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Average Distance Functions and Their Applications

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Average Distance Functions and Their Applications

A Dissertation submitted in partial satisfaction
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by

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ABSTRACT OF THE DISSERTATION

Average Distance Functions and Their Applications

by

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University of California, Riverside, June 2012
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This thesis explores families of metric spaces. It has two parts. First, the Kuratowski embedding is an isometric embedding of a metric space, M, into $L^\infty(M)$. We extend this map to a family of maps by averaging over metric ϵ–balls. The image of M under this map can be regarded as a deformation of M inside $L^\infty(M)$. After restricting our metric spaces to Riemannian manifolds, we explore how curvature affects this deformation. Furthermore, we give a complete description of the deformation of S^n. Second, we prove a diffeomorphsim stability theorem. The smallest r so that a metric r-ball covers a metric space M is called the radius of M. The volume of a metric r-ball in the space form of constant curvature k is an upper bound for the volume of any Riemannian manifold with sectional curvature k and radius r. We show that when such a manifold has volume almost equal to this upper bound, it is diffeomorphic to a sphere or a real projective space.

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Chapter 1

Introduction

In this paper we use Banach geometry to explore relations between Riemannian manifolds with Ricci curvature bounded from below. The following fact motivated the line of reasoning found within; namely, if (M, d) is a metric space, then the mapping $x \mapsto d(x, \cdot) \in L^\infty(M)$ is an isometric embedding. The proof is a simple consequence of the definition of the L^∞-norm and the $\triangle$-inequality. If we ignore set theory, then L^∞ can be viewed as a space containing all metric spaces, and it is reasonable to suspect that "similar" metric spaces are close in L^∞. If our spaces have additional structure, then this notion makes sense. For example, if (M, d) is a complete, separable metric space, then one can use a countably dense subset to approximate (M, d). In particular, complete Riemannian manifolds with Ricci curvature bounded below by a positive constant k are compact by a classical theorem of Myers. However, if we are considering such manifolds, we are ignoring a lot of additional tools available to us if we stop at compactness. For example, all Riemannian manifolds are metric-measure spaces. By using measure, we can define average distance function(al)s on "nice" metric-measure space–like Riemannian manifolds–and view their images in L^∞ as perturbations of M. The Riemannian case will be the main focus of the first part of this thesis. However, we acknowledge that we can be a bit more general than this, and so we begin with some basic definitions.

Let $(\mathcal{M}, d_H)$ denote the collection of compact subsets equipped with the Hausdorff metric. If you would prefer to be concrete, then consider $\mathcal{M}$ to be induced from a compact Riemannian manifold M, but again, the definition below can be extended to more general (compact) metric spaces. Using the Hausdoff measure we can partition $(\mathcal{M}, d_H)$ into ordered pairs: $(\mathcal{M}^s, d_H)$, the collection of s-dimensional subsets of $(\mathcal{M}, d_H)$. From this we define the averaging map,

$$\eta^s : (\mathcal{M}^s, d_H) \longrightarrow L^\infty(M) \ ,$$

via the following cases:

$s > 0$: $\eta^s(A)(\cdot) = \frac{1}{\mathcal{H}^s(A)} \int_{z \in A} d(\cdot, z) d\mathcal{H}^s(A) \equiv ⨍_{z \in A} d(\cdot, z) d\mathcal{H}^s(A)$

$s = 0$: $\eta^0(A(\cdot)) = \sup\{\frac{1}{k} \sum_{i=1}^{k} d(\cdot, b_i) : \ b_i \in B, \ B \subset A$ with finite cardinality, $|B| = k\}$.

That they are all elements of $L^\infty(M)$ for all $s \geq 0$ is fairly clear in this context as $d(\cdot, z) \leq diam(M)$.

For each $\epsilon \geq 0$ we have a map

$$U_\epsilon : (\mathcal{M}, d_H) \longrightarrow (\mathcal{M}, d_H)$$

called ϵ-thickening, defined by

$$U_\epsilon(A) = \{x : d(x, A) \leq \epsilon\}.$$

With the subsets $(\mathcal{M}^s, d_H)$ defined above, observe that

$U_\epsilon : (\mathcal{M}^s, d_H) \longrightarrow (\mathcal{M}^n, d_H);\ \forall\, s,\ \epsilon > 0$

$U_0 \equiv \mathrm{id}_{\mathcal{M}}$.

For the rest of the paper, we shall restrict our attention to compact, connected Riemannian n–manifolds. Let M be such a manifold. Additionally, we wish to impose some curvature conditions. So let $\mathcal{R}^d_{k,v}$ denote the collection of Riemannian n-manifolds with upper radius bound d, lower volume bound v, and lower Ricci curvature bound k (or, more precisely $k(n-1)$, but we will be loose with this language). The radius of a Riemannian manifold being:

$$rad(M) = \min_{p \in M} \max_{x \in M} d(p, x).$$

Furthermore, the following notation is employed throughout the first chapter:

1.1 Notation and Conventions for Chapters 2-4

1. $vol_n(\cdot)$ denotes the canonical Riemannian measure, i.e. n-dimensional Hausdorff measure. When n is the dimension of M, volume generally will appear as: $vol(\cdot)$. When we make volume comparisons between, say the sphere and an arbitrary manifold, M, $v_n(\cdot)$ or $v(\cdot)$ will be used to denote the volume of the n-sphere, and $vol(\cdot)$ will denote the volume of M.

2. We will use a variety of notations for open metric balls. They will appear as $B(x, \epsilon)$ or by $B_{x,\epsilon}$. When the radius is fixed, they will appear as B_x. Closed metric balls will be denoted by $D(x, \epsilon)$, $D_{x,\epsilon}$, or D_x. Similar notation will be used for boundaries; namely, they will appear as $\partial B(x, \epsilon)$, ∂B_x, or $\partial B_{x,\epsilon}$.

3. $Diam(M) = \max_x \max_y d(x, y)$ is the diameter of a metric space.

4. The complement to a set A is denoted by A^c.

5. The notation-A°-denotes the interior of A.

6. We use the notation commonly employed by analysts for averaging, namely

$$\frac{1}{vol(B_x)} \int_{z \in B_x} d(w, z) d\mu$$

will appear as:

$$\fint_{z \in B_x} d(w, z) d\mu.$$

7. Expressions like

$$\int_{z \in B_y} d(\cdot, z) d\mu \quad \text{and} \quad \fint_{z \in B_y} d(\cdot, z) d\mu$$

will appear a lot in this document. To streamline notation, we break from tradition and suppress the measure, $d\mu$, for the most part; there are only a handful of moments in which expressing the measure is important. The measure will always be the standard Riemannian (Hausdorff) measure. In addition, the variable "z" will be our canonical variable of integration. Hence, the above integrals will commonly appear as

$$\int_{B_y} d(\cdot, z) \quad \text{and} \quad \fint_{B_y} d(\cdot, z).$$

8. $\bar{M}_k$ will denote space forms. A space form is a complete, simply connected Riemannian n–manifold of constant sectional curvature k.

9. As stated above, $\mathcal{R}^d_{k,v}$ denotes the collection of Riemannian n–manifolds with upper radius bound d, lower volume bound v, and lower Ricci curvature bound k.

10. Elements of $\bar{M}_k$ will be written with bars: $\bar{x}$. Elements of an arbitrary member of $\mathcal{R}^d_{k,v}$ will be written with plain text.

11. A segment is a geodesic that is distance minimizing.

12. The segment domain at a point $x \in M$ of the Riemannian exponential map will be denoted by seg_x.

    Explicitly, $seg_x = \{v \in T_xM : t \mapsto exp(tv)$ is a segment $\forall t \in [0, 1]\}$.

13. The cut locus of a point x will appear as $cutloc_x$. In terms of the segment domain, the cut locus is $seg_x \setminus seg_x^\circ$. In particular, the "distance from x" function, d_x, is smooth inside the cut locus.

Chapter 2

The Space of Averages

We now fix the dimension of M to be n, and adopt the following notation for averaging that will be used throughout the rest of our discussion on averaging and the space of averages (defined below).

Definition 1

As the dimension is assumed fixed, denote the composition $\eta^n \circ U_\epsilon : M \longrightarrow L^\infty(M)$ *by* η_ϵ*. Thus* $\eta_\epsilon : M \longrightarrow L^\infty(M)$ *is a map defined by*

$$x \longmapsto \eta_\epsilon(x) : M \longrightarrow L^\infty(M)$$

$$\eta_\epsilon(x)(y) := \fint_{z \in B(x,\epsilon)} dist(y, z)$$

if $\epsilon > 0$*, and in particular,*

$$x \longmapsto \eta_0(x) : M \longrightarrow L^\infty(M)$$

$$\eta_0(x)(y) := d(x, y)$$

is the Kuratowski embedding; i.e. the standard embedding of M into $L^\infty(M)$ commonly written as

$$x \longmapsto d_x \ \ (= d(\cdot, x)),$$

where $d : M \times M \longrightarrow \mathbb{R}$ is the metric distance.

We explore some basic properties of average functionals. First, the following "pointed Lipschitz" relation holds (naturally).

Property 2

For any $\epsilon \geq 0$ and any (fixed) $y \in M$

$$w \longmapsto \eta_\epsilon(y)(w)$$

is a 1-Lipschitz map.

Proof.

$$|\fint_{B_y} d(x_1, z) - \fint_{B_y} d(z, x_2)\big| = |\fint_{B_y} d(x_1, z) - d(z, x_2)\big| \leq d(x_1, x_2)$$

■

Lemma 3

If $B_x \neq B_y$, then $\eta_\epsilon(x) \neq \eta_\epsilon(y)$.

Proof.

Let us first observe that if B_x and B_y are disjoint, then the distance from any point in B_y to x is greater than ϵ. Hence,

$$\fint_{B_y} d(x, z) > \epsilon$$

and

$$\fint_{B_x} d(x,z) < \epsilon$$

Therefore $\eta_\epsilon(x) \neq \eta_\epsilon(y)$.

Now consider the general case in which $B_x \neq B_y$ and $B_x \cap B_y \neq \emptyset$. Without loss of generality, we can assume that $vol(B_y) \geq vol(B_x)$. We need to show that there exists a point in which $\eta_\epsilon(x)(w) \neq \eta_\epsilon(y)(w)$. For this, first notice that for any two metric balls, B_u and B_v, and any point $a \in M$, we can write

$$\fint_{B_u} d(a,z) = \frac{1}{vol(B_u)}\big(\int_{B_u \setminus (B_v \cap B_u)} d(a,z) + \int_{B_v \cap B_u} d(a,z)\big).$$

Now take the center of B_x, and observe that the following two inequalities hold:

$$\fint_{B_y} d(x,z) > \Big(\frac{vol(B_y) - vol(B_x \cap B_y)}{vol(B_y)}\Big)\epsilon + \frac{1}{vol(B_y)} \int_{B_x \cap B_y} d(x,z)$$

and

$$\fint_{B_x} d(x,z) < \Big(\frac{vol(B_x) - vol(B_x \cap B_y)}{vol(B_x)}\Big)\epsilon + \frac{1}{vol(B_x)} \int_{B_x \cap B_y} d(x,z).$$

We finish the proof by showing that:

$$\Big(\frac{vol(B_y) - vol(B_x \cap B_y)}{vol(B_y)} - \frac{vol(B_x) - vol(B_x \cap B_y))}{vol(B_x)}\Big)\epsilon \geq \Big(\frac{1}{vol(B_x)} - \frac{1}{vol(B_y)}\Big)c_0 \quad (2.1)$$

where, for computational simplicity, we have

$$c_0 = \int_{B_x \cap B_y} d(x,z)$$

We do not use any sophisticated method to show this–the easiest way is to work backwards, noting that each line is algebraically equivalent to the preceding one. Indeed, equation 2.1 is the same as:

$$\Big(\big(vol(B_y)-vol(B_x\cap B_y)\big)vol(B_x)-\big(vol(B_x)-vol(B_y\cap B_x)\big)vol(B_y)\Big)\epsilon \geq \Big(vol(B_y)-vol(B_x)\Big)c_0,$$

$$\big(vol(B_x \cap B_y)vol(B_y) - vol(B_y \cap B_x)vol(B_x)\big)\epsilon \geq \big(vol(B_y) - vol(B_x)\big)c_0,$$

$$vol(B_x \cap B_y)\epsilon \geq c_0 = \int_{B_x\cap B_y} d(x,z),$$

$$\epsilon \geq ⨍_{B_x\cap B_y} d(x,z).$$

As this last line is true, we conclude that

$$⨍_{B_y} d(x,z) > ⨍_{B_x} d(x,z),$$

i.e.

$$B_x \neq B_y \implies \eta_\epsilon(x) \neq \eta_\epsilon(y).$$

■

Lemma 4

If $B_x = B_y$, then $x = y$ as long as $\epsilon < rad(M)$; i.e. the mapping $x \mapsto B_x$ is injective for $\epsilon < rad(M) < Diam(M)$.

Proof.

Suppose that $B_x = B_y$. We have that $d(w,x) = \epsilon = d(w,y)$ for any point $w \in \partial B_x = \partial B_y$. If $D_{x,rad(M)} \neq M$, there exists a point $p \notin cutloc_x$, which can be joined to x by a minimal, unit-speed geodesic (often referred to as a segment). Let

w be the point that intersects ∂B_x along the segment back to x. In particular, w is in the interior of a segment, and hence not a conjugate point. At such a point, the exponential map is a local diffeomorphism, and accordingly, there is a neighborhood U_w so that $U_w \cap \partial B_x$ is a smooth, $n-1$ dimensional submanifold. If $\gamma(t)$ is the segment that connects w and x, starting at $\gamma(0) = w$, then by 1^{st} variation, $\gamma(t)$ leaves the boundary, $\partial B_x = \partial B_y$, at a right angle. After travelling a distance of ϵ, $\gamma(t)$ must arrive at x. However, since $U_w \cap \partial B_x = U_w \cap \partial B_y$, the geodesic joining y to w must also meet the boundary at a right angle. But geodesics are determined by there initial conditions; hence this geodesic must be $\gamma(t)$ as well. As $d(w, y) = \epsilon$, it follows that $x = y$. ■

Corollary 5

The mapping $x \longmapsto \eta_\epsilon(x)$ is injective as long as $\epsilon < rad(M)$.

Note that, up until this point, we have not used the L^∞–norm in any meaningful way. We now turn to the geometric structure of this metric space.

Definition 6

For fixed ϵ, the set $\{\eta_\epsilon(x) : x \in M\}$ equipped with the L^∞–sup norm is called the space of ϵ averages, or simply the space of averages, for short. This metric space will be denoted by M_ϵ.

With this language in place, we can state a corollary to the above lemmas.

Corollary 7

The space of averages is a manifold homeomorphic to M for each $\epsilon < rad(M)$. Moreover, the upper bound of $rad(M)$ is optimal.

Proof.

Any continuous, injective map from a compact space to a Hausdorff space is a embedding. The fact that the radius is an optimal upper bound follows by considering the sphere. If preferred, one can look ahead to theorem 16 or to the comments after corollary 11. ■

We also have the following relation between M and M_ϵ for $M \in \mathcal{R}^d_{k,v}$.

Lemma 8

For any $x, y \in M$ we have that

$$\|\eta_\epsilon(x) - \eta_0(y)\|_\infty = \eta_\epsilon(x)(y)$$

Proof.

$$\|\eta_\epsilon(x) - \eta_0(y)\|_\infty = \sup_w \left| \left(\fint_{B_x} d(w,z) \right) - d(w,y) \right| =$$

$$\sup_w \left| \fint_{B_x} \Big(d(w,z) - d(w,y) \Big) \right| = \fint_{B_x} dist(z,y) = \eta_\epsilon(x)(y).$$

where the last inequality follows from monotonicity and the $\triangle$-inequality. ■

2.1 Curvature Comparisons

In order to establish the next few results, we will have to introduce some machinery. The Bishop-Gromov volume comparison is a very important tool in Riemannian geometry. It states that the volume of balls does not increase faster than the volume of balls in the model space. More precisely, if $Ric(M) \geq k(n-1)$, then for any $x \in M$

$$\frac{vol(B_{x,r})}{v(B_{\bar{x},r})}$$

is a non-increasing function of r, with limit equal to 1 as $r \to 0$.

Next, one can show that $\frac{d}{d\epsilon} vol(B(x,\epsilon)) = vol_{n-1}(\partial B(x,\epsilon))$. Indeed, the proof is straightforward.

Proposition 9

Let M be Riemannian, and let $B(x,\epsilon) \subset M$. Then

$$\frac{d}{d\epsilon} vol(B(x,\epsilon)) = vol_{n-1}(\partial B(x,\epsilon)).$$

Proof.

$$\frac{d}{d\epsilon} vol(B(x,\epsilon)) = \frac{d}{d\epsilon} \int_{B(\vec{0}_x,\epsilon)\cap seg_x} (dexp_p)_z d\mu = \frac{d}{d\epsilon} \int_0^\epsilon \int_{S^{n-1}(t)\cap seg_x} (dexp_p)_z dudt$$

$$= \int_{S^{n-1}(\epsilon)\cap seg_x} (dexp_p)_z du = vol(\partial B_{x,\epsilon}).$$

■

With these facts, we can show the following.

Lemma 10

Let $x \in M$ and $\bar{x} \in \bar{M}_k$, where $\bar{M}_k$ is a space form. If $Ric(M) > k(n-1)$, then for any ϵ,

$$\eta_\epsilon(\bar{x})(\bar{x}) \geq \eta_\epsilon(x)(x).$$

with equality if and only if $\epsilon = 0$.

Proof.

First, observe that we can rewrite the average in the following way via Cavalieri's principle:

$$\eta_\epsilon(x)(x) = \frac{1}{vol(B(x,\epsilon))} \int_0^\epsilon t vol_{n-1}(\partial B(x,t))dt;$$

and additionally, using the above fact this is equivalent to

$$\frac{1}{vol(B(x,\epsilon))}\int_0^\epsilon t\frac{d}{dt}vol(B(x,t))dt,$$

which begs the use of integration by parts. Indeed, we have that

$$\frac{1}{vol(B(x,\epsilon))}\int_0^\epsilon t\frac{d}{dt}vol(B(x,t))dt = \frac{1}{vol(B(x,\epsilon))}\Big(\epsilon vol(B(x,\epsilon)) - \int_0^\epsilon vol(B(x,t))dt\Big)$$

$$= \epsilon - \int_0^\epsilon \frac{vol(B(x,t))}{vol(B(x,\epsilon))}dt = \int_0^\epsilon \Big(1 - \frac{vol(B(x,t))}{vol(B(x,\epsilon))}\Big)dt.$$

The integrand is comparable via the Bishop-Gromov comparison theorem. In particular, it states that the following inequality holds whenever $t \leq \epsilon$:

$$\frac{vol(B(x,t))}{v(B(\bar{x},t))} \geq \frac{vol(B(x,\epsilon))}{v(B(\bar{x},\epsilon))},$$

with equality if and only if $M = \bar{M}_k$. Therefore, with our assumptions

$$1 - \frac{v(B(\bar{x},t))}{v(B(\bar{x},\epsilon))} > 1 - \frac{vol(B(x,t))}{vol(B(x,\epsilon))}.$$

The computation holds regardless of the space $M \in \mathcal{R}^d_{k,v}$. Hence, we proven our claim:

$$\text{if } \epsilon \neq 0, \text{ then } \eta_\epsilon(\bar{x})(\bar{x}) > \eta_\epsilon(x)(x).$$

■

We record the following useful expression for $\frac{d}{d\epsilon}\eta_\epsilon(\bar{x})(\bar{x})$. We write it as a corollary to the last lemma, but we recognize that it is more of a consequence of the decomposition in the previous proof, than of the statement itself.

Corollary 11

$$\frac{d}{d\epsilon}\eta_\epsilon(x)(x) = \frac{vol_{n-1}(\partial B_x)\big(\epsilon - \eta_\epsilon(x)(x)\big)}{vol(B_x)}.$$

Proof.

The formula follows from basic calculus. Simply rewrite the integrand of the average as $tvol_{n-1}(\partial B(x,t))$. Then using this, one can differentiate $\eta_\epsilon(x)(x)$ using the quotient rule and the fundamental theorem of calculus as follows:

$$\frac{d}{d\epsilon}\eta_\epsilon(x)(x) = \frac{d}{d\epsilon}\frac{1}{vol(B(x,\epsilon))}\int_0^\epsilon tvol_{n-1}(\partial B(x,t))dt = \frac{vol_{n-1}(\partial B_x)\big(\epsilon - \eta_\epsilon(x)(x)\big)}{vol(B_x)}.$$

■

We also note here that the derivative is positive as long as $0 < \epsilon < rad(M)$.

We offer the following loose interpretation of the results in this section: one can view the above expressions as a type of flow inequality. Take, for example, $\bar{M}_k = S^n$. Observe that for S^n, the expression

$$\eta_\pi(\bar{x})(\bar{w}) = \frac{1}{v(S^n)}\int_{\bar{z}\in S^n} d(\bar{w},\bar{z})$$

is actually independent of $\bar{x}$. This does not hold for arbitrary $M \in \mathcal{R}^d_{k,v}$ because, in general, $rad(M) \neq Diam(M)$ and $vol(B(x,\epsilon))$ is variable in x. However, we still have the above result for given x and $\bar{x}$, namely,

$$\|\eta_\epsilon(\bar{x}) - \eta_0(\bar{x})\|_\infty = \eta_\epsilon(\bar{x})(\bar{x}) \geq \eta_\epsilon(x)(x) = \|\eta_\epsilon(x) - \eta_0(x)\|_\infty.$$

As ϵ increases, each $\eta_\epsilon(x)$ flows to the point $\eta_{rad(x)}(x)$, where $rad(x) = \max_y d(y,x)$.

Each $\eta_\epsilon(\bar{x})$ is displaced from $\eta_0(\bar{x})$ further than any $\eta_\epsilon(x)$ is from $\eta_0(x)$. On the whole, we know that M_ϵ flows through manifolds inside L^∞ (of course, through the appropriate range of ϵ). In particular, for the sphere, S^n_ϵ flows to a point (since $\eta_\pi(\bar{x})$ is independent of $\bar{x}$). This suggests that S^n_ϵ is contracting as ϵ increases. We will turn towards this question in what follows next.

Chapter 3

Preliminary Lipschitz Estimates

We begin with the following.

Observation 12

Let M be a compact Riemannian homogeneous space. If $\Lambda_{x,y}$ denotes the set of isometries mapping $x \mapsto y$, then

$$\|\eta_\epsilon(x) - \eta_\epsilon(y)\|_\infty < \fint_{z \in B_x} d(z, \tau(z))$$

for any $\tau \in \Lambda_{x,y}$.

Proof.

Observe that for any $w \in M$ and $\tau \in \Lambda_{x,y}$

$$\left|\fint_{B_x} d(w,z) - \fint_{B_y} d(w,z)\right| = \left|\fint_{B_x} d(w,z) - d(w,\tau(z))\right| < \fint_{B_x} d(z,\tau(z))$$

■

The quantity, $d(w, \tau(w))$ is called the displacement of an isometry. Of course, the above becomes more interesting when we can estimate the displacement. Note that we can always take $\sup_{z \in B(x,\epsilon)} \tau(z)$ for an upper bound, and then minimize over $\tau \in \Lambda_{x,y}$. In any case, we present a few examples. In the simplest case we can show:

Example 13 (Euclidean Space)

Let $x,\ y \in \mathbb{R}^n$.

$$\|\eta_\epsilon(x) - \eta_\epsilon(y)\|_\infty = d(x,y)$$

Proof.

This follows from the fact that translations are contained in $\Lambda_{x,y}$, and that the displacement of a translation is constant. In particular, we can find τ in which the displacement is the same as the distance, $d(x,y)$. ■

It is interesting to note that in the case of Euclidean space, $\mathbb{R}^n_\epsilon$ is isometric to $\mathbb{R}^n$ for any ϵ.

A Lipschitz relation holds on S^n.

Example 14

Let $\bar{x}, \bar{y} \in S^n$, then

$$\|\eta_\epsilon(\bar{x}) - \eta_\epsilon(\bar{y})\|_\infty \leq d(\bar{x},\bar{y})$$

Proof.

The idea of the proof is as follows. Any orientation-preserving isometry that rotates along the great circle, $C_{\bar{x}\bar{y}}$, containing $\bar{x}$, $\bar{y}$ and fixes the orthogonal complement has maximal displacement along that geodesic, $\gamma_{\bar{x}\bar{y}}$. If the reader is comfortable with that statement, without proof, then skip the rest. If not, then we proceed with the following.

Fix $\bar{x}, \bar{y} \in S^n$ and let $\tau \in SO(n+1)$ be an orientation-preserving rotation sending $\bar{x} \mapsto \bar{y}$ that fixes the orthogonal complement. Let $\bar{w} \in S^n$ be arbitrary. If $\bar{w} \in C_{\bar{x}\bar{y}}$, then there is nothing to show. In the other case, by first variation, the minimizing geodesic that connects $\bar{w}$ to $C_{\bar{x}\bar{y}}$ meets at a right angle at some point, $\bar{w}_{\bar{x}} \in C_{\bar{x}\bar{y}}$. Let $\bar{n}$ be the closest (to $\bar{w}$) fixed point of τ (at distance $\frac{\pi}{2}$ from $C_{\bar{x}\bar{y}}$) that is on the geodesic connecting $\bar{w}_{\bar{x}}$ to $\bar{w}$. The geodesic triangle formed by the vertices $\bar{w}_{\bar{x}}, \tau(\bar{w}_{\bar{x}}) := \bar{w}_{\bar{y}}$, and $\bar{n}$ satisfies:

1) $\bar{w} \in \gamma_{\bar{w}_{\bar{x}}\bar{n}}$

2) $\tau(\bar{w}) \in \gamma_{\bar{w}_{\bar{y}}\bar{n}}$

3) $d(\bar{n}, \bar{w}) = d(\bar{n}, \tau(\bar{w}))$

where $\gamma_{..}$ denotes the geodesics joining the above points in S^n.

By basic spherical geometry, we conclude that $d(\bar{x}, \bar{y}) = d(\bar{w}_{\bar{w}}, \tau(\bar{w}_{\bar{x}})) \geq d(\bar{w}, \tau(\bar{w}))$. This establishes the inequality. ■

We can now prove a result about the filling radius of S^n_ϵ. For definitions and some background, consult [37] and [16].

Corollary 15

The metric space of averages induced by the n-sphere satisfies the following inequality:

$$FilRad(S^n_\epsilon) \leq FilRad(S^n).$$

Proof.

The spread decreases under distance decreasing maps. In addition, from [37], $FilRad(S^n_\epsilon) \leq \frac{1}{2}Spread(S^n_\epsilon)$, and in the case of the sphere, $FilRad(S^n) = \frac{1}{2}Spread(S^n)$.

■

We will, in fact, give another proof of the above corollary. Namely, in [37] it was shown that all members in the collection of Riemannian manifolds with positive sectional curvature bounded away from zero have smaller filling radii than the model sphere. At this point it isn't obvious that the S^n_ϵ are members of that collection, namely that they are *Riemannian* manifolds. In the next chapter, we turn to the structure of S^n_ϵ and prove that each S^n_ϵ is isometric to a round n-sphere as long as $\epsilon < Diam(S^n)$.

Chapter 4

The Structure of S^n_ϵ

The main theorem of this section is the following.

Theorem 16

If $\epsilon \in [0, Diam(S^n))$, then S^n_ϵ is a round n-sphere with $Diam(S^n_\epsilon) = \eta_\epsilon(\bar{x})(\bar{x}_a) - \eta_\epsilon(\bar{x}_a)(\bar{x}_a)$. Here we have chosen to write $\bar{x}_a$ instead of $-\bar{x}$ for the antipode of $\bar{x} \in S^n$. At the critical value of $\epsilon = Diam(S^n)$, S^n_ϵ is a point in $L^\infty(M)$.

The proof follows from the following list of facts associated with averaging on the sphere. Specifically, it follows from one important observation: the standard $SO(n+1)$ action on S^n can be extended to the space of averages.

Lemma 17 *Let $\epsilon \in [0, Diam(S^n))$.*

(a) $\eta_\epsilon(\bar{x})(\bar{x}) = \eta_\epsilon(\bar{y})(\bar{y})$ *for any* $\bar{x}, \bar{y} \in S^n$.

(b) $\eta_\epsilon(\bar{x})(\bar{y}) = \eta_\epsilon(\bar{y})(\bar{x})$ *for any* $\bar{x}, \bar{y} \in S^n$.

(c) $\eta_\epsilon(\bar{x})(\bar{y}) + \eta_\epsilon(\bar{x}_a)(\bar{y}) = \pi$ *for any* $\bar{y} \in S^n$ *and any* $\bar{x}$, $\bar{x}_a$ *at maximal distance.*

(d) Combining c) and b) we get the following rule that's reminiscent of the $\triangle$-inequality:

$\eta_\epsilon(\bar{x})(\bar{y}) + \eta_\epsilon(\bar{y})(\bar{x}_a) = \pi$.

(e) $\min\limits_{\bar{w}} \eta_\epsilon(\bar{x})(\bar{w}) = \eta_\epsilon(\bar{x})(\bar{x})$, *and* $\|\eta_\epsilon(\bar{x})\|_\infty = \eta_\epsilon(\bar{x})(\bar{x}_a)$ *where* $\bar{x}$, $\bar{x}_a$ *are points at maximal distance.*

(f) $\|\eta_\epsilon(\bar{x}) - \eta_\epsilon(\bar{x}_a)\|_\infty = Diam(S^n_\epsilon)$.

(g) *There is a transitive* $SO(n+1)$ *action on the space of averages:* $SO(n+1) \times S^n_\epsilon \longrightarrow S^n_\epsilon$ *given by:* $O \cdot \eta_\epsilon(x) = \eta_\epsilon(O(x))$.

(h) *The action defined in g) is isometric; i.e if* $O \in SO(n+1)$, *then*

$$\|\eta_\epsilon(O(x)) - \eta_\epsilon(O(y))\|_\infty = \|\eta_\epsilon(x) - \eta_\epsilon(y)\|_\infty$$

(i) *The isotropy group of* $\eta_\epsilon(\bar{x})$ *is the same as* $\bar{x} \in S^n$, *namely,* $SO(n)$.

Proof.

Parts (a) and (b) are obvious. The statements are change of variables formulas written in our η_ϵ notation. However, we will show one example of this in:

Part (c):

By computation, we have:

$$\eta_\epsilon(\bar{x})(\bar{y}) + \eta_\epsilon(\bar{x}_a)(\bar{y}) = \frac{1}{vol(B_{\bar{x}})} \int_{B_{\bar{x}}} d(\bar{y}, \bar{z}) + \frac{1}{vol(B_{\bar{x}_a})} \int_{B_{\bar{x}_a}} d(\bar{y}, \bar{z}) =$$

$$\frac{1}{vol(B_{\bar{x}})} \Big(\int_{B_{\bar{x}}} d(\bar{y}, \bar{z}) + \int_{B_{\bar{x}}} d(\bar{y}, \bar{z}_a) \Big) = \frac{\pi \cdot vol(B_{\bar{x}})}{vol(B_{\bar{x}})} = \pi.$$

The equality from the second to the third expression is a change of variables using the antipodal map.

Part (d):

See the statement in the lemma.

Part (e):

Consider $B(\bar{w},\epsilon)$ and $B(\bar{x},\epsilon)$. Then notice that by symmetry:

$$\int_{B_{\bar{x}} \bigcap B_{\bar{w}}} d(\bar{x},\bar{z}) = \int_{B_{\bar{w}} \bigcap B_{\bar{x}}} d(\bar{w},\bar{z}).$$

Thus,

$$\begin{aligned}
\eta_\epsilon(\bar{x})(\bar{w}) &= \frac{1}{vol(B_{\bar{x}})}\Big(\int_{B_{\bar{x}}\setminus B_{\bar{w}}} d(\bar{w},\bar{z}) + \int_{B_{\bar{x}} \bigcap B_{\bar{w}}} d(\bar{w},\bar{z})\Big) \\
&\geq \frac{1}{vol(B_{\bar{x}})}\Big(\int_{B_{\bar{x}}\setminus B_{\bar{w}}} d(\bar{x},\bar{z}) + \int_{B_{\bar{x}} \bigcap B_{\bar{w}}} d(\bar{w},\bar{z})\Big) \\
&= \frac{1}{vol(B_{\bar{x}})}\Big(\int_{B_{\bar{x}}\setminus B_{\bar{w}}} d(\bar{x},\bar{z}) + \int_{B_{\bar{x}} \bigcap B_{\bar{w}}} d(\bar{x},\bar{z})\Big) \\
&= \eta_\epsilon(\bar{x})(\bar{x}).
\end{aligned}$$

This establishes the minimum. To get the maximum, use part (c) and write

$$\min_{\bar{w}} \eta_\epsilon(\bar{x}_a)(\bar{w}) = \min_{\bar{w}} \big(\pi - \eta_\epsilon(\bar{x})(\bar{w})\big) = \pi - \max_{\bar{w}} \eta_\epsilon(\bar{x})(\bar{w}).$$

From this we see that $\bar{w} = \bar{x}_a$, and hence $\|\eta_\epsilon(\bar{x})\|_\infty = \eta_\epsilon(\bar{x})(\bar{x}_a)$.

Part (f):

By combining parts (a), (b), and (e) we have that for arbitrary $\bar{x}$, $\bar{y}$, and $\bar{w} \in S^n$

$$\big|\eta_\epsilon(\bar{x})(\bar{w}) - \eta_\epsilon(\bar{y})(\bar{w})\big| = \big|\eta_\epsilon(\bar{w})(\bar{x}) - \eta_\epsilon(\bar{w})(\bar{y})\big| \leq \max_{\bar{v}} \eta_\epsilon(\bar{w})(\bar{v}) - \min_{\bar{v}} \eta_\epsilon(\bar{w})(\bar{v})$$

$$= \eta_\epsilon(\bar{w})(\bar{w}_a) - \eta_\epsilon(\bar{w})(\bar{w}) = \eta_\epsilon(\bar{x})(\bar{x}_a) - \eta_\epsilon(\bar{x}_a)(\bar{x}_a).$$

Therefore,

$$\|\eta_\epsilon(\bar{x}) - \eta_\epsilon(\bar{x}_a)\|_\infty = \eta_\epsilon(\bar{x})(\bar{x}_a) - \eta_\epsilon(\bar{x}_a)(\bar{x}_a) = Diam(S^n_{ave}).$$

Part (g):

To be explicit, the action is

$$O \cdot \eta_\epsilon(\bar{x})(\bar{w}) = \fint_{\bar{z} \in B_{O(\bar{x})}} d(\bar{w}, \bar{z}) = \eta_\epsilon(O(\bar{x}))(\bar{w});\ \ O \in SO(n+1).$$

However, we also make note here that, of course,

$$\fint_{\bar{z} \in B_{O(\bar{x})}} d(\bar{w}, \bar{z}) \equiv \fint_{\bar{z} \in B_{\bar{x}}} d(\bar{w}, O(\bar{z})).$$

By inspection, the action is transitive (since it is transitive on S^n).

Part (h):

This follows from the fact that $SO(n+1)$ acts by isometries (on S^n). Using the last expression in part (g) we have that:

$$O \cdot \eta_\epsilon(\bar{x})(O(\bar{w})) - O \cdot \eta_\epsilon(\bar{y})(O(\bar{w})) = \fint_{\bar{z} \in B_{\bar{x}}} d(O(\bar{w}), O(\bar{z})) - \fint_{\bar{z} \in B_{\bar{y}}} d(O(\bar{w}), O(\bar{z}))$$

$$= \fint_{\bar{z} \in B_{\bar{x}}} d(\bar{w}, \bar{z}) - \fint_{\bar{z} \in B_{\bar{y}}} d(\bar{w}, \bar{z}) = \eta_\epsilon(\bar{x})(\bar{w}) - \eta_\epsilon(\bar{y})(\bar{w})$$

So, if the last expression obtains a maximum at w, then the first expression obtains that same value at $O(\bar{w})$. The other inequality follows by applying O^{-1}. Therefore, our group action is isometric (on the space of averages).

Part (i):

By corollary 5, we have that the map $\bar{x} \longmapsto \eta_\epsilon(\bar{x})$ is injective for $\epsilon \in [0, \pi)$.

Thus,

$$O \cdot \eta_\epsilon(\bar{x}) = \eta_\epsilon(O(\bar{x})) = \eta_\epsilon(\bar{x}) \text{ iff } O(\bar{x}) = \bar{x}.$$

Therefore, the isotropy group is $SO(n)$. ■

Remark 18

For the rest of the paper, we will frequently make use of lemma 17 in our computations without any comment.

Proof of Theorem 16.

From the above work, we know that $SO(n+1)/SO(n) = SO(n+1) \cdot \eta_\epsilon(x)$. The fact that the round n-sphere is isometric to $SO(n+1)/SO(n)$ is a fairly standard theorem in Riemannian geometry, and our above work shows that S_ϵ^n is isometric to $SO(n+1)/SO(n)$ (in particular, parts (g)–(i)). Finally, the explicit formula for the diameter is contained in the proof of part (f) of lemma 17. ■

4.1 S_ϵ^n Distance Estimates and Formulas

We will now take a moment to write down a few consequences of the above theorem and lemma. First, we know how to define the "closest point map" from S^n to S_ϵ^n. Admittedly, it is not a remarkable result on its own, but it does provide us with a picture of how S_ϵ^n "contracts" as we increase ϵ.

Corollary 19

$$\inf_{\bar{y}} \|\eta_\epsilon(\bar{x}) - \eta_0(\bar{y})\|_\infty = \|\eta_\epsilon(\bar{x}) - \eta_0(\bar{x})\|_\infty = \eta_\epsilon(\bar{x})(\bar{x}).$$

Proof.

Use lemma 8 along with (half of) part (e) of lemma 17. ■

Second, using 1^{st} variation we can deduce that

Lemma 20

For any $\bar{x}, \bar{y} \in S^n$

$$\|\eta_\epsilon(\bar{x}) - \eta_\epsilon(\bar{y})\|_\infty = \eta_\epsilon(\bar{x})(\bar{y}) - \eta_\epsilon(\bar{y})(\bar{y})$$

Proof.

What we'd like to show is that, for arbitrary $\bar{w} \in S^n$, $\eta_\epsilon(\bar{x})(\bar{w}) - \eta_\epsilon(\bar{y})(\bar{w}) \leq \eta_\epsilon(\bar{x})(\bar{y}) - \eta_\epsilon(\bar{y})(\bar{y})$. Our argument will be one in which we compare Riemannian sums, or more precisely, their integrands; however, we will first spend some time simplifying the set up.

Towards this end, consider $\eta_\epsilon(\bar{x})(\bar{w}) - \eta_\epsilon(\bar{y})(\bar{w})$ for arbitrary $\bar{w} \in S^n$. Without loss of generality, assume that $\pi \geq d(\bar{w}, \bar{x}) \geq d(\bar{w}, \bar{y}) \geq 0$ so that

$$\eta_\epsilon(\bar{x})(\bar{w}) - \eta_\epsilon(\bar{y})(\bar{w}) \geq 0.$$

In addition, using lemma 17, we write

$$\eta_\epsilon(\bar{x})(\bar{w}) - \eta_\epsilon(\bar{y})(\bar{w}) = \eta_\epsilon(\bar{w})(\bar{x}) - \eta_\epsilon(\bar{w})(\bar{y}).$$

Observe that the value of $\eta_\epsilon(\bar{w})(\bar{y})$ is invariant under the $SO(n)$ action that fixes $\bar{w} \in S^n$. So choose an element that, in addition to fixing $\bar{w}$, rotates the geodesic segment connecting $\bar{w}$ to $\bar{y}$ onto the geodesic connecting $\bar{w}$ to $\bar{x}$. Next, let $\tau_t \in SO(n+1)$ be a parametrized family of rotations, continuously parametrized by $t \in [0, 1]$, so that $\tau_0 = id_{S^n}$ and $\tau_1(\bar{w}) = \bar{y}$; and additionally, each τ_t leaves invariant the great circle containing $\bar{x}, \bar{y}$, and $\bar{w}$. A picture of our set up is displayed above.

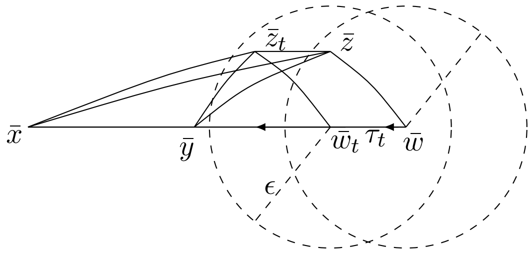

Figure 4.1: Sphere Arrangement

If we denote the lengths of geodesic segments joining $\bar{x}$ to $\bar{z}_t(= \tau_t(\bar{z}))$ and $\bar{y}$ to $\bar{z}_t$ by $l_{\bar{x}\bar{z}_t}$ and $l_{\bar{y}\bar{z}_t}$ respectively, then the quantity we want to discuss is:

$$\frac{dl_{\bar{x}\bar{z}_t}}{dt} - \frac{dl_{\bar{y}\bar{z}_t}}{dt} = -\cos(\angle\bar{z}\bar{z}_t\bar{x}) - \big(-\cos(\angle\bar{z}\bar{z}_t\bar{y})\big).$$

where the notation $\angle\bar{u}\bar{v}\bar{s}$ denotes angle with vertex $\bar{v}$ (and the above is the 1^{st} variation formula). Our hypotheses imply that the above picture is correct for arbitrary $\bar{z} \in B(\bar{w}, \epsilon)$, namely that $\angle\bar{z}\bar{z}_t\bar{x} \geq \angle\bar{z}\bar{z}_t\bar{y}$ with equality if and only if $\bar{z}$ is on the great circle containing $\bar{x}$ and $\bar{y}$. The negative cosine function is monotonically increasing on $[0, \pi]$; hence, $-\cos(\angle\bar{z}\bar{z}_t\bar{x}) \geq -\cos(\angle\bar{z}\bar{z}_t\bar{y})$. Thus, for all $t \in [0, 1]$,

$$\int_0^t \frac{dl_{\bar{x}\bar{z}_r}}{dr} dr \geq \int_0^t \frac{dl_{\bar{y}\bar{z}_r}}{dr} dr \ , \implies$$

$$l_{\bar{x}\bar{z}_t} - l_{\bar{x}\bar{z}_0} \geq l_{\bar{y}\bar{z}_t} - l_{\bar{y}\bar{z}_0}, \implies \quad l_{\bar{x}\bar{z}_t} - l_{\bar{y}\bar{z}_t} \geq l_{\bar{x}\bar{z}_0} - l_{\bar{y}\bar{z}_0}.$$

Now $l_{\bar{x}\bar{z}_t} - l_{\bar{y}\bar{z}_t}$ is nothing more than our integrand, $d(\bar{x}, \bar{z}_t) - d(\bar{y}, \bar{z}_t)$; i.e.

$$d(\bar{x}, \bar{z}_0) - d(\bar{y}, \bar{z}_0) \leq d(\bar{x}, \bar{z}_t) - d(\bar{y}, \bar{z}_t) = d(\bar{x}, \tau_t(\bar{z})) - d(\bar{y}, \tau_t(\bar{z})).$$

In particular, we have shown that for any $\bar{w} \in S^n$

$$\eta_\epsilon(\bar{w})(\bar{x}) - \eta_\epsilon(\bar{w})(\bar{y}) < \big(\tau_1 \cdot \eta_\epsilon(\bar{w})\big)(\bar{x}) - \big(\tau_1 \cdot \eta_\epsilon(\bar{w})\big)(\bar{y}) = \eta_\epsilon(\bar{y})(\bar{x}) - \eta_\epsilon(\bar{y})(\bar{y});$$

that is to say,

$$\eta_\epsilon(\bar{x})(\bar{w}) - \eta_\epsilon(\bar{y})(\bar{w}) \leq \eta_\epsilon(\bar{x})(\bar{y}) - \eta_\epsilon(\bar{y})(\bar{y}).$$

■

Finally, we have an exact formula for our Lipschitz constant (recall example 14). In fact, we can prove something stronger.

Theorem 21

$$\|\eta_\epsilon(\bar{x}) - \eta_\epsilon(\bar{y})\|_\infty = d(\bar{x}, \bar{y}) \frac{\|\eta_\epsilon(\bar{x}) - \eta_\epsilon(\bar{x}_a)\|_\infty}{\pi};$$

i.e. $\bar{x} \longmapsto \eta_\epsilon(\bar{x})$ *is a homothety, with explicit scaling constant given by:*

$$\bar{l}_\epsilon = \frac{\|\eta_\epsilon(\bar{x}) - \eta_\epsilon(\bar{x}_a)\|_\infty}{\pi} \leq 1,$$

with equality iff $\epsilon = 0$. *Moreover,*

$$\frac{d}{d\epsilon}\bar{l}_\epsilon = -\left(\frac{2vol_{n-1}(\partial B_{\bar{x}})(\epsilon - \eta_\epsilon(\bar{x}_a)(\bar{x}_a))}{\pi vol(B_{\bar{x}})}\right)$$

Proof.

The first equation follows from the fact that length, s, on a n-sphere of radius r is given by $s = r\theta$, where θ denotes the distance on the unit sphere, $S^n(1)$.

The next claim follows from a fairly lengthy, but straightforward, computation.

$$\pi \frac{d}{d\epsilon}\bar{l}_\epsilon = \frac{d}{d\epsilon}\|\eta_\epsilon(\bar{x}) - \eta_\epsilon(\bar{x}_a)\|_\infty = \frac{d}{d\epsilon}\Big(\eta_\epsilon(\bar{x})(\bar{x}_a) - \eta_\epsilon(\bar{x}_a)(\bar{x}_a)\Big) =$$

$$\left(\frac{1}{vol(B_{\bar{x}})}\cdot\frac{d}{d\epsilon}\Big(\int_{B_{\bar{x}}} d(\bar{x}_a,\bar{z})-\int_{B_{\bar{x}_a}} d(\bar{x}_a,\bar{z})\Big)+\frac{d}{d\epsilon}\frac{1}{vol(B_{\bar{x}})}\cdot\Big(\int_{B_{\bar{x}}} d(\bar{x}_a,\bar{z})-\int_{B_{\bar{x}_a}} d(\bar{x}_a,\bar{z})\Big)\right)$$

Let $A(x,\epsilon,\delta)$ denote the annulus with inner radius ϵ and outer radius δ. We compute:

$$\frac{d}{d\epsilon}\Big(\int_{B_{\bar{x}}} d(\bar{x}_a,\bar{z}) - \int_{B_{\bar{x}_a}} d(\bar{x}_a,\bar{z})\Big) = \lim_{h\to 0}\frac{1}{h}\Big(\int_{A(\bar{x},\epsilon,\epsilon+h)} d(\bar{x}_a,\bar{z}) - \int_{A(\bar{x}_a,\epsilon,\epsilon+h)} d(\bar{x}_a,\bar{z})\Big)$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\Big(\int_{\partial B_{\bar{x}}}\int_0^h \pi - (\epsilon+t)\Big) - \Big(\int_{\partial B_{\bar{x}_a}}\int_0^h \epsilon+t\Big)\right) = \lim_{h\to 0}\frac{1}{h}\Big(\int_{\partial B_{\bar{x}}} \pi h - 2\epsilon h - h^2\Big)$$

Note that we're suppressing the radius and that $\partial B_{\bar{x}}$ is really $\partial B_{\bar{x},\epsilon+h}$. Additionally, our notation is a bit cumbersome. Technically, we need to check a similar limit involving $A(\bar{x},\epsilon-h,\epsilon)$, but it works out the same. In any case, we have that

$$\frac{d}{d\epsilon}\Big(\int_{B_{\bar{x}}} d(\bar{x}_a,\bar{z}) - \int_{B_{\bar{x}_a}} d(\bar{x}_a,\bar{z})\Big) = vol_{n-1}(\partial B_{\bar{x}})\big(\pi - 2\epsilon\big)$$

Recalling our comments made prior to lemma 10, we have

$$\frac{d}{d\epsilon}\Big(\frac{1}{vol(B_{\bar{x}})}\Big) = -\Big(\frac{1}{vol(B_{\bar{x}})}\Big)^2\frac{d}{d\epsilon}\Big(vol(B_x)\Big) = -\Big(\frac{1}{vol(B_{\bar{x}})}\Big)^2\Big(vol_{n-1}(\partial B_{\bar{x}})\Big)$$

Hence, by putting things together, we see that

$$\frac{d}{d\epsilon}\Big(\eta_\epsilon(\bar{x})(\bar{x}_a) - \eta_\epsilon(\bar{x}_a)(\bar{x}_a)\Big) = \frac{vol_{n-1}(\partial B_{\bar{x}})(\pi - 2\epsilon - Diam(S^n_\epsilon))}{vol(B_{\bar{x}})}$$

We can make some more simplifications. From lemma 17, we know that $\eta_\epsilon(\bar{x})(\bar{y}) + \eta_\epsilon(\bar{x}_a)(\bar{y}) = \pi$, so that

$$Diam(S^n_\epsilon) = \eta_\epsilon(\bar{x})(\bar{x}_a) - \eta_\epsilon(\bar{x}_a)(\bar{x}_a) = (\pi - \eta_\epsilon(\bar{x}_a)(\bar{x}_a)) - \eta_\epsilon(\bar{x}_a)(\bar{x}_a) = \pi - 2\eta_\epsilon(\bar{x}_a)(\bar{x}_a)$$

Therefore,

$$\pi \frac{d}{d\epsilon}\bar{l}_\epsilon = \frac{2vol_{n-1}(\partial B_{\bar{x}})(\eta_\epsilon(\bar{x}_a)(\bar{x}_a) - \epsilon))}{vol(B_{\bar{x}})}$$

and

$$\frac{d}{d\epsilon}\bar{l}_\epsilon = -\left(\frac{2vol_{n-1}(\partial B_{\bar{x}})(\epsilon - \eta_\epsilon(\bar{x}_a)(\bar{x}_a))}{\pi vol(B_{\bar{x}})}\right). \tag{4.1}$$

■

Corollary 22

$$-\frac{\pi}{2}\frac{d}{d\epsilon}\bar{l}_\epsilon = \frac{d}{d\epsilon}\eta_\epsilon(\bar{x})(\bar{x}) \tag{4.2}$$

and hence,

$$\|\eta_\epsilon(\bar{x}) - \eta_\epsilon(\bar{y})\|_\infty = d(\bar{x}, \bar{y})\left(1 - \frac{2\eta_\epsilon(\bar{x})(\bar{x})}{\pi}\right)$$

Proof.

From equation 4.1 the left-hand side is

$$\frac{vol_{n-1}(\partial B_{\bar{x}})(\epsilon - \eta_\epsilon(\bar{x}_a)(\bar{x}_a)))}{vol(B_{\bar{x}})},$$

and one can see that this is the same as the right-hand side by referring back to corollary 11.

The second expression can be obtained by combining the equation for the scaling constant in theorem 21 with lemma 17, or from integrating formula 4.2, noting that $\bar{l}_0 = 1$. ■

Remark 23

Looking back to the statement of theorem 21, notice that as $\epsilon \geq \eta_\epsilon(\bar{x}_a)(\bar{x}_a)$ *we* *have that:*

$$\frac{d}{d\epsilon}\|\eta_\epsilon(\bar{x}) - \eta_\epsilon(\bar{y})\|_\infty \leq 0, \text{ with equality if and only if } \epsilon = 0 \text{ or } \pi.$$

One can read off the two zeros from the formula. Indeed, $\frac{d}{d\epsilon}\bar{l}_\epsilon\Big|_{\epsilon=0} = 0$ *occurs because both* ϵ *and* $\eta_0(\bar{x}_a)(\bar{x}_a)$ *are zero, and* $\frac{d}{d\epsilon}\bar{l}_\epsilon\Big|_{\epsilon=\pi} = 0$ *as* $vol_{n-1}(\partial B_{\bar{x}}) = 0$. *This conforms with the description given earlier; i.e. the fact that* $S^n_\epsilon \subset L^\infty(S^n)$ *is a n-sphere that collapses to a point as* $\epsilon \to \pi$.

It is, of course, interesting to notice that $\frac{d}{d\epsilon}\bar{l}_\epsilon$ *has a critical point on* $[0, \pi]$. *Convexity might be important, however the expression for the second derivative of* $\bar{l}_\epsilon$ *is horrendous, so we will be content with the above.*

4.2 Growth Estimates

Lemma 24

Let $\bar{x}, \bar{w} \in S^n$ *and* $\epsilon \in [0, \pi)$. *If*

1) $d(\bar{w}, \bar{x}) = \frac{\pi}{2}$, *then* $\frac{d}{d\epsilon}\eta_\epsilon(\bar{w})(\bar{x}) = 0$

2) $d(\bar{w}, \bar{x}) < \frac{\pi}{2}$, *then* $\frac{d}{d\epsilon}\eta_\epsilon(\bar{w})(\bar{x}) > 0$

3) $d(\bar{w}, \bar{x}) > \frac{\pi}{2}$, *then* $\frac{d}{d\epsilon}\eta_\epsilon(\bar{w})(\bar{x}) < 0$.

Proof.

Recall part (c) of lemma 17:

$$\eta_\epsilon(\bar{x})(\bar{w}) + \eta_\epsilon(\bar{x}_a)(\bar{w}) = \pi,$$

where $\bar{x}_a$ is antipodal to $\bar{x}$. So if $d(\bar{w}, \bar{x}) = \frac{\pi}{2}$ for $\bar{w}$ and $\bar{x}$, then

$$\frac{d}{d\epsilon}\eta_\epsilon(\bar{x})(\bar{w}) = -\frac{d}{d\epsilon}\eta_\epsilon(\bar{x}_a)(\bar{w}).$$

However, $d(\bar{w}, \bar{x}_a) = \frac{\pi}{2}$, and hence $\eta_\epsilon(\bar{x}_a)(\bar{w}) = \eta_\epsilon(\bar{x})(\bar{w})$. Thus

$$\frac{d}{d\epsilon}\eta_\epsilon(\bar{x})(\bar{w}) = -\frac{d}{d\epsilon}\eta_\epsilon(\bar{x})(\bar{w}),$$

proving 1).

Before we proceed with the proofs of 2) and 3), notice that these statements are actually equivalent. Namely,

$$\frac{d}{d\epsilon}\eta_\epsilon(\bar{x})(\bar{w}) = -\frac{d}{d\epsilon}\eta_\epsilon(\bar{x}_a)(\bar{w})$$

so that if $d(\bar{w}, \bar{x}) < \frac{\pi}{2}$ for arbitrary $\bar{w}, \bar{x} \in S^n$, then $-\frac{d}{d\epsilon}\eta_\epsilon(\bar{w})(\bar{x}_a)$ remains negative, and visa versa. So assume that $\bar{w}, \bar{x}$ satisfy $d(\bar{w}, \bar{x}) < \frac{\pi}{2}$. Now, by theorems 14 and 21 and lemma 20, we can write

$$\frac{d}{d\epsilon}\eta_\epsilon(\bar{x})(\bar{y}) = \frac{d(\bar{x}, \bar{y})}{\pi}\frac{d}{d\epsilon}Diam(S^n_\epsilon) + \frac{d}{d\epsilon}\eta_\epsilon(\bar{y})(\bar{y}),$$

as well as,

$$\frac{d}{d\epsilon}Diam(S^n_\epsilon) = \frac{d}{d\epsilon}\eta_\epsilon(\bar{y}_a)(\bar{y}) - \frac{d}{d\epsilon}\eta_\epsilon(\bar{y})(\bar{y}) = -2\frac{d}{d\epsilon}\eta_\epsilon(\bar{y})(\bar{y}).$$

Therefore,

$$\frac{d}{d\epsilon}\eta_\epsilon(\bar{x})(\bar{y}) > 0,$$

if and only if,

$$\frac{d}{d\epsilon}\eta_\epsilon(\bar{y})(\bar{y}) > -\frac{d(\bar{x},\bar{y})}{\pi}\frac{d}{d\epsilon}Diam(S^n_\epsilon) = 2\frac{d(\bar{x},\bar{y})}{\pi}\frac{d}{d\epsilon}\eta_\epsilon(\bar{y})(\bar{y}),$$

if and only if,

$$1 > \frac{2}{\pi} \text{ or } \frac{\pi}{2} > d(\bar{x},\bar{y}).$$

As this was our hypothesis, we conclude our result. ■

Corollary 25

$\eta_\pi(\bar{x})(\bar{y}) = \frac{\pi}{2}$ for all $\bar{x}$, $\bar{y} \in S^n$ and for any $n \in \mathbb{N}$. Hence, for all $\bar{x}, \bar{y} \in S^n$, $\eta_\epsilon(\bar{x})(\bar{y}) \to \frac{\pi}{2}$ as $\epsilon \to \pi$.

Proof.

We know that if $d(\bar{x},\bar{w}) = \frac{\pi}{2}$, then

$$\eta_\epsilon(\bar{x})(\bar{w}) = \frac{\pi}{2}$$

for all ϵ, and in particular, for $\epsilon = \pi$. Since $\eta_\pi(\bar{x})(\bar{w})$ is independent of $\bar{x}$ and $\bar{w}$, the values must be the same; i.e. $\eta_\pi(\bar{x})(\bar{y}) = \frac{\pi}{2}$ for all $\bar{x}, \bar{y} \in S^n$.

The second part follows from continuity and the previous lemma. ■

4.3 Distance Between the Levels of S^n_ϵ

Lemma 26

Suppose that $\delta > \epsilon$, then

$$\|\eta_\delta(\bar{x}) - \eta_\epsilon(\bar{y})\|_\infty = \|\eta_\delta(\bar{x}) - \eta_\delta(\bar{y})\|_\infty + \|\eta_\delta(\bar{y}) - \eta_\epsilon(\bar{y})\|_\infty,$$

and additionally,

$$\|\eta_\delta(\bar{x}) - \eta_\epsilon(\bar{y})\|_\infty = \eta_\delta(\bar{x})(\bar{y}) - \eta_\epsilon(\bar{y})(\bar{y}).$$

Proof.

In the course of showing that $\|\eta_\delta(\bar{y}) - \eta_\epsilon(\bar{y})\|_\infty = \eta_\delta(\bar{y})(\bar{y}) - \eta_\epsilon(\bar{y})(\bar{y})$, both results will appear. The method of proof is fairly straightforward. Indeed, the following inequalities are equivalent:

$$\begin{aligned} \|\eta_\delta(\bar{w}) - \eta_\delta(\bar{y})\|_\infty < \|\eta_\epsilon(\bar{w}) - \eta_\epsilon(\bar{y})\|_\infty \text{ (theorem 21)} \\ \eta_\delta(\bar{w})(\bar{y}) - \eta_\delta(\bar{y})(\bar{y}) < \eta_\epsilon(\bar{w})(\bar{y}) - \eta_\epsilon(\bar{y})(\bar{y}) \\ \eta_\delta(\bar{y})(\bar{w}) - \eta_\delta(\bar{y})(\bar{y}) < \eta_\epsilon(\bar{y})(\bar{w}) - \eta_\epsilon(\bar{y})(\bar{y}) \\ \eta_\delta(\bar{y})(\bar{w}) - \eta_\epsilon(\bar{y})(\bar{w}) < \eta_\delta(\bar{y})(\bar{y}) - \eta_\epsilon(\bar{y})(\bar{y}); \end{aligned}$$

i.e. the map $w \longmapsto \eta_\delta(\bar{y})(\bar{w}) - \eta_\epsilon(\bar{y})(\bar{w})$ is maximal at $\bar{y}$.

To finish the result, write

$$\|\eta_\delta(\bar{x})-\eta_\epsilon(\bar{y})\|_\infty = \|\eta_\delta(\bar{x})-\eta_\epsilon(\bar{y})+\eta_\delta(\bar{y})-\eta_\delta(\bar{y})\|_\infty \leq \|\eta_\delta(\bar{x})-\eta_\delta(\bar{y})\|_\infty+\|\eta_\delta(\bar{y})-\eta_\epsilon(\bar{y})\|_\infty.$$

That is to say,

$$\sup_{\bar{w}} \left|\eta_\delta(\bar{x})(\bar{w}) - \eta_\epsilon(\bar{y})(\bar{w})\right| \leq \sup_{\bar{w}} \left|\eta_\delta(\bar{x})(\bar{w}) - \eta_\delta(\bar{y})(\bar{w})\right| + \sup_{\bar{w}} \left|\eta_\delta(\bar{y})(\bar{w}) - \eta_\epsilon(\bar{y})(\bar{w})\right|.$$

Equality now follows because both functions, $\eta_\delta(\bar{x})-\eta_\delta(\bar{y})$ and $\eta_\delta(\bar{y})-\eta_\epsilon(\bar{y})$, are positive and maximal at $\bar{y}$. (These last statements follow from lemma 20 and corollary 11). ■

Corollary 27

Suppose $\delta > \epsilon$. If

1. *$d(\bar{x},\bar{y}) > \frac{\pi}{2}$, then* $\|\eta_\delta(\bar{x}) - \eta_\epsilon(\bar{y})\|_\infty < \|\eta_\epsilon(\bar{x}) - \eta_\epsilon(\bar{y})\|_\infty$.

2. *$d(\bar{x},\bar{y}) < \frac{\pi}{2}$, then* $\|\eta_\epsilon(\bar{x}) - \eta_\epsilon(\bar{y})\|_\infty < \|\eta_\delta(\bar{x}) - \eta_\epsilon(\bar{y})\|_\infty$.

3. *$d(\bar{x},\bar{y}) = \frac{\pi}{2}$, then* $\|\eta_\delta(\bar{x}) - \eta_\epsilon(\bar{y})\|_\infty = \|\eta_\epsilon(\bar{x}) - \eta_\epsilon(\bar{y})\|_\infty$.

Proof.

This follows from our derivative estimates made in lemma 24. Namely, if $d(\bar{x},\bar{y}) > \frac{\pi}{2}$, then $\eta_\epsilon(\bar{x})(\bar{y}) \geq \eta_\delta(\bar{x})(\bar{y})$. If $d(\bar{x},\bar{y}) < \frac{\pi}{2}$, then the opposite relation holds. If $d(\bar{x},\bar{y}) = \frac{\pi}{2}$, recall that $\eta_\epsilon(\bar{x})(\bar{y}) = \frac{\pi}{2}$, regardless of ϵ. ■

Chapter 5

The Diffeomorphism Type of Manifolds with Almost Maximal Volume

5.1 Introduction

Any closed Riemannian n–manifold M has a lower bound for its sectional curvature, $k \in \mathbb{R}$. This gives an upper bound for the volume of any metric ball $B(x, r) \subset M$,

$$vol(B(x, r)) \leq vol(\mathcal{D}_k^n(r)),$$

where $\mathcal{D}_k^n(r)$ is an r–ball in the n–dimensional, simply connected space form of constant curvature k. If $rad(M)$ is the smallest number r such that a metric r–ball covers M, it follows that

$$vol(M) \leq vol(\mathcal{D}_k^n(rad(M))).$$

Recall that the invariant $rad(M)$ can alternatively be defined as

$$rad(M) = \min_{p \in M} \max_{x \in M} d(p, x).$$

In the event that $vol(M)$ is almost equal to $vol(\mathcal{D}^n_k(rad(M)))$, we determine the diffeomorphism type of M.

Theorem 28

Given $n \in \mathbb{N}, k \in \mathbb{R}$, and $r > 0$, there is an $\varepsilon > 0$ so that every closed Riemannian n–manifold M with

$$\begin{aligned} sec(M) &\geq k, \\ rad(M) &\leq r, \text{ and} \\ vol(M) &\geq vol(\mathcal{D}^n_k(r)) - \varepsilon \end{aligned} \tag{5.1}$$

is diffeomorphic to S^n or $\mathbb{R}P^n$.

Grove and Petersen obtained the same result with *diffeomorphism* replaced by *homeomorphism* in [12]. They also showed that for any $\varepsilon > 0$ and $M = S^n$ or $\mathbb{R}P^n$ there are Riemannian metrics that satisfy (5.1) except when $k > 0$ and $r \in \left(\frac{1}{2}\frac{\pi}{\sqrt{k}}, \frac{\pi}{\sqrt{k}}\right)$.

For $k > 0$ and $r \in \left(\frac{1}{2}\frac{\pi}{\sqrt{k}}, \frac{\pi}{\sqrt{k}}\right)$, Grove and Petersen also computed the optimal upper volume bound for the class of manifolds with

$$sec(M) \geq k \quad \text{and} \quad rad(M) \leq r. \tag{5.2}$$

It is strictly less than $vol(\mathcal{D}^n_k(r))$ [12]. For $k > 0$ and $r \in \left(\frac{1}{2}\frac{\pi}{\sqrt{k}}, \frac{\pi}{\sqrt{k}}\right)$, manifolds satisfying (5.2) with almost maximal volume are already known to be diffeomorphic to spheres [14]. The main theorem in [26] gives the same result when $r = \frac{\pi}{\sqrt{k}}$.

For $k > 0$ and $r = \frac{\pi}{\sqrt{k}}$, the maximal volume $vol(\mathcal{D}_1^n\left(\frac{\pi}{\sqrt{k}}\right))$ is realized by the n-sphere with constant curvature k. For $k > 0$ and $r = \frac{\pi}{2\sqrt{k}}$, the maximal volume $vol(\mathcal{D}_1^n\left(\frac{\pi}{2\sqrt{k}}\right))$ is realized by $\mathbb{R}P^n$ with constant curvature k. Apart from these cases, there are no Riemannian manifolds M satisfying (5.2) and $vol(M) = vol(\mathcal{D}_k^n(r))$. Rather, the maximal volume is realized by one of the following two types of Alexandrov spaces. [12]

Example 29 ***(Crosscap)***

The constant curvature k Crosscap, $C_{k,r}^n$, is the quotient of $\mathcal{D}_k^n(r)$ obtained by identifying antipodal points on the boundary. Thus $C_{k,r}^n$ is homeomorphic to $\mathbb{R}P^n$. There is a canonical metric on $C_{k,r}^n$ that makes this quotient map a submetry. The universal cover of $C_{k,r}^n$ is the double of $\mathcal{D}_k^n(r)$. If we write this double as $\mathbb{D}_k^n(r) := \mathcal{D}_k^n(r)^+ \cup_{\partial\mathcal{D}_k^n(r)^\pm} \mathcal{D}_k^n(r)^-$, then the free involution

$$A : \mathbb{D}_k^n(r) \longrightarrow \mathbb{D}_k^n(r)$$

that gives the covering map $\mathbb{D}_k^n(r) \longrightarrow C_{k,r}^n$ is

$$A : (x, +) \longmapsto (-x, -),$$

where the sign in the second entry indicates whether the point is in $\mathcal{D}_k^n(r)^+$ or $\mathcal{D}_k^n(r)^-$.

Example 30 ***(Purse)***

Let $R : \mathcal{D}_k^n(r) \to \mathcal{D}_k^n(r)$ be reflection in a totally geodesic hyperplane H through *the center of* $\mathcal{D}_k^n(r)$. *The Purse,* $P_{k,r}^n$, *is the quotient space*

$$\mathcal{D}_k^n(r)\,/\,\{v \sim R(v)\}\,,\ \textit{provided } v \in \partial\mathcal{D}_k^n(r)\,.$$

Alternatively, we let $\{\mathcal{HD}_k^n(r)\}^+ \cup \{\mathcal{HD}_k^n(r)\}^- = D_k^n(r)$ *be the decomposition of* $\mathcal{D}_k^n(r)$ *into the two half disks on either side of* H. *Then* $P_{k,r}^n$ *is isometric to the double of* $\{\mathcal{HD}_k^n(r)\}^+$.

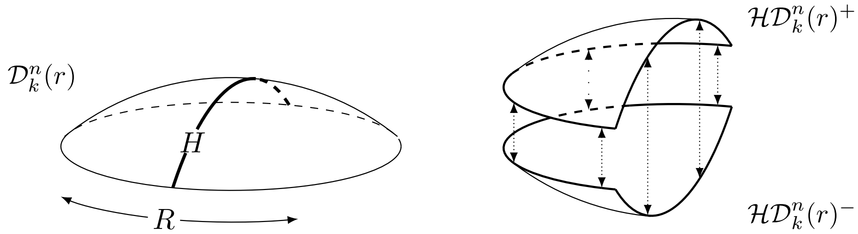

Figure 5.1: Two equivalent constructions of $P_{1,r}^2$

Let $\{M_i\}_{i=1}^{\infty}$ be a sequence of closed n-manifolds satisfying $sec(M) \geq k$ and $rad(M) \leq r$ and $\{vol(M_i)\}$ converging to $vol(\mathcal{D}_k^n(r))$ where $r \leq \frac{\pi}{2\sqrt{k}}$ if $k > 0$. Grove and Petersen showed that $\{M_i\}$ has a subsequence that converges to either $C_{k,r}^n$ or $P_{k,r}^n$ in the Gromov-Hausdorff topology [12]. The main theorem follows by combining this with the following *diffeomorphism stability theorems.*

Theorem 31

Let $\{M_i\}_{i=1}^{\infty}$ *be a sequence of closed Riemannian* n*–manifolds with* $sec(M_i) \geq k$ *so that*

$$M_i \longrightarrow C_{k,r}^n$$

in the Gromov-Hausdorff topology. Then all but finitely many of the M_i*s are diffeomorphic to* $\mathbb{R}P^n$.

Note 1

After finishing this paper, we were informed that theorem 5.4 can be derived from theorem 6.1 in [20].

Theorem 32

Let $\{M_i\}_{i=1}^{\infty}$ be a sequence of closed Riemannian n–manifolds with sec$(M_i) \geq k$ so that

$$M_i \longrightarrow P_{k,r}^n$$

in the Gromov-Hausdorff topology. Then all but finitely many of the M_is are diffeomorphic to S^n.

Remark 33

One can get Theorem 32 for the case $k = 1$ and $r > \operatorname{arccot}\left(\frac{1}{\sqrt{n-3}}\right)$ as a corollary of Theorem C in [15]. Theorem 31 when $k = 1$ and $r = \frac{\pi}{2}$ follows from the main theorem in [39] and the fact that $C_{1,\frac{\pi}{2}}^n$ is $\mathbb{R}P^n$ with constant curvature 1 . With minor modifications of our proof, the hypothesis sec$(M_i) \geq k$ in Theorems 31 and 32 can replaced, except in one case, with an arbitrary uniform lower curvature bound. The exceptional case, is Theorem 31 in dimension 4, specifically in Proposition 67. For ease of notation, we have written all of the proofs for $\{M_i\}_{i=1}^{\infty}$ with sec$(M_i) \geq k$ converging to $C_{k,r}^n$ or $P_{k,r}^n$.

Section 2 introduces notations and conventions used throughout the remainder of this document. Section 3 is review of necessary tools from Alexandrov geometry. Section 4 develops machinery and proves Theorem 31 in the case when $n \neq 4$. Theorem 31 in dimension 4 is proven in Section 5, and Theorem 32 is proven in Section 6.

Throughout the remainder of the paper, we assume without loss of generality, by rescaling if necessary, that $k = -1, 0$ or 1.

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5.2 Notations and Conventions for Chapter 5

We assume a basic familiarity with Alexandrov spaces, including but not limited to [1]. Let X be an n–dimensional Alexandrov space and $x, p, y \in X$.

Notation 34

1. *We call minimal geodesics in X* segments. *We denote by px a segment in X with endpoints p and x.*

2. *We let Σ_p and T_pX denote the space of directions and tangent cone at p, respectively.*

3. *For $v \in T_pX$ we let γ_v be the segment whose initial direction is v.*

4. *Following [30], $\Uparrow_x^p \subset \Sigma_x$ will denote the set of directions of segments from x to p, and $\uparrow_x^p \in \Uparrow_x^p$ denotes the direction of a single segment from x to p.*

5. *We let $\sphericalangle(x, p, y)$ denote the angle of a hinge formed by px and py and $\tilde{\sphericalangle}(x, p, y)$ denote the corresponding comparison angle.*

6. *Following [26], we let* $\tau : \mathbb{R}^k \to \mathbb{R}_+$ *be any function that satisfies*

$$\lim_{x_1,\dots,x_k \to 0} \tau\left(x_1, \dots, x_k\right) = 0,$$

and abusing notation we let $\tau : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}$ *be any function that satisfies*

$$\lim_{x_1,\dots,x_k \to 0} \tau\left(x_1, \dots, x_k | y_1, \dots, y_n\right) = 0,$$

provided that $y_1, \dots, y_n$ *remain fixed.*

When making an estimate with a function τ *we implicitly assert the existence of such a function for which the estimate holds.*

7. *We denote by* $\mathbb{R}^{1,n}$ *the Minkowski space* $(\mathbb{R}^{n+1}, g)$, *where* g *is the semi-Riemannian metric defined by*

$$g = -dx_0^2 + dx_1^2 + \cdots + dx_n^2$$

for coordinates $(x_0, x_1, \cdots, x_n)$ *on* $\mathbb{R}^{n+1}$.

8. *We reserve* $\{e_j\}_{j=0}^m$ *for the standard orthonormal basis in both euclidean and Minkowski space.*

9. *We use two isometric models for hyperbolic space,*

$$H_+^n := \left\{ (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} \middle| -(x_0)^2 + (x_1)^2 + \cdots + (x_n)^2 = -1,\ x_0 > 0 \right\}$$

and

$$H_-^n := \left\{ (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} \middle| -(x_0)^2 + (x_1)^2 + \cdots + (x_n)^2 = -1,\ x_0 < 0 \right\}.$$

10. We obtain explicit double disks, $\mathbb{D}^n_k(r) := \mathcal{D}^n_k(r)^+ \cup_{\partial\mathcal{D}^n_k(r)^\pm} \mathcal{D}^n_k(r)^-$*, by viewing* $\mathcal{D}^n_k(r)^+$ *and* $\mathcal{D}^n_k(r)^-$ *explicitly as*

$$\mathcal{D}^n_k(r)^+ := \left[\begin{array}{ll} \left\{ z \in H^n_+ \subset \mathbb{R}^{1,n} \middle| \operatorname{dist}_{H^n_+}(e_0, z) \le r \right\} & \text{if } k=-1 \\ \left\{ z \in \{e_0\} \times \mathbb{R}^n \subset \mathbb{R}^{n+1} \middle| \operatorname{dist}_{\mathbb{R}^{n+1}}(e_0, z) \le r \right\} & \text{if } k=0 \\ \left\{ z \in S^n \subset \mathbb{R}^{n+1} \middle| \operatorname{dist}_{S^n}(e_0, z) \le r \right\} & \text{if } k=1, \end{array}\right.$$

and

$$\mathcal{D}^n_k(r)^- := \left[\begin{array}{ll} \left\{ z \in H^n_- \subset \mathbb{R}^{1,n} \middle| \operatorname{dist}_{H^n_-}(-e_0, z) \le r \right\} & \text{if } k=-1 \\ \left\{ z \in \{-e_0\} \times \mathbb{R}^n \subset \mathbb{R}^{n+1} \middle| \operatorname{dist}_{\mathbb{R}^{n+1}}(-e_0, z) \le r \right\} & \text{if } k=0 \\ \left\{ z \in S^n \subset \mathbb{R}^{n+1} \middle| \operatorname{dist}_{S^n}(-e_0, z) \le r \right\} & \text{if } k=1. \end{array}\right.$$

Since $r < \frac{\pi}{2}$ when $k = 1$, $\mathcal{D}^n_k(r)^+$ and $\mathcal{D}^n_k(r)^-$ are disjoint in all three cases.

5.3 Basic Tools From Alexandrov Geometry

The notion of strainers [1] in an Alexandrov space forms the core of the calculus arguments used to prove our main theorem. In this section, we review this notion and its relevant consequences. In some sense the idea can be traced back to [26], and some of the ideas that we review first appeared in other sources such as [36] and [40].

Definition 35

Let X *be an Alexandrov space. A point* $x \in X$ *is said to be* (n, δ, r)*–strained* *by the strainer* $\{(a_i, b_i)\}_{i=1}^n \subset X \times X$ *provided that for all* $i \neq j$ *we have*

$$\widetilde{\sphericalangle}(a_i, x, b_j) > \tfrac{\pi}{2} - \delta, \quad \widetilde{\sphericalangle}(a_i, x, b_i) > \pi - \delta,$$

$$\widetilde{\sphericalangle}(a_i, x, a_j) > \tfrac{\pi}{2} - \delta, \quad \widetilde{\sphericalangle}(b_i, x, b_j) > \tfrac{\pi}{2} - \delta, \text{ and}$$

$$\min_{i=1,\ldots,n} \{d(\{a_i, b_i\}, x)\} > r.$$

We say a metric ball $B \subset X$ *is an* (n, δ, r)*–strained neighborhood with strainer* $\{a_i, b_i\}_{i=1}^n$ *provided every point* $x \in B$ *is* (n, δ, r)*–strained by* $\{a_i, b_i\}_{i=1}^n$.

The following is observed in [40].

Proposition 36

Let X *be a compact* n*-dimensional Alexandrov space. Then the following are equivalent.*

1 *There is a (sufficiently small)* $\eta > 0$ *so that for every* $p \in X$

$$d_{GH}\left(\Sigma_p, S^{n-1}\right) < \eta.$$

2 *There is a (sufficiently small)* $\delta > 0$ *and an* $r > 0$ *such that* X *is covered by finitely many* (n, δ, r)*–strained neighborhoods.*

Theorem 37 *([1] Theorem 9.4)*

Let X *be an* n*–dimensional Alexandrov space with curvature bounded from below. Let* $p \in X$ *be* (n, δ, r)*–strained by* $\{(a_i, b_i)\}_{i=1}^n$. *Provided* δ *is small enough, there is a* $\rho > 0$ *such that the map* $f : B(p, \rho) \to \mathbb{R}^n$ *defined by*

$$f(x) = (d(a_1, x), d(a_2, x), \ldots, d(a_n, x))$$

is a bi-Lipschitz embedding with Lipschitz constants in $(1-\tau(\delta,\rho), 1+\tau(\delta,\rho))$.

If every point in X is (n,δ,r)–strained, we can equip X with a C^1–differentiable structure defined by Otsu and Shioya in [27]. The charts will be smoothings of the map from the theorem above and are defined as follows: Let $x \in X$ and choose $\sigma > 0$ so that $B(x,\sigma)$ is (n,δ,r)–strained by $\{a_i,b_i\}_{i=1}^n$. Define $d_{i,x}^\eta : B(x,\sigma) \to \mathbb{R}$ by

$$d_{i,x}^\eta(y) = \frac{1}{vol(B(a_i,\eta))} \int_{z \in B(a_i,\eta)} d(y,z).$$

Then $\varphi_x^\eta : B(x,\sigma) \to \mathbb{R}^n$ is defined by

$$\varphi_x^\eta(y) = (d_{1,x}^\eta(y), \ldots, d_{n,x}^\eta(y)). \tag{5.3}$$

Of course, these are nothing more than the averages considered in part 1. We apologize for the notation change mid-document. However, the perspective has changed. Before, we studied a metric space of averages–the space of average maps, and the functionals were the objects of study. Here, they are tools. We care about the manifolds, not the coordinates.

If B is (n,δ,r)–strained by $\{a_i,b_i\}_{i=1}^n$, any choice of $2n$–directions $\left\{\left(\uparrow_x^{a_i}, \uparrow_x^{b_i}\right)\right\}_{i=1}^n$ where $x \in B$ will be called a set of straining directions for Σ_x. As in, [1, 40], we say an Alexandrov space Σ with $\operatorname{curv}\Sigma \geq 1$ is globally (m,δ)-strained by pairs of subsets $\{A_i,B_i\}_{i=1}^m$ provided

$$|d(a_i,b_j) - \tfrac{\pi}{2}| < \delta, \quad d(a_i,b_i) > \pi - \delta,$$
$$|d(a_i,a_j) - \tfrac{\pi}{2}| < \delta, \quad |d(b_i,b_j) - \tfrac{\pi}{2}| < \delta$$

for all $a_i \in A_i$, $b_i \in B_i$ and $i \neq j$.

Theorem 38 *([1] Theorem 9.5, cf also [26] Section 3)*

Let Σ be an $(n-1)$–dimensional Alexandrov space with curvature ≥ 1. Suppose Σ is globally strained by $\{A_i, B_i\}$. There is a map $\tilde{\Psi} : \mathbb{R}^n \longrightarrow S^{n-1}$ so that $\Psi : \Sigma \to S^{n-1}$ defined by

$$\Psi(x) = \tilde{\Psi} \circ (d(A_1, x), d(A_2, x), \ldots, d(A_n, x))$$

is a bi-Lipschitz homeomorphisms with Lipshitz constants in $(1 - \tau(\delta), 1 + \tau(\delta))$.

Remark 39

The description of $\tilde{\Psi} : \mathbb{R}^n \longrightarrow S^{n-1}$ in [1] is explicit but is geometric rather than via a formula. Combining the proof in [1] with a limiting argument, one can see that the map Ψ can be given by

$$\Psi(x) = \left(\sum \cos^2(d(A_i, x))\right)^{-1/2} (\cos(d(A_1, x)), \ldots, \cos(d(A_n, x))).$$

In particular, the differentials of $\varphi_x^\eta : B(x, \sigma) \subset X \longrightarrow \varphi(B(x, \sigma))$ are almost isometries.

Next we state a powerful lemma showing that for an (n, δ, r) strained neighborhood, angle and comparison angle almost coincide for geodesic hinges with one side in this neighborhood and the other reaching a strainer.

Lemma 40 *([1] Lemma 5.6)*

Let $B \subset X$ be $(1, \delta, r)$–strained by (y_1, y_2). For any $x, z \in B$

$$|\tilde{\sphericalangle}(y_1, x, z) + \tilde{\sphericalangle}(y_2, x, z) - \pi| < \tau(\delta, d(x, z)|r)$$

In particular, for $i = 1, 2,$

$$|\sphericalangle(y_i, x, z) - \tilde{\sphericalangle}(y_i, x, z)| < \tau(\delta, d(x, z)|r).$$

Corollary 41

Let $B \subset X$ be $(1, \delta, r)$–strained by (a, b). Let $\{X^\alpha\}_{\alpha=1}^\infty$ be a sequence of Alexandrov spaces with $\operatorname{curv} X^\alpha \geq k$ such that $X^\alpha \longrightarrow X$. For $x, z \in B$, suppose that $a^\alpha, b^\alpha, x^\alpha, z^\alpha \in X^\alpha$ converge to $a, b, x,$ and z respectively. Then

$$|\sphericalangle(a^\alpha, x^\alpha, z^\alpha) - \sphericalangle(a, x, z)| < \tau(\delta, d(x, z), \tau(1/\alpha | d(x, z)) \mid r).$$

Proof.

The convergence $X^\alpha \longrightarrow X$ implies that we have convergence of the corresponding comparison angles. The result follows from the previous lemma. ∎

Lemma 42

Let $B \subset X$ be (n, δ, r)–strained by $\{(a_i, b_i)\}_{i=1}^n$. Let $\{X^\alpha\}_{\alpha=1}^\infty$ have $\operatorname{curv} X^\alpha \geq k$ and suppose that $X_\alpha \longrightarrow X$. Let $\{(\gamma_{1,\alpha}, \gamma_{2,\alpha})\}_{\alpha=1}^\infty$ be a sequence of geodesic hinges in the X^α that converge to a geodesic hinge (γ_1, γ_2) with vertex in B. Then

$$\left|\sphericalangle\left(\gamma_{1,\alpha}'(0), \gamma_{2,\alpha}'(0)\right) - \sphericalangle\left(\gamma_1'(0), \gamma_2'(0)\right)\right| < \tau(\delta, \tau(1/\alpha | \operatorname{len}(\gamma_1), \operatorname{len}(\gamma_2)) \mid r).$$

Remark 43

Note that without the strainer, $\liminf_{\alpha \to \infty} \sphericalangle\left(\gamma_{1,\alpha}'(0), \gamma_{2,\alpha}'(0)\right) \geq \sphericalangle(\gamma_1'(0), \gamma_2'(0))$ *[11], [1].*

Proof.

Apply the previous corollary with $x^\alpha = \gamma_{1,\alpha}(0)$, $z^\alpha = \gamma_{1,\alpha}(\varepsilon)$, $x^\alpha \to x$, and $z^\alpha \to z$ to conclude

$$\left| \sphericalangle(\Uparrow_{x^\alpha}^{a_i^\alpha}, \gamma'_{1,\alpha}(0)) - \sphericalangle(\Uparrow_x^{a_i}, \gamma'_1(0)) \right| < \tau\left(\delta, d(x,z), \tau\left(1/\alpha | d(x,z)\right) \mid r\right).$$

Similar reasoning with $x^\alpha = \gamma_{2,\alpha}(0)$, $z^\alpha = \gamma_{2,\alpha}(\varepsilon)$, $x = \lim_{\alpha\to\infty} x^\alpha$, and $z = \lim_{\alpha\to\infty} z^\alpha$ gives

$$\left| \sphericalangle(\Uparrow_{x^\alpha}^{a_i^\alpha}, \gamma'_{2,\alpha}(0)) - \sphericalangle(\Uparrow_x^{a_i}, \gamma'_2(0)) \right| < \tau\left(\delta, d(x,z), \tau\left(1/\alpha | d(x,z)\right) \mid r\right).$$

Since $d(x,z)$ may be as small as we please, the result then follows from Theorem 38. $\blacksquare$

Lemma 44 *([40] Lemma 1.8.2)*

Let $\{(a_i,b_i)\}_{i=1}^n$ be an (n,δ,r)–strainer for $B \subset X$. For any $x \in B$ and $\mu > 0$, let Σ_x^μ be the set of directions $v \in \Sigma_x$ so that $\gamma_v|_{[0,\mu]}$ is a segment. For any sufficiently small $\mu > 0$, Σ_x^μ is $\tau(\delta,\mu)$–dense in Σ_x.

Corollary 45

Suppose $X^\alpha \longrightarrow X$, $\{(a_i,b_i)\}_{i=1}^n$ is an (n,δ,r)–strainer for $B \subset X$, and (n,δ,r)–strainers $\{(a_i^\alpha,b_i^\alpha)\}_{i=1}^n$ for $B^\alpha \subset X^\alpha$ satisfy

$$\left(\{(a_i^\alpha,b_i^\alpha)\}_{i=1}^n, B^\alpha\right) \longrightarrow \left(\{(a_i,b_i)\}_{i=1}^n, B\right).$$

For any fixed $\mu > 0$ and any sequence of directions $\{v^\alpha\}_{a=1}^\infty \subset \Sigma_{x^\alpha}$ with $x^\alpha \in B^\alpha$, there is a sequence $\{w^\alpha\}_{a=1}^\infty \subset \Sigma_{x^\alpha}^\mu$ with

$$\sphericalangle(w^\alpha, v^\alpha) < \tau(\delta,\mu)$$

so that a subsequence of $\{\gamma_{w^\alpha}\}_{\alpha=1}^{\infty}$ *converges to a geodesic* $\gamma : [0, \mu] \longrightarrow X$.

From Arzela-Ascoli and Hopf-Rinow, we conclude

Proposition 46

Let X *be an Alexandrov space and* $p, q \in X$. *For any* $\varepsilon > 0$, *there is a* $\delta > 0$ *so that for all* $x \in B(p, \delta)$ *and all* $y \in B(q, \delta)$ *and any segment* xy, *there is a segment* pq *so that*

$$\mathrm{dist}(xy, pq) < \varepsilon.$$

We end this section by showing that convergence to a compact Alexandrov space X without collapse implies the convergence of the corresponding universal covers, provided $|\pi_1(X)| < \infty$. For our purposes, when $X = C^n_{k,r}$, it would be enough to use [33] or [7].

The key tools are Perelman's Stability and Local Structure Theorems and the notion of first systole, which is the length of the shortest closed non-contractible curve. Perelman's proof of the Local Structure Theorem can be found in [29], this result is also a corollary to his Stability Theorem, whose proof is published in [17].

Theorem 47

Let $\{X_i\}_{i=1}^{\infty}$ *be a sequence of* n*–dimensional Alexandrov spaces with a uniform lower curvature bound converging to a compact,* n*–dimensional Alexandrov space* X. *If the fundamental group of* X *is finite, then*

1 *A subsequence of the universal covers,* $\{\tilde{X}_i\}_{i=1}^{\infty}$, *of* $\{X_i\}_{i=1}^{\infty}$ *converges to the universal cover,* $\tilde{X}$, *of* X.

2 *A subsequence of the deck action by* $\pi_1(X_i)$ *on* $\{\tilde{X}_i\}_{i=1}^{\infty}$ *converges to the deck action of* $\pi_1(X)$ *on* $\tilde{X}$.

Proof.

In [29], Perelman shows X is locally contractible. Let $\{U_j\}_{j=1}^n$ be an open cover of X by contractible sets and let μ be a Lebesgue number of this cover. By Perelman's Stability Theorem, there are $\tau\left(\frac{1}{i}\right)$–Hausdorff approximations

$$h_i : X \longrightarrow X_i$$

that are also homeomorphisms. Therefore, if i is sufficiently large, $\{h_i(U_j)\}_{j=1}^n$ is an open cover for X_i by contractible sets with Lebesgue number $\mu/2$. It follows that the first systoles of the X_is are uniformly bounded from below by μ. Since the minimal displacement of the deck transformations by $\pi_1(X_i)$ on $\tilde{X}_i \longrightarrow X_i$ is equal to the first systole of X_i, this displacement is also uniformly bounded from below by μ. By precompactness, a subsequence of $\{\tilde{X}_i\}$ converges to a length space Y. From Proposition 3.6 of [7], a subsequence of the actions $\left(\tilde{X}_i, \pi_1(X_i)\right)$ converges to an isometric action by some group G on Y. By Theorem 2.1 in [6], $X = Y/G$. Since the displacements of the (nontrivial) deck transformations by $\pi_1(X_i)$ on $\tilde{X}_i \longrightarrow X_i$ are uniformly bounded from below, the action by G on Y is properly discontinuous. Hence $Y \longrightarrow Y/G = X$ is a covering space of X. By the Stability Theorem, Y is simply connected, so Y is the universal cover of X. ■

Remark 48

When the X_i are Riemannian manifolds, one can get the uniform lower bound for the systoles of the X_is from the generalized Butterfly Lemma in [10]. The same argument also works in the Alexandrov case but requires Perelman's critical point theory, and hence is no simpler than what we presented above.

Lens spaces show that without the noncollapsing hypothesis this result is false even in constant curvature.

5.4 Cross Cap Stability

The main step to prove Theorem 31 is the following.

Theorem 49

Let $\{M^\alpha\}_{\alpha=1}^\infty$ be a sequence of closed Riemannian n–manifolds with $sec(M^\alpha) \geq k$ so that

$$M^\alpha \longrightarrow C^n_{k,r}$$

in the Gromov-Hausdorff topology. Let $\tilde{M}^\alpha$ be the universal cover of M^α. Then for all but finitely many α, there is a C^1 embedding

$$\tilde{M}^\alpha \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\}$$

that is equivariant with respect to the deck transformations of $\tilde{M}^\alpha \longrightarrow M^\alpha$ and the Z_2–action on R^{n+1} generated by $-id$.

Two and three manifolds have unique differential structures up to diffeomorphism; so in dimensions two and three Theorems 31 and 49 follow from the main result of [12]. We give the proof in dimension 4 in section 6. Until then, we assume that $n \geq 5$.

Proof of Theorem 31 modulo Theorem 49.

By Perelman's Stability Theorem all but finitely many $\{\tilde{M}^\alpha\}_{\alpha=1}^\infty$ are homeomorphic to S^n (cf [12]). Combining this with Theorem 49 and Brown's Theorem 9.7 in [24] gives an H–cobordism between the embedded image of $\tilde{M}^\alpha \subset \mathbb{R}^{n+1}$ and the standard S^n. Modding out by $\mathbb{Z}_2$, we see that M^α and $\mathbb{R}P^n$ are H–cobordant. Since the

Whitehead group of $\mathbb{Z}_2$ is trivial ([19], [25], p. 373), any H–cobordism between M^α and $\mathbb{R}P^n$ is an S–cobordism and hence a product, which completes the proof. [2, 23, 34] $\blacksquare$

The proof of Theorem 31 does not exploit any a priori differential structure on the Crosscap. Instead we exploit a model embedding of the double disk

$$\mathbb{D}^n_k(r) \hookrightarrow \mathbb{R}^{n+1},$$

whose restriction to either half, $\mathcal{D}^n_k(r)^+$ or $\mathcal{D}^n_k(r)^-$, is the identity on the last n–coordinates. By describing the identity $\mathcal{D}^n_k(r) \longrightarrow \mathcal{D}^n_k(r)$ in terms of distance functions, we then argue that this embedding can be lifted to all but finitely many of a sequence $\{M^\alpha\}$ converging to $\mathbb{D}^n_k(r)$.

The Model Embedding

Let $A : \mathbb{D}^n_k(r) \to \mathbb{D}^n_k(r)$ be the free involution mentioned in Example 29. For $z \in \mathbb{D}^n_k(r)$, we define $f_z : \mathbb{D}^n_k(r) \to \mathbb{R}$ by

$$f_z(x) = h_k \circ \operatorname{dist}(A(z), x) - h_k \circ \operatorname{dist}(z, x) \tag{5.4}$$

where $h_k : \mathbb{R} \to \mathbb{R}$ is defined as

$$h_k(x) = \begin{cases} \frac{1}{2\sinh r}\cosh(x) & \text{if } k = -1 \\ \frac{x^2}{4r} & \text{if } k = 0 \\ \frac{1}{2\sin r}\cos(x) & \text{if } k = 1. \end{cases}$$

Recall that we view $\mathcal{D}^n_k(r)^\pm$ as metric r-balls centered at $p_0 = e_0$ and $A(p_0) =$

$-e_0$ in either $H^n_{\pm}$, $\{\pm e_0\} \times \mathbb{R}^n$, or S^n. For $i = 1, 2, \ldots, n$ we set

$$p_i := \begin{cases} \cosh(r)e_0 + \sinh(r)e_i & \text{if } k = -1 \\ e_0 + re_i & \text{if } k = 0 \\ \cos(r)e_0 - \sin(r)e_i & \text{if } k = 1. \end{cases} \tag{5.5}$$

The functions $\{f_i\}_{i=1}^n := \{f_{p_i}\}_{i=1}^n$ are then restrictions of the last n–coordinate functions of $\mathbb{R}^{n+1}$ to $\mathcal{D}^n_k(r)^{\pm}$. We set $f_0 := f_{p_0}$. In contrast to $f_1, \ldots, f_n$, our f_0 is not a coordinate function. On the other hand its gradient is well defined everywhere on $\mathbb{D}^n_k(r) \setminus \{p_0, A(p_0)\}$, even on $\partial\mathcal{D}^n_k(r)^+ = \partial\mathcal{D}^n_k(r)^-$ where it is normal to $\partial\mathcal{D}^n_k(r)^+ = \partial\mathcal{D}^n_k(r)^-$.

Define $\Phi : \mathbb{D}^n_k(r) \to \mathbb{R}^{n+1}$, by

$$\Phi = (f_0, f_1, f_2, \cdots, f_n),$$

and observe that

Proposition 50

Φ is a continuous, $\mathbb{Z}_2$–equivariant embedding.

Proof.

Write $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ and let $\pi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be projection. Since $f_1, f_2, \cdots, f_n$ are coordinate functions, the restrictions

$$\pi \circ \Phi|_{\mathcal{D}^n_k(r)^{\pm}} : \mathcal{D}^n_k(r)^{\pm} \longrightarrow \mathbb{R}^n$$

are both the identity. From this and the definition of f_0, we conclude that Φ is one–to–one. Since $\mathbb{D}^n_k(r)$ is compact, it follows that Φ is an embedding. The $\mathbb{Z}_2$–equivariance is immediate from definition 5.4. ■

Lifting the Model Embedding

To start the proof of Theorem 49 let $\{M^{\alpha}\}_{\alpha=1}^{\infty}$ be a sequence of closed Riemannian n–manifolds with $sec(M^{\alpha}) \geq k$ so that

$$M^{\alpha} \longrightarrow C_{k,r}^{n},$$

and we let $\{\tilde{M}^{\alpha}\}_{\alpha=1}^{\infty}$ denote the corresponding sequence of universal covers. From Theorem 47, a subsequence of $\{\tilde{M}^{\alpha}\}_{\alpha=1}^{\infty}$ together with the deck transformations $\tilde{M}^{\alpha} \longrightarrow M^{\alpha}$ converge to $(\mathbb{D}_{k}^{n}(r), A)$. For all but finitely many α, $\pi_{1}(M^{\alpha})$ is isomorphic to $\mathbb{Z}_{2}$. We abuse notation and call the nontrivial deck transformation of $\tilde{M}^{\alpha} \longrightarrow M^{\alpha}$, A.

First we extend definition 5.4 by letting $f_{z}^{\alpha} : \tilde{M}^{\alpha} \to \mathbb{R}$ be defined by

$$f_{z}^{\alpha}(x) = h_{k} \circ \operatorname{dist}(A(z), x) - h_{k} \circ \operatorname{dist}(z, x). \tag{5.6}$$

Let $p_{i}^{\alpha} \in \tilde{M}^{\alpha}$ converge to $p_{i} \in \mathbb{D}_{k}^{n}(r)$, and for some $d > 0$ define $f_{i,d}^{\alpha} : \tilde{M}^{\alpha} \to \mathbb{R}$ by

$$f_{i,d}^{\alpha}(x) = \frac{1}{vol(B(p_{i}^{\alpha}, d))} \int_{q^{\alpha} \in B(p_{i}^{\alpha}, d)} f_{q^{\alpha}}^{\alpha}(x). \tag{5.7}$$

Differentiation under the integral gives

Proposition 51

The $f_{i,d}^{\alpha}$ are C^{1} and $\left|\nabla f_{i,d}^{\alpha}\right| \leq 2$.

We now define $\Phi_{d}^{\alpha} : \tilde{M}^{\alpha} \to \mathbb{R}^{n+1}$ by

$$\Phi_{d}^{\alpha} = \left(f_{0,d}^{\alpha}, f_{1,d}^{\alpha}, f_{2,d}^{\alpha}, \cdots, f_{n,d}^{\alpha}\right).$$

As $\alpha \to \infty$ and $d \to 0$, Φ_d^α converges to Φ in the Gromov–Hausdorff sense. Since Φ is an embedding it follows that Φ_d^α is one–to–one in the large. More precisely,

Proposition 52

For any $\nu > 0$, if α is sufficiently large and d is sufficiently small, then

$$\Phi_d^\alpha (x) \neq \Phi_d^\alpha (y),$$

provided $\operatorname{dist}(x, y) > \nu$.

Since the $\mathbb{Z}_2$-equivariance of Φ_d^α immediately follows from definition 5.7, all that remains to prove Theorem 49 is the following proposition:

Proposition 53

There is a $\rho > 0$ so that Φ_d^α is one to one on all ρ–balls, provided that α is sufficiently large and d is sufficiently small.

This is a consequence of Key Lemma 55 (stated below), whose statement and proof occupy the remainder of this section.

Uniform Immersion

The proof of the Inverse Function Theorem in [31] gives

Theorem 54 *(Quantitative Immersion Theorem)*

Let

$$\mathbb{R}^n_{\hat{i}} := \{(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{n+1})\} \subset \mathbb{R}^{n+1}$$

and let

$$P_{\hat{i}} : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n_{\hat{i}}$$

be orthogonal projection.

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ be a C^1 map so that for some $a \in \mathbb{R}^n$, $\lambda > 0$, and $\rho > 0$, there is an $i \in \{1, \ldots, n+1\}$ so that

$$|d\,(P_{\hat{\imath}} \circ F)_a\,(v)| \geq \lambda\,|v|$$

and

$$|d\,(P_{\hat{\imath}} \circ F)_a\,(v) - d\,(P_{\hat{\imath}} \circ F)_x\,(v)| < \frac{\lambda}{2}\,|v|$$

for all $x \in B\,(a, \rho)$ and $v \in \mathbb{R}^n$, then $(P_{\hat{\imath}} \circ F)\,|_{B(a,\rho)}$ is a one–to–one, open map.

We note that every space of directions to $\mathbb{D}^n_k(r)$ is isometric to S^{n-1}. By proposition 36, there are $r, \delta > 0$ so that every point in the double disk has a neighborhood B that is (n, δ, r)–strained. If $B \subset \mathbb{D}^n_k(r)$ is (n, δ, r)–strained by $\{a_i, b_i\}_{i=1}^n$, by continuity of comparison angles, we may assume there are sets $B^\alpha \subset \tilde{M}^\alpha$ (n, δ, r)–strained by $\{a_i^\alpha, b_i^\alpha\}_{i=1}^n$ such that

$$\left(\{(a_i^\alpha, b_i^\alpha)\}_{i=1}^n, B^\alpha\right) \longrightarrow \left(\{(a_i, b_i)\}_{i=1}^n, B\right).$$

Given $x^\alpha \in B^\alpha$, we let $\varphi^\eta_{x^\alpha}$ be as in 5.3.

To prove Proposition 53 it suffices to prove the following.

Key Lemma 55

There is a $\lambda > 0$ and $\rho > 0$ so that for all $x^\alpha \in \tilde{M}^\alpha$ there is an $i_{x^\alpha} \in \{0, 1, \ldots, n\}$ such that the function $F := \Phi^\alpha_d \circ (\varphi^\eta_{x^\alpha})^{-1}$ satisfies

1.

$$\left|d(P_{\hat{\imath}_{x^\alpha}} \circ F)_{\varphi^\eta_{x^\alpha}(x^\alpha)}\,(v)\right| > \lambda\,|v|$$

and

2.

$$\left| d\left(P_{\hat{\imath}_{x^\alpha}} \circ F\right)_{\varphi^{\eta}_{x^\alpha}(y)}(v) - d\left(P_{\hat{\imath}_{x^\alpha}} \circ F\right)_{\varphi^{\eta}_{x^\alpha}(x^\alpha)}(v) \right| < \frac{\lambda}{2}|v|$$

for all $y \in B(x^\alpha, \rho)$ and $v \in \mathbb{R}^n$, provided that α is sufficiently large and d and η are sufficiently small.

We show in the next subsection that part 1 of Key Lemma 55 holds, and in the following subsection we show that part 2 holds.

Lower bound on the differential

We begin by illustrating that the first part of the key lemma holds for the model embedding.

Lemma 56

There is a $\lambda > 0$ so that for all $v \in T\mathbb{D}^n_k(r)$ there is a $j(v) \in \{0, 1, \ldots, n\}$ so that

$$\left| D_v f_{j(v)} \right| > \lambda |v|.$$

Proof.

Recall that the double disk $\mathbb{D}^n_k(r)$ is the union of two copies of $\mathcal{D}^n_k(r)$ that we call $\mathcal{D}^n_k(r)^+$ and $\mathcal{D}^n_k(r)^-$—glued along their common boundary—that throughout this section we call $\mathcal{S} := \partial\mathcal{D}^n_k(r)^{\pm}$.

If $x \in \mathbb{D}^n_k(r) \setminus \mathcal{S}$, then for $i \neq 0$, ∇f_i is unambiguously defined; moreover,

$$\{\nabla f_i(x)\}_{i=1}^n$$

is an orthonormal basis. Thus the lemma certainly holds on $\mathbb{D}^n_k(r) \setminus \mathcal{S}$.

For $x \in \mathcal{S}$ and $i \in \{1, \ldots, n\}$, we can think of the gradient of f_i as multivalued. More precisely, for $x \in \mathcal{S}$, we view

$$\mathcal{S} \subset \mathcal{D}_k^n(r)^{\pm} \subset \begin{cases} H_{\pm}^n & \text{if } k = -1 \\ \{\pm e_0\} \times \mathbb{R}^n & \text{if } k = 0 \\ S^n & \text{if } k = 1 \end{cases}$$

and define $\nabla f_i^{\pm}$ to be the gradient at x of the coordinate function that extends f_i to either $H_{\pm}^n, \{\pm e_0\} \times \mathbb{R}^n$, or S^n.

From definition 5.4, for any $v \in T_x \mathbb{D}_k^n(r)$

$$D_v f_i = \begin{cases} \left\langle \nabla f_i^{+}, v \right\rangle & \text{if } v \text{ is inward to } \mathcal{D}_k^n(r)^{+} \\ \left\langle \nabla f_i^{-}, v \right\rangle & \text{if } v \text{ is inward to } \mathcal{D}_k^n(r)^{-}. \end{cases}$$

Notice that the projections of ∇f_i^{+} and ∇f_i^{-} onto $T_x \mathcal{S}$ coincide, so for $v \in T_x \mathcal{S}$ we have $D_v f_i = \left\langle \nabla f_i^{+}, v \right\rangle = \left\langle \nabla f_i^{-}, v \right\rangle$. As $\left\{\nabla f_i^{+}\right\}_{i=1}^{n}$ is an orthonormal basis, the lemma holds for $v \in T\mathcal{S}$ and hence also for v in a neighborhood U of $T\mathcal{S} \subset T\mathbb{D}_k^n(r)|_{\mathcal{S}}$. Since ∇f_0 is well defined on $\mathcal{S}$ and normal to $\mathcal{S}$, for any unit $v \in T\mathbb{D}_k^n(r)|_{\mathcal{S}} \setminus U$, we have $|D_v f_0| > 0$. The lemma follows from the compactness of the set of unit vectors in $T\mathbb{D}_k^n(r)|_{\mathcal{S}} \setminus U$. ■

Notice that at p_k and $A(p_k)$ the gradients of f_k and f_0 are colinear. Using this we conclude

Addendum 57

Let p_k be any of $p_1, \ldots p_n$. There is an $\varepsilon > 0$ so that for all $x \in B(p_k, \varepsilon) \cup B(A(p_k), \varepsilon)$ and all $v \in T_x \mathbb{D}_k^n(r)$, the index $j(v)$ in the previous lemma can be chosen to be different from k.

Lemma 58

There is a $\lambda > 0$ so that for all $v \in T_x\mathbb{D}^n_k(r)$ there is a $j\,(v) \in \{0, 1, \ldots, n\}$ so that

$$|D_v f_z| > \lambda\,|v|$$

for all $z \in B(p_{j(v)}, d)$, provided d is sufficiently small.

Proof.

If not then for each $i = 0, 1, \ldots, n$ there is a sequence $\{z_i^j\}_{j=1}^{\infty} \subset \mathbb{D}^n_k(r)$ with $\mathrm{dist}(z_i^j, p_i) < \frac{1}{j}$ and a sequence of unit $v^j \in T_{x^j}\mathbb{D}^n_k(r)$ so that

$$\left|D_{v^j} f_{z_i^j}\right| < \frac{1}{j}.$$

Choose the segments $x^j z_i^j$ and $x^j A\left(z_i^j\right)$ so that

$$\begin{aligned} \sphericalangle\left(\uparrow_{x^j}^{z_i^j},\ v^j\right) &= \sphericalangle\left(\Uparrow_{x^j}^{z_i^j},\ v^j\right) \text{ and} \\ \sphericalangle\left(\uparrow_{x^j}^{A(z_i^j)},\ v^j\right) &= \sphericalangle\left(\Uparrow_{x^j}^{A(z_i^j)},\ v^j\right). \end{aligned}$$

After passing to subsequences, we have $v^j \to v$, $x^j \to x$ and

$$\begin{aligned} x^j z_i^j &\to xp_i \\ x^j A\left(z_i^j\right) &\to xA\left(p_i\right), \end{aligned}$$

for some choice of segments xp_i and $xA(p_i)$. Using Lemma 42 and Corollary 45 we conclude

$$\begin{aligned}\left|\sphericalangle\left(\uparrow_{x^j}^{z_i^j}, v^j\right) - \sphericalangle\left(\uparrow_x^{p_i}, v\right)\right| &< \tau\left(\delta, \tau\left(\frac{1}{j}\middle|\operatorname{dist}(x, p_i)\right)\right), \\ \left|\sphericalangle\left(\uparrow_{x^j}^{A\left(z_i^j\right)}, v^j\right) - \sphericalangle\left(\uparrow_x^{A(p_i)}, v\right)\right| &< \tau\left(\delta, \tau\left(\frac{1}{j}\middle|\operatorname{dist}(x, A(p_i))\right)\right).\end{aligned} \tag{5.8}$$

If $x \notin \mathcal{S}$, then the segments xp_i and $xA(p_i)$ are unambiguously defined, and so the previous inequality and the hypothesis $\left|D_{v^j} f_{z_i^j}\right| < \frac{1}{j}$, contradict the previous lemma and its addendum.

If $x \in \mathcal{S}$ and $v \in T_x\mathcal{S}$, then

$$\sphericalangle\left(\uparrow_x^{p_i}, v\right) \text{ and } \sphericalangle\left(\uparrow_x^{A(p_i)}, v\right)$$

are independent of the choice of the segments xp_i and $xA(p_i)$, so the hypothesis $\left|D_{v^j} f_{z_i^j}\right| < \frac{1}{j}$ together with the Inequalities 5.8 contradict the previous lemma and its addendum. Thus our result holds for $v \in T\mathcal{S}$ and hence also for v in a neighborhood U of $T\mathcal{S} \subset T\mathbb{D}_k^n(r)|_{\mathcal{S}}$.

For a unit vector $v \in T\mathbb{D}_k^n(r)|_{\mathcal{S}} \setminus U$, we saw in the proof of the previous lemma that for some $\lambda > 0$

$$|D_v f_0| > \lambda. \tag{5.9}$$

For $x \in \mathcal{S}$, we have unique segments xp_0 and $xA(p_0)$, so the hypothesis $\left|D_{v^j} f_{z_i^j}\right| < \frac{1}{j}$ and inequalities 5.8 contradict Inequality 5.9. ■

Combining the proof of the previous lemma with Addendum 57, we get

Addendum 59

Let p_k be any of $p_1, \ldots p_n$. There is an $\varepsilon > 0$ so that for all $x \in B(p_k, \varepsilon) \cup B(A(p_k), \varepsilon)$ and all $v \in T_x \mathbb{D}_k^n(r)$, the index $j(v)$ in the previous lemma can be chosen to be different from k.

Lemma 60

There is a $\lambda > 0$ so that for all $v \in T\tilde{M}^\alpha$ there is a $j(v) \in \{0, 1, \ldots, n\}$ so that

$$D_v f^\alpha_{j(v),d} > \lambda |v|,$$

provided α is sufficiently large and d is sufficiently small.

Proof.

If the lemma were false, then there would be a sequence of unit vectors $\{v^\alpha\}_{\alpha=1}^\infty$ with $v^\alpha \in T_{x^\alpha}\tilde{M}^\alpha$ such that for all i,

$$\left|D_{v^\alpha} f^\alpha_{i,d}\right| < \tau\left(\frac{1}{\alpha}, d\right).$$

Let $\lim_{\alpha\to\infty} x^\alpha = x \in \mathbb{D}_k^n(r)$. By Corollary 45, for any $\mu > 0$ there is a sequence $\{w^\alpha\}_{\alpha=1}^\infty$ with $w^\alpha \in \Sigma^\mu_{x^\alpha}$ such that

$$\sphericalangle(v^\alpha, w^\alpha) < \tau(\delta, \mu).$$

Since $\left|\nabla f^\alpha_{i,d}\right| \leq 2$,

$$\left|D_{w^\alpha} f^\alpha_{i,d}\right| < \tau\left(\delta, \mu, \frac{1}{\alpha}, d\right) \tag{5.10}$$

for all i. After passing to a subsequence, we conclude that $\left\{\gamma_{w^\alpha}|_{[0,\mu]}\right\}_{\alpha=1}^\infty$ converges to a segment $\gamma_w|_{[0,\mu]}$. By the previous lemma, there is a $\lambda > 0$ and a $j(w)$ so that for all

$z \in B(p_{j(w)}, d)$,

$$|D_w f_z| > \lambda |w|, \tag{5.11}$$

provided d is small enough. Moreover, by Addendum 59 we may assume that

$$\begin{aligned} \operatorname{dist}\left(x, p_{j(w)}\right) &> 100d > \mu \text{ and} \\ \operatorname{dist}\left(x, A\left(p_{j(w)}\right)\right) &> 100d > \mu. \end{aligned} \tag{5.12}$$

By the Mean Value Theorem, there is a $z^{\alpha}_{j(w)} \in B\left(p^{\alpha}_{j(w)}, d\right)$ with

$$D_{w^{\alpha}} f^{\alpha}_{z^{\alpha}_{j(w)}} = D_{w^{\alpha}} f^{\alpha}_{j(w),d}. \tag{5.13}$$

Choose segments $x^{\alpha} z^{\alpha}_{j(w)}$ and $x^{\alpha} A(z^{\alpha}_{j(w)})$ in $\tilde{M}^{\alpha}$ so that

$$\begin{aligned} \sphericalangle\left(\uparrow_{x^{\alpha}}^{z^{\alpha}_{j(w)}}, w^{\alpha}\right) &= \sphericalangle\left(\Uparrow_{x^{\alpha}}^{z^{\alpha}_{j(w)}}, w^{\alpha}\right) \text{ and} \\ \sphericalangle\left(\uparrow_{x^{\alpha}}^{A(z^{\alpha}_{j(w)})}, w^{\alpha}\right) &= \sphericalangle\left(\Uparrow_{x^{\alpha}}^{A(z^{\alpha}_{j(w)})}, w^{\alpha}\right). \end{aligned}$$

After passing to a subsequence, we may assume that for some $z_{j(w)} \in B(p_{j(w)}, d)$, $x^{\alpha} z^{\alpha}_{j(w)}$ and $x^{\alpha} A(z^{\alpha}_{j(w)})$ converge to segments $x z_{j(w)}$ and $xA(z_{j(w)})$, respectively. By Lemma 42,

$$\begin{aligned} \left|\sphericalangle(\uparrow_{x^{\alpha}}^{z^{\alpha}_{j(w)}}, \gamma'_{w^{\alpha}}(0)) - \sphericalangle(\uparrow_{x}^{z_{j(w)}}, \gamma'_{w}(0))\right| &< \tau\left(\delta, \tau\left(1/\alpha|\mu, \operatorname{dist}\left(x, z_{j(w)}\right)\right)\right) \\ \left|\sphericalangle(\uparrow_{x^{\alpha}}^{A(z^{\alpha}_{j(w)})}, \gamma'_{w^{\alpha}}(0)) - \sphericalangle(\uparrow_{x}^{A\left(z_{j(w)}\right)}, \gamma'_{w}(0))\right| &< \tau\left(\delta, \tau\left(1/\alpha|\mu, \operatorname{dist}\left(x, A\left(z_{j(w)}\right)\right)\right)\right). \end{aligned}$$

Combining the previous two sets of displays with 5.12

$$\left|D_{w^{\alpha}} f^{\alpha}_{z^{\alpha}_{j(w)}} - D_w f_{z_{j(w)}}\right| < \tau\left(\delta, \tau\left(1/\alpha|\mu\right)\right). \tag{5.14}$$

So by Equation 5.13,

$$\left| D_{w^\alpha} f^\alpha_{j(w),d} - D_w f_{z_{j(w)}} \right| < \tau\left(\delta, \tau\left(1/\alpha|\mu\right)\right),$$

but this contradicts Inequalities 5.10 and 5.11. ■

The first claim of Key Lemma 55 follows by combining the previous lemma with the fact that the differentials of the $\varphi^\eta_{x^\alpha}$'s are almost isometries.

Remark 61

Note that when x^α is close to p_k or $A\left(p_k\right)$, the desired estimate

$$\left| d(P_{\hat{i}_{x^\alpha}} \circ F)_{\varphi^\eta_{x^\alpha}(x^\alpha)}\left(v\right) \right| > \lambda \left|v\right|$$

holds with $P_{\hat{i}_{x^\alpha}} = P_{\hat{k}}$. This follows from Addendum 59 and the proof of the previous lemma.

Equicontinuity of Differentials

In this subsection, we establish the second part of the key lemma. If x^α is not close to one of the $p_k s$ or $A\left(p_k\right) s$ we will show the stronger estimate

$$\left| d\left(F\right)_{\varphi^\eta_{x^\alpha}(y)}\left(v\right) - d\left(F\right)_{\varphi^\eta_{x^\alpha}(x^\alpha)}\left(v\right) \right| < \frac{\lambda}{2}\left|v\right|. \tag{5.15}$$

So at such points, the second part of the key lemma holds with *any* choice of coordinate projection $P_{\hat{i}_{x^\alpha}}$.

For x^α close to p_k or $A\left(p_k\right)$, we will show

$$\left| d\left(P_{\hat{k}} \circ F\right)_{\varphi^\eta_{x^\alpha}(y)}\left(v\right) - d\left(P_{\hat{k}} \circ F\right)_{\varphi^\eta_{x^\alpha}(x^\alpha)}\left(v\right) \right| < \frac{\lambda}{2}\left|v\right|, \tag{5.16}$$

where λ is the constant whose existence was established in the previous section. Together with remark 61, this will establish the key lemma.

Suppose $B \subset \mathbb{D}^n_k(r)$ is (n,δ,r)–strained by $\{(a_i,b_i)\}_{i=1}^n$. Let $x,y \in B$ and let

$$\varphi^\eta : B \longrightarrow \mathbb{R}^n$$

be the map defined in 5.3 and [27]. Set

$$P_{x,y} := (d\varphi^\eta)_y^{-1} \circ (d\varphi^\eta)_x : T_x\mathbb{D}^n_k(r) \to T_y\mathbb{D}^n_k(r).$$

It follows that $P_{x,y}$ is a $\tau(\delta,\eta)$–isometry.

Lemma 62

Let $B \subset \mathbb{D}^n_k(r)$ be (n,δ,r)–strained by $\{(a_i,b_i)\}_{i=1}^n$. Given $\varepsilon > 0$ and $x \in B$, there is a $\rho(x,\varepsilon) > 0$ so that the following holds.

For all $k \in \{0,1,\ldots,n\}$, there is a subset $E_{k,x} \subset \{B(p_k,d) \cup B(A(p_k),d)\}$ with measure $\mu(E_{k,x}) < \varepsilon$ so that for all $z \in B(p_k,d) \setminus E_{k,x}$, all $y \in B(x,\rho(x,\varepsilon))$, and all $v \in \Sigma_x$,

$$\begin{aligned} \left|\sphericalangle\left(v,\uparrow_x^z\right) - \sphericalangle\left(P_{x,y}(v),\uparrow_y^z\right)\right| &< \tau\left(\varepsilon,\delta,\eta\middle|\operatorname{dist}(x,z)\right) \text{ and} \\ \left|\sphericalangle\left(v,\uparrow_x^{A(z)}\right) - \sphericalangle\left(P_{x,y}(v),\uparrow_y^{A(z)}\right)\right| &< \tau\left(\varepsilon,\delta,\eta\middle|\operatorname{dist}(x,A(z))\right). \end{aligned}$$

Proof.

Let $C_x = \{z | z \in \operatorname{Cutlocus}(x) \text{ or } A(z) \in \operatorname{Cutlocus}(x)\}$ and set

$$E_{k,x} = B(C_x,\nu) \cap \{B(p_k,d) \cup B(A(p_k),d)\}.$$

Choose $\nu > 0$ so that $\mu(E_{k,x}) < \varepsilon$.

By Proposition 46, for each $z \in B(p_k, d) \setminus E_{k,x}$, there is a $\rho(x, z, \varepsilon)$ so that for all $y \in B(x, \rho(x, z, \varepsilon))$ and any choice of segment zy,

$$\mathrm{dist}(zx, zy) < \varepsilon,$$

where zx is the unique segment from z to x.

Making $\rho(x, z, \varepsilon)$ smaller and using Corollary 41, it follows that for any $\tilde{a}_i, \bar{a}_i \in B(a_i, \eta)$,

$$\begin{aligned} \left|\sphericalangle\left(\Uparrow_x^{\tilde{a}_i}, \uparrow_x^z\right) - \sphericalangle\left(\Uparrow_y^{\bar{a}_i}, \uparrow_y^z\right)\right| &< \tau(\delta, \varepsilon, \eta | \mathrm{dist}(x, z), \mathrm{dist}(y, z)) \\ &= \tau(\delta, \varepsilon, \eta | \mathrm{dist}(x, z)). \end{aligned}$$

It follows that

$$\left|(d\varphi^\eta)_x(\uparrow_x^z) - (d\varphi^\eta)_y\left(\uparrow_y^z\right)\right| < \tau(\delta, \varepsilon, \eta | \mathrm{dist}(x, z)),$$

and hence

$$\sphericalangle\left(P_{x,y}(\uparrow_x^z), \uparrow_y^z\right) = \sphericalangle\left((d\varphi^\eta)_y^{-1} \circ (d\varphi^\eta)_x(\uparrow_x^z), \left(\uparrow_y^z\right)\right) < \tau(\delta, \varepsilon, \eta | \mathrm{dist}(x, z)).$$

So for any $v \in \Sigma_x$,

$$\begin{aligned} \left|\sphericalangle(v, \uparrow_x^z) - \sphericalangle\left(P_{x,y}(v), \uparrow_y^z\right)\right| &\leq |\sphericalangle(v, \uparrow_x^z) - \sphericalangle(P_{x,y}(v), P_{x,y}(\uparrow_x^z))| + \\ &\quad \left|\sphericalangle(P_{x,y}(v), P_{x,y}(\uparrow_x^z)) - \sphericalangle\left(P_{x,y}(v), \uparrow_y^z\right)\right| \\ &< \tau(\delta, \eta) + \tau(\varepsilon, \delta, \eta | \mathrm{dist}(x, z)) \\ &= \tau(\varepsilon, \delta, \eta | \mathrm{dist}(x, z)). \end{aligned}$$

Using Proposition 46 and the precompactness of $B(p_k, d) \setminus E_{k,x}$, we can then choose $\rho(x, z, \varepsilon)$ to be independent of $z \in B(p_k, d) \setminus E_{k,x}$. A similar argument gives the second inequality. ∎

Corollary 63

Given any $\varepsilon > 0$, there is a $\rho(\varepsilon) > 0$ so that for any $x \in \mathbb{D}^n_k(r)$, $y \in B(x, \rho(\varepsilon))$, and $z \in B(p_i, d) \setminus E_{i,x}$, we have

$$\left|D_v f_z - D_{P_{x,y}(v)} f_z\right| < \tau\left(\varepsilon, \delta, \eta | \mathrm{dist}\,(z, x)\,, \mathrm{dist}\,(A\,(z)\,, x)\right)$$

for all unit vectors $v \in \Sigma_x$.

Proof.

Since $\mathbb{D}^n_k(r)$ is compact, the $\rho(\varepsilon, x)$ from the previous lemma can be chosen to be independent of x.

Given $x \in \mathbb{D}^n_k(r)$, $y \in B(x, \rho(\varepsilon))$, and $v \in \Sigma_x$, choose segments yz and $yA(z)$ so that

$$\begin{aligned} \sphericalangle\left(\uparrow^z_y, P_{x,y}(v)\right) &= \sphericalangle\left(\Uparrow^z_y, P_{x,y}(v)\right) \text{ and} \\ \sphericalangle\left(\uparrow^{A(z)}_y, P_{x,y}(v)\right) &= \sphericalangle\left(\Uparrow^{A(z)}_y, P_{x,y}(v)\right). \end{aligned}$$

Since the segments xz and $xA(z)$ are unique, the result follows from the formula for directional derivatives of distance functions, the previous lemma, and the chain rule. ∎

We can lift a strainer from $\mathbb{D}^n_k(r)$ to any $\tilde{M}^\alpha$ if $\mathrm{dist}_{GH}\left(\tilde{M}^\alpha, \mathbb{D}^n_k(r)\right)$ is sufficiently small. So if x^α and y^α are sufficiently close, we define

$$P_{x^\alpha, y^\alpha} := (d\varphi^\eta)^{-1}_{y^\alpha} \circ (d\varphi^\eta)_{x^\alpha} : T_{x^\alpha}\tilde{M}^\alpha \to T_{y^\alpha}\tilde{M}^\alpha.$$

Lemma 64

Let i be in $\{0,\ldots,n\}$. There is a $\rho>0$ so that for any $x^{\alpha}\in\tilde{M}^{\alpha}$, any $y^{\alpha}\in B(x^{\alpha},\rho)$, and any unit $v^{\alpha}\in T_{x^{\alpha}}\tilde{M}^{\alpha}$ we have

$$\left|D_{v^{\alpha}}f_{i,d}^{\alpha}-D_{P_{x^{\alpha},y^{a}}(v^{\alpha})}f_{i,d}^{\alpha}\right|<\tau\left(\rho,\frac{1}{\alpha},\delta,\eta|\mathrm{dist}\left(x^{\alpha},p_{i}^{\alpha}\right),\mathrm{dist}\left(x^{\alpha},A\left(p_{i}^{\alpha}\right)\right)\right),$$

provided d is sufficiently small.

Proof.

If not, then for any $\rho>0$ and some $i=0,1,\ldots,n$, there would be a sequence of points $x^{\alpha}\to x\in\mathbb{D}_{k}^{n}(r)$, a sequence of unit vectors $\{v^{\alpha}\}_{\alpha=1}^{\infty}$ and a constant $C>0$ that is independent of α, δ, and η so that

$$\begin{aligned}\left|D_{v^{\alpha}}f_{i,d}^{\alpha}-D_{P_{x^{\alpha},y^{a}}(v^{\alpha})}f_{i,d}^{\alpha}\right| &\geq C,\\ \mathrm{dist}\left(x,p_{i}\right) &\geq C,\ \text{and}\\ \mathrm{dist}\left(x,A\left(p_{i}\right)\right) &\geq C \end{aligned}\tag{5.17}$$

for some $y^{\alpha}\in B(x^{\alpha},\rho)$. Choose $\varepsilon>0$ and take $\rho<\rho(\varepsilon)$ where $\rho(\varepsilon)$ is from the previous corollary. We assume $B\left(x,\rho(\varepsilon)\right)$ is (n,δ,r)–strained. Let $y=\lim y^{\alpha}$ and $\mu>0$ be sufficiently small. By corollary 45, there are sequences $\{w^{\alpha}\}_{\alpha=1}^{\infty}\in\Sigma_{x^{\alpha}}^{\mu}$ and $\{\tilde{w}^{\alpha}\}_{\alpha=1}^{\infty}\in\Sigma_{y^{\alpha}}^{\mu}$ so that

$$\begin{aligned}\sphericalangle\left(v^{\alpha},w^{\alpha}\right) &< \tau\left(\delta,\mu\right)\\ \sphericalangle\left(P_{x^{\alpha},y^{a}}\left(w^{\alpha}\right),\tilde{w}^{\alpha}\right) &< \tau\left(\delta,\mu\right)\end{aligned}\tag{5.18}$$

and subsequences $\{\gamma_{w^{\alpha}}\}_{\alpha=1}^{\infty}$ and $\{\gamma_{\tilde{w}^{\alpha}}\}_{\alpha=1}^{\infty}$ converging to segments γ_{w} and $\gamma_{\tilde{w}}$ that are parameterized on $[0, \mu]$. Since $|\nabla f_{i,d}^{\alpha}| \leq 2$, we may assume for a possibly smaller constant C that

$$\left|D_{w^{\alpha}} f_{i,d}^{\alpha} - D_{\tilde{w}^{\alpha}} f_{i,d}^{\alpha}\right| \geq C.$$

Thus for some $z^{\alpha} \in B(p_i^{\alpha}, d)$ with $\mathrm{dist}_{\mathrm{Haus}}(z^{\alpha}, E_{i,x}) > 2\nu$,

$$|D_{w^{\alpha}} f_{z^{\alpha}}^{\alpha} - D_{\tilde{w}^{\alpha}} f_{z^{\alpha}}^{\alpha}| \geq \frac{C}{2}. \tag{5.19}$$

Passing to a subsequence, we have $z^{\alpha} \to z \in B(p_i, d) \setminus E_{i,x}$. As in the proof of Lemma 60 (Inequality 5.14), we have

$$\begin{aligned} |D_{w^{\alpha}} f_{z^{\alpha}}^{\alpha} - D_w f_z| &< \tau(\delta, \tau(1/\alpha|\mu)) \text{ and} \\ |D_{\tilde{w}^{\alpha}} f_{z^{\alpha}}^{\alpha} - D_{\tilde{w}} f_z| &< \tau(\delta, \tau(1/\alpha|\mu)). \end{aligned}$$

Thus,

$$\begin{aligned} |D_{w^{\alpha}} f_{z^{\alpha}}^{\alpha} - D_{\tilde{w}^{\alpha}} f_{z^{\alpha}}^{\alpha}| &\leq |D_{w^{\alpha}} f_{z^{\alpha}}^{\alpha} - D_w f_z| + |D_w f_z - D_{\tilde{w}} f_z| + |D_{\tilde{w}} f_z - D_{\tilde{w}^{\alpha}} f_{z^{\alpha}}^{\alpha}| \\ &< |D_w f_z - D_{\tilde{w}} f_z| + \tau(\delta, \tau(1/\alpha|\mu)) \\ &\leq \left|D_w f_z - D_{P_{x,y}(w)} f_z\right| + \left|D_{P_{x,y}(w)} f_z - D_{\tilde{w}} f_z\right| + \tau(\delta, \tau(1/\alpha|\mu)) \\ &\leq \tau(\varepsilon, \delta, \mu, \eta, \tau(1/\alpha|\mu)) \end{aligned}$$

by the previous corollary and Inequalities 5.17 and 5.18. Choosing $\varepsilon, \delta, \eta,\ \mu$, and $1/\alpha$ small enough, we have a contradiction to 5.19. ■

The previous lemma, together with the definitions of Φ_d^{α}, $(\varphi^{\eta})^{-1}$ and $P_{x^{\alpha}, y^{a}}$ establishes the estimates 5.15 and 5.16 and hence the second part of Key Lemma, completing the proof of Theorem 31, except in dimension 4.

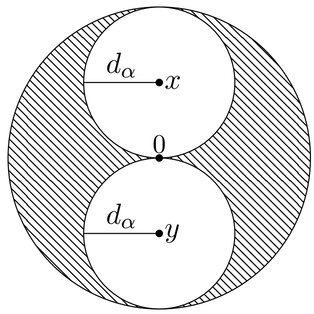

Figure 5.2: The model $\mathcal{D}_k^n(2d_\alpha)$.

5.5 Recognizing $\mathbb{R}P^4$

To prove Theorem 31 in dimension 4, we exploit the following corollary of the fact that $\mathrm{Diff}_+(S^3)$ is connected [3].

Corollary 65

Let M be a smooth 4–manifold obtained by smoothly gluing a 4–disk to the boundary of the nontrivial 1–disk bundle over $\mathbb{R}P^3$. Then M is diffeomorphic to $\mathbb{R}P^4$.

To see that our M^αs have this structure, we use standard triangle comparison and argue as we did in the part of Section 5.4 titled "Lower Bound on Differential" to conclude

Proposition 66

For any fixed $\rho_0 > 0$, $f_{0,d}^\alpha$ does not have critical points on $M^\alpha \setminus \{B(p_0^\alpha, \rho_0) \cup B(A(p_0^\alpha), \rho_0)\}$, and $\nabla f_{0,d}^\alpha$ is gradient-like for $\mathrm{dist}(A(p_0^\alpha), \cdot)$ and $-\mathrm{dist}(p_0^\alpha, \cdot)$, provided α is sufficiently large and d is sufficiently small.

Finally, using Swiss Cheese Volume Comparison (see 1.1 in [12]) we will show

Proposition 67

There is a $\rho_0 > 0$ so that $\mathrm{dist}(p_0^\alpha, \cdot)$ does not have critical points in $B(p_0^\alpha, \rho_0)$, provided α is sufficiently large.

Proof.

Since $vol(M^{\alpha}) \to vol(\mathcal{D}_k^n(r))$, $vol(B(p_0^{\alpha}, r)) \to vol(\mathcal{D}_k^n(r))$. Via Swiss Cheese Volume Comparison (see 1.1 in [12]) we shall see that the presence of a critical point close to p_0^{α} contradicts $vol(B(p_0^{\alpha}, r)) \to vol(\mathcal{D}_k^n(r))$. Suppose q_{α} is critical for $\mathrm{dist}(p_0^{\alpha}, \cdot)$, and $\mathrm{dist}(p_0^{\alpha}, q_{\alpha}) = d_{\alpha} \to 0$. Let x, y be points in $\partial\mathcal{D}_k^n(d_{\alpha})$ at maximal distance. By Swiss Cheese Comparison and 1.4 in [12],

$$\begin{aligned}
vol\left(B\left(q_{\alpha}, 2d_{\alpha}\right) \setminus B\left(p_0^{\alpha}, d_{\alpha}\right)\right) &\leq vol\left(\mathcal{D}_k^n\left(2d_{\alpha}\right) \setminus \left\{B\left(x, d_{\alpha}\right) \cup B\left(y, d_{\alpha}\right)\right\}\right) \\
&= vol\left(\mathcal{D}_k^n\left(2d_{\alpha}\right)\right) - 2vol\left(\mathcal{D}_k^n\left(d_{\alpha}\right)\right).
\end{aligned}$$

Since

$$vol(B(p_0^{\alpha}, d_{\alpha})) \leq vol(\mathcal{D}_k^n(d_{\alpha})),$$

we conclude

$$\begin{aligned}
vol\left(B\left(q_{\alpha}, 2d_{\alpha}\right)\right) &\leq vol\left(\mathcal{D}_k^n\left(2d_{\alpha}\right)\right) - vol\left(\mathcal{D}_k^n\left(d_{\alpha}\right)\right) \\
&< \kappa \cdot vol(\mathcal{D}_k^n(2d_{\alpha}))
\end{aligned}$$

for some $\kappa \in (0, 1)$. By relative volume comparison for $\rho \geq 2d_{\alpha}$,

$$\kappa > \frac{vol(B(q_{\alpha}, 2d_{\alpha}))}{vol(\mathcal{D}_k^n(2d_{\alpha}))} \geq \frac{vol(B(q_{\alpha}, \rho))}{vol(\mathcal{D}_k^n(\rho))}$$

or

$$\kappa \cdot vol(\mathcal{D}_k^n(\rho)) > vol(B(q_{\alpha}, \rho)).$$

Since

$$
\begin{aligned}
B\left(p_0^\alpha, r\right) &\subset B\left(q_\alpha, r+d_\alpha\right), \\
vol(B\left(p_0^\alpha, r\right)) &< \kappa \cdot vol(\mathcal{D}_k^n\left(r+d_\alpha\right)).
\end{aligned}
$$

Letting $d_\alpha \rightarrow 0$, we conclude that

$$
vol(B\left(p_0^\alpha, r\right)) < \kappa \cdot vol(\mathcal{D}_k^n\left(r\right)),
$$

a contradiction. ■

An identical argument shows

Proposition 68

There is a $\rho_0 > 0$ so that $\operatorname{dist}\left(A\left(p_0^\alpha\right), \cdot\right)$ does not have critical points in $B\left(A\left(p_0^\alpha\right), \rho\right)$, provided α is sufficiently large.

Combining the previous three propositions, we see that $(f_{0,d}^\alpha)^{-1}\left(0\right)$ is diffeomorphic to S^3. By Geometrization, $(f_{0,d}^\alpha)^{-1}\left(0\right) / \left\{\mathrm{id}, A\right\}$ is diffeomorphic to $\mathbb{R}P^3$. If ρ_0 is as in Proposition 66, it follows that $(f_{0,d}^\alpha)^{-1}([-\rho_0, \rho_0]) / \left\{\mathrm{id}, A\right\}$ is the nontrivial 1–disk bundle over $\mathbb{R}P^3$. $\tilde{M}^\alpha \setminus (f_{0,d}^\alpha)^{-1}([-\rho_0, \rho_0])$ consists of two smooth 4–disks that get interchanged by A. Thus M^α has the structure of Corollary 65 and is hence diffeomorphic to $\mathbb{R}P^4$.

Remark 69

The proof of Perelman's Parameterized Stability Theorem [17] can substitute for Geometrization to allow us to conclude that $f^{-1}\left(0\right) / \left\{\mathrm{id}, A\right\}$ is homeomorphic and therefore diffeomorphic to $\mathbb{R}P^3$. The need to cite the proof rather than the theorem stems *from the fact that the definition of admissible functions in [17] excludes* $f^{\alpha}_{0,d}$. *It is straightforward (but tedious) to see that the proof goes through for an abstract class that includes* $f^{\alpha}_{0,d}$.

The fact that $\mathbb{R}P^4$ *admits exotic differential structures can be seen by combining [18] with either [4] or [5].*

5.6 Purse Stability

We let Γ^n denote the group of twisted n–spheres. Recall that there is a filtration

$$\{e\} \subset \Gamma^n_{n-1} \subset \cdots \subset \Gamma^n_1 = \Gamma^n$$

by subgroups, which are called Gromoll groups [9]. Rather than using the definition of the Γ^n_qs from [9], we use the equivalent notion from Theorem D in [15].

Definition 70

Let

$$f : S^{q-1} \times S^{n-q} \longrightarrow S^{q-1} \times S^{n-q}$$

be a diffeomorphism that satisfies

$$p_{q-1} \circ f = p_{q-1},$$

where

$$p_{q-1} : S^{q-1} \times S^{n-q} \longrightarrow S^{q-1}$$

is projection to the first factor. Then Γ^n_q *consists of those smooth manifolds that are* *diffeomorphic to*

$$D^q \times S^{n-q} \cup_f S^{q-1} \times D^{n-q+1}. \tag{5.20}$$

Theorem 71

Let $\{M^\alpha\}_{\alpha=1}^{\infty}$ be a sequence of closed, Riemannian n–manifolds with

$$sec(M^\alpha) \geq k$$

so that

$$M_\alpha \longrightarrow P^n_{k,r}$$

in the Gromov-Hausdorff topology. Then for α sufficiently large, $M_\alpha \in \Gamma^n_{n-1}$.

Notice that a diffeomorphism $f : S^{n-2} \times S^1 \longrightarrow S^{n-2} \times S^1$ so that $p_{n-2} \circ f = p_{n-2}$ gives rise to an element of $\pi_{n-2}\left(\mathrm{Diff}_+\left(S^1\right)\right)$. If two such diffeomorphisms give the same homotopy class, then the construction 5.20 yields diffeomorphic manifolds (cf [15]). Since the group of orientation preserving diffeomorphisms of the circle deformation retracts to $SO\left(2\right)$, it follows that for $n \geq 4$, $\Gamma^n_{n-1} = \{e\}$. Since $\Gamma^n = \{e\}$ for $n = 1, 2, 3,$ we have $\Gamma^n_{n-1} = \{e\}$ for all n. Thus all but finitely many of the M^αs in Theorem 71 are diffeomorphic to S^n, and to prove Theorem 32 it suffices to prove Theorem 71.

The Model Submersion

Recall that we view $\mathcal{D}_k^n(r)$ as a metric r-ball centered at $p_0 = e_0$ in either $H_+^n \subset \mathbb{R}^{1,n}$, $\{e_0\} \times \mathbb{R}^n \subset \mathbb{R}^{n+1}$, or $S^n \subset \mathbb{R}^{n+1}$, and we defined

$$p_i := \begin{cases} \cosh(r)e_0 + \sinh(r)e_i & \text{if } k = -1 \\ e_0 + re_i & \text{if } k = 0 \\ \cos(r)e_0 - \sin(r)e_i & \text{if } k = 1. \end{cases}$$

We let the totally geodesic hyperplane $H \subset \mathcal{D}_k^n(r)$ that defines $P_{k,r}^n$ be the one containing $p_0, p_1, \ldots, p_{n-1}$. We denote the singular subset of $P_{k,r}^n$ by $\mathcal{S}$, that is, $\mathcal{S}$ is the copy of S^{n-2} which is the boundary of the $(n-1)$–disk $\mathcal{D}_k^n(r) \cap H$. Thus $\{p_i\}_{i=1}^{n-1} \subset \mathcal{S}$.

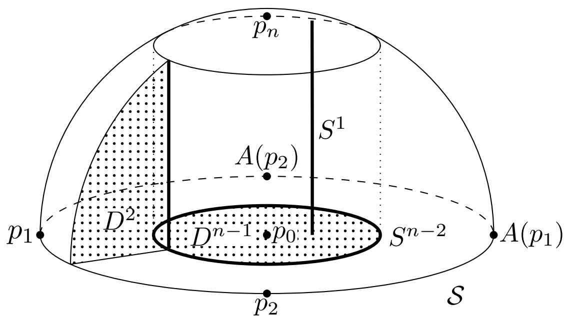

Figure 5.3: One side of $P_{k,r}^n$ for $n = 3$ and $k = 0$.

As the antipodal map $A : \mathcal{D}_k^n(r) \longrightarrow \mathcal{D}_k^n(r)$ commutes with the reflection R in H, it induces a well-defined involution of $P_{k,r}^n$, which we also call A. Note that $A : P_{k,r}^n \longrightarrow P_{k,r}^n$ restricts to the antipodal map of $\mathcal{S}$ and fixes the circle at maximal distance from $\mathcal{S}$.

For $i = 1, \ldots, n-1$, we view $\mathcal{S} \subset \mathcal{D}_k^n(r)$ and define f_i as in 5.4

$$f_i(x) := h_k \circ \operatorname{dist}(A(p_i), x) - h_k \circ \operatorname{dist}(p_i, x).$$

We let $\Psi : P^n_{k,r} \longrightarrow \mathbb{R}^{n-1}$ be defined by

$$\Psi = (f_1, f_2, \ldots, f_{n-1})\,.$$

Lifting The Model Submersion

Let $\{M^\alpha\}_{\alpha=1}^{\infty}$ be a sequence of closed, Riemannian n–manifolds with

$$sec(M^\alpha) \geq k$$

so that

$$M_\alpha \longrightarrow P^n_{k,r}.$$

In contrast to the situation for the Crosscap, the isometry $A : P^n_{k,r} \longrightarrow P^n_{k,r}$ need not lift to an isometry of M^α. We nevertheless let $A : M^\alpha \longrightarrow M^\alpha$ denote any map that is Gromov-Hausdorff close to $A : P^n_{k,r} \longrightarrow P^n_{k,r}$.

As before, we define $f^\alpha_{i,d} : M^\alpha \longrightarrow \mathbb{R}$ by

$$f^\alpha_{i,d}(x) = \int_{z\in B\left(A\left(p_i^\alpha\right),d\right)} h_k \circ \mathrm{dist}\,(z,x) - \int_{z\in B\left(p_i^\alpha,d\right)} h_k \circ \mathrm{dist}\,(z,x)\,. \tag{5.21}$$

We let $\Psi^\alpha_d : M^\alpha \longrightarrow \mathbb{R}^{n-1}$ be defined by

$$\Psi^\alpha_d = (f^\alpha_{1,d}, \ldots, f^\alpha_{n-1,d}).$$

The Handles

We identify $\mathbb{R}^{n-1}$ with

$$\mathbb{R}^{n-1} \equiv \operatorname{span}\{e_1, \ldots, e_{n-1}\} \subset \begin{cases} \mathbb{R}^{1,n} & \text{if } k = -1 \\ \mathbb{R}^{n+1} & \text{if } k = 0 \\ \mathbb{R}^{n+1} & \text{if } k = 1. \end{cases}$$

For small $\varepsilon > 0$, we set

$$\begin{aligned} E_0(\varepsilon) &:= (\Psi)^{-1}(D^{n-1}(0, r-\varepsilon)), \\ E_0^{\alpha}(\varepsilon) &:= (\Psi_d^{\alpha})^{-1}(D^{n-1}(0, r-\varepsilon)), \\ E_1(\varepsilon) &:= (\Psi)^{-1}(\overline{A^{n-1}(0, r-\varepsilon, 2r)}), \text{ and} \\ E_1^{\alpha}(\varepsilon) &:= (\Psi_d^{\alpha})^{-1}(\overline{A^{n-1}(0, r-\varepsilon, 2r)}), \end{aligned}$$

where $\overline{A^{n-1}(0, r-\varepsilon, 2r)}$ is the closed annulus in $\mathbb{R}^{n-1}$ centered at 0 with inner radius $r-\varepsilon$ and outer radius $2r$, and $D^{n-1}(0, r-\varepsilon)$ is the closed ball in $\mathbb{R}^{n-1}$ centered at 0 with radius $r-\varepsilon$.

Theorem 71 is a consequence of the next two lemmas.

Key Lemma 72

For any sufficiently small $\varepsilon > 0$,

$$\Psi_d^{\alpha} : E_0^{\alpha}(\varepsilon) \longrightarrow D^{n-1}(0, r-\varepsilon)$$

is a trivial S^1–bundle, provided α is sufficiently large and d is sufficiently small.

Let $\operatorname{pr} : \overline{A^{n-1}(0, r-\varepsilon, 2r)} \to \partial\left(D^{n-1}(0, r-\varepsilon)\right) = S^{n-2}$ be radial projection and set

$$g := \mathrm{pr} \circ \Psi : E_1(\varepsilon) \to \partial\left(D^{n-1}(0, r-\varepsilon)\right)$$

$$g_d^\alpha := \mathrm{pr} \circ \Psi_d^\alpha : E_1^\alpha(\varepsilon) \to \partial\left(D^{n-1}(0, r-\varepsilon)\right).$$

Key Lemma 73

There is an $\varepsilon > 0$ so that

$$g_d^\alpha : E_1^\alpha(\varepsilon) \longrightarrow \partial\left(D^{n-1}(0, r-\varepsilon)\right)$$

is a trivial D^2–bundle over $\partial\left(D^{n-1}(0, r-\varepsilon)\right) = S^{n-2}$, provided α is sufficiently large and d is sufficiently small.

Since every space of directions of $P_{k,r}^n$ contains an isometrically embedded, totally geodesic copy of S^{n-3}, and every space of directions of $P_{k,r}^n \setminus \mathcal{S}$ contains an isometrically embedded, totally geodesic copy of S^{n-1}, we get the following. (Cf Proposition 36.)

Proposition 74

There are $r, \delta > 0$ so that every point in the purse $P_{k,r}^n$ has a neighborhood B that is $(n-2, \delta, r)$–strained.

For any neighborhood U of $\mathcal{S}$, there are $r, \delta > 0$ so that every point in $P_{k,r}^n \setminus U$ has a neighborhood B that is (n, δ, r)–strained.

Remark 75

For $x \in \mathcal{S}$, the strainer $\{(a_i, b_i)\}_{i=1}^{n-2}$ can be chosen to lie in $\mathcal{S}$.

Because the $f_i : P_{k,r}^n \longrightarrow \mathbb{R}$ are coordinate functions, $\Psi|_{\mathcal{D}_k^n(r) \cap H}$ differs from the identity by a translation. Using this and ideas from Section 5.4, we will be able to prove

Proposition 76

There is a neighborhood U of $\mathcal{S} \subset P^n_{k,r}$ so that for any family of open sets $U^\alpha \subset M^\alpha$ with $U^\alpha \to U$, $g^\alpha_d|_{U^\alpha}$ is a submersion, provided α is sufficiently large and d is sufficiently small.

We will show that our key lemmas hold for any $\varepsilon > 0$ so that

$$\Psi^{-1}\left(\overline{A^{n-1}(0, r-\varepsilon, r)}\right) \subset U.$$

Since $\{f_i\}_{i=1}^{n-1}$ are the $(n-1)$–coordinate functions for the standard embedding of $\mathcal{S} = S^{n-2} \subset \mathbb{R}^{n-1}$, we have

Lemma 77

There is a $\lambda > 0$ so that for all $v \in T\mathcal{S}$, there is an j so that the j^{th}–component function of g satisfies

$$|D_v(g_j)| > \lambda |v|.$$

As in Section 5.4, we have

Addendum 78

Let p_k be any of $p_1, \ldots p_{n-1}$. There is an $\varepsilon > 0$ so that for all $x \in B(p_k, \varepsilon) \cup B(A(p_k), \varepsilon)$ and all $v \in T_x\mathcal{S}$, the index j in the previous lemma can be chosen to be different from k.

To lift Lemma 77 to the M^αs, we need an analog of $T\mathcal{S}$ within each M^α, or better a notion of g^α_d–almost horizontal for each $U^\alpha \subset M^\alpha$. To achieve this, cover

$\mathcal{S}$ by a finite number of $(n-2,\delta,r)$–strained neighborhoods $B\subset P^n_{k,r}$ with strainers $\{(a_i,b_i)\}_{i=1}^{n-2}\subset\mathcal{S}$. Let U be the union of this finite collection, and let $U^\alpha\subset M^\alpha$ converge to U.

Given $x^\alpha\in U^\alpha$, we now define a g_d^α–almost horizontal space at x^α as follows. Let B^α be a $(n-2,\delta,r)$–strained neighborhood for x^α with strainers $\{(a_i^\alpha,b_i^\alpha)\}_{i=1}^{n-2}$ that converge

$$\left(B^\alpha,\{(a_i^\alpha,b_i^\alpha)\}_{i=1}^{n-2}\right)\longrightarrow\left(B,\{(a_i,b_i)\}_{i=1}^{n-2}\right),$$

where $\left(B,\{(a_i,b_i)\}_{i=1}^{n-2}\right)$ is part of our finite collection of $(n-2,\delta,r)$–strained neighborhoods for points in $\mathcal{S}\subset P^n_{k,r}$. We set

$$H_{x^\alpha}^{g_d^\alpha}:=\mathrm{span}_{i\in\{1,\ldots,n-2\}}\left\{\uparrow_{x^\alpha}^{a_i^\alpha}\right\},$$

where $\uparrow_{x^\alpha}^{a_i^\alpha}$ is the direction of *any* segment from x^α back to a_i^α. Regardless of this choice, $H_{x^\alpha}^{g_d^\alpha}$ satisfies the following Lemma, from which Proposition 76 follows.

Lemma 79

There is a $\lambda>0$ so that for all $x^\alpha\in U^\alpha$ and all $v\in H_{x^\alpha}^{g_d^\alpha}$, there is an j so that the j^{th}–component function of g_d^α satisfies

$$\left|D_v\left((g_d^\alpha)_j\right)\right|>\lambda\,|v|\,,$$

provided U and d are sufficiently small and α is sufficiently large. In particular, $g_d^\alpha|_{U^\alpha}$ is a submersion.

Proof.

Let $x_\alpha\to x$, and for all $j=1,\ldots,n-1$, let $z_j^\alpha\to z_j\in B\left(p_j,d\right)$. If $x_\alpha z_j^\alpha$

converges to xz_j, then by Corollary 41,

$$\left| \sphericalangle \left(\Uparrow_{x^\alpha}^{a_i^\alpha}, \Uparrow_{x^\alpha}^{z_j^\alpha} \right) - \sphericalangle \left(\Uparrow_x^{a_i}, \Uparrow_x^{zj} \right) \right| < \tau \left(\delta, 1/\alpha | \mathrm{dist} \left(x, z_j \right) \right).$$

Similarly for a sequence of segments $x_\alpha A \left(z_j^\alpha \right)$ converging to $xA \left(z_j \right)$, we have

$$\left| \sphericalangle \left(\Uparrow_{x^\alpha}^{a_i^\alpha}, \Uparrow_{x^\alpha}^{A\left(z_j^\alpha\right)} \right) - \sphericalangle \left(\Uparrow_x^{a_i}, \Uparrow_x^{A(z_j)} \right) \right| < \tau \left(\delta, 1/\alpha | \mathrm{dist} \left(x, A \left(z_j \right) \right) \right).$$

Arguing as in the proof of Lemma 60, we have for all i and j,

$$\left| D_{\Uparrow_{x^\alpha}^{a_i^\alpha}} \left(g_d^\alpha \right)_j - D_{\Uparrow_x^{a_i}} \left(g \right)_j \right| < \tau \left(\delta, d, 1/\alpha | \mathrm{dist} \left(x, p_j \right), \mathrm{dist} \left(x, A \left(p_j \right) \right) \right).$$

Since $v \in H_{x^\alpha}^{g_d^\alpha} = \mathrm{span}_{i \in \{1, \ldots, n-2\}} \left\{ \Uparrow_{x^\alpha}^{a_i^\alpha} \right\}$, the lemma follows from the previous display together with Lemma 77, Addendum 78, and the hypothesis that U is sufficiently small.

■

Let $p_n \in \mathcal{D}_k^n \left(r \right)$ be as in 5.5, and let $Q : \mathcal{D}_k^n \left(r \right) \longrightarrow P_{k,r}^n$ be the quotient map. We abuse notation and call $Q \left(p_n \right)$, p_n. We define $f_n : P_{k,r}^n \to \mathbb{R}$ by

$$f_n(x) := h_k \circ \mathrm{dist} \left(\left(p_n \right), x \right) - h_k \circ d(p_0, x).$$

With a slight modification of the proof of Proposition 36, we get

Lemma 80

There are $\delta, r > 0$ so that for all $x \in E_0 \left(\varepsilon/2 \right)$ there is an (n, δ, r)–strainer $\{(a_i, b_i)\}_{i=1}^n$ with

$$\{(a_i, b_i)\}_{i=1}^{n-1} \subset f_n^{-1} \left(l \right)$$

for some $l \in \mathbb{R}$.

We cover $E_0(\varepsilon/2)$ by a finite number of such (n, δ, r)–strained sets and make

Definition 81

For $x \in E_0(\varepsilon/2)$, *set*

$$H_x^{\Psi} := \mathrm{span}_{i \in \{1,\dots,n-1\}} \{\uparrow_x^{a_i}\},$$

where $\{(a_i, b_i)\}_{i=1}^{n-1}$ *is as in the previous lemma.*

Since $\Psi : E_0(\varepsilon/2) \longrightarrow D^{n-1}(r - \varepsilon/2)$ is simply orthogonal projection, we have

Lemma 82

There is a $\lambda > 0$ *so that for all* $x \in E_0(\varepsilon/2)$ *and all* $v \in H_x^{\Psi}$, *there is an* i *so that*

$$|D_v f_i| > \lambda |v|.$$

To lift this lemma to the M^{α}s, we need a notion of Ψ_d^{α}–almost horizontal for each M^{α}. Given $z^{\alpha} \in E_0^{\alpha}(\varepsilon/2)$, we define a Ψ_d^{α}–almost horizontal space at z^{α} as follows. Let B^{α} be a (n, δ, r)–strained neighborhood for z^{α} with strainers $\{(a_i^{\alpha}, b_i^{\alpha})\}_{i=1}^{n}$ that converge

$$(B^{\alpha}, \{(a_i^{\alpha}, b_i^{\alpha})\}_{i=1}^{n}) \longrightarrow (B, \{(a_i, b_i)\}_{i=1}^{n}),$$

where $(B, \{(a_i, b_i)\}_{i=1}^{n})$ is part of our finite collection of (n, δ, r)–strained neighborhoods for points in $E_0(\varepsilon/2)$ that comes from Lemma 80. We set

$$H_{z^{\alpha}}^{\Psi_d^{\alpha}} := \mathrm{span}_{i \in \{1,\dots,n-1\}} \left\{\uparrow_{z^{\alpha}}^{a_i^{\alpha}}\right\},$$

where $\Uparrow_{z^\alpha}^{a_i^\alpha}$ is the direction of *any* segment from z^α back to a_i^α. Regardless of this choice, $H_{z^\alpha}^{\Psi_d^\alpha}$ satisfies the following Lemma, whose proof is nearly identical to the proof of Lemma 60.

Lemma 83

There is a $\lambda > 0$ so that for all $z^\alpha \in E_0^\alpha(\varepsilon/2)$ and all $v \in H_{z^\alpha}^{\Psi_d^\alpha}$, there is an $i \in \{1, \ldots, n-1\}$ so that

$$\left|D_v f_{i,d}^\alpha\right| > \lambda |v|,$$

provided α is sufficiently large and d is sufficiently small. In particular, $\Psi_d^\alpha|_{E_0^\alpha(\varepsilon/2)}$ is a submersion.

Proposition 84

$E_1^\alpha(\varepsilon)$ is homeomorphic to $S^{n-2} \times D^2$, and $E_0^\alpha(\varepsilon)$ is homeomorphic to $D^{n-1} \times S^1$, provided α is sufficiently large and d is sufficiently small.

Proof.

First we show that $E_0^\alpha(\varepsilon)$ is connected. By the Stability Theorem [17], we have homeomorphisms $h_\alpha : P_k^n(r) \longrightarrow M^\alpha$ that are also Gromov–Hausdorff approximations (cf [10], [12] and [29]). Thus for α sufficiently large, we have

$$E_0^\alpha(\varepsilon) \subset h_\alpha\left(E_0(\varepsilon/2)\right).$$

Let $\rho^\alpha : M^\alpha \longrightarrow \mathbb{R}$ be defined by

$$\rho^\alpha(x) := \left|\Psi_d^\alpha(x)\right|.$$

Since $\Psi_d^\alpha|_{E_0^\alpha(\varepsilon/2)}$ is a submersion, it follows that ρ^α does not have critical points on

$E_0^\alpha(\varepsilon/2) \backslash E_0^\alpha(2\varepsilon)$. By construction, the flow lines of $\nabla\rho^\alpha$ are transverse to the boundary of $E_0^\alpha(\varepsilon)$ and hence can be used to move $h_\alpha(E_0(\varepsilon/2))$ onto $E_0^\alpha(\varepsilon)$. It follows that $E_0^\alpha(\varepsilon)$ is connected.

Since $\Psi_d^\alpha|_{E_0^\alpha(\varepsilon)}$ is a proper submersion, it is a fiber bundle with contractible base $D^{n-1}(0, r-\varepsilon)$. Since the fiber is 1–dimensional and the total space is connected, we conclude that $E_0^\alpha(\varepsilon)$ is homeomorphic to $D^{n-1} \times S^1$.

We choose a homeomorphism $h_0 : E_0(\varepsilon/2) \longrightarrow E_0^\alpha(\varepsilon/2)$ so that

$$\begin{tikzcd} E_0(\varepsilon/2) \arrow[rr, "h_0"] \arrow[dr, "\Psi_d"'] & & E_0^\alpha(\varepsilon/2) \arrow[dl, "\Psi_d^\alpha"] \\ & D^{n-1} & \end{tikzcd}$$

commutes. Using the proof of the Gluing Theorem ([17], Theorem 4.6), we construct a homeomorphism $h : P_k^n(r) \longrightarrow M^\alpha$ so that

$$h = \begin{cases} h_0 & \text{on } E_0(\varepsilon) \\ h_\alpha & \text{on } E_1(\varepsilon/4). \end{cases}$$

It follows that $h(E_1(\varepsilon)) = E_1^\alpha(\varepsilon)$. Since $E_1(\varepsilon)$ is homeomorphic to $S^{n-2} \times D^2$, the result follows. ■

Proof of Key Lemma 73.

By Proposition 76, $g_d^\alpha : E_1^\alpha(\varepsilon) \longrightarrow \partial D^{n-1}(0, r-\varepsilon) = S^{n-2}$ is a submersion. Since g_d^α is proper, g_d^α is a fiber bundle with two-dimensional fiber F. From the long exact homotopy sequence and Proposition 84, we conclude that F is a 2–disk. For $n \neq 4$, every D^2–bundle over S^{n-2} is trivial by Theorem 1 of [21]. When $n = 4$, $E_1^\alpha(\varepsilon)$ is a D^2–bundle over S^2 whose total space is homeomorphic to $S^2 \times D^2$. It follows for example from [35] that $E_1^\alpha(\varepsilon)$ is trivial in all cases, completing the proof of Key Lemma 73. ■

Proof of Key Lemma 72.

Since $\Psi_d^\alpha|_{E_0^\alpha(\varepsilon)}$ is a proper submersion, $\left(E_0^\alpha(\varepsilon), \Psi_d^\alpha\right)$ is a fiber bundle over $D^{n-1}(0, r-\varepsilon)$ with one-dimensional fiber F. Since $E_0^\alpha(\varepsilon)$ is also homeomorphic to $D^{n-1} \times S^1$, it follows that the fiber is S^1. The base is contractible, so the bundle is trivial. This completes the proof of Key Lemma 72 and hence the proofs of Theorems 71 and 32, establishing our Main Theorem. ■

Double Disk Stability

The proof of Theorem 31 also yields

Corollary 85

Let $\{M_i\}_{i=1}^{\infty}$ be a sequence of closed Riemannian n–manifolds with $sec(M_i) \geq k$ so that

$$M_i \longrightarrow \mathbb{D}_k^n(r)$$

in the Gromov-Hausdorff topology. Then all but finitely many of the M_is are diffeomorphic to S^n.

Proof.

In contrast to Theorem 49, we do not necessarily have an isometric involution of the M_is. Instead, we let $A : M_i \longrightarrow M_i$ be any map which is Gromov-Hausdorff close to $A : \mathbb{D}_k^n(r) \longrightarrow \mathbb{D}_k^n(r)$. We then define $f_{i,d}^\alpha : M_i \longrightarrow \mathbb{R}$ as in 5.21 and proceed as in the proof of Theorem 31. ■

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