Title
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Publication Date
1964-07-01

UCRL-11555

University of California

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Berkeley, California

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Submitted to Physical Review

UCRL-11555

UNIVERSITY OF CALIFORNIA
Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

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ABSTRACT

The solution to Chew's modified strip-approximation N/D equations is discussed. A Wiener-Hopf resolvent kernel appearing in the solution is evaluated explicitly. The solution to the N/D equation is studied as a function of the phase shift at the strip boundary.

I. INTRODUCTION

Chew and Chew and Jones have recently proposed a new method of solving the pion-pion problem.[1,2] The method is based on the strip approximation. It differs, however, from the original work of Chew and Frautschi[3] in considering, as input, double spectral functions convenient for describing the exchange of Regge trajectories and in applying unitarity through the N/D method rather than the Mandelstam iteration procedure.[4] The equations are able to include, effectively, two contributions in addition to those from particle and resonance exchange: the effects of (a) continuum exchange and (b) inelastic processes in the direct channel at energies higher than the strip width. The incorporation of these effects complicates the partial-wave N/D equations in two ways: (a) the Born term evaluation requires double integrals (instead of single integrals as in the case of non-zero-width resonance exchange); (b) the integral equation for N is not Fredholm.

The purpose of this work is to discuss the modified equation for N and its solution. In Section II we review Chew's method of solving the modified equation. In Section III we evaluate an integral to find an explicit, and convenient, form for a Wiener-Hopf resolvent kernel. In Section IV we discuss the properties of the kernel and the solution as a function of a parameter important in the scheme, the phase shift at the strip boundary. We show, in particular, that the effective attraction available from varying this parameter is limited.

II. THE MODIFIED EQUATIONS

We first recall the results of Ref. 2. The amplitude $B_\ell(s)$ satisfies the dispersion relation

$$B_\ell(s) = B_\ell^V(s) + \int_4^{s_1} ds' \, \mathrm{Im}\, B_\ell(s') / (s' - s), \tag{1}$$

where s_1 is the strip width and B^V is the generalized "potential". Assuming elastic unitarity for $4 < s < s_1$, we have

$$\mathrm{Im}\, B_\ell(s) = |B_\ell(s)|^2 \rho_\ell(s) \tag{2}$$

with

$$\rho_\ell(s) = \sqrt{(s-4)/s}\, \left[(s-4)/4\right]^\ell . \tag{3}$$

Equations (1) and (2) are equivalent to

$$B_\ell(s) = N_\ell(s)/D_\ell(s),$$

$$D_\ell(s) = 1 - \frac{1}{\pi}\int \rho_\ell(s')\, N_\ell(s')/(s' - s), \tag{4}$$

$$N_\ell(s) = B_\ell^V(s) + \frac{1}{\pi}\int \frac{B_\ell^V(s') - B_\ell^V(s)}{s' - s}\, \rho_\ell(s')\, N_\ell(s') . \tag{5}$$

From Eq. (1) we see that, as $s \to s_1$, we have

$$B_\ell^V(s) \to \frac{-1}{\pi} \operatorname{Im} B_\ell^V(s_1) \log (s_1 - s), \tag{6}$$

hence the integral Eq. (5) is not Fredholm. Chew[2] has shown that the solution of Eq. (5) can be written as

$$N_\ell(s) = \int O_\ell(s, s') N_\ell^O(s') ds', \tag{7}$$

where N_ℓ^O satisfies the Fredholm equation

$$N_\ell^O(s) = B_\ell^V(s) + \int_4^{s_1} ds' K_\ell'(s, s') N_\ell^O(s'). \tag{8}$$

$K_\ell'(s,s')$ is given by

$$K_\ell'(s,s') = \int K_\ell(s,s'') O_\ell(s'',s'),$$

$$K_\ell(s,s'') = \frac{1}{\pi} \left\{ \frac{B_\ell^V(s') - B_\ell^V(s)}{s' - s} \rho_\ell(s') + \frac{\lambda_\ell}{\pi} \frac{\log(s_1 - s') - \log(s_1 - s)}{s' - s} \right\}, \tag{9}$$

where

$$\lambda_\ell = \rho_\ell(s_1) \,\mathrm{Im}\, B_\ell^V(s_1) . \tag{10}$$

We write $\lambda_\ell = \sin^2 \pi a_\ell$; in the case of elastic unitarity $\pi a_\ell = \delta_\ell(s_1)$, where $\delta_\ell(s_1)$ is the phase shift at s_1 .

The quantity $O_\ell(s_1, s')$ is a Wiener-Hopf resolvent kernel and is given by

$$O_\ell(s_1 s') = \Theta_\ell(x(s), x(s'))/(s_1 - s'), \tag{11}$$

where

$$x(s) = \log \left[(s_1 - 4)/(s_1 - s)\right]$$

and

$$\Theta_\ell(x, x') = (4\pi^2 i)^{-1} \int_C dk \int_{C'} dk' \frac{e^{ik'x' - ikx}}{k' - k} \frac{\phi_{1\ell}(k)}{\phi_{2\ell}(k')} . \tag{12}$$

The contours C and C' are shown in Fig. 1. Finally, the quantities ϕ_1 and ϕ_2 satisfy the relation

$$\phi_{2\ell}(k)/\phi_{1\ell}(k) = \gamma(k) = 1 - \lambda_\ell/\sin^2 \pi ik \tag{13}$$

and are given by

$$\phi_{1\ell}(k) = \Gamma(-ik + a_\ell)\,\Gamma(-ik - a_\ell)/\Gamma^2(-ik), \tag{14}$$

$$\phi_{2\ell}(k) = 1/\phi_{1\ell}(i - k) . \tag{15}$$

The poles and zeros of ϕ_1 and ϕ_2 are discussed in Ref. 2 and are shown in Fig. 1 .

Chew's prescription for solving Eq. (5) is then as follows:[2] (a) the transformation (9) is made, (b) the Fredholm equation (8) is solved, and (c) the transformation (7) is made. Kreps[5] has pointed out that from Eq. (8) one can find a slightly different procedure in which: (a) the transformation

$$\tilde{\tilde{B}}_\ell^{V}(s) = \int ds'\, O_\ell(s, s') B_\ell^{V}(s') \tag{7'}$$

is made, (b) the transformation

$$K_\ell''(s, s') = \int ds''\, O_\ell(s,s'')\, K_\ell(s'',s') \tag{9'}$$

is made, and (c) the Fredholm equation

$$N_\ell(s) = \tilde{B}_\ell^{V}(s) + \int ds' K_\ell''(s,s')\, N_\ell(s') \tag{8'}$$

is solved. Neither prescription is superior for numerical solution, since both involve the same number of transformations.

We note that inelasticity below s_1 may easily be introduced formally into the above treatment. If Eq. (2) is replaced by

$$\mathrm{Im}\, B_\ell(s) = |B_\ell(s)|^2 \rho_\ell(s) R_\ell(s) \qquad (2')$$

with

$$R_\ell(s) = \sigma_\ell^{\mathrm{total}}(s)/\sigma_\ell^{\mathrm{elastic}}(s)$$

then Eqs. (4) through (14) are changed only by the replacement of ρ_ℓ by $\rho_\ell{}'(s) = \rho_\ell(s) R_\ell(s)$. Note that, although the quantity λ_ℓ is still given by $\lambda_\ell = \sin^2 \pi a_\ell$ (and is real), the identification of πa_ℓ as the phase shift at s_1 is no longer valid. A drawback associated with replacing Eq. (2) by Eq. (2') is the current lack of any method for calculating R_ℓ.

III. THE WIENER-HOPF RESOLVENT KERNEL

We now evaluate the double integral of Eq. (12). First we note that $\gamma(k)$ has zeros at $k = k_n^{\pm}$, where

$$k_n^{\pm} = i(n \pm a). \qquad (16)$$

The residue of $1/\gamma$ at $k_n^{\pm}$ is given by

$$\text{Res } 1/\gamma\ (k)|k_n^{\pm} = \mp(2\pi i)^{-1} \tan \pi a . \qquad (17)$$

Eliminating $\phi_{1\ell}(k')$ from Eq. (12) by means of Eq. (13) and using Eq. (17) gives for $\theta_\ell(x,x')$, when the k' contour is closed above,

$$\theta(x,x') = (2\pi)^{-1} \int_C dk\ \mathrm{Exp}\left[i\ k(x' - x)\right] / \gamma(k)$$

$$+ (4\pi^2\ i\ )^{-1}\ \tan \pi a \int_C dk\ \phi_1(k)\ \mathrm{Exp}\left[-i\ k\ x\right]$$

$$\times \sum_{n=1}^{\infty} \left\{ \mathrm{Exp}\left[i\ k_n^-\ x'\right] / \left[(k_n^- - k)\phi_1\ (k_n^-)\right] - \mathrm{Exp}\left[i\ k_n^+\ x'\right] \right.$$

$$\left. / \left[(k_n^+ - k)\ \phi_1(k_n^+)\right] \right\} . \qquad (18)$$

The first integral on the right-hand side of Eq. (18) can be evaluated by closing the contour C in the upper half plane for $x' > x$ or in the lower half plane for $x' < x$. It gives

$$\delta(x' - x) + \tan \pi a \, \theta_A(x,x'),$$

where

$$\theta_A(x,x') = \pi^{-1} \sinh a(x' - x) e^{-(x'-x)} \Big/ \left[1 - \ell^{-(x'-x)}\right]. \tag{19}$$

The delta-function term arises from the fact that $1/\gamma \to 1$ at ∞ so that, for $x \approx x'$, the integrand must be written as $1 + (1/\gamma - 1)$. where the first term gives the delta-function and the residue theorem can be applied to the second. For the second integral we close the contour C in the lower half plane, obtaining

$$\theta_3(x,x') = -i(x^1/2\pi)^2 \sum_{n=1}^{\infty} \sum_{m=0}^{-\infty}$$

$$\times \left\{ \frac{\phi_2(k_m^+)}{\phi_1(k_n^-)} \, \frac{e^{i(k_n^- x' - k_m^+ x)}}{k_n^- - k_m^+} \right.$$

$$\frac{\phi_2(k_m^-)}{\phi_1(k_n^-)} \, \frac{e^{i(k_n^- x' - k_m^- x)}}{k_n^- - k_m^-} \qquad \text{(cont.)}$$

$$- \frac{\phi_2(k_m^+)}{\phi_1(k_n^+)} \frac{e^{i(k_n^+ x' - k_m^+ x)}}{k_n^+ - k_m^+}$$

$$\left. + \frac{\phi_2(k_m^-)}{\phi_1(k_n^+)} \frac{e^{i(k_n^+ x' - k_m^- x)}}{k_n^+ - k_m^-} \right\} . \qquad (20)$$

We thus have for θ

$$\theta_\ell(x,x') = \delta(x' - x) + \tan \pi a\, \theta_A(x,x') + \tan \pi^2 a\, \theta_B(x,x') . \qquad (21)$$

At this point we may note several properties of θ . From Eq. (21), we see that as $a_\ell \rightarrow 0$

$$\theta_\ell(x,x') \rightarrow \delta(x' - x) .$$

This agrees with the fact that, for $\lambda = 0$, Eq. (5) is Fredholm. From (19) and (20) we see that the quantities θ_A and θ_B individually have the proper asymptotic behavior in x and x' discussed in Ref. 1 $\left[e^{ax}, e^{(-1+a)x'}\right]$. It is also possible to see that θ_A is positive and θ_B is negative, for positive a_ℓ less than 1/2 . The above form for θ is readily amenable to digital computer computations and has been used in numerical work to be published separately.

IV. BEHAVIOR IN a_ℓ

We begin by noting that (for positive N) the sign of the contribution from the integral term in Eq. (5) is just that of the average value of $dB_\ell{}^V(s)/ds$. Assuming, for simplicity, that this term is small, consider the case of a_ℓ near 0. Using the Kreps form Eq. (8'), neglecting $\tan^2 \pi a\, \theta_B$ compared to $\tan \pi a\, \theta_A$, and dropping the integral term yields

$$N(s) \cong B_\ell{}^V(s) + \tan \pi a \int \theta_A(s,s')\, B_\ell{}^V(s')\, ds'. \quad (22)$$

Thus a small positive phase shift at s_1 yields a small attraction, and a negative phase shift a repusion, to a positive Born term. We expect the phase shift at s_1 to be positive generally, since the boundary condition at s_1 matches the low-energy amplitude to the high-energy Regge form in which phase shifts go to zero from positive values. Chew has emphasized the multiplicative nature of the correction arising from the boundary condition at s_1: for a_ℓ not too large, N is proportional to its value for zero a_ℓ. The remarks above obtain also for nonzero a_ℓ; the Kreps kernel, Eq. (8'), is a result of multiplicative correction of the same nature as that of the inhomogeneous term.

Note that Regge trajectories are found by solving Eq. (5) for a range of values of ℓ and that, as pointed out in Ref. 1, the end point of a trajectory occurs at a value of ℓ for which there is a homogeneous solution of Eq. (5). The solution of Eq. (5) will be much more sensitive to changes in a_ℓ near the end point of a trajectory than far from an eigenvalue of the homogeneous equation. These remarks imply that the effect of increasing a_ℓ uniformly is to add a constant to the function $\ell = \alpha(s)$.

We next turn to the case of a_ℓ approaching 1/2 . Referring to Eq. (12) and Fig. 1, we see that, since $\phi_{1\ell}(k)$ is analytic above $k = ia$ and $\phi_{2\ell}(k')$ has no zeros below $k' = (1-a_\ell)$, the contours C and C' are not pinched by the coalescing of the poles k_n^+ and k^-_{n+1} as $a_\ell \to 1/2$. Thus $\Theta(x,x',a_\ell)$ is analytic at $a_\ell = 1/2$. This result may be seen explicitly from the expression for Θ_A and Θ_B of Eqs. (4) and (5). From Eq. (19) we have that, as $a_\ell \to 1/2$,

$$\tan \pi a\, \Theta_A(x,x') \to (2\pi^2)^{-1} e^{-(x'-x)/2} \times \left[(1/2 - a)^{-1} - (x' - x)\coth \frac{x' - x}{2} + \cdots\right]. \tag{2}$$

The limit for Θ_B is not quite so simple. As a approaches 1/2 we have $\phi_2(k_0^+) \to 0$, and $\phi_1(k_1^-) \to 0$. The following limits occur in Eq. (20): the <u>(m,n)</u>th term in the fourth series, the <u>(m+1,n)</u>th term in the third series, the <u>(m,n+1)</u>th term in the second series, and the <u>(m+1,n+1)</u>th term in the first series all approach the same value.

These two properties yield the result that the only singularity from $\tan^2 \pi a \theta_B$ arises from the (m=0,n=1)th term of the first series. The contribution from this term is

$$\tan^2 \pi a \theta_B^{(0,1)}(x,x') \to (2\pi^2)^{-1} e^{-(x'-x)} \left[(1/2-a)^{-1} + 8\,\ell n\, 2 + x + x' + \cdots\right]. \tag{24}$$

In finding Eq. (24) we have used the relation

$$\Gamma'(1)/\Gamma(1) - \Gamma'(1/2)/\Gamma(1/2) = 2\ell n\, 2 .$$

The other terms in Eq. (24) also contribute to $\theta(x,x',a_\ell = 1/2)$ but are less important in the limits $x,x' \to \infty$. We thus have

$$\lim_{\substack{x\to\infty \\ x'\to\infty \\ a_\ell \to 1/2}} \theta_\ell(x,x') = (2\pi^2)^{-1} \ell^{-(x'-x)/2} \times \left[8\,\ell n\, 2 + x' + x - (x'-x)\coth\frac{x'-x}{2}\right]. \tag{25}$$

The analyticity of θ at $a = 1/2$ is a very desirable property. Although we expect the strip boundary to be large enough so that the phase shift there is below $\pi/2$,[6] the absence of a singularity in a_ℓ precludes too great a sensitivity to the precise value of the phase shift. Thus a weakly attractive potential cannot be made to give a resonance merely by choosing $\delta(s_1)$ near enough to $\pi/2$.

In concluding, we point out a mechanism for the solution's being insensitive to the exact value of the strip boundary s_1. If s_1 is increased we see from Eq. (4) that resonance energies will be lowered. At larger s, however, $\delta(s)$ is expected to be smaller, decreasing the effective attraction. These two compensating changes will tend to yield a smaller N, a smaller dD/ds, and hence a constant resonance position and width $\left[\propto N/(dD/ds)\right]$.

ACKNOWLEDGMENT

The author is grateful to Professor Geoffrey F. Chew for many helpful conversations. He also wishes to thank Dr. David L. Judd for his hospitality at the Lawrence Radiation Laboratory.

*This work was done under the auspices of the U. S. Atomic Energy Commission.

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3. G.F. Chew and S. C. Frautschi, Phys. Rev. 124, 264 (1961).

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FIGURE CAPTION

Fig. 1. Contours C and C' and zeros of $\gamma(k)$.

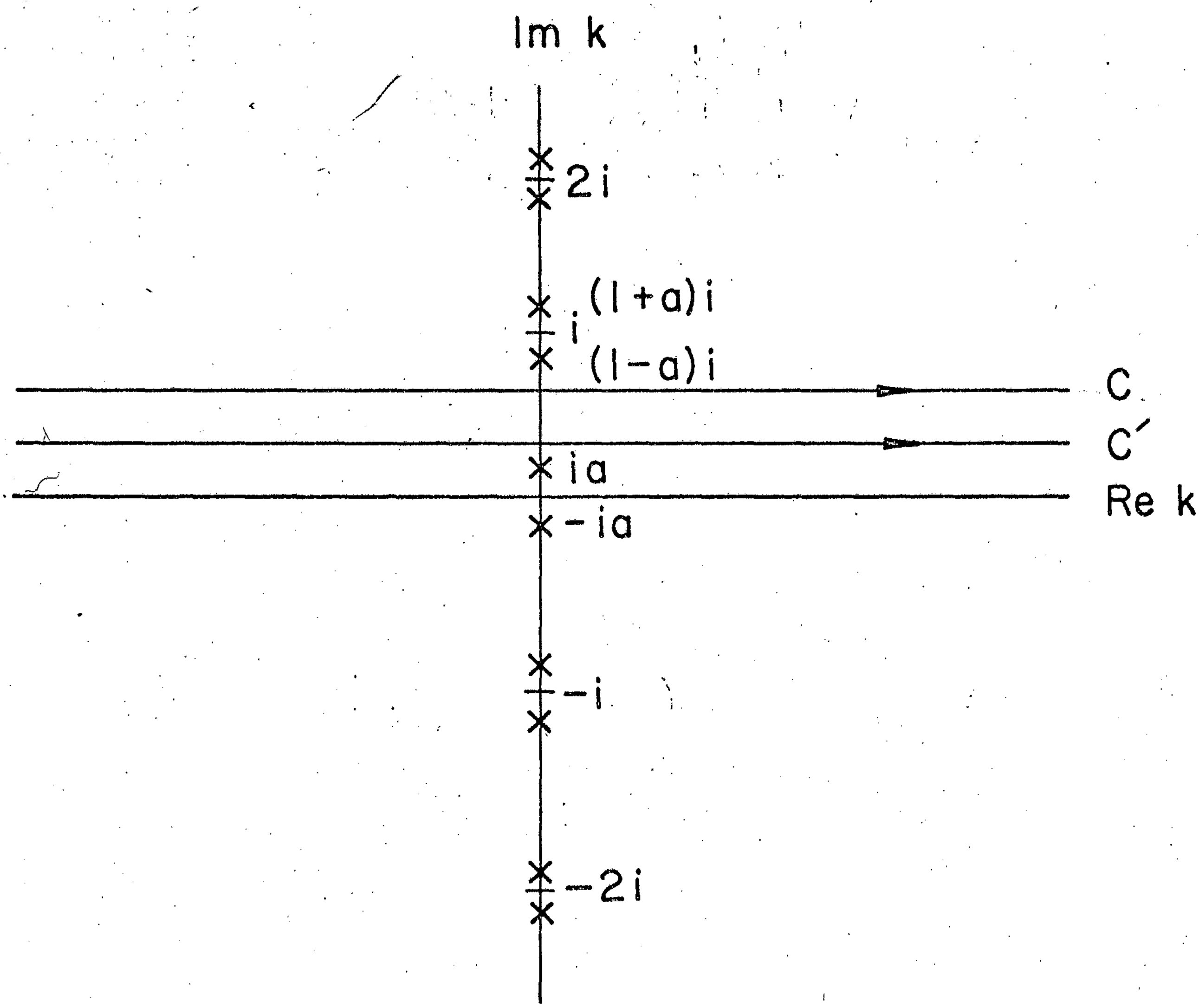

Fig. 1

MUB-3562