecd$ with lead term fbd. This must be divisible by the lead term of some polynomial $r \in F$. But this lead term cannot be bd (by its presence in p), hence it must be either bf or df. In case it is bf then F contains a polynomial of the form $r = bf - n_1n_2$. Now $n_1, n_2 \in \{a, c, d, e\}$ by primality. And part b) of this lemma allows us to further say $n_1, n_2 \in \{c, e\}$ which along with squarefreeness implies that $r = bf - ce$ as required. In case the term is df, similar considerations show that parts a) - d) will not allow any n_1n_2.

In the case of 7 variables, restrict F to the subring $k[a, b, c, d, e, f, g]$. Now

$$p = ae - bd, \ q = af - cg.$$

Computing an S-pair with the lex term order ($a > b > d >$ all other variables): we obtain $fbd - ebg$ with lead term fbd. This must be divisible by the lead term of some polynomial $r \in F$. But this lead term cannot be bd (by its presence in p), hence it must be either bf or df. By symmetry we can assume that it is df and that F contains a polynomial of the form $df - n_1n_2$. Now $n_1, n_2 \in \{a, b, c, e, g\}$ by primality, and part b) restricts us further to $n_1, n_2 \in \{c, e, g\}$. And since cg already appears, we can conclude that $n_1n_2 = eg$ or ec (and again by symmetry, we may assume $n_1n_2 = ec$. But now notice that q and r are two polynomials whose supports intersect, and involve only 6 variables. Hence, by the previous part of this proof, we can conclude that F contains a polynomial $s = gd - ae$. But this is a contradiction by Lemma 3.3. □

Proof of Theorem 3.1. Suppose that $|F| > 1$. Since F is irreducible, it must contain two polynomials whose supports intersect. By Lemma 3.5 we can conclude that F contains polynomials of the form:

$$p_1 = ae - bd, \ p_2 = af - cd, \ p_3 = bf - ce$$

up to coefficients. However, computing an S-pair with the lex order on ($a > b > \cdots > f$) we obtain:

$$S(p_1, p_2) = bdf - cde \text{ (with some nonzero coefficients)}$$

which after reducing by p_3 we obtain either zero, or a constant multiple of cde. In the latter case, in order to continue the algorithm, we would have to have another polynomial in F whose lead term divided cde. By the presence of p_1, p_2, p_3, the terms cd and ce are prohibited. And by Lemma 3.5 b), de is also prohibited. Hence, this S-pair must reduce to zero after only two subtractions.

This means in fact, that the polynomials are precisely determinants of some matrix

$$\begin{pmatrix} \lambda_1 a & \lambda_2 b & \lambda_3 c \\ \mu_1 d & \mu_2 e & \mu_3 f \end{pmatrix}$$

for some nonzero constants λ_i, μ_i.

To complete the proof, suppose that $F \neq \{p_1, p_2, p_3\}$. Since F is irreducible, one polynomial $p_4 \in F$ must share a variable with say, p_1. Renaming variables if necessary, say that variable is a. Then (ignoring constants for the moment) by applying the proof of Lemma 3.5 e), to the polynomials $p_1 = ae - bd$ and $p_4 = ah - m_1 m_2$ we can conclude that F contains the minors of the matrix

$$\begin{pmatrix} \lambda_1 a & \lambda_2 b & \lambda_4 g \\ \mu_1 d & \mu_2 e & \mu_4 h \end{pmatrix}.$$

Applying this technique to p_2 and p_4 as well, shows that we in fact get all 2×2 minors of the full matrix

$$\begin{pmatrix} \lambda_1 a & \lambda_2 b & \lambda_3 c & \lambda_4 g \\ \mu_1 d & \mu_2 e & \mu_3 f & \mu_4 h \end{pmatrix}.$$

Inductively we continue this process until we obtain all of F. □

Notice that in our proof, every term order we used was a Lex term order, and we ended up a with a prime ideal. Hence, we have the following:

Corollary 3.6. *If F is a set of prime quadratic binomials that minimally generate an ideal. Then the following are equivalent:*

1. *F is a Gröbner basis with respect to every Lex term order.*

2. *F is a Gröbner basis with respect to every term order.*

3. *F generates a prime ideal and the irreducible robust components of F are generic determinantal ideals and hypersurfaces.*

Corollary 3.7. *If X is a generic $k \times n$ matrix, and F is the set of 2×2 minors, then F is a universal Gröbner basis if and only if $k = 2$.*

Remark 3.8. *It is almost the case that* every *irreducibly robust component is determinantal. Indeed, every prime binomial is up to rescaling either $xy - zw$ or $x^2 - yz$. The former is determinantal. Thus the only possible non-determinantal robust component is $\{x^2 - yz\}$.*

3.3 From Determinants to Lawrence Ideals

Encouraged by the result of the previous section, it is natural to ask to what extent robustness classifies generic determinantal ideals. Indeed, it is easy to see that the ideal of minors of any $2 \times n$ matrix whose entries are relatively prime monomials will be robust. There are two questions we consider:

Question 3.9.

1. *If I is a robust toric ideal, is I generated by the 2×2 minors of some matrix of monomials?*

2. *Precisely which matrices of monomials provide robust ideals of 2×2 minors?*

The answer to the first question is negative. Examples are provided by *Lawrence ideals*, studied in [Stu96]. If I is any toric ideal, with corresponding variety $X \subset \mathbb{P}^{n-1}$, then the ideal J corresponding to the re-embedding of X in $(\mathbb{P}^1)^n$ is called the Lawrence lifting of I. Its ideal is generated by polynomials of the following form:

$$J_L = \left(\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}} - \mathbf{x}^{\mathbf{b}}\mathbf{y}^{\mathbf{a}} \mid \mathbf{a} - \mathbf{b} \in L\right) \ \subset \ S = k[x_1, \dots, x_n, y_1, \dots, y_n],$$

where L is a sublattice of $\mathbb{Z}^n$ and k is a field. Here $\mathbf{a} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ for $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$. Binomial ideals of the form J_L are called *Lawrence ideals*. The following result is Theorem 7.1 in [Stu96].

Proposition 3.10. *The following sets of binomials in a Lawrence ideal J_L coincide:*

a) *Any minimal set of binomial generators of J_L.*

b) *Any reduced Gröbner basis for J_L.*

c) *The universal Gröbner basis for J_L.*

d) *The Graver basis for J_L.*

Hence Lawrence ideals provide a large source of robust toric ideals, and naturally include the class of generic determinantal ideals. Given this, it is natural to rephrase the first part of Question 3.9 as:

Question 3.11. *Does robustness characterize Lawrence ideals?*

Again the answer is negative.

Example 3.12. *The ideal*

$$\mathtt{I} = (\mathtt{b}^2\mathtt{e} - \mathtt{a}^2\mathtt{f}, \mathtt{bc}^2 - \mathtt{adf}, \mathtt{ac}^2 - \mathtt{bde}, \mathtt{c}^4 - \mathtt{d}^2\mathtt{ef})$$

in the polynomial ring $\mathbb{Q}[a,b,c,d,e,f]$ *is robust but not Lawrence. This example was found using the software* Macaulay2 *and* Gfan *[Sti, Jen]. It is the toric ideal* I_L *corresponding to the lattice defined by the kernel of*

$$L = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 3 & 1 \end{pmatrix}$$

Given this counterexample we ask

Question 3.13. *Is there a nice combinatorial description of robust toric ideals?*

Remark 3.14. *Heuristically, it is very easy to find robust ideals that are not Lawrence by starting with a Lawrence ideal given by* J_L*. This lattice* L *gives rise to a lattice* $\tilde{L} \subset \mathbb{N}^{2n}$ *such that* $J_L = I_{\tilde{L}}$*. By modifying* $\tilde{L}$ *slightly, it is very often the case that the resulting toric ideal is robust (though often non-homogeneous). The ubiquity of these examples computationally suggests that a nice combinatorial description of robustness may require imposing further hypotheses.*

3.4 Matrices of Monomials

In this section we answer Question 3.9.2.

Theorem 3.15. *Suppose that* X_i, Y_j *are monomials of degree at least 1 in some given set of variables* $\mathcal{U} = \{u_1, u_2, \ldots, u_d\}$*. Let*

$$A = \begin{pmatrix} X_1 & X_2 & \cdots & X_n \\ Y_1 & Y_2 & \cdots & Y_n \end{pmatrix},$$

where $n \geqslant 3$ *and suppose that the set* F *of* 2×2*-minors* $X_iY_j - X_jY_i$*,* $i \neq j$ *consists of irreducible binomials. Then* F *is robust if and only if all the monomials* X_i, Y_j *are relatively prime.*

The proof is technical, so we begin by fixing notation. Since we will assume that each $X_iY_j - X_jY_i$ is prime for all $i \neq j$, then, $\gcd(X_i, X_j) = \gcd(Y_i, Y_j) = \gcd(X_i, Y_i) = 1$ for all $i \neq j$. Therefore, if we define

$$z_{ij} = \gcd(X_i, Y_j),$$

then, we can write

$$X_i = x_i \prod_{j \neq i} z_{ij} \quad \text{and} \quad Y_j = y_j \prod_{i \neq j} z_{ij}.$$

Thus,

$$\gcd(x_i, x_j) = \gcd(y_i, y_j) = 1 \text{ for all } i \neq j \tag{i}$$

$$\gcd(z_{ij}, z_{kl}) = 1 \text{ whenever } i \neq k \text{ or } j \neq l, \tag{ii}$$

$$\gcd(x_i, z_{kl}) = 1 \text{ if } i \neq k \text{ and } \gcd(y_j, z_{kl}) = 1 \text{ if } j \neq l. \tag{iii}$$

Our goal is to show that $z_{ij} = 1$ for all $i \neq j$.

Lemma 3.16. *If $z_{12} \neq 1$ then $n \geqslant 4$ and for each $m \geqslant 3$, there exist permutations $i_m, l_m \neq 1, 2, m$ and $j_m, k_m \neq 1, 2$ and term orders $>_1$ and $>_2$ such that*

$$X_{i_m} Y_{j_m} \mid X_2 Y_m^2 \frac{X_1}{z_{12}} \text{ and } X_{i_m} Y_{j_m} >_1 X_{j_m} Y_{i_m} \tag{iv}$$

$$X_{k_m} Y_{l_m} \mid X_m^2 Y_1 \frac{Y_2}{z_{12}} \text{ and } X_{k_m} Y_{l_m} >_2 X_{l_m} Y_{k_m}. \tag{iv'}$$

Moreover,

$$X_{i_m} = x_{i_m} z_{i_m m} \text{ and } x_{i_m} | z_{i_m m},$$

$$Y_{l_m} = y_{l_m} z_{m l_m} \text{ and } y_{l_m} | z_{m l_m}.$$

Proof. We will build $<_1$ and $<_2$ in several steps. To begin, take a lex term order $>$ where the variables in z_{12} are first. Consider the S-pair:

$$S\left(\underline{X_1 Y_m} - X_m Y_1, \underline{X_m Y_2} - X_2 Y_m\right) =$$

$$= \frac{\operatorname{lcm}(X_1 Y_m, X_m Y_2)}{X_1 Y_m}(X_1 Y_m - X_m Y_1) - \frac{\operatorname{lcm}(X_1 Y_m, X_m Y_2)}{X_m Y_2}(X_m Y_2 - X_2 Y_m) =$$

$$= X_m \frac{Y_2}{z_{12}}(X_1 Y_m - X_m Y_1) - Y_m \frac{X_1}{z_{12}}(X_m Y_2 - X_2 Y_m) =$$

$$= X_2 Y_m^2 \frac{X_1}{z_{12}} - X_m^2 Y_1 \frac{Y_2}{z_{12}}.$$

Since all of the variables in $X_2 Y_m^2 \frac{X_1}{z_{12}} - X_m^2 Y_1 \frac{Y_2}{z_{12}}$ are different from the variables in z_{12}, then, there exist term orders $>_1$ and $>_2$ refining $>$ for which $X_2 Y_m^2 \frac{X_1}{z_{12}}$ is the leading term for $>_1$ and $X_m^2 Y_1 \frac{Y_2}{z_{12}}$ is the leading term for $>_2$.

Consider first $>_1$: $X_2 Y_m^2 \frac{X_1}{z_{12}} >_1 X_m^2 Y_1 \frac{Y_2}{z_{12}}$. Since the $X_i Y_j - X_j Y_i$ form a Gröbner basis with respect to $>_1$, there exist $i_m \neq j_m$ such that

$$X_{i_m} Y_{j_m} \mid X_2 Y_m^2 \frac{X_1}{z_{12}} \text{ and } X_{i_m} Y_{j_m} >_1 X_{j_m} Y_{i_m} \tag{iv}$$

in this ordering. If $j_m = 2$, then, $z_{12} \mid Y_2 \mid X_2 Y_m^2 \frac{X_1}{z_{12}}$, which is not true. If $j_m = 1$, then, since $z_{12} \mid X_1 \mid X_1 Y_{i_m}$ and $z_{12} \nmid X_{i_m} Y_1$ and z_{12} was chosen to have its variables first in $>_1$, then, $X_{j_m} Y_{i_m} = X_1 Y_{i_m} >_1 X_{i_m} Y_1 = X_{i_m} Y_{j_m}$, which is not true by (iv). Thus, $j_m \geqslant 3$. Similarly, we can deduce that $i_m \geqslant 3$. Thus, $i_m, j_m \geqslant 3$.

Since $i_m \geqslant 3$, $\gcd(X_1, X_{i_m}) = \gcd(X_2, X_{i_m}) = 1$, and we are assuming that $X_i, Y_i \neq 1$ for all i and $X_{i_m} \mid X_2 Y_m^2 \frac{X_1}{z_{12}}$ then, $X_{i_m} \mid Y_m^2$. Since $\gcd(X_m, Y_m) = 1$, $i_m \neq m$ either. Thus, we have that $i_m \neq 1, 2, m$ and , $j_m \geqslant 3$. So, in particular, if $n = 3$, we already have a contradiction. We assume now that $n \geqslant 4$. Since $X_{i_m} = x_{i_m} \prod_{j \neq i_m} z_{i_m j}$ and $Y_m = y_m \prod_{k \neq m} z_{km}$, properties (i) and (iii) allow us to conclude that $X_{i_m} = x_{i_m} z_{i_m m}$ and $x_{i_m} \mid z_{i_m m}$.

Going back to when we chose the ordering $>_1$, consider now the ordering $>_2$ for which $X_m^2 Y_1 \frac{Y_2}{z_{12}} >_2 X_2 Y_m^2 \frac{X_1}{z_{12}}$. By a symmetric argument we find that there exist $k_m \geqslant 3, l_m \neq 1, 2, m$ such that $X_{k_m} Y_{l_m} \mid X_m^2 Y_1 \frac{Y_2}{z_{12}}$ and, thus, $Y_{l_m} = y_{l_m} z_{m l_m}$ with $y_{l_m} \mid z_{m l_m}$.

Hence, there exist $i_m, l_m \neq 1, 2, m$ and $j_m, k_m \neq 1, 2$ such that $X_{i_m} = x_{i_m} z_{i_m m}$, $x_{i_m} \mid z_{i_m m}$ and $Y_{l_m} = y_{l_m} z_{m l_m}$, $y_{l_m} \mid z_{m l_m}$. □

Lemma 3.17. *If $z_{12} \neq 1$, then, n is even and we can rearrange the numbers $1, .., n$ so that for each $i \leqslant \frac{n}{2}$,*

$$X_{2i} = x_{2i} z_{2i,2i+1} \text{ and } Y_{2i} = y_{2i} z_{2i+1,2i}$$

$$X_{2i+1} = x_{2i+1} z_{2i+1,2i} \text{ and } Y_{2i+1} = y_{2i+1} z_{2i,2i+1}$$

and $x_{2i}, y_{2i+1} \mid z_{2i,2i+1}$ and $x_{2i+1}, y_{2i} \mid z_{2i+1,2i}$.

Proof. By property (ii) and Lemma 4.2 we have that $m \mapsto i_m$ and $m \mapsto l_m$ are permutations on $\{3, .., n\}$ with no fixed points. Thus, for each $i \geqslant 3$, there exists $m \geqslant 3$ such that $m \neq i$ and $i = i_m$, $X_i = x_i z_{im}$ with $x_i \mid z_{im}$ and for each $l \geqslant 3$, there exists $m \geqslant 3$, $m \neq l$ such that $l = l_m$ and $Y_l = y_l z_{ml}$.

Fix $m' \neq 1, 2$. Then, $X_{i_{m'}} = x_{i_{m'}} z_{i_{m'} m'}$ and $x_{i_{m'}} \mid z_{i_{m'} m'}$. Since we assumed that $X_{i_{m'}} \neq 1$, then, $z_{i_{m'} m'} \neq 1$. So now, repeating the whole argument with m' and $i_{m'}$ instead of with 1 and 2 (recall that $i_{m'} \neq 1, 2, m'$), we would get similar permutations on $\{1, \ldots, n\} \setminus \{m', i_{m'}\}$. But, by property (ii), these permutations have to agree with the permutations $m \mapsto i_m$ and $m \mapsto l_m$ from above on the set $\{1, \ldots, n\} \setminus \{1, 2, m', i_{m'}\}$. Thus, $(1, 2)$ and $(m', i_{m'})$ will be transpositions in all of these permutations (and, in particular, $i_{m'} = l_{m'}$).

Since we can run the above argument with any $m' \neq 1, 2$, we have that the permutations $m \mapsto i_m$ and $m \mapsto l_m$ agree and are composed of transpositions (m, i_m). In particular, n is even and, after rearranging the numbers from $\{1, .., n\}$ so that $i_{2k} = 2k + 1$ and, thus, $i_{2k+1} = 2k$ for all $k = 1, .., n/2$, our matrix A looks as follows:

$$A = \begin{pmatrix} x_1 z_{12} & x_2 z_{21} & x_3 z_{34} & x_4 z_{43} & \cdots \\ y_1 z_{21} & y_2 z_{12} & y_3 z_{43} & y_4 z_{34} & \cdots \end{pmatrix}.$$

The rest of the statement of the Lemma follows from Lemma 4.2. □

Proof of Theorem: ($\Rightarrow$): It suffices to show that $z_{ij} = 1$ for all i, j. Without loss of generality, assume that $z_{12} \neq 1$.

By Lemma 4.2 we have that

$$X_{i_m} Y_{j_m} \mid X_2 Y_m^2 \frac{X_1}{z_{12}} \quad \text{and} \quad X_{k_m} Y_{l_m} \mid X_m^2 Y_1 \frac{Y_2}{z_{12}}$$

for all $m \geqslant 3$. But by (the proof of) Lemma 4.3 we know that the above hold when we substitute 1 and 2 with any m' and $i_{m'}$ such that $m \neq i_{m'}, m'$, i.e.

$$X_{i_m} Y_{j_m} \mid X_{i_{m'}} Y_m^2 \frac{X_{m'}}{z_{m' i_{m'}}} \quad \text{and} \quad X_{k_m} Y_{l_m} \mid X_m^2 Y_{m'} \frac{Y_{i_{m'}}}{z_{m' i_{m'}}}$$

Rewriting out the expressions in the form $X_{i_m} = x_{i_m} z_{i_m m}$ and $Y_{l_m} = y_{l_m} z_{m l_m}$ and then canceling repeating factors, we get that

$$x_{i_m} y_{j_m} z_{l_{j_m} j_m} \mid x_{i_{m'}} z_{i_{m'} m'} y_m^2 z_{i_m m} x_{m'} \quad \text{and} \quad x_{k_m} z_{k_m l_{k_m}} y_{l_m} \mid x_m^2 z_{m l_m} y_{m'} z_{i_{m'} m'} y_{i_{m'}}$$

for all $m' \neq m, i_m$. Therefore, we have that for every $m' \neq m, i_m$

$$z_{l_{j_m} j_m} \mid x_{i_{m'}} z_{i_{m'} m'} y_m^2 z_{i_m m} x_{m'} \quad \text{and} \quad z_{k_m l_{k_m}} \mid x_m^2 z_{m l_m} y_{m'} z_{i_{m'} m'} y_{i_{m'}}$$

Noting that $x_{i_{m'}} \mid z_{i_{m'} m'}$ and $x_{m'} \mid z_{m' l_{m'}}$ and, similarly for $y_{m'}, y_{i_{m'}}$, switching m' with $i_{m'}$, and using (ii), shows us that

$$z_{l_{j_m} j_m} \mid y_m^2 z_{i_m m} \quad \text{and} \quad z_{k_m l_{k_m}} \mid x_m^2 z_{m l_m}$$

Again, by property (ii), the only way for this to happen is if $j_m = k_m = m$. In that case, we have that

$$x_{i_m} y_m z_{l_m m} \mid x_{i_{m'}} z_{i_{m'} m'} y_m^2 z_{i_m m} x_{m'} \quad \text{and} \quad x_m z_{m l_m} y_{l_m} \mid x_m^2 z_{m l_m} y_{m'} z_{i_{m'} m'} y_{i_{m'}}$$

Again, by (i),(ii), and (iii), and by switching m' and $i_{m'}$, we have that

$$x_{i_m} y_m z_{i_m m} \mid y_m^2 z_{i_m m} \quad \text{and} \quad x_m z_{m l_m} y_{l_m} \mid x_m^2 z_{m l_m}$$

After cancelations,

$$x_{i_m} \mid y_m \quad \text{and} \quad y_{l_m} \mid x_m.$$

Thus, $x_{i_m} = y_{l_m} = 1$. Since i and l are permutations, we have that $x_m = y_m$ for all m. Thus, our matrix looks like this

$$A = \begin{pmatrix} z_{12} & z_{21} & z_{34} & z_{43} & \cdots \\ z_{21} & z_{12} & z_{43} & z_{34} & \cdots \end{pmatrix}.$$

But then, we have that, for example, $X_1 Y_2 - X_2 Y_1 = z_{12}^2 - z_{21}^2$, which is not prime! Contradiction! Thus, $z_{ij} = 1$ for all $i \neq j$.

Thus, $z_{12} = 1$ and by symmetry, $z_{ij} = 1$ for all $i \neq j$ and $\gcd(X_i, Y_j) = 1$ for all $i \neq j$. Combined with the assumptions of the theorem statement, we get that $\gcd(X_i, X_j) = \gcd(X_i, Y_i) = \gcd(Y_i, Y_j) = 1$ for all $i \neq j$, that is, all the entries of A are pairwise relatively prime.

($\Leftarrow$): Assume that the entries of A are pairwise relatively prime. Let $<$ be any monomial term order. To show that the 2×2 minors of A are a Gröbner basis with respect to $<$, we just need to show that all S-pairs reduce to zero. By the result of Bernstin-Sturmfels-Zelevinsky, such a reduction is guaranteed to exist for a generic matrix $X = (x_{ij})$. Since all entries of A are relatively prime, it is clear that such a reduction will extend simply by the ring map: $x_{ij} \mapsto X_{ij}$. □

3.5 Robustness of Higher Betti Numbers

Our interest in robust ideals originated with the following classical inequality:

$$\beta_i(S/I) \leq \beta_i(S/\operatorname{in}_< I) \quad \text{for all } i. \tag{3.18}$$

It is natural to ask for which ideals and term orders equality holds (for all i). In the setting of determinantal ideals, Conca et al proved in [CHT06] that the ideal of maximal minors of a generic matrix has some initial ideal with this property. They also gave examples of determinantal ideals for which no initial ideal has this property. In a different vein, Conca, Herzog and Hibi showed in [CHH04] that if the generic initial ideal $\mathrm{G}in(I)$ has $\beta_i(I) = \beta_i(\mathrm{G}in(I))$ for some $i > 0$, then $\beta_k(I)$ and $\beta_k(\mathrm{G}in(I))$ also agree for $k \geqslant i$.

Our interest was to instead approach the inequality 3.18 in a universal setting, i.e. to consider when equality holds for *all* term orders $<$. In this case we say that I has *robust Betti numbers*. The following result is due to the first author [Boo12]

Theorem 3.19. *If $I := I_k(X)$ is the ideal of maximal minors of a generic $k \times n$ matrix X and $<$ is any term order, then*

$$\beta_{ij}(S/\operatorname{in}_< I) = \beta_{ij}(S/I) \quad \text{for all } i, j. \tag{3.20}$$

In particular, every initial ideal is a Cohen-Macaulay, squarefree monomial ideal with a linear free resolution. Further, the resolution can be obtained from the Eagon-Northcott complex by taking appropriate lead terms of each syzygy.

A combination of Theorems 3.19 and 3.1 yields

Corollary 3.21. *Let I be a toric ideal generated in degree two. If I is robust, then I has robust Betti numbers.*

Our original hope with this project was that all robust toric ideals had robust Betti numbers. Unfortunately, the situation seems much more delicate.

Example 3.22. *Using Gfan [Jen], we were able to check that the Lawrence ideal J_L corresponding to the lattice L given by the matrix*

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 7 & 8 \end{pmatrix}$$

has initial ideals with different Betti numbers.

3.6 Acknowledgments

Many of the results in this paper were discovered via computations using Macaulay 2 [Sti] and Gfan [Jen]. We thank David Eisenbud and Bernd Sturmfels for many useful discussions. Finally we thank Seth Sullivant for introducing us to Lawrence ideals and starting us on the path toward Example 3.12.

Chapter 4

Closures of Linear Spaces

This chapter presents joint work with Federico Ardila.

4.1 Introduction

If $L \subset \mathbb{A}^n$ is a d-dimensional linear space over an infinite field k, then its closure in $\mathbb{P}^n$ is trivially a projective linear space. Its homogeneous ideal is generated by $n-d$ linear forms, and is perhaps one of the simplest projective varieties. However, this is only just one of many possible closures! In this paper we consider the next simplest case: the closure $\widetilde{L} \subset (\mathbb{P}^1)^n$ of a linear space $L \subset \mathbb{A}^n$ induced by the embedding $\mathbb{A}^n \hookrightarrow (\mathbb{P}^1)^n$. This case is already quite interesting; it exhibits several elegant algebraic properties, and many of its algebraic invariants can be determined directly from the matroid of L.

Closures of linear spaces

Let X be an affine variety in affine space $\mathbb{A}^n$. Choose a frame $F = \{\langle e_1\rangle, \ldots, \langle e_n\rangle\}$ where the e_i form a basis of $\mathbf{k}^n$ and $\langle\ \rangle$ denotes linear span. This allows us to identify $\mathbb{A}^n$ with $\mathbb{A}^1 \times \cdots \times \mathbb{A}^1$. The usual embedding of $\mathbb{A}^1$ into $\mathbb{P}^1$ by adding a single point at infinity then gives us an embedding $\mathbb{A}^n \to (\mathbb{P}^1)^n$. We will study the scheme-theoretic closure $\widetilde{X}$ of X in $(\mathbb{P}^1)^n$ induced by this embedding.

For the remainder of the paper, we fix a choice of coordinates, and let $S = k[x_1, \ldots, x_n]$. If $I \subset S$ is the ideal of polynomials vanishing at I, then the ideal $I(\widetilde{X}) \subset k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ of $\widetilde{X}$ is given by

$$I(\widetilde{X}) := (f^h \mid f \in I),$$

where f^h is the total homogenization of f, obtained by substituting x_i with x_i/y_i in f and clearing denominators. In general, it seems quite difficult to find a canonical presentation of the ideal $I(\widetilde{X})$, or to determine its algebraic invariants, such as the degree, number of generators, or multigraded Betti numbers. However, we show that when $X = L$ is a linear subspace (or an affine subspace), all of these questions have elegant answers in terms of the matroid of L.

The choice of a basis $\{e_i\}$ gives an embedding $\pi : \mathrm{Gr}(d, n) \to \mathbb{P}(\wedge^d \mathbf{k}^n)$ mapping a vector subspace L of $\mathbf{k}^n$ to its Plücker vector $\pi(L)$ in $\mathbb{P}(\wedge^d \mathbf{k}^n)$. Although the coordinates of $\pi(L)$ depend on the choice of basis, the set of coordinate hyperplanes containing $\pi(L)$ only depends on the frame F. This set can be identified with the *matroid* M of L: for a d-subset S of $[n]$, the hyperplane H_S contains $\pi(L)$ if and only if $[n] - S$ is a basis of M. More explicitly, if A is an $(n-d) \times n$ matrix whose rows generate the ideal I when regarded as linear forms, then M is the set of linearly independent $(n-d)$-subsets of columns of A.

Our main result is that the matroid M completely determines several important combinatorial invariants of the closure $\widetilde{L}$:

Theorem 4.1. *Let $L \subset \mathbb{A}^n$ be an affine linear space and let $\widetilde{L}$ be its closure via the embedding $\phi : \mathbb{A}^n \subset (\mathbb{P}^1)^n$. Let M be the matroid of L. The following invariants depend only on M: the $\mathbb{Z}^n$ multi-degree of $\widetilde{L}$, the number of minimal generators of the defining ideal $I(\widetilde{L})$, the graded Betti numbers of $I(\widetilde{L})$, and the set of initial ideals of $I(\widetilde{L})$ (with respect to all term orders).*

Theorem 4.2 provides a detailed description of all these invariants. This theorem also holds in the slightly more general context of linear spaces not containing the origin. For simplicity, we delay that treatment to Section 4.6.

Our results on closures of linear spaces

In this section we state our main theorems more precisely, and illustrate them with an example. We will briefly introduce the relevant definitions in Example 4.4, and discuss them more carefully in Section 4.2.

Having fixed coordinates, we can associate to L its linear matroid M. We give the ring $S = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ a $\mathbb{Z}^n$-grading given by

$$\deg x_i = \deg y_i = e_i$$

where e_i is the ith unit vector in $\mathbb{Z}^n$. We need $\deg x_i = \deg y_i$ for our ideal to be homogeneous. The following theorem shows that the structure of M determines several algebraic and combinatorial invariants of $I(\widetilde{L})$:

Theorem 4.2. *Let $L \subset \mathbb{A}^n$ be a d-dimensional linear space and let $\widetilde{L} \subset (\mathbb{P}^1)^n$ be the closure of L induced by the embedding $\mathbb{A}^n \subset (\mathbb{P}^1)^n$. Let M be the matroid of L; it has rank $r = n - d$. Then:*

(a) The homogenized cocircuits of $I(L)$ minimally generate the ideal $I(\widetilde{L})$.

(b) The homogenized cocircuits of $I(L)$ form a universal Gröbner basis for $I(\widetilde{L})$, which is reduced under any term order.

(c) The $\mathbb{Z}^n$-multi-degree of $\widetilde{L}$ is $\sum_B t_{b_1} \cdots t_{b_r}$ summing over all bases $B = \{b_1, \ldots, b_r\}$.

(d) There are at most $r! \cdot b$ distinct initial ideals of $I(\widetilde{L})$, where b is the number of bases.

(e) The primary decomposition of an initial ideal $I_< := \mathrm{in}_< I(\widetilde{L})$ is given by:

$$I_< = \bigcap_{B \text{ basis}} \langle\, x_e \,:\, e \in IA_<(B)\,,\, y_e \,:\, e \in IP_<(B) \rangle$$

where $B = IA_<(B) \sqcup IP_<(B)$ is the partition of B into internally active and passive elements with respect to $<$.

Theorem 4.3. *The non-zero multigraded Betti numbers of $S/I(\widetilde{L})$ are precisely:*

$$\beta_{i,\vec{a}}(S/I(\widetilde{L})) = |\mu(F, \widehat{1})|$$

for each flat F of M, where $i = r - r(F)$, and $\vec{a} = e_{[n]-F}$. Here μ is the Möbius function of the lattice of flats of M.

To illustrate, we include the following running example:

Example 4.4. *To the linear ideal*

$$I = \langle x_1 + x_2 + x_6,\ x_2 - x_3 + x_5,\ x_3 + x_4 \rangle\,.$$

given by $r = 3$ independent equations in $n = 6$ variables, and the corresponding linear subspace $L \subset \mathbf{k}^6$ of dimension $d = n - r = 3$, we associate the $r \times n$ matrix whose rows correspond to our 3 equations:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

We regard the columns of A as a point configuration in $\mathbb{P}^{r-1} = \mathbb{P}^2$, respectively, as shown in Figure 4.1. The affine dependence relations among the points correspond to the linear dependence relations among the columns of the matrix. A different generating set for I would give a different point configuration with the same affine dependence relations.

It is known [Stu96, Prop. 1.6] that the minimal universal Gröbner basis of I is given by the cocircuits *of I: the linear forms in L using an inclusion-minimal set of variables.*

$$I = \langle x_1 + x_2 + x_6,\ x_1 + x_3 - x_5 + x_6,\ x_1 - x_4 - x_5 + x_6,\ x_2 - x_3 + x_5,\ x_2 + x_4 + x_5,\ x_3 + x_4 \rangle$$

We identify the cocircuits with their support sets $126, 1356, 1456, 235, 245,$ *and* 34*. They are the complements of the* hyperplanes $345, 24, 23, 146, 136,$ *and* 1256 *spanned by A. Theorem*

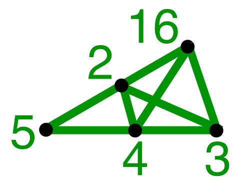


Figure 4.1: A point configuration $A \subset \mathbb{P}^2$ corresponding to the linear ideal I.

4.2(a,b) claims that the homogenized cocircuits minimally generate $\widetilde{I}$, and give a universal Gröbner basis:

$$\widetilde{I} = \langle x_1y_2y_6 + y_1x_2y_6 + y_1y_2x_6,\ x_1y_3y_5y_6 + y_1x_3y_5y_6 - y_1y_3x_5y_6 + y_1y_3y_5x_6, \ldots, x_3y_4 + y_3x_4\rangle$$

The bases *of A are the maximal independent sets of A; they correspond to the non-zero maximal minors of A, and hence to the non-zero Plücker coordinates of L. The* 13 *bases of A are*

$$\mathcal{B} = \{123, 124, 134, 135, 145, 234, 235, 236, 245, 246, 346, 356, 456\}.$$

Theorem 4.2(c) states that the multidegree of $\widetilde{L}$ is

$$\text{mdeg}\ \widetilde{L} = t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + \cdots + t_4t_5t_6.$$

Theorem 4.2(d) says that $I(\widetilde{L})$ has at most $(6-3)! \cdot 13 = 78$ initial ideals. Using the software Gfan [Jen] one can check that it actually has 72 *initial ideals.*

Theorem 4.2(e) tells us the primary decomposition of the initial ideal $I_< = in_< I(\widetilde{L})$ with respect to any linear order $<$. For the natural order $1 < 2 < 3 < 4 < 5 < 6$ we get

$$\begin{aligned} I^c(M,<) \;=\; & \langle x_1,x_2,x_3\rangle \cap \langle x_1,x_2,y_4\rangle \cap \langle x_1,y_3,y_4\rangle \cap \langle x_1,x_3,y_5\rangle \cap \langle y_1,y_4,y_5\rangle \cap \\ & \langle y_2,y_3,y_4\rangle \cap \langle y_2,x_3,y_5\rangle \cap \langle x_2,x_3,y_6\rangle \cap \langle y_2,y_4,y_5\rangle \cap \langle x_2,y_4,y_6\rangle \cap \\ & \langle y_3,y_4,y_6\rangle \cap \langle x_3,y_5,y_6\rangle \cap \langle y_4,y_5,y_6\rangle . \end{aligned}$$

We have a primary component $\langle z_b : b \in B\rangle$ for each basis B, where z_b equals x_b or y_b depending on whether b is internally active *or* passive *in B. For each $b \in B$ consider the cocircuit $D(B,b)$, which consists of the points not on the hyperplane spanned by $B - b$. If b is the smallest element of $D(B,b)$ then b is said to be* active *in B, and $z_b = x_b$. Otherwise, b is* passive *in B and $z_b = y_b$.*

For example, the basis 235 *contributes the primary component $\langle y_2, x_3, y_5\rangle$ because* 2 *is internally passive (*2 *is not the smallest element in $D(235,2) = 126$), 3 is internally active (3 is smallest in $D(235,3) = 34$), and 5 is internally passive (5 is not smallest in $D(235,5) =$ 1456).*

Theorem 4.3 is best understood pictorially. The flats *of M are the affine subspaces spanned by the points in A. They are partially ordered by inclusion. Recursively define* *the numbers $\mu(F, \widehat{1})$ by $\mu(\widehat{1}, \widehat{1}) = 1$ and $\mu(F, \widehat{1}) = -\sum_{G>F} \mu(G, \widehat{1})$ for $G \neq \widehat{1}$, where $\widehat{1}$ is the maximal flat. These numbers are shown circled in Figure 4.2, and they give the non-zero multigraded Betti numbers of S/I:*

$$\beta_{0,\emptyset} = 1$$
$$\beta_{1,34} = \beta_{1,245} = \beta_{1,235} = \beta_{1,1456} = \beta_{1,1356} = \beta_{1,126} = 1$$
$$\beta_{2,2345} = \beta_{2,13456} = \beta_{2,12456} = \beta_{2,12356} = 2, \qquad \beta_{2,12346} = 1$$
$$\beta_{3,123456} = 4$$

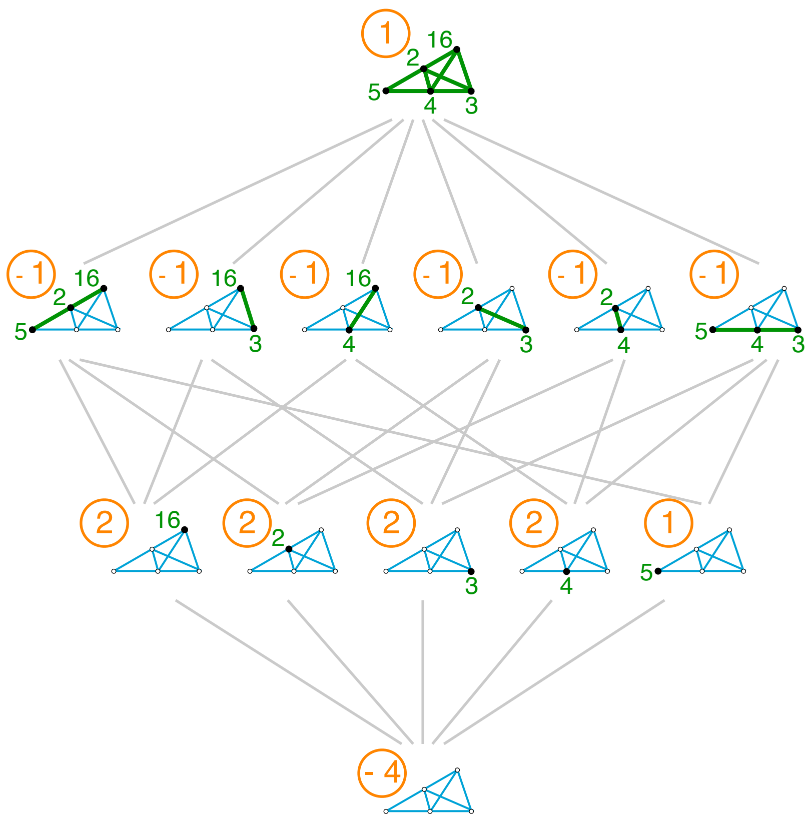


Figure 4.2: The Möbius function $\mu(F, \widehat{1})$ of the lattice of flats M encodes the non-zero multigraded Betti numbers of $I(\widetilde{L})$

From this we can read off the standard $\mathbb{Z}$-graded Betti table of S/I

I	1	6	9	4
	1	–	–	–
	–	1	–	–
	–	3	2	–
	–	2	7	4

The equality for $i = 1$ in Theorem 4.3 is implied by the fact that $I(\widetilde{L})$ is robust. To see this, notice that the flats of rank $r-1$ are the hyperplanes, which correspond to the complements of the cocircuits.

Corollary 4.5. *If L is a linear space the ideal $I(\widetilde{L})$ and all of its initial ideals are Cohen-Macaulay.*

Our results on matroids

Our analysis of the closure $\widetilde{L}$ of a linear space $L \subset \mathbb{A}^n$ in $(\mathbb{P}^1)^n$ gives rise to some constructions and results in matroid theory of independent interest.

Fix an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ and let $\Delta = \mathrm{conv}\{e_1, \ldots, e_n\}$ be the standard simplex in $\mathbb{R}^n$. For each subset $S \subseteq [n]$ consider the indicator vector $e_S = \sum_{s \in S} e_s$ and the face $\Delta_S = \mathrm{conv}\{e_s : s \in S\}$ of Δ. For a matroid M on $[n]$ consider the polytope

$$O_M = \sum_{D \text{ cocircuit of } M} \Delta_D,$$

where the *Minkowski sum* of $P, Q \subset \mathbb{R}^n$ is $P + Q := \{p + q : p \in P, q \in Q\}$.

Theorem 4.6. *If a matroid M on $[n]$ has rank r, then the polytope O_M*

(a) is given by $\sum_{i=1}^n x_i = D$ where D is the number of cocircuits of M, and the inequalities $\sum_{i \in S} x_i \leq D(S)$ for $S \subseteq [n]$, where $D(S)$ is the number of cocircuits intersecting S,

(b) has dimension $n - c$ where c is the number of connected components of M,

(c) has the matroid polytope $P_M = \mathrm{conv}\{e_B : B \text{ basis}\}$ as a Minkowski summand,

(d) has at most $r! \cdot b$ vertices, where b is the number of bases.

We are also led to the study of an interesting simplicial complex. Let M be a matroid and let $<$ be a linear order on the ground set S. Consider the $2|E|$-element set $\{x_e, y_e : e \in E\}$, and identify subsets and monomials, and write

$$x_A y_B := \{x_a : a \in A\} \cup \{y_b : b \in B\}.$$

Theorem 4.7. *There is a simplicial complex $B_<(M)$ on $\{x_e, y_e : e \in E\}$ such that*

1. *The facets of $B_<(M)$ are the sets $x_{B \cup EP(B)} y_{B \cup EA(B)}$ for each basis B.*
2. *The minimal non-faces are $x_{\min C} y_{C - \min C}$ for each circuit C.*

Related results

Closures of linear spaces are closely related to reciprocal varieties. A reciprocal variety may be thought of as a different homogenization obtained by $x_i \mapsto 1/x_i$. In [PS06] Proudfoot and Speyer proved that reciprocal varieties are generated minimally by universal Gröbner bases and that their degree can be computed as a Tutte polynomial. We originally became interested in closures of linear spaces because of this universal Gröbner basis property and a well-known result for toric ideals. If X is any affine toric variety, then its closure $\widetilde{X}$ in $(\mathbb{P}^1)^n$ is minimally generated by a universal Gröbner basis. The variety $\widetilde{X}$ is called the Lawrence lifting of X (see [Stu96]).

Originally we hoped that such a result might hold more generally for affine varieties, but the following example illustrates that it does not, and the situation is quite subtle. This example also illustrates that, contrary to the case of closures in $\mathbb{P}^n$, there is no simple numerical relationship between the number of generators of $I(X)$ and $I(\widetilde{X})$, even in terms of Gröbner bases.

Example 4.8. *Let* $I = (x_1 + x_2 + x_3, x_1 + x_3 + x_4, x_1^2 + x_2^2 + x_1x_4)$

	I	$I(\widetilde{X})$
number of generators	3	12
size of a reduced Gröbner basis	3	14 *or* 15
size of a universal Gröbner basis	8	21

Ideals minimally generated by universal Gröbner bases, called *robust ideals* in [BR13], are by no means a common occurrence. Even in the toric case, this condition is very strong, yet a complete classification is unknown. Nonetheless, robust ideals have cropped up in many classical situations, see ([Boo12, BR13, CNG13, PS06, SZ93])

4.2 Preliminaries from matroid theory

The toolkit of matroid theory is ideally suited to study the geometric and algebraic invariants in this project. Matroid theory can be approached from many equivalent points of view. This can make the theory confusing at first; different papers often use very different definitions of a matroid. However, in the long run, the existence of these "cryptomorphisms" is an extremely powerful feature of the theory. This project illustrates this point very well; many different matroid theoretic concepts appear naturally, as Example 4.4 clearly shows. In this section we introduce these concepts in more detail; they will play a fundamental role in what follows.

One definition of a matroid.

Definition 3. *A* matroid $M = (E, \mathcal{I})$ *consists of a ground set* E *and a family* $\mathcal{I}$ *of sets of* E*, called the* independent sets *of* M*, which satisfy the following axioms:*

(I1) The empty set is independent.
(I2) A subset of an independent set is independent.
(I3) If X and Y are independent and $|X| < |Y|$, then there exists $y \in Y - X$ such that $X \cup y$ is independent.

Matroid theory can be thought of as a combinatorial theory of independence. The prototypical example is the family of *linear* or *realizable matroids*, which arise from linear independence. If E is a set of vectors in a vector space V, then the linearly independent subsets of E form a matroid.

In a matroid M, a *circuit* is a minimal dependent set. A *basis* is a maximal independent set. All bases of M have the same size, which is called the *rank* $r(M)$ of M. Similarly, all maximal independent subsets of any set $S \subseteq E$ have the same size, which is called the rank $r(S)$.

In our running Example 4.4, the bases and circuits are:

$$\begin{aligned}\mathcal{B} &= \{123, 124, 134, 135, 145, 234, 235, 236, 245, 246, 346, 356, 456\},\\ \mathcal{C} &= \{16, 125, 256, 345, 1234, 2346\}.\end{aligned}$$

The independent sets are the subsets of the bases.

Duality

If $\mathcal{B}$ is the collection of bases of a matroid M, then $\mathcal{B}^* := \{E - B \,:\, B \in \mathcal{B}\}$ is also the collection of bases of a matroid, called the *dual matroid* M^*. If M is the matroid of a set A of n vectors which generate $\mathbf{k}^d$, then one can find a set of n vectors which generate $\mathbf{k}^{n-d}$ whose matroid is M^*. Figure 4.3 shows a point configuration dual to the one of Example 4.4. The reader may check that the bases of A^* are precisely the complements of the bases of A.

A *circuit* of M^* is called a *cocircuit* of M. It can also be characterized as a minimal set D whose removal decreases the rank of M; that is, $r(E - D) < r$. The cocircuits in Example 4.4 are

$$\nabla = \{34, 126, 235, 245, 1356, 1456\}.$$

In A^* they are the minimally dependent sets. In A they are the complements of the hyperplanes spanned by A.

The following technical lemma will be very useful to us.

Lemma 4.9. *[Oxl92] If C is a circuit and D is a cocircuit of M, then $|C \cap D| \neq 1$.*

If M is a matroid on E and $A \subset E$ then there are matroids $M \backslash A = M_{E-A}$ and M/A on $E - A$, called the *deletion* and *contraction* of A in M, whose independent sets are

$$\begin{aligned}\mathcal{I}(M \backslash A) &= \{I \in \mathcal{I}(M) \,:\, I \subseteq E - A\}\\ \mathcal{I}(M/A) &= \{I - B_A \,:\, I \in \mathcal{I}(M), B_A \subseteq I\}\end{aligned}$$

Figure 4.3: The point configuration $A \subset \mathbb{P}^{r-1} = \mathbb{P}^2$ and a dual configuration $A^* \subset \mathbb{P}^{n-r-1} = \mathbb{P}^2$.

where B_A is a basis of A. Deletion and contraction are dual operations:

$$(M/A)^* = M^* \backslash A.$$

If M comes from a set S of vectors in a vector space V, then $M \backslash A$ corresponds to deleting the vectors in A, while M/A corresponds to the images of the vectors in $V/\text{span}(A)$.

The matroid of a linear ideal.

Fix a choice of a standard basis for $\mathbf{k}^n$. Let L be an r-dimensional subspace of $\mathbf{k}^n$ and let $I(L) \subset \mathbf{k}[x_1, \ldots, x_n]$ be its defining linear ideal. There is one particularly useful generating set for $I(L)$, which we now describe. For each linear form f in $I(L)$ consider its *support* $\text{supp}(f) \subseteq [n]$ consisting of those i such that x_i has a nonzero coefficient in f. Among these, consider the set ∇ of inclusion-minimal supports; these are called the *cocircuits* of $I(L)$. They are the cocircuits of a matroid $M(L)$, called the *matroid of* L.[1] Notice that for each cocircuit D there is a unique linear form f (up to scalar multiplication) in $I(L)$ with $\text{supp}(f) = D$, so there is no ambiguity in calling this form f a cocircuit as well.

Proposition 4.10. *[Stu96, Prop. 1.6] The cocircuits of the linear ideal $I(L)$ form a universal Gröbner basis for $I(L)$.*

Linear matroids are precisely the matroids of linear ideals. As we explained in Example 4.4, if B is a matrix whose rows generate $I(L)$, one may easily check that the linear matroid on the columns of B equals the matroid of L.

Matroid duality can then be seen as a generalization of duality of subspaces. Our chosen basis for $\mathbf{k}^n$ determines a dual basis for the dual vector space $(\mathbf{k}^n)^*$. If $L^\perp \subset (\mathbf{k}^n)^*$ is the orthogonal complement of our vector space L, then the matroid of $L^\perp$ is dual to the matroid of L.

Basis activities

Proposition 4.11. *[Cra69] Given a basis B and an element $x \notin B$, there is a unique circuit $C = C(B, x)$ contained in $B \cup x$. It is called the* fundamental circuit *of B and x, and is*

[1] Some authors define the *matroid of* L to be the dual matroid $M(L)^*$.

given by:

$$C(B,x) = \{y \in E \,:\, B - x \cup y \text{ is a basis}\}$$

Notice that $x \in C(B,x)$.

Given a basis B and an element $y \in B$, there is a unique cocircuit $D = D(B,y)$ contained in $E - B \cup y$. It is called the fundamental cocircuit *of B and y, and is given by:*

$$D(B,y) = \{x \in E \,:\, B - x \cup y \text{ is a basis}\}$$

Notice that $y \in D(B,y)$.

Definition 4. *Consider a matroid M and a linear order $<$ on its ground set. Let B be a basis of M. We say that an element $e \notin B$ is* externally active *if it is the smallest element in $C(B,e)$, and it is* externally passive *otherwise. Let $EA_<(B)$ and $EP_<(B)$ be the sets of externally active and externally passive elements with respect to B and $<$.*[2]

Similarly, we say that an element $i \in B$ is internally active *if it is the smallest element in $D(B,i)$, and it is* internally passive *otherwise. We write $IA_<(B)$ and $IP_<(B)$ for the sets of internally active and internally passive elements with respect to B and $<$.*

We will need the following beautiful result by Crapo:

Theorem 4.12. *[Cra69] Let M be a matroid on the ground set S and let $<$ be a linear order on S. Every subset A of S can be uniquely written in the form $A = B \cup E - I$ for some basis B, some subset $E \subseteq EA(B)$, and some subset $I \subseteq IA(B)$. Equivalently, the intervals $[B - IA(B), B \cup EA(B)]$ form a partition of the poset 2^S of subsets of S ordered by inclusion.*

Lattice of flats and Möbius function

A *flat* F of a matroid M is a subset which is maximal for its rank; that is, a set such that $r(F \cup f) = r(F) + 1$ for all $f \notin F$. The flats of rank $r - 1$ are called *hyperplanes*. In the case that interests us, when M is the linear matroid of a set of vectors $E \subset \mathbf{k}^n$, the flats of M correspond to the subspaces spanned by subsets of E. The *lattice of flats* L_M is the poset of flats ordered by containment; it is in fact a lattice, graded by rank. The flats in Example 4.4 are

$$L_M = \{\emptyset, 16, 2, 3, 4, 5, 1256, 136, 146, 23, 24, 345, 123456\},$$

as illustrated in Figure 4.2.

The *Möbius function* of L_M is the map $\mu : Int(L_M) \to \mathbb{Z}$ from the intervals of L_M to $\mathbb{Z}$ characterized by $\mu(x,x) = 1$ for all $x \in L_M$ and $\sum_{x \leq z \leq y} \mu(z,y) = 0$ for all $x < y$.[3] The *Möbius number* of M is $\mu(M) = \mu(\widehat{0}, \widehat{1})$, where $\widehat{0}$ and $\widehat{1}$ are the minimum and maximum elements of L_M. Figure 4.2 shows the value of $\mu(F, \widehat{1})$ next to each flat F of M.

[2] When the choice of the order $<$ is clear, we will omit the subscript and write simply $EA(B)$ and $EP(B)$.

[3] It is more common to demand that $\sum_{x \leq z \leq y} \mu(x,z) = 0$ for all $x < y$, but it may be shown that these two conditions are equivalent.

Independence complexes and cocircuit ideals

To a matroid M on the ground set E one associates a simplicial complex

$$IN(M) = \{I \subseteq E : I \text{ is independent in } M\}$$

called the *independence complex* of M. For us, the independence complex of M^* is more relevant.

Theorem 4.13. *[Bjö92, Theorem 7.8.1] If M is a matroid of rank r on $[n]$, then*

$$H_i(IN(M^*); \mathbb{Z}) = \begin{cases} \mathbb{Z}^{|\mu(M)|}, & \text{if } i = n - r - 1 \text{ and } M \text{ has no loops} \\ 0, & \text{otherwise.} \end{cases}.$$

Recall that the *Stanley-Reisner ideal* of a simplicial complex Δ on a set $\{x_1, \ldots, x_n\}$ is the ideal $I_\Delta = \langle x_{i_1} x_{i_2} \cdots x_{i_k} : \{i_1, \ldots, i_k\} \text{ is not a face of } \Delta\rangle$ of $\mathbf{k}[x_1, \ldots, x_n]$. The *Stanley-Reisner ring* is $\mathbf{k}[x_1, \ldots, x_n]/I_\Delta$. Since the minimal non-faces of $IN(M^*)$ are the circuits of M^*, which are the cocircuits of M, the Stanley–Reisner ideal of $IN(M^*)$ is the *cocircuit ideal*

$$I_{IN(M^*)} = \left\langle \prod_{c \in C} x_c \, : C \text{ is a cocircuit of } M \right\rangle.$$

It is known [MS05, Theorem 1.7] that the components of the primary decomposition of a squarefree monomial ideal I_Δ are in bijection with the facets of Δ; each facet F corresponds to the primary component $\langle x_f : f \notin F\rangle$. Since the facets of $IN(M^*)$ are the bases of M^*, we get that the primary decomposition of $I_{IN(M^*)}$ is

$$I_{IN(M^*)} = \bigcap_{B \text{ basis}} \langle x_b : b \in B\rangle.$$

In our running Example 4.4 we have

$$\begin{aligned} I_{IN(M^*)} &= \langle x_1x_2x_6,\ x_2x_3x_5,\ x_2x_4x_5,\ x_3x_4,\ x_1x_3x_5x_6,\ x_1x_4x_5x_6\rangle \\ &= \langle x_1, x_2, x_3\rangle \cap \langle x_1, x_2, x_4\rangle \cap \langle x_1, x_3, x_4\rangle \cap \langle x_1, x_3, x_5\rangle \cap \langle x_1, x_4, x_5\rangle \cap \\ &\quad \langle x_2, x_3, x_4\rangle \cap \langle x_2, x_3, x_5\rangle \cap \langle x_2, x_3, x_6\rangle \cap \langle x_2, x_4, x_5\rangle \cap \langle x_2, x_4, x_6\rangle \cap \\ &\quad \langle x_3, x_4, x_6\rangle \cap \langle x_3, x_5, x_6\rangle \cap \langle x_4, x_5, x_6\rangle \end{aligned}$$

Now we recall Hochster's formula for the Betti numbers of a squarefree monomial ideal:

Theorem 4.14. *[MS05, Corollary 5.12] The nonzero Betti numbers of the Stanley–Reisner ring I_Δ lie only in squarefree degrees σ, and*

$$\beta_{i-1,\sigma}(I_\Delta) = \dim_{\mathbf{k}} \widetilde{H}^{|\sigma|-i-1}(\Delta|_\sigma)$$

Let us apply this formula to $\Delta = IN(M^*)$ and $\sigma = E - A$ for a subset $A \subset E$. We have that $\Delta|_{E-A} = IN(M^*|(E-A)) = IN(M^* \backslash A)$. Notice that $(M^* \backslash A)^* = M/A$ has no loops if and only if A is a flat of M. Also $r(M/A) = r - r(A)$ and $\mu(M/A) = \mu(A, \widehat{1})$. Combining these observations with Theorem 4.13, we obtain the following result.

Theorem 4.15. *The only nonzero Betti numbers of the cocircuit ideal $I_{IN(M^*)}$ of M are*

$$\beta_{r-r(A), e_{E-A}}(I_{IN(M^*)}) = |\mu(A, \widehat{1})|$$

for the flats A of M.

4.3 The polytope.

Let $\Delta = \operatorname{conv}\{e_1, \ldots, e_n\}$ be the standard simplex in $\mathbb{R}^n$, and for each $I \subseteq [n]$ let

$$\Delta_I = \operatorname{conv}\{e_i \,:\, i \in I\}.$$

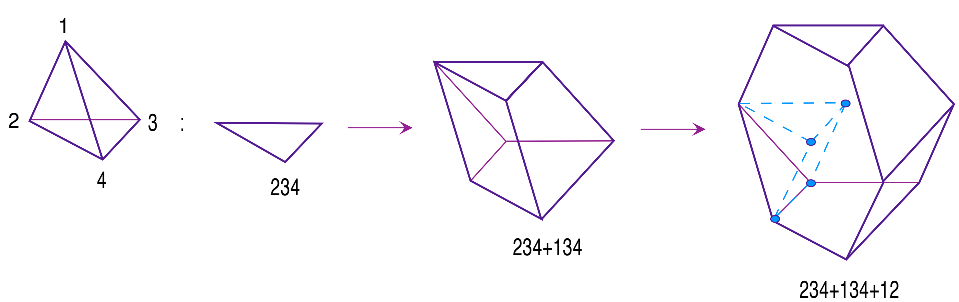
For a matroid M on the ground set $[n]$, consider the polytope defined by the Minkowski sum

$$O_M = \sum_{D \text{ cocircuit of } M} \Delta_D,$$

where the *Minkowski sum* of $P, Q \subset \mathbb{R}^n$ is $P + Q := \{p + q \,:\, p \in P, q \in Q\}$.

We will see that these polytopes O_M are related to *matroid polytopes*, which are much better known and understood; see, for example [Edm03, GGMS87]. The vertices of the matroid polytope P_M of M are the vectors $e_B = e_{b_1} + \cdots e_{b_r}$ for each basis $B = \{b_1, \ldots, b_r\}$ of M. The *connected components* of M are the equivalence classes for the equivalence relation where $a \sim b$ if $a, b \in C$ for some circuit C. It is known that $\dim P_M = n - c$ where c is the number of connected components of M.

Figure 4.3 shows the polytope $O_M = \Delta_{12} + \Delta_{134} + \Delta_{234}$ for the matroid M with bases 12, 13, 14, 23, and 24. For comparison, the matroid polytope P_M is shown inside O_M in dotted lines.



Theorem 4.6. *If a matroid M on $[n]$ has rank r, then the polytope O_M*

(a) is given by $\sum_{i=1}^{n} x_i = D$ where D is the number of cocircuits of M, and the inequalities $\sum_{i \in S} x_i \leq D(S)$ for $S \subseteq [n]$, where $D(S)$ is the number of cocircuits intersecting S,

(b) has dimension $n - c$ where c is the number of connected components of M,

(c) has the matroid polytope $P_M = \operatorname{conv}\{e_B : B \text{ basis}\}$ as a Minkowski summand,

(d) has at most $r! \cdot b$ vertices, where b is the number of bases.

Before we prove this theorem, it is useful to recall some basic facts about generalized permutahedra [PRW08]. The *permutahedron* Π_n is the convex hull of the $n!$ permutations of $\{1, \ldots, n\}$ in $\mathbb{R}^n$; its normal fan is the braid arrangement formed by the hyperplanes $x_i = x_j$ for $i \neq j$. A *generalized permutahedron* is a polytope P obtained from Π_n by moving the vertices (possibly identifying some of them) while preserving the edge directions. This is equivalent to requiring that the normal fan of P is a coarsening of the braid arrangement.

Every permutahedron is of the form

$$P_n(\{z_I\}) = \{(t_1, \ldots, t_n) \in \mathbb{R}^n : \sum_{i=1}^{n} t_i = z_{[n]}, \sum_{i \in I} t_i \leq z_I \text{ for all } I \subseteq [n]\}$$

where z_I is a real number for each $I \subseteq [n]$, and $z_\emptyset = 0$. The vector $(z_I)_{I \subseteq [n]}$ is *submodular*; that is, $z_I + z_J \geq z_{I \cup J} + z_{I \cap J}$ for all subsets I and J of $[n]$. Furthermore, this is a bijection between generalized permutahedra and points in the submodular cone in $\mathbb{R}^{2^n}$ defined by the submodular inequalities. [AA11, MPS$^+$09, PRW08, Sch03]. This shows that generalized permutahedra are essentially the same as *polymatroids*, which predate them.

There is an alternative description of generalized permutahedra. Every Minkowski sum of simplices of the form Δ_I is a generalized permutahedron [PRW08] and, conversely, every generalized permutahedron can be expressed as a *signed* sum of such simplices.[ABD10] This automatically implies that O_M is a generalized permutahedron. Also, P_M is the generalized permutahedron $P_n(r(I))_{I \subset [n]}$ where $r(I)$ is the rank of I in the matroid. [ABD10, Sch03]

Proof of Theorem 4.6. For a polytope $P \in \mathbb{R}^n$ and a linear functional $w \in (\mathbb{R}^n)^*$, let $(P)_w$ be the face of P minimized by w.

(a) Since O_M is a generalized permutahedron, we have $O_M = P_n(z_I)$ for some vector z_I. Since $\Delta_D = P_n(z_I^D)$ where z_I^D is 1 if $I \cap D \neq \emptyset$ and 0 otherwise, the result follows from the fact that $P_n(z_I) + P_n(z'_I) = P_n(z_I + z'_I)$.

(b) From the Minkowski sum expression for O_M it is clear that the edge directions of O_M are precisely the edge directions of the various Δ_D. These are the vectors of the form $e_c - e_d$ where c and d are in the same cocircuit; that is, in the same connected component of M^*. Their span is the subspace given by the equations $\sum_{i \in K_a} x_i = 0$ for the connected components K_a of M^*, which are also the connected components of M. The result follows.

(c) When M is the matroid of a linear ideal I, this claim is related to (but not implied by) Proposition 4.10 and the fact that the matroid polytope is a state polytope of I. [Stu96, Proposition 2.11] We proceed as follows.

We know that $O_M = P_n(D(I))_{I\subset[n]}$ and $P_M = P_n(r(I))_{I\subset[n]}$, where r is the rank function of M. We claim that $q(I) = D(I) - r(I)$ is a submodular function; it will then follow that $Q = P_n(q(I))_{I\subset[n]}$ is a generalized permutahedron such that $O_M = P_M + Q$.

Let $\delta_q(S,a,b) = -q(S\cup a\cup b) + q(S\cup a) + q(S\cup b) - q(S)$ for $S \subset [n]$ and $a, b \in [n] - S$; define δ_D and δ_r analogously. We will prove that δ_q is always non-negative; this property of "local submodularity" of $q(I)$ implies its submodularity.

Assume contrariwise that $\delta_q(S,a,b) < 0$. Notice that δ_D is non-negative because D is submodular, and δ_r equals 0 or 1 because $r(S\cup s) - r(S) = 0$ or 1 for $s \notin S$. Therefore, to have $\delta_q(S,a,b) = \delta_D(S,a,b) - \delta_r(S,a,b) < 0$, we must have

$$\delta_D(S,a,b) = 0, \qquad \delta_r(S,a,b) = 1. \tag{4.16}$$

To have $\delta_r(S,a,b) = 1$, we must have $r(S) = s$ and $r(S\cup a) = r(S\cup b) = r(S\cup a\cup b) = s+1$ for some s. One easily checks that $\delta_D(S,a,b) = 0$ is the number of cocircuits containing a and b and not intersecting S. Since hyperplanes are the complements of cocircuits, every hyperplane $H \supset S$ must contain either a or b. If a hyperplane $H \supset S$ contained a but not b, submodularity would imply $1 = r(H\cup b) - r(H) \le r(S\cup a\cup b) - r(S\cup a) = 0$. Therefore every hyperplane $H \supset S$ must contain both a and b, and every hyperplane of M/S contains both a and b. This is only possible if a and b are loops in M/S, which contradicts that $r(S\cup a) = r(S) + 1$.

(d) Since the normal fan of O_M coarsens the braid arrangement,

$$\{(O_M)_\pi \,:\, \pi \text{ is a permutation of } [n]\}$$

is a complete list of the vertices of O_M, possibly with repetitions. The π-minimal vertex is

$$(O_M)_\pi = \sum_{D \text{ cocircuit of } M} (\Delta_D)_\pi = \sum_{D \text{ cocircuit of } M} e_{\min_\pi(D)} = (d_1^\pi, \ldots, d_n^\pi)$$

where d_i^π is the number of cocircuits of M whose π-smallest element is i.

Next we observe that the support of any vertex $(O_M)_\pi$ of O_M is a basis of M; more specifically,

$$\operatorname{supp}(O_M)_\pi = B_\pi \tag{4.17}$$

where $B = B_\pi$ denotes the *π-minimal basis* of M, which minimizes $\sum_{b\in B}\pi(b)$. This basis is unique by the greedy algorithm for matroids. The claim (4.17) follows from a variant of the greedy algorithm for matroids due to Tarjan [Koz91, Theorem 2.7], called the *blue rule*. To construct the π-minimum basis B_π, one successively chooses a cocircuit with no blue elements, and colors its smallest element blue. One does this repeatedly, in any order, until it is no longer possible. In the end, the set of blue elements is the basis B_π. Clearly the blue elements are precisely those i such that $d_i^\pi \neq 0$.

Finally, it remains to observe that for each π, the vertex $(O_M)_\pi$ is determined uniquely by M and the relative order of $\pi(B_\pi)$. To see this, notice that d_i^π is the number of cocircuits D of M such that $\pi(i)$ is the smallest element of $\pi(B_\pi \cap D)$. This number only depends on the matroid M, the basis B_π, and the relative order of $\pi(B_\pi)$. Since there are b choices for B_π and $r!$ choices for the relative order of $\pi(B_\pi)$, the desired result follows. □

4.4 The simplicial complex and the primary decomposition.

Let M be a matroid and let $<$ be a linear order on the ground set E. We will build a simplicial complex on the $2|E|$-element set $\{x_e, y_e : e \in E\}$. We will identify subsets and monomials, and write

$$x_A y_B := \{x_a : a \in A\} \cup \{y_b : b \in B\}.$$

Theorem 4.18. *There is a simplicial complex $B_<(M)$ on $\{x_e, y_e : e \in E\}$ such that*

1. *The facets of $B_<(M)$ are the sets $x_{B\cup EP(B)} y_{B \cup EA(B)}$ for each basis B.*
2. *The minimal non-faces are $x_{\min C} y_{C - \min C}$ for each circuit C.*

Proof. We need to prove that, for $S, T \subseteq E$

$$x_S y_T \subseteq x_{B\cup EP(B)} y_{B\cup EA(B)} \text{ for some basis } B$$
$$\text{if and only if}$$
$$x_S y_T \not\supseteq x_{\min C} y_{C-\min C} \text{ for all circuits } C.$$

First we prove the forward direction. Assume, contrariwise, that $x_S y_T \subseteq x_{B\cup EP(B)} y_{B\cup EA(B)}$ for some basis B and $x_S y_T \supseteq x_{\min C} \cup y_{C-\min C}$ for some circuit C. Then

$$x_{\min C} y_{C-\min C} \subseteq x_{B\cup EP(B)} y_{B\cup EA(B)}.$$

Let $\min C = c$. Since $c \in B \cup EP(B)$, there are two cases:

If $c \in B$: Let $D = D(B, c)$ be the fundamental cocircuit. Then $c \in C \cap D$ and, since $|C \cap D| \neq 1$, we can find another element $d \in C \cap D$. Since $d \in D(B, c)$, we have $c \in C(B, d)$; and $c < d$, so d is not externally active in B. Also, $d \in D(B, c)$ implies that $d \notin B$. Therefore $d \notin B \cup EA(B)$. This contradicts that $C - \min C \subseteq B \cup EA(B)$.

If $c \in EP(B)$: We can find an element $d \in C(B, c)$ with $d < c$. Now $d \in C(B, c)$ implies $c \in D(B, d) =: D$, so $c \in C \cap D$. Again, this means we can find another $e \in C \cap D$. Since $e \in C$ and $c = \min C$, we have $c < e$, and therefore $d < e$. Now, $e \in D$ implies that $e \notin B$. Also $e \in D(B, d)$ implies that $d \in C(B, e)$; and $d < e$ then implies that $e \notin EA(B)$. Therefore $e \notin B \cup EA(B)$. Again, this contradicts that $C - \min C \subseteq B \cup EA(B)$. This completes the proof of the forward direction.

To prove the backward direction, assume that $x_S y_T \not\supseteq x_{B\cup EP(B)} y_{B\cup EA(B)}$ for all bases B. We need to show that $x_S y_T \supseteq x_{\min C} y_{C-\min C}$ for some circuit C.

By 4.12 we can write $T = B \cup E - I$ for some basis B, some subset $E \subseteq EA(B)$, and some subset $I \subseteq IA(B)$. Then $T \subseteq B \cup EA(B)$, so $S \not\subseteq B \cup EP(B)$. Therefore we can find $s \in S$ with $s \notin B \cup EP(B)$; that is, $s \in EA(B)$.

Let $C = C(B, s)$. We claim that $x_S y_T \supseteq x_{\min C} y_{C-\min C}$. Since $s \in EA(B)$, $s = \min C$, so $S \supseteq \min C$. It remains to show that $T \supseteq C - \min C$. Assume, contrariwise, that $d \in C - \min C$ but $d \notin T$. Since $d \in C - \min C$, $d \in B$. Since $d \notin T = B \cup E - I$, this implies that $d \in I$, so d is internally active in B. Therefore d is the smallest element in $D(B, d)$. But $d \in C(B, s)$ implies that $s \in D(B, d)$, which implies that $s > d$. This contradicts that $s = \min C$. The desired result follows. □

Theorem 4.19. *The primary decomposition of the ideal*

$$C(M,<) = \left\langle x_{c_1} y_{c_2} y_{c_3} \cdots y_{c_k} \,:\, C = \{c_1, \ldots, c_k\} \text{ is a circuit of } M \text{ and } c_1 = \min_{<} C \right\rangle$$

is

$$C(M,<) = \bigcap_{B \text{ basis of } M} \langle\, x_e \,:\, e \in EA_<(B)\,,\, y_e \,:\, e \in EP_<(B) \rangle$$

Proof. By Theorem 4.18.1, $C(M,<)$ is the Stanley-Reisner ideal of the simplicial complex $B_<(M)$ of Theorem 4.18. Theorem 4.18.2 then implies the desired primary ideal decomposition. □

This simplicial complex which is closely related to two important simplicial complexes from matroid theory. If we set $y_i = x_i$, the above ideal is the Stanley-Reisner ideal of the independence complex of M, whose facets are the bases of M. If we set $x_i = 1$, we get the Stanley-Reisner ideal of the broken circuit complex of M, whose facets are the "nbc-bases" of M.

4.5 Proofs of our Main Theorems

Having built up the necessary combinatorial background, we now use algebraic and geometric tools to complete the proof of our main theorems.

One of our goals is to show that the set $G = \{f_C^h\}$ of homogenized circuits is a *universal Gröbner basis* (UGB); that is, a Gröbner basis for $\widetilde{I}$ with respect to any term order. One key tool is the following: If two ideals share the same codimension and degree and one contains the other, then under suitably nice conditions we can say they are equal.

Theorem 4.20. *Let $L \subset \mathbb{A}^n$ be a d-dimensional linear space and let $\widetilde{L} \subset (\mathbb{P}^1)^n$ be the closure of L induced by the embedding $\mathbb{A}^n \subset (\mathbb{P}^1)^n$. Let M be the matroid of L; it has rank $r = n - d$. Then:*

(a) The homogenized cocircuits of $I(L)$ minimally generate the ideal $I(\widetilde{L})$.

(b) The homogenized cocircuits of $I(L)$ form a universal Gröbner basis for $I(\widetilde{L})$, which is reduced under any term order.

(c) The $\mathbb{Z}^n$-multi-degree of $\widetilde{L}$ is $\sum_B t_{b_1} \cdots t_{b_r}$ summing over all bases $B = \{b_1, \ldots, b_r\}$.

(d) There are at most $r! \cdot b$ distinct initial ideals of $I(\widetilde{L})$, where b is the number of bases.

(e) The primary decomposition of an initial ideal $I_< := \mathrm{in}_< I(\widetilde{L})$ is given by:

$$I_< = \bigcap_{B \text{ basis}} \langle\, x_e \,:\, e \in IA_<(B)\,,\, y_e \,:\, e \in IP_<(B) \rangle$$

where $B = IA_<(B) \sqcup IP_<(B)$ is the partition of B into internally active and passive elements with respect to $<$.

Proof of (c). We compute the multi-degree of $\widetilde{L}$ using a geometric argument. Let $\Delta \subset [n]$ be of cardinality $r = n - d$, where $d = \dim L$. Consider the linear subspace $Z_\Delta \subset (\mathbb{P}^1)^n = \prod_{i=1}^n (\mathbb{P}^1)_i$ (where we give subindices to the various $\mathbb{P}^1$s to distinguish them) given by

$$Z_\Delta = \prod_{i \in \Delta} (\mathbb{P}^1)_i \times \prod_{i \notin \Delta} q_i$$

where $q_i \in (\mathbb{P}^1)_i$ is a general point. If X is a subvariety of $(\mathbb{P}^1)^n$ of codimension r then denote by $m(Z_\Delta, X)$ the intersection multiplicity of X with Z_Δ. By the genericity of Z_Δ this will simply be the number of points in the intersection counted with multiplicity. Then the *multi-degree* of X is defined to be the sum

$$\mathrm{mdeg}\ X = \sum m(Z_\Delta, X) t_{\Delta_1} \cdots t_{\Delta_r},$$

where $\Delta = \{\Delta_1, \ldots, \Delta_r\}$ ranges over all subsets of $[n]$ of size r. We will prove that the intersection multiplicities are

$$m(Z_\Delta, \widetilde{L}) = \begin{cases} 1 & \text{if } \Delta \text{ is a basis of } M \text{, and} \\ 0 & \text{otherwise.} \end{cases} \tag{4.21}$$

Notice that the n lines in $(\mathbb{P}^1)^n$ have self-intersection zero, so the multidegree of $\widetilde{L}$ is a squarefree polynomial of degree r in $\mathbb{Z}[t_1, \ldots, t_n]$. Thus when we prove (4.21), our formula for mdeg $\widetilde{L}$ will follow.

So let Δ be a basis, and without loss of generality, say $\Delta = \{1, \ldots, r\}$. In the affine patch where no coordinate y_i equals zero, we are working in the original affine space, so it is clear that $Z_\Delta \cap \widetilde{L} = Z_\Delta \cap L$ is a single point. Since the points q_i in Z_Δ are general, we may suppose that they lie in the affine patch, so we can assume the coordinate $y_i \neq 0$ for

$i > r$. Now let $i \leq k$. Notice that since Δ is a basis, there is a cocircuit D containing i whose support is contained in $\{i\} \cup \{r+1, \ldots, n\}$; in fact, this is the fundamental cocircuit $D(\Delta, i)$. If y_i were equal to zero, then the equation $f_D^h = 0$ would force $x_i = 0$, which is impossible. Hence all intersections must occur in the affine patch, and $m(Z_\Delta, \widetilde{L}) = 1$.

On the other hand, if $\Delta = \{i_1, \ldots, i_r\}$ is not a basis, then there is a cocircuit D which is disjoint from Δ. This means that Z_Δ does not meet the hypersurface defined by f_D. Hence $\widetilde{L}$ does not meet Z_Δ and therefore $m(Z_\Delta, \widetilde{L}) = 0$. The desired result follows. □

Proof of (e). Let $<$ be a monomial term order on $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$. We begin with a remark:

Remark 4.22. *If $<$ is a monomial term order, it is sufficient for Gröbner computations to assume that $<$ is given by a weight order w on the $2n$ variables $x_1, \ldots, x_n, y_1, \ldots, y_n$. Since all of the polynomials in $I(\widetilde{L})$ are multi-homogeneous, the term order w' given by*

$$\begin{aligned} w'(x_i) &= w(x_i) - w(y_i) \\ w'(y_i) &= 0 \end{aligned}$$

will pick out the same initial terms on $I(\widetilde{X})$ as w. Thus we can assume that the weights on the y variables are all zero.

Let $G = \{f_C^h\}$ denote the set of homogenized cocircuits of M, and let $D(M, <) = \mathrm{in}_< G$ denote the ideal generated by the leading terms of the polynomials in $G = \{f_C^h\}$. Notice that each term of a given f_C^h has degree one in the x-variables and is homogeneous. Thus by the remark, to determine the leading term of f_C^h it is sufficient to know only the linear order on the x_is. Therefore

$$\mathrm{in}_< G = \left\langle x_{d_1} y_{d_2} y_{d_3} \cdots y_{d_k} \ : \ D = \{d_1, \ldots, d_k\} \text{ is a cocircuit of } M \text{ and } d_1 = \min_< D \right\rangle.$$

When applied to the dual matroid M^*, Theorem 4.19 says that

$$\mathrm{in}_< G = \bigcap_{B \text{ basis of } M} \langle x_e \ : \ e \in IA_<(B) \, , \, y_e \ : \ e \in IP_<(B) \rangle \, . \tag{4.23}$$

which implies that

$$\text{mdeg } \mathrm{in}_< G = \sum_{B \text{ basis}} t_{b_1} \cdots t_{b_r} = \text{mdeg } I(\widetilde{L}).$$

We also have that

$$\text{mdeg } I(\widetilde{L}) = \text{mdeg } \mathrm{in}_< I(\widetilde{L}).$$

since multi-degree is preserved by flat degenerations. Therefore we have an inclusion

$$\mathrm{in}_< G \subset \mathrm{in}_< I(\widetilde{L})$$

where both ideals have the same multidegree. Since the smaller ideal is reduced and equidimensional, it follows that they are equal. □

Proof of (a) and (b). Since $\operatorname{in}_< G = \operatorname{in}_< I(\widetilde{L})$ for any $<$, $G = \{f_C^h\}$ is a universal Gröbner basis for $I(\widetilde{L})$. To see it is reduced for each term order, just notice that no term divides another, because no cocircuit contains another. Since no term of G divides any other, the element of G are linearly independent over $\mathbf{k}$, so they minimally generate $\langle G\rangle = I(\widetilde{L})$. This proves (a) and (b). □

Proof of (d). Now we prove part (d) - that the number of distinct initial ideals of $\operatorname{in}_< I(\widetilde{L})$ is at most $r! \cdot b$ where r is the codimension of $I(L)$ and b is the number of bases of M. Let $<$ be a term order, and consider the initial ideal $J = \operatorname{in}_< I(\widetilde{L})$. Since the cocircuits form a Gröbner basis, and they are linear in the y-variables, $<$ is determined by the linear order on the y_i. The y-support of the generators of J is precisely $Y_B := \{y_b : b \in B\}$ for some basis B. We claim that for each B there are at most $r!$ ideals J whose y-support is the set Y_B. A term order yielding an initial ideal with y-support Y_B can only depend on the relative order of the elements of Y_B since no other terms are ever selected as leading terms. There are at most $r!$ ways to order these r elements. □

Theorem 4.3. *The non-zero multigraded Betti numbers of $S/I(\widetilde{L})$ are precisely:*

$$\beta_{i,\vec{a}}(S/I(\widetilde{L})) = |\mu(F, \widehat{1})|$$

for each flat F of M, where $i = r - r(F)$, and $\vec{a} = e_{[n]-F}$. They are also equal to the multigraded Betti numbers of $S/\operatorname{in}_< I(\widetilde{L})$ for any $<$. Here μ is the Möbius function of the lattice of flats of M.

Proof. As we already remarked, the initial ideal

$$\operatorname{in}_< I(\widetilde{L}) = \left\langle x_{d_1} y_{d_2} y_{d_3} \cdots y_{d_k} \,:\, D = \{d_1, \ldots, d_k\} \text{ is a cocircuit of } M \text{ and } d_1 = \min_< D \right\rangle$$

is closely related to the Stanley-Reisner ideal

$$I_{IN(M^*)} = \langle x_{d_1} x_{d_2} \cdots x_{d_k} \,:\, D = \{d_1, \ldots, d_k\} \text{ is a cocircuit of } M\rangle$$

of the independence complex $IN(M^*)$ of the dual matroid M^*. More precisely, the second is obtained from the first by setting $y_i = x_i$. In fact, this substitution is equivalent to taking $I^c(M, <)$ modulo a regular sequence. This follows immediately from the primary decomposition of $\operatorname{in}_< I(\widetilde{L})$ given by Theorem 4.19, together with the following lemma:

Lemma 4.24. *Let I be a squarefree monomial ideal in $S = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ satisfying*

(P1) For each i, no associated prime of I contains both x_i and y_i, and

(P2) No minimal generator of I contains a product of the form $x_i y_i$.

Then

$$\{x_1 - y_1, \ldots, x_n - y_n\}$$

is a regular sequence on S/I.

Proof. Notice that (P1) implies that $x_1 - y_1$ is a regular element on S/I. We now form the ideal

$$I' = I \otimes S/(x_1 - y_1)$$

which we realize as an ideal in the polynomial ring $S/(x_1)$ via the substitution $x_1 \mapsto y_1$. We claim that I' has properties $(P1), (P2)$ and then the proof will be complete by induction.

First, the minimal generators of I' are precisely the generators of I after the substitution $x_1 \mapsto y_1$. Thus (P2) is satisfied.

Now denote the primary decomposition of I as $I = \cap P_i$. By P_i' we will denote the ideal P_i after the substitution $x_1 \mapsto y_1$. We claim that

$$I' = \cap P_i'.$$

Since substitution is just a ring map, the inclusion $I' \subset \cap P_i'$ holds.

Conversely, suppose that f is a minimal generator of $\cap(P_i')$. Notice that f does not involve x_1. We have two cases:

Case 1: y_1 does not divide f. In this case, f is actually in I and hence membership of f in (P_i) guarantees membership in (P_i') since the factors of f relevant to ideal membership do not change under our substitution.

Case 2: y_1 divides f, say $f = y_1 g$. Consider the element $h = x_1 f$. Since h is divisible by both x_1 and y_1, and since f is in $\cap(P_i')$, we know h is in fact in each ideal P_i. Thus $h = x_1 y_1 g \in I$. But since I has no minimal generators by (P1) divisible by $x_1 y_1$ we know that either $x_1 g$ or $y_1 g$ must be in I. Under the substitution, both of these elements will be sent to f, so that $f \in I'$.

Now I' satisfies (P1) by construction and the proof is complete. □

With Lemma 4.24 at hand, we are now ready to prove Theorem 4.3. As taking initial ideals is a flat degeneration we have:

$$\beta_{i,\vec{a}}(S/I(\widetilde{L})) \leq \beta_{i,\vec{a}}(S/(\text{in}_< I(\widetilde{L}))).$$

The only way that this inequality can be strict, however, is if $\beta_{i,\vec{a}}(S/(\text{in}_< I(\widetilde{L})))$ is nonzero for some $\vec{a}$ and for two successive values of i. In this case, it is possible that these Betti numbers form a *successive cancellation* and do not appear in the resolution of $S/I(\widetilde{L})$. However, since $\{x_1 - y_1, \ldots, x_n - y_n\}$ is a regular sequence by the previous Lemma, we know that the Betti numbers of $S/(\text{in}_< I(\widetilde{L}))$ are equal to those of the independence ideal $I_{IN(M^*)}$. By Theorem 4.15, we see that no such successive cancellations are possible. □

4.6 On Homogeneity

Throughout, we assumed that the linear space L was actually a vector subspace of $\mathbf{k}^n$. This is a minor assumption, but nonetheless, the nonhomogeneous case has some interesting applications.

In this section, suppose that L is defined by the matrix equation

$$A \cdot \vec{x} = \vec{b}.$$

As before we associate to A a matroid $M = M(L)$. This matroid has a set of cocircuits $\mathcal{D}$ which we will identify with the (unique up to scalar multiple) linear equations $f \in I(L)$ with minimal support. It will also be useful to consider the matroid M_{hom} associated to the matrix $(A \mid (-b))$. We denote its set of cocircuits as $\mathcal{D}_{hom}$ and identify these with homogeneous linear polynomials in the variables $x_1, \ldots, x_n, x_0$.

The degree and generators

The proof of Theorem 4.2(c) carries through unchanged to show that

Proposition 4.25.

$$\text{mdeg } \widetilde{L} = \sum_{b \in B} t_{b_1} \cdots t_{b_k}$$

where the sum is taken over all bases $b = \{b_1, \ldots, b_k\}$ of $M(L)$. □

It is again the case that the set

$$\mathcal{D}^h = \{f^h \mid f \in \mathcal{D}\}.$$

is a minimal generating set for $I(\widetilde{L})$. To see this, we only need to notice that under the lexicographic monomial order $x_1 > \cdots > x_n > y_1 > \cdots > y_n$, the initial terms of each element of $\mathcal{D}^h$ are independent of $\vec{b}$. In fact, they are the same as the leading terms in the case when $\vec{b} = 0$. Hence the ideal these monomials generate has a primary decomposition given by Theorem 4.19. Thus we can conclude via the argument in Theorem 4.2(ab) that $\mathcal{D}^h$ is indeed a minimal generating set (and a Gröbner basis under this term order) for $I(\widetilde{L})$. It is still true that $\mathcal{D}$ is also a universal Gröbner basis for $I(\widetilde{L})$ as we show in the next section.

The initial ideals

In the homogeneous case, an initial ideal of $I(\widetilde{L})$ is determined by the linear order on the original x-variables. This is no longer true in the non-homogeneous setting. The difference is that we now have terms which do not involve any x's. An example is illustrative:

Example 4.26. *Consider the linear ideal*

$$I = \langle x_1 - x_2 + x_6 + a,\ x_2 + x_3 - x_5 + b,\ x_3 - x_4 + c \rangle .$$

where a, b, c are parameters. For any choice of parameters, the closure $I(\widetilde{L})$ is generated by the homogenization of the six co-circuits with support in $\{126, 1356, 1456, 235, 245, 34\}$.

The multidegree of $\widetilde{L}$ *is given by the thirteen bases of the matroid determined by* A*. For comparison we introduce the ideal*

$$I_{hom} = \langle x_1 - x_2 + x_6 + a \cdot x_0,\ x_2 + x_3 - x_5 + b \cdot x_0,\ x_3 - x_4 + c \cdot x_0 \rangle \,.$$

in seven variables. This ideal defines a linear space L_{hom}*. The initial ideals of* $I(\widetilde{L_{hom}})$ *are closely related to those of* $I(\widetilde{L})$*. The following table gives some numbers:*

(a,b,c)	number of initial ideals of $I(\widetilde{L})$	number of initial ideals of $I(\widetilde{L_{hom}})$
(0,0,0)	72	72
(1,1,1)	111	150
(1,1,2)	114	156

Table 4.1: Number of initial ideals

For homogeneous spaces, we proved that the number of initial ideals of $I(\widetilde{L})$ is at most $r! \cdot b$ where $r = n - d$ was the codimension of L and b was the number of bases of $M(L)$. This bound is visibly false in the non-homogeneous case as shown in Table 4.1. The correct bound if $r! \cdot b_{hom}$ where b_{hom} is the number of bases for M_{hom}. In fact, we have the following.

Proposition 4.27. *If* L *is a non-homogeneous space, then each initial ideal of* $I(\widetilde{L})$ *is a localization of an initial ideal of* $I(\widetilde{L_{hom}})$*. In particular the number of initial ideals of* $I(\widetilde{L})$ *is at most the number of initial ideals of* $I(\widetilde{L_{hom}})$*.*

Proof. Let $<$ be a monomial term order on $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$. By homogeneity we can assume this order comes from a weight vector w with

$$w(x_1) > w(x_2) > \cdots > w(x_n), \qquad w(y_i) = 0 \text{ for all } i.$$

We extend $<$ to a term order on $k[x_0, \ldots, x_n, y_0, \ldots, y_n]$ by assigning $w(x_0) = w(y_0) = 0$. This essentially ensures that these new variables do not affect any leading term computations.

We will now compare $\mathcal{D}$ to $\mathcal{D}_{hom}$. If $f \in \mathcal{D}$ has a nonzero constant term, then there is a linear polynomial $f' \in \mathcal{D}_{hom}$ with support equal to $\text{supp}(f) \cup \{x_0\}$. Since $\mathcal{D}_{hom}$ is a universal Gröbner basis by Theorem 4.2, its initial ideal with respect to $<$ is generated by the leading terms of each element of $\mathcal{D}_{hom}$. Call this ideal J. Our claim is that

$$\text{in}_< I(\widetilde{L}) = J(x_0 = 1, y_0 = 1).$$

First notice that the equality

$$\text{in}_< f = \text{in}_< f'|_{x_0 = y_0 = 1}$$

shows that that $\text{in}_< I(\widetilde{L}) \subset J(x_0 = 1, y_0 = 1)$. To show the other inclusion it suffices to show that that the multi-degree of $J(x_0 = 1, y_0 = 1)$ is equal to that of $I(\widetilde{L})$. That is, that its primary components correspond to the bases of M. The decomposition of J is given by Theorem 4.19, and setting x_0, y_0 to 1 is equivalent to ignoring those components that contain x_0 or y_0. These correspond to the bases of M_{hom} that contain 0. Thus the only components that survive are those that correspond to bases of M. □

Bibliography

[AA11] M. Aguiar and F. Ardila, *The hopf monoid of generalized permutahedra*, Preprint (2011).

[ABD10] Federico Ardila, Carolina Benedetti, and Jeffrey Doker, *Matroid polytopes and their volumes*, Discrete Comput. Geom. **43** (2010), no. 4, 841–854.

[Bjö92] Anders Björner, *The homology and shellability of matroids and geometric lattices*, In Matroid applications, Cambridge Univ. Press, 1992, pp. 226–283.

[Boo12] Adam Boocher, *Free resolutions and sparse determinantal ideals*, Math. Res. Lett. **19** (2012), no. 4, 805–821.

[BR07] Winfried Bruns and Tim Römer, *h-vectors of Gorenstein polytopes*, J. Combin. Theory Ser. A **114** (2007), no. 1, 65–76.

[BR13] Adam Boocher and Elina Robeva, *Robust toric ideals*, Preprint (2013), Available at http://arxiv.org/abs/1304.0603.

[BS12] Mats Boij and Jonas Söderberg, *Betti numbers of graded modules and the multiplicity conjecture in the non-Cohen-Macaulay case*, Algebra Number Theory **6** (2012), no. 3, 437–454. MR 2966705

[BZ93] David Bernstein and Andrei Zelevinsky, *Combinatorics of maximal minors*, J. Algebraic Combin. **2** (1993), no. 2, 111–121.

[CHH04] Aldo Conca, Jürgen Herzog, and Takayuki Hibi, *Rigid resolutions and big Betti numbers*, Comment. Math. Helv. **79** (2004), no. 4, 826–839. MR 2099124 (2005k:13025)

[CHT06] Aldo Conca, Serkan Hoşten, and Rekha R. Thomas, *Nice initial complexes of some classical ideals*, Algebraic and geometric combinatorics, Contemp. Math., vol. 423, Amer. Math. Soc., Providence, RI, 2006, pp. 11–42.

[CNG13] Aldo Conca, Emanuela De Negri, and Elisa Gorla, *Universal groebner bases for maximal minors*, Preprint (2013), Available at http://arxiv.org/abs/1302.4461.

[Cra69] Henry H. Crapo, *The Tutte polynomial*, Aequationes Mathematicae **3** (1969), no. 3, 211–229 (English).

[Edm03] Jack Edmonds, *Submodular functions, matroids, and certain polyhedra*, Combinatorial Optimization (Michael Junger, Gerhard Reinelt, and Giovanni Rinaldi, eds.), Lecture Notes in Computer Science, vol. 2570, Springer Berlin Heidelberg, 2003, pp. 11–26 (English).

[GGMS87] I.M Gelfand, R.M Goresky, R.D MacPherson, and V.V Serganova, *Combinatorial geometries, convex polyhedra, and schubert cells*, Advances in Mathematics **63** (1987), no. 3, 301 – 316.

[GM82] M. Giusti and M. Merle, *Singularités isolées et sections planes de variétés déterminantielles. II. Sections de variétés déterminantielles par les plans de coordonnées*, Algebraic geometry (La Rábida, 1981), Lecture Notes in Math., vol. 961, Springer, Berlin, 1982, pp. 103–118. MR 708329 (86e:32013)

[GMN13] E. Gorla, J. C. Migliore, and U. Nagel, *Gröbner bases via linkage*, J. Algebra **384** (2013), 110–134. MR 3045153

[HH11] Jürgen Herzog and Takayuki Hibi, *Monomial ideals*, Graduate Texts in Mathematics, vol. 260, Springer-Verlag London Ltd., London, 2011. MR 2724673

[Jen] Anders N. Jensen, *Gfan, a software system for Gröbner fans and tropical varieties*, Available at http://home.imf.au.dk/jensen/software/gfan/gfan.html.

[JW07] Jakob Jonsson and Volkmar Welker, *A spherical initial ideal for Pfaffians*, Illinois J. Math. **51** (2007), no. 4, 1397–1407.

[Koz91] Dexter Kozen, *The design and analysis of algorithms*, Springer-Verlag, New York, 1991.

[MPS+09] Jason Morton, Lior Pachter, Anne Shiu, Bernd Sturmfels, and Oliver Wienand, *Convex rank tests and semigraphoids*, SIAM J. Discrete Math. **23** (2009), no. 3, 1117–1134. MR 2538642 (2011b:62126)

[MS05] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. MR 2110098 (2006d:13001)

[Oxl92] J. G. Oxley, *Matroid theory*, Oxford University Press, New York, 1992.

[Pee11] Irena Peeva, *Graded syzygies*, Algebra and Applications, vol. 14, Springer-Verlag London Ltd., London, 2011.

[PRW08] Alex Postnikov, Victor Reiner, and Lauren Williams, *Faces of generalized permutohedra.*, Doc. Math., J. DMV **13** (2008), 207–273 (English).

[PS06] Nicholas Proudfoot and David Speyer, *A broken circuit ring*, Beiträge Algebra Geom. **47** (2006), no. 1, 161–166. MR 2246531 (2007c:13029)

[Sch03] A. Schrijver, *Combinatorial optimization - polyhedra and efficiency*, Springer, 2003.

[Sti] Michael E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at `http://www.math.uiuc.edu/Macaulay2/`.

[Stu96] Bernd Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996. MR 1363949 (97b:13034)

[SZ93] Bernd Sturmfels and Andrei Zelevinsky, *Maximal minors and their leading terms*, Adv. Math. **98** (1993), no. 1, 65–112. MR 1212627 (94h:52020)

[ZNZN00] Rahim Zaare-Nahandi and Rashid Zaare-Nahandi, *Gröbner basis and free resolution of the ideal of 2-minors of a* $2 \times n$ *matrix of linear forms*, Comm. Algebra **28** (2000), no. 9, 4433–4453. MR 1772515 (2001f:13020)