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# Matrix Factorisation of Morse-Bott functions

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## Abstract

For a function  $W \in \mathbb{C}[X]$  on a smooth algebraic variety  $X$  with Morse-Bott critical locus  $Y \subset X$ , Kapustin, Rozansky and Saulina [KRS] suggest that the associated matrix factorisation category  $\text{MF}(X; W)$  should be equivalent to the differential graded category of 2-periodic coherent complexes on  $Y$  (with a topological twist from the normal bundle of  $Y$ ). I confirm their conjecture in the special case when the first neighbourhood of  $Y$  in  $X$  is split, and establish the corrected general statement. The answer involves the full Gerstenhaber structure on Hochschild cochains. This note was inspired by the failure of the conjecture, observed by Pomerleano and Preygel [PP], when  $X$  is a general one-parameter deformation of a  $K3$  surface  $Y$ .

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## 1. Statements

Associated to a regular function  $W$  on a smooth algebraic variety  $X$  is a differential super<sup>1</sup>-category  $\text{MF}(X; W)$  of *matrix factorisations* [O1, O2, LP, P]. For quasi-projective  $X$ , objects in  $\text{MF}$  are represented by pairs of vector bundles and maps,  $d_0 : E_0 \rightrightarrows E_1 : d_1$ , with  $d_1 \circ d_0 = W \cdot \text{Id}_{E_0}$  and  $d_0 \circ d_1 = W \cdot \text{Id}_{E_1}$ . Restricted away from the zero-fiber  $W^{-1}(0)$ , and (less obviously) away from the critical locus  $Y$  of  $W$  within  $W^{-1}(0)$ , the category is quasi-equivalent to 0. For more general  $X$ , the global category can be obtained by patching local objects via quasi-isomorphisms [LP, O2].

(1.1) *Split Morse-Bott case.* When  $W$  has a single, Morse critical point  $y$ ,  $\text{MF}(X; W)$  is quasi-equivalent to the category of Clifford super-modules based on  $T_y X$ , with the Hessian form  $\partial^2 W$  of  $W$  (Proposition 2.3). According to the parity of  $\dim X$ , Bott periodicity [ABS] reduces us to the category of super-vector spaces, or to super-modules over the rank one Clifford algebra  $\text{Cliff}(1)$ .

A tempting generalisation to Morse-Bott functions  $W$  with critical locus  $Y \in W^{-1}(0)$  would assert the equivalence of  $\text{MF}(X; W)$  with  $\text{DSCoh}(Y)$ , the differential super-category of 2-periodic complexes of coherent  $\mathcal{O}_Y$ -modules (or  $\mathcal{O}_Y \otimes \text{Cliff}(1)$ , if  $Y$  has odd co-dimension). A topological correction to this guess arises from the first two normal Stiefel-Whitney classes of  $Y$  in  $X$ . These classes assemble with the co-dimension parity into an element  $\tau = (\pm, w_1, w_2^c)$  of the super-Brauer group [DK] and define the  $\tau$ -twisted differential super-category  $\text{DSCoh}^\tau(Y)$  (see §2.1). The first result is

**Theorem 1.** *If the first neighbourhood of  $Y$  in  $X$  is split, then  $\text{MF}(X; W) \equiv \text{DSCoh}^\tau(Y)$ .*

This was conjectured in [KRS], §B without the splitting assumption, supported by physics arguments. However, the equivalence *does* depend on the splitting, and while Theorem 1 does generalise to the non-split case (Theorem 2 below), the stated equivalence fails.

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<sup>1</sup>We use the word *grading* for a  $\mathbb{Z}$ -grading and *super* for a  $\mathbb{Z}/2$ -grading.

(1.2) *Presentation of  $X$ .* We describe a germ of  $X$  near  $Y$  as a deformation of (a germ within) its normal bundle  $\nu : N \rightarrow Y$ . In the holomorphic setting, this is controlled by Kodaira-Spencer-Kuranishi theory, and for convenience, I also encode it in the Dolbeault model. Algebraic information may be lost when  $Y$  is not projective; but the relevant deformation theory is formal (we only need the formal neighbourhood of  $Y$  in  $X$  for the matrix factorisation category) and is thus controlled by the algebraic Hochschild complex. The reader may substitute algebraic models of this complex for Dolbeault forms; the statements and their proof can be adjusted, at the price of making the answer less explicit.

There is no higher cohomology along  $\nu$ , so a full Kodaira-Spencer deformation datum  $\varphi$  for the germ of  $X$  lives in  $\Omega^{(0,1)}(Y; \nu_* \mathcal{T}(N))$ , the Dolbeault forms on  $Y$  with coefficients in the push-down of the holomorphic tangent sheaf  $\mathcal{T}(N)$  of the total space  $N$ . (Only the formal germ of  $\varphi$  along  $Y$  is relevant.) As we wish to fix the zero-section  $Y$  and the normal bundle  $N$ , we choose a  $\varphi$  which vanishes on  $Y$  and whose vertical vector component vanishes quadratically. (The normal derivative of the vertical component of  $\varphi$  lives in  $\Omega^{(0,1)}(Y; N^\vee \otimes N)$ ; if not exact, it deforms  $N$ .)

Integrability of  $\varphi$  and holomorphy of  $W$  on  $X$  are expressed using the Schouten bracket  $\{-, -\}$ :

$$\bar{\partial}\varphi + \frac{1}{2}\{\varphi, \varphi\} = 0 \quad (\text{Maurer-Cartan}), \quad \bar{\partial}_\varphi(W) := \bar{\partial}W + \{\varphi, W\} = 0.$$

The special form of  $\varphi$  forces  $W$  to be  $\bar{\partial}$ -holomorphic along the fibres of  $\nu$  (but  $\nu$  itself is not holomorphic for  $\bar{\partial}_\varphi$ , unless  $\varphi$  is fully vertical). The normal Hessian form  $\partial^2 W$  at  $Y$  is  $\bar{\partial}$ -holomorphic on  $N$ .

(1.3) *Local simplification.* Locally, over small Stein open subsets  $U \subset Y$ , we can choose holomorphic Morse coordinates  $\{t_i\}$  on  $X$  normal to  $U$ . Then,  $W = \frac{1}{2} \sum t_i^2$ , while the vector component of  $\varphi$  is purely horizontal with respect to the  $t_i$ . The Morse coordinates are also  $\bar{\partial}$ -holomorphic along each fibre of  $\nu$ . The algebraic analogue holds formally near an affine  $U$ .

(1.4) *The general Morse-Bott case.* According to Kontsevich's formality theorem [DTT], formal deformations of  $\text{DSCoh}(Y)$  as a super-category are controlled by the Dolbeault-Hochschild complex  $\Omega^{(0,\bullet)}(Y; \Lambda^\bullet TY)$ : to wit, the formal, even Maurer-Cartan solutions therein. Deformations of the twisted versions  $\text{DSCoh}^\tau(Y)$  are controlled in the same way, because  $\tau$  is a locally constant twist. The Morse-Bott matrix factorisation question is answered by an even Maurer-Cartan element  $\Phi_c \in \Omega^{(0,\bullet)}(Y; \Lambda^\bullet TY)$  describing how  $\text{MF}(X; W)$  differs from  $\text{DSCoh}^\tau(Y)$ . Let us find it.

The shifted cotangent bundles  $T^\vee Y[1]$ ,  $T^\vee N[1]$  are (shifted) holomorphic symplectic manifolds, related by a Lagrangian correspondence  $L(\nu) := \nu^* T^\vee Y[1]$  which is induced from the projection  $N \xrightarrow{\nu} Y$ . Extend all structure sheaves quasi-isomorphically by replacing  $\mathcal{O}_Y$  with its Dolbeault resolution. The DG algebras of functions become

$$\Omega^{(0,\bullet)}(Y; \Lambda^\bullet \mathcal{T}(Y)), \quad \Omega^{(0,\bullet)}(Y; \nu_* \Lambda^\bullet \mathcal{T}(N)), \quad \Omega^{(0,\bullet)}(Y; \nu_* \mathcal{O}(N) \otimes \Lambda^\bullet \mathcal{T}(Y)) \quad (1.5)$$

localising holomorphically along the fibres of the projection to  $Y$ , smoothly on the base, and carrying the  $\bar{\partial}$ -differential. If needed, we emphasize the DG structure by appending  $\bar{\partial}$ . Thus,  $\Phi := W + \varphi$  is a function on  $(T^\vee N[1], \bar{\partial})$ . Restrict  $\Phi$  to the (co-isotropic) DG-submanifold  $(L(\nu), \bar{\partial})$ . As  $W$  is Morse-Bott and  $\varphi$  is nilpotent, the critical locus  $C \subset L(\nu)$  of  $\Phi$  along the projection to  $T^\vee Y[1]$  is  $C^\infty$  isomorphic to the base, and the critical value  $\Phi_c$  becomes a function on  $(T^\vee Y[1], \bar{\partial})$ .

**1.6 Proposition.**  $\Phi_c$  is a Maurer-Cartan element of  $\bigoplus_{p>1} \Omega^{(0,p)}(Y; \Lambda^p \mathcal{T}(Y))$ .

**Theorem 2.**  $\text{MF}(X; W)$  is equivalent to the deformation of  $\text{DSCoh}^\tau(Y)$  by  $\Phi_c$ .

1.7 *Remark.* These results have conceptual counterparts in Proposition 3.2 and Theorem 3 below.

1.8 *Remark.* The normal derivative of  $\varphi$  at  $Y$  is  $\bar{\partial}$ -closed in  $\Omega^{(0,1)}(Y; N^\vee \otimes TY)$ , and its class in  $\text{Ext}_Y^1(N; TY)$  obstructs the splitting the first neighbourhood of  $Y$  in  $X$ . In the split case, we can remove it by a gauge transformation. Once  $\varphi$  vanishes *quadratically* at  $Y$ ,  $C$  is the zero-section  $T^\vee Y[1]$  and  $\Phi_c = 0$ , so Theorem 2 implies Theorem 1.

*1.9 Example.* Take  $N = Y \times \text{Spec } \mathbb{C}[[t]]$ ,  $W = t^2/2$ , and describe  $X$  as a deformation of  $N$  by the Maurer-Cartan path  $\varphi(t) = \sum_{n \geq 1} \varphi_n t^n$ ,  $\varphi_n \in \Omega^{(0,1)}(Y; TY)$ . The critical point equation

$$\Phi'(t) = t + \varphi'(t) = 0$$

may be solved degree-by-degree, as  $\deg t = 0$  and  $\deg \varphi = 2$ . We get the solution and critical value

$$\begin{aligned} t_c &= -\varphi_1 + 2\varphi_1\varphi_2 - 4\varphi_1\varphi_2^2 + 3\varphi_1^2\varphi_3 + O(8), \\ \Phi_c &= W(t_c) + \varphi(t_c) = -\frac{1}{2}\varphi_1^2 + \varphi_1^2\varphi_2 - 2\varphi_1^2\varphi_2^2 - \varphi_1^3\varphi_3 + O(10). \end{aligned}$$

Universally,  $\Phi_c$  is a power series in the  $\varphi_n$ , but on a fixed  $Y$  it truncates to  $\varphi$ -degree  $\dim Y$ . One can see that  $\varphi_n$  first appears in a monomial of  $\varphi$ -degree  $n + 1$ , so  $\Phi_c$  only depends on the neighbourhood of  $Y$  of order  $(\dim Y - 1)$ .

Since  $\Phi_c$  starts in degree 4, it vanishes when  $Y$  is a curve. For a surface, we only see  $-\frac{1}{2}\varphi_1^2$ , representing a Dolbeault class in  $H^2(Y; \Lambda^2 TY)$ . For a  $K3$  surface, this gives a single number obstructing the [KRS] conjecture. This obstruction was found by Pomerleano and Preygel [PP].

## 2. Refreshers

I collect in this section some quick and basic background facts that may help the reader.

*(2.1) Clifford bundles.* Let  $Y$  be a variety,  $N \rightarrow Y$  a vector bundle and  $W \in \mathbb{C}[N]$  a regular function which is non-degenerately quadratic along the fibres of  $N$ . The bundle of super-algebras  $\text{Cliff}(N, W)$  over  $Y$  is generated over  $\mathcal{O}_Y$  by sections of  $N$ , declared to be odd, with relations  $\sigma\sigma' + \sigma'\sigma = \partial^2 W / \partial\sigma\partial\sigma'$ . This algebra is invertible over  $\mathcal{O}_Y$  modulo Morita equivalence, with inverse  $\text{Cliff}(N, -W)$ . Invertibility gives an isomorphism between the deformation spaces (and stacks) of the categories of super-complexes of coherent  $\mathcal{O}_Y$ -modules and  $\text{Cliff}(N, W)$  super-modules.

If a global  $\text{Spin}^c$ -module  $S^\pm$  for  $\text{Cliff}(N)$  exists (the *projective* bundle always does), then its endomorphism algebra  $\mathcal{E}$  is  $\mathcal{O}_Y$  or  $\mathcal{O}_Y \otimes \text{Cliff}(1)$ , according to the parity of  $\dim N$ , and  $S^\pm$  gives a Morita equivalence of  $\text{Cliff}(N, W)$  with  $\mathcal{E}$ . The existence of  $S^\pm$  is obstructed by Stiefel-Whitney classes of the orthogonal bundle  $N$ , specifically  $w_1 \in H^1(Y; \mathbb{Z}/2)$  and the image  $w_2^\mathcal{E}$  of  $w_2$  in  $H^2(Y; \mathcal{O}^\times)$ ; however, the same Morita argument shows that the category of super  $\text{Cliff}(N)$ -modules depends on  $N$  only via the Brauer twist  $\tau = (\pm, w_1, w_2^\mathcal{E})$ , and we denote it by  $\text{DSCoh}^\tau(Y)$ .

The twisted category can also be built as follows. Locally, equivalences between super-modules for  $\text{Cliff}(N)$  and  $\mathcal{E}$  are mediated by a  $\text{Spin}$  module  $S^\pm$ . On overlaps, the orthogonal group acts *projectively* on  $S^\pm$ , with projective cocycle classified by  $w_2^\mathcal{E}$ . This gives a non-trivial topological action on the Clifford module categories, patching them to  $\text{DSCoh}^\tau(Y)$ , rather than  $\text{DSCoh}(Y)$ .

*(2.2) The case of vector bundles.* The relation between matrix factorisations of quadratic functions and Clifford modules is summarised in the following *Thom isomorphism*, a special case of Theorem 1, deforming the classical equivalence between symmetric and exterior algebra modules.

**2.3 Proposition.** *The DS category  $\text{MF}(N; W)$  is quasi-equivalent to  $\text{DSCoh}^\tau(Y)$ .*

*Proof.* The equivalence is given by the Atiyah-Bott-Shapiro Thom class [ABS]: the graded-projective Spinor bundle  $S^\pm$  of  $N$  splits locally into its even and odd parts, interchanged globally by  $w_1 \in H^1(Y; \mathbb{Z}/2)$ . Pulled back to the total space  $N$ ,  $S^\pm$  carries a pair  $d_0 : S^+ \rightleftharpoons S^- : d_1$  of endomorphisms, which at a point  $n \in N$  are the Clifford multiplications by the vector  $n$ . This curved complex is an object in  $\text{MF}(N; W) \otimes \text{DSCoh}^\tau(Y)$  and defines a pair of adjoint functors between the two categories, which can be checked to induce a local Morita equivalences between the local respective categories over  $Y$ , and therefore a global quasi-equivalence. (This is a version of *Knörrer periodicity* [O1].)  $\square$

*2.4 Remark.* We can use, instead of the  $\text{Spin}$  module, the Clifford bundle version of the Thom class, with fiber  $\text{Cliff}(N; W)$  and right Clifford multiplications. This a bi-module gives a Morita equivalence between  $\text{MF}(N; W)$  and  $\text{Cliff}(N; W)$ -modules, and deforms the traditional Koszul duality between symmetric and exterior algebra modules.

(2.5) *L<sub>∞</sub> structures and maps.* Let  $V$  be a (cohomologically) graded vector space and denote by  $V[1]$  the same space with the grading shifted down by one. An  $L_∞$  structure on  $V$  is a degree-one vector field  $B$  on  $V[1]$  whose Lie action squares to zero,  $\mathcal{L}_B \circ \mathcal{L}_B = 0$ . Classically,  $V$  is a Lie algebra and the purely quadratic vector field  $B$  is one-half of the Lie bracket  $V^{\wedge 2} \rightarrow V$ ; the null-square condition is the Jacobi identity. Slightly more generally, a linear component of  $B$  is a differential on  $V$ . An  $L_∞$  map  $\mu : (V, B) \rightarrow (V', B')$  is a graded (but not necessarily linear) map  $V \rightarrow V'$  compatible with the vector fields. As a loop space,  $V[1]$  is naturally based at 0, so one usually requires that  $B$  vanishes at the origin and  $\mu(0) = 0$  and defines  $\mu$  to be a quasi-isomorphism if  $d\mu$  is so on tangent spaces at  $0 \in V$ ; this amounts to completing  $V$  at 0. (In Theorem 3 below, we secretly base the first spaces at the element  $\frac{1}{2}\partial^2 W$ , using a translation.)

The *Koszul-Chevalley complex*  $\text{Chev}^*(V, B)$  of  $V, B$  is the commutative DG algebra of functions (usually, Taylor series at 0) on  $V$  with differential  $\mathcal{L}_B$ , and the notion of  $L_∞$  quasi-isomorphism reduces to that of commutative DG algebras in this localisation. The DG space  $\text{Spec Chev}(V, B)$  is the derived moduli stack of solutions of the Maurer-Cartan equation  $\partial v + \frac{1}{2}[v, v] = 0$  in  $V$  in the case of DG Lie algebras,<sup>2</sup> modulo formal gauge transformations. In the case when  $V$  is the (shifted down by one) Hochschild cochain complex of a linear category, with its  $L_∞$  structure, a formal but fundamental result identifies this with the moduli stack of formal deformations of the category.

(2.6) *Kontsevich formality (after Tamarkin).* The Hochschild-Kostant-Rosenberg theorem identifies  $HH^*$  of a regular variety with the Gerstenhaber algebra of its polyvector fields. The  $E_2$  Hochschild cohomology of the latter, which classifies its formal deformations  $[\mathbb{F}]$ , is just  $\mathbb{C}$ , so DG enhancements of this same Gerstenhaber algebra is classified by the de Rham group  $H^3(\mathbb{C}^\times)$  (and vanishes locally). Since there is no such characteristic class of complex manifolds, polyvector fields are the local model for the deformation Gerstenhaber complex, and its Dolbeault resolution is the correct global model. (The same absence of characteristic classes makes the statement equivariant under the group of local coordinate changes.) This makes our computation of the deformation class intrinsically meaningful. Unfortunately, there is no universal flabby differential Gerstenhaber resolution in the algebraic category, and the Dolbeault resolution leads to a more explicit answer.

### 3. Deformation theory proof

In the special case of the normal bundle, Theorem 1 is a Koszul duality between Clifford module and matrix factorisation categories. An isomorphism between their respective (derived) deformation moduli stacks underlies Theorem 2, which asserts a bijection between the complex points of these stacks. This isomorphism comes from an  $L_∞$  quasi-isomorphisms between the controlling differential super-Lie algebras (in fact, one of DS Gerstenhaber algebras). I will spell it out geometrically and algebraically. Geometrically, the critical locus  $C$  is a super-submanifold *not* of  $(T^\vee N[1], \bar{\partial})$ , but of the variant  $(T^\vee N[1], \bar{\partial} + \Phi)$  with the same super-algebra, but modified differential  $\bar{\partial} + \{\Phi, -\}$ . This is quasi-isomorphic to the holomorphic DS manifold  $T^\vee X[1]$  with differential  $\{W, -\}$ . Before spelling out this refinement of Proposition 1.6, we verify the original statement, since we need the computation.

*Proof of Proposition 1.6.* Consider first the special case of Example 1.9. The  $(p, p)$  form of  $\Phi_c$  is clear. Next,  $\Phi'(t_c) = 0$ , so  $\Phi(t) = \Phi_c + O(t - t_c)^2$ , and then

$$\bar{\partial}\Phi_c + \frac{1}{2}\{\Phi_c, \Phi_c\} = \bar{\partial}(\Phi(t)) + \frac{1}{2}\{\Phi(t), \Phi(t)\} + O(t - t_c). \quad (3.1)$$

On the right, the term written out vanishes identically in  $t$ , so setting  $t = t_c$  shows that the left side is zero. For general  $X$ , the statement is local over  $Y$ , so we repeat this in Morse coordinates (§1.3).  $\square$

**3.2 Proposition.** *The space  $C$ , with DS algebra of functions  $\Omega^{(0, \bullet)}(Y; \Lambda^\bullet \mathcal{T}(Y))$  and differential  $\bar{\partial} + \{\Phi_c, -\}$ , is a super-submanifold of  $(T^\vee N[1], \bar{\partial} + \Phi)$ . The embedding is a quasi-equivalence.*

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<sup>2</sup>The zero-locus of  $B$ , in general.

*Proof.* The statement is local and we can use the simplified coordinates of §1.3, assuming for notational ease that only one such coordinate  $t$  is present. Substitute  $\Phi'(t) = (t - t_c)\Phi''(t_c) + O(t - t_c)^2$  in the identity  $\bar{\partial}\Phi'(t) + \{\Phi(t), \Phi'(t)\} \equiv 0$  to get

$$(\bar{\partial}(t - t_c) + \{\Phi(t), (t - t_c)\}) \cdot \Phi''(t_c) = O(t - t_c).$$

Since  $\Phi''(t_c)$  is invertible, it follows that the ideal  $(t - t_c)$  is closed under  $\bar{\partial} + \{\Phi(t), -\}$ .

A local  $\bar{\partial}$ -holomorphic frame  $n$  of  $N$  defines a function  $H_n$  on  $(T^\vee N[1], \bar{\partial})$ , which cuts out  $L(\nu)$  within  $T^\vee N[1]$  and gives the second generator of the ideal  $\mathcal{I}_C$  of  $C$ . The associated Hamiltonian flow  $\{H_n, -\}$  is vertical translation by  $n$ ; therefore,  $\bar{\partial}H_n + \{\Phi, H_n\} = -\partial\Phi/\partial n$  vanishes on the critical locus  $C$ , and so  $\mathcal{I}_C$  is invariant under  $\bar{\partial}_{\Phi(t)}$ .

The same old relation  $\Phi(t) - \Phi_c = O(t - t_c)^2$  shows that  $\bar{\partial}_{\Phi(t)}$  becomes  $\bar{\partial}_{\Phi_c} \bmod \mathcal{I}_C$ .

Quasi-equivalence is checked locally, in a holomorphic product presentation of  $X$  near  $U \subset Y$ ; the complex  $(\Lambda^\bullet \mathcal{T}(X), \{W, -\})$  then resolves the skyscraper sheaf  $\Lambda^\bullet \mathcal{T}(Y)$  on  $U$ .<sup>3</sup>  $\square$

(3.3) *The  $L_\infty$  equivalence.* Denote by  $W_2$  the quadratic part of  $W$ . For the Hochschild complexes of  $\text{DSCoh}^\tau(Y)$  and  $\text{MF}(N, W_2)$ , we use the first two Dolbeault function spaces in (1.5), save that  $\bar{\partial}$  in the second is replaced by  $\bar{\partial} + \{W_2, -\}$ . The relevant  $L_\infty$  algebras are their down-shifts by one (with the Schouten bracket). A non-linear map  $\chi$  between these is defined by sending a function  $\eta$  on  $T^\vee N[1]$  to the critical value  $\Phi_c$  along the projection  $L(\nu) \rightarrow T^\vee Y[1]$  of (the restriction to  $L(\nu)$  of)  $\Phi := W_2 + \eta$ . Here, we treat a degree-zero component of  $\eta$  as a small deformation, so that  $W_2$  is the leading term.

**Theorem 3.**  $\chi$  is an  $L_\infty$  quasi-isomorphism. It is equivalent to the one induced on formal deformation stacks by the Koszul quasi-equivalence  $\text{MF}(N, W_2) \equiv \text{DSCoh}^\tau(Y)$  of categories.

Quasi-isomorphy having been seen in Proposition 3.2, we must check the  $L_\infty$  property and lift  $\chi$  to the universal bundles of deformed categories over the Maurer-Cartan deformation stacks.

*The  $L_\infty$  property.* In a holomorphic local frame of  $N$  with Morse coordinates  $\{t_i\}$ , we project one coordinate at a time, reducing the verification to a single  $t$ . At a point  $\eta(t) = \varphi(t) + \psi(t) \cdot \partial/\partial t$ ,

$$d\chi : (\delta\varphi, \delta\psi) \mapsto (t_c + \varphi'(t_c)) \cdot \delta t_c + \delta\varphi(t_c) = \delta\varphi(t_c),$$

having called  $t_c$  the critical point of  $\Phi(t) := \frac{1}{2}t^2 + \varphi(t)$  and  $\delta t_c$  its first variation.

At a general *even* point  $\eta(t) = \varphi(t) + \psi(t) \cdot \partial/\partial t$  (valued in a super-commutative algebra  $\mathbb{C}[\varepsilon_i]$ , so that the coefficients in  $\varphi$  are even and those in  $\psi$  odd), the value of the structural vector field is

$$\bar{\partial}\varphi(t) + \bar{\partial}\psi(t) \cdot \partial/\partial t - t\psi(t) + \frac{1}{2}[\varphi, \varphi](t) + \psi(t)\psi'(t)\partial/\partial t - \varphi'(t)\psi(t) + [\varphi, \psi](t)\partial/\partial t,$$

and applying  $d\chi$  and using criticality of  $t_c$  gives

$$(\bar{\partial}\varphi)(t_c) + \frac{1}{2}[\varphi, \varphi](t_c) - (t_c + \varphi'(t_c)) \cdot \psi(t_c) = (\bar{\partial}\varphi)(t_c) + \frac{1}{2}[\varphi, \varphi](t_c). \quad (3.4)$$

The right side agrees with  $\bar{\partial}\Phi(t) + \frac{1}{2}\{\Phi(t), \Phi(t)\}$  evaluated at  $t = t_c$ , and formula (3.1) leads us to  $\bar{\partial}\Phi_c + \frac{1}{2}\{\Phi_c, \Phi_c\}$ , which is the value of the structural vector field at  $\Phi_c$ .  $\square$

*Matching the deformed categories.* In a local Morse frame of  $N$ , the constant map  $\Phi_c \mapsto (\eta(t) \equiv \Phi_c)$  provides a left inverse to  $\chi$ . The Thom isomorphism (Proposition 2.3) identifies  $\text{DSCoh}^\tau(U)$  with  $\text{MF}(N, W_2)$ , by tensoring with the ABS Thom class. Local frames differ by orthogonal gauge transformations, whose (projective) action on  $S^\pm$  renders this identification (2.3) gauge equivariant.<sup>4</sup>

A generalised deformation  $(U, \varphi)$ ,  $\varphi \in \Omega^{(0, \bullet)}(U; \Lambda^\bullet \mathcal{T}_U)$  gives, in each Morse frame of  $N$ , a product deformation of  $N$  and a matching Thom equivalence of its MF category with  $\text{DSCoh}^\tau(U, \varphi)$ . Now, the stabiliser, in the group of gauge transformations, of the Maurer-Cartan class of this product

<sup>3</sup>This last identification is only valid locally: globally, we deform  $Y$ .

<sup>4</sup>We act on  $\text{DSCoh}^\tau(Y)$  via the  $\text{Pin}^c$  central extension, §2.1.

deformation of  $N$  acts on the Thom bimodule in the usual (projective) manner. This extends the gauge equivariance of the local equivalence between the base category  $DSCoh^\tau(U, \varphi)$  and the deformed MF categories, and assembles, on overlaps between the various  $U \subset Y$ , to a global equivalence between universally deformed categories.  $\square$

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