UNIVERSITY OF CALIFORNIA
Los Angeles

Approximations in Operator Theory and Free Probability

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Paul Daniel Skoufranis

2014

ABSTRACT OF THE DISSERTATION

Approximations in Operator Theory
and Free Probability

by

Paul Daniel Skoufranis
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2014
Professor Dimitri Y. Shlyakhtenko, Chair

We will investigate several related problems in Operator Theory and Free Probability. The notion of an exact C*-algebra is modified to reduced free products where it is shown, by examining another exact sequence of Toeplitz-Pimsner Algebras, that every C*-algebra is freely exact. This enables a discussion of strongly convergent random variables where we show that strong convergence is preserved under reduced free products. We will also analyze the distributions of freely independent random variables where it is shown that the distribution of a non-trivial polynomial in freely independent semicircular variables is atomless and has an algebraic Cauchy transform. These results are obtained by considering an analogue of the Strong Atiyah Conjecture for discrete groups and by considering algebraic formal power series in non-commuting variables respectively. More information about the distributions of operators will be obtained by examining when normal operators are limits of nilpotent operators in various C*-algebras including von Neumann algebras and unital, simple, purely infinite C*-algebras. The main techniques used to examine when a normal operator is a limit of nilpotent operators come from known matrix algebra results along with the projection structures of said algebras. Finally, using specific information about norm convergence of nilpotent operators, we will examine the closed unitary and similarity orbits of normal operators in von Neumann algebras and unital, simple, purely infinite C*-algebras.

The dissertation of Paul Daniel Skoufranis is approved.

Eric D'Hoker

Sorin Popa

Edward G. Effros

Dimitri Y. Shlyakhtenko, Committee Chair

University of California, Los Angeles

2014

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

I would like to thank my thesis advisor Dimitri Shlyakhtenko for all of his advice and support these past five years. Due to his council, my research projects have flourished and I would not have succeeded in my program without his aid. I would also like to thank Dimitri Shlyakhtenko for his permission for our joint paper [72] to be included in this dissertation. Dimitri Shlyakhtenko's contribution to [72] was having the main idea behind our result in which we were together able to fill in the details. I would also like to acknowledge that parts of this dissertation were supported financially by the National Science and Engineering Research Council of Canada's post-graduate fellowship program.

VITA

2007	Teacher/Tutor/Teaching Assistant, Bronte College of Canada, Mississauga, Ontario.
2009	B.Math. (Pure Mathematics, Physics Minor, Cooperative Program), University of Waterloo.
2009	Awarded NSERC PGS M Research Scholarship.
2010	Awarded NSERC PGS D3 Research Scholarship.
2010–2011	Teaching Assistant, Mathematics Department, UCLA. Taught discussion sections of MATH 3C, MATH 31B, MATH 33A, and MATH 115A.
2012–2013	Boot Camp Teaching Assistant, Mathematics Department, UCLA. Taught discussion sections of the Linear Algebra portion of the Mathematics Department Summer Program for new graduate students.
2013	Course Instructor, Mathematics Department, UCLA. Taught lecture sections of MATH 32B.
2013–2014	Teaching Assistant Coordinator, Mathematics Department, UCLA. Taught discussion sections of MATH 495 to train new graduate students to teach college mathematics.

PUBLICATIONS

I. Charlesworth, B. Nelson, and P. Skoufranis, *On Two-Faced Families of Non-Commutative Random Variables*, preprint (2014), 26 pages.

D. Shlyakhtenko and P. Skoufranis, *Freely Independent Random Variables with Non-Atomic Distributions*, accepted to Transactions of the American Mathematics Society (2014), 25 pages.

P. Skoufranis, *Closed Unitary and Similarity Orbits of Normal Operators in Purely Infinite C^*-Algebras*, Journal of Functional Analysis, 265 (2013), no. 3, 474-506.

P. Skoufranis, *Normal Limits of Nilpotent Operators in C^*-Algebras*, accepted to Journal of Operator Theory (2013), 23 pages.

P. Skoufranis, *Normal Limits of Nilpotent Operators in von Neumann Algebras*, Integral Equations and Operator Theory, 77 (2013), no. 3, 407-439.

P. Skoufranis, *On a Notion of Exactness for Reduced Free Products of C^*- Algebras*, accepted to Journal für die reine und angewandte Mathematik (2013), 25 pages.

K. Hare and P. Skoufranis, *The Smoothness of Orbital Measures on Exceptional Lie Groups*, Journal of Lie Theory, 21 (2011), no. 4, 987-1007.

CHAPTER 1

Introduction and Background

In this introduction, a brief outline of background material essential to the comprehension of the mathematics contained in this dissertation will be provided. This introduction is not meant to be fully comprehensive so we advise the interested reader to consult alternative material as necessary.

1.1 Classical Probability

The main focus of classical probability theory is the study and manipulation of random variables.

Definition 1.1.1. A measure space is a pair $(\mathcal{X}, \Omega)$ where $\mathcal{X}$ is a set, Ω is a σ-algebra of subsets of $\mathcal{X}$ (that is, Ω consists of subsets of $\mathcal{X}$ including the empty set $\emptyset$ and $\mathcal{X}$ such that Ω is closed under countable unions and complementation).

Definition 1.1.2. A probability space is a triple $(\mathcal{X}, \Omega, \mu)$ where $(\mathcal{X}, \Omega)$ is a measure space and $\mu : \Omega \to [0, 1]$ is a probability measure on Ω.

Definition 1.1.3. Given a probability space $(\mathcal{X}, \Omega, \mu)$ and a measure space $(\mathcal{X}', \Omega')$, a random-variable X on $(\mathcal{X}, \Omega, \mu)$ to $(\mathcal{X}', \Omega')$ is a measurable function $X : \mathcal{X} \to \mathcal{X}'$ (that is, if $S \in \Omega'$, then $X^{-1}(S) := \{t \in \mathcal{X} \mid X(t) \in S\} \in \Omega$).

Given a real-valued random variable X, we can define a new probability measure μ_X on Ω' such that

$$\mu_X(S) := \mu(\{t \in \mathcal{X} \mid X(t) \in S\})$$

for all $S \in \Omega'$.

Definition 1.1.4. Let X be a real-valued random variable. The measure μ_X described above is called the measure associated to the random variable X or the probability distribution of X.

Definition 1.1.5. In this setting, for a subset S of Ω', we define the probability that X is in S by

$$\mathrm{Prob}(X \in S) := \mu_X(S).$$

In particular, for an subset $S \subseteq \Omega'$,

$$\mathrm{Prob}(X \in S) = \int_S 1 \, d\mu_X(t).$$

Definition 1.1.6. Given a measureable function f on $(\mathcal{X}', \Omega')$, we defined the expected value of $f(X)$ to be the quantity

$$\mathbb{E}(f(X)) = \int_{\mathcal{X}'} f(t) \, d\mu_X(t).$$

The map $\mathbb{E}$ taking a measurable function f on $(\mathcal{X}', \Omega')$ to its expected value is called the expectation map.

For the most part, we will restrict our attention to random variables X such that $\mathcal{X} = \mathcal{X}' = \mathbb{R}$ and Ω and Ω' are the Borel subset of $\mathbb{R}$. We will further assume that μ_X is compactly supported (that is, there exists $a, b \in \mathbb{R}$ such that $\mu_X((-\infty, a)) = 0$ and $\mu_X((b, \infty)) = 0$). This restriction implies the polynomial functions are μ_X-integrable and thus allow us to make the following definition.

Definition 1.1.7. Let X be a real-valued random variable on $\mathbb{R}$ with associated measure μ_X. For $n \in \mathbb{N} \cup \{0\}$, the n^{th} moment of X, denoted m_n^X, is defined by

$$m_n^X := \int_{\mathbb{R}} t^n \, d\mu_X(t).$$

Notice, by definitions, that $m_0^X = 1$ and $\mathbb{E}(X) = m_1^X$ (which is also called the expectation of X). Another important quantity in classical probability theory is the following.

Definition 1.1.8. The variance of X, denoted $\mathrm{Var}(X)$, is defined to be

$$\mathrm{Var}(X) := m_2^X - \left(m_1^X\right)^2.$$

The main reason we restricted our attention to random variables X with compact support is that the moment sequence $\left(m_n^X\right)_{n\geq 1}$ completely characterizes X. Indeed, since μ_X is compactly supported, the polynomial functions are dense in the space of continuous functions on the support of μ_X and completely characterize μ_X as a measure on $\mathbb{R}$ (that is, if Y is another random variable such that $\left(m_n^X\right)_{n\geq 1} = \left(m_n^Y\right)_{n\geq 1}$ then $\mu_X = \mu_Y$). Thus, instead of considering the measure associated with a random variable, we can consider the sequence of real numbers $\left(m_n^X\right)_{n\geq 1}$.

One important way of viewing the moments is via the following definition.

Definition 1.1.9. Let μ be a compactly supported probability measure on $\mathbb{R}$. The Cauchy transform of μ, denoted G_μ, is the function defined on $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-t}\, d\mu(t).$$

Remarks 1.1.10. Given a real-valued random variable X, notice that

$$\begin{aligned} G_{\mu_X}(z) &= \textstyle\int_{\mathbb{R}} \frac{1}{z-t}\, d\mu_X(t) \\ &= \textstyle\int_{\mathbb{R}} \frac{1}{z} \frac{1}{1-\frac{t}{z}}\, d\mu_X(t) \\ &= \textstyle\int_{\mathbb{R}} \sum_{n\geq 0} \frac{t^n}{z^{n+1}}\, d\mu_X(t) \\ &= \textstyle\frac{1}{z} + \sum_{n\geq 1} \frac{m_n^X}{z^{n+1}}. \end{aligned}$$

Note that the above computation only makes sense analytically if we have $\left|\frac{t}{z}\right| < 1$ for all t in the support of μ_X. Alternatively we can define G_{μ_X} as a formal power series via the expression

$$G_{\mu_X}(z) = \frac{1}{z} + \sum_{n\geq 1} \frac{m_n^X}{z^{n+1}}.$$

In particular, we see that

$$\frac{1}{z} G_{\mu_X}\left(\frac{1}{z}\right) = 1 + \sum_{n\geq 1} m_n^X z^n$$

in which case G_{μ_X} completely encapsulates all of the moments and thus the random variable X.

The main focus of classical probability theory is the study of groups of random variables with a specific property.

Definition 1.1.11. Let $X := (X_1, \dots, X_d) : \mathbb{R} \to \mathbb{R}^d$ be a random variable defined by

$$X(t) = (X_1(t), \dots, X_d(t))$$

for all $t \in \mathbb{R}$ where $X_1, \dots, X_d : \mathbb{R} \to \mathbb{R}$ are random variables (in this setting, μ_X is called the joint distribution of $X_1, \dots, X_d$). We say that $X_1, \dots, X_d$ are independent if

$$\mathrm{Prob}(X \in [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]) = \prod_{j=1}^{d} \mathrm{Prob}(X_j \in [a_j, b_j])$$

for every $[a_1, b_1], [a_2, b_2], \dots, [a_d, b_d] \subseteq \mathbb{R}$.

Remarks 1.1.12. It is elementary to see that if $X_1, \dots, X_d$ are independent random variables, then the joint distribution satisfies

$$\mu_X = \mu_{X_1} \times \mu_{X_2} \times \cdots \times \mu_{X_d}$$

(that is, μ_X must be the product measure of $\mu_{X_1}, \dots, \mu_{X_d}$).

As polynomials in one variable determined the probability distribution of a single random variable, polynomials in multiple variables determine the probability distribution of a random variable $X : \mathbb{R} \to \mathbb{R}^d$

Definition 1.1.13. Let $X := (X_1, \ldots, X_d) : \mathbb{R} \to \mathbb{R}^d$ be a random variable defined by

$$X(t) = (X_1(t), \ldots, X_d(t))$$

for all $t \in \mathbb{R}$ where $X_1, \ldots, X_d : \mathbb{R} \to \mathbb{R}$ are random variables. For a polynomial in d-variables $p(x_1, \ldots, x_d)$, we defined the expected value of $p(X_1, \ldots, X_d)$ to be

$$\mathbb{E}(p(X_1, \ldots, X_n)) = \int_{\mathbb{R}^d} p(t_1, \ldots, t_d)\, d\mu_X(t_1, \ldots, t_d).$$

For $\ell_1, \ldots, \ell_d \in \mathbb{N} \cup \{0\}$, the $(\ell_1, \ell_2, \ldots, \ell_d)$-moment of $X_1, \ldots, X_d$ is defined to be

$$m^{X_1,\ldots,X_d}_{\ell_1,\ldots,\ell_d} := \mathbb{E}(X_1^{\ell_1} X_2^{\ell_2} \cdots X_d^{\ell_d}).$$

Remarks 1.1.14. In the case that $X_1, \ldots, X_d$ are independent, we see that

$$\begin{aligned} m^{X_1,\ldots,X_d}_{\ell_1,\ldots,\ell_d} &= \int_{\mathbb{R}^d} t_1^{\ell_1} \cdots t_d^{\ell_d}\, d\mu_X(t_1, \ldots, t_d) \\ &= \int_{\mathbb{R}^d} t_1^{\ell_1} \cdots t_d^{\ell_d}\, d\mu_{X_1}(t_1) d\mu_2(t_2) \cdots d\mu_d(t_d) \\ &= \prod_{j=1}^d m^{X_j}_{\ell_j}. \end{aligned}$$

Thus, in the case of independent random variables, the moments of the joint probability distribution are easily obtained from the moments of the individual distributions.

One distribution that plays an essential role in classical probability theory and independent random variables is the following.

Definition 1.1.15. The normalized Gaussian distribution is the measure μ_{Gaus} on $\mathbb{R}$ defined by

$$\mu_{\text{Gaus}}([a,b]) = \frac{1}{2\pi} \int_a^b e^{-\frac{x^2}{2}}\, dx$$

for all $[a,b] \subseteq \mathbb{R}$.

The following theorem is one of the central theorems in classical probability theory.

Theorem 1.1.16 (Central Limit Theorem)**.** *Let $X_1, X_2, \ldots$ be independent, identically distributed random variables (that is, X_j are all independent and have the same probability distributions) with $\mathbb{E}(X_j) = 0$ and $\mathrm{Var}(X_j) = 1$ for all $j \in \mathbb{N}$. Then, for all $[a, b] \subseteq \mathbb{R}$,*

$$\lim_{n\to\infty} \mathrm{Prob}\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n} X_j \in [a, b]\right) = \mu_{\mathrm{Gaus}}([a, b]).$$

One idea related to the above theory is the question, "If X_1 and X_2 are independent random variables, what is the distribution of $X_1 + X_2$?" Examining the above, we see for each $n \in \mathbb{N}$ that

$$\mathbb{E}((X_1 + X_2)^n) = \int_{\mathbb{R}}\int_{\mathbb{R}} (t_1 + t_2)^n \, d\mu_{X_1}(t_1) d\mu_{X_2}(t_2) = \int_{\mathbb{R}} t^n \, d(\mu_{X_1} * \mu_{X_2})$$

where $\mu_{X_1} * \mu_{X_2}$ is the convolution measure of μ_{X_1} and μ_{X_2} (which may be uniquely defined via the above formula).

1.2 C*-Algebras

The notion of a C*-algebra is an essential concept in the study of Operator Theory and Operator Algebras. For a complete introduction to C*-algebras, we refer the reader to [21]. Before discussing C*-algebras, we begin with some basic definitions.

Definition 1.2.1. Let V be a vector space over the complex numbers. A norm on V is a function $\|\cdot\| : V \to [0, \infty)$ such that

1. $\|v\| = 0$ if and only if $v = \vec{0}$,
2. $\|\lambda v\| = |\lambda| \, \|v\|$ whenever $v \in V$ and $\lambda \in \mathbb{C}$,
3. $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

A normed linear space is a pair $(V, \|\cdot\|)$ where V is a vector space over the complex numbers and $\|\cdot\|$ is a norm on V.

Remarks 1.2.2. Even though a normed linear space is a pair $(V, \|\cdot\|)$, we will often say that V is a normed linear space meaning that V comes equipped with a fixed canonical norm.

Definition 1.2.3. A normed linear space $\mathfrak{X}$ is said to be complete if whenever $(x_n)_{n\geq 1}$ is a sequence of elements in $\mathfrak{X}$ with the property that for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \epsilon$$

for all $n, m \geq N$ (such a sequence is said to be Cauchy), then there exists an $x \in \mathfrak{X}$ such that for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$\|x - x_n\| < \epsilon$$

for all $n \geq N$ (in which case we write $x = \lim_{n\to\infty} x_n$). That is, a norm linear space is complete if every Cauchy sequence converges. A complete normed linear space is called a Banach space.

Definition 1.2.4. A Banach algebra is a Banach space $\mathfrak{A}$ equipped with an algebra structure over the complex numbers such that

$$\|AB\| \leq \|A\| \, \|B\|$$

for all $A, B \in \mathfrak{A}$. That is, a Banach algebra is a normed algebra over the complex numbers that is complete and whose norm is submultiplicative.

There is significant theory dedicated to Banach algebras that applies to C*-algebras. For the purposes of this dissertation, we will focus only on said theory in the context of C*-algebras. In order to define a C*-algebra, we will need the following.

Definition 1.2.5. Let $\mathcal{A}$ be an algebra over the complex numbers. An involution on $\mathcal{A}$ is a function $* : \mathcal{A} \to \mathcal{A}$ such that

1. $(A^*)^* = A$ for all $A \in \mathcal{A}$ (i.e. $*$ is idempotent),

2. $(A+B)^* = A^* + B^*$ for all $A, B \in \mathcal{A}$ (i.e. $*$ is additive),

3. $(\lambda A)^* = \overline{\lambda} A^*$ for all $A \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ (i.e. combining with (2), $*$ is conjugate linear), and

4. $(AB)^* = B^* A^*$ for all $A, B \in \mathcal{A}$ (i.e. $*$ is antimultiplicative).

Definition 1.2.6. A C*-algebra is a Banach algebra $\mathfrak{A}$ together with an involution $* : \mathfrak{A} \to \mathfrak{A}$ such that

$$\|A^* A\| = \|A\|^2$$

for all $A \in \mathfrak{A}$. The above equation is called the C*-equation or the C*-identity.

Remarks 1.2.7. Given a *-algebra, there is at most one C*-norm on said algebra.

Example 1.2.8. The complex numbers $\mathbb{C}$ is a C*-algebra when equipped with its usual algebra structure, the absolute value as its norm, and complex conjugation as its involution.

Example 1.2.9. Let X be a compact Hausdorff space. The continuous functions on X, denoted $C(X)$, is a C*-algebra when equipped with the algebra structure given by pointwise addition and multiplication, with

$$\|f\|_\infty := \sup\{|f(x)| \mid x \in X\}$$

as its norm, and pointwise complex conjugation as its involution.

Example 1.2.10. Let $\mathcal{M}_n(\mathbb{C})$ denote the set of n by n matrices with entries in the complex numbers. Then $\mathcal{M}_n(\mathbb{C})$ is a C*-algebra when equipped with matrix addition and matrix multiplication, with the operator norm (see Remarks 1.2.16) as its norm, and the conjugate transpose as its involution.

To fully understand the above example and construct more examples, we consider the following.

Definition 1.2.11. Let V be a vector space over the complex numbers. An inner product on V is a function $\langle \cdot, \cdot \rangle_V : V \times V \to \mathbb{C}$ such that

1. $\langle v, v\rangle_V \geq 0$ for all $v \in V$,
2. for $v \in V$, $\langle v, v\rangle_V = 0$ implies $v = \vec{0}$,
3. $\langle \lambda v + w, x\rangle_V = \lambda\langle v, x\rangle_V + \langle w, x\rangle_V$ for all $v, w, x \in V$ and $\lambda \in \mathbb{C}$, and
4. $\langle x, \lambda v + w\rangle_V = \overline{\lambda}\langle x, v\rangle_V + \langle x, w\rangle_V$ for all $v, w, x \in V$ and $\lambda \in \mathbb{C}$.

An inner product space is a pair $(V, \langle \cdot, \cdot \rangle_V)$ where V is a vector space over the complex numbers and $\langle \cdot, \cdot \rangle_V$ is an inner product on V.

Remarks 1.2.12. Even though an inner product space is a pair $(V, \langle \cdot, \cdot \rangle_V)$, we will often say that V is an inner product space meaning that V comes equipped with a fixed canonical inner product.

Remarks 1.2.13. If V is an inner product space, it is easy to see that the function $\| \cdot \| : V \to [0, \infty)$ defined by

$$\|v\| = \sqrt{\langle v, v\rangle}$$

is a norm on V.

Definition 1.2.14. A Hilbert space $\mathcal{H}$ is an inner product space that is complete with respect to the norm defined in Remarks 1.2.13.

Definition 1.2.15. Let $\mathcal{H}$ be a Hilbert space. A linear map $T : \mathcal{H} \to \mathcal{H}$ is said to be bounded if

$$\sup\{\|T\xi\| \mid \xi \in \mathcal{H}, \|\xi\| \leq 1\} < \infty.$$

We will denote the set of bounded linear maps on a Hilbert space $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$.

Remarks 1.2.16. Given a Hilbert space $\mathcal{H}$, there is a canonical norm on $\mathcal{B}(\mathcal{H})$, known as the operator norm, defined by

$$\|T\| := \sup\{\|T\xi\| \mid \xi \in \mathcal{H}, \|\xi\| \leq 1\}$$

for all $T \in \mathcal{B}(\mathcal{H})$. It is possible to show that $\mathcal{B}(\mathcal{H})$ is complete with respect to the operator norm and thus a Banach space. Moreover, it is not difficult to show that if $T, S \in \mathcal{B}(\mathcal{H})$ then

$$\|TS\| \leq \|T\| \, \|S\|$$

and thus $\mathcal{B}(\mathcal{H})$ is a Banach algebra.

Given an element $T \in \mathcal{B}(\mathcal{H})$, it is possible to find a (unique) element $T^* \in \mathcal{B}(\mathcal{H})$, called the adjoint of T, such that

$$\langle T^*\xi, \eta \rangle_{\mathcal{H}} = \langle \xi, T\eta \rangle_{\mathcal{H}}$$

for all $\xi, \eta \in \mathcal{H}$. The map that takes an operator T to the adjoint of T is an involution on $\mathcal{B}(\mathcal{H})$. In fact, $\mathcal{B}(\mathcal{H})$ is then a C*-algebra when equipped with its Banach algebra structure and the adjoint as its involution.

Remarks 1.2.17. It is not difficult to see that if $\mathfrak{A}$ is a norm closed *-subalgebra of $\mathcal{B}(\mathcal{H})$, then $\mathfrak{A}$ is a C*-algebra. In fact, every C*-algebra can be represented this way (see Theorem 1.2.34).

Example 1.2.18. Let $\mathcal{H}$ be a Hilbert space and let $\mathfrak{K}$ denote the set of compact operators on $\mathcal{H}$ (that is, $\mathfrak{K}$ is the closure of the finite rank operators on $\mathcal{H}$). The Calkin algebra, denoted $\mathcal{Q}(\mathcal{H})$, is the quotient algebra $\mathcal{B}(\mathcal{H})/\mathfrak{K}$. The Calkin algebra can be shown to be a C*-algebra.

There has been significant study of elements of a C*-algebra. To discuss some of this theory, we consider the following definitions.

Definition 1.2.19. Let $\mathfrak{A}$ be a unital C*-algebra (that is, there exists an element $I_{\mathfrak{A}} \in \mathfrak{A}$ known as the identity of $\mathfrak{A}$ such that $AI_{\mathfrak{A}} = A = I_{\mathfrak{A}}A$ for all $A \in \mathfrak{A}$). An element $A \in \mathfrak{A}$ is said to be invertible if there exists an element $B \in \mathfrak{A}$ such that $AB = BA = I_{\mathfrak{A}}$.

Remarks 1.2.20. Given a unital C*-algebra $\mathfrak{A}$, we will denote the set of invertible elements by $\mathfrak{A}^{-1}$. It is not difficult to see that $\mathfrak{A}^{-1}$ contains $I_{\mathfrak{A}}$ and is a group under multiplication.

Moreover, it is possible to show that $\mathfrak{A}^{-1}$ is an open subset of $\mathfrak{A}$. This implies the connected component of the identity of $\mathfrak{A}^{-1}$, denoted $\mathfrak{A}_0^{-1}$, is an open subgroup of $\mathfrak{A}^{-1}$ containing $I_{\mathfrak{A}}$.

Remarks 1.2.21. Given a non-unital C*-algebra $\mathfrak{A}$, there is a canonical way to construct a C*-algebra $\widetilde{\mathfrak{A}}$, called the unitization of $\mathfrak{A}$, such that $\mathfrak{A}$ is a maximal ideal in $\widetilde{\mathfrak{A}}$.

Definition 1.2.22. Let $\mathfrak{A}$ be a unital C*-algebra and let $A \in \mathfrak{A}$. The spectrum of A, denoted $\sigma(A)$, is the set

$$\sigma(A) := \{\lambda \in \mathbb{C} \mid \lambda I_{\mathfrak{A}} - A \notin \mathfrak{A}^{-1}\}.$$

For a non-unital C*-algebra $\mathfrak{A}$ and an element $A \in \mathfrak{A}$, we define the spectrum of A, denoted $\sigma(A)$, to be the spectrum of A when we view A as an element of the unitization $\widetilde{\mathfrak{A}}$ of $\mathfrak{A}$.

Remarks 1.2.23. For an element A of a C*-algebra $\mathfrak{A}$, $\sigma(A)$ is always a compact subset of $\mathbb{C}$. Moreover, given two C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathfrak{A} \subseteq \mathfrak{B}$, if $A \in \mathfrak{A} \subseteq \mathfrak{B}$ then the spectrum of A viewed as an element of $\mathfrak{A}$ is the same as the spectrum of A viewed as an element of $\mathfrak{B}$.

Remarks 1.2.24. Let $q : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H})$ be the canonical quotient map from $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra. For an operator $T \in \mathcal{B}(\mathcal{H})$, the essential spectrum of T, denoted $\sigma_e(T)$, is $\sigma_e(T) := \sigma(q(T))$.

With the above definitions in hand, we can now discuss various types of operators of C*-algebras.

Definition 1.2.25. Let $\mathfrak{A}$ be a C*-algebra. An element $T \in \mathfrak{A}$ is said to be:

1. normal if $T^*T = TT^*$.

2. self-adjoint if $T^* = T$.

3. positive if T is normal and $\sigma(T) \subseteq [0, \infty)$.

4. a projection if $T^2 = T^* = T$.

5. a unitary if $\mathfrak{A}$ is unital and $T^*T = TT^* = I_{\mathfrak{A}}$.

The set of normal elements of $\mathfrak{A}$ will be denoted $\mathrm{Nor}(\mathfrak{A})$, the set of self-adjoint elements of $\mathfrak{A}$ will be denote $\mathfrak{A}_{\mathrm{sa}}$, the set of positive elements of $\mathfrak{A}$ will be denoted $\mathfrak{A}_+$, and the set of unitary elements of $\mathfrak{A}$ will be denoted $\mathcal{U}(\mathfrak{A})$. We will write $A \geq 0$ whenever $A \in \mathfrak{A}_+$.

Remarks 1.2.26. It is not difficulty to show that $\mathfrak{A}_+ \subseteq \mathfrak{A}_{\mathrm{sa}} \subseteq \mathrm{Nor}(\mathfrak{A})$, that every projection in $\mathfrak{A}$ is positive, and $\mathcal{U}(\mathfrak{A}) \subseteq \mathrm{Nor}(\mathfrak{A})$.

It turns out the structure theory of normal operators inside a C*-algebra is very nice. To understand such structure, consider the following.

Definition 1.2.27. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras. An algebra homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ is said to be a *-homomorphism if $\pi(A^*) = \pi(A)^*$ for all $A \in \mathfrak{A}$. In the case that $\mathfrak{B} = \mathcal{B}(\mathcal{H})$, a *-homomorphism is also called a representation.

Remarks 1.2.28. It is possible to show that any *-homomorphism between C*-algebras is a contractive map.

Definition 1.2.29. Two C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are said to be isomorphic if there exists a bijective *-homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$.

The structure theory of normal operators in C*-algebras is now apparent.

Theorem 1.2.30 (The Continuous Functional Calculus for Normal Operators)**.** *Let $\mathfrak{A}$ be a C*-algebra and let $N \in \mathfrak{A}$ be a normal operator. Let $C^*(N)$ denote the (abelian) C*-subalgebra of $\mathfrak{A}$ generated by N and N^*. Then $C^*(N)$ is isomorphic to $C(\sigma(N))$.*

Remarks 1.2.31. Note that Theorem 1.2.30 implies that if N is a normal operator on a C*-algebra $\mathfrak{A}$ and f is a continuous function on $\sigma(N)$, then it makes sense to consider $f(N)$.

In order to comprehend and study C*-algebras, it is essential to understand the theory of the representations of a C*-algebra. The following definitions and theorems allow just that.

Definition 1.2.32. Let $\mathfrak{A}$ be a C*-algebra. A state on $\mathfrak{A}$ is a linear functional $\varphi : \mathfrak{A} \to \mathbb{C}$ such that $\|\varphi\| = 1$ and $\varphi(A) \geq 0$ whenever $A \in \mathfrak{A}_+$.

Remarks 1.2.33. If $\mathfrak{A}$ is a unital C*-algebra and φ is a state on $\mathfrak{A}$, then $\varphi(I_{\mathfrak{A}}) = 1$.

Theorem 1.2.34 (GNS Construction)**.** *Let $\mathfrak{A}$ be a (unital) C*-algebra. For every state φ on $\mathfrak{A}$ there exists a (unital) representation $\pi_\varphi : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_\varphi)$ for some Hilbert space $\mathcal{H}_\varphi$ and a unit vector $\xi_\varphi \in \mathcal{H}_\varphi$ such that*

$$\varphi(A) = \langle \pi_\varphi(A)\xi_\varphi, \xi_\varphi \rangle_{\mathcal{H}_\varphi}$$

for all $A \in \mathfrak{A}$. The triple $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)$ is called the GNS representation of φ. In particular, given any C-algebra $\mathfrak{A}$ there exists a Hilbert space $\mathcal{H}$ and an injective (also called faithful) representation $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ for $\mathfrak{A}$.*

In particular, Theorem 1.2.34 allows us to view any C*-algebra as a C*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. This allows us various constructions of new C*-algebras.

Definition 1.2.35. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. The tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$, denoted $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the Hilbert space completion of the algebraic tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$, denoted $\mathcal{H}_1 \odot \mathcal{H}_2$, under the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2}$ such that

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} = \langle \xi_1, \xi_2 \rangle_{\mathcal{H}_1} \langle \eta_1, \eta_2 \rangle_{\mathcal{H}_2}$$

for all $\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \in \mathcal{H}_1 \odot \mathcal{H}_2$.

Remarks 1.2.36. It is possible to show that if $T \in \mathcal{B}(\mathcal{H}_1)$ and $S \in \mathcal{B}(\mathcal{H}_2)$ then there exists a unique element $T \otimes S \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that

$$(T \otimes S)(\xi \otimes \eta) = T\xi \otimes S\eta$$

for all $\xi \in \mathcal{H}_1$ and $\eta \in \mathcal{H}_2$. Furthermore, it is possible to show that

$$\|T \otimes S\| \leq \|T\| \, \|S\|$$

and

$$(T \otimes S)^* = T^* \otimes S^*.$$

Definition 1.2.37. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be C*-algebras and, for each $j \in \{1, 2\}$, let $\pi_j : \mathfrak{A}_j \to \mathcal{B}(\mathcal{H}_j)$ be a faithful representation of $\mathfrak{A}_j$. The minimal (or spacial) tensor product of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, denoted $\mathfrak{A}_1 \otimes_{\min} \mathfrak{A}_2$ (or $\mathfrak{A}_1 \otimes_\sigma \mathfrak{A}_2$) is the C*-completion of the image of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ under the map

$$\pi : \mathfrak{A}_1 \odot \mathfrak{A}_2 \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

defined by

$$\pi(A_1 \otimes A_2) = \pi_1(A_1) \otimes \pi_2(A_2)$$

for all $A_1 \in \mathfrak{A}_1$ and $A_2 \in \mathfrak{A}_2$.

Remarks 1.2.38. A priori it appears that the minimal tensor product of two C*-algebras depends on the representations of those C*-algebras on Hilbert spaces. It is a technically difficult proof to show that this is not the case.

Another way to construct other C*-algebras is the following which leads to an important concept in the theory of C*-algebras.

Remarks 1.2.39. Given a Hilbert space $\mathcal{H}$, the set of n by n matrices with entries from $\mathcal{B}(\mathcal{H})$, denoted $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$, has a canonical *-algebra structure. However, it is possible to show that $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ is isomorphic to $\mathcal{B}(\mathcal{H}^{\oplus n})$ as *-algebras (where, given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathcal{H}_1 \oplus \mathcal{H}_2$ is the Hilbert space given by the vector space direct sum of $\mathcal{H}_1$ and $\mathcal{H}_2$ with the inner product

$$\langle (\xi_1, \eta_1), (\xi_2, \eta_2) \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} = \langle \xi_1, \xi_2 \rangle_{\mathcal{H}_1} + \langle \eta_1, \eta_2 \rangle_{\mathcal{H}_2}$$

for all $\xi_1, \xi_2 \in \mathcal{H}_1$ and $\eta_1, \eta_2 \in \mathcal{H}_2$, and the direct sum of more that two Hilbert spaces is defined recursively (where a completion must be taken in the case there are infinitely many Hilbert spaces)). Hence $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ is a C*-algebra. Furthermore, given a C*-algebra $\mathfrak{A}$

we can view $\mathfrak{A}$ as a C*-subalgebra of $\mathcal{B}(\mathcal{H})$ and thus we can view $\mathcal{M}_n(\mathfrak{A})$ as a *-algebra of $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$. It is possible to show that $\mathcal{M}_n(\mathfrak{A})$ is complete with respect to the norm induced by $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ and thus a C*-algebra.

Definition 1.2.40. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras. A linear map $\varphi : \mathfrak{A} \to \mathfrak{B}$ is said to be a completely positive map if for each $n \in \mathbb{N}$ the map $\varphi_n : \mathcal{M}_n(\mathfrak{A}) \to \mathcal{M}_n(\mathfrak{B})$ defined by

$$\varphi_n([A_{i,j}]) = [\varphi(A_{i,j})]$$

for all $[A_{i,j}] \in \mathcal{M}_n(\mathfrak{A})$ is such that

$$\varphi_n([A_{i,j}]) \geq 0$$

whenever $[A_{i,j}] \in \mathcal{M}_n(\mathfrak{A})$ is such that $[A_{i,j}] \geq 0$.

Example 1.2.41. It is not difficult to show that every *-homomorphism between C*-algebras is completely positive as is every state on a C*-algebra.

For more on completely positive maps, see [51].

1.3 Purely Infinite C*-Algebras

One important concept in C*-algebra theory is the study of the projections of a C*-algebra. The main tool for comparing projections is the following.

Definition 1.3.1. Let $\mathfrak{A}$ be a C*-algebra and let $P_1, P_2 \in \mathfrak{A}$ be projections. We write $P_1 \leq P_2$ if $P_1P_2 = P_2P_1 = P_1$. We say that P_1 and P_2 are Murray-von Neumann equivalent if there exists an element $V \in \mathfrak{A}$ (called a partial isometry) such that $P_1 = V^*V$ and $P_2 = VV^*$.

Example 1.3.2. Two projections $P_1, P_2 \in \mathcal{M}_n(\mathbb{C})$ are Murray-von Neumann equivalent if and only if $\mathrm{rank}(P_1) = \mathrm{rank}(P_2)$.

Example 1.3.3. Two projections $P_1, P_2 \in \mathcal{B}(\mathcal{H})$ are Murray-von Neumann equivalent if and only if $\mathrm{rank}(P_1) = \mathrm{rank}(P_2)$.

Remarks 1.3.4. It is not difficult to see that Murray-von Neumann equivalence of projections is an equivalence relation.

Some important classes of projections are as follows.

Definition 1.3.5. Let $\mathfrak{A}$ be a C*-algebra. A projection $P \in \mathfrak{A}$ is said to be infinite if there exists a projection $P_1 \in \mathfrak{A}$ such that $P_1 \leq P$, $P_1 \neq P$, and P and P_1 are Murray-von Neumann equivalent. A projection $P \in \mathfrak{A}$ is said to be finite if P is not infinite.

Definition 1.3.6. Let $\mathfrak{A}$ be a C*-algebra. A projection $P \in \mathfrak{A}$ is said to be properly infinite if there exists projections $P_1, P_2 \in \mathfrak{A}$ such that $P_1 + P_2 \leq P$ and P_1, P_2, and P are all Murray-von Neumann equivalent.

Example 1.3.7. A projection $P \in \mathcal{B}(\mathcal{H})$ is infinite if and only if P is properly infinite if and only if $\text{rank}(P)$ is infinite.

Remarks 1.3.8. If $\mathfrak{A}$ is a unital C*-algebra with an infinite projection, then the identity of $\mathfrak{A}$ is infinite. To see this, we notice that if P is an infinite projection in $\mathfrak{A}$, then there exists a projection $P_1 \in \mathfrak{A}$ such that $PP_1 = P_1P = P_1$, $P_1 \neq P$, and P and P_1 are Murray-von Neumann equivalent. It is easy to see that $P_1 + (I_{\mathfrak{A}} - P)$ is a projection in $\mathfrak{A}$ that does not equal $I_{\mathfrak{A}}$ yet is Murray-von Neumann equivalent to $I_{\mathfrak{A}}$. Hence $I_{\mathfrak{A}}$ is an infinite projection.

One important class of C*-algebras can be described as follows.

Definition 1.3.9. Let $\mathfrak{A}$ be a C*-algebra. A C*-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is said to be hereditary if whenever $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ are positive operators such that $0 \leq A \leq B$, then $A \in \mathfrak{A}$.

Definition 1.3.10. A C*-algebra $\mathfrak{A}$ is said to be simple if the only closed ideals of $\mathfrak{A}$ are $\{0\}$ and $\mathfrak{A}$.

Definition 1.3.11. Let $\mathfrak{A}$ be a unital, simple C*-algebra. We say that $\mathfrak{A}$ is purely infinite if every non-zero hereditary C*-subalgebra of $\mathfrak{A}$ has an infinite projection.

Example 1.3.12. For an infinite dimensional Hilbert space $\mathcal{H}$, the Calkin algebra $\mathcal{Q}(\mathcal{H})$ can be shown to be a unital, simple, purely infinite C*-algebra.

Example 1.3.13. The universal C*-algebra generated by two operator V_1 and V_2 such that $V_1^*V_1 = I = V_2^*V_2$ and $V_1V_1^* + V_2V_2^* = I$ is called the Cuntz algebra and is denoted $\mathcal{O}_2$. It is possible to show that $\mathcal{O}_2$ is a unital, simple, purely infinite C*-algebra.

Remarks 1.3.14. Notice that if $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra and $P \in \mathfrak{A}$ is a non-zero projection, then $P\mathfrak{A}P$ is a hereditary C*-subalgebra of $\mathfrak{A}$ and thus contains an infinite projection. Since P is the identity element of $P\mathfrak{A}P$, Remarks 1.3.8 implies that P is an infinite projection in $P\mathfrak{A}P$ and thus is an infinite projection in $\mathfrak{A}$. Hence every non-zero projection in a unital, simple, purely infinite C*-algebra is infinite.

In fact, the projection structure of unital, simple, purely infinite C*-algebras is even more elaborate than the above remarks describes. For proofs of the following theorems, see [21, Chapter V].

Theorem 1.3.15. *If $\mathfrak{A}$ is a simple C*-algebra and P is an infinite projection in $\mathfrak{A}$, then for every $n \in \mathbb{N}$ there exists projections $P_1, \ldots, P_n \in \mathfrak{A}$ such that $P_1, \ldots, P_n, P$ are all Murray-von Neumann equivalent and $\sum_{j=1}^n P_j \leq P$. Hence every infinite projection in $\mathfrak{A}$ is properly infinite.*

Theorem 1.3.16. *Let $\mathfrak{A}$ be a simple C*-algebra and let $P_1, P_2 \in \mathfrak{A}$ be projections. If P_1 is infinite then P_2 is Murray-von Neumann equivalent to a subprojection of P_1.*

Theorem 1.3.17. *Every unital, simple, purely infinite C*-algebra $\mathfrak{A}$ has real rank zero. That is, the set of self-adjoint elements of $\mathfrak{A}$ with a finite number of points in their spectrum are dense in $\mathfrak{A}_{\mathrm{sa}}$.*

Theorem 1.3.17 is nice as it allows self-adjoint operators in a unital, simple, purely infinite C*-algebra to be approximated by self-adjoint elements with a finite number of points in their spectrum (which is nice as the Continuous Functional Calculus implies some interesting results). The same concept can be discussed for normal operators.

Definition 1.3.18. Let $\mathfrak{A}$ be a unital C*-algebra. We say that $\mathfrak{A}$ has the finite normal property (property (FN)) if every normal operator in $\mathfrak{A}$ is the limit of normal operators from

$\mathfrak{A}$ with finite spectrum. We say that $\mathfrak{A}$ has the weak finite normal property (property weak (FN)) if every normal operator $N \in \mathfrak{A}$ such that $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$ is the limit of normal operators from $\mathfrak{A}$ with finite spectrum.

Theorem 1.3.19 ([40, Theorem 4.4])**.** *Every unital, simple, purely infinite C*-algebra has property weak (FN).*

Remarks 1.3.20. Given a unital C*-algebra $\mathfrak{A}$, there are two abelian groups $K_0(\mathfrak{A})$ and $K_1(\mathfrak{A})$ that encapsulate information about the projection structure and unitary operator structure of $\mathfrak{A}$ respectively. For a unital, simple, purely infinite C*-algebra $\mathfrak{A}$, $K_0(\mathfrak{A})$ and $K_1(\mathfrak{A})$ are very nice. For more information, see [16].

1.4 Von Neumann Algebras

The class of von Neumann algebras plays an important role in the theory of C*-algebras due to the additional properties held by said algebras. We begin the definitions of said algebras.

Definition 1.4.1. Let $\mathcal{H}$ be a Hilbert space. The weak operator topology on $\mathcal{B}(\mathcal{H})$, abbreviated WOT, is the topology on $\mathcal{B}(\mathcal{H})$ where a net $(T_\lambda)_{\lambda\in\Lambda}$ converges in the WOT to an operator $T \in \mathcal{B}(\mathcal{H})$ if and only if

$$\lim_{\Lambda} \langle T_\lambda \xi, \eta\rangle_{\mathcal{H}} = \langle T\xi, \eta\rangle_{\mathcal{H}}$$

for all $\xi, \eta \in \mathcal{H}$.

Definition 1.4.2. A C*-subalgebra $\mathfrak{M}$ of $\mathcal{B}(\mathcal{H})$ is said to be a von Neumann algebra if $\mathfrak{M}$ is closed in the weak operator topology; that is, if $(t_\lambda)_\Lambda$ is a net of operators from $\mathfrak{M}$ that converge in the weak operator topology to an operator $T \in \mathcal{B}(\mathcal{H})$, then $T \in \mathfrak{M}$.

Example 1.4.3. It is clear that $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra for every Hilbert space $\mathcal{H}$.

Example 1.4.4. Let (X, μ) be a measure space. It is possible to show that $L_\infty(X, \mu)$, the essentially bounded functions on (X, μ), is a von Neumann subalgebra of $\mathcal{B}(L_2(X, \mu))$ via the representation

$$(M_f g)(x) = f(x)g(x)$$

for all $f \in L_\infty(X, \mu)$ and $g \in L_2(X, \mu)$.

Example 1.4.5. Let G be a group (which we view as equipped with the discrete topology) and let $\ell_2(G)$ denote the Hilbert space with $\{\delta_g\}_{g \in G}$ as an orthonormal basis. For each $h \in G$, define $\lambda(h) \in \mathcal{B}(\mathcal{H})$ to be the operator defined by

$$\lambda(h)\delta_g = \delta_{hg}$$

for all $g \in G$. The C*-algebra generated by $\{\lambda(h) \mid h \in G\}$ is called the reduced group C*-algebra of G and is denoted $C^*_{\text{red}}(G)$. The weak operator topology closure of $C^*_{\text{red}}(G)$ is then a von Neumann algebra called the group von Neumann algebra of G and is denoted $L(G)$.

Example 1.4.6. Let $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathfrak{N} \subseteq \mathcal{B}(\mathcal{K})$ be von Neumann algebras. The tensor product of $\mathfrak{M}$ and $\mathfrak{N}$, denoted $\mathfrak{M}\overline{\otimes}\mathfrak{N}$, is the von Neumann subalgebra of $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ obtained by taking the weak operator topology closure of $\mathfrak{M} \otimes_{\min} \mathfrak{N}$.

One important class of von Neumann algebras are described below.

Definition 1.4.7. Let $\mathfrak{M}$ be a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$. The commutant of $\mathfrak{M}$ in $\mathcal{B}(\mathcal{H})$, denoted $\mathfrak{M}'$ is the set

$$\mathfrak{M}' := \{T \in \mathcal{B}(\mathcal{H}) \mid TA = AT \text{ for all } A \in \mathfrak{M}\}.$$

Definition 1.4.8. A von Neumann subalgebra $\mathfrak{M}$ of $\mathcal{B}(\mathcal{H})$ is said to be a factor if $\mathfrak{M} \cap \mathfrak{M}' = \mathbb{C}I_{\mathcal{H}}$.

Remarks 1.4.9. Factors are an important class of von Neumann algebras as every von Neumann algebra can be written as a direct integral of factors. For a more detailed exposition on this, see [35].

It turns out that von Neumann algebras have a plethora of projection operators and the structure of said projections aids in classifying von Neumann algebras. In particular, there are three main types of von Neumann algebras cleverly called type I, type II, and type III von Neumann algebras. Type I von Neumann algebras can be further subdivided type I_n von Neumann algebras (where n is a cardinal number) with a type I_n von Neumann algebra being called finite if n is a natural number. Type II von Neumann algebras can be further subdivided into type II_1 von Neumann algebras (also called finite type II von Neumann algebras) and type II_∞ von Neumann algebras.

It is possible to show that every von Neumann algebra is the direct sum of type I_n, type II_1, type II_∞, and type III von Neumann algebras. Furthermore, each von Neumann algebra of a fixed type is the direct integral of factors of the same type.

Finite von Neumann algebras (which correspond to the von Neumann algebras described above where the identity projection is finite) are particular interesting because of the following objects.

Definition 1.4.10. Let $\mathfrak{A}$ be a C*-algebra. A state τ on $\mathfrak{A}$ is said to be tracial if

$$\tau(AB) = \tau(BA)$$

for all $A, B \in \mathfrak{A}$.

Example 1.4.11. Let G be a discrete group. Using the notation of Example 1.4.5, we define $\tau_G : L(G) \to \mathbb{C}$ by

$$\tau_G(T) = \langle T\delta_e, \delta_e \rangle_{\ell_2(G)}$$

for all $T \in L(G)$. It is not difficult to verify that τ_G is a tracial state on $L(G)$ (in fact, τ_G is faithful).

Remarks 1.4.12. It is possible to show that every finite von Neumann algebra (that is, one that is a sum of finite type I and finite type II von Neumann algebras) has a faithful tracial state τ (that is, the GNS representation of τ is injective). Finite type I von Neumann algebras look like matrices of continuous functions whereas type II_1 von Neumann algebras are more mysterious. In particular, type II_1 factors behave like continuous analogues of matrix algebras and many experts believe type II_1 factors are the correct setting to study linear algebra.

Remarks 1.4.13. The projection structure of factors is very nice. Every type I_n factor is isomorphic to $\mathcal{M}_n(\mathbb{C})$ where we understand the projection structure. Every type I_∞ factor is isomorphic to $\mathcal{B}(\mathcal{H})$ where we also understand the projection structure. If $\mathfrak{M}$ is a type III factor, then every non-zero projection in $\mathfrak{M}$ is properly infinite and any two non-zero projections are Murray-von Neumann equivalent. If $\mathfrak{M}$ is a type II_1 factor, then there is a unique faithful tracial state τ on $\mathfrak{M}$ and two projections $P_1, P_2 \in \mathfrak{M}$ are Murray-von Neumann equivalent if and only if $\tau(P_1) = \tau(P_2)$. If $\mathfrak{M}$ is a type II_∞ factor, there is a type II_1 factor $\mathfrak{N}$ such that $\mathfrak{M} = \mathfrak{N}\overline{\otimes}\mathcal{B}(\mathcal{H})$ for some infinite dimensional Hilbert space $\mathcal{H}$ and the projection structure of $\mathfrak{M}$ can be induced from this isomorphism.

1.5 Free Probability

In [77], Voiculescu introduce the notion of free probability with the goal of solving the following open question.

Question 1.5.1. *For each $n \in \mathbb{N}$, let $\mathbb{F}_n$ be the free group on n generators. For $n, m \in \mathbb{N}\setminus\{1\}$ does $L(\mathbb{F}_n) \simeq L(\mathbb{F}_m)$ imply $n = m$?*

In doing so, Voiculescu created a non-commutative probability theory now known as free probability. The following serves as motivation for how to take classical probability theory and derive a non-commutative probability theory.

Remarks 1.5.2. Let $X_1, \ldots, X_d$ be independent real-valued random variables on $\mathbb{R}$ with compact support. Then for each $j \in \{1, \ldots, d\}$ it is then possible to view each X_j as an element of $\mathcal{B}(L_2(\mu_{X_j}))$ by the formula

$$(X_j(f))(t) := tf(t)$$

for all $t \in \mathbb{R}$ and $f \in L_2(\mu_{X_j})$. If $\xi_j \in L_2(\mu_{X_j})$ is the constant function one (that is $\xi_j(t) = 1$ for all $t \in \mathbb{R}$), we easily see that

$$\langle X_j^n \xi_j, \xi_j \rangle_{L_2(\mu_{X_j})} = \int_{\mathbb{R}} t^n \, d\mu_j(t) = \mathbb{E}(X_j^n)$$

for all $n \in \mathbb{N}$. Therefore, if we consider the unit vector

$$\xi_0 := \xi_1 \otimes \cdots \otimes \xi_d \in L_2(\mu_{X_1}) \otimes \cdots \otimes L_2(\mu_{X_d})$$

and the operators

$$T_1, \ldots, T_d \in \mathcal{B}(L_2(\mu_{X_1})) \otimes \cdots \otimes \mathcal{B}(L_2(\mu_{X_d}))$$

defined by

$$T_j = I_{L_2(\mu_{X_1})} \otimes \cdots \otimes I_{L_2(\mu_{X_{j-1}})} \otimes X_j \otimes I_{L_2(\mu_{X_{j+1}})} \otimes \cdots \otimes I_{L_2(\mu_{X_d})}$$

(where X_j appears as the j^{th} element in the tensor) for all $j \in \{1, \ldots, d\}$, then it is easy to see for each $\ell_1, \ldots, \ell_d \in \mathbb{N} \cup \{0\}$ that

$$\left\langle \left(T_1^{\ell_1} \cdots T_d^{\ell_d} \right) \xi_0, \xi_0 \right\rangle_{L_2(\mu_{X_1}) \otimes \cdots \otimes L_2(\mu_{X_d})} = m_{\ell_1, \ldots, \ell_d}^{X_1, \ldots, X_d}.$$

Thus the operators $T_1, \ldots, T_d$ completely describe the joint distribution of $X_1, \ldots, X_d$

Note that our operators are in

$$\mathcal{B}(L_2(\mu_{X_1})) \otimes \cdots \otimes \mathcal{B}(L_2(\mu_{X_d})) \subseteq \mathcal{B}(L_2(\mu_{X_1}) \otimes \cdots \otimes L_2(\mu_{X_d})) = \mathcal{B}(L_2(\mu))$$

where $\mu = \mu_{X_1} \times \cdots \times \mu_{X_d}$. Thus, saying that $X_1, \ldots, X_d$ are independent random variables is equivalent to saying that when we view $X_1, \ldots, X_d$ as operators on a Hilbert space with the correct joint distribution, the Hilbert space decomposes as a tensor product of smaller Hilbert spaces where $X_1, \ldots, X_d$ act on different tensor products corresponding to their individual moments.

Notice that the operators $T_1, \ldots, T_d$ in the above remarks commute (that is, $T_jT_k = T_kT_j$ for all $j, k \in \{1, \ldots, d\}$). Thus the notion of independence in classical probability theory can be viewed as a commutativity property. As, in algebra, tensor products correspond to commutativity and free products correspond to non-commutativity, we desire a way to represent our operators as free products of operators. This leads us to the following definitions.

Definition 1.5.3. For $j \in \{1, \ldots, d\}$, let $\mathcal{H}_j$ be a Hilbert space, let $\xi_j \in \mathcal{H}_j$ be a unit vector, and let $\mathcal{H}_j^0 := \mathcal{H}_j \ominus \mathbb{C}\xi_j$. The free product of $\mathcal{H}_1, \ldots, \mathcal{H}_d$ with respect to the unit vectors $\xi_1, \ldots, \xi_d$ is the Hilbert space

$$*_{j=1}^d(\mathcal{H}_j, \xi_j) := \mathbb{C}\xi_0 \oplus \left(\bigoplus_{\substack{n \geq 1, \{j_k\}_{k=1}^n \subseteq \{1, \ldots, d\}, \\ j_k \neq j_{k+1} \text{ for } k \in \{1, \ldots, n-1\}}} \mathcal{H}_{j_1}^0 \otimes \mathcal{H}_{j_2}^0 \otimes \cdots \otimes \mathcal{H}_{j_n}^0 \right).$$

(the unit vector ξ_0 is called the distinguished unit vector). In the case $d = 2$, we will write $(\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)$ instead of $*_{j=1}^2(\mathcal{H}_j, \xi_j)$.

Remarks 1.5.4. For $*_{j=1}^d(\mathcal{H}_j, \xi_j)$, the distinguished vector ξ_0 can be viewed as an amalgamation of all of the ξ_j's at once.

Next we desire to determine how operators should act on $*_{j=1}^d(\mathcal{H}_j, \xi_j)$.

Construction 1.5.5. For $j \in \{1, \ldots, d\}$ let $\mathcal{A}_j$ be a unital algebra, let $\pi_j : \mathcal{A}_j \to \mathcal{B}(\mathcal{H}_j)$ be a faithful, unital representation, and let $\xi_j \in \mathcal{H}_j$ be a unit vector (often one can take $\mathcal{A}_j \subseteq \mathcal{B}(\mathcal{H}_j)$ and when $\mathcal{A}_j$ are $*$-algebras, we take π_j to be $*$-representations). There is a canonical action of each $\mathcal{A}_j$ on $*_{j=1}^d(\mathcal{H}_j, \xi_j)$. To define this action let $\mathcal{H}(j)$ be the smallest Hilbert subspace of $*_{j=1}^d(\mathcal{H}_j, \xi_j)$ containing the distinguished vector along with all direct summands $\mathcal{H}_{j_1}^0 \otimes \cdots \otimes \mathcal{H}_{j_n}^0$ of arbitrary length with $j_1 = j$. Then there exists a canonical isomorphism $U_j : \mathcal{H}_j \otimes \mathcal{H}(j) \to *_{j=1}^d(\mathcal{H}_j, \xi_j)$ defined by

$$U_j : \begin{cases} \mathbb{C}\xi_j \otimes \mathbb{C}\xi_0 & \mathbb{C}\xi_0 \\ \mathcal{H}_j^0 \otimes \mathbb{C}\xi_0 & \mathcal{H}_j^0 \\ \mathbb{C}\xi_j \otimes \mathcal{H}_{j_1}^0 \otimes \mathcal{H}_{j_2}^0 \otimes \cdots \otimes \mathcal{H}_{j_n}^0 & \mathcal{H}_{j_1}^0 \otimes \mathcal{H}_{j_2}^0 \otimes \cdots \otimes \mathcal{H}_{j_n}^0 \\ \mathcal{H}_j^0 \otimes \mathcal{H}_{j_1}^0 \otimes \mathcal{H}_{j_2}^0 \otimes \cdots \otimes \mathcal{H}_{j_n}^0 & \mathcal{H}_j^0 \otimes \mathcal{H}_{j_1}^0 \otimes \mathcal{H}_{j_2}^0 \otimes \cdots \otimes \mathcal{H}_{j_n}^0 \end{cases} \quad \overset{\simeq}{\to}$$

where U_j is the canonical isomorphism in each of the four parts listed. We define the action of $\mathcal{A}_j$ on $*_{j=1}^d(\mathcal{H}_j, \xi_j)$ by $A\zeta := U(\pi_j(A) \otimes Id)U^*\zeta$ for all $A \in \mathcal{A}_j$ and for all $\zeta \in *_{j=1}^d(\mathcal{H}_j, \xi_j)$.

Definition 1.5.6. With the above notation and construction, the algebra generated by $\mathcal{A}_1, \ldots, \mathcal{A}_d$ on $\mathcal{B}\left(*_{j=1}^d(\mathcal{H}_j, \xi_j)\right)$ is called the reduced free product of $\mathcal{A}_1, \ldots, \mathcal{A}_d$ with respect to $\pi_1, \ldots, \pi_d$ and is denoted $*_{j=1}^d(\mathcal{A}_j, \pi_j, \xi_j)$. In the case $\mathcal{A}_1, \ldots, \mathcal{A}_d$ are C*-algebras, we will also use $*_{j=1}^d(\mathcal{A}_j, \pi_j, \xi_j)$ to denote the C*-subalgebra of $\mathcal{B}\left(*_{j=1}^d(\mathcal{H}_j, \xi_j)\right)$ generated by $\mathcal{A}_1, \ldots, \mathcal{A}_d$.

Remarks 1.5.7. Given real-valued random variables $X_1, \ldots, X_d$ on $\mathbb{R}$, if for each $j \in \{1, \ldots, d\}$ we view $X_j \in \mathcal{B}(L_2(\mu_{X_j}))$, let $\mathcal{A}_j$ be the subalgebra of $\mathcal{B}(L_2(\mu_{X_j}))$ generated by X_j, and consider $*_{j=1}^d(\mathcal{A}_j, \pi_j, \xi_j)$ where π_j is the inclusion representation of ξ_j is the constant function one in $L_2(\mu_{X_j})$ (see Remarks 1.5.2), it is possible to see that elements of $*_{j=1}^d(\mathcal{A}_j, \pi_j, \xi_j)$ satisfy an interesting relation with respect to the state

$$\varphi(T) := \langle T\xi_0, \xi_0\rangle_{*_{j=1}^d(L_2(\mu_{X_j}), \xi_j)}.$$

Indeed this leads us to the following definitions.

Definition 1.5.8. A non-commutative probability spaces is a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(I_{\mathcal{A}}) = 1$.

Remarks 1.5.9. In free probability, the linear function φ plays the role of the expectation map $\mathbb{E}$ does in classical probability.

Definition 1.5.10. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_d$ of $\mathcal{A}$ are said to be free with respect to φ if

$$\varphi(A_1 A_2 \cdots A_n) = 0$$

whenever $n \in \mathbb{N}$, $j_1, \ldots, j_n \in \{1, \ldots, d\}$ are such that $j_k \neq j_{k+1}$ for all $k \in \{1, \ldots, n-1\}$, and $A_k \in \mathcal{A}_{j_k}$ for all $k \in \{1, \ldots, n\}$ are such that $\varphi(A_k) = 0$.

Similarly, operators $A_1, \ldots, A_d \in \mathcal{A}$ are said to be free with respect to φ if the algebras generated by each individual A_k are free with respect to φ.

Example 1.5.11. Let $X_1, \ldots, X_d$ be real-valued random variables on $\mathbb{R}$. Consider the non-commutative probability space $\left(*_{j=1}^{d}(\mathcal{A}_j, \pi_j, \xi_j), \varphi\right)$ as in Remarks 1.5.7. Then $X_1, \ldots, X_d$ are free with respect to φ.

Remarks 1.5.12. The advantage of considering a non-commutative probability space $(\mathcal{A}, \varphi)$ instead of a probability space $(\mathcal{X}, \Omega, \mu)$ is the fact that elements of $\mathcal{A}$ need not commute which leads to a more interesting structure. Indeed, due to the non-commutativity of $(\mathcal{A}, \varphi)$, the correct analogue for the moments is more complicated.

Definition 1.5.13. Let $A_1, \ldots, A_d$ be elements of a non-commutative probability space $(\mathcal{A}, \varphi)$. For each $n \in \mathbb{N}$ and $j_1, \ldots, j_n \in \{1, \ldots, d\}$, the $(j_1, \ldots, j_n)$-moment of $A_1, \ldots, A_d$ is

$$m_{(j_1, \ldots, j_n)}^{A_1, \ldots, A_d} = \varphi(A_{j_1} \cdots A_{j_n}).$$

Remarks 1.5.14. In general, due to the non-commutative structures, given freely independent elements $A_1, \ldots, A_d \in (\mathcal{A}, \varphi)$ the $(j_1, \ldots, j_n)$-moment of $A_1, \ldots, A_d$ does not simply depend on how many j_k's are equal to ℓ for each $\ell \in \{1, \ldots, d\}$ as does for independent random variables (see Remarks 1.1.14). Indeed, if X_1 and X_2 are freely independent with respect to φ, it need not be the case that $\varphi(X_1 X_2 X_1 X_2)$ and $\varphi(X_1 X_1 X_2 X_2)$ agree. For example, suppose $\varphi(X_1) = 0 = \varphi(X_2)$ and $\varphi(X_1^2) = 1 = \varphi(X_2^2)$. Then

$$\varphi(X_1X_2X_1X_2) = 0$$

by the freeness of X_1 and X_2. However, again by the freeness of X_1 and X_2,

$$\begin{aligned} 0 &= \varphi\left(\left(X_1^2 - \varphi\left(X_1^2\right)\right)\left(X_2^2 - \varphi\left(X_2^2\right)\right)\right) \\ &= \varphi(X_1X_1X_2X_2) - 2\varphi(X_1^2)\varphi(X_2^2) + \varphi(X_1^2)\varphi(X_2^2) \end{aligned}$$

so $\varphi(X_1X_1X_2X_2) = \varphi(X_1^2)\varphi(X_2^2) = 1 \neq \varphi(X_1X_2X_1X_2)$.

Remarks 1.5.15. Given random variables $X_1, \ldots, X_d$ that are freely independent with respect to φ, the trick of considering $X_j^n - \varphi(X_j^n)$ along with a recursive argument may be used to show that the joint moments of $X_1, \ldots, X_d$ depend only on the individual moments of each X_j.

As the Gaussian distribution distribution (see Definition 1.1.15) plays a central role in classical probability theory, the following distribution lies at the centre of free probability.

Definition 1.5.16. The normalized semicircular distribution centred at zero is the measure μ_{semi} on $[-2, 2]$ defined by

$$\mu_{\text{semi}}([a, b]) = \frac{1}{2\pi} \int_a^b \sqrt{4 - x^2}\, dx$$

for all $[a, b] \subseteq [-2, 2]$.

Remarks 1.5.17. It is a simple computation to show that the n^{th}-moment of μ_{semi} is zero if n is odd and otherwise, if $n = 2k$, the $2k^{\text{th}}$-moment of μ_{semi} is the k^{th} Catalan number $c_k := \frac{1}{k+1}\binom{2k}{k}$.

The following theorem is then the free analogue of Theorem 1.1.16.

Theorem 1.5.18 (Free Central Limit Theorem)**.** *Let $X_1, X_2, \ldots$ be freely independent, identically distributed random variables in a non-commutative probability space $(\mathcal{A}, \varphi)$ with*

$\varphi(X_j) = 0$ and $\varphi(X_j^2) = 1$ for all $j \in \mathbb{N}$. Then, for all $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \varphi\left(\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n} X_j\right)^m\right) = \int_{-2}^{2} t^m \, d\mu_{\text{semi}}(t).$$

Remarks 1.5.19. As in classical probability theory, it is possible to determine the distribution of $X_1 + X_2$ when X_1 and X_2 are freely independent random variables. Indeed this was first done in [78] where the answer is the free additive convolution of μ_{X_1} and μ_{X_2}, denoted $\mu_{X_1} \boxplus \mu_{X_2}$.

1.6 Exact C*-Algebras

The notion of an exact C*-algebra has played a fundamental role in the theory of C*-algebras and has been well-studied by Kirchberg, Wassermann, and others (see [37] and [83]). Exact C*-algebras are generally well-behaved and many of the common and interesting examples of C*-algebras are exact. In addition, the property that a C*-algebra is exact is preserved under many common operations such as taking subalgebras, taking direct sums, taking minimal tensor products, and taking reduced free products (for example, see [14] and the references therein). Over the years many equivalent definitions of an exact C*-algebra have been developed and the most common are listed in the following theorem.

Theorem 1.6.1 (Due to Kirchberg, Wassermann, and others; see [14] for the proof of the first three equivalences)**.** *Let $\mathfrak{B}$ be a C*-algebra. Then the following are equivalent:*

1. *For every Hilbert space $\mathcal{H}$ and faithful representation $\sigma : \mathfrak{B} \to \mathcal{B}(\mathcal{H})$ there exist nets $(\varphi_\lambda : \mathfrak{B} \to \mathcal{M}_{n_\lambda}(\mathbb{C}))_\Lambda$ and $(\psi_\lambda : \mathcal{M}_{n_\lambda}(\mathbb{C}) \to \mathcal{B}(\mathcal{H}))_\Lambda$ of contractive, completely positive maps such that*

$$\lim_\Lambda \|\sigma(B) - \psi_\lambda(\varphi_\lambda(B))\| = 0$$

   for all $B \in \mathfrak{B}$.

2. *For every exact sequence of C^*-algebras $0 \to \mathfrak{J} \xrightarrow{i} \mathfrak{A} \xrightarrow{q} (\mathfrak{A}/\mathfrak{J}) \to 0$ the sequence*

$$0 \to \mathfrak{J} \otimes_{\min} \mathfrak{B} \xrightarrow{i \otimes Id_{\mathfrak{B}}} \mathfrak{A} \otimes_{\min} \mathfrak{B} \xrightarrow{q \otimes Id_{\mathfrak{B}}} (\mathfrak{A}/\mathfrak{J}) \otimes_{\min} \mathfrak{B} \to 0$$

is exact.

3. *For any sequence $(\mathfrak{A}_n)_{n\geq 1}$ of unital C^*-algebras the *-homomorphism*

$$\left(\frac{\prod_{n\geq 1} \mathfrak{A}_n}{\bigoplus_{n\geq 1} \mathfrak{A}_n}\right) \odot \mathfrak{B} \to \frac{\left(\prod_{n\geq 1} \mathfrak{A}_n\right) \otimes_{\min} \mathfrak{B}}{\left(\bigoplus_{n\geq 1} \mathfrak{A}_n\right) \otimes_{\min} \mathfrak{B}}$$

defined by

$$\left((A_n)_{n\geq 1} + \bigoplus_{n\geq 1} \mathfrak{A}_n\right) \otimes B \mapsto (A_n)_{n\geq 1} \otimes B + \left(\bigoplus_{n\geq 1} \mathfrak{A}_n\right) \otimes_{\min} \mathfrak{B}$$

is continuous with respect to the minimal tensor norm on $\left(\frac{\prod_{n\geq 1} \mathfrak{A}_n}{\bigoplus_{n\geq 1} \mathfrak{A}_n}\right) \odot \mathfrak{B}$.

4. *If $\mathfrak{A}_n$ and $\mathfrak{A}$ are unital C^*-algebras, $k \in \mathbb{N}$, $\{A_i\}_{i=1}^k \subseteq \mathfrak{A}$, and $\{A_{i,n}\}_{i=1}^k \subseteq \mathfrak{A}_n$ are such that $\|p(A_1, \ldots, A_k)\|_{\mathfrak{A}} = \limsup_{n\to\infty} \|p(A_{1,n}, \ldots, A_{k,n})\|_{\mathfrak{A}_n}$ for every polynomial p in k non-commuting variables and their complex conjugates, then for all $B_1, \ldots, B_k \in \mathfrak{B}$*

$$\left\|\sum_{i=1}^k A_i \otimes B_i\right\|_{\mathfrak{A}\otimes_{\min}\mathfrak{B}} = \limsup_{n\to\infty} \left\|\sum_{i=1}^k A_{i,n} \otimes B_i\right\|_{\mathfrak{A}_n\otimes_{\min}\mathfrak{B}}.$$

If one of the above conditions holds then $\mathfrak{B}$ is said to be an exact C^*-algebra.

The proof of the equivalence of the third and fourth conditions is non-standard yet simple and thus is presented below.

Lemma 1.6.2. *For any C^*-algebra $\mathfrak{B}$ and any sequence of unital C^*-algebras $(\mathfrak{A}_n)_{n\geq 1}$ there exists an injective *-homomorphism*

$$\Phi : \frac{\left(\prod_{n\geq 1} \mathfrak{A}_n\right) \otimes_{\min} \mathfrak{B}}{\left(\bigoplus_{n\geq 1} \mathfrak{A}_n\right) \otimes_{\min} \mathfrak{B}} \to \frac{\prod_{n\geq 1}(\mathfrak{A}_n \otimes_{\min} \mathfrak{B})}{\bigoplus_{n\geq 1}(\mathfrak{A}_n \otimes_{\min} \mathfrak{B})}$$

defined by

$$\Phi\left((A_n)_{n\geq 1}\otimes B+\left(\bigoplus_{n\geq 1}\mathfrak{A}_n\right)\otimes_{\min}\mathfrak{B}\right):=(A_n\otimes B)_{n\geq 1}+\bigoplus_{n\geq 1}(\mathfrak{A}_n\otimes_{\min}\mathfrak{B})$$

for all $(A_n)_{n\geq 1}\in\prod_{n\geq 1}\mathfrak{A}_n$ *and* $B\in\mathfrak{B}$.

Proof. Consider the map $\pi_0:\left(\prod_{n\geq 1}\mathfrak{A}_n\right)\odot\mathfrak{B}\to\prod_{n\geq 1}(\mathfrak{A}_n\otimes_{\min}\mathfrak{B})$ defined by

$$\pi_0((A_n)_{n\geq 1}\otimes B):=(A_n\otimes B)_{n\geq 1}.$$

It is easy to verify that π_0 is well-defined, continuous, and isometric with respect to the minimal tensor products and thus induces a injective *-homomorphism

$$\pi:\left(\prod_{n\geq 1}\mathfrak{A}_n\right)\otimes_{\min}\mathfrak{B}\to\prod_{n\geq 1}(\mathfrak{A}_n\otimes_{\min}\mathfrak{B}).$$

Clearly

$$\pi\left(\left(\bigoplus_{n\geq 1}\mathfrak{A}_n\right)\otimes_{\min}\mathfrak{B}\right)\subseteq\bigoplus_{n\geq 1}(\mathfrak{A}_n\otimes_{\min}\mathfrak{B}).$$

Therefore the *-homomorphism

$$\Phi:\frac{\left(\prod_{n\geq 1}\mathfrak{A}_n\right)\otimes_{\min}\mathfrak{B}}{\left(\bigoplus_{n\geq 1}\mathfrak{A}_n\right)\otimes_{\min}\mathfrak{B}}\to\frac{\prod_{n\geq 1}(\mathfrak{A}_n\otimes_{\min}\mathfrak{B})}{\bigoplus_{n\geq 1}(\mathfrak{A}_n\otimes_{\min}\mathfrak{B})}$$

as described in the statement of the lemma exists.

To see Φ is injective, suppose $T\in\left(\prod_{n\geq 1}\mathfrak{A}_n\right)\otimes_{\min}\mathfrak{B}$ and

$$\pi(T)\in\bigoplus_{n\geq 1}(\mathfrak{A}_n\otimes_{\min}\mathfrak{B}).$$

Let $(B_\lambda)_\Lambda$ be a C*-bounded approximate identity for $\mathfrak{B}$. For each $n \in \mathbb{N}$ and $\lambda \in \Lambda$ let

$$E_{n,\lambda} := (I_{\mathfrak{A}_1}, I_{\mathfrak{A}_2}, \cdots, I_{\mathfrak{A}_n}, 0, 0, \cdots) \otimes B_\lambda \in \left(\bigoplus_{n\geq 1} \mathfrak{A}_n\right) \otimes_{\min} \mathfrak{B}.$$

Define a partial ordering on $\mathbb{N} \times \Lambda$ by $(n, \lambda) \leq (m, \lambda')$ if and only if $n \leq m$ and $\lambda \leq \lambda'$. It is easy to verify that $(E_{n,\lambda})_{\mathbb{N}\times\Lambda}$ is a C*-bounded approximate identity for $\left(\bigoplus_{n\geq 1} \mathfrak{A}_n\right) \otimes_{\min} \mathfrak{B}$ and $(\pi(E_{n,\lambda}))_{\mathbb{N}\times\Lambda}$ is a C*-bounded approximate identity for $\bigoplus_{n\geq 1}(\mathfrak{A}_n \otimes_{\min} \mathfrak{B})$. Whence

$$\lim_{\mathbb{N}\times\Lambda} \|\pi(TE_{n,\lambda} - T)\| = \lim_{\mathbb{N}\times\Lambda} \|\pi(T)\pi(E_{n,\lambda}) - \pi(T)\| = 0.$$

Since π is isometric, $\lim_{\mathbb{N}\times\Lambda} \|TE_{n,\lambda} - T\| = 0$ so

$$T = \lim_{\mathbb{N}\times\Lambda} TE_{n,\lambda} \in \left(\bigoplus_{n\geq 1} \mathfrak{A}_n\right) \otimes_{\min} \mathfrak{B}.$$

Thus $\ker(\pi) = \left(\bigoplus_{n\geq 1} \mathfrak{A}_n\right) \otimes_{\min} \mathfrak{B}$ so Φ is injective. □

Proof that the third and fourth statements of Theorem 1.6.1 are equivalent. Let

$$T = \sum_{i=1}^{k} A_i \otimes B_i \in \left(\frac{\prod_{n\geq 1} \mathfrak{A}_n}{\bigoplus_{n\geq 1} \mathfrak{A}_n}\right) \odot \mathfrak{B}$$

be arbitrary. For all $i \in \{1, \ldots, k\}$ there exists $A_{i,n} \in \mathfrak{A}_n$ such that

$$\|p(A_1, \ldots, A_k)\|_{\mathfrak{A}} = \limsup_{n\to\infty} \|p(A_{1,n}, \ldots, A_{k,n})\|_{\mathfrak{A}_n}$$

for every polynomials p in k non-commutating variables and their complex conjugates (that is, choose a lifting of each A_i). If $\mathfrak{B}$ satisfies the fourth statement of Theorem 1.6.1 then

$$\begin{aligned}
\|T\| &= \limsup_{n\to\infty} \left\|\sum_{i=1}^k A_{i,n} \otimes B_i\right\| \\
&= \left\|\left(\sum_{i=1}^k A_{i,n} \otimes B_i\right)_{n\geq 1} + \bigoplus_{n\geq 1} (\mathfrak{A}_n \otimes_{\min} \mathfrak{B})\right\| \\
&= \left\|\sum_{i=1}^k (A_{i,n})_{n\geq 1} \otimes B_i + \left(\bigoplus_{n\geq 1} \mathfrak{A}_n\right) \otimes_{\min} \mathfrak{B}\right\|
\end{aligned}$$

where the last equality follows from Lemma 1.6.2. Thus the fourth statement of Theorem 1.6.1 implies the third statement.

For the other direction, suppose $\mathfrak{B}$ satisfies the third statement in Theorem 1.6.1. Let $\mathfrak{A}_n$ and $\mathfrak{A}$ be unital C*-algebras, let $k \in \mathbb{N}$, let $A_1, \ldots, A_k \in \mathfrak{A}$, and let $\{A_{i,n}\}_{i=1}^k \subseteq \mathfrak{A}_n$ be such that

$$\|p(A_1, \ldots, A_k)\|_{\mathfrak{A}} = \limsup_{n\to\infty} \|p(A_{1,n}, \ldots, A_{k,n})\|_{\mathfrak{A}_n}$$

for every non-commutative polynomials p in k-variables and their complex conjugates. We may assume that $\mathfrak{A} = \overline{^*\text{-alg}(A_1, \ldots, A_k)}$ by properties of the minimal tensor product.

Fix $B_1, \ldots, B_k \in \mathfrak{B}$. The third equivalence in Theorem 1.6.1 implies that the canonical inclusion

$$\left(\frac{\prod_{n\geq 1} \mathfrak{A}_n}{\bigoplus_{n\geq 1} \mathfrak{A}_n}\right) \odot \mathfrak{B} \to \frac{\left(\prod_{n\geq 1} \mathfrak{A}_n\right) \otimes_{\min} \mathfrak{B}}{\left(\bigoplus_{n\geq 1} \mathfrak{A}_n\right) \otimes_{\min} \mathfrak{B}}$$

is continuous with respect to the minimal tensor product and extends to an injective inclusion on the minimal tensor product. By the assumptions on $\mathfrak{A}$, $\mathfrak{A} \subseteq (\prod_{n\geq 1} \mathfrak{A}_n)/(\bigoplus_{n\geq 1} \mathfrak{A}_n)$ via the identification of A_i with $(A_{i,n})_{n\geq 1} + \bigoplus_{n\geq 1} \mathfrak{A}_n$. Thus

$$\begin{aligned}
\left\|\sum_{i=1}^k A_i \otimes B_i\right\| &= \left\|\sum_{i=1}^k \left((A_{i,n})_{n\geq 1} + \left(\bigoplus_{n\geq 1} \mathfrak{A}_n\right)\right) \otimes B_i\right\| \\
&= \left\|\sum_{i=1}^k (A_{i,n})_{n\geq 1} \otimes B_i + \left(\bigoplus_{n\geq 1} \mathfrak{A}_n\right) \otimes_{\min} \mathfrak{B}\right\| \\
&= \left\|\sum_{i=1}^k (A_{i,n})_{n\geq 1} \otimes B_i + \bigoplus_{n\geq 1} (\mathfrak{A}_n \otimes_{\min} \mathfrak{B})\right\|
\end{aligned}$$

(where the last equality follows from Lemma 1.6.2) so

$$\left\|\sum_{i=1}^{k} A_i \otimes B_i\right\|_{\mathfrak{A}\otimes_{\min}\mathfrak{B}} = \limsup_{n\to\infty}\left\|\sum_{i=1}^{k} A_{i,n} \otimes B_i\right\|_{\mathfrak{A}_n\otimes_{\min}\mathfrak{B}}$$

as desired. $\square$

1.7 Strong Atiyah Conjecture for Groups

The Strong Atiyah Conjecture for Groups was introduced via a question in [8] where Atiyah asked whether the analytic L_2-Betti numbers of certain Riemannian G-manifolds were always rational. We will briefly outline the material pertaining to the Strong Atiyah Conjecture related to this dissertation (for a more comprehensive treatment, see [44]).

Definition 1.7.1. Let G be a discrete group and let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{e_n\}_{n\geq 1}$. For a positive operator $T \in \mathcal{B}(\ell_2(G)\otimes\mathcal{H})$, we define the von Neumann trace of T to be the element of $[0,\infty]$ defined by

$$\mathrm{tr}_{L(G)}(T) := \sum_{n\geq 1}\langle T(\delta_e\otimes e_n), \delta_e\otimes e_n\rangle_{\ell_2(G)\otimes\mathcal{H}}.$$

Definition 1.7.2. Let G be a discrete group, let $\mathcal{H}$ be a finite dimensional Hilbert space, let $P_{\mathcal{M}}$ be a projection in $L(G)\overline{\otimes}\mathcal{B}(\mathcal{H})$, and let $\mathcal{M}\subseteq \ell_2(G)\otimes\mathcal{H}$ denote the range of $P_{\mathcal{M}}$. We define the von Neumann dimension of $\mathcal{M}$ to be

$$\dim_{L(G)}(\mathcal{M}) := \mathrm{tr}_{L(G)}(P_{\mathcal{M}}) \in [0,\infty).$$

Example 1.7.3. If $G=\{1\}$, then $L(G)=\mathbb{C}$ and $\mathrm{tr}_{L(G)}$ is the standard trace on $\ell_2(G)\otimes\mathcal{H}\simeq\mathcal{H}$ for any Hilbert space $\mathcal{H}$. Thus the von Neumann dimension of a closed subspace is the complex dimension of the subspace.

Example 1.7.4. Let $G=\mathbb{Z}$. Then we can view $L(G)$ as $L_\infty(\mathbb{T})$ acting on $L_2(\mathbb{T})$ as in Remarks 1.5.2. Therefore, if $X\subseteq\mathbb{T}$ is a measureable subset, the characteristic function of

X, denoted χ_X, is a projection in $L_\infty(\mathbb{T})$ and thus its image defines a closed subspace of $L_2(\mathbb{T})$. In this case

$$\mathrm{tr}_{L(G)}(\chi_X) = \langle \chi_X \delta_e, \delta_e \rangle = \int_{\mathbb{T}} \chi_X(x) dm(x) = m(X)$$

where $m(X)$ is the Lebesgue measure of X. Thus it possible to obtain a closed subspace with von Neumann dimension to be any number in $[0, 1]$.

With the notion of von Neumann dimension complete, we can begin to examine the Strong Atiyah Conjecture.

Definition 1.7.5. Let G be a discrete group and let $\mathcal{FIN}(G)$ denote the set of all finite subgroups of G. We denote by $\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$ the (additive) subgroup of $\mathbb{Q}$ generated by the set of rational numbers $\frac{1}{n}$ where $n = |H|$ for some element $H \in \mathcal{FIN}(G)$.

Conjecture 1.7.6 (Strong Atiyah Conjecture)**.** *A discrete group G satisfies the strong Atiyah Conjecture if*

$$\dim_{L(G)}(\ker(\lambda_n(A))) \in \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$$

for any matrix $A \in \mathcal{M}_n(\mathbb{C}G)$.

Since the only finite subgroup of a free group is the trivial group, the Strong Atiyah Conjecture for the free groups reduces to the following.

Theorem 1.7.7 (Strong Atiyah Conjecture for Free Groups)**.** *If $\mathbb{F}_m$ is the free group generated by $m \in \mathbb{N} \cup \{\infty\}$ elements, then*

$$\dim_{L(\mathbb{F}_m)}(\ker(\lambda_n(A))) \in \mathbb{Z}$$

for any matrix $A \in \mathcal{M}_n(\mathbb{C}\mathbb{F}_m)$.

Unfortunately, it is known that Conjecture 1.7.6 is false.

Example 1.7.8. The lamplighter group L is the group

$$L := \left(\bigoplus_{n\in\mathbb{Z}} \mathbb{Z}_2 \right) \rtimes \mathbb{Z}$$

where the semidirect product is taken with respect to the shift automorphism on $\bigoplus_{n\in\mathbb{Z}} \mathbb{Z}_2$ sending $(g_n)_{n\geq 1}$ to $(g_{n-1})_{n\geq 1}$. Let $e_0 \in \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}_2$ be the elements whose entries are all zero except for the entry at zero (which then must be 1) and let S denote the generator of $\mathbb{Z}$ in L. It is then easy to see that $\{e_0, S\}$ generates L as a group.

Let $M : \ell_2(G) \to \ell_2(G)$ be the operator defined by left multiplication by $\frac{1}{4}(e_0S + S + (e_0S)^{-1} + S^{-1})$. Then it is possible to show that the von Neumann dimension of the kernel of M with respect to the group L is $\frac{1}{3}$ yet every finite subgroup of L has cardinality of the form 2^n so $\frac{1}{3} \notin \frac{1}{|\mathcal{FIN}(L)|}\mathbb{Z}$. For references, see [44, Theorem 10.23].

However, Theorem 1.7.7 is true. For a simple proof, see [44]. Alternatively, Theorem 3.3.1 provides a more general result and thus a complicated proof that Theorem 1.7.7 is true.

1.8 Single Operator Theory on Hilbert Spaces

Single Operator Theory on Hilbert spaces is a large area of functional analysis dedicated to determining properties and approximations of operators on Hilbert spaces. We will briefly outline some results in Single Operator Theory.

Definition 1.8.1. Let $\mathfrak{A}$ be a C*-algebra. An element $T \in \mathfrak{A}$ is said to be nilpotent if there exists an $n \in \mathbb{N}$ such that $T^n = 0$. An element $T \in \mathfrak{A}$ is said to be quasinilpotent if $\sigma(T) = \{0\}$. We will denote the set of nilpotent elements of $\mathfrak{A}$ by $\mathrm{Nil}(\mathfrak{A})$ and the set of quasinilpotent elements of $\mathfrak{A}$ by $\mathrm{QuasiNil}(\mathfrak{A})$.

It is not difficult to see that $\mathrm{Nil}(\mathcal{M}_n(\mathbb{C})) = \mathrm{QuasiNil}(\mathcal{M}_n(\mathbb{C}))$ and both of these sets are closed in the operator topology. However, if $\mathcal{H}$ is an infinite dimensional Hilbert space, $\mathrm{Nil}(\mathcal{B}(\mathcal{H})) \neq \mathrm{QuasiNil}(\mathcal{B}(\mathcal{H}))$ and neither set is closed in the operator topology. In [27, Problem 7] Halmos posed the question, "Is every quasinilpotent operator in $\mathcal{B}(\mathcal{H})$ the norm limit of nilpotent operators?" An affirmative answer to this question was first given in [7], a subsequent proof was given in [6], and a simpler proof was given in [3].

However Halmos realized that his question was 'wrong' in the sense that there are non-quasinilpotent operators in $\mathcal{B}(\mathcal{H})$ that are in the closure of the nilpotent operators (see [28]). This led to the question, "What is the closure of the set of nilpotent operators in $\mathcal{B}(\mathcal{H})$?" The answer to this question was first given in [4]:

Theorem 1.8.2 ([4], see [32, Theorem 5.1] for a proof)**.** *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is a norm limit of nilpotent operators from $\mathcal{B}(\mathcal{H})$ if and only if the following conditions are satisfied:*

1. *The spectrum of T is connected and contains zero.*

2. *The essential spectrum of T is connected and contains zero.*

3. *The Fredholm index of $\lambda I_{\mathcal{H}} - T$ is zero for all $\lambda \in \mathbb{C}$ such that $\lambda I_{\mathcal{H}} - T$ is semi-Fredholm.*

A significant amount of work on this problem was done by Herrero (see [29], [30], and [31]). In particular, before [4], Herrero proved the following interesting result that is a specific case of Theorem 1.8.2.

Theorem 1.8.3 ([29, Theorem 7], also see [32, Proposition 5.6])**.** *Let $N \in \mathcal{B}(\mathcal{H})$ be a normal operator. Then the following are equivalent:*

1. *N is a norm limit of nilpotent operators from $\mathcal{B}(\mathcal{H})$.*

2. *N is a norm limit of quasinilpotent operators from $\mathcal{B}(\mathcal{H})$.*

3. *The spectrum of N is connected and contains zero.*

Another interesting proof of Theorem 1.8.3 was given in [26]. In fact, the techniques for one direction of the proof of Theorem 1.8.3 implies the following.

Lemma 1.8.4. *Let $\mathfrak{A}$ be a C^*-algebra and let $T \in \mathfrak{A}$ be a limit of quasinilpotent operators from $\mathfrak{A}$. Then the spectrum of T is connected and contains zero.*

Proof. If $T \in \mathfrak{A}$ is a limit of quasinilpotent operators from $\mathfrak{A}$ then zero must be in the spectrum of T since quasinilpotent operators are not invertible and the set of invertible elements of $\mathfrak{A}$ is an open set. Furthermore, if the spectrum of T is not connected, the spectrum of T would be contained in the union of two disjoint open sets. By the semicontinuity of the spectrum, this would imply that any sequence of elements from $\mathfrak{A}$ converging to T must eventually have spectrum contained in both open sets. As the spectrum of a quasinilpotent operator is a singleton, a contradiction is reached. $\square$

While studying this problem in $\mathcal{B}(\mathcal{H})$, a solution to the same question for the Calkin algebra was developed.

Theorem 1.8.5 ([32, Theorem 5.34])**.** *Let $\mathcal{B}(\mathcal{H})$ be the bounded linear operators on a complex, separable Hilbert space $\mathcal{H}$, let $\mathfrak{A}$ be the Calkin algebra, let $q : \mathcal{B}(\mathcal{H}) \to \mathfrak{A}$ be the canonical quotient map, and let $T \in \mathcal{B}(\mathcal{H})$. Then $q(T)$ is a norm limit of nilpotent operators from $\mathfrak{A}$ if and only if the essential spectrum of T is connected and contains zero and the Fredholm index of $\lambda I_{\mathcal{H}} - T$ is zero for all λ such that $\lambda I_{\mathcal{H}} - T$ is semi-Fredholm.*

For an excellent summary of the above work, see [32] and [2].

Definition 1.8.6. Let $\mathfrak{A}$ be a unital C*-algebra and let $A \in \mathfrak{A}$. The unitary orbit of A, denoted $\mathcal{U}(A)$, is the set

$$\mathcal{U}(A) := \{UAU^* \in \mathfrak{A} \mid U \in \mathcal{U}(\mathfrak{A})\}.$$

The similarity orbit of A, denoted $\mathcal{S}(A)$, is the set

$$\mathcal{S}(A) := \left\{VAV^{-1} \in \mathfrak{A} \mid V \in \mathfrak{A}^{-1}\right\}.$$

Remarks 1.8.7. Notice if $B \in \mathfrak{A}$ then $B \in \mathcal{U}(A)$ if and only if $A \in \mathcal{U}(B)$ and $B \in \mathcal{S}(A)$ if and only if $A \in \mathcal{S}(B)$. We will denote $B \in \mathcal{U}(A)$ by $A \sim_u B$ and we will denote $B \in \mathcal{S}(A)$ by $A \sim B$. Clearly $\sim_u$ and $\sim$ are equivalence relations.

Example 1.8.8. Given two normal matrices $N_1, N_2 \in \mathcal{M}_n(\mathbb{C})$, $N_1 \sim_u N_2$ if and only if N_1 and N_2 have the same eigenvalues (counting multiplicities).

Example 1.8.9. Given two matrices $A, B \in \mathcal{M}_n(\mathbb{C})$, $A \sim B$ if and only if A and B have the same Jordan Normal Form.

Example 1.8.10. If $N_1, N_2 \in \mathcal{B}(\mathcal{H})$ are normal operators and $\mathcal{H}$ is infinite dimensional, it is possible that $\sigma(N_1) = \sigma(N_2)$ yet N_1 and N_2 are not unitarily equivalent. The main issue comes from the fact that it is possible that N_1 has eigenvalues whereas N_2 has no eigenvalues and thus N_1 and N_2 cannot be unitarily equivalent. Thus, for an arbitrary C*-algebra, we should consider slightly different objects.

Remarks 1.8.11. We will use $\overline{\mathcal{U}(A)}$ and $\overline{\mathcal{S}(A)}$ to denote the norm closures in $\mathfrak{A}$ of the unitary and similarity orbits of A respectively. Note if $B \in \overline{\mathcal{U}(A)}$ then $A \in \overline{\mathcal{U}(B)}$ and $B \in \overline{\mathcal{S}(A)}$. If $B \in \overline{\mathcal{U}(A)}$ we will say that A and B are approximately unitarily equivalent in $\mathfrak{A}$ and will write $A \sim_{au} B$. Clearly $\sim_{au}$ is an equivalence relation. Furthermore if A is a normal operator and $A \sim_{au} B$ then B is a normal operator. If $B \in \overline{\mathcal{S}(A)}$ then it is not necessary that $A \in \overline{\mathcal{S}(B)}$ and B need not be normal if A is normal. However if $B \in \overline{\mathcal{S}(A)}$ and $C \in \overline{\mathcal{S}(B)}$ then $C \in \overline{\mathcal{S}(A)}$.

For $\mathcal{B}(\mathcal{H})$ and $\mathcal{Q}(\mathcal{H})$, the answer to the question of when two normal operators are approximately unitarily equivalent is well known (and amazing mathematics).

Theorem 1.8.12 (Weyl-von Neumann-Berg Theorem)**.** *Let $N_1, N_2 \in \mathcal{B}(\mathcal{H})$ be normal operators. Then $N_1 \sim_{au} N_2$ if and only if*

1. *$\sigma(N_1) = \sigma(N_2)$, and*

2. *if $\lambda \in \sigma(N_1)$ is an isolated point,* $\dim(\ker(\lambda I_{\mathcal{H}} - N_1)) = \dim(\ker(\lambda I_{\mathcal{H}} - N_2))$.

Theorem 1.8.13 (Brown-Douglas-Fillmore Theorem; see [12])**.** *Let $N_1, N_2 \in \mathcal{Q}(\mathcal{H})$ be normal operators. Then $N_1 \sim_{au} N_2$ if and only if*

1. *$\sigma_e(N_1) = \sigma_e(N_2)$, and*

2. *if $\lambda \notin \sigma(N_1)$ then $\lambda I - N_1$ and $\lambda I - N_2$ are in the same connect component of $\mathfrak{A}^{-1}$.*

Closed similarity orbits are more complicated than closed unitary orbits.

Remarks 1.8.14. It is an easy application of the semicontinuity of the spectrum to show that if $A, B \in \mathfrak{A}$ are such that $B \in \overline{\mathcal{S}(A)}$ then $\sigma(A) \subseteq \sigma(B)$ and $\sigma(A)$ intersects every connected component of $\sigma(B)$. Thus $\sigma(A) = \sigma(B)$ whenever $A, B \in \mathfrak{A}$ are approximately unitarily equivalent.

An almost complete classification (which excludes certain pathelogical examples) of the closed similarity orbit of an arbitrary bounded linear operator on a complex, infinite dimensional Hilbert space was announced in [5, Theorem 1] and a proof was given in [2, Theorem 9.2]. An easy modification of the proof of [5, Theorem 1] led to an almost complete classification of the closed similarity orbit of an arbitrary operator in the Calkin algebra (announced in [5, Theorem 2] and proved in [2, Theorem 9.3]). The following is a reduction of these results to normal operators in the Calkin algebra.

Theorem 1.8.15 ([5, Theorem 2], see [2, Theorem 9.3] for a proof)**.** *Let N and M be normal operators in the Calkin algebra. Then $N \in \overline{\mathcal{S}(M)}$ if and only if*

1. *$\sigma_e(M) \subseteq \sigma_e(N)$,*

2. *each component of $\sigma_e(N)$ intersects $\sigma_e(M)$,*

3. *the Fredholm index of $\lambda I - M$ and $\lambda I - N$ agree for all $\lambda \notin \sigma_e(N)$, and*

4. *if $\lambda \in \sigma_e(N)$ is not isolated in $\sigma_e(N)$, the component of λ in $\sigma_e(N)$ contains some non-isolated point of $\sigma_e(M)$.*

CHAPTER 2

Free Exactness and Strong Convergence

In this chapter, which is based on the author's work in [73], we will analyze how the second and fourth equivalences in Theorem 1.6.1 can be adapted to the context of reduced free products. In Section 2.1 we will modify the second equivalence in Theorem 1.6.1 by replacing the minimal tensor product with the reduced free product. First we will demonstrate a way to take the reduced free product of a short exact sequence of C*-algebras against a fixed C*-algebra. Our main result is that every C*-algebra is 'freely exact'; that is, taking the reduced free product of a short exact sequence of C*-algebras against a fixed C*-algebra preserves exactness. This will be accomplished by embedding these short sequences into short exact sequences involving Toeplitz-Pimsner algebras (Section 2.2) and restricting back to our original sequences (Section 2.3).

In Section 2.4 we will analyze how the fourth equivalence of Theorem 1.6.1 can be adapted to the context of reduced free products. It will be demonstrated that the conclusion of fourth equivalence of Theorem 1.6.1 holds when the minimal tensor product is replaced with the reduced free product for any C*-algebra. This will be accomplished by first proving the result for the C*-algebra generated by a finite number of free creation operators (previously proven in the appendix of [47] due to Shlyakhtenko), then for exact C*-algebras, and finally for arbitrary C*-algebras.

2.1 Construction of Sequence of Free Product C*-Algebras

The purpose of this section is to replace the tensor products with reduced free products in the second equivalence in Theorem 1.6.1 and examine the result. We begin by describing a reduced free product analog of taking the tensor product of an exact sequence with a fixed C*-algebra. Most typical results for the reduced free product of C*-algebras requires the states used in the construction to have faithful GNS representations and thus hinders the consideration of quotient maps. The solutions is to go straight to the construction of the reduced free product of two C*-algebras.

Construction 2.1.1. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be unital C*-algebras, let $\mathfrak{J}$ be an ideal of $\mathfrak{A}_1$, let $\pi_{1,0} : \mathfrak{A}_1/\mathfrak{J} \to \mathcal{B}(\mathcal{H}_{1,0})$, $\pi_{1,1} : \mathfrak{A}_1 \to \mathcal{B}(\mathcal{H}_{1,1})$, and $\pi_2 : \mathfrak{A}_2 \to \mathcal{B}(\mathcal{H}_2)$ be unital representations such that $\pi_{1,0}$ and π_2 are faithful and, if $\mathcal{H}_1 := \mathcal{H}_{1,0} \oplus \mathcal{H}_{1,1}$ and $q : \mathfrak{A}_1 \to \mathfrak{A}_1/\mathfrak{J}$ is the canonical quotient map, $\pi_1 := (\pi_{1,0} \circ q) \oplus \pi_{1,1} : \mathfrak{A}_1 \to \mathcal{B}(\mathcal{H}_1)$ is faithful, and let $\xi_1 \in \mathcal{H}_{1,0}$ and $\xi_2 \in \mathcal{H}_2$ be unit vectors. Consider the reduced free products $(\mathfrak{A}_1/\mathfrak{J}, \pi_{1,0}, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2)$ and $(\mathfrak{A}_1, \pi_1, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2)$. Let $\langle \mathfrak{J} \rangle_{\mathfrak{A}_1 * \mathfrak{A}_2}$ denote the closed ideal of $(\mathfrak{A}_1, \pi_1, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2)$ generated by $\mathfrak{J}$.

By the construction of the free product of Hilbert spaces, $(\mathcal{H}_{1,0}, \xi_1) * (\mathcal{H}_2, \xi_2)$ can be viewed canonically as a Hilbert subspace of $(\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)$. Since $(\mathfrak{A}_1, \pi_1, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2)$ acts on $(\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)$ and $(\mathfrak{A}_1/\mathfrak{J}, \pi_{1,0}, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2)$ acts on $(\mathcal{H}_{1,0}, \xi_1) * (\mathcal{H}_2, \xi_2)$, by considering the action of $(\mathfrak{A}_1, \pi_1, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2)$ on $(\mathcal{H}_{1,0}, \xi_1) * (\mathcal{H}_2, \xi_2) \subseteq (\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)$ it is easily seen that $(\mathcal{H}_{1,0}, \xi_1) * (\mathcal{H}_2, \xi_2)$ is an invariant subspace of $(\mathfrak{A}_1, \pi_1, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2)$ and $(\mathfrak{A}_1/\mathfrak{J}, \pi_{1,0}, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2)$ is the compression of $(\mathfrak{A}_1, \pi_1, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2)$ to this subspace. Thus there is a well-defined surjective *-homomorphism

$$\pi : (\mathfrak{A}_1, \pi_1, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2) \to (\mathfrak{A}_1/\mathfrak{J}, \pi_{1,0}, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2)$$

defined by

$$\pi(T) := P_{(\mathcal{H}_{1,0}, \xi_1) * (\mathcal{H}_2, \xi_2)} T|_{(\mathcal{H}_{1,0}, \xi_1) * (\mathcal{H}_2, \xi_2)}$$

where $P_{(\mathcal{H}_{1,0},\xi_1)*(\mathcal{H}_2,\xi_2)}$ is the orthogonal projection onto $(\mathcal{H}_{1,0},\xi_1)*(\mathcal{H}_2,\xi_2)$.

If $J \in \mathfrak{J}$ then it is easily seen that $J|_{(\mathcal{H}_{1,0},\xi_1)*(\mathcal{H}_2,\xi_2)} = 0$. Therefore the algebraic ideal generated by $\mathfrak{J}$ in $(\mathfrak{A}_1,\pi_1,\xi_1)*(\mathfrak{A}_2,\pi_2,\xi_2)$ is in the kernel of π and thus $\langle\mathfrak{J}\rangle_{\mathfrak{A}_1*\mathfrak{A}_2} \subseteq \ker(\pi)$. Hence we can consider the sequence of C*-algebras

$$0 \to \langle\mathfrak{J}\rangle_{\mathfrak{A}_1*\mathfrak{A}_2} \xrightarrow{i} (\mathfrak{A}_1,\pi_1,\xi_1)*(\mathfrak{A}_2,\pi_2,\xi_2) \xrightarrow{\pi} (\mathfrak{A}_1/\mathfrak{J},\pi_{1,0},\xi_1)*(\mathfrak{A}_2,\pi_2,\xi_2) \to 0$$

where i is the inclusion map. Clearly i is injective, π is surjective, and $\langle\mathfrak{J}\rangle_{\mathfrak{A}_1*\mathfrak{A}_2} \subseteq \ker(\pi)$. Hence the sequence is exact if and only if $\ker(\pi) \subseteq \langle\mathfrak{J}\rangle_{\mathfrak{A}_1*\mathfrak{A}_2}$; that is there is no element of $(\mathfrak{A}_1,\pi_1,\xi_1)*(\mathfrak{A}_2,\pi_2,\xi_2) \setminus \langle\mathfrak{J}\rangle_{\mathfrak{A}_1*\mathfrak{A}_2}$ that is zero on the copy of $(\mathcal{H}_{1,0},\xi_1)*(\mathcal{H}_2,\xi_2)$ inside $(\mathcal{H}_1,\xi_1)*(\mathcal{H}_2,\xi_2)$.

The requirements on $\pi_{1,0}$, π_1, and π_2 are necessary to ensure we are considering objects related directly to $\mathfrak{A}_1/\mathfrak{J}$, $\mathfrak{A}_1$, and $\mathfrak{A}_2$. The conditions on $\pi_{1,0}$, π_1, π_2, ξ_1, and ξ_2 are also designed so the vectors ξ_1 and ξ_2 give rise to vector states on our C*-algebras. Moreover $\pi_{1,0}$, π_1, and π_2 are assumed to be unital so the C*-algebras under consideration are truly reduced free products of C*-algebras. Finally the consideration of $(\mathfrak{A}_1,\pi_1,\xi_1)*(\mathfrak{A}_2,\pi_2,\xi_2)$ was necessary to ensure the *-homomorphism π existed.

Our main goal is to prove the following result.

Theorem 2.1.2. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be unital C*-algebras, let $\mathfrak{J}$ be an ideal of $\mathfrak{A}_1$, let $\pi_{1,0} : \mathfrak{A}_1/\mathfrak{J} \to \mathcal{B}(\mathcal{H}_{1,0})$, $\pi_{1,1} : \mathfrak{A}_1 \to \mathcal{B}(\mathcal{H}_{1,1})$, and $\pi_2 : \mathfrak{A}_2 \to \mathcal{B}(\mathcal{H}_2)$ be unital representations such that $\pi_{1,0}$ and π_2 are faithful and, if $\mathcal{H}_1 := \mathcal{H}_{1,0} \oplus \mathcal{H}_{1,1}$ and $q : \mathfrak{A}_1 \to \mathfrak{A}_1/\mathfrak{J}$ is the canonical quotient map, $\pi_1 := (\pi_{1,0} \circ q) \oplus \pi_{1,1} : \mathfrak{A}_1 \to \mathcal{B}(\mathcal{H}_1)$ is faithful, and let $\xi_1 \in \mathcal{H}_{1,0}$ and $\xi_2 \in \mathcal{H}_2$ be unit vectors. Under these assumptions, the sequence of C*-algebras*

$$0 \to \langle\mathfrak{J}\rangle_{\mathfrak{A}_1*\mathfrak{A}_2} \xrightarrow{i} (\mathfrak{A}_1,\pi_1,\xi_1)*(\mathfrak{A}_2,\pi_2,\xi_2) \xrightarrow{\pi} (\mathfrak{A}_1/\mathfrak{J},\pi_{1,0},\xi_1)*(\mathfrak{A}_2,\pi_2,\xi_2) \to 0$$

is exact.

The proof of Theorem 2.1.2 will be completed at the end of Section 2.3.

2.2 An Exact Sequence of Toeplitz-Pimsner Algebras

In this section we will examine a short exact sequence of C*-algebras involving Toeplitz-Pimsner algebras.

Construction 2.2.1. Extending the notation of Construction 2.1.1, let $\mathcal{H}_{2,0} := \mathcal{H}_2$ and let $\pi_{2,0} := \pi_2 : \mathfrak{A}_2 \to \mathcal{B}(\mathcal{H}_{2,0})$. For all $i \in \{1, 2\}$ and $n \in \mathbb{N}$ let

$$\mathcal{L}_{i,n} := \mathcal{H}_{i_1} \otimes \cdots \otimes \mathcal{H}_{i_n}$$

where $\{i_k\}_{k=1}^n \subseteq \{1, 2\}$, $i_1 = i$, and $i_k \neq i_{k+1}$ for $k \in \{1, \ldots, n-1\}$. Let $\mathcal{K} := \mathcal{K}_1 \oplus \mathcal{K}_2$ where

$$\mathcal{K}_i := \bigoplus_{n \in \mathbb{N}} \mathcal{L}_{i,n}.$$

Let $S \in \mathcal{B}(\mathcal{K})$ be the isometry defined by

$$S(\eta_1 \otimes \cdots \otimes \eta_n) = \xi_j \otimes \eta_1 \otimes \cdots \otimes \eta_n \in \mathcal{L}_{j,n+1}$$

for all $\eta_1 \otimes \cdots \otimes \eta_n \in \mathcal{L}_{i,n}$ where $i \neq j$ and $i, j \in \{1, 2\}$. Notice that $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ has a faithful representation on $\mathcal{K}$ given by

$$(A_1 \oplus A_2)(\eta_1 \otimes \cdots \otimes \eta_n) = (\pi_i(A_i)\eta_1) \otimes \eta_2 \otimes \cdots \otimes \eta_n$$

for all $\eta_1 \otimes \cdots \otimes \eta_n \in \mathcal{L}_{i,n}$, $A_j \in \mathfrak{A}_j$, and $i \in \{1, 2\}$. Let $C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$ denote the C*-subalgebra of $\mathcal{B}(\mathcal{K})$ generated by $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ and S. The C*-algebra $C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$ is called a Toeplitz-Pimsner C*-algebra (usually it is required that π_1 and π_2 are faithful GNS representations).

Similarly, by considering the Hilbert space $\mathcal{K}_0 := \mathcal{K}_{1,0} \oplus \mathcal{K}_{2,0}$ where

$$\mathcal{K}_{i,0} := \bigoplus_{\substack{n \geq 1, \{i_k\}_{k=1}^n \subseteq \{1,2\}, i_1 = i \\ i_k \neq i_{k+1} \text{ for } k \in \{1,\ldots,n-1\}}} \mathcal{H}_{i_1,0} \otimes \cdots \otimes \mathcal{H}_{i_n,0}$$

for $i \in \{1,2\}$ and an isometry $S_0 \in \mathcal{B}(\mathcal{K}_0)$ in the same manner as $S \in \mathcal{B}(\mathcal{K})$, we can construct a second Toeplitz-Pimsner algebra $C^*((\mathfrak{A}_1/\mathfrak{J}) \oplus \mathfrak{A}_2, S_0)$. Notice $\mathcal{K}_0$ may be viewed canonically as a Hilbert subspace of $\mathcal{K}$ since $\mathcal{H}_{1,0} \subseteq \mathcal{H}_1$ and $\mathcal{H}_{2,0} = \mathcal{H}_2$. By considering the actions of $\mathfrak{A}_1$, $\mathfrak{A}_2$, S, and S^*, it is easy to see that $\mathcal{K}_0$ is a reducing subspace of $C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$. By restricting the compression map of $\mathcal{K}$ onto $\mathcal{K}_0$ to $C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$ we obtain a surjective *-homomorphism

$$\pi' : C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S) \to C^*((\mathfrak{A}_1/\mathfrak{J}) \oplus \mathfrak{A}_2, S_0).$$

Let $\langle \mathfrak{J} \rangle_{\mathfrak{A}_1 \oplus \mathfrak{A}_2}$ be the ideal of $C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$ generated by $\mathfrak{J} \subseteq \mathfrak{A}_1$. Since $\pi_{1,0}(J) = 0$ for all $J \in \mathfrak{J}$, it is clear that $\langle \mathfrak{J} \rangle_{\mathfrak{A}_1 \oplus \mathfrak{A}_2} \subseteq \ker(\pi')$.

The main result of this section is the following.

Theorem 2.2.2. *With the notation as in Construction 2.2.1, the sequence*

$$0 \to \langle \mathfrak{J} \rangle_{\mathfrak{A}_1 \oplus \mathfrak{A}_2} \to C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S) \xrightarrow{\pi'} C^*((\mathfrak{A}_1/\mathfrak{J}) \oplus \mathfrak{A}_2, S_0) \to 0$$

is exact.

The proof of Theorem 2.2.2 will be completed through a sequence of easily verifiable lemmas.

Lemma 2.2.3. *For all $i, j \in \{1,2\}$ with $i \neq j$ and $A, B \in \mathfrak{A}_i$,*

$$S^*AS = \langle A\xi_i, \xi_i \rangle_{\mathcal{H}_i} P_{\mathcal{K}_j}, \quad ASB = 0, \quad AS^*B = 0, \quad P_{\mathcal{K}_j} SA = SA, \quad \textit{and} \quad AS^*P_{\mathcal{K}_j} = AS^*$$

where $P_{\mathcal{K}_j}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{K}_j$. Thus the span of all operators of the form

$$(A_1S)\cdots(A_nS)A_{n+1}(S^*A_{n+2})\cdots(S^*A_{n+m+1})$$

where $n, m \geq 0$, $\{i_k\}_{k=1}^{n+m+1} \subseteq \{1,2\}$, $i_k \neq i_{k+1}$ for $k \in \{1,\ldots,n+m\}$, and $A_k \in \mathfrak{A}_{i_k}$ is dense in $C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$.

Proof. Fix $i \in \{1,2\}$, let $j \in \{1,2\} \setminus \{i\}$, and let $A \in \mathfrak{A}_i$. If $\zeta \in \mathcal{K}_i$ then $S\zeta \in \mathcal{K}_2$ so $S^*AS\zeta = 0$. However, if $\eta_1 \otimes \cdots \otimes \eta_n \in \mathcal{L}_{j,n}$ then

$$\begin{aligned} S^*AS(\eta_1 \otimes \cdots \otimes \eta_n) &= S^*A(\xi_i \otimes \eta_1 \otimes \cdots \otimes \eta_n) \\ &= S^*((A\xi_i) \otimes \eta_1 \otimes \cdots \otimes \eta_n) \\ &= \langle A\xi_i, \xi_i\rangle_{\mathcal{H}_i}\eta_1 \otimes \cdots \otimes \eta_n \end{aligned}$$

Whence, by linearity and density, $S^*AS = \langle A\xi_i, \xi_i\rangle_{\mathcal{H}_i} P_{\mathcal{K}_j}$.

Fix $i \in \{1,2\}$, $j \in \{1,2\} \setminus \{i\}$, and $A, B \in \mathfrak{A}_i$. To see $ASB = 0$ notice $A(\mathcal{L}_{j,n}) = \{0\}$ and $B(\mathcal{L}_{j,n}) = \{0\}$ for all n. However $S(B(\mathcal{L}_{i,n})) \subseteq \mathcal{L}_{j,n+1}$ and thus $ASB = 0$. Similarly $AS^*B = 0$. To see $P_{\mathcal{K}_j}SA = SA$ notice $SA(\mathcal{L}_{j,n}) = \{0\}$ and $SA(\mathcal{L}_{i,n}) \subseteq \mathcal{L}_{j,n+1} \subseteq \mathcal{K}_j$. Hence $P_{\mathcal{K}_j}SA = SA$. Similarly $AS^*P_{\mathcal{K}_j} = AS^*$.

Using the fact that $\mathrm{alg}(\mathfrak{A}_1, \mathfrak{A}_2, S, S^*)$ is dense in $C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$, $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ is unital, the fact that $P_{\mathcal{K}_j} \in \mathfrak{A}_j$ for all $j \in \{1,2\}$, and the above results, we obtain that the desired span is dense in $C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$. ☐

The next step in the proof is to define a action of the unit circle $\mathbb{T}$ on $\mathcal{B}(\mathcal{K})$. For each $\theta \in [0, 2\pi)$ define $U_\theta \in \mathcal{B}(\mathcal{K})$ by

$$U_\theta(\eta_1 \otimes \cdots \otimes \eta_n) = e^{-n\theta\sqrt{-1}}\eta_1 \otimes \cdots \otimes \eta_n$$

for all $\eta_1 \otimes \cdots \otimes \eta_n \in \mathcal{L}_{i,n}$. It is clear that U_θ is a unitary operator with $U_\theta^* = U_{-\theta}$ and $U_\theta U_\beta = U_{\theta+\beta}$ (where we view $\theta + \beta \mod 2\pi$). Notice each U_θ defines a *-homomorphism

$\alpha_\theta : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$ by

$$\alpha_\theta(T) = U_\theta^* T U_\theta$$

for all $T \in \mathcal{B}(\mathcal{K})$.

Lemma 2.2.4. *If $T \in C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$ and $\alpha_\theta(T) = T$ for all $\theta \in [0, 2\pi)$ then*

$$T\left(\mathcal{L}_{i,n}\right) \subseteq \mathcal{L}_{i,n}$$

for all $i \in \{1, 2\}$ and for all $n \in \mathbb{N}$.

Proof. First it is clear that

$$\left(\bigoplus_{n=1 \mod 2} \mathcal{L}_{1,n}\right) \oplus \left(\bigoplus_{n=2 \mod 2} \mathcal{L}_{2,n}\right)$$

and

$$\left(\bigoplus_{n=1 \mod 2} \mathcal{L}_{2,n}\right) \oplus \left(\bigoplus_{n=2 \mod 2} \mathcal{L}_{1,n}\right)$$

are reducing subspaces of $C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$ since each is invariant under $\mathfrak{A}_1$, $\mathfrak{A}_2$, S, and S^*.

Suppose otherwise that there exists an $i \in \{1, 2\}$, an $m \in \mathbb{N}$, and an $h \in \mathcal{L}_{i,m}$ so that $T(h) \notin \mathcal{L}_{i,m}$. Without loss of generality suppose

$$\mathcal{L}_{i,m} \subseteq \left(\bigoplus_{n=1 \mod 2} \mathcal{L}_{1,n}\right) \oplus \left(\bigoplus_{n=2 \mod 2} \mathcal{L}_{2,n}\right).$$

Thus $T(h) \in (\bigoplus_{n=1 \mod 2} \mathcal{L}_{1,n}) \oplus (\bigoplus_{n=2 \mod 2} \mathcal{L}_{2,n})$. Write

$$T(h) = \bigoplus_{j \geq 1} h_j$$

where $h_j \in \mathcal{L}_{1,j}$ if j is odd and $h_j \in \mathcal{L}_{2,j}$ when j is even. Since $T(h) \notin \mathcal{L}_{i,m}$, there exists a

$k \in \mathbb{N} \setminus \{m\}$ such that $h_k \neq 0$. However

$$\begin{aligned}
\bigoplus_{j\geq 1} h_j = T(h) &= \alpha_\theta(T)h \\
&= U_{-\theta}TU_\theta h \\
&= U_{-\theta}Te^{-m\theta\sqrt{-1}}h \\
&= e^{-m\theta\sqrt{-1}}U_{-\theta}\left(\bigoplus_{j\geq 1} h_j\right) = \bigoplus_{j\geq 1}(e^{-(m-j)\theta\sqrt{-1}})h_j
\end{aligned}$$

for all $\theta \in [0, 2\pi)$. Therefore $h_k = e^{-(m-k)\theta\sqrt{-1}}h_k$ for all $\theta \in [0, 2\pi)$. As $k \neq m$ and $h_k \neq 0$, this is an impossibility. $\square$

Lemma 2.2.5. *For all $\theta \in [0, 2\pi)$ and all $A \in \mathfrak{A}_1 \oplus \mathfrak{A}_2$, $\alpha_\theta(S) = e^{\theta\sqrt{-1}}S$ and $\alpha_\theta(A) = A$. Therefore $\theta \mapsto \alpha_\theta(T)$ is a continuous map for all $T \in C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$. Hence the map $\mathcal{E} : C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S) \to C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$ given by*

$$\mathcal{E}(T) := \frac{1}{2\pi}\int_0^{2\pi} \alpha_\theta(T)\, d\theta$$

is a well-defined, contractive linear map with the property that $\alpha_\theta(\mathcal{E}(T)) = \mathcal{E}(T)$ for all $T \in C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$ and $\theta \in [0, 2\pi)$.

Proof. The fact that $\alpha_\theta(A) = A$ for all $\theta \in [0, 2\pi)$ and $A \in \mathfrak{A}_1 \oplus \mathfrak{A}_2$ comes from the fact that each $\mathcal{L}_{i,n}$ is an invariant subspace of $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ and thus $U_\theta \in (\mathfrak{A}_1 \oplus \mathfrak{A}_2)'$ (the commutant of $\mathfrak{A}_1 \oplus \mathfrak{A}_2$). Notice for each $i \in \{1, 2\}$ and each $\eta_1 \otimes \cdots \otimes \eta_n \in \mathcal{L}_{i,n}$ that

$$\begin{aligned}
\alpha_\theta(S)(\eta_1 \otimes \cdots \otimes \eta_n) &= U_{-\theta}S\left(e^{-n\theta\sqrt{-1}}\eta_1 \otimes \cdots \otimes \eta_n\right) \\
&= U_{-\theta}\left(e^{-n\theta\sqrt{-1}}\xi_j \otimes \eta_1 \otimes \cdots \otimes \eta_n\right) \\
&= e^{(n+1)\theta\sqrt{-1}}e^{-n\theta\sqrt{-1}}\xi_j \otimes \eta_1 \otimes \cdots \otimes \eta_n \\
&= e^{\theta\sqrt{-1}}S(\eta_1 \otimes \cdots \otimes \eta_n)
\end{aligned}$$

where $j \in \{1, 2\} \setminus \{i\}$. Whence $\alpha_\theta(S) = e^{\theta\sqrt{-1}}S$ by linearity and density.

To see $\theta \mapsto \alpha_\theta(T)$ is a continuous map for all $T \in C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$, notice the result holds for all $T \in \mathrm{alg}(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S, S^*)$ by the above results. Since each α_θ is a contraction and

$\mathrm{alg}(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S, S^*)$ is dense in $C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$, the result follows.

The fact that $\mathcal{E}$ is a well-defined, contractive linear map is then trivial and the fact that $\alpha_\theta(\mathcal{E}(T)) = \mathcal{E}(T)$ for all $T \in C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$ follows from the fact that $\alpha_\theta \circ \alpha_\beta = \alpha_{\theta+\beta}$ (as $U_\theta U_\beta = U_{\theta+\beta}$) and the fact that the Lebesgue measure on the unit circle is translation invariant. $\square$

If $T \in C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$ is of the form

$$T = (A_1 S) \cdots (A_n S) A_{n+1} (S^* A_{n+2}) \cdots (S^* A_{n+m+1})$$

where $n, m \geq 0$, $\{i_k\}_{k=1}^{n+m+1} \subseteq \{1, 2\}$, $i_k \neq i_{k+1}$ for $k \in \{1, \ldots, n+m\}$, and $A_k \in \mathfrak{A}_{i_k}$, then $\mathcal{E}(T) = 0$ whenever $n \neq m$ and $\mathcal{E}(T) = T$ whenever $n = m$.

Lemma 2.2.6. *For all $T \in C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$ and $n \in \mathbb{N}$ define*

$$\Sigma_n(T) := \sum_{j=0}^{n} \left(1 - \frac{j}{n+1}\right) (S^*)^j \mathcal{E}(S^j T) + \sum_{j=1}^{n} \left(1 - \frac{j}{n+1}\right) \mathcal{E}(T (S^*)^j) S^j.$$

Then $\lim_{n \to \infty} \|T - \Sigma_n(T)\| = 0$.

Proof. Notice for all $T \in C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S)$ that

$$\begin{aligned} \Sigma_n(T) &= \tfrac{1}{2\pi} \int_0^{2\pi} \left(\sum_{j=0}^n \left(1 - \tfrac{j}{n+1}\right) (S^*)^j \alpha_\theta(S^j T) + \sum_{j=1}^n \left(1 - \tfrac{j}{n+1}\right) \alpha_\theta(T (S^*)^j) S^j \right) d\theta \\ &= \tfrac{1}{2\pi} \int_0^{2\pi} \left(\sum_{j=-n}^n \left(1 - \tfrac{|j|}{n+1}\right) e^{j\theta\sqrt{-1}} \right) \alpha_\theta(T) d\theta \\ &= \tfrac{1}{2\pi} \int_0^{2\pi} \sigma_n(\theta) \alpha_\theta(T) d\theta \end{aligned}$$

where $\sigma_n(\theta) := \sum_{j=-n}^n \left(1 - \frac{|j|}{n+1}\right) e^{j\theta\sqrt{-1}}$ is Fejér's kernel. Recall $\left\{\frac{1}{2\pi}\sigma_n(\theta) d\theta\right\}_{n \geq 1}$ define probability measures on $\mathbb{T}$ that converge weak* (from $C(\mathbb{T})$) to the point mass at 0. Thus

$$\|\Sigma_n(T)\| \leq \frac{1}{2\pi} \int_0^{2\pi} \sigma_n(\theta) \|\alpha_\theta(T)\| \, d\theta = \|T\|$$

for all $n \in \mathbb{N}$. Since $\mathcal{E}$ and thus Σ_n is linear for all $n \in \mathbb{N}$ and each Σ_n is a contraction, it suffices to prove the result on a set whose span is dense; namely

$$\{A_1SA_2S\cdots A_nSBS^*C_1S^*\cdots S^*B_m \mid n,m\geq 0, A_i,B,C_j\in\mathfrak{A}_1\oplus\mathfrak{A}_2\}$$

by Lemma 2.2.3.

To complete the proof, notice if $T = A_1SA_2S\cdots A_nSBS^*C_1S^*\cdots S^*B_m$ where $n,m\geq 0$ and $A_i,B,C_j\in\mathfrak{A}_1\oplus\mathfrak{A}_2$ then

$$\mathcal{E}(S^kT) = \begin{cases} S^kT & \text{if } k+n=m \\ 0 & \text{otherwise}\end{cases}$$

and

$$\mathcal{E}(T(S^*)^k) = \begin{cases} T(S^*)^k & \text{if } n=m+k \\ 0 & \text{otherwise}\end{cases}$$

Whence $\Sigma_k(T) = \left(1-\frac{|n-m|}{k+1}\right)T$ for all $k\geq |n-m|$ which clearly converges to T as $k\to\infty$. $\square$

To prove Theorem 2.2.2 it suffices to prove $\ker(\pi')\subseteq\langle\mathfrak{J}\rangle_{\mathfrak{A}_1\oplus\mathfrak{A}_2}$. It is trivial to prove that $\mathcal{E}\left(\langle\mathfrak{J}\rangle_{\mathfrak{A}_1\oplus\mathfrak{A}_2}\right)\subseteq\langle\mathfrak{J}\rangle_{\mathfrak{A}_1\oplus\mathfrak{A}_2}$, $\mathcal{E}(\ker(\pi'))\subseteq\ker(\pi')$, and $\mathcal{E}\left(\langle\mathfrak{J}\rangle_{\mathfrak{A}_1\oplus\mathfrak{A}_2}\right)\subseteq\mathcal{E}(\ker(\pi'))$. Using these facts and Lemma 2.2.6 gives the following reduction of our problem.

Lemma 2.2.7. *If* $\mathcal{E}\left(\langle\mathfrak{J}\rangle_{\mathfrak{A}_1\oplus\mathfrak{A}_2}\right)=\mathcal{E}(\ker(\pi'))$ *then* $\ker(\pi')=\langle\mathfrak{J}\rangle_{\mathfrak{A}_1\oplus\mathfrak{A}_2}$.

Proof. By Construction 2.2.1 it suffices to show that $\ker(\pi')\subseteq\langle\mathfrak{J}\rangle_{C^*(\mathfrak{A}_1\oplus\mathfrak{A}_2,S)}$. Let $T\in\ker(\pi')$. Recall $T=\lim_{n\to\infty}\Sigma_n(T)$ by Lemma 2.2.6. Moreover $S^kT\in\ker(\pi')$ and $T(S^*)^k\in\ker(\pi')$ for all $k\geq 0$ as $T\in\ker(\pi')$. Whence

$$\mathcal{E}(S^kT),\mathcal{E}(T(S^*)^k)\in\mathcal{E}(\ker(\pi'))=\mathcal{E}\left(\langle\mathfrak{J}\rangle_{C^*(\mathfrak{A}_1\oplus\mathfrak{A}_2,S)}\right)\subseteq\langle\mathfrak{J}\rangle_{C^*(\mathfrak{A}_1\oplus\mathfrak{A}_2,S)}$$

for all $k\geq 0$. This implies that $\Sigma_n(T)\in\langle\mathfrak{J}\rangle_{C^*(\mathfrak{A}_1\oplus\mathfrak{A}_2,S)}$ for all n and thus $T\in\langle\mathfrak{J}\rangle_{C^*(\mathfrak{A}_1\oplus\mathfrak{A}_2,S)}$. $\square$

To begin the process of showing $\mathcal{E}\left(\langle\mathfrak{J}\rangle_{\mathfrak{A}_1\oplus\mathfrak{A}_2}\right) = \mathcal{E}(\ker(\pi'))$ we will examine the structure of $\langle\mathfrak{J}\rangle_{\mathfrak{A}_1\oplus\mathfrak{A}_2}$. Notice if $J \in \mathfrak{J}$ then $JS = 0$ as $J\xi_1 = 0$. Whence $S^*J = 0$ for all $J \in \mathfrak{J}$. Therefore, using the property that the algebraic ideal generated by $\mathfrak{J}$ in $\mathrm{alg}(\mathfrak{A}_1, \mathfrak{A}_2, S, S^*)$ is dense in $\langle\mathfrak{J}\rangle_{\mathfrak{A}_1\oplus\mathfrak{A}_2}$ and the results and ideas from Lemma 2.2.3, the span of all operators of the form

$$(A_nS)\cdots(A_1S)J(S^*B_1)\cdots(S^*B_m)$$

where $n, m \geq 0$, $J \in \mathfrak{J}$, $A_i, B_j \in \mathfrak{A}_1$ if $i, j = 0 \mod 2$, and $A_i, B_j \in \mathfrak{A}_2$ if $i, j = 1 \mod 2$ is dense in $\langle\mathfrak{J}\rangle_{\mathfrak{A}_1\oplus\mathfrak{A}_2}$.

For each $n \geq 0$ and $k \in \{1, 2\}$, let $\mathfrak{A}_{k,(n)}$ be the span of all operators of the form

$$(A_nS)\cdots(A_1S)A(S^*B_1)\cdots(S^*B_n)$$

where $A \in \mathfrak{A}_k$, $A_i, B_j \in \mathfrak{A}_1$ if $i, j \neq k \mod 2$, and $A_i, B_j \in \mathfrak{A}_2$ if $i, j = k \mod 2$. Let $\mathfrak{J}_{(n)}$ denote the subset of $\mathfrak{A}_{1,(n)}$ consisting of all operators of the above form with $A \in \mathfrak{J}$.

Lemma 2.2.8. *The span of* $\bigcup_{n\geq 0}\left(\mathfrak{A}_{1,(n)} \cup \mathfrak{A}_{2,(n)}\right)$ *is dense in* $\mathcal{E}(C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S))$ *and the span of* $\bigcup_{n\geq 0}\mathfrak{J}_{(n)}$ *is dense in* $\mathcal{E}\left(\langle\mathfrak{J}\rangle_{\mathfrak{A}_1\oplus\mathfrak{A}_2}\right)$.

Proof. We will only prove the first claim as the second follows verbatim with the aid of Lemma 2.2.3. It is clear $\bigcup_{n\geq 0}\mathfrak{J}_{(n)} \subseteq \mathcal{E}\left(\langle\mathfrak{J}\rangle_{C^*(\mathfrak{A}_1\oplus\mathfrak{A}_2,S)}\right)$.

Let $T \in \mathcal{E}\left(\langle\mathfrak{J}\rangle_{C^*(\mathfrak{A}_1\oplus\mathfrak{A}_2,S)}\right)$ and let $\epsilon > 0$. As $T \in \langle\mathfrak{J}\rangle_{C^*(\mathfrak{A}_1\oplus\mathfrak{A}_2,S)}$ there exists an

$$R \in \operatorname{span}\left\{ (A_nS)\cdots(A_1S)J(S^*B_1)\cdots(S^*B_m) \;\middle|\; \begin{array}{l} n, m \geq 0, J \in \mathfrak{J}, \\ A_i, B_j \in \mathfrak{A}_1 \text{ if } i, j = 0 \mod 2, \\ A_i, B_j \in \mathfrak{A}_2 \text{ if } i, j = 1 \mod 2 \end{array} \right\}$$

such that $\|T - R\| < \epsilon$. Then $\mathcal{E}(T) = T$ since $T \in \mathcal{E}\left(\langle\mathfrak{J}\rangle_{C^*(\mathfrak{A}_1\oplus\mathfrak{A}_2,S)}\right)$ and thus

$$\|T - \mathcal{E}(R)\| = \|\mathcal{E}(T - R)\| \leq \|T - R\| < \epsilon.$$

Clearly

$$\mathcal{E}\left(\operatorname{span}\left\{(A_nS)\cdots(A_1S)J(S^*B_1)\cdots(S^*B_m)\;\middle|\;\begin{array}{l} n,m\geq 0, J\in\mathfrak{J}, \\ A_i,B_j\in\mathfrak{A}_1 \text{ if } i,j=0 \mod 2, \\ A_i,B_j\in\mathfrak{A}_2 \text{ if } i,j=1 \mod 2\end{array}\right\}\right)$$
$$=\operatorname{span}\left\{(A_nS)\cdots(A_1S)J(S^*B_1)\cdots(S^*B_n)\;\middle|\;\begin{array}{l} n\geq 0, J\in\mathfrak{J}, \\ A_i,B_j\in\mathfrak{A}_1 \text{ if } i,j=0 \mod 2, \\ A_i,B_j\in\mathfrak{A}_2 \text{ if } i,j=1 \mod 2\end{array}\right\}$$
$$=\operatorname{span}\left(\bigcup_{n\geq 0}\mathfrak{J}_{(n)}\right).$$

Thus $\mathcal{E}(R)\in\overline{\operatorname{span}\left(\bigcup_{n\geq 0}\mathfrak{J}_{(n)}\right)}$ which completes the claim. $\square$

Now we will examine how $\mathfrak{A}_{1,(n)}$, $\mathfrak{A}_{2,(n)}$, and $\mathfrak{J}_{(n)}$ act on $\mathcal{K}$. We will only discuss $\mathfrak{A}_{1,(n)}$ and $\mathfrak{J}_{(n)}$ as the analysis of $\mathfrak{A}_{2,(n)}$ will be similar.

Since $\mathcal{E}(T)=T$ for all $T\in\mathfrak{A}_{1,(n)}$ $\left(T\in\mathfrak{J}_{(n)}\right)$, $T(\mathcal{L}_{i,m})\subseteq\mathcal{L}_{i,m}$ for all $i\in\{1,2\}$ and $m\in\mathbb{N}$ by Lemma 2.2.4. Fix

$$T=(A_nS)\cdots(A_1S)A(S^*B_1)\cdots(S^*B_n)$$

where $A_i,B_j\in\mathfrak{A}_1$ if $i,j=0\mod 2$, $A_i,B_j\in\mathfrak{A}_2$ if $i,j=1\mod 2$, and $A\in\mathfrak{A}_1$ ($A\in\mathfrak{J}$). If $k\in\{1,2\}$ and $k=n\mod 2$ then $T(\mathcal{L}_{k,m})=\{0\}$ for all $m\in\mathbb{N}$. Fix $k\in\{1,2\}$ with $k\neq n\mod 2$ and let $\ell\in\{1,2\}\setminus\{k\}$. Then it is possible to show that for all $\eta=\eta_1\otimes\cdots\otimes\eta_{n+1}\in\mathcal{L}_{k,n+1}$

$$T(\eta)=\langle\eta_1\otimes\cdots\otimes\eta_n,\zeta\rangle_{\mathcal{L}_{k,n}}(\omega\otimes A\eta_{n+1})$$

where

$$\zeta:=B_n^*\xi_k\otimes B_{n-1}^*\xi_\ell\otimes\cdots\otimes B_2^*\xi_1\otimes B_1^*\xi_2$$

and

$$\omega:=A_n\xi_k\otimes A_{n-1}\xi_\ell\otimes\cdots\otimes A_2\xi_1\otimes A_1\xi_2.$$

Moreover T acts as the zero operator on elements of $\mathcal{L}_{k,m}$ for all $m \leq n$ since the m^{th} S^* will act on an element of $\mathcal{L}_{1,1} \oplus \mathcal{L}_{2,1}$. Finally if $m \geq n+1$ notice $\mathcal{L}_{k,m} = \mathcal{L}_{k,n+1} \otimes \mathcal{L}_{k,m-n-1}$ if $n+1$ is even and $\mathcal{L}_{k,m} = \mathcal{L}_{k,n+1} \otimes \mathcal{L}_{\ell,m-n-1}$ if $n+1$ is odd. Therefore, if $R = T|_{\mathcal{L}_{k,n+1}}$, it is easy to see that T acts on $\mathcal{L}_{k,m}$ as $R \otimes I_{\mathcal{L}_{k,m-n-1}}$ when $n+1$ is even and $m \geq n+1$ and T acts on $\mathcal{L}_{k,m}$ as $R \otimes I_{\mathcal{L}_{\ell,m-n-1}}$ when $n+1$ is odd and $m \geq n+1$.

For $i \in \{1,2\}$ let $\mathcal{N}_i = \overline{\mathfrak{A}_i \xi_i}$ which is a Hilbert subspace of $\mathcal{H}_{i,0} \subseteq \mathcal{H}_i$ containing ξ_i. Thus, for a fixed $k \in \{1,2\}$ with $k \neq n \mod 2$ and with $\ell \in \{1,2\} \setminus \{k\}$, by restricting to $\mathcal{L}_{k,n}$ and taking limits of elements of $\mathfrak{A}_{1,(n)}$ $\left(\mathfrak{J}_{(n)}\right)$ over $A_1, \ldots, A_n, B_1, \ldots, B_n$ where $A_i, B_j \in \mathfrak{A}_1$ if $i, j = 0 \mod 2$ and $A_i, B_j \in \mathfrak{A}_2$ if $i, j = 1 \mod 2$, every operator in $\mathcal{B}(\mathcal{L}_{k,n})$ of the form

$$\eta_1 \otimes \cdots \otimes \eta_{n+1} \mapsto \langle \eta_1 \otimes \cdots \otimes \eta_n, \zeta_1 \otimes \cdots \otimes \zeta_n \rangle_{\mathcal{L}_{k,n}} (\zeta_1' \otimes \cdots \otimes \zeta_n' \otimes A\eta_{n+1})$$

where $\zeta_i, \zeta_j' \in \mathcal{N}_k$ if $i, j = 1 \mod 2$, $\zeta_i, \zeta_j' \in \mathcal{N}_\ell$ if $i, j = 0 \mod 2$, and $A \in \mathfrak{A}_1$ ($A \in \mathfrak{J}$) may be obtained.

Lastly, by describing the action of $\mathfrak{A}_{2,(n)}$ on $\mathcal{K}$, it is possible to show that if $T \in \mathfrak{A}_{1,(n)}$, $R \in \mathfrak{A}_{2,(n)}$, $k, \ell \in \{1,2\}$, $k = n \mod 2$, and $\ell \neq n \mod 2$ then the actions of T and R are completely determined by their actions on $\mathcal{L}_{1,n+1} \oplus \mathcal{L}_{2,n+1}$ with $T(\mathcal{L}_{k,m}) = \{0\}$ for all $m \in \mathbb{N}$, $R(\mathcal{L}_{\ell,m}) = \{0\}$ for all $m \in \mathbb{N}$, and

$$\|T + R\| = \max\{\left\|T|_{\mathcal{L}_{\ell,n+1}}\right\|, \left\|R|_{\mathcal{L}_{k,n+1}}\right\|\}.$$

The above structure will be important as we will consider the restriction of $\mathcal{E}\left(\langle \mathfrak{J} \rangle_{\mathfrak{A}_1 \oplus \mathfrak{A}_2}\right)$ and $\mathcal{E}(\ker(\pi'))$ to the subspaces $\mathcal{L}_{1,n} \oplus \mathcal{L}_{2,n}$ of $\mathcal{K}$. For $m, n \in \mathbb{N}$ and $i \in \{1,2\}$ let $P_{i,m}$ be the orthogonal projection of $\mathcal{K}$ onto $\mathcal{L}_{i,m}$, let P_m be the orthogonal projection onto $\mathcal{L}_{1,m} \oplus \mathcal{L}_{2,m}$, and let $Q_n := \sum_{j=1}^n P_j$ which is the orthogonal projection of $\mathcal{K}$ onto $\bigoplus_{k=1}^n (\mathcal{L}_{1,k} \oplus \mathcal{L}_{2,k})$. Therefore, using the above discussion, $\mathfrak{A}_{i,(n)} P_m = \{0\}$ for all $m \leq n$ and if $m > n+1$ then each element $T \in \mathfrak{A}_{1,(n)} \cup \mathfrak{A}_{2,(n)}$ acts on $\mathcal{L}_{j,m}$ by $(P_{j,n+1} T P_{j,n+1}) \otimes I$ where I is the appropriate identity. Using the P_n's and the above information about the actions of $\mathfrak{A}_{1,(n)}$, $\mathfrak{A}_{2,(n)}$, and

$\mathfrak{J}_{(n)}$ on $\mathcal{L}_{1,n} \oplus \mathcal{L}_{2,n}$, we obtain the following.

Lemma 2.2.9. *The set* $Q_n \left(\operatorname{span}\left(\bigcup_{m<n} \mathfrak{J}_{(m)}\right)\right) Q_n$ *is dense in* $Q_n \mathcal{E}(\ker(\pi'))Q_n$ *for all* $n \in \mathbb{N}$.

Proof. Clearly $Q_n \left(\operatorname{span}\left(\bigcup_{m<n} \mathfrak{J}_{(m)}\right)\right) Q_n \subseteq Q_n \mathcal{E}(\ker(\pi'))Q_n$ for all $n \in \mathbb{N}$. Thus it suffices to show that each element of $\mathcal{E}(\ker(\pi'))$ can be approximated uniformly on $Q_m\mathcal{K}$ by an element of $\operatorname{span}\left(\bigcup_{m<n} \mathfrak{J}_{(m)}\right)$. We proceed by induction on n.

Let $T \in \mathcal{E}(\ker(\pi'))$ and let $\epsilon > 0$. As $T \in \mathcal{E}(C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S))$ Lemma 2.2.8 implies that there exists an $m \in \mathbb{N}$, $T_{1,j} \in \mathfrak{A}_{1,(j)}$, and $T_{2,j} \in \mathfrak{A}_{2,(j)}$ such that $\left\| T - \sum_{i=1}^{2} \sum_{j=0}^{m} T_{i,j} \right\| < \epsilon$. Therefore

$$\|P_1 T P_1 - P_1 T_{1,0} P_1 - P_1 T_{2,0} P_1\| = \left\| P_1 \left(T - \sum_{i=1}^{2} \sum_{j=0}^{m} T_{i,j} \right) P_1 \right\| < \epsilon.$$

Note $T_{1,0} \in \mathfrak{A}_{1,(0)} = \mathfrak{A}_1$ and $T_{2,0} \in \mathfrak{A}_{2,(0)} = \mathfrak{A}_2$. However $P_1 T P_1(\mathcal{H}_{1,0}) = \{0\}$ and $P_1 T_{2,0} P_1(\mathcal{H}_{1,0}) = \{0\}$. Whence

$$\|T_{1,0}h\| = \|P_1 T P_1 h - P_1 T_{1,0} P_1 h - P_1 T_{2,0} P_1 h\| \leq \epsilon \|h\|$$

for all $h \in \mathcal{H}_{1,0}$. Since $\mathfrak{A}_1$ acts on $\mathcal{H}_{1,0}$ via $\pi_{1,0} \circ q$, $\|\pi_{1,0}(q(T_0))\| < \epsilon$. Thus $\|q(T_0)\|_{\mathfrak{A}_1/\mathfrak{J}} < \epsilon$ so there exists a $J \in \mathfrak{J}$ such that $\|T_{1,0} - J\| < \epsilon$. Similarly $P_1 T P_1(\mathcal{H}_2) = \{0\}$ and $T_{1,0}(\mathcal{H}_2) = \{0\}$. Hence $\|T_{2,0}\| < \epsilon$. Thus $J \in \mathfrak{J}_{(0)}$ and

$$\|P_1 T P_1 - P_1 J P_1\| \leq \|P_1 T P_1 - P_1 T_{1,0} P_1 - P_1 T_{2,0} P_1\| + \|T_{2,0}\| + \|T_{1,0} - J\| < 3\epsilon$$

as desired.

Suppose the result is true for some $n \geq 1$. By the inductive hypothesis there exists an $R \in \operatorname{span}\left(\bigcup_{m<n} \mathfrak{J}_{(m)}\right)$ such that $\|Q_n T Q_n - Q_n R Q_n\| < \epsilon$ and thus

$$\left\| Q_n R Q_n - Q_n \left(\sum_{i=1}^{2} \sum_{j=0}^{n-1} T_{i,j} \right) Q_n \right\| < 2\epsilon.$$

By the above discussions and by considering direct sums,

$$\|Q_{n+1}TQ_{n+1} - Q_{n+1}RQ_{n+1} - Q_{n+1}T_{1,n}Q_{n+1} - Q_{n+1}T_{2,n}Q_{n+1}\| < 3\epsilon.$$

Thus it suffices to approximate $T_{1,n} + T_{2,n} \in \mathfrak{A}_{1,(n)} + \mathfrak{A}_{2,(n)}$ uniformly on $Q_{n+1}\mathcal{K}$ with an element of $\mathfrak{J}_{(n)}$. Since elements of $\mathfrak{A}_{1,(n)}$, $\mathfrak{A}_{2,(n)}$ and $\mathfrak{J}_{(n)}$ are zero when restricted to $Q_n\mathcal{K}$, it suffices to perform the approximation on $\mathcal{L}_{1,n+1} \oplus \mathcal{L}_{2,n+1}$. Moreover, as $T_{1,n}$ and $T_{2,n}$ act on orthogonal spaces and map into orthogonal spaces, it suffices to consider each operator separately.

As in the base case T, $T_{1,n}$, and R vanish on the domain of $T_{2,n}$ giving the estimate $\|T_{2,n}\| < 3\epsilon$. To approximate $T_{1,n}$ with an element of $\mathfrak{J}_{(n)}$ fix $k \in \{1,2\}$ with $k \neq n \mod 2$ and let $\ell \in \{1,2\} \setminus \{k\}$. Hence $T_{1,n}$ is completely determined by its action on $\mathcal{L}_{k,n+1}$ and

$$\|P_{k,n+1}TP_{k,n+1} - P_{k,n+1}RP_{k,n+1} - P_{k,n+1}T_{1,n}P_{k,n+1}\| < 3\epsilon$$

by the above inequality and the fact that $P_{k,n+1}T_{2,n}P_{k,n+1} = 0$ by the above discussion. Write

$$T_{1,n} = \sum_{q=1}^{p} \left(A_n^{(q)}S\right) \cdots \left(A_1^{(q)}S\right) A^{(q)} \left(S^* B_1^{(q)}\right) \cdots \left(S^* B_n^{(q)}\right),$$

where $A_i^{(q)}, B_j^{(q)} \in \mathfrak{A}_1$ if $i,j = 0 \mod 2$, $A_i^{(q)}, B_j^{(q)} \in \mathfrak{A}_2$ if $i,j = 1 \mod 2$, and $A^{(q)} \in \mathfrak{A}_1$ for all $q \in \{1,\ldots,p\}$. View $T_{1,n} \in \mathcal{B}(\mathcal{L}_{k,n+1})$ as in the previous discussion. By applying the Gram-Schmidt Orthogonalization Process, we can then write $T_{1,n}$ as the map

$$T_{1,n}(\eta_1 \otimes \cdots \otimes \eta_{n+1}) = \sum_{i=1}^{p}\sum_{j=1}^{p} \langle \eta_1 \otimes \cdots \otimes \eta_n, \zeta_j\rangle (\omega_i \otimes A_{i,j}\eta_{n+1})$$

where $\{A_{i,j}\}_{i,j=1}^{p} \subseteq \mathrm{span}(\{A^{(q)} \mid q \in \{1,\ldots,p\}\}) \subseteq \mathfrak{A}_1$ and

$$\zeta_j, \omega_i \in \mathcal{N}_{i_1} \otimes \mathcal{N}_{i_2} \otimes \cdots \otimes \mathcal{N}_{i_n}$$

(where for $i \in \{1,2\}$ $\mathcal{N}_i = \overline{\mathfrak{A}_i \xi_i}$, $i_j = k$ if j is odd, and $i_j = \ell$ if j is even (and automatically $i_n = 2$)) are such that $\{\zeta_j\}_{j=1}^p$ and $\{\omega_i\}_{i=1}^p$ are orthonormal sets.

To examine the norm of $T_{1,n}$ in this form, suppose $\zeta \in \mathcal{L}_{k,n+1}$ is such that $\|\zeta\| \leq 1$. Then we can write $\zeta = \sum_{j=1}^p \zeta_j \otimes \eta_j + \sum_{\gamma \in \Gamma} \zeta_\gamma \otimes \eta_\gamma$ where $\{\zeta_\gamma\}_{\gamma\in\Gamma} \subseteq \mathcal{L}_{k,n}$ extends $\{\zeta_j\}_{j=1}^p$ to an orthonormal basis of $\mathcal{L}_{k,n}$ and $\eta_j, \eta_\gamma \in \mathcal{H}_1$. Thus $\sum_{j=1}^p \|\eta_j\|^2 \leq \|\zeta\|^2 \leq 1$ and

$$\begin{aligned} \|T_{1,n}\zeta\|_{\mathcal{L}_{k,n+1}} &= \left\| \sum_{j=1}^p \sum_{i=1}^p \|\zeta_j\|^2 \omega_i \otimes A_{i,j}\eta_j \right\|_{\mathcal{L}_{k,n+1}} \\ &= \left\| \sum_{i=1}^p \omega_i \otimes \left(\sum_{j=1}^p A_{i,j}\eta_j \right) \right\|_{\mathcal{L}_{k,n+1}} \\ &= \left(\sum_{i=1}^p \left\| \sum_{j=1}^p A_{i,j}\eta_j \right\|_{\mathcal{H}_1}^2 \right)^{\frac{1}{2}} \qquad (*). \end{aligned}$$

This final expression is directly related to the norm of $[A_{i,j}] \in \mathcal{M}_p(\mathfrak{A}_1)$. Indeed recall that $\mathfrak{A}_1$ is acting on $\mathcal{H}_1 = \mathcal{H}_{1,0} \oplus \mathcal{H}_{1,1}$ via $(\pi_{1,0} \circ q) \oplus \pi_{1,1}$ and define $\sigma_p : \mathcal{M}_p(\mathfrak{A}_1) \to \mathcal{B}\left(\mathcal{H}_1^{\oplus p}\right)$ by

$$\sigma_p([A'_{i,j}])(h_1 \oplus \cdots \oplus h_\ell) = \bigoplus_{i=1}^p \left(\sum_{j=1}^p A'_{i,j} h_j \right)$$

for all $[A'_{i,j}] \in \mathcal{M}_p(\mathfrak{A}_1)$. Clearly σ_p is a faithful representation of $\mathcal{M}_p(\mathfrak{A}_1)$ since $(\pi_{1,0} \circ q) \oplus \pi_{1,1}$ is a faithful representation of $\mathfrak{A}_1$. Notice $\sigma_p([A'_{i,j}])$ is zero on $\mathcal{H}_{1,0}^{\oplus p} \subseteq \mathcal{H}_1^{\oplus p}$ if and only if each $A'_{i,j}$ is zero on $\mathcal{H}_{1,0}$ if and only if $[A'_{i,j}] \in \mathcal{M}_p(\mathfrak{J})$. Since $\mathcal{M}_p(\mathfrak{J})$ is an ideal of $\mathcal{M}_p(\mathfrak{A}_1)$, $\mathcal{H}_{1,0}^{\oplus p}$ is a reducing subspace for $\sigma_p(\mathcal{M}_p(\mathfrak{A}_1))$, and $\mathcal{M}_p(\mathfrak{J}) = \ker\left(\sigma_p|_{\mathcal{H}_{1,0}^{\oplus p}}\right)$, we obtain that $\sigma_p|_{\mathcal{H}_{1,0}^{\oplus p}}$ is a faithful representation of $\mathcal{M}_p(\mathfrak{A}_1)/\mathcal{M}_p(\mathfrak{J}) \simeq \mathcal{M}_p(\mathfrak{A}_1/\mathfrak{J})$.

Using the fact that $P_{k,n+1}TP_{k,n+1} - P_{k,n+1}RP_{k,n+1}$ is zero on

$$\mathcal{H}_{i_1,0} \otimes \mathcal{H}_{i_2,0} \otimes \cdots \otimes \mathcal{H}_{i_n,0} \otimes \mathcal{H}_{i_{n+1},0}$$

where $i_j = k$ if j is odd and $i_j = \ell$ if j is even (so $i_{n+1} = 1$ automatically) (as $T \in \mathcal{E}(\ker(\pi'))$ and $R \in \operatorname{span}\left(\bigcup_{m<n} \mathfrak{J}_{(m)}\right)$), we obtain that

$$\left\| P_{n+1}T_{1,n}P_{n+1}|_{\mathcal{H}_{i_1,0}\otimes\mathcal{H}_{i_2,0}\otimes\cdots\otimes\mathcal{H}_{i_n,0}\otimes\mathcal{H}_{i_{n+1},0}} \right\| < 3\epsilon.$$

As $\omega_i, \zeta_j \in \mathcal{H}_{i_1,0} \otimes \mathcal{H}_{i_2,0} \otimes \cdots \otimes \mathcal{H}_{i_n,0}$ for all $i, j \in \{1, \ldots, p\}$, using $(*)$ we see that

$$\left(\sum_{i=1}^{p} \left\| \sum_{j=1}^{p} A_{i,j} \eta_j \right\|_{\mathcal{H}_{1,0}}^2 \right)^{\frac{1}{2}} < 3\epsilon$$

for all $\eta_1, \ldots, \eta_p \in \mathcal{H}_{1,0}$ with $\sum_{j=1}^{p} \|\eta_j\|^2 \leq 1$. Hence $\left\| \sigma_p([A_{i,j}])|_{\mathcal{H}_{1,0}^{\oplus p}} \right\| < 3\epsilon$. Since $\sigma_p|_{\mathcal{H}_{1,0}^{\oplus p}}$ is a faithful representation of $\mathcal{M}_p(\mathfrak{A}_1)/\mathcal{M}_p(\mathfrak{J}) \simeq \mathcal{M}_p(\mathfrak{A}_1/\mathfrak{J})$, there exists a $[J_{i,j}] \in \mathcal{M}_p(\mathfrak{J})$ such that $\|[A_{i,j}] - [J_{i,j}]\|_{\mathcal{M}_p(\mathfrak{A}_1)} < 3\epsilon$. Thus

$$\left(\sum_{i=1}^{p} \left\| \sum_{j=1}^{p} (A_{i,j} - J_{i,j}) \eta_j \right\|_{\mathcal{H}_1}^2 \right)^{\frac{1}{2}} < 3\epsilon$$

for all $\eta_1, \ldots, \eta_\ell \in \mathcal{H}_1$ with $\sum_{j=1}^{p} \|\eta_j\|^2 \leq 1$.

Define $R' \in \mathcal{B}(\mathcal{L}_{k,n+1})$ by

$$R'(\eta_1 \otimes \cdots \otimes \eta_{n+1}) = \sum_{i=1}^{p} \sum_{j=1}^{p} \langle \eta_1 \otimes \cdots \otimes \eta_n, \zeta_j \rangle (\omega_i \otimes J_{i,j} \eta_{n+1})$$

and extend by linearity and density. Note $(*)$ implies that R' is a bounded linear map and

$$\|R' - P_{k,n+1} T_{1,n} P_{k,n+1}\|_{\mathcal{B}(\mathcal{L}_{k,n+1})} < 3\epsilon.$$

As

$$\zeta_j, \omega_i \in \mathcal{N}_{i_1} \otimes \mathcal{N}_{i_2} \otimes \cdots \otimes \mathcal{N}_{i_n},$$

the above discussion implies that there exists a $R_0 \in \mathfrak{J}_{(n)}$ such that

$$\|R' - P_{k,n+1} R_0 P_{k,n+1}\|_{\mathcal{B}(\mathcal{L}_{k,n+1})} < \epsilon.$$

Hence

$$\|P_{k,n+1} R_0 P_{k,n+1} - P_{k,n+1} T_{1,n} P_{k,n+1}\| < 4\epsilon.$$

Since $R_0 \in \mathfrak{J}_{(n)}$ and $T_{1,n} \in \mathfrak{A}_{1,(n)}$, we obtain that

$$\|Q_{n+1}R_0Q_{n+1} - Q_{n+1}T_{1,n}Q_{n+1}\| < 4\epsilon.$$

By combining all of our approximations

$$\|Q_{n+1}TQ_{n+1} - Q_{n+1}(R + R_0)Q_{n+1}\| < 10\epsilon$$

and, as $R + R_0 \in \mathrm{span}\left(\bigcup_{m<n+1} \mathfrak{J}_{(m)}\right)$, the result follows. □

The above result shows we can approximate elements of $\mathcal{E}(\ker(\pi'))$ uniformly on $Q_n\mathcal{K}$ by elements of $\mathrm{span}\left(\bigcup_{m<n} \mathfrak{J}_{(m)}\right)$. The following result shows that this is enough to prove the assumptions of Lemma 2.2.7.

Lemma 2.2.10. *Let $T \in \mathcal{E}(C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S))$ and let $\epsilon > 0$. There exists an $n \in 2\mathbb{N}$ such that*

$$\left\|P_{j,m}TP_{j,m} - P_{j,n}TP_{j,n} \otimes I_{\mathcal{L}_{j,m-n}}\right\|_{\mathcal{B}(\mathcal{L}_{j,m})} < \epsilon$$

for all $m \geq n$ and $j \in \{1, 2\}$ (where $\mathcal{L}_{j,m} \simeq \mathcal{L}_{j,n} \otimes \mathcal{L}_{j,m-n}$ canonically).

Proof. This result follows from the fact that

$$\mathrm{span}\left(\bigcup_{n\geq 1}(\mathfrak{A}_{1,(n)} \cup \mathfrak{A}_{2,(n)})\right)$$

is dense in

$$\mathcal{E}(C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S))$$

and the result hold for this algebraic span. □

Proof of Theorem 2.2.2. Recall $\mathcal{E}\left(\langle\mathfrak{J}\rangle_{\mathfrak{A}_1\oplus\mathfrak{A}_2}\right) \subseteq \mathcal{E}(\ker(\pi'))$ by the above discussions. Let

$T \in \mathcal{E}(\ker(\pi'))$ and let $\epsilon > 0$. By Lemma 2.2.10 there exists an $n \in 2\mathbb{N}$ so that

$$\left\| P_{j,m} T P_{j,m} - P_{j,n} T P_{j,n} \otimes I_{\mathcal{L}_{j,m-n}} \right\|_{\mathcal{B}(\mathcal{L}_{j,m})} < \epsilon$$

for all $m \geq n$ and $j \in \{1, 2\}$. By Lemma 2.2.9 there exists an $R \in \operatorname{span}\left(\bigcup_{m<n} \mathfrak{J}_{(m)}\right)$ such that $\|Q_n(T - R)Q_n\| < \epsilon$. As $T, R \in \mathcal{E}(C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S))$

$$\|T - R\| = \sup_{m \geq 1} \max_{j \in \{1,2\}} \|P_{j,m}(T - R)P_{j,m}\|$$

by Lemma 2.2.4 and the above inequality implies $\|P_{j,m}(T - R)P_{j,m}\|_{\mathcal{L}_{j,m}} < \epsilon$ for all $m \leq n$ and $j \in \{1, 2\}$. Thus $\|P_{j,n} T P_{j,n} - P_{j,n} R P_{j,n}\| < \epsilon$ for all $j \in \{1, 2\}$. However, since $R \in \operatorname{span}\left(\bigcup_{m<n} \mathfrak{J}_{(m)}\right)$, $P_{j,m} R P_{j,m} = P_{j,n} R P_{j,n} \otimes I_{\mathcal{L}_{j,m-n}}$ for all $m \geq n$ and $j \in \{1, 2\}$ (as n is even) and thus

$$\|P_{j,m} T P_{j,m} - P_{j,m} R P_{j,m}\| \leq 2\epsilon$$

for all $m \geq n$ and $j \in \{1, 2\}$. Whence $\|T - R\| \leq 2\epsilon$. As $R \in \langle \mathfrak{J} \rangle_{\mathfrak{A}_1 \oplus \mathfrak{A}_2}$, we obtain that $T \in \mathcal{E}\left(\langle \mathfrak{J} \rangle_{\mathfrak{A}_1 \oplus \mathfrak{A}_2}\right)$. Hence $\mathcal{E}(\ker(\pi')) = \mathcal{E}\left(\langle \mathfrak{J} \rangle_{\mathfrak{A}_1 \oplus \mathfrak{A}_2}\right)$ so $\ker(\pi') = \langle \mathfrak{J} \rangle_{\mathfrak{A}_1 \oplus \mathfrak{A}_2}$ by Lemma 2.2.7. □

2.3 Every C*-Algebra is Freely Exact

In this section we will complete the proof of Theorem 2.1.2. By Theorem 2.2.2 we know certain short sequences of C*-algebras are exact and we will use the proof of [14, Theorem 4.8.2] to construct a commutative diagram of short sequences. The proof of [14, Theorem 4.8.2] is concrete and allows us to demonstrate that the compression of $\langle \mathfrak{J} \rangle_{\mathfrak{A}_1 \oplus \mathfrak{A}_2}$ corresponds with the description of $\langle \mathfrak{J} \rangle_{\mathfrak{A}_1 * \mathfrak{A}_2}$ developed in Discussion 2.3.1. The remainder of the proof is then trivial.

Remarks 2.3.1. Using the notation of Construction 2.1.1, we desire to determine the structure of $\langle \mathfrak{J} \rangle_{\mathfrak{A}_1 * \mathfrak{A}_2}$ inside $(\mathfrak{A}_1, \pi_1, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2)$. For $i \in \{1, 2\}$ let

$$\mathfrak{A}_i^0 := \{A \in \mathfrak{A}_i \mid \langle A\xi_i, \xi_i \rangle_{\mathcal{H}_i} = 0\}$$

so $\mathfrak{A}_i = \mathbb{C} I_{\mathfrak{A}_i} + \mathfrak{A}_i^0$. Thus, by the algebraic properties of ideals, it is clear that the span of all operators of the form

$$A_1 B_1 \cdots A_n B_n J B'_m A'_m \cdots B'_1 A'_1$$

where $n, m \geq 0$, $A_i, A'_j \in \mathfrak{A}_1^0 \cup \{I_{\mathfrak{A}_1}\}$, $B_i, B'_j \in \mathfrak{A}_2^0 \cup \{I_{\mathfrak{A}_2}\}$, and $J \in \mathfrak{J}$ is dense in $\langle \mathfrak{J} \rangle_{\mathfrak{A}_1 * \mathfrak{A}_2}$. Notice $\mathfrak{J} \subseteq \mathfrak{A}_1^0$. Using the fact that the identity elements of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are the identity element of $(\mathfrak{A}_1, \pi_1, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2)$ when viewed as elements of $(\mathfrak{A}_1, \pi_1, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2)$, we can further assume that $A_i, A'_j \in \mathfrak{A}_1^0$ whenever $i, j \geq 2$ and $B_i, B'_j \in \mathfrak{A}_2^0$ as whenever an $I_{\mathfrak{A}_1}$ or $I_{\mathfrak{A}_2}$ occurs we can reduce the length of the product, multiply two elements of the alternate $\mathfrak{A}_i$, write this element in the form $\mathbb{C} I_{\mathfrak{A}_i} + \mathfrak{A}_i^0$, write the new operator as the sum of two operators, and continue the reduction process.

We desire to describe the action of each operator in the above span on $(\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)$. If

$$T = A_1 B_1 \cdots A_n B_n J B'_m A'_m \cdots B'_1 A'_1$$

where $n, m \geq 0, \{A_i\}_{i=1}^n, \{A'_j\}_{j=1}^m \subseteq \mathfrak{A}_1^0$, $\{B_i\}_{i=1}^n, \{B'_j\}_{j=1}^m \subseteq \mathfrak{A}_2^0$, and $J \in \mathfrak{J}$, it is easy to verify that T is non-zero only on the direct summand

$$\left(\bigoplus_{k \geq m+1} (\mathcal{H}_1^0 \otimes \mathcal{H}_2^0)^{\otimes k} \right) \oplus \left(\bigoplus_{k \geq m} (\mathcal{H}_1^0 \otimes \mathcal{H}_2^0)^{\otimes k} \otimes \mathcal{H}_1^0 \right) \subseteq (\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2),$$

if $\omega_1 \in (\mathcal{H}_1^0 \otimes \mathcal{H}_2^0)^{\otimes m} \otimes \mathcal{H}_1^0$ and

$$\omega_2 \in \bigcup_{k \geq 1} \left((\mathcal{H}_2^0 \otimes \mathcal{H}_1^0)^{\otimes k} \cup (\mathcal{H}_2^0 \otimes \mathcal{H}_1^0)^{\otimes k} \otimes \mathcal{H}_2^0 \right)$$

then $T(\omega_1 \otimes \omega_2) = T(\omega_1) \otimes \omega_2$. Moreover, a proof of the above facts reveals that if

$$\eta := \eta_1 \otimes \zeta_1 \otimes \cdots \otimes \eta_m \otimes \zeta_m \otimes \eta_{m+1} \in (\mathcal{H}_1^0 \otimes \mathcal{H}_2^0)^{\otimes m} \otimes \mathcal{H}_1^0$$

then

$$T(\eta) = \langle \eta_1 \otimes \zeta_1 \otimes \cdots \otimes \eta_m \otimes \zeta_m, \omega_1 \rangle_{(\mathcal{H}_1^0 \otimes \mathcal{H}_2^0)^{\otimes m}} (\omega_2 \otimes J\eta_{m+1})$$

where

$$\omega_1 := (A_1')^* \xi_1 \otimes (B_1')^* \xi_2 \otimes \cdots \otimes (A_m')^* \xi_1 \otimes (B_m')^* \xi_2$$

and

$$\omega_2 := (A_1 \xi_1) \otimes (B_1 \xi_2) \otimes \cdots \otimes (A_n \xi_1) \otimes (B_n \xi_2).$$

The cases where $A_1 = I_{\mathfrak{A}_1}$ and/or $A_1' = I_{\mathfrak{A}_1}$ are similar.

To embed the sequences under consideration in Theorem 2.1.2 into an exact sequence from Theorem 2.2.2, we will use the following notation and maps.

Notation 2.3.2. Let $\mathfrak{A}_{i,1} := \mathfrak{A}_i$ for $i = 1, 2$, let $\mathfrak{A}_{1,0} := \mathfrak{A}_1/\mathfrak{J}$, let $\mathfrak{A}_{2,0} := \mathfrak{A}_2$, and let $S_1 := S$. Using the notation of Construction 2.2.1, for $j = 0, 1$ let

$$P_j := I - S_j^2 (S_j^*)^2 \in C^*(\mathfrak{A}_{1,j} \oplus \mathfrak{A}_{2,j}, S_j)$$

and let

$$U_j := P_j (S_j + S_j^*) P_j \in C^*(\mathfrak{A}_{1,j} \oplus \mathfrak{A}_{2,j}, S_j).$$

For $j \in \{0, 1\}$ and $i \in \{1, 2\}$ define the unital, completely positive maps

$$\psi_{i,j} : \mathfrak{A}_{i,j} \to P_j C^*(\mathfrak{A}_{1,j} \oplus \mathfrak{A}_{2,j}, S_j) P_j$$

by

$$\psi_{i,j}(A) = P_j A P_j + U_j A U_j$$

for all $A \in \mathfrak{A}_{i,j}$.

Lemma 2.3.3. *There exists a unital, completely positive map*

$$\Psi : (\mathfrak{A}_1, \pi_1, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2) \to P_1 C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S) P_1$$

such that

$$\Psi(A_1 \cdots A_n) = \psi_{i_1,1}(A_1) \cdots \psi_{i_n,1}(A_n)$$

whenever $A_k \in \mathfrak{A}^0_{i_k}$, $\{i_k\}_{k=1}^n \subseteq \{1,2\}$, *and* $i_k \neq i_{k+1}$ *for all* $k \in \{1, \ldots, n-1\}$. *Moreover there exists a* **-homomorphism*

$$\sigma : C^*(\Psi((\mathfrak{A}_1, \pi_1, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2))) \to (\mathfrak{A}_1, \pi_1, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2)$$

such that $\sigma \circ \Psi = Id_{(\mathfrak{A}_1,\pi_1,\xi_1)*(\mathfrak{A}_2,\pi_2,\xi_2)}$. *In fact* σ *is the compression map of* $\mathcal{B}(\mathcal{K})$ *to* $\mathcal{B}(\mathcal{K}_{1,1})$ *where* $\mathcal{K}_{1,1} \subseteq \mathcal{K}$ *is a Hilbert space isomorphic to* $(\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)$.

Similarly there exists a unital, completely positive map

$$\Psi_0 : (\mathfrak{A}_{1,0}, \pi_{1,0}, \xi_1) * (\mathfrak{A}_{2,0}, \pi_{2,0}, \xi_2) \to P_0 C^*(\mathfrak{A}_{1,0} \oplus \mathfrak{A}_{2,0}, S_0) P_0$$

such that

$$\Psi_0(A_1 \cdots A_n) = \psi_{i_1,0}(A_1) \cdots \psi_{i_n,0}(A_n)$$

whenever $A_k \in \mathfrak{A}_{i_k,0}$ *are such that* $\langle A_k \xi_{i_k}, \xi_{i_k} \rangle_{\mathcal{H}_{i_k,0}} = 0$, $\{i_k\}_{k=1}^n \subseteq \{1,2\}$, *and* $i_k \neq i_{k+1}$ *for all* $k \in \{1, \ldots, n-1\}$.

Proof. The proof of the above result is contained in [14, Theorem 4.8.2]. Note that the proof in [14] is done under the assumptions that π_1, π_2, and $\pi_{1,0}$ are the faithful representations corresponding to a GNS construction. However these assumptions are not used in the proof.

For the purpose of Lemma 2.3.5, we remark that the Hilbert subspace $\mathcal{K}_{1,1}$ of $\mathcal{K}$ is the subspace

$$\mathcal{H}_1 \oplus \left(\bigoplus_{n \geq 0} \mathcal{H}_1 \otimes (\mathcal{H}_2^0 \otimes \mathcal{H}_1^0)^{\otimes n} \otimes \mathcal{H}_2^0 \otimes \mathcal{H}_1 \right)$$

and is isomorphic to $(\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)$ via the standard identifications $\mathbb{C}\xi_1 \otimes \mathcal{H}_2^0 \simeq \mathcal{H}_2^0$ and $\mathcal{H}_2^0 \otimes \mathbb{C}\xi_1 \simeq \mathcal{H}_2^0$. $\square$

Lemma 2.3.4. *With Ψ and Ψ_0 as in Lemma 2.3.3, the diagram*

$$\begin{array}{ccc} (\mathfrak{A}_1, \pi_1, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2) & \xrightarrow{\pi} & (\mathfrak{A}_1/\mathfrak{J}, \pi_{1,0}, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2) \\ \downarrow \Psi & & \downarrow \Psi_0 \\ C^*(\mathfrak{A}_1 \oplus \mathfrak{A}_2, S) & \xrightarrow{\pi'} & C^*((\mathfrak{A}_1/\mathfrak{J}) \oplus \mathfrak{A}_2, S_0) \end{array}$$

commutes.

Proof. Recall that the span of $I_{(\mathfrak{A}_1,\pi_1,\xi_1)*(\mathfrak{A}_2,\pi_2,\xi_2)}$ and

$$\{A_1 \cdots A_n \mid A_k \in \mathfrak{A}_{i_k}^0, \{i_k\}_{k=1}^n \subseteq \{1,2\}, i_k \neq i_{k+1} \text{ for all } k \in \{1, \ldots, n-1\}\}$$

is dense in $(\mathfrak{A}_1, \pi_1, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2)$ and thus it suffices to verify the diagram commutes on these operators. Using the properties of Ψ and Ψ_0 from Lemma 2.3.3 and the fact that

$$\psi_{i,0}(\pi(A)) = P_0\pi(A)P_0 + U_0\pi(A)U_0 = \pi'(P_1AP_1 + U_1AU_1) = \pi'(\psi_{i,1}(A))$$

for all $i \in \{1, 2\}$ and $A \in \mathfrak{A}_i$, the result follows. $\square$

The final technical challenge of the proof of Theorem 2.1.2 is the following.

Lemma 2.3.5. *Let $\sigma : \mathcal{B}(\mathcal{K}) \to \mathcal{B}((\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2))$ be the compression map from Lemma 2.3.3. If $T \in \langle \mathfrak{J} \rangle_{\mathfrak{A}_1 \oplus \mathfrak{A}_2}$ then $\sigma(T) \in \langle \mathfrak{J} \rangle_{\mathfrak{A}_1 * \mathfrak{A}_2}$.*

Proof. By the above discussions and the notation in 2.3.2, it is easy to see that the span of

$$(A_nS) \cdots (A_1S)J(S^*B_1) \cdots (S^*B_m)$$

where $n, m \geq 0$, $A_i, B_j \in \mathfrak{A}_1^0 \cup \{I_{\mathfrak{A}_1}\}$ if i, j are even, $A_i, B_j \in \mathfrak{A}_2^0 \cup \{I_{\mathfrak{A}_2}\}$, and $J \in \mathfrak{J}$ is dense in $\langle \mathfrak{J} \rangle_{\mathfrak{A}_1 \oplus \mathfrak{A}_2}$ and thus it suffices to show that the compression of each of these operators to

$\mathcal{K}$ corresponds to an operator in $\langle \mathfrak{J} \rangle_{\mathfrak{A}_1 * \mathfrak{A}_2}$ as described in Discussion 2.3.1. It is not difficult to show that the compression of one of these operators is zero unless n and m are even, $A_i, B_j \in \mathfrak{A}_2^0$ if i, j are odd, and $A_i, B_j \subseteq \mathfrak{A}_1^0$ if i, j are even with $i < n$ and $j < m$. Moreover, when the compression is non-zero, the above operator corresponds to the operator in $\langle \mathfrak{J} \rangle_{\mathfrak{A}_1 * \mathfrak{A}_2}$ described by removing the S's and S^*'s in the above expression. $\square$

Proof of Theorem 2.1.2. Suppose $T \in ((\mathfrak{A}_1, \pi_1, \xi_1) * (\mathfrak{A}_2, \pi_2, \xi_2)) \cap \ker(\pi)$. Therefore

$$\pi'(\Psi(T)) = \Psi_0(\pi(T)) = 0$$

by Lemma 2.3.4. By Theorem 2.2.2 $\Psi(T) \in \langle \mathfrak{J} \rangle_{\mathfrak{A}_1 \oplus \mathfrak{A}_2}$. Therefore $\sigma(\Psi(T)) \in \langle \mathfrak{J} \rangle_{\mathfrak{A}_1 * \mathfrak{A}_2}$ by Lemma 2.3.5. However $T = \sigma(\Psi(T))$ by Lemma 2.3.3 so $T \in \langle \mathfrak{J} \rangle_{\mathfrak{A}_1 * \mathfrak{A}_2}$ as desired. $\square$

2.4 Strong Convergence is Preserved by Free Products

With the modification to the second equivalence of Theorem 1.6.1 complete, we turn our attention to developing the analog of the fourth equivalence of Theorem 1.6.1 in the context of reduced free products. We begin with a definition.

Definition 2.4.1. Let $\left\{X_i^{(k)}\right\}_{i=1}^n$ and $\{X_i\}_{i=1}^n$ be generators for the non-commutative probability spaces $(\mathfrak{A}_k, \tau_k)$ and $(\mathfrak{A}, \tau)$ respectively. We say that $\left\{X_i^{(k)}\right\}_{i=1}^n$ converge strongly to $\{X_i\}_{i=1}^n$ if

1. $\limsup_{k\to\infty} \left\| p\left(X_1^{(k)}, \ldots, X_n^{(k)}\right)\right\|_{\mathfrak{A}_k} = \|p(X_1, \ldots, X_n)\|_{\mathfrak{A}}$, and
2. $\lim_{k\to\infty} \tau_k\left(p\left(\left(X_1^{(k)}, \ldots, X_n^{(k)}\right)\right)\right) = \tau(p(X_1, \ldots, X_n))$

for all $p \in \mathbb{C}\langle t_1, \ldots, t_n\rangle$.

The following is the adaptation of the fourth equivalence of Theorem 1.6.1 to reduced free products and is a generalization of the appendix of [47] due to Shlyakhtenko (where, if $\mathfrak{A}_i$ are C*-algebras with states φ_i that have faithful GNS representations, $(\mathfrak{A}_1, \varphi_1) * (\mathfrak{A}_2, \varphi_2)$ is the reduced free product, $\varphi_1 * \varphi_2$ is the vector state on $(\mathfrak{A}_1, \varphi_1) * (\mathfrak{A}_2, \varphi_2)$ corresponding to the distinguished vector, $\mathbb{C}\langle t_1, \ldots, t_n \rangle$ denotes set of all complex polynomials in n non-commuting variables and their complex conjugates, and a pair $(\mathfrak{A}, \tau)$ is said to be a non-commutative probability space if $\mathfrak{A}$ is a unital C*-algebra and τ is a state on $\mathfrak{A}$ with a faithful GNS representation):

Theorem 2.4.2. *Let* $\left\{X_i^{(k)}\right\}_{i=1}^n$, $\left\{Y_i^{(k)}\right\}_{i=1}^m$, $\{X_i\}_{i=1}^n$, *and* $\{Y_i\}_{i=1}^m$ *be generators for the non-commutative probability spaces* $(\mathfrak{A}_k, \tau_k)$, $(\mathfrak{B}_k, \varphi_k)$, $(\mathfrak{A}, \tau)$, *and* $(\mathfrak{B}, \varphi)$ *respectively. If* $\left\{X_i^{(k)}\right\}_{i=1}^n$ *converge strongly to* $\{X_i\}_{i=1}^n$ *and* $\left\{Y_i^{(k)}\right\}_{i=1}^n$ *converge strongly to* $\{Y_i\}_{i=1}^n$, *then* $\left\{X_i^{(k)}\right\}_{i=1}^n \cup \left\{Y_i^{(k)}\right\}_{i=1}^n$ *converge strongly to* $\{X_i\}_{i=1}^n \cup \{Y_i\}_{i=1}^n$ *(where the later operators are in* $(\mathfrak{A}_k, \tau_k) * (\mathfrak{B}_k, \varphi_k)$ *and* $(\mathfrak{A}, \tau) * (\mathfrak{B}, \varphi)$*).*

By examining the fourth equivalence of Theorem 1.6.1, it can easily be seen that the above result is connected with the notion of an exact C*-algebra by replacing tensor products with reduced free products. To begin the proof of Theorem 2.4.2, we note one inequality is trivially implied by verifying that the assumptions imply convergence in L_2-norms.

Lemma 2.4.3. *With the assumptions and notation of Theorem 2.4.2,*

$$\lim_{k\to\infty} (\tau_k * \varphi_k)\left(p\left(X_1^{(k)}, \ldots, X_n^{(k)}, Y_1^{(k)}, \ldots, Y_m^{(k)}\right)\right) = (\tau * \varphi)(p(X_1, \ldots, X_n, Y_1, \ldots, Y_m))$$

and

$$\liminf_{k\to\infty} \left\|p\left(X_1^{(k)}, \ldots, X_n^{(k)}, Y_1^{(k)}, \ldots, Y_m^{(k)}\right)\right\| \geq \|p(X_1, \ldots, X_n, Y_1, \ldots, Y_m)\|$$

for all $p \in \mathbb{C}\langle t_1, \ldots, t_{n+m}\rangle$.

Proof. First we claim if $p \in \mathbb{C}\langle t_1, \ldots, t_{n+m}\rangle$ is arbitrary then

$$\lim_{k\to\infty} (\tau_k * \varphi_k)\left(p\left(X_1^{(k)}, \ldots, X_n^{(k)}, Y_1^{(k)}, \ldots, Y_m^{(k)}\right)\right) = (\tau * \varphi)(p(X_1, \ldots, X_n, Y_1, \ldots, Y_m)).$$

To see this notice by the same arguments as used in Discussion 2.3.1 that $p(t_1, \ldots, t_{n+m})$

can be written as

$$p(t_1, \ldots, t_{n+m}) = \sum_{\ell=1}^{N} \prod_{w=1}^{z_\ell} p_{\ell,w}(t_1, \ldots, t_n) q_{\ell,w}(t_{n+1}, \ldots, t_{n+m})$$

where $\tau(p_{\ell,w}(X_1, \ldots, X_n)) = 0$ and $\varphi(q_{\ell,w}(Y_1, \ldots, Y_m)) = 0$ for all $w \in \{1, \ldots, z_\ell\}$ and $\ell \in \{1, \ldots, N\}$ except possible for possible $p_{\ell,1}$ and q_{ℓ,z_ℓ} which can be constant functions. Thus $(\tau * \varphi)(p(X_1, \ldots, X_n, Y_1, \ldots, Y_m))$ is

$$\sum_{\ell=1}^{N} \prod_{w=1}^{z_\ell} \tau(p_{\ell,w}(X_1, \ldots, X_n))\varphi(q_{\ell,w}(Y_1, \ldots, Y_m))$$

by freeness and $(\tau_k * \varphi_k)\left(p\left(X_1^{(k)}, \ldots, X_n^{(k)}, Y_1^{(k)}, \ldots, Y_m^{(k)}\right)\right)$ is

$$\sum_{\ell=1}^{N}(\tau_k * \varphi_k)\left(\prod_{w=1}^{z_\ell} p_{\ell,w}\left(X_1^{(k)}, \ldots, X_n^{(k)}\right) q_{\ell,w}\left(Y_1^{(k)}, \ldots, Y_m^{(k)}\right)\right)$$

by linearity. In the case that $\tau(p_{\ell,w}(X_1, \ldots, X_n)) = 0$ and $\varphi(q_{\ell,w}(Y_1, \ldots, Y_m)) = 0$ for all $w \in \{1, \ldots, z_\ell\}$, notice the product from 1 to z_ℓ of

$$\left(p_{\ell,w}\left(X_1^{(k)}, \ldots, X_n^{(k)}\right) - \tau_k\left(p_{\ell,w}\left(X_1^{(k)}, \ldots, X_n^{(k)}\right)\right) I_{\mathfrak{A}_k}\right) q_{\ell,w}\left(Y_1^{(k)}, \ldots, Y_m^{(k)}\right)$$

can be written as

$$\left(\prod_{w=1}^{z_\ell} p_{\ell,w}\left(X_1^{(k)}, \ldots, X_n^{(k)}\right) q_{\ell,w}\left(Y_1^{(k)}, \ldots, Y_m^{(k)}\right)\right) + T_k$$

where T_k is the sum of products of elements in

$$\left\{p_{\ell,w}\left(X_1^{(k)}, \ldots, X_n^{(k)}\right), \tau_k\left(p_{\ell,w}\left(X_1^{(k)}, \ldots, X_n^{(k)}\right)\right), q_{\ell,w}\left(Y_1^{(k)}, \ldots, Y_m^{(k)}\right)\right\}_{w=1}^{z_\ell}$$

where each product contains at least one $\tau_k\left(p_{\ell,w}\left(X_1^{(k)}, \ldots, X_n^{(k)}\right)\right)$ and $T_{k'}$ can be obtained from T_k by exchanging the index k with k'. It is straightforward to show that

$$(\tau_k * \varphi_k)\left(\prod_{w=1}^{z_\ell} p_{\ell,w}\left(X_1^{(k)}, \ldots, X_n^{(k)}\right) q_{\ell,w}\left(Y_1^{(k)}, \ldots, Y_m^{(k)}\right)\right) + (\tau_k * \varphi_k)(T_k) = 0.$$

Since $\tau(p_{\ell,w}(X_1, \ldots, X_n)) = 0$ and $\varphi(q_{\ell,w}(Y_1, \ldots, Y_m)) = 0$ for all $w \in \{1, \ldots, z_\ell\}$, the assumptions of the lemma imply

$$\lim_{k\to\infty} (\tau_k * \varphi)(T_k) = 0$$

as every term in used in T_k is bounded by the first and second assumptions of Theorem 2.4.2 and

$$\lim_{k\to\infty} \tau_k\left(p_{\ell,w}\left(X_1^{(k)}, \ldots, X_n^{(k)}\right)\right) = \tau(p_{\ell,w}(X_1, \ldots, X_n)) = 0$$

for all $w \in \{1, \ldots, z_\ell\}$ by the third assumption of Theorem 2.4.2 . As similar computations hold when $p_{\ell,1}$ and/or q_{ℓ,z_ℓ} are constants (by possibly using the $Y^{(k)}$ instead of the $X^{(k)}$ and the fourth assumption of Theorem 2.4.2), the claim has been proven.

For each $k \in \mathbb{N}$ let $\|T\|_{2,\tau_k*\varphi_k} = (\tau_k * \varphi)(T^*T)^{\frac{1}{2}}$ for all $T \in (\mathfrak{A}_k, \tau_k) * (\mathfrak{B}_k, \varphi_k)$ and let $\|T\|_{2,\tau*\varphi} = (\tau * \varphi)(T^*T)^{\frac{1}{2}}$ for all $T \in (\mathfrak{A}, \tau) * (\mathfrak{B}, \varphi)$. By considering the construction of the reduced free product, for a fixed polynomial $p \in \mathbb{C}\langle t_1, \ldots, t_{n+m}\rangle$ the norm $\|p(X_1, \ldots, X_n, Y_1, \ldots, Y_m)\|$ agrees with

$$\sup\left\{|(\tau * \varphi)((p \cdot p_1 \cdot p_2)(X_1, \ldots, X_n, Y_1, \ldots, Y_m))|\right\}$$

where the supremum is taken over all $p_i \in \mathbb{C}\langle t_1, \ldots, t_{n+m}\rangle$ with

$$\|p_i(X_1, \ldots, X_n, Y_1, \ldots, Y_m)\|_{2,\tau*\varphi} < 1.$$

As a similar expression holds for $\left\|p\left(X_1^{(k)}, \ldots, X_n^{(k)}, Y_1^{(k)}, \ldots, Y_m^{(k)}\right)\right\|$, the result follows. $\square$

Remarks 2.4.4. Using the notation in Theorem 2.4.2, let

$$\mathfrak{D} := \frac{\prod_{k\geq 1}\left((\mathfrak{A}_k, \tau_k) * (\mathfrak{B}_k, \varphi_k)\right)}{\bigoplus_{k\geq 1}\left((\mathfrak{A}_k, \tau_k) * (\mathfrak{B}_k, \varphi_k)\right)}$$

and let $q : \prod_{k\geq 1}\left((\mathfrak{A}_k, \tau_k) * (\mathfrak{B}_k, \varphi_k)\right) \to \mathfrak{D}$ be the canonical quotient map. Consider the C*-subalgebra $\mathfrak{C}$ of $\mathfrak{D}$ generated by

$$\left\{ q\left(\left(X_j^{(k)}\right)_{k\geq 1}\right) \mid j \in \{1, \ldots, n\}\right\} \bigcup \left\{ q\left(\left(Y_j^{(k)}\right)_{k\geq 1}\right) \mid j \in \{1, \ldots, m\}\right\}.$$

Lemma 2.4.3 tells us that there exists a surjective *-homomorphism $\Psi : \mathfrak{C} \to (\mathfrak{A}, \tau) * (\mathfrak{B}, \varphi)$ defined by

$$\Psi\left(q\left(\left(X_j^{(k)}\right)_{k\geq 1}\right)\right) = X_j \qquad \text{and} \qquad \Psi\left(q\left(\left(Y_j^{(k)}\right)_{k\geq 1}\right)\right) = Y_j.$$

Moreover Ψ is an isomorphism if and only if

$$\limsup_{k\to\infty} \left\| p\left(X_1^{(k)}, \ldots, X_n^{(k)}, Y_1^{(k)}, \ldots, Y_m^{(k)}\right)\right\| \leq \|p(X_1, \ldots, X_n, Y_1, \ldots, Y_m)\|$$

for all polynomials $p \in \mathbb{C}\langle t_1, \ldots, t_{n+m}\rangle$. Thus Theorem 2.4.2 is true if and only if Ψ is an isomorphism. The question of whether Ψ is an isomorphism can be considered as a modification of the third equivalence of Theorem 1.6.1.

Our next goal is to prove Theorem 2.4.2 provided that $Y_j^{(k)} = Y_j$ for all $k \in \mathbb{N}$ and $j \in \{1, \ldots, m\}$, $\varphi_k = \varphi$ for all $k \in \mathbb{N}$, and $\mathfrak{B}$ is an exact C*-algebra. To do this we reprove the following known results from the appendix of [47] that prove Theorem 2.4.2 when the Y_j are free creation operators on a Fock space.

Lemma 2.4.5. *Let $\mathfrak{A}$ be a unital C*-algebra with a state τ with a faithful GNS representation and let $\mathfrak{B}$ be the universal C*-algebra generated by $\mathfrak{A}$ and elements $L_1, \ldots, L_n$ satisfying $L_i^* A L_j = \delta_{i,j}\tau(A)$ for all $A \in \mathfrak{A}$ (where $\delta_{i,j}$ is the Kronecker delta function). Let ψ be the* *linear functional on* $^*\text{-alg}(\mathfrak{A}, \{L_j\}_{j=1}^n)$ *defined by* $\psi|_{\mathfrak{A}} = \tau$ *and*

$$\psi(A_0 L_{i_1} A_1 \cdots A_{k-1} L_{i_k} A_k A'_0 L^*_{j_1} A'_1 \cdots A'_{\ell-1} L^*_{j_\ell} A'_\ell) = 0$$

whenever $A_1, \ldots, A_k, A'_1, \ldots, A'_\ell \in \mathfrak{A}$ *and at least one of* k *and* ℓ *is non-zero. Then* ψ *extends to a state on* $\mathfrak{B}$ *having a faithful GNS representation. Moreover, if* $(\mathfrak{A}, \tau) * (\mathcal{E}, \phi)$ *where* $(\mathcal{E}, \phi)$ *is the* C^**-algebra generated by* n *free creation operators* $\ell_1, \ldots, \ell_n$ *on the full Fock space* $\mathcal{F}(\mathbb{C}^n)$ *and* ϕ *is the vacuum expectation, there exists an isomorphism* $\Phi : (\mathfrak{B}, \psi) \to (\mathfrak{A}, \tau) * (\mathcal{E}, \phi)$ *such that* $\Phi(A) = A$ *for all* $A \in \mathfrak{A}$ *and* $\Phi(L_j) = \ell_j$ *for all* $j \in \{1, \ldots, m\}$.

Proof. Let $\left(\widehat{\mathfrak{B}}, \widehat{\psi}\right)$ be the reduced free product $(\mathfrak{A}, \tau) * (\mathcal{E}, \phi)$. By [71, Corollary 2.5] $\ell_i A \ell_j^* = \delta_{i,j}\tau(A)$ for all $A \in \mathfrak{A}$ and

$$\widehat{\psi}(A_0 \ell_{i_1} A_1 \cdots A_{k-1} \ell_{i_k} A_k A'_0 \ell^*_{j_1} A'_1 \cdots A'_{\ell-1} \ell^*_{j_\ell} A'_\ell) = 0$$

whenever $A_1, \ldots, A_k, A'_1, \ldots, A'_\ell \in \mathfrak{A}$ and at least one of k and ℓ is non-zero. Hence, by the universal property of $\mathfrak{B}$, there exists a *-homomorphism $\Phi : \mathfrak{B} \to \widehat{\mathfrak{B}}$ such that $\psi = \widehat{\psi} \circ \Phi$.

To complete the lemma it suffices to prove Φ is injective. However, by [52] (and by applying the same 'Fourier series'-like argument as in Section 2.2), it suffices to check that the linear span of $\{AL_i^* B L_j C \mid i, j \in \{1, \ldots, n\}, A, B, C \in \mathfrak{A}\}$ is dense in $\mathfrak{A}$ and that there exists a homomorphism $\alpha : \{z \in \mathbb{C} \mid |z| = 1\} \to \text{Hom}\left(\widehat{\mathfrak{B}}\right)$ such that $\alpha_z(A) = A$ for all $A \in \mathfrak{A}$ and $\alpha_z(\ell_j) = z\ell_j$ for all $j \in \{1, \ldots, n\}$. However the first claim is trivial by taking $i = j$, $B = I_{\mathfrak{A}} = C$. Since it is trivial to verify that there exists a homomorphism $\alpha : \{z \in \mathbb{C} \mid |z| = 1\} \to \text{Hom}\left(\widehat{\mathfrak{B}}\right)$ such that $\alpha_z(\ell_j) = z\ell_j$ for all $j \in \{1, \ldots, n\}$, taking the free product with the identity map on $\mathfrak{A}$ will complete the lemma. $\square$

Lemma 2.4.6. *Theorem 2.4.2 is true with the additional assumptions that* $Y_j^{(k)} = Y_j$ *for all* $k \in \mathbb{N}$ *and* $j \in \{1, \ldots, m\}$, $\varphi_k = \varphi$ *for all* $k \in \mathbb{N}$, $\mathfrak{B}$ *is the* C^**-algebra generated by* m *creation operators* $\ell_1, \ldots, \ell_m$ *on a Fock space, and* φ *is the vector state of the vacuum vector.*

Proof. Consider the C*-algebra

$$\mathfrak{D} := \frac{\prod_{k\geq 1}\left((\mathfrak{A}_k, \tau_k) * (C^*(\ell_1, \ldots, \ell_m), \varphi)\right)}{\bigoplus_{k\geq 1}\left((\mathfrak{A}_k, \tau_k) * (C^*(\ell_1, \ldots, \ell_n), \varphi)\right)}.$$

Let $q : \prod_{k\geq 1}\left((\mathfrak{A}_k, \tau_k) * (C^*(\ell_1, \ldots, \ell_m), \varphi)\right) \to \mathfrak{D}$ be the canonical quotient map and let

$$X'_j := q\left(\left(X_j^{(k)}\right)_{k\geq 1}\right) \text{ and } L_i := q\left((\ell_i)_{k\geq 1}\right)$$

for all $j \in \{1, \ldots, n\}$ and $i \in \{1, \ldots, m\}$. Notice, by the first assumption of Theorem 2.4.2, $\mathfrak{A}$ is isomorphic to the C*-subalgebra of $\mathfrak{D}$ generated by $\{X'_j\}_{j=1}^n$. Let $\mathfrak{C}$ be the C*-subalgebra of $\mathfrak{D}$ generated by $\mathfrak{A}$ and $\{L_j\}_{j=1}^m$. By Remarks 2.4.4 there exists a *-homomorphism $\Psi : \mathfrak{C} \to (\mathfrak{A}, \tau) * (C^*(\ell_1, \ldots, \ell_m), \varphi)$ such that $\Psi(X'_j) = X_j$ for all $j \in \{1, \ldots, n\}$ and $\Psi(L_j) = \ell_j$ for all $j \in \{1, \ldots, m\}$.

We claim that Ψ is an isomorphism. To see this, we note by the third assumption of Theorem 2.4.2 that

$$L_i^* p(X'_1, \ldots, X'_n) L_j = \delta_{i,j} \tau(p(X'_1, \ldots, X'_n)) I_{\mathfrak{A}}$$

for all polynomials $p \in \mathbb{C}\langle t_1, \ldots, t_n\rangle$. Hence, by Lemma 2.4.5 and by universality, there exists a *-homomorphism $\Phi : (\mathfrak{A}, \tau) * (C^*(\ell_1, \ldots, \ell_m), \varphi) \to \mathfrak{C}$ such that $\Phi(X_j) = X'_j$ for all $j \in \{1, \ldots, n\}$ and $\Phi(\ell_j) = L_j$ for all $j \in \{1, \ldots, m\}$. Hence Ψ is invertible with inverse Φ. Thus the result follows from Remarks 2.4.4. □

To prove Theorem 2.4.2 provided that $Y_j^{(k)} = Y_j$ for all $k \in \mathbb{N}$ and $j \in \{1, \ldots, m\}$, $\varphi_k = \varphi$ for all $k \in \mathbb{N}$, and $\mathfrak{B}$ is an exact C*-algebra we will make use of the following result that provides an embedding of the reduced free product of two C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ into a reduced free product involving $\mathfrak{A} \otimes_{\min} \mathfrak{B}$.

Lemma 2.4.7. *Let $(\mathfrak{A}, \varphi)$ and $(\mathfrak{B}, \psi)$ be non-commutative probability spaces, let ℓ_1 be the unilateral forward shift on $\ell_2(\mathbb{N})$, let $\{e_n\}_{n\geq 1}$ be the standard orthonormal basis for $\ell_2(\mathbb{N})$,* *and let $\phi : C^*(\ell_1) \to \mathbb{C}$ be defined by $\phi(T) = \langle Te_1, e_1\rangle$ for all $T \in C^*(\ell_1)$. There exists a unitary $U \in C^*(\ell_1)$ (independent of $\mathfrak{A}$ and $\mathfrak{B}$) and an injective *-homomorphism*

$$\Psi : (\mathfrak{A}, \varphi) * (\mathfrak{B}, \psi) \to (\mathfrak{A} \otimes_{\min} \mathfrak{B}, \varphi \otimes \psi) * (C^*(\ell_1), \phi)$$

such that $\Psi(A) = A \otimes I_{\mathfrak{B}}$ and $\Psi(B) = U^*(I_{\mathfrak{A}} \otimes B)U$ for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$.

Proof. See [22, Proposition 4.2]. □

Lemma 2.4.8. *Theorem 2.4.2 is true under the additional assumptions that $Y_j^{(k)} = Y_j$ for all $k \in \mathbb{N}$ and $j \in \{1, \ldots, m\}$, $\varphi_k = \varphi$ for all $k \in \mathbb{N}$, and $\mathfrak{B}$ is an exact C*-algebra.*

Proof. Since $\mathfrak{B}$ is exact, by the fourth equivalence of Theorem 1.6.1 and by the first assumption of Theorem 2.4.2, we obtain that

$$\limsup_{k\to\infty} \left\|p\left(X_1^{(k)} \otimes I, \ldots, X_n^{(k)} \otimes I, I \otimes Y_1, \ldots, I \otimes Y_m\right)\right\|_{\mathfrak{A}_k \otimes_{\min} \mathfrak{B}}$$

is

$$\|p(X_1 \otimes I, \ldots, X_n \otimes I, I \otimes Y_1, \ldots, I \otimes Y_m)\|_{\mathfrak{A} \otimes \mathfrak{B}}$$

for all $p \in \mathbb{C}\langle t_1, \ldots, t_{n+m}\rangle$. By the structure of the states on the tensor products and by the third assumption of Theorem 2.4.2,

$$\lim_{k\to\infty} (\tau_k \otimes \varphi)\left(p\left(X_1^{(k)} \otimes I, \ldots, X_n^{(k)} \otimes I, I \otimes Y_1, \ldots, I \otimes Y_m\right)\right)$$

is

$$(\tau \otimes \varphi)(p(X_1 \otimes I, \ldots, X_n \otimes I, I \otimes Y_1, \ldots, I \otimes Y_m))$$

for all $p \in \mathbb{C}\langle t_1, \ldots, t_{n+m}\rangle$. Therefore Lemma 2.4.6 implies the limit of

$$\left\|p\left(X_1^{(k)} \otimes I, \ldots, X_n^{(k)} \otimes I, I \otimes Y_1, \ldots, I \otimes Y_m, T\right)\right\|_{(\mathfrak{A}_k \otimes_{\min} \mathfrak{B}, \tau_k \otimes \varphi) * (C^*(\ell_1), e_1)}$$

as $k \to \infty$ is

$$\|p(X_1 \otimes I, \ldots, X_n \otimes I, I \otimes Y_1, \ldots, I \otimes Y_m, T)\|_{(\mathfrak{A}\otimes_{\min}\mathfrak{B},\tau\otimes\varphi)*(C^*(\ell_1),e_1)}$$

for all $p \in \mathbb{C}\langle t_1, \ldots, t_{n+m+1}\rangle$ and for all $T \in C^*(\ell_1)$. By using $T = U$ where U is a unitary as in Lemma 2.4.7 and by viewing $(\mathfrak{A}_k, \tau_k) * (\mathfrak{B}, \varphi)$ and $(\mathfrak{A}, \tau) * (\mathfrak{B}, \varphi)$ as C*-subalgebras of $(\mathfrak{A}_k \otimes_{\min} \mathfrak{B}, \tau_k \otimes \varphi) * (C^*(\ell_1), e_1)$ and $(\mathfrak{A} \otimes_{\min} \mathfrak{B}, \tau \otimes \varphi) * (C^*(\ell_1), e_1)$ respectively, the result follows. $\square$

Just as Lemma 2.4.8 upgraded Lemma 2.4.6 to exact C*-algebras by use of Lemma 2.4.7 and tensor products, we will use Lemma 2.4.8 along with the following lemma involving direct sums to prove Theorem 2.4.2.

Lemma 2.4.9. *For $i \in \{1, 2\}$ let $(\mathfrak{A}_i, \tau_i)$ be non-commutative probability spaces. Let $\tau : \mathfrak{A}_1 \oplus \mathfrak{A}_2 \to \mathbb{C}$ be the state given by*

$$\tau(A_1 \oplus A_2) = \frac{1}{2}(\tau_1(A_1) + \tau_2(A_2))$$

for all $A_1 \in \mathfrak{A}_1$ and $A_2 \in \mathfrak{A}_2$.

Let $\mathcal{O}_2$ be the Cuntz algebra, let $\mathcal{F}_2$ be the canonical CAR C*-subalgebra of $\mathcal{O}_2$, let $\tau' : \mathcal{F}_2 \to \mathbb{C}$ be the unique normalized trace on $\mathcal{F}_2$, let $\mathcal{E} : \mathcal{O}_2 \to \mathcal{F}_2$ be the canonical conditional expectation of $\mathcal{O}_2$ onto $\mathcal{F}_2$, and let $\sigma := \tau' \circ \mathcal{E} : \mathcal{O}_2 \to \mathbb{C}$. Note σ is a faithful state.

Let $\mathfrak{C}$ be any C*-algebra with a state ρ such that there exists a unitary $U \in \mathfrak{C}$ such that $\rho|_{C^*(U)}$ is faithful, $\rho(U) = 0$, and the GNS representation of $\mathfrak{C}$ with respect to ρ is faithful. Then there exists an injective *-homomorphism

$$\pi : (\mathfrak{A}_1, \tau_1) * (\mathfrak{A}_2, \tau_2) \to ((\mathfrak{A}_1 \oplus \mathfrak{A}_2) \otimes \mathcal{O}_2, \tau \otimes \sigma) * (\mathfrak{C}, \rho)$$

and elements $X, Y, Z, W \in C^*(I \otimes \mathcal{O}_2, \mathfrak{C}) \subseteq ((\mathfrak{A}_1 \oplus \mathfrak{A}_2) \otimes \mathcal{O}_2, \tau \otimes \sigma) * (\mathfrak{C}, \rho)$ independent of the choice of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ such that $\pi(A_1) = X(A_1 \oplus 0)Y$ for all $A_1 \in \mathfrak{A}_1$ and $\pi(A_2) = Z(0 \oplus A_2)W$

for all $A_2 \in \mathfrak{A}_2$.

Proof. See [13, Lemma 5.6]. $\square$

Proof of Theorem 2.4.2. For each $k \in \mathbb{N}$ define the state $\psi_k : \mathfrak{A}_k \oplus \mathfrak{B}_k \to \mathbb{C}$ by

$$\psi_k(A \oplus B) = \frac{1}{2}(\tau_k(A) + \varphi_k(B))$$

for all $A \in \mathfrak{A}_k$ and $B \in \mathfrak{B}_k$ and define the state $\psi : \mathfrak{A} \oplus \mathfrak{B} \to \mathbb{C}$ by

$$\psi(A \oplus B) = \frac{1}{2}(\tau(A) + \varphi(B))$$

for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. By the first and second assumptions of Theorem 2.4.2, it is clear that

$$\limsup_{k\to\infty} \left\| p\left(X_1^{(k)} \oplus 0, \ldots, X_n^{(k)} \oplus 0, 0 \oplus Y_1^{(k)}, \ldots, 0 \oplus Y_m^{(k)}\right) \right\|_{\mathfrak{A}_k \oplus \mathfrak{B}_k}$$

is

$$\|p(X_1 \oplus 0, \ldots, X_n \oplus 0, 0 \oplus Y_1, \ldots, 0 \oplus Y_m)\|_{\mathfrak{A}\oplus\mathfrak{B}}$$

for all $p \in \mathbb{C}\langle t_1, \ldots, t_{n+m}\rangle$ and by the third and fourth assumptions of Theorem 2.4.2

$$\lim_{k\to\infty} \psi_k\left(p\left(X_1^{(k)} \oplus 0, \ldots, X_n^{(k)} \oplus 0, 0 \oplus Y_1^{(k)}, \ldots, 0 \oplus Y_m^{(k)}\right)\right)$$

is

$$\psi(p(X_1 \oplus 0, \ldots, X_n \oplus 0, 0 \oplus Y_1, \ldots, 0 \oplus Y_m))$$

for all $p \in \mathbb{C}\langle t_1, \ldots, t_{n+m}\rangle$,

Let S_1 and S_2 be two isometries that generated the Cuntz algebra. Since $\mathcal{O}_2$ is exact, by viewing $\mathfrak{A}_k \oplus \mathfrak{B}_k \in (\mathfrak{A}_k \oplus \mathfrak{B}_k) \otimes_{\min} \mathcal{O}_2$ and $\mathfrak{A} \oplus \mathfrak{B} \in (\mathfrak{A} \oplus \mathfrak{B}) \otimes_{\min} \mathcal{O}_2$ canonically, the fourth equivalence of Theorem 1.6.1 implies that

$$\limsup_{k\to\infty} \left\| p\left(X_1^{(k)} \oplus 0, \ldots, X_n^{(k)} \oplus 0, 0 \oplus Y_1^{(k)}, \ldots, 0 \oplus Y_m^{(k)}, S_1, S_2\right) \right\|_{(\mathfrak{A}_k \oplus \mathfrak{B}_k)\otimes_{\min}\mathcal{O}_2}$$

is

$$\|p\left(X_1 \oplus 0, \ldots, X_n \oplus 0, 0 \oplus Y_1, \ldots, 0 \oplus Y_m, S_1, S_2\right)\|_{(\mathfrak{A} \oplus \mathfrak{B}) \otimes_{\min} \mathcal{O}_2}$$

for all $p \in \mathbb{C}\langle t_1, \ldots, t_{n+m+2}\rangle$.

Let σ be the faithful state from Lemma 2.4.9. Therefore

$$\lim_{k \to \infty} (\psi_k \otimes \sigma)\left(p\left(X_1^{(k)} \oplus 0, \ldots, X_n^{(k)} \oplus 0, 0 \oplus Y_1^{(k)}, \ldots, 0 \oplus Y_m^{(k)}, S_1, S_2\right)\right)$$

is

$$(\psi \otimes \sigma)\left(p\left(X_1 \oplus 0, \ldots, X_n \oplus 0, 0 \oplus Y_1, \ldots, 0 \oplus Y_m, S_1, S_2\right)\right)$$

for all $p \in \mathbb{C}\langle t_1, \ldots, t_{n+m+2}\rangle$ by the structure of the tensor products of states.

Let $\mathfrak{C} = \mathcal{M}_2(\mathbb{C})$, let ρ be the faithful normalized trace on $\mathfrak{C}$, and let

$$U := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since $\mathfrak{C}$ can be generated by a single operator free from ρ, Lemma 2.4.8 implies if $T \in \mathcal{M}_2(\mathbb{C})$ and $p \in \mathbb{C}\langle t_1, \ldots, t_{n+m+3}\rangle$ then the norms of

$$p\left(X_1^{(k)} \oplus 0, \ldots, X_n^{(k)} \oplus 0, 0 \oplus Y_1^{(k)}, \ldots, 0 \oplus Y_m^{(k)}, S_1, S_2, T\right)$$

in $((\mathfrak{A}_k \oplus \mathfrak{B}_k) \otimes_{\min} \mathcal{O}_2, \psi_k \otimes \sigma) * (\mathcal{M}_2(\mathbb{C}), \rho)$ converges to the norm of

$$p\left(X_1 \oplus 0, \ldots, X_n \oplus 0, 0 \oplus Y_1, \ldots, 0 \oplus Y_m, S_1, S_2, T\right)$$

in $((\mathfrak{A} \oplus \mathfrak{B}) \otimes_{\min} \mathcal{O}_2, \psi \otimes \sigma) * (\mathcal{M}_2(\mathbb{C}), \rho)$. Therefore the result clearly follows by the embedding properties given by Lemma 2.4.9. $\square$

Combining Theorem 2.1.2, Theorem 2.4.2, and Remarks 2.4.4, we have the following analog of Lemma 1.6.2 for reduced free products.

Corollary 2.4.10. *Suppose that* $\left\{X_i^{(k)}\right\}_{i=1}^n$, $\{X_i\}_{i=1}^n$, *and* $\{Y_i\}_{i=1}^m$ *are generators for the non-commutative probability spaces* $(\mathfrak{A}_k, \tau_k)$, $(\mathfrak{A}, \tau)$, *and* $(\mathfrak{B}, \varphi)$ *respectively and that*

1. $\limsup_{k\to\infty} \left\|q\left(X_1^{(k)}, \ldots, X_n^{(k)}\right)\right\|_{\mathfrak{A}_k} = \|q(X_1, \ldots, X_n)\|_{\mathfrak{A}}$ *and*
2. $\lim_{k\to\infty} \tau_k\left(q\left(\left(X_1^{(k)}, \ldots, X_n^{(k)}\right)\right)\right) = \tau(q(X_1, \ldots, X_n))$

for all $q \in \mathbb{C}\langle t_1, \ldots, t_n\rangle$. *Let* $\mathfrak{D}$ *be the unital* C^**-subalgebra of* $\prod_{k\geq 1} \mathfrak{A}_k$ *generated by*

$$\left\{\left(X_i^{(k)}\right)_{k\geq 1}\right\}_{i=1}^n$$

and let $\mathfrak{J} := \mathfrak{D} \cap \left(\bigoplus_{k\geq 1} \mathfrak{A}_k\right)$. *Then* $\mathfrak{J}$ *is an ideal of* $\mathfrak{D}$ *such that* $\mathfrak{D}/\mathfrak{J} \simeq \mathfrak{A}$.

Let $\sigma : \mathfrak{B} \to \mathcal{B}(\mathcal{K})$ *be the GNS representation of* φ *with unit cyclic vector* η, *let* $\pi_0 : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_0)$ *be the GNS representation of* τ *with unit cyclic vector* ξ, *let* $\pi_1 : \mathfrak{D} \to \mathcal{B}(\mathcal{H}_1)$ *be a faithful, unital representation, let* $q : \mathfrak{D} \to \mathfrak{A}$ *be the canonical quotient map, and let* $\pi := (\pi_0 \circ q) \oplus \pi_1 : \mathfrak{D} \to \mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}_1)$ *which is a faithful, unital representation. Then there exists an injective* **-homomorphism*

$$\Phi : \frac{(\mathfrak{D}, \pi, \xi) * (\mathfrak{B}, \sigma, \eta)}{\langle \mathfrak{J}\rangle_{\mathfrak{D}*\mathfrak{B}}} \to \frac{\prod_{k\geq 1}((\mathfrak{A}_k, \tau_k) * (\mathfrak{B}, \varphi))}{\bigoplus_{k\geq 1}((\mathfrak{A}_k, \tau_k) * (\mathfrak{B}, \varphi))}$$

such that

$$\Phi\left(\left(X_i^{(k)}\right)_{k\geq 1} + \langle \mathfrak{J}\rangle_{\mathfrak{D}*\mathfrak{B}}\right) = \left(X_i^{(k)}\right)_{k\geq 1} + \bigoplus_{k\geq 1}((\mathfrak{A}_k, \tau_k) * (\mathfrak{B}, \varphi))$$

for all $i \in \{1, \ldots, n\}$ *and*

$$\Phi(B + \langle \mathfrak{J}\rangle_{\mathfrak{D}*\mathfrak{B}}) = (B)_{k\geq 1} + \bigoplus_{k\geq 1}((\mathfrak{A}_k, \tau_k) * (\mathfrak{B}, \varphi))$$

for all $B \in \mathfrak{B}$.

Recently in [55], Pisier has developed a direct proof of Theorem 2.4.2 using the non-commutative Khintchine inequalities developed in [59]. Our original proof of Theorem 2.4.2

did not allow the Y-variables to vary and it was observed that the above proof works in this setting after [55].

CHAPTER 3

Freely Independent Random Variables with Non-Atomic Distributions

In this chapter, which is based on the author's joint paper with his advisor [72], we examine the distributions of non-commutative polynomials of non-atomic, freely independent random variables. In particular, we obtain an analogue of the Strong Atiyah Conjecture for free groups thus proving that the measure of each atom of any $n \times n$ matricial polynomial of non-atomic, freely independent random variables is an integer multiple of n^{-1}. In addition, we show that the Cauchy transform of the distribution of any matricial polynomial of freely independent semicircular variables is algebraic and thus the polynomial has a distribution that is real-analytic except at a finite number of points.

3.1 Summary of Main Results on Distributions of Non-Atomic Random Variables

One of the essential themes in the study of free probability [82] and its applications to random matrix theory is to determine specific properties of the spectral distribution of a fixed (matricial) polynomial in freely independent random variables. For example, some of the earliest work in free probability theory concerns free convolution, which is the study of the distribution of the polynomial $P(X, Y) = X + Y$ in two freely independent random variables. In particular, the recent paper [10] of Belinschi, Mai, and Speicher uses an analytic theory for operator-valued additive free convolution and Anderson's self-adjoint linearization trick to provide an algorithm for determining distributions of arbitrary polynomials. Combining the previously known results from [54], [25], [1], and [66] along with the results contained in this dissertation, we obtain the following summary of the known properties of distributions of matrices whose entries are polynomials in several free variables (or, equivalently, polynomials in free variables having matricial coefficients).

Theorem 3.1.1. *Let $X_1, \ldots, X_n$ be normal, freely independent random variables and let $[p_{i,j}]$ be an $\ell \times \ell$ matrix whose entries are non-commuting polynomials in n variables and their adjoints such that $[p_{i,j}(X_1, \ldots, X_n)]$ is normal. Then*

1. *if there exists $\{d_j\}_{j=1}^n \subseteq \mathbb{N}$ such that the measure of each atom in the probability distribution of X_j is an integer multiple of $\frac{1}{d_j}$, then the measure of each atom in the probability distribution of $[p_{i,j}(X_1, \ldots, X_n)]$ is an integer multiple of $\frac{1}{d\ell}$ where $d := \prod_{j=1}^n d_j$.*

In particular,

2. *if the probability distribution of each X_j is non-atomic, then the measure of each atom in the probability distribution of $[p_{i,j}(X_1, \ldots, X_n)]$ is an integer multiple of $\frac{1}{\ell}$.*

If, in addition, $X_1, \ldots, X_n$ are freely independent semicircular variables or freely independent Haar unitaries and $[p_{i,j}(X_1, \ldots, X_n)]$ is self-adjoint, then

3. *the spectrum of $[p_{i,j}(X_1, \ldots, X_n)]$ is a union of at most ℓ disjoint sets each of which is either a closed interval or a point, and*

4. *the measure of each connected subset of the spectrum of $[p_{i,j}(X_1, \ldots, X_n)]$ is a multiple of $\frac{1}{\ell}$.*

Furthermore, if μ is the spectral distribution of $[p_{i,j}(X_1, \ldots, X_n)]$, if K is the support of μ, and if G_μ is the Cauchy transform of μ, then

5. *G_μ is an algebraic formal power series and thus*

6. *there exists a finite subset A of $\mathbb{R}$ such that if I is a connected component of $\mathbb{R} \setminus A$ and $\mu|_I$ is the restriction of μ to I, then $\mu|_I = 0$ whenever $I \setminus K \neq \emptyset$ and if $I \subseteq K$, then $\mu|_I$ has probability density function $\mathrm{Im}(g)|_I$ where g is an analytic function defined on*

$$W := \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < \delta\} \setminus \bigcup_{a \in A} \{a - it \mid t \in [0, \infty)\}$$

for some $\delta > 0$ such that g agrees with G_μ on $\{z \in \mathbb{C} \mid 0 < \mathrm{Im}(z) < \delta\}$ and for each $a \in A$ there exists an $N \in \mathbb{N}$ and an $\epsilon > 0$ such that $(z-a)^N g(z)$ admits an expansion on $W \cap \{z \in \mathbb{C} \mid |z - a| < \epsilon\}$ as a convergent power series in $r_N(z-a)$ where $r_N(z)$ is the analytic N^{th}-root of z defined with branch $\mathbb{C} \setminus \{-it \mid t \in [0, \infty)\}$.

Finally, if the support of μ is contained in $[0, \infty)$, then

7. $\lim_{\epsilon \to 0} \int_\epsilon^1 \ln(t)\, d\mu(t) > -\infty$.

In this theorem, by a polynomial in $X_1, \ldots, X_n$ we mean a fixed element of the $*$-algebra generated by $X_1, \ldots, X_n$.

Parts (3) and (4) of Theorem 3.1.1 follow directly from [54, Corollary 3.2] which computes the K-groups of $C^*_{\mathrm{red}}(\mathbb{F}_n)$, the reduced group C*-algebras of the free groups. The characterization of the K_0-group immediately implies that the normalized trace of any projection in $\mathcal{M}_\ell(C^*_{\mathrm{red}}(\mathbb{F}_n))$ is an integer multiple of ℓ^{-1}. Notice that part (4) of Theorem 3.1.1 does not imply part (2) of Theorem 3.1.1 in the setting of part (4) as atoms may occur inside a closed interval of the spectrum. Alternatively, these results were obtained using random matrix techniques in [25].

Notice that part (2) of Theorem 3.1.1 applies when $X_1, \ldots, X_n$ are freely independent semicircular variables. Since freely independent semicircular variables describe the non-commutative law of certain independent large random matrices (see [82]) we obtain the following application to random matrix theory.

For each $N \in \mathbb{N}$ let $X_1(N), \ldots, X_n(N)$ be self-adjoint Gaussian random matrices (or, more generally, matrices with independent, identically distributed entries satisfying certain moment conditions; see [82] or [34] for details) and let p be an arbitrary non-constant non-commutative polynomial in n variables which is self-adjoint in the sense that $Y(N) = p(X_1(N), \dots, X_n(N))$ is always a self-adjoint matrix. Let μ_N be the empirical spectral measure of $Y(N)$ (that is, $\mu_N[a, b]$ is the average proportion of eigenvalues of $Y(N)$ which lie in $[a, b]$).

Corollary 3.1.2. *With the notation as above, the measures μ_N converge to a non-atomic limiting measure μ.*

Indeed, by a result of Voiculescu (see [82] or [34]), it is known that μ_N converges weakly to a measure μ that is the law of $p(X_1, \dots, X_n)$ where $X_1, \dots, X_n$ are freely independent semicircular variables. Thus part (2) of Theorem 3.1.1 implies that μ has no atoms provided p is non-constant.

The motivation for the proof of Theorem 3.1.1 part (2) stems from the knowledge that the statement of the theorem holds by the Strong Atiyah Conjecture for the free groups in the case when $X_1, \dots, X_n$ are freely independent Haar unitaries. The Strong Atiyah Conjecture (motivated by the work in [8] and proved for a class of groups that includes free groups by Linnell in [43]; also see [44] and references therein) states that the kernel projection of an arbitrary matrix with entries taken from the group ring $\mathbb{C}\mathbb{F}_n$ of a free group on n generators must have integer von Neumann trace. To prove our theorem, we consider the analogue of the Strong Atiyah Conjecture for $*$-subalgebras of a tracial von Neumann algebra. We call this notion the Strong Atiyah Property (since it is known that the Strong Atiyah Conjecture does not hold even for arbitrary group algebras; see [24] or [44] for example). It is not hard to see that the Strong Atiyah Property holds for $*$-algebras generated by a single normal element with non-atomic spectral measure. Our main result states that the Strong Atiyah Property for $*$-algebras is stable under taking free products (in the sense of free probability theory [82]) with the group algebra of a free group. Our proof closely follows [67] with the main difference of being adapted for free products of algebras and not groups. Using this result, we are able to conclude that the Strong Atiyah Property holds for any $*$-algebra generated by $X_1, \dots, X_n$ provided that X_j are free and each has a non-atomic distribution.

The proof that part (5) of Theorem 3.1.1 is true in the case $X_1, \ldots, X_n$ are freely independent Haar unitaries is contained in the proof of [66, Theorem 3.6]. In Section 3.5 we will adapt the proof of [66, Theorem 3.6] to the semicircular case (see Theorem 3.5.3). The main idea of the proof is to use the fact that if a certain tracial map on formal power series in a single variable with coefficients in a tracial $*$-algebra $\mathcal{A}$ maps rational formal power series to algebraic formal power series, then the Cauchy transform of a measure associated to a self-adjoint element of $\mathcal{A}$ is algebraic (see Lemma 3.5.6). The proof that the tracial map is as desired in the case $\mathcal{A}$ is generated by semicircular variables follows from demonstrating that a specific formal power series in non-commuting variables is algebraic via a specific property of the semicircular variables (see Lemma 3.5.11).

It is an interesting question whether the Cauchy transform of any polynomial in freely independent random variable $X_1, \ldots, X_n$ is algebraic provided the Cauchy transform of each X_j is algebraic.

The question of whether the Cauchy transform of a measure is an algebraic power series as in part (5) of Theorem 3.1.1 has previously been studied in particular cases. For example [56, Example 3.8] demonstrates that the Cauchy transform of the quarter-circular distribution is not algebraic. Furthermore [56, Corollary 9.5] demonstrates that if μ and ν are compactly supported probability measures on $\mathbb{R}$ which have algebraic Cauchy transforms and are the weak limits of the empirical spectral measures of $N \times N$ random matrices, then the free additive convolution $\mu \boxplus \nu$ (see [78]) is algebraic. Moreover, [56, Corollary 9.6] demonstrates that if, in addition, μ and ν have support contained in the positive real axis, then the free multiplicative convolution $\mu \boxtimes \nu$ (see [79]) is algebraic. This question was also considered in [1] for limit laws of certain random matrices. In fact a result much like ours was hinted at in that paper. Using [1, Theorem 2.9] we see that part (6) of Theorem 3.1.1 is implied by part (5) of Theorem 3.1.1. In particular, part (6) of Theorem 3.1.1 directly provides information about the probability density function of μ by the Stietjes inversion formula.

Finally, in Section 3.5, we will prove part (7) of Theorem 3.1.1 by following the proof of [66, Theorem 3.6] which demonstrates that if the Cauchy transform of a measure is algebraic, then the Novikov-Shubin invariants of the measure are non-zero. Our interest in part (7) of Theorem 3.1.1 comes from the following question: if p is an arbitrary, non-constant, self-adjoint polynomial in n free semicircular variables, must it be the case that the free entropy (as defined in [80]) of p is finite? Indeed elementary arguments may be used to show that if S is a semicircular variable and p is a non-constant polynomial such that $p(S)$ is self-adjoint, then the spectral measure of $p(S)$ has finite free entropy. Further evidence that this must be true comes from a strengthened version of part (2) of Theorem 3.1.1 for matrices of the form $[p_{i,j}]$ where each $p_{i,j} \in \mathrm{alg}(S_1, \ldots, S_n) \otimes \mathrm{alg}(S_1, \ldots, S_n)$, which we prove below. In particular, it follows that the vector of non-commutative difference quotients $JP := [\partial_1 P, \ldots, \partial_n P]$ (see [81]) has maximal rank whenever P is a non-constant, non-commutative polynomial in n free semicircular variables.

Given the success of [10] in providing an algorithm for determining the distributions of (matricial) polynomials in semicircular variables, it would also be of interest if an alternate proof of Theorem 3.1.1 could be constructed using the ideas and techniques from [10].

3.2 The Atiyah Property for Tracial $*$-Algebras

In this section we will introduce the notion of the Atiyah Property for tracial $*$-algebra. In addition, several examples of tracial $*$-algebras that satisfy the Atiyah Property, which will be of use in Section 3.3, will be provided.

If $\ell \in \mathbb{N}$ and τ is a linear functional on an algebra $\mathcal{A}$, then τ_ℓ will denote the linear functional on $\mathcal{M}_\ell(\mathcal{A})$ given by

$$\tau_\ell([A_{i,j}]) = \sum_{i=1}^{\ell} \tau(A_{i,i})$$

for all $[A_{i,j}] \in \mathcal{M}_\ell(\mathcal{A})$. Notice that if τ is tracial (that is, $\tau(AB) = \tau(BA)$ for all $A, B \in \mathcal{A}$), then τ_ℓ is tracial.

Definition 3.2.1. Let $\mathcal{A}$ be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, let τ be a vector state that is tracial on $\mathcal{A}$, and let Γ be an additive subgroup of $\mathbb{R}$ containing $\mathbb{Z}$. We say that $(\mathcal{A}, \tau)$ has the Atiyah

Property with group Γ if for any $n, m \in \mathbb{N}$ and $A \in \mathcal{M}_{m,n}(\mathcal{A})$ the kernel of the induced operator $L_A : \mathcal{H}^{\oplus n} \to \mathcal{H}^{\oplus m}$ given by $L_A(\xi) = A\xi$ satisfies $\tau_m(\ker(L_A)) \in \Gamma$. We say that $(\mathcal{A}, \tau)$ has the Strong Atiyah Property if $(\mathcal{A}, \tau)$ has the Atiyah Property with group $\mathbb{Z}$.

Of course the case that $\Gamma = \mathbb{R}$ is of no interest in the above definition. By the fact that $\ker(L_A) = \ker(L_{A^*A})$, it suffices to consider $n = m$ in the above definition. In this case it is easy to see that $\ker(L_A) = \overline{\mathrm{Im}(L_{A^*})}$ so we may replace kernels with images in the above definition. Furthermore, if $\mathcal{A}$ is equipped with a C*-norm and τ is faithful on the C*-algebra generated by $\mathcal{A}$, the tracial representation of $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ clearly does not matter.

It is clear that if G is a group that satisfies the Strong Atiyah Conjecture (e.g. any free group) and τ_G is the canonical tracial state on $L(G)$ (the group von Neumann algebra), then $(\mathbb{C}G, \tau_G)$ has the Strong Atiyah Property. The following provides examples of a tracial $*$-algebras that have the Atiyah Property. In particular, the following result implies that the tracial $*$-algebra generated by a single semicircular variable has the Strong Atiyah Property with respect to the canonical tracial state (see [82] or [34]).

Lemma 3.2.2. *Let μ be a compactly supported probability measure on $\mathbb{C}$. Let Γ be the topological closure of the additive subgroup of $\mathbb{R}$ generated by 1 and the measures of the atoms of μ and let $(\mathcal{A}, \tau)$ be the tracial $*$-subalgebra of $L_\infty(\mu) \subseteq \mathcal{B}(L_2(\mu))$ generated by multiplication by polynomials with trace*

$$\tau(M_p) = \int_{\mathbb{C}} p \, d\mu.$$

Then $(\mathcal{A}, \tau)$ has the Atiyah Property with group Γ.

Proof. Let δ_t denote the point-mass measure at $t \in \mathbb{C}$. Then we can write

$$\mu = \nu + \sum_t \alpha_t \delta_t$$

where ν is a non-atomic, compactly supported measure on $\mathbb{C}$ and $\alpha_t \in \Gamma$ for all t. Therefore $\nu(\mathbb{C}) \in \Gamma$ by the construction of Γ.

To see that $(\mathcal{A}, \tau)$ has the Atiyah Property with group Γ, fix $\ell \in \mathbb{N}$ and let $[A_{i,j}] \in \mathcal{M}_\ell(\mathcal{A})$. Viewing each $A_{i,j}$ as a polynomial, we can view $[A_{i,j}]$ as a measureable function from $\mathbb{C}$ to $\mathcal{M}_\ell(\mathbb{C})$. Moreover, if P is the projection onto the image of $[A_{i,j}]$ (which is in the von Neumann algebra generated by $\mathcal{M}_\ell(\mathcal{A})$ and thus is in $L_\infty(\mu)\overline{\otimes}\mathcal{M}_\ell(\mathbb{C})$) and $P_t \in \mathcal{M}_\ell(\mathbb{C})$ is the projection onto the image of $[A_{i,j}(t)]$, it is elementary to see that $P(t) = P_t$ μ-almost everywhere. Hence

$$\tau_\ell(P) = \int_{\mathbb{C}} \mathrm{tr}(P(t))\, d\mu(t) = \int_{\mathbb{C}} \mathrm{rank}([A_{i,j}(t)])\, d\mu(t).$$

Recall the rank of a matrix $M \in \mathcal{M}_\ell(\mathbb{C})$ may be obtained by computing the maximum size of a submatrix with non-zero determinant. However, the pointwise determinant of submatrices of $[A_{i,j}(t)]$ is a polynomial in t and thus is either zero everywhere or non-zero except at a finite number of points. Hence we obtain that $t \mapsto \mathrm{rank}([A_{i,j}(t)])$ is an integer-valued function that is constant except at a finite number of points which may or may not be atoms of μ. It is then easy to deduce that $\tau_\ell(P)$ is an integer-valued combination of elements of Γ and thus lies in Γ. $\square$

Extending these integration techniques, we obtain the following result for the product of measures on $\mathbb{C}$. Notice that the tracial $*$-algebra constructed is the tensor product of tracial $*$-algebras from Lemma 3.2.2.

Lemma 3.2.3. *Let $n \in \mathbb{N}$ and let $\{\mu_j\}_{j=1}^n$ be non-atomic, compactly supported probability measures on $\mathbb{C}$. Let μ be the product measure of $\{\mu_j\}_{j=1}^n$ and let $(\mathcal{A}, \tau)$ be the tracial $*$-algebra generated by multiplication by the coordinate functions $\{x_j\}_{j=1}^n$ with trace*

$$\tau(M_f) = \int_{\mathbb{C}^n} f\, d\mu.$$

Then $(\mathcal{A}, \tau)$ has the Strong Atiyah Property .

Proof. We claim that if $p(x_1, \ldots, x_n)$ is a polynomial and V is the zero set of $p(x_1, \ldots, x_n)$, then $\mu(V) \in \{0, 1\}$ and $\mu(V) = 1$ only occurs when $p(x_1, \ldots, x_n)$ is the zero polynomial. To prove this claim, we proceed by induction on n with the case $n = 1$ following from Lemma 3.2.2. Suppose the claim holds for $n - 1$. Let $p(x_1, \ldots, x_n)$ be any polynomial and let ν be the product measure of $\{\mu_j\}_{j=1}^{n-1}$. Clearly the claim is trivial if $p(x_1, \ldots, x_n)$ is the zero polynomial so suppose $p(x_1, \ldots, x_n)$ is not the zero polynomial. For each $t \in \mathbb{C}$ let

$$V_t := \{(x_1, \ldots, x_{n-1}) \in \mathbb{C}^n \mid p(x_1, \ldots, x_{n-1}, t) = 0\}.$$

Therefore the zero set of $p(x_1, \ldots, x_n)$ is $\bigcup_{t \in \mathbb{C}} V_t$ and $\nu(V_t) \in \{0, 1\}$ for each $t \in \mathbb{C}$ by the induction hypothesis. If $\nu(V_t) = 1$, then $p(x_1, \ldots, x_{n-1}, t)$ must be the zero polynomial which implies $x_n - t$ divides $p(x_1, \ldots, x_n)$ since we can write

$$p(x_1, \ldots, x_n) = \sum_{k=1}^{n-1} \sum_{i_k \geq 0} p_{i_1, \ldots, i_{n-1}}(x_n) x_1^{i_1} \cdots x_{n-1}^{i_{n-1}}$$

where $p_{i_1, \ldots, i_{n-1}}$ are polynomials and if $p_{i_1, \ldots, i_{n-1}}(t) \neq 0$ for at least one $i_1, \ldots, i_{n-1}$, then clearly $p(x_1, \ldots, x_{n-1}, t)$ would not be the zero polynomial. By degree arguments there are at most a finite number of $t \in \mathbb{C}$ such that $x_n - t$ divides $p(x_1, \ldots, x_n)$ so $\nu(V_t) = 0$ except for a finite number of $t \in \mathbb{C}$. Since μ_n contains no atoms, by integrating using Fubini's Theorem we easily obtain that the zero set of $p(x_1, \ldots, x_n)$ has zero μ-measure as desired.

To see that $(\mathcal{A}, \tau)$ has the Strong Atiyah Property, fix $\ell \in \mathbb{N}$ and let $[A_{i,j}] \in \mathcal{M}_\ell(\mathcal{A})$. Thus each $A_{i,j}$ is a multivariable polynomial. If P is the projection onto the image of $[A_{i,j}]$, then, as in the proof of Lemma 3.2.2, we obtain that

$$\tau_\ell(P) = \int_{\mathbb{C}^n} \operatorname{rank}([A_{i,j}(t_1, \ldots, t_n)]) \, d\mu(t_1, \ldots, t_n).$$

Since the rank of a matrix can be determined by computing the largest non-zero determinant of a submatrix and since the determinant of any submatrix of $[A_{i,j}(x_1, \ldots, x_n)]$ is a polynomial in $x_1, \ldots, x_n$ whose zero set either has zero or full μ-measure, the result is complete. $\square$

Next we endeavour to extend the above result to include compactly supported probability measures on $\mathbb{R}$ that have atoms. We will only focus on measures with atoms that lie in certain subgroups of $\mathbb{Q}$ since the main result of Section 3.3 will also only apply to these groups.

To discuss such measures, for an additive subgroup Γ of $\mathbb{Q}$ and a $d \in \mathbb{N}$ we define

$$\frac{1}{d}\Gamma := \left\{ \frac{1}{d} g \mid g \in \Gamma \right\},$$

which is clearly an additive subgroup of $\mathbb{Q}$ that contains $\mathbb{Z}$ if Γ contains $\mathbb{Z}$. As such, the following result is trivial.

Lemma 3.2.4. *Let $(\mathcal{A}, \tau)$ be a tracial $*$-algebra that has the Atiyah Property with group Γ and let $\ell \in \mathbb{N}$. Then $(\mathcal{M}_\ell(\mathcal{A}), \frac{1}{\ell}\tau_\ell)$ has the Atiyah Property with group $\frac{1}{\ell}\Gamma$.*

Theorem 3.2.5. *Let $n \in \mathbb{N}$ and let $\{\mu_j\}_{j=1}^n$ be compactly supported probability measures on $\mathbb{C}$. Let μ be the product measure of $\{\mu_j\}_{j=1}^n$ and let $(\mathcal{A}, \tau)$ be the tracial $*$-algebra generated by multiplication by the coordinate functions $\{x_j\}_{j=1}^n$ with trace*

$$\tau(M_f) = \int_{\mathbb{C}^n} f \, d\mu.$$

Suppose for each $j \in \{1, \ldots, n\}$ there exists a $d_j \in \mathbb{N}$ such that the atoms of μ_j have measures contained in $\frac{1}{d_j}\mathbb{Z}$. If $d := \prod_{j=1}^n d_j$, then $(\mathcal{A}, \tau)$ has the Atiyah Property with group $\frac{1}{d}\mathbb{Z}$.

Proof. By assumptions, for each $j \in \{1, \ldots, n\}$ we can write

$$\mu_j = \mu_j'' + \sum_k \frac{\alpha_k}{d_j} \delta_{t_k}$$

where δ_t represents the point-mass probability measure at t, the sum is finite, $\alpha_k \in \mathbb{N}$, $t_{k_1} \neq t_{k_2}$ if $k_1 \neq k_2$, and μ_j'' is an non-atomic measure. Notice $\mu_j''(\mathbb{C}) \in \frac{1}{d_j}\mathbb{Z}$. Let $\mu_j' := \frac{1}{\mu''(\mathbb{C})}\mu_j''$ if $\mu_j'' \neq 0$ and let μ_j' be the Lebesgue measure on $[0, 1]$ if $\mu_j'' = 0$. Therefore the tracial $*$-algebra generated by polynomials acting on $L_2(\mu_j)$ can represented a tracial $*$-algebra of diagonal matrices in $\mathcal{M}_{d_j}(\mathcal{B}(L_2(\mu_j')))$ (with respect to the canonical normalized matrix trace) where the polynomial x maps to the matrix with x appearing on the diagonal $d_j\mu_j''(\mathbb{C})$ times and each t_k appearing on the diagonal α_k times.

Let μ' be the product measure of $\{\mu_j'\}_{j=1}^n$ and let $(\mathcal{A}_{\mu'}, \tau_{\mu'})$ be the tracial $*$-algebra generated by multiplication by the coordinate functions $\{x_j\}_{j=1}^n$ with trace $\tau_\mu(M_f) = \int_{\mathbb{C}^n} f\, d\mu'$. By taking tensor products of the tracial $*$-algebras generated by polynomials acting on $L_2(\mu_j)$, it is easily seen using the above representations that $(\mathcal{A}, \tau)$ can be represented in the tracial $*$-algebra $(\mathcal{M}_d(\mathcal{A}_{\mu'}), \frac{1}{d}(\tau_{\mu'})_d)$. Since Lemma 3.2.3 implies $(\mathcal{A}_{\mu'}, \tau_{\mu'})$ has the Strong Atiyah Property, Lemma 3.2.4 implies $(\mathcal{M}_d(\mathcal{A}_{\mu'}), \frac{1}{d}(\tau_{\mu'})_d)$ has the Atiyah Property with group $\frac{1}{d}\mathbb{Z}$ completing the proof. $\square$

3.3 Atiyah Property for Freely Independent Random Variables

The goal of this section is to use the Atiyah Property for tracial $*$-algebras to gain information about the distributions of matricial polynomials of freely independent random variables. In particular, Theorem 3.3.1 will enable the extensions of the results from Section 3.2 to the non-commutative setting as seen in Theorem 3.3.4 thus completing the proof of part (1) of Theorem 3.1.1. The proof of Theorem 3.3.1, which is based on the proof of [67, Proposition 3] (or the updated version [68, Proposition 6.1]), will be postponed until the next section in order to focus on the applications of Theorem 3.3.1.

Recall that given unital $*$-algebras $\mathcal{A}_i \subseteq \mathcal{B}(\mathcal{H}_i)$ with vector states τ_i that are tracial on $\mathcal{A}_i$, we can consider the $*$-subalgebra $\mathcal{A}_1 * \mathcal{A}_2$ inside the reduced free product C*-algebra $(\mathcal{B}(\mathcal{H}_1), \tau_1) * (\mathcal{B}(\mathcal{H}_2), \tau_2)$ generated by $\mathcal{A}_1$ and $\mathcal{A}_2$. The canonical vector state $\tau_1 * \tau_2$ is then a tracial state on $\mathcal{A}_1 * \mathcal{A}_2$ (see [82] or [34]). Similarly we can consider the $*$-subalgebra $\mathcal{A}_1 \odot \mathcal{A}_2$ inside the C*-algebra $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ generated by $T \otimes I_{\mathcal{H}_2}$ and $I_{\mathcal{H}_1} \otimes S$ for all $T \in \mathcal{A}_1$ and $S \in \mathcal{A}_2$. With this notation, it is easy to state the following technical result.

Theorem 3.3.1. *Let $n \in \mathbb{N}$, let $\mathbb{F}_n$ be the free group on n generators, let $\mathbb{C}\mathbb{F}_n$ be the group $*$-algebra equipped with the C*-norm defined by the left regular representation, and let $\tau_{\mathbb{F}_n}$ be the canonical trace on $L(\mathbb{F}_n)$. Let $\mathcal{A}$ and $\mathcal{B}$ be $*$-subalgebras of the tracial von Neumann algebras* *with separable preduals $(\mathfrak{M}, \tau_{\mathfrak{M}})$ and $(\mathfrak{N}, \tau_{\mathfrak{N}})$ respectively. Suppose that $(\mathcal{A} \odot \mathcal{B}, \tau_{\mathfrak{M}} \overline{\otimes} \tau_{\mathfrak{N}})$ has the Atiyah Property with group $\frac{1}{d}\mathbb{Z}$ for some $d \in \mathbb{N}$. Then $((\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}, (\tau_{\mathfrak{M}} * \tau_{\mathbb{F}_n}) \overline{\otimes} \tau_{\mathfrak{N}})$ has the Atiyah Property with group $\frac{1}{d}\mathbb{Z}$.*

Clearly Theorem 3.3.1 implies the following two results.

Corollary 3.3.2. *If $\mathcal{A}$ and $\mathcal{B}$ are as in Theorem 3.3.1 and $n, m \in \mathbb{N}$, then $((\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot (\mathcal{B} * \mathbb{C}\mathbb{F}_m), (\tau_{\mathfrak{M}} * \tau_{\mathbb{F}_n}) \overline{\otimes} (\tau_{\mathfrak{N}} * \tau_{\mathbb{F}_m}))$ has the Atiyah Property with group $\frac{1}{d}\mathbb{Z}$.*

Proof. This is a simple application of Theorem 3.3.1 twice using $\mathcal{A} = \mathcal{B}$ and $\mathcal{B} = \mathcal{A} * \mathbb{C}\mathbb{F}_n$ the second time. □

Corollary 3.3.3. *Let $\mathcal{A}$ be a $*$-subalgebra of a tracial von Neumann algebra with separable predual $(\mathfrak{M}, \tau_{\mathfrak{M}})$. Suppose $(\mathcal{A}, \tau_{\mathfrak{M}})$ has the Atiyah Property with group $\frac{1}{d}\mathbb{Z}$ for some $d \in \mathbb{N}$. Then $(\mathcal{A} * \mathbb{C}\mathbb{F}_n, \tau_{\mathfrak{M}} * \tau_{\mathbb{F}_n})$ has the Atiyah Property with group $\frac{1}{d}\mathbb{Z}$.*

Proof. Take $\mathcal{B} = \mathbb{C}$ in Theorem 3.3.1. □

Using Theorem 3.3.1 along with the examples of Section 3.2, we obtain the following result which provides important information about the spectral distributions of matricial polynomials of normal, freely independent random variables.

Theorem 3.3.4. *Let $n \in \mathbb{N}$ and let $X_1, \ldots, X_n$ be normal, freely independent random variables with probability measures μ_j as distribution respectively. Suppose for each $j \in \{1, \ldots, n\}$ there exists a $d_j \in \mathbb{N}$ such that the atoms of μ_j have measures contained in $\frac{1}{d_j}\mathbb{Z}$. If $\mathcal{A}$ is the unital $*$-algebra generated by $X_1, \ldots, X_n$ (obtained by taking a reduced free product of tracial $*$-algebras), τ is the canonical trace on $\mathcal{A}$, and $d := \prod_{j=1}^{n} d_j$, then $(\mathcal{A}, \tau)$ has the Atiyah Property with group $\frac{1}{d}\mathbb{Z}$.*

Furthermore, if $[p_{i,j}]$ is an $\ell \times \ell$ matrix whose entries are non-commutative polynomials in n variables and their adjoints such that $[p_{i,j}(X_1, \ldots, X_n)]$ is normal, then the measure of any atom of the spectral distribution of $[p_{i,j}(X_1, \ldots, X_n)]$ with respect to the normalized trace $\frac{1}{\ell}\tau_\ell$ is in $\frac{1}{d\ell}\mathbb{Z}$.

Proof. Let μ be the product measure of $\{\mu_j\}_{j=1}^n$ and let $(\mathcal{A}_0, \tau_0)$ be the tracial $*$-algebra generated by multiplication by the coordinate functions $\{x_j\}_{j=1}^n$ on $L_2(\mu)$ with trace $\tau_0(M_f) = \int_{\mathbb{C}^n} f\, d\mu$. Clearly each X_j has a representation in $\mathcal{A}_0$ as multiplication by the coordinate function x_j so we will view $X_j \in \mathcal{A}_0$ for all $j \in \{1, \dots, n\}$. Let $U := \lambda(1)$ be the canonical generating unitary operator for $L(\mathbb{Z})$. Then it is easy to see that X_1, UX_2U^*, …, $U^nX_n(U^n)^*$ are freely independent in $\mathcal{A}_0 * \mathbb{C}\mathbb{Z}$ with respect to the trace $\tau_0 * \tau_{\mathbb{Z}}$. However, since $(\mathcal{A}_0, \tau_0)$ has the Atiyah Property with group $\frac{1}{d}\mathbb{Z}$ by Theorem 3.2.5, $(\mathcal{A}_0 * \mathbb{C}\mathbb{Z}, \tau_0 * \tau_{\mathbb{Z}})$ has the Atiyah Property with group $\frac{1}{d}\mathbb{Z}$ by Theorem 3.3.1. Hence $(\mathcal{A}, \tau)$ has the Atiyah Property with group $\frac{1}{d}\mathbb{Z}$ by taking the canonical isomorphism of tracial $*$-algebras.

Next suppose that $[p_{i,j}]$ is an $\ell \times \ell$ matrix whose entries are non-commutative polynomials in n variables and their adjoints such that $[p_{i,j}(X_1, \dots, X_n)]$ is normal and the spectral distribution of $[p_{i,j}(X_1, \dots, X_n)]$ has an atom. By translation we may assume that this atom occurs at zero and thus corresponds to the kernel projection of $[p_{i,j}(X_1, \dots, X_n)]$. Since $(\mathcal{A}, \tau)$ has the Atiyah Property with group $\frac{1}{d}\mathbb{Z}$ we obtain that the measure of the atom is in $\frac{1}{d\ell}\mathbb{Z}$. □

As an application of the above result, we recall that Voiculescu developed in [78] the notion of the additive free product of measures in which if $\{X_j\}_{j=1}^n$ are self-adjoint, freely independent random variables with probability measures μ_j as distribution respectively, then the additive free product measure $\mu := \mu_1 \boxplus \cdots \boxplus \mu_n$ is the distribution of $X_1 + \cdots + X_n$ in the reduced free product C*-algebra. Hence Theorem 3.3.4 implies the following specific case of [11, Theorem 7.4].

Corollary 3.3.5 (see [11, Theorem 7.4])**.** *If $n \in \mathbb{N}$ and $\{\mu_j\}_{j=1}^n$ are non-atomic, compactly supported probability measures on $\mathbb{R}$, then $\mu_1 \boxplus \cdots \boxplus \mu_n$ has no atoms.*

Proof. Since each μ_j contains no atoms, we can apply Theorem 3.3.4 to conclude that $\mu := \mu_1 \boxplus \cdots \boxplus \mu_n$ may only have atoms in $\mathbb{Z}$. Since μ is a probability measure, if μ has an atom, then μ must be a point-mass measure which would imply that $X_1 + \cdots + X_n = \alpha I$ for some $\alpha \in \mathbb{R}$ contradicting the fact that X_1, …, X_n are freely independent. □

To complete this section, we can extend Theorem 3.3.4 to tensor products of tracial $*$-algebras generated by self-adjoint, freely independent random variables.

Corollary 3.3.6. *Let $n, m \in \mathbb{N}$ and let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be collections of normal, freely independent random variables with probability measures μ_j and ν_k as distribution respectively. Let $(\mathcal{A}, \tau_{\mathcal{A}})$ and $(\mathcal{B}, \tau_{\mathcal{B}})$ be the tracial $*$-algebras generated by the reduced free products of $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_m\}$ respectively. Suppose for each $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$ there exists a $d_j, d'_k \in \mathbb{N}$ such that the atoms of μ_j and ν_k have measures contained in $\frac{1}{d_j}\mathbb{Z}$ and $\frac{1}{d'_k}\mathbb{Z}$ respectively. If*

$$d := \prod_{j=1}^{n} d_j \cdot \prod_{k=1}^{m} d'_k,$$

then $(\mathcal{A} \otimes \mathcal{B}, \tau_{\mathcal{A}} \overline{\otimes} \tau_{\mathcal{B}})$ has the Atiyah Property with group $\frac{1}{d}\mathbb{Z}$.

Proof. Let μ be the product measure of $\{\mu_j\}_{j=1}^n$ and let ν be the product measure of $\{\nu_k\}_{k=1}^m$. Let $(\mathcal{A}_0, \tau_{\mathcal{A},0})$ be the tracial $*$-algebra generated by multiplication by the coordinate functions $\{x_j\}_{j=1}^n$ on $L_2(\mu)$ with trace $\tau_{\mathcal{A},0}(M_f) = \int_{\mathbb{C}^n} f \, d\mu$ and let $(\mathcal{B}_0, \tau_{\mathcal{B},0})$ be the tracial $*$-algebra generated by multiplication by the coordinate functions $\{y_k\}_{k=1}^m$ on $L_2(\nu)$ with trace $\tau_{\mathcal{B},0}(M_f) = \int_{\mathbb{C}^m} f \, d\nu$. Therefore $(\mathcal{A}_0 \odot \mathcal{B}_0, \tau_{\mathcal{A},0} \overline{\otimes} \tau_{\mathcal{B},0})$ has the Atiyah Property with group $\frac{1}{d}\mathbb{Z}$ by Theorem 3.2.5. The remainder of the proof follows the proof of Theorem 3.3.4 by an application of Corollary 3.3.2. $\square$

Notice that Corollary 3.3.6 has the following interesting application. For any $n, m \in \mathbb{N}$ let $P_1, \ldots, P_m \in \mathcal{A} := \mathrm{alg}(S_1, \ldots, S_n)$ be polynomials in n free semicircular variables $S_1, \ldots, S_n$ and let ∂_j be the non-commutative difference quotient derivations (see [81]). Let $JP := [\partial_i P_j]_{ij}$ which is an $n \times m$ matrix with entries in $\mathcal{A} \otimes \mathcal{A}$. The matrix JP is the non-commutative Jacobian of $P := (P_1, \ldots, P_m)$. We define the rank of JP to be the (non-normalized) trace of its image projection in $\mathcal{M}_n(W^*(\mathcal{A} \otimes \mathcal{A}))$.

Corollary 3.3.7. *With the above notation,* $\mathrm{rank}(JP) \in \{0, 1, \ldots, \min(m, n)\}$*. In particular, if $\{P_j\}_{j=1}^m$ are not all constant, then* $\mathrm{rank}(JP) \geq 1$.

3.4 Proof of Theorem 3.3.1

This section is devoted to the proof of Theorem 3.3.1, which underlies all results of Section 3.3. Our proof is essentially the same as the argument of Schick in [67] adapted for the case of algebras. This proof has themes similar to those used in [58, Lemma 10.43], which makes use of the notion of a Fredholm module to show that the free groups satisfy the Strong Atiyah Conjecture. The idea of applying Fredholm modules has its roots in a proof of the Kadison Conjecture for free groups on two generators from [15].

Proof of Theorem 3.3.1. Let $\mathcal{H} := L_2(\mathfrak{M}, \tau_{\mathfrak{M}})$. Thus $\mathfrak{M}$ has left and right actions on $\mathcal{H}$. Similarly, let $\mathcal{K} := L_2(\mathfrak{N}, \tau_{\mathfrak{N}})$. For a right-$(\mathfrak{M}\overline{\otimes}\mathfrak{N})^{\oplus \ell}$ invariant subspace $\mathcal{L}$ of $(\mathcal{H} \otimes \mathcal{K})^{\oplus \ell}$, we define

$$\dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\mathcal{L}) := \mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(Q) = (\tau_{\mathfrak{M}}\overline{\otimes}\tau_{\mathfrak{N}})_\ell(Q)$$

where Q is the orthogonal projection onto $\mathcal{L}$ (which is an element of $\mathcal{M}_\ell(\mathfrak{M}\overline{\otimes}\mathfrak{N})$ acting on the left).

For later convenience we desire to construct a certain isomorphism of Hilbert spaces that commonly appears in the proof that $\mathbb{F}_n$ satisfies the Strong Atiyah Conjecture. We desire a bijection

$$\psi : \{\delta_h \mid h \in \mathbb{F}_n \setminus \{e\}\} \to \{\delta_h \otimes e_i \mid h \in \mathbb{F}_n, i \in \{1, \ldots, n\}\}.$$

(where $\{e_i\}_{i=1}^n$ are the canonical orthonormal basis for $\mathbb{C}^n$) as this will clearly produce a unitary operator

$$\Psi : \ell_2(\mathbb{F}_n) \ominus (\mathbb{C}\delta_e) \to \ell_2(\mathbb{F}_n) \otimes \mathbb{C}^n.$$

Let $\{u_i\}_{i=1}^n$ be generators for $\mathbb{F}_n$. Consider the Cayley graph of $\mathbb{F}_n$ with edges $\{g, gu_i\}$. For each $h \in \mathbb{F}_n \setminus \{e\}$ let $e(h)$ be the first edge of the geodesic from h to e. Thus we may write $e(h) = \{\psi_0(h), \psi_0(h)u_{r(h)}\}$ for some $r(h) \in \{1, \ldots, n\}$. Thus if we define

$$\psi(\delta_h) := \delta_{\psi_0(h)} \otimes e_{r(h)},$$

we clearly obtain a bijection.

Let λ denote the left regular representation of $\mathbb{F}_n$ on $\ell_2(\mathbb{F}_n)$. We claim that Ψ has the property that for each $T \in \mathbb{C}\mathbb{F}_n$ the set of $\{\delta_h\}_{h\in\mathbb{F}_n\setminus\{e\}}$ such that $\Psi(\lambda(T)\delta_h)$ does not make sense (i.e. $\langle\lambda(T)\delta_h, \delta_e\rangle \neq 0$) or

$$\Psi(\lambda(T)\delta_h) \neq (\lambda(T) \otimes I_{\mathbb{C}^n})\Psi(\delta_h)$$

is finite. To see this notice for fixed $g, h \in \mathbb{F}_n$ the only way that $\lambda(g)(\delta_h) \notin \ell_2(\mathbb{F}_n) \ominus (\mathbb{C}\delta_e)$ is if $gh = e$ and the only way that $\Psi(\lambda(g)\delta_h) \neq (\lambda(g) \otimes I_{\mathbb{C}^n})\Psi(\delta_h)$ can occur is if when reducing gh a term from g cancels the second-last letter in h (which occurs for a finite number of h for a given g). Thus the claim follows by the linearity of Ψ.

Let $\{\zeta_j\}_{j\in\mathbb{Z}}$ be any orthonormal basis for $\mathcal{K}$ with ζ_0 a trace vector. We claim we may assume that there exists an orthonormal basis $\{\xi_j\}_{j\in\mathbb{Z}}$ of $\mathcal{H}$ such that ξ_0 is a trace vector and

$$\{k \in \mathbb{Z} \mid \langle T\xi_j, \xi_k\rangle_{\mathcal{H}} \neq 0\}$$

is finite for each $j \in \mathbb{Z}$ and $T \in \mathcal{A}$. To see this, we first may assume that $\mathcal{A}$ is finitely generated by self-adjoint operators $\{A_k\}_{k=1}^m$ since we need only check the Atiyah Property for one matrix with entries in $(\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}$ at a time and a finite number of elements of $\mathcal{A}$ will appear. If $\{\xi'_j\}_{j\in\mathbb{Z}}$ is any orthonormal basis of $\mathcal{H}$ with $\xi'_0 = \xi_0$ a trace vector, then the desired basis will be produced by applying the Gram-Schmidt Orthogonalization Process to

$$\{A_{i_1} \cdots A_{i_m}\xi'_j \mid j \in \mathbb{Z}, m \in \mathbb{N} \cup \{0\}, \{i_k\}_{k=1}^m \subseteq \{1, \ldots, n\}\}$$

starting with ξ'_0.

Recall $(\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}$ acts on $((\mathcal{H}, \xi_0) * (\ell_2(\mathbb{F}_n), \delta_e)) \otimes \mathcal{K}$ and

$$(\mathcal{H}, \xi_0) * (\ell_2(\mathbb{F}_n), \delta_e) = \mathbb{C}\xi_0 \oplus \left(\bigoplus \mathbb{C}\left(\xi_{j_1} \otimes \delta_{g_1} \otimes \cdots\right)\right) \oplus \left(\bigoplus \mathbb{C}\left(\delta_{g_1} \otimes \xi_{j_1} \otimes \cdots\right)\right)$$

(where $\xi_0 = \delta_e$) where all the tensors in the direct sums have finite length (ending at any point), alternate between basis elements of $\mathcal{H}$ and $\ell_2(\mathbb{F}_n)$, $j_k \in \mathbb{N}$, $i_k \in \mathbb{Z}\setminus\{0\}$, and $g_k \in \mathbb{F}_n \setminus \{e\}$. Notice that the union of the vectors used in the above definition of $(\mathcal{H}, \xi_0) * (\ell_2(\mathbb{F}_n), \delta_e)$ is an orthonormal basis for $(\mathcal{H}, \xi_0) * (\ell_2(\mathbb{F}_n), \delta_e)$. For convenience of notation, $\xi_0 \otimes \delta_{g_1} \otimes \cdots := \delta_{g_1} \otimes \cdots$, $\cdots \otimes \delta_{g_m} \otimes \xi_0 := \cdots \otimes \delta_{g_m}$, and $\cdots \otimes \xi_{j_m} \otimes \delta_e = \cdots \otimes \xi_{j_m}$.

Define the Hilbert spaces

$$\mathcal{L}_+ := ((\mathcal{H}, \xi_0) * (\ell_2(\mathbb{F}_n), \delta_e)) \otimes \mathcal{K} \quad \text{and} \quad \mathcal{L}_- := (\mathcal{L}_+ \otimes \mathbb{C}^n \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{K}).$$

Notice that $(\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}$ has a canonical left action on $\mathcal{L}_+$ and thus induces a canonical left action on $\mathcal{L}_-$ by letting an operator $T \in (\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}$ act via $(T \otimes I_{\mathbb{C}^n} \otimes I_{\mathcal{H}}) \oplus 0$. Thus we may view $\mathcal{L}_+$ and $\mathcal{L}_-$ as left $(\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}$-modules. Similarly, $\mathfrak{M}\overline{\otimes}\mathfrak{N}$ has a canonical right action on $\mathcal{H} \otimes \mathcal{K}$ and thus on $\mathcal{L}_+$ by

$$(\cdots \otimes \delta_{g_m} \otimes \xi_{j_m} \otimes \zeta)T = \cdots \otimes \delta_{g_m} \otimes ((\xi_{j_m} \otimes \zeta)T)$$

for all $\zeta \in \mathcal{K}$. Hence $\mathcal{L}_+$ is also a right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-module. It is clear that the right action of $\mathfrak{M}\overline{\otimes}\mathfrak{N}$ and the left action of $(\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}$ on $\mathcal{L}_+$ commute.

We desire to construct a bijection ϕ between the canonical basis elements of $\mathcal{L}_+$ and $\mathcal{L}_-$ which will induce a unitary operator $\Phi : \mathcal{L}_+ \to \mathcal{L}_-$. It is clear that if $\Lambda := \Lambda_0 \cup \Lambda'$ where $\Lambda_0 = \{\xi_j \otimes \zeta_{j'}\}_{j,j' \in \mathbb{Z}}$ and

$$\Lambda' := \left\{ (\xi_{j_0} \otimes \delta_{g_1} \otimes \cdots \otimes \delta_{g_m} \otimes \xi_{j_m}) \otimes \zeta_{j'} \;\middle|\; \begin{array}{c} m \geq 1, \{g_k\}_{k=1}^m \in \mathbb{F}_n \setminus \{e\}, \\ j_0, j_m, j' \in \mathbb{Z}, \{j_k\}_{k=1}^{m-1} \subseteq \mathbb{Z} \setminus \{0\} \end{array} \right\},$$

then Λ is an orthonormal basis of $\mathcal{L}_+$. Furthermore

$$\Theta := \{0 \oplus (\xi_j \otimes \zeta_{j'})\}_{j,j' \in \mathbb{Z}} \cup \{(\eta \otimes e_i \otimes \xi_j) \oplus 0 \mid \eta \in \Lambda, j \in \mathbb{Z}, i \in \{1, \ldots, n\}\}$$

is an orthonormal basis of $\mathcal{L}_-$. Define $\phi : \Lambda \to \Theta$ by defining $\phi|_{\Lambda_0}$ via

$$\phi(\xi_j \otimes \zeta_{j'}) = 0 \oplus (\xi_j \otimes \zeta_{j'})$$

for all $j, j' \in \mathbb{Z}$ and by defining $\phi|_{\Lambda'}$ via the following rule: for

$$\eta = (\xi_{j_0} \otimes \delta_{g_1} \otimes \cdots \otimes \delta_{g_m} \otimes \xi_{j_m}) \otimes \zeta_{j'} \in \Lambda$$

define

$$\phi(\eta) = (((\xi_{j_0} \otimes \delta_{g_1} \otimes \cdots \otimes \delta_{g_{m-1}} \otimes \xi_{j_{m-1}} \otimes \delta_{\psi_0(g_m)}) \otimes \zeta_{j'}) \otimes e_{r(g_m)} \otimes \xi_{j_m}) \oplus 0$$

(where if $\psi_0(g_m) = e$, we reduce the length of the first tensor by removing δ_e). Since Ψ is a bijection on the given basis elements, it is elementary to verify that ϕ is a bijection and thus induces a Hilbert space isomorphism $\Phi : \mathcal{L}_+ \to \mathcal{L}_-$.

Define a right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-module structure on $\mathcal{L}_-$ by defining $\eta T := \Phi((\Phi^{-1}(\eta))T)$ for all $T \in \mathfrak{M}\overline{\otimes}\mathfrak{N}$ and $\eta \in \mathcal{L}_-$. It is easy to see that

$$((\eta \otimes e_i \otimes \xi_k) \oplus (\xi_j \otimes \zeta_{j'}))(T \otimes S) = (\eta(I_{\mathcal{H}} \otimes S) \otimes e_i \otimes \xi_k T) \oplus (\xi_j T \otimes \zeta_{j'} S)$$

for all $T \in \mathfrak{M}$ and $S \in \mathfrak{N}$. Hence $0 \oplus (\mathcal{H} \otimes \mathcal{K})$ and $(\mathcal{L}_+ \otimes \mathbb{C}^n \otimes \mathcal{H}) \oplus 0$ are a right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-invariant subspace of $\mathcal{L}_-$. It is clear that the right action of $\mathfrak{M}\overline{\otimes}\mathfrak{N}$ on $\mathcal{L}_-$ commutes with the left action of $(\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}$ on $\mathcal{L}_-$.

Define Ξ to be the union of $\{\xi_0 \otimes \zeta_0\}$ with

$$\left\{ (\xi_{j_0} \otimes \delta_{g_1} \otimes \cdots \otimes \xi_{j_{m-1}} \otimes \delta_{g_m}) \otimes \zeta_0 \;\middle|\; \begin{array}{c} m \geq 1, \{g_k\}_{k=1}^m \in \mathbb{F}_n \setminus \{e\}, \\ j_0 \in \mathbb{Z}, \{j_k\}_{k=1}^{m-1} \subseteq \mathbb{Z} \setminus \{0\} \end{array} \right\}.$$

It is clear that Ξ is a set of orthonormal vectors in $\mathcal{L}_+$ each of which generates a one-$\mathfrak{M}\overline{\otimes}\mathfrak{N}$-dimensional right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-submodule of $\mathcal{L}_+$ that are pairwise orthogonal and whose union is dense in $\mathcal{L}_+$ (as ξ_0 and ζ_0 are cyclic vectors for the right actions). By the definition of Φ it is clear that $\Phi(\Xi)$ is a set of orthonormal vectors in $\mathcal{L}_-$ each of which generates a one-$\mathfrak{M}\overline{\otimes}\mathfrak{N}$-dimensional right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-submodule of $\mathcal{L}_-$ that are pairwise orthogonal and whose union is dense in $\mathcal{L}_-$.

We claim if $T \in (\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}$, then

$$\{\xi \in \Xi \mid \langle T\xi, \xi_j \otimes \zeta_{j'}\rangle_{\mathcal{L}_+} \neq 0 \text{ for some } j, j' \in \mathbb{Z} \text{ or } \Phi(T(\xi)) \neq T(\Phi(\xi))\}$$

is a finite subset (containing ξ_0). By linearity it suffices to prove the claim when T is a product of elements from $\mathcal{A}\cup\mathcal{B}\cup\{\lambda(h)\}_{h\in\mathbb{F}_n}$. First we will prove the claim when $T \in \mathcal{A}\cup\mathcal{B}$. However, it clearly follows that $\langle T\xi, \xi_j \otimes \zeta_{j'}\rangle_{\mathcal{L}_+} \neq 0$ for some $j, j' \in \mathbb{Z}$ or $\Phi(T(\xi)) \neq T\Phi(\xi)$ only if $\xi = \xi_0 \otimes \zeta_0$.

Next we will prove the claim for $T \in \{\lambda(h)\}_{h\in\mathbb{F}_n\setminus\{e\}}$. Fix $h \in \mathbb{F}_n$, fix $T = \lambda(h)$, and fix

$$\xi = \xi_{j_0} \otimes \delta_{g_1} \otimes \cdots \xi_{j_{m-1}} \otimes \delta_{g_m} \otimes \zeta_0 \in \Xi \setminus \{\xi_0 \otimes \zeta_0\}.$$

If $m > 1$ or $j_0 \neq 0$, then $\langle T\xi, \xi_j \otimes \zeta_{j'}\rangle = 0$ for all $j, j' \in \mathbb{Z}$ and $\Phi(T(\xi)) = T(\Phi(\xi))$ are clear. Otherwise $\xi = \delta_{g_1} \otimes \zeta_0$ and it clear that $\langle T\xi, \xi_j \otimes \zeta_{j'}\rangle \neq 0$ for some $j, j' \in \mathbb{Z}$ only if $hg_1 = e$ and $\Phi(T(\xi)) = T(\Phi(\xi))$ unless $\Psi(T\delta_{g_1}) \neq (T \otimes I_{\mathbb{C}^n})\Psi(\delta_{g_1})$. Since the number of such g_1 is finite, the claim holds in this case.

Next notice for any element $\xi \in \Xi$ and any element T of $\mathcal{A} \cup \{\lambda(h)\}_{h\in\mathbb{F}_n}$ that $T\xi$ is a finite linear combination of elements of $\Xi \cup \{\xi_j \otimes \zeta_0\}_{j\in\mathbb{Z}}$ by the choice of the orthonormal basis $\{\xi_j\}_{j\in\mathbb{Z}}$. Furthermore, for any element $\xi \in \Xi \cup \{\xi_j \otimes \zeta_0\}_{j\in\mathbb{Z}}$ and any element T of $\mathcal{A} \cup \{\lambda(h)\}_{h\in\mathbb{F}_n}$ there are only a finite number of elements η of Ξ such that $\langle T\eta, \xi\rangle_{\mathcal{L}_+} \neq 0$. Therefore if $T_1, \ldots, T_n \in \mathcal{A} \cup \{\lambda(h)\}_{h\in\mathbb{F}_n}$, then the set of all $\xi \in \Xi$ such that $\langle T_1 \cdots T_n\xi, \xi_j \otimes \zeta_{j'}\rangle_{\mathcal{L}_+} \neq 0$ for some $j, j' \in \mathbb{Z}$, $\langle T_2 \cdots T_n\xi, \xi_j \otimes \zeta_{j'}\rangle_{\mathcal{L}_+} \neq 0$ for some $j, j' \in \mathbb{Z}$, or $\Phi(T_1 \cdots T_n\xi) \neq T_1\Phi(T_2 \cdots T_n\xi)$ is finite. Thus the claim then follows by recursion and the fact that the $\mathcal{B}$-operator commute with elements of $\mathcal{A} * \mathbb{C}\mathbb{F}_n$ and with Φ.

The above construction show that we have two representations of $(\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}$ that differ by a $\mathcal{A} \odot \mathcal{B}$-finite rank operator. In order to complete the proof, we need a way to analyze the trace of such operators. Fix $\ell \in \mathbb{N}$ and fix

$$A := [A_{i,j}] \in \mathcal{M}_\ell((\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}).$$

The left actions of $(\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}$ on $\mathcal{L}_\pm$ allows A to act on $\mathcal{L}_\pm^{\oplus \ell}$. Let $A_\pm$ be the left action of A on $\mathcal{L}_\pm^{\oplus \ell}$ and let $P_\pm \in \mathcal{B}(\mathcal{L}_\pm^{\oplus \ell})$ be the projection onto the image of $A_\pm$. Thus we desire to show that $((\tau_{\mathfrak{M}} * \tau_{\mathbb{F}_n})\overline{\otimes}\tau_{\mathfrak{N}})_\ell(P_+) \in \frac{1}{d}\mathbb{Z}$. Since the right action of $\mathfrak{M}\overline{\otimes}\mathfrak{N}$ on $\mathcal{L}_\pm$ commutes with the left action of $(\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}$, we easily obtain that all operators under consideration commute with the diagonal right action of $\mathfrak{M}\overline{\otimes}\mathfrak{N}$ on these spaces.

Notice that there are only finitely many elements of $(\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}$ that appear in A. For each of these elements T, we recall that

$$\{\xi \in \Xi \mid \langle T\xi, \xi_j \otimes \zeta_{j'}\rangle_{\mathcal{L}_+} \neq 0 \text{ for some } j, j' \in \mathbb{Z} \text{ or } \Phi(T(\xi)) \neq T(\Phi(\xi))\}$$

is finite. Let $\mathcal{L}_{+,0}$ be the finite $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-dimensional right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-submodule of $\mathcal{L}_+$ spanned by the vectors that appear in the above set for at least one $T \in (\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}$ appearing in A. Thus $\mathcal{L}_{+,c} := \mathcal{L}_+ \ominus \mathcal{L}_{+,0}$ is a right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-submodule of $\mathcal{L}_+$.

Let $\mathcal{L}_{-,c} := \Phi(\mathcal{L}_{+,c})$, which is a right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-submodule of $\mathcal{L}_-$. Therefore, since $\mathcal{L}_{+,0}$ contained all $\xi \in \Xi$ where $\Phi(T(\xi)) \neq T(\Phi(\xi))$ for some $T \in (\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}$ appearing in A and since the right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-actions commutes with the left action of T and with Φ, we clearly obtain that

$$A_+|_{\mathcal{L}_{+,c}} = \Phi^{-1} \circ A_- \circ \Phi|_{\mathcal{L}_{+,c}}.$$

By progressively adding the right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-submodule of $\mathcal{L}_+$ generated by a single element of Ξ we can choose an increasing sequence

$$\mathcal{L}_{+,0} \subset \mathcal{L}_{+,1} \subset \mathcal{L}_{+,2} \subset \cdots \subset \mathcal{L}_+$$

of finite $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-dimensional right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-submodules of $\mathcal{L}_+$ such that

$$\mathcal{L}_+ = \overline{\bigcup_{j\geq 0} \mathcal{L}_{+,j}}.$$

Let $\mathcal{L}_{-,j} := \Phi(\mathcal{L}_{+,j})$ for all $j \in \mathbb{N} \cup \{0\}$. Hence each $\mathcal{L}_{-,j}$ is a right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-submodule of $\mathcal{L}_-$ generated by a finite number of elements of $\Phi(\Xi)$. Notice that $\Lambda_0 \subseteq \mathcal{L}_{+,0}$ so $0 \oplus (\mathcal{H} \otimes \mathcal{K}) \subseteq \mathcal{L}_{-,0}$. By construction, it is clear that

$$A_\pm(\mathcal{L}_\pm^{\oplus\ell}) = \overline{\bigcup_{j\geq 0} A_\pm(\mathcal{L}_{\pm,j}^{\oplus\ell})}.$$

For each $j \in \mathbb{N} \cup \{0\}$ let $P_{\pm,j}$ be the orthogonal projections onto $\overline{A_\pm(\mathcal{L}_{\pm,j}^{\oplus\ell})}$.

Since only finitely many elements of $(\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}$ appear in A, by our selection right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-modules generated by elements of Ξ we see that A_+ has finite propagation; that is, for every $j \in \mathbb{N}$ there exists an $n_j \in \mathbb{N}$ such that $A_+(\mathcal{L}_{+,j}^{\oplus\ell}) \subseteq \mathcal{L}_{+,n_j}^{\oplus\ell}$. Indeed an element of $\mathcal{B}$ does not modify the submodule, $\{\lambda(h)\}_{h\in\mathbb{F}_n}$ permutes the elements of Ξ, and an element of $\mathcal{A}$ maps an element of Ξ to at most a finite-$\mathfrak{M}\overline{\otimes}\mathfrak{N}$-dimensional $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-module by the choice of the basis $\{\xi_j\}_{j\in\mathbb{Z}}$. Similarly, as the left action of $(\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}$ on $\mathcal{L}_-$ has the same form and the right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-modules $\mathcal{L}_{-,j}$ are generated by elements of $\Phi(\Xi)$, A_- also has propagation so we may assume that $A_-(\mathcal{L}_{-,j}^{\oplus\ell}) \subseteq \mathcal{L}_{-,n_j}^{\oplus\ell}$ by choosing n_j sufficiently large.

The above allows us to view $A_\pm(\mathcal{L}_{\pm,j}^{\oplus\ell})$ as images of rectangular matrices with entries in $\mathcal{A} \odot \mathcal{B}$ acting on the left from $(\mathcal{H} \otimes \mathcal{K})^{\oplus q_j}$ to $(\mathcal{H} \otimes \mathcal{K})^{\oplus p_j}$ for some appropriate choice of q_j and p_j. Indeed an element from $\mathbb{C}\mathbb{F}_n$ acting on an element of Ξ or $\Phi(\Xi)$ acts as a scalar matrix since $\{\lambda(h)\}_{h\in\mathbb{F}_n}$ sends the right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-basis vectors Ξ and $\Phi(\Xi)$ to scalar multiples of other elements of Ξ and $\Phi(\Xi)$ respectively. Furthermore, each element $T \in \mathcal{A}$ acts by the usual left action of $\mathcal{A}$ on $\mathcal{H} \subseteq \mathcal{L}_+$ (which corresponds to the action of $\mathcal{A} \otimes I_\mathcal{K}$ on the right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-module generated by $\xi_0 \otimes \zeta_0 \in \Xi$) and otherwise act by sending the other elements of Ξ and every element of $\Phi(\Xi)$ to a finite linear combination of elements of Ξ and $\Phi(\Xi)$ respectively and thus can be viewed as scalar matrices on these right $\mathfrak{M}$-modules. Furthermore, it is clear that an element of $\mathcal{B}$ acts via $I_{\mathcal{H}} \otimes \mathcal{B}$ on each of the one-$\mathfrak{M}\overline{\otimes}\mathfrak{N}$-dimensional right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-modules spanned by an element of Ξ or $\Phi(\Xi)$. Thus the claim follows. Therefore, since $\mathcal{A} \odot \mathcal{B}$ has the Atiyah Property with group $\frac{1}{d}\mathbb{Z}$, we obtain that

$$\mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(P_{\pm,j}) = \dim_{\mathfrak{M}}(A_{\pm}(\mathcal{L}_{\pm,j}^{\oplus \ell})) \in \frac{1}{d}\mathbb{Z}.$$

Notice that

$$\overline{A_{\pm}(\mathcal{L}_{\pm,0}^{\oplus \ell})},\ \overline{A_{\pm}(\mathcal{L}_{\pm,c}^{\oplus \ell})},\ \text{and each } \overline{A_{\pm}((\mathcal{L}_{\pm,j} \cap \mathcal{L}_{+,c})^{\oplus \ell})}$$

are all closed right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-modules (note $\mathcal{L}_{\pm,j} \cap \mathcal{L}_{\pm,c} = \mathcal{L}_{\pm,j} \ominus \mathcal{L}_{\pm,0}$). We claim that

$$\begin{aligned}&\dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}\left(\overline{A_{\pm}(\mathcal{L}_{\pm,0}^{\oplus \ell}) \cap \overline{A_{\pm}(\mathcal{L}_{\pm,c}^{\oplus \ell})}}\right)\\ &= \lim_{j\to\infty} \dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}\left(\overline{A_{\pm}(\mathcal{L}_{\pm,0}^{\oplus \ell}) \cap \overline{A_{\pm}((\mathcal{L}_{\pm,j} \cap \mathcal{L}_{\pm,c})^{\oplus \ell})}}\right).\end{aligned}$$

To see this, it suffices by the continuity of von Neumann dimension (see [44, proof of Theorem 1.12]) to show that

$$\overline{A_{\pm}(\mathcal{L}_{\pm,0}^{\oplus \ell}) \cap \overline{A_{\pm}(\mathcal{L}_{\pm,c}^{\oplus \ell})}} = \overline{\bigcup_{j\geq 0} A_{\pm}(\mathcal{L}_{\pm,0}^{\oplus \ell}) \cap \overline{A_{\pm}((\mathcal{L}_{\pm,j} \cap \mathcal{L}_{\pm,c})^{\oplus \ell})}}.$$

To see this, notice one inclusion is trivial. For the other inclusion, recall that $A_{\pm}$ has finite propagation so there exists an $n_0 \in \mathbb{N}$ such that $A_{\pm}(\mathcal{L}_{\pm,0}^{\oplus \ell}) \subseteq \mathcal{L}_{\pm,n_0}^{\oplus \ell}$ so

$$\begin{aligned}A_{\pm}(\mathcal{L}_{\pm,0}^{\oplus \ell}) \cap \overline{A_{\pm}(\mathcal{L}_{\pm,c}^{\oplus \ell})} &= A_{\pm}(\mathcal{L}_{\pm,0}^{\oplus \ell}) \cap \mathcal{L}_{\pm,n_0}^{\oplus \ell} \cap \overline{A_{\pm}(\mathcal{L}_{\pm,c}^{\oplus \ell})}\\ &= A_{\pm}(\mathcal{L}_{\pm,0}^{\oplus \ell}) \cap \mathcal{L}_{\pm,n_0}^{\oplus \ell} \cap \left(\overline{\textstyle\bigcup_{j\geq 1} A_{\pm}((\mathcal{L}_{\pm,j} \cap \mathcal{L}_{\pm,c})^{\oplus \ell})}\right).\end{aligned}$$

We claim that

$$\mathcal{L}_{\pm,n_0}^{\oplus \ell} \cap \left(\overline{\bigcup_{j\geq 1} A_{\pm}((\mathcal{L}_{\pm,j} \cap \mathcal{L}_{\pm,c})^{\oplus \ell})}\right) = \mathcal{L}_{\pm,n_0}^{\oplus \ell} \cap \overline{A_{\pm}((\mathcal{L}_{\pm,m} \cap \mathcal{L}_{\pm,c})^{\oplus \ell})}$$

for some sufficiently large $m \in \mathbb{N}$. Specifically, to choose m, we notice, by the same arguments that Φ almost commutes with the left actions, that there exists an $m \in \mathbb{N}$ such that if $\eta \in \mathcal{L}_{\pm,m+k} \ominus \mathcal{L}_{\pm,m}$ for any $k \geq 1$, then every entry of A applied to η is orthogonal to $\mathcal{L}_{\pm,n_0}$ (that is, there are a finite number of elements η of Ξ for which there is an entry T in A such that $T\eta$ has non-zero inner product with an element of $\mathcal{L}_{\pm,n_0} \cap \Xi$). To see the above equality for this $m \in \mathbb{N}$, we notice that one inclusion is trivial. For the other inclusion, fix

$$\xi \in \mathcal{L}_{\pm,n_0}^{\oplus\ell} \cap \left(\overline{\bigcup_{j\geq 1} A_\pm((\mathcal{L}_{\pm,j} \cap \mathcal{L}_{\pm,c})^{\oplus\ell})}\right).$$

Thus there exists $\eta_j \in (\mathcal{L}_{\pm,c} \cap \mathcal{L}_{\pm,j})^{\oplus\ell}$ such that $\xi = \lim_{j\to\infty} A_\pm \eta_j$. Therefore, if P is the projection of $\mathcal{L}_{\pm,c}^{\oplus\ell}$ onto $(\mathcal{L}_{\pm,m} \cap \mathcal{L}_{\pm,c})^{\oplus\ell}$, then

$$A\eta_j = A(P\eta_j) + \omega_j$$

where $\omega_j \in (\mathcal{L}_{\pm,n_0}^{\oplus\ell})^\perp$. Therefore, since

$$\lim_{j\to\infty} A_\pm \eta_j = \xi \in \mathcal{L}_{\pm,n_0}^{\oplus\ell},$$

we obtain that $\lim_{j\to\infty} \omega_j = 0$ and $\xi = \lim_{j\to\infty} A(P\zeta_j)$ where $P\zeta_j \in (\mathcal{L}_{\pm,m} \cap \mathcal{L}_{\pm,c})^{\oplus\ell}$ as desired. Hence the claim is complete. Thus

$$\begin{array}{rcl} A_\pm(\mathcal{L}_{\pm,0}^{\oplus\ell}) \cap \overline{A_\pm(\mathcal{L}_{\pm,c}^{\oplus\ell})} & = & A_\pm(\mathcal{L}_{\pm,0}^{\oplus\ell}) \cap \mathcal{L}_{\pm,n_0}^{\oplus\ell} \cap \overline{A_\pm((\mathcal{L}_{\pm,m} \cap \mathcal{L}_{\pm,c})^{\oplus\ell})} \\ & = & A_\pm(\mathcal{L}_{\pm,0}^{\oplus\ell}) \cap \overline{A_\pm((\mathcal{L}_{\pm,m} \cap \mathcal{L}_{\pm,c})^{\oplus\ell})} \\ & \subseteq & \overline{\bigcup_{j\geq 0} A_\pm(\mathcal{L}_{\pm,0}^{\oplus\ell}) \cap \overline{A_\pm((\mathcal{L}_{\pm,j} \cap \mathcal{L}_{\pm,c})^{\oplus\ell})}} \end{array}$$

which completes the claim.

Let $P_{\pm,c}$ to be the orthogonal projections onto $\overline{A_\pm(\mathcal{L}_{\pm,c}^{\oplus\ell})}$ and for each $j \in \mathbb{N} \cup \{0\}$ let $P_{\pm,j,c}$ be the orthogonal projection onto $\overline{A_\pm(\mathcal{L}_{\pm,j} \cap \mathcal{L}_{\pm,c})^{\oplus\ell}}$. Notice that $P_{\pm,c}$ and each $P_{\pm,j,c}$ need not be in the von Neumann algebra generated by $\mathcal{M}_\ell((\mathcal{A} * \mathbb{CF}_n) \odot \mathcal{B})$ but do commute with the right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-action on their respective spaces. Since

$$A_+|_{\mathcal{L}_{+,c}} = \Phi^{-1} \circ A_- \circ \Phi|_{\mathcal{L}_{+,c}},$$

we obtain that $P_{+,j,c} = \Phi^{-1} \circ P_{-,j,c} \circ \Phi$ for all $j \in \mathbb{N} \cup \{0\}$ and $P_{+,c} = \Phi^{-1} \circ P_{-,c} \circ \Phi$. Hence

$$\langle P_{+,c}\eta, \eta\rangle_{\mathcal{L}_+^{\oplus \ell}} = \langle P_{-,c}\Phi(\eta), \Phi(\eta)\rangle_{\mathcal{L}_-^{\oplus \ell}} \text{ and } \langle P_{+,j,c}\eta, \eta\rangle_{\mathcal{L}_+^{\oplus \ell}} = \langle P_{-,j,c}\Phi(\eta), \Phi(\eta)\rangle_{\mathcal{L}_-^{\oplus \ell}}$$

for all $j \in \mathbb{N} \cup \{0\}$ and $\eta \in \mathcal{L}_+^{\oplus \ell}$.

Let $Q_\pm := P_\pm - P_{\pm,c}$ and for each $j \in \mathbb{N} \cup \{0\}$ define $Q_{\pm,j} := P_{\pm,j} - P_{\pm,j,c}$. Clearly these are projections onto the complements of smaller projections in larger projections. We claim that

$$\mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(Q_\pm) = \lim_{j\to\infty} \mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(Q_{\pm,j}).$$

To begin, let A_0 denote the restriction of $A_\pm$ to $\mathcal{L}_{\pm,0}^{\oplus \ell}$. We claim for each fixed $j \in \mathbb{N}$ that

$$0 \longrightarrow \ker(Q_{\pm,j}A_0) \longrightarrow \mathcal{L}_{\pm,0}^{\oplus \ell} \xrightarrow{Q_{\pm,j}A_0} \overline{\mathrm{Im}(Q_{\pm,j})} \longrightarrow 0$$

is a weakly exact sequence (that is, the images are dense in the kernels). To see this, it suffices to check weak exactness at $\overline{\mathrm{Im}(Q_{\pm,j})}$. It is clear that $Q_{\pm,j}(A_\pm(\mathcal{L}_{\pm,j}^{\oplus \ell}))$ is dense in $\overline{\mathrm{Im}(Q_{\pm,j})}$. However

$$A_\pm(\mathcal{L}_{\pm,j}^{\oplus \ell}) = A_\pm(\mathcal{L}_{\pm,0}^{\oplus \ell}) + A_\pm((\mathcal{L}_{\pm,j} \cap \mathcal{L}_{\pm,c})^{\oplus \ell})$$

and it is clear that $Q_{\pm,j}(A((\mathcal{L}_{\pm,j} \cap \mathcal{L}_{\pm,c})^{\oplus \ell})) = 0$. Thus $Q_{\pm,j}(A_\pm(\mathcal{L}_{\pm,0}^{\oplus \ell})) = Q_{\pm,j}(A_\pm(\mathcal{L}_{\pm,j}^{\oplus \ell}))$ is dense in $\overline{\mathrm{Im}(Q_{\pm,j})}$. Since each term in the weak exact sequence is a right $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-module and weak exact sequence preserve $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-dimension (see [44, proof of Theorem 1.12]), we obtain that

$$\dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\mathcal{L}_{\pm,0}^{\oplus \ell}) = \dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\mathrm{Im}(Q_{\pm,j})) + \dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\ker(Q_{\pm,j}A_0))$$

(which are all finite as $\dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\mathcal{L}_{\pm,0}^{\oplus\ell})$ is finite by construction). Furthermore, it is clear that

$$\ker(Q_{\pm,j}A_0) = \{\eta \in \mathcal{L}_{\pm,0}^{\oplus\ell} \mid Q_{\pm,j}A_0\eta = 0\}.$$

Hence the sequence

$$0 \longrightarrow \ker(A_0) \longrightarrow \ker(Q_{\pm,j}A_0) \xrightarrow{A_0} \overline{A_\pm(\mathcal{L}_{\pm,0}^{\oplus\ell}) \cap \ker(Q_{\pm,j})} \longrightarrow 0$$

is weakly exact. This implies the sequence

$$0 \longrightarrow \ker(A_0) \longrightarrow \ker(Q_{\pm,j}A_0) \xrightarrow{A_0} \overline{A_\pm(\mathcal{L}_{\pm,0}^{\oplus\ell}) \cap \overline{A_\pm((\mathcal{L}_{\pm,j} \cap \mathcal{L}_{\pm,c})^{\oplus\ell})}} \longrightarrow 0$$

is a weakly exact sequence since it is elementary to verify that

$$\overline{A_\pm(\mathcal{L}_{\pm,0}^{\oplus\ell}) \cap \ker(Q_{\pm,j})} = \overline{A_\pm(\mathcal{L}_{\pm,0}^{\oplus\ell}) \cap \overline{A_\pm((\mathcal{L}_{\pm,j} \cap \mathcal{L}_{\pm,c})^{\oplus\ell})}}.$$

Hence we obtain that

$$\begin{aligned} & \dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\ker(Q_{\pm,j}A_0)) \\ = \; & \dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\ker(A_0)) + \dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}\left(\overline{A_\pm(\mathcal{L}_{\pm,0}^{\oplus\ell}) \cap \overline{A_\pm((\mathcal{L}_{\pm,j} \cap \mathcal{L}_{\pm,c})^{\oplus\ell})}}\right). \end{aligned}$$

By combining the two above dimension equations we obtain that

$$\begin{aligned} \dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\mathrm{Im}(Q_{\pm,j})) = \; & \dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\mathcal{L}_{\pm,0}^{\oplus\ell}) - \dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\ker(A_0)) \\ & -\dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}\left(\overline{A_\pm(\mathcal{L}_{\pm,0}^{\oplus\ell}) \cap \overline{A_\pm((\mathcal{L}_{\pm,j} \cap \mathcal{L}_{\pm,c})^{\oplus\ell})}}\right) \end{aligned}$$

for each $j \in \mathbb{N}$. Similarly, by repeating the same arguments we obtain that

$$\begin{aligned} \dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\mathrm{Im}(Q_{\pm})) = \; & \dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\mathcal{L}_{\pm,0}^{\oplus\ell}) - \dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\ker(A_0)) \\ & -\dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}\left(\overline{A_\pm(\mathcal{L}_{\pm,0}^{\oplus\ell}) \cap \overline{A_\pm((\mathcal{L}_{\pm,c})^{\oplus\ell})}}\right). \end{aligned}$$

Therefore, as all the terms in the above dimension equations are finite (in fact bounded by

$\dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\mathcal{L}_{\pm,0}^{\oplus\ell}))$,

$$\begin{aligned}\mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(Q_\pm) &= \dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\mathrm{Im}(Q_\pm)) \\ &= \lim_{j\to\infty}\dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\mathrm{Im}(Q_{\pm,j})) = \lim_{j\to\infty}\mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(Q_{\pm,j}).\end{aligned}$$

We will now use Ξ and $\Phi(\Xi)$ to compute traces. For each $\eta \in \Xi$ and $i \in \{1, \ldots, \ell\}$ let

$$\eta_i = (0, 0, \ldots, 0, \eta, 0, \ldots, 0) \in \mathcal{L}_+^{\oplus\ell}$$

where η is in the i^{th} spot and similarly let

$$\phi(\eta_i) = (0, \ldots, 0, \phi(\eta), 0, \ldots, 0) \in \mathcal{L}_-^{\oplus\ell}.$$

Since Ξ and $\Phi(\Xi)$ are orthonormal $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-bases for $\mathcal{L}_+$ and $\mathcal{L}_-$ respectively, we easily obtain that

$$\mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(Q_+) = \sum_{\eta\in\Xi}\sum_{i=1}^{\ell}\langle Q_+\eta_i, \eta_i\rangle_{\mathcal{L}_+^{\oplus\ell}}$$

and

$$\mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(Q_-) = \sum_{\eta\in\Xi}\sum_{i=1}^{\ell}\langle Q_-\phi(\eta_i), \phi(\eta_i)\rangle_{\mathcal{L}_-^{\oplus\ell}}.$$

Furthermore, we notice if $\eta = \xi_0 \otimes \zeta_0 \in \Xi$, then

$$\sum_{i=1}^{\ell}\langle P_+\eta_i, \eta_i\rangle_{\mathcal{L}_+^{\oplus\ell}} = ((\tau_{\mathfrak{M}} * \tau_{\mathbb{F}_n})\overline{\otimes}\tau_{\mathfrak{N}})_\ell(P_+)$$

whereas

$$\sum_{i=1}^{\ell}\langle P_-\phi(\eta_i), \phi(\eta_i)\rangle_{\mathcal{L}_-^{\oplus\ell}} = \sum_{i=1}^{\ell} 0 = 0$$

by the definition of A_- and P_-. Finally, we claim that

$$\sum_{i=1}^{\ell}\langle P_+\eta_i, \eta_i\rangle_{\mathcal{L}_+^{\oplus\ell}} - \langle P_-\phi(\eta_i), \phi(\eta_i)\rangle_{\mathcal{L}_-^{\oplus\ell}} = 0$$

for all $\eta \in \Xi \setminus \{\xi_0 \otimes \zeta_0\}$. To see this, suppose

$$\eta = (\xi_{j_0} \otimes \delta_{g_1} \otimes \cdots \otimes \delta_{g_m}) \otimes \zeta_0 \in \Xi \setminus \{\xi_0 \otimes \zeta_0\}.$$

Then, by considering the above expression of $\phi(\eta)$ and the right action of $L(\mathbb{F}_n)$ on $(\mathcal{H}, \xi_0) * (\ell_2(\mathbb{F}_n), \delta_e)$, there exists a unitary operator $U_\eta \in L(\mathbb{F}_n)$ such that U_η commutes with the left actions of $\mathfrak{M}$, $L(\mathbb{F}_n)$, and $\mathfrak{N}$ on $\mathcal{L}_+ \otimes \mathbb{C}^n \otimes \mathcal{H}$ such that $U_\eta \phi(\eta) = \eta \otimes e_{i_0} \otimes \xi_0$ for some $i_0 \in \{1, \ldots, n\}$. Since every element $T \in (\mathcal{A} * \mathbb{C}\mathbb{F}_n) \odot \mathcal{B}$ acts on $\mathcal{L}_-$ via $(T \otimes I_{\mathbb{C}^n} \otimes I_{\mathcal{H}}) \oplus 0_{\mathcal{H}\otimes\mathcal{K}}$, P_- is $(P_+ \otimes I_{\mathbb{C}^n} \otimes I_{\mathcal{H}}) \oplus 0_{(\mathcal{H}\otimes\mathcal{K})^{\oplus \ell}}$ so

$$\begin{aligned}
\textstyle\sum_{i=1}^{\ell} \langle P_- \phi(\eta_i), \phi(\eta_i)\rangle_{\mathcal{L}_-^{\oplus \ell}} &= \textstyle\sum_{i=1}^{\ell} \langle P_- U_\eta^*(\eta \otimes e_{i_0} \otimes \xi_0), U_\eta^*(\eta \otimes e_{i_0} \otimes \xi_0)\rangle_{\mathcal{L}_-^{\oplus \ell}} \\
&= \textstyle\sum_{i=1}^{\ell} \langle P_-(\eta \otimes e_{i_0} \otimes \xi_0), \eta \otimes e_{i_0} \otimes \xi_0\rangle_{\mathcal{L}_-^{\oplus \ell}} \\
&= \textstyle\sum_{i=1}^{\ell} \langle P_+ \eta_i, \eta_i\rangle_{\mathcal{L}_+^{\oplus \ell}}
\end{aligned}$$

as claimed. Hence

$$\sum_{\eta \in \Xi} \sum_{i=1}^{\ell} \left(\langle P_+ \eta_i, \eta_i\rangle_{\mathcal{L}_+^{\oplus \ell}} - \langle P_- \phi(\eta_i), \phi(\eta_i)\rangle_{\mathcal{L}_-^{\oplus \ell}} \right) = (\tau * \tau_{\mathbb{F}_n})_\ell(P_+).$$

Thus the proof will be complete if the left-hand side of the above equation is in $\frac{1}{d}\mathbb{Z}$.

To begin we notice for all $\eta \in \Xi$ and $i \in \{1, \ldots, \ell\}$ that

$$\begin{aligned}
& \langle P_+ \eta_i, \eta_i\rangle_{\mathcal{L}_+^{\oplus \ell}} - \langle P_- \phi(\eta_i), \phi(\eta_i)\rangle_{\mathcal{L}_-^{\oplus \ell}} \\
= \; & \langle P_{+,c} \eta_i, \eta_i\rangle_{\mathcal{L}_+^{\oplus \ell}} - \langle P_{-,c} \phi(\eta_i), \phi(\eta_i)\rangle_{\mathcal{L}_-^{\oplus \ell}} \\
& + \langle Q_+ \eta_i, \eta_i\rangle_{\mathcal{L}_+^{\oplus \ell}} - \langle Q_- \phi(\eta_i), \phi(\eta_i)\rangle_{\mathcal{L}_-^{\oplus \ell}} \\
= \; & 0 + \langle Q_+ \eta_i, \eta_i\rangle_{\mathcal{L}_+^{\oplus \ell}} - \langle Q_- \phi(\eta_i), \phi(\eta_i)\rangle_{\mathcal{L}_-^{\oplus \ell}}.
\end{aligned}$$

Similarly, we obtain for all $\eta \in \Xi$, $i \in \{1, \ldots, \ell\}$, and $j \in \mathbb{N}$ that

$$\begin{aligned}
& \langle P_{+,j}\eta_i, \eta_i\rangle_{\mathcal{L}_+^{\oplus\ell}} - \langle P_{-,j}\phi(\eta_i), \phi(\eta_i)\rangle_{\mathcal{L}_-^{\oplus\ell}} \\
= \;& \langle P_{+,c,j}\eta_i, \eta_i\rangle_{\mathcal{L}_+^{\oplus\ell}} - \langle P_{-,c,j}\phi(\eta_i), \phi(\eta_i)\rangle_{\mathcal{L}_-^{\oplus\ell}} \\
& +\langle Q_{+,j}\eta_i, \eta_i\rangle_{\mathcal{L}_+^{\oplus\ell}} - \langle Q_{-,j}\phi(\eta_i), \phi(\eta_i)\rangle_{\mathcal{L}_-^{\oplus\ell}} \\
= \;& 0 + \langle Q_{+,j}\eta_i, \eta_i\rangle_{\mathcal{L}_+^{\oplus\ell}} - \langle Q_{-,j}\phi(\eta_i), \phi(\eta_i)\rangle_{\mathcal{L}_-^{\oplus\ell}}
\end{aligned}$$

Since $\mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(P_{\pm,j}) = \dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(A_\pm(\mathcal{L}_{\pm,j}^{\oplus\ell})) \in \frac{1}{d}\mathbb{Z}$ for all $j \in \mathbb{N}$, and since $Q_{\pm,j}$ have finite $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-rank (bounded by $\dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\mathcal{L}_{+,0}^{\oplus\ell})$), the following computation is valid:

$$\begin{aligned}
& \mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(Q_{+,j}) - \mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(Q_{-,j}) \\
& = \textstyle\sum_{\eta\in\Xi}\sum_{i=1}^{\ell} \langle Q_{+,j}\eta_i, \eta_i\rangle_{\mathcal{L}_+^{\oplus\ell}} - \langle Q_{-,j}\phi(\eta_i), \phi(\eta_i)\rangle_{\mathcal{L}_-^{\oplus\ell}} \\
& = \textstyle\sum_{\eta\in\Xi}\sum_{i=1}^{\ell} \langle P_{+,j}\eta_i, \eta_i\rangle_{\mathcal{L}_+^{\oplus\ell}} - \langle P_{-,j}\phi(\eta_i), \phi(\eta_i)\rangle_{\mathcal{L}_-^{\oplus\ell}} \\
& = \mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(P_{+,j}) - \mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(P_{-,j}) \in \tfrac{1}{d}\mathbb{Z}.
\end{aligned}$$

Therefore, since Q_+ and Q_- have finite $\mathfrak{M}\overline{\otimes}\mathfrak{N}$-rank (bounded above by $\dim_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(\mathcal{L}_{+,0}^{\oplus\ell})$), we obtain that

$$\mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(Q_+) - \mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(Q_-) = \lim_{j\to\infty} \mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(Q_{+,j}) - \mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(Q_{-,j}) \in \frac{1}{d}\mathbb{Z}.$$

Hence

$$\begin{aligned}
((\tau_{\mathfrak{M}} * \tau_{\mathbb{F}_n})\overline{\otimes}\tau_{\mathfrak{N}})_\ell(P_+) \;&=\; \textstyle\sum_{\eta\in\Xi}\sum_{i=1}^{\ell} \left(\langle P_+\eta_i, \eta_i\rangle_{\mathcal{L}_+^{\oplus\ell}} - \langle P_-\phi(\eta_i), \phi(\eta_i)\rangle_{\mathcal{L}_-^{\oplus\ell}}\right) \\
&=\; \textstyle\sum_{\eta\in\Xi}\sum_{i=1}^{\ell} \left(\langle Q_+\eta_i, \eta_i\rangle_{\mathcal{L}_+^{\oplus\ell}} - \langle Q_-\phi(\eta_i), \phi(\eta_i)\rangle_{\mathcal{L}_-^{\oplus\ell}}\right) \\
&=\; \mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(Q_+) - \mathrm{tr}_{\mathfrak{M}\overline{\otimes}\mathfrak{N}}(Q_-) \in \tfrac{1}{d}\mathbb{Z}
\end{aligned}$$

which completes the proof. □

3.5 Algebraic Cauchy Transforms of Polynomials in Semicircular Variables

In this section we will demonstrate that the Cauchy transform of any self-adjoint matricial polynomial of semicircular variables is algebraic (see Theorem 3.5.3). Knowing that the Cauchy transform of a measure is algebraic provides information about the spectral distribution of operators as seen in Theorem 3.1.1. To begin, we recall the notion of a formal power series in commuting variables.

Definition 3.5.1. Let $n \in \mathbb{N}$ and let $X = \{z_1, \ldots, z_n\}$. For a ring R, a formal power series in commuting variables X with coefficients in R is a map $P : (\mathbb{N} \cup \{0\})^n \to R$ which we will write as

$$P = \sum_{j=0}^{n} \sum_{k_j \geq 0} P(k_1, \ldots, k_n) z_1^{k_1} \cdots z_n^{k_n}.$$

A formal power series P is called a polynomial if $P(k_1, \ldots, k_n) = 0$ except for a finite number of n-tuples $(k_1, \ldots, k_n)$. The set of all formal power series with coefficients in R will be denoted $R[[X]]$ and the set of all polynomials with coefficients in R will be denoted $R[X]$.

The set of formal power series over a ring R can be given a ring structure. Indeed, if addition on $R[[X]]$ is defined coordinate-wise and the product of $P, Q \in R[[X]]$ is defined via the rule

$$(P + Q)(k_1, \ldots, k_n) = \sum_{j=0}^{n} \sum_{\ell_j=0}^{k_j} P(\ell_1, \ldots, \ell_n) Q(k_1 - \ell_1, \ldots, k_n - \ell_n),$$

it is elementary to verify that $R[[X]]$ is a ring. Clearly $R[X]$ is a subring of $R[[X]]$ which enables us to construct the quotient field of $R[X]$. The quotient field of $R[X]$ will be denoted $R(X)$.

With the above definitions, we have the following definition essential to this section.

Definition 3.5.2. Let $n \in \mathbb{N}$, let $X = \{z_1, \ldots, z_n\}$, and let R be an integral domain. A formal power $P \in R[[X]]$ is said to be algebraic if there exists an $m \in \mathbb{N}$ and $\{q_j\}_{j=0}^{m} \subseteq R(X)$

not all zero such that

$$\sum_{j=0}^{m} q_j P^j = 0.$$

Equivalently, by clearing denominators, we can require $\{q_j\}_{j=0}^m \subseteq R[X]$. The set of all algebraic elements of $R[[X]]$ is denoted $R_{\text{alg}}[[X]]$.

Our main interest lies in demonstrating that certain formal power series relating to measures are algebraic. In particular, given a compactly supported probability measure μ, we saw in Remarks 1.1.10 implies G_μ has a Laurent expansion that defines a formal power series in $\mathbb{C}[[\{\frac{1}{z}\}]]$. Thus it makes sense to ask whether G_μ is algebraic.

In order to state the main result of this section, we will need some additional notation. Let $\mathfrak{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. Let $A \in \mathfrak{M}$ be a fixed self-adjoint operator. Since A is a self-adjoint element in a von Neumann algebra, for each $t \in \mathbb{R}$ let $E_A(t) \in \mathfrak{M}$ be the spectral projection of A onto $(-\infty, t]$. The cumulative density function of A, denoted F_A, is the function on $[-\|A\|, \|A\|]$ defined by $F_A(t) = \tau(E_A(t))$. Clearly F_A is a right continuous function that is bounded above by 1. In turn, F_A defines the spectral measure of A, denoted μ_A, by the equation

$$\mu_A((t_1, t_2]) = F_A(t_2) - F_A(t_1).$$

Notice that μ_A is a Borel probability measure supported on $[-\|A\|, \|A\|]$. Recall the spectral measure has the unique property that if f is a continuous function on the spectrum of A, then

$$\tau(f(A)) = \int_0^{\|A\|} f(t)\, d\mu_A(t).$$

With the above notation, we have the following important result which provides information about spectral distributions as indicated in Section 3.1.

Theorem 3.5.3. *Let $n, \ell \in \mathbb{N}$, let $S_1, \ldots, S_n$ be freely independent semicircular variables, let $\mathcal{A}$ be the $*$-algebra generated by $S_1, \ldots, S_n$, and let $A \in \mathcal{M}_\ell(\mathcal{A})$ be a fixed self-adjoint operator. The Cauchy transform of the spectral measure of A is algebraic.*

In order to prove Theorem 3.5.3 we will mimic the proof of [66, Theorem 3.6] which proves said result when $S_1, \ldots, S_n$ are replaced with freely independent Haar unitaries. In order to mimic the proof in [66], we recall another type of formal power series in commuting variables.

Definition 3.5.4. Let S be a ring and let R be a subring of S. It is said that R is rationally closed in S if for every matrix with entries in R which is invertible when viewed as a matrix with entries in S, the entries of the inverse lies in R.

The rational closure of R in S, denoted $\mathcal{R}(R \subseteq S)$, is the smallest subring of S containing R that is rationally closed.

For an arbitrary ring R and finite set X, the rational closure $\mathcal{R}(R[X] \subseteq R[[X]])$ is called the ring of rational power series over R and is denoted $R_{\mathrm{rat}}[[X]]$.

It turns out that the key to showing the Cauchy transform G_{μ_A} is algebraic for all positive matrices A with entries in a tracial $*$-algebra is intrinsically related to the following map.

Definition 3.5.5. Let $\mathfrak{M}$ be a finite von Neumann algebra with faithful, normal, tracial state τ. The tracial map on formal power series in one variable is the map $Tr_{\mathfrak{M}} : \mathfrak{M}[[\{z\}]] \to \mathbb{C}[[\{z\}]]$ defined by

$$Tr_{\mathfrak{M}}\left(\sum_{n\geq 0} T_n z^n\right) = \sum_{n\geq 0} \tau(T_n) z^n.$$

In particular, the beginning of the proof of [66, Theorem 3.6] demonstrates the following.

Lemma 3.5.6. *Let $\mathfrak{M}$ be a finite von Neumann algebra with faithful, normal, tracial state τ and let $\mathcal{A}$ be a subalgebra of $\mathfrak{M}$. If*

$$Tr_{\mathfrak{M}}(\mathcal{A}_{\mathrm{rat}}[[\{z\}]]) \subseteq \mathbb{C}_{\mathrm{alg}}[[\{z\}]],$$

then the Cauchy transform G_{μ_A} is algebraic for every positive matrix $A \in \mathcal{M}_\ell(\mathcal{A})$ and any $\ell \in \mathbb{N}$.

Proof. As in the proof of [66, Theorem 3.6], for an arbitrary $\ell \in \mathbb{N}$ and positive matrix $A \in \mathcal{M}_\ell(\mathcal{A})$, the entries of $z(I_{\mathcal{M}_\ell(\mathcal{A})} - Az)^{-1}$ (which can be viewed as an element of $\mathcal{M}_\ell(\mathcal{A})[[\{z\}]]$ by expanding the result when $\|A\|\,|z| < 1$) lie in the rational closure $\mathcal{A}_{\text{rat}}[[\{z\}]]$. By assumption, the formal power series

$$q(z) := Tr_{\mathcal{M}_\ell(\mathfrak{M})}\left(z(I_{\mathcal{M}_n(\mathcal{A})} - Az)^{-1}\right) = \sum_{j=1}^{\ell} Tr_{\mathfrak{M}}((z(I_{\mathcal{M}_\ell(\mathcal{A})} - Az)^{-1})_{jj})$$

is an element of $\mathbb{C}_{\text{alg}}[[\{z\}]]$. Thus $q(z^{-1})$ is an element of $\mathbb{C}_{\text{alg}}[[\{\frac{1}{z}\}]]$. If $\tau_{\mathcal{M}_\ell(\mathfrak{M})}$ is the canonical trace on $\mathcal{M}_\ell(\mathfrak{M})$, it is well-known that

$$G_{\mu_A}(z) = \tau_{\mathcal{M}_\ell(\mathfrak{M})}((zI_{\mathcal{M}_n(\mathcal{A})} - A)^{-1}) = q(z^{-1})$$

in the domain $\{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0, |z| > \|A\|\}$. Hence $G_{\mu_A} \in \mathbb{C}_{\text{alg}}[[\{\frac{1}{z}\}]]$ as desired. $\square$

Thus the proof of Theorem 3.5.3 will be complete provided the assumptions of Lemma 3.5.6 can be verified. Following [66], it is necessary to examine formal power series in non-commuting variables.

Definition 3.5.7. Let X be a finite set (which will be called an alphabet) and let $W(X)$ denote the set of all words with letters in X. The empty word will be denoted by e. For a ring R, a formal power series with non-commuting variables X with coefficients in R is a map $P : W(X) \to R$ which we will write as

$$P = \sum_{w \in W(X)} P(w)w.$$

A formal power series P is called a polynomial $P(w) = 0$ except for a finite number of words $w \in W(X)$. The set of all formal power series with coefficients in R will be denoted $R\langle\langle X\rangle\rangle$ and the set of all polynomials with coefficients in R will be denoted $R\langle X\rangle$.

The set of formal power series over a ring R can be given a ring structure. Indeed, if addition on $R\langle\langle X\rangle\rangle$ is defined coordinate-wise, and multiplication is defined via the rule

$$\left(\sum_{w\in W(X)} P(w)w\right)\cdot\left(\sum_{w\in W(X)} Q(w)w\right) = \sum_{w\in W(X)}\left(\sum_{u,v\in W(X),uv=w} P(u)Q(v)\right)w$$

(notice that for each $w \in W(X)$ there are a finite number of pairs $u, v \in W(X)$ such that $w = uv$), it is elementary to verify that $R\langle\langle X\rangle\rangle$ is a ring. Thus it makes sense to consider the rational closure of $R\langle X\rangle$ inside $R\langle\langle X\rangle\rangle$ which will be denoted $R_{\mathrm{rat}}\langle\langle X\rangle\rangle$.

As with formal power series in commuting variables, there is a notion of an algebraic formal power series in non-commuting variables. The definition of such a formal power series is more technical than in the commutative case and is based on the following definition.

Definition 3.5.8 (Schützenberger)**.** Let $X := \{x_1, \ldots, x_n\}$ be an alphabet and let $Z := \{z_1, \ldots, z_m\}$ be an alphabet disjoint from X. A proper algebraic system over a ring R is a set of equations $z_i = p_i(x_1, \ldots, x_n, z_1, \ldots, z_m)$ for $i \in \{1, \ldots, m\}$ where each p_i is an element of $R\langle X \cup Z\rangle$ that has no constant term nor term of the form αz_j where $\alpha \in R$ and $j \in \{1, \ldots, n\}$.

A solution to a proper algebraic system is an m-tuple $(P_1, \ldots, P_m) \in R\langle\langle X\rangle\rangle^m$ such that $P_j(e) = 0$ and $p_j(x_1, \ldots, x_n, P_1, \ldots, P_m) = P_j$ for all $j \in \{1, \ldots, m\}$.

Definition 3.5.9. A formal power series $P \in R\langle\langle X\rangle\rangle$ is said to be algebraic if $P - P(e)e$ is a component of the solution of a proper algebraic system. The set all algebraic formal power series in $R\langle\langle X\rangle\rangle$ will be denoted by $R_{\mathrm{alg}}\langle\langle X\rangle\rangle$.

In order to prove the assumptions of Lemma 3.5.6 hold in the context of Theorem 3.5.3, the proof of [66, Theorem 2.19(ii)] will be mimicked. To do so, it is necessary to show that a certain formal power series in non-commuting variables is algebraic. The following formula involving traces of words of semicircular variables plays a crucial role.

Lemma 3.5.10 (See [81, Section 3])**.** *Let $n \in \mathbb{N}$, let $S_1, \ldots, S_n$ be freely independent semi-circular variables (with second moments 1), let $\mathcal{A}$ be the $*$-algebra generated by $S_1, \ldots, S_n$,* *let τ be the canonical trace on $\mathcal{A}$, and let $X := \{x_1, \ldots, x_n\}$ be an alphabet. For each $j \in \{1, \ldots, n\}$ and $w \in W(X)$,*

$$\tau(S_j w(S_1, \ldots, S_n)) = \sum_{u,v \in W(X), w = u x_j v} \tau(u(S_1, \ldots, S_n))\tau(v(S_1, \ldots, S_n))$$

where, for a word $w_0 \in W(X)$, $w_0(S_1, \ldots, S_n)$ is the element of $\mathcal{A}$ obtained by substituting S_i for x_i.

Lemma 3.5.11. *With the notation as in Lemma 3.5.10, the formal power series $P_{semi} \in \mathbb{C}\langle\langle X \rangle\rangle$ defined by*

$$P_{semi} := \sum_{w \in W(X)} \tau(w(S_1, \ldots, S_n))w$$

is algebraic.

Proof. By Lemma 3.5.10 we easily obtain that

$$\begin{aligned}
& P_{semi} - e \\
= \;& \textstyle\sum_{j=1}^{n} \sum_{w \in W(X)} \tau(S_j w(S_1, \ldots, S_n)) x_j w \\
= \;& \textstyle\sum_{j=1}^{n} \sum_{w,u,v \in W(X), w = u x_j v} \tau(u(S_1, \ldots, S_n))\tau(v(S_1, \ldots, S_n)) x_j u x_j v \\
= \;& \textstyle\sum_{j=1}^{n} \sum_{u,v \in W(X)} \tau(u(S_1, \ldots, S_n))\tau(v(S_1, \ldots, S_n)) x_j u x_j v \\
= \;& \textstyle\sum_{j=1}^{n} x_j P_{semi} x_j P_{semi}.
\end{aligned}$$

Hence it is elementary to verify that $P_{semi} - e$ is a solution to the proper algebraic system

$$z = \sum_{j=1}^{n} x_j z x_j z + x_j^2 z + x_j z x_j + x_j^2.$$

Thus P_{semi} is algebraic by definition. $\square$

Using Lemma 3.5.11 it is easy to verify the proof of [66, Theorem 2.19(ii)] generalizes enough to complete the proof of Theorem 3.5.3. We will only sketch the changes to the proof of [66, Theorem 2.19(ii)] as it nearly follows verbatim.

Proof of Theorem 3.5.3. Let $\mathfrak{M}$ be the von Neumann algebra generated by $S_1, \ldots, S_n$. By Lemma 3.5.6 it suffices to show that the tracial map on formal power series $Tr_{\mathfrak{M}} : \mathfrak{M}[[\{z\}]] \to \mathbb{C}[[\{z\}]]$ has the property that

$$Tr_{\mathfrak{M}}(\mathcal{A}_{\mathrm{rat}}[[\{z\}]]) \subseteq \mathbb{C}_{\mathrm{alg}}[[\{z\}]].$$

Let $S := \{x_1, \ldots, x_n\}$ be an alphabet. As in the proof of [66, Theorem 2.19(ii)], there is a canonical way to view

$$(\mathbb{C}\langle S\rangle)_{\mathrm{rat}}[[\{z\}]] \subseteq (\mathbb{C}(z))_{\mathrm{rat}}\langle\langle S\rangle\rangle.$$

Consider the injective homomorphisms $\pi : W(S) \to \mathcal{A}$ uniquely defined by $\pi(x_j) = S_j$ for all $j \in \{1, \ldots, n\}$. Clearly π extends to a homomorphism $\pi : \mathbb{C}\langle S\rangle \to \mathcal{A}$ and thus also extends to a homomorphism $\pi : (\mathbb{C}\langle S\rangle)[[\{z\}]] \to \mathcal{A}[[\{z\}]]$ by applying π coordinate-wise.

Let $P \in \mathcal{A}_{\mathrm{rat}}[[\{z\}]]$ be arbitrary. Using algebraic properties, the proof of [66, Theorem 2.19(ii)] implies that

$$P \in \pi\left((\mathbb{C}\langle S\rangle)_{\mathrm{rat}}[[\{z\}]]\right).$$

Choose $\overline{P} \in (\mathbb{C}\langle S\rangle)_{\mathrm{rat}}[[\{z\}]] \subseteq (\mathbb{C}(z))_{\mathrm{rat}}\langle\langle S\rangle\rangle$ such that $\pi(\overline{P}) = P$. Recall that

$$P_{semi} := \sum_{w \in W(S)} \tau(w(S_1, \ldots, S_n))w \in \mathbb{C}_{\mathrm{alg}}\langle\langle S\rangle\rangle \subseteq (\mathbb{C}(z))_{\mathrm{alg}}\langle\langle S\rangle\rangle$$

by Lemma 3.5.11. Hence the Haadamard Product

$$\overline{P} \odot P_{semi} := \sum_{w \in W(S)} \overline{P}(w)P_{semi}(w)w = \sum_{w \in W(S)} \tau(w(S_1, \ldots, S_n))\overline{P}(w)w$$

is an element of $(\mathbb{C}(z))_{\mathrm{alg}}\langle\langle S\rangle\rangle$ by a theorem of Schützenberger from [69].

Since $\overline{P} \odot P_{semi} \in (\mathbb{C}(z))_{\mathrm{alg}}\langle\langle S\rangle\rangle$, if we substitute $1 \in \mathbb{C}$ for every element of S we obtain a well-defined power series in $\mathbb{C}[[\{z\}]]$. Indeed if

$$P = \sum_{m\geq 0} p_m(S_1, \ldots, S_n)z^m$$

for some non-commutative polynomials p_m in n variables, then

$$\overline{P} = \sum_{m\geq 0}(p_m(x_1, \ldots, x_n) + q_m(x_1, \ldots, x_n))z^m$$

for some non-commutative polynomials q_m in n variables such that $q_m(S_1, \ldots, S_n) = 0$. Hence

$$\overline{P} \odot P_{semi} = \sum_{w\in W(S)} \tau(w(S_1, \ldots, S_n)) \left(\sum_{m\geq 0}(coef(p_m, w) + coef(q_m, w))z^m\right) w$$

where $coef(p, w)$ is the element of $\mathbb{C}$ that is the coefficient of w in p. Therefore, by replacing each w with the scalar 1, we obtain

$$\begin{aligned} & \textstyle\sum_{w\in W(S)} \tau(w(S_1, \ldots, S_n)) \left(\sum_{m\geq 0}(coef(p_m, w) + coef(q_m, w))z^m\right) \\ = \;& \textstyle\sum_{m\geq 0} \tau\left(\sum_{w\in W(S)}(coef(p_m, w) + coef(q_m, w))w(S_1, \ldots, S_n)\right) z^m \\ = \;& \textstyle\sum_{m\geq 0} \tau\left(p_m(S_1, \ldots, S_n) + q_m(S_1, \ldots, S_n)\right) z^m \\ = \;& \textstyle\sum_{m\geq 0} \tau(p_m(S_1, \ldots, S_n))z^m = Tr_{\mathfrak{M}}(P) \end{aligned}$$

as desired. Thus the proof of [66, Theorem 2.19(ii)] implies that $Tr_{\mathfrak{M}}(P)$ is an element of $\mathbb{C}_{\mathrm{alg}}[[\{z\}]]$ as desired. □

With the proof of Theorem 3.5.3 complete, we turn our attention to further information that Sauer's results from [66] imply. The main purpose of [66] was to show the rationality and positivity of the Novikov-Shubin invariant for matrices with entries in the group algebra of a virtually free group. In particular, the Novikov-Shubin invariants are well-defined for any finite, tracial von Neumann algebra.

Definition 3.5.12. Let $\mathfrak{M}$ be a finite von Neumann algebra with faithful, normal, tracial state τ. For a positive operator $A \in \mathfrak{M}$ with cumulative spectral distribution F_A, the Novikov-Shubin invariant $\alpha(A) \in [0, \infty] \cup \{\infty^+\}$ of A is defined as

$$\alpha(A) := \begin{cases} \liminf_{t \to 0^+} \frac{\ln(F_A(t) - F_A(0))}{\ln(t)} & \text{if } F_A(t) > F_A(0) \text{ for all } t > 0 \\ \infty^+ & \text{otherwise} \end{cases}.$$

For a positive operator A in a finite von Neumann algebra $\mathfrak{M}$, it is easy to see that $\alpha(A) = \infty^+$ implies that zero is isolated in the spectrum of A. Furthermore, if $\alpha(A) = \lambda \in [0, \infty)$, then $F_A(t) - F_A(0)$ behaves like t^λ as t tends to zero.

The Novikov-Shubin invariants are of interest in the context of Theorem 3.5.3 due to the following result which is directly implied by the proof of [66, Theorem 3.6].

Lemma 3.5.13 (See [66, Theorem 3.6] for a proof)**.** *Let $\mathfrak{M}$ be a finite von Neumann algebra with faithful, normal, tracial state τ. Let $A \in \mathfrak{M}$ be a positive operator and let μ_A be the spectral measure of A. If the Cauchy transform G_{μ_A} is algebraic, then the Novikov-Shubin invariant $\alpha(A)$ is a non-zero rational number or ∞^+.*

The Novikov-Shubin invariants are of interest in terms of determining the decay of the spectral density function at zero due to the following result.

Lemma 3.5.14 (See [44, Theorem 3.14(4)])**.** *Let $\mathfrak{M}$ be a finite von Neumann algebra with faithful, normal, tracial state τ. If $A \in \mathfrak{M}$ is a positive operator and F_A is the spectral density function of A, then*

$$\lim_{\epsilon \to 0} \int_\epsilon^{\|A\|} \frac{1}{t}(F_A(t) - F_A(0))\, dt < \infty$$

provided $\alpha(A) \neq 0$.

Proof. If $\alpha(A) = \infty^+$, then $F_A(t) - F_A(0)$ is a right continuous function bounded that is zero on a neighbourhood of zero. Hence the result follows. If $\alpha(A) \in (0, \infty]$, then it is trivial to verify from Definition 3.5.12 that there exists a $\delta > 0$ and an $\lambda \in (0, \alpha(A))$ such that

$F(t) - F(0) \leq t^{\lambda}$ for all $0 \leq t \leq \delta$. Hence

$$0 \leq \int_0^{\delta} \frac{1}{t}(F_A(t) - F_A(0))\, dt \leq \int_0^{\delta} t^{\lambda-1}\, dt < \infty.$$

Thus the result follows as $F_A(t) - F_A(0)$ is a right continuous function bounded. $\square$

Furthermore, the following result provides information on how to extract information from the conclusion of Lemma 3.5.14 to obtain information about integrating logarithms against the spectral measure.

Lemma 3.5.15 (See [44, Lemma 3.15(1)])**.** *Let $\mathfrak{M}$ be a finite von Neumann algebra with faithful, normal, tracial state τ. If $A \in \mathfrak{M}$ is a positive operator, F_A is the spectral density function of A, and μ_A is the spectral measure of A, then*

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{\|A\|} \frac{1}{t}(F_A(t) - F_A(0))\, dt < \infty$$

if and only if

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{\|A\|} \ln(t)\, d\mu_A(t) > -\infty.$$

Combining the above results, we obtain the following.

Theorem 3.5.16. *Let $n, \ell \in \mathbb{N}$, let $X_1, \ldots, X_n$ be freely independent semicircular variables or freely independent Haar unitaries, and let $\mathcal{A}$ be the $*$-algebra generated by $X_1, \ldots, X_n$. Then*

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{\|A\|} \ln(t)\, d\mu_A(t) > -\infty$$

for all positive $A \in \mathcal{M}_\ell(\mathcal{A}) \setminus \{0\}$. Furthermore, if μ_A does not have an atom at zero (e.g. when $\ell = 1$ by Theorem 3.3.4), then

$$\int_0^{\|A\|} \ln(t)\, d\mu_A(t) > -\infty.$$

CHAPTER 4

Normal Limits of Nilpotent Operators in C*-Algebras

As the question of when an operator is a norm limit of nilpotent operators from $\mathcal{B}(\mathcal{H})$ has been solved, it is natural to ask whether this question can be phrased in the context of an arbitrary C*-algebra. In particular, the above work raises an interesting question: "Given a C*-algebra $\mathfrak{A}$, what is the closure of the nilpotent and quasinilpotent operators of $\mathfrak{A}$?"

Due to the existence and elegance of multiple proofs of Theorem 1.8.3, it is natural rephrase the above question in the context of C*-algebras; that is, "Given an arbitrary C*-algebra $\mathfrak{A}$ and a normal operator $N \in \mathfrak{A}$, can simple conditions be given to determine whether N is a norm limit of nilpotent or quasinilpotent operators from $\mathfrak{A}$?" Although the GNS construction implies $\mathfrak{A}$ can be embedded faithfully into the bounded linear operators on a Hilbert space, Theorem 1.8.3 does not provided the answer to this question as the image of $\mathfrak{A}$ need not contain the necessary nilpotent or quasinilpotent operators. However, a solution to this question can be easily obtained in several particular cases. For example, this question is easily solved for abelian C*-algebras and has a solution in the case of the Calkin algebra as demonstrated in Theorem 1.8.5.

In this chapter, which is based on the author's work from [74] and [75], we will investigate the intersection of the normal operators with the norm closure of the nilpotent operators in various C*-algebras. Since von Neumann algebras behave in a similar manner to $\mathcal{B}(\mathcal{H})$, our first goal is to study this question for von Neumann algebras. Section 4.1 will completely answer this question for type I von Neumann algebras. For a type I_∞ von Neumann algebra as above, the answer is as expected; a normal operator is a limit of nilpotent operators if and only if it is pointwise the limit of nilpotent operators (see Theorem 4.1.5). Section 4.2

will then generalize the above results to the type III von Neumann algebra setting. The conclusions of Theorem 1.8.3 are shown to hold for every type III factor (see Proposition 4.2.1) and, by using the fact that every type III von Neumann algebra is a direct integral of type III factors, it is obtain that a normal operator in a type III von Neumann algebra is a limit of nilpotent operators if and only if it is pointwise the limit of nilpotent operators (see Theorem 4.2.2).

A solution to the above question in a type II von Neumann algebra appears to be a difficult task. Section 4.3 will provide restrictions for when a normal operator can be a limit of nilpotent operators in a C*-algebra with a faithful tracial state. In particular, for a type II_1 von Neumann algebra Corollary 4.3.8 implies that no non-zero self-adjoint operator is a limit of nilpotent operators and Theorem 4.3.13 implies a large class of normal operators cannot by limits of nilpotent operators. However Section 4.4 shows that normal operators in type II_1 factors with spectrum equal to the closed unit disk whose spetral distributions are absolutely continuous and rotationally invariant are limits of nilpotent operators (see Theorem 4.4.6). Section 4.5 will be devoted to the discussion of type II_∞ factors where approximations appear to be simpler and will pose a possible method for obtaining a solution.

There are many other questions related to the nilpotent operators in $\mathcal{B}(\mathcal{H})$. For example, in [29, Corollary 6] Herrero showed that every normal operator in $\mathcal{B}(\mathcal{H})$ was the norm limit of operators that are sums of two nilpotent operators. More recently [48] gives an excellent overview of the results pertaining to the span of nilpotent operators with nilpotency index two. In particular [48, Theorem 5.2] shows that if $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ is a weakly closed, unital C*-algebra with infinite multiplicity (i.e. $\mathfrak{M} \simeq \mathfrak{M}\overline{\otimes}\mathcal{B}(\mathcal{H})$) then every element of $\mathfrak{M}$ is the sum of eight nilpotent operators with nilpotency index at most two.

Section 4.6 will examine when a normal operator is in the closure of the span of the nilpotent operators in a von Neumann algebra. In particular [29, Corollary 6] will be shown to generalize to type I and type III von Neumann algebras as well as type II_∞ factors. This later result is evidence that the question of when normal operators can be limits of nilpotent operators in type II_∞ factors may be the same as in the type I and type III setting.

In [29] Herrero also examined the distance from an arbitrary fixed projection $P \in \mathcal{B}(\mathcal{H})$ to the nilpotent and quasinilpotent operators. In particular [29, Corollary 9] shows that these distances were equal and either 0, 1, or $\frac{1}{2}$. Additional work has been done to obtain bounds for the distance from a rank one projection in the $n \times n$ matrices to the nilpotent $n \times n$ matrices (see [45] and [46]). Section 4.7 will be devoted to extending [29, Corollary 9] to von Neumann algebras. In particular, [29, Corollary 9] generalizes to type I and type III von Neumann algebras as well as type II_∞ von Neumann algebras.

With the study of these problems for von Neumann algebras taken as far as possible, Section 4.8 will examine these questions in the context of unital, simple, purely infinite C*-algebras. As unital, simple, purely infinite C*-algebras have a plethora of projections with particular structure similar to that of von Neumann algebras, a complete solution to our problem will be obtained for said algebras (see Theorem 4.8.6). In particular, as the Calkin algebra is a unital, simple, purely infinite C*-algebra, Section 4.8 will generalize Theorem 1.8.5. Section 4.8 will also examine auxiliary questions such as the closure of the span of nilpotent opertors and the distance from a projection to the nilpotent operators in any unital, simple, purely infinite C*-algebra.

Section 4.9 will examine this question in the context of AFD C*-algebras. AFD C*-algebras are one generalization of finite dimensional C*-algebras and thus it is surprising that the closure of nilpotent operators in said algebras is incredible complex. In particular, Section 4.9 relates the norm closure of the nilpotent operators in AFD C*-algebras to the asymptotic behaviour of nilpotent matrices as the dimension of the matrices are allowed to increase and will demonstrate the existence of AFD C*-algebras with non-zero normal operators in the closure of the nilpotent operators.

Section 4.10 will generalize a construction from [57] to demonstrate that there exists a separable, nuclear, quasidiagonal C*-algebra where every operator is a norm limit of nilpotent operators. The cone of this C*-algebra is then AF-embeddable and it will be demonstrated this cone has also has the property that every operator is a norm limit of nilpotent operators.

4.1 Type I von Neumann Algebras

In this section we will determine when a normal operator in a type I von Neumann algebra with separable predual is a norm limit of nilpotent operators. We will begin with finite type I von Neumann algebras where the results trivially follow from the known results on matrix algebras.

Proposition 4.1.1. *Let $\mathfrak{M}$ be a finite type I von Neumann algebra. Then*

$$\mathrm{Nor}(\mathfrak{M}) \cap \overline{\mathrm{QuasiNil}(\mathfrak{M})} = \{0\}.$$

Proof. Since $\mathfrak{M}$ is a finite type I von Neumann algebra there exist compact Hausdorff spaces X_n such that $\mathfrak{M} \subseteq \prod_{n\geq 1} \mathcal{M}_n(C(X_n))$. Therefore, since an element of $\prod_{n\geq 1} \mathcal{M}_n(C(X_n))$ is quasinilpotent only if each direct summand is quasinilpotent, the proof will be complete by showing

$$\mathrm{Nor}(\mathcal{M}_n(C(X_n))) \cap \overline{\mathrm{QuasiNil}(\mathcal{M}_n(C(X_n)))} = \{0\}$$

for each $n \in \mathbb{N}$. However this result follows from the fact that every normal (quasinilpotent) element of $\mathcal{M}_n(C(X_n))$ must be normal (quasinilpotent) at each element of X_n and the only normal matrix that is a limit of quasinilpotent matrices is the zero matrix. □

To deal with type I_∞ von Neumann algebras with separable predual, we recall that every such algebra has the form $L_\infty(X, \mathcal{B}(\mathcal{H}))$ for some Radon measure space (X, μ). For a normal operator $N \in L_\infty(X, \mathcal{B}(\mathcal{H}))$ to be a limit of nilpotent operators, it is clear that N must be the pointwise limit of nilpotent operators almost everywhere. The difficultly in the converse lies in the fact that the integral of nilpotent operators need not be nilpotent if the degrees of nilpotency are unbounded. This issue will be resolved by Lemma 4.1.4 which was motivated by [26]. We will begin with the following useful observation that is implied by [26].

Lemma 4.1.2. *Let $D \in \mathrm{Nor}(\mathcal{B}(\mathcal{H}))$ be such that $\sigma(D) = \{\lambda_0, \lambda_1, \ldots, \lambda_k\}$ where $\lambda_0 = 0$ and*

$\lambda_i \neq \lambda_j$ if $i \neq j$. If the essential spectrum of D agrees with the spectrum of D then

$$\mathrm{dist}(D, \mathrm{Nil}(\mathcal{B}(\mathcal{H}))) \leq \frac{1}{2} \min_{T \in \mathcal{T}} \max_{e \in \mathcal{E}(T)} \mathrm{length}(e)$$

where $\mathcal{T}$ is the set of all trees with vertices $\{\lambda_0, \lambda_1, \ldots, \lambda_k\}$ and straight lines for edges, $\mathcal{E}(T)$ is the set of edges of a tree $T \in \mathcal{T}$, and $\mathrm{length}(e)$ *is the Euclidean length of a straight line e connecting λ_i to λ_j.*

Proof. If $D = 0$, the result is trivial. Otherwise let

$$\delta := \min_{T \in \mathcal{T}} \max_{e \in \mathcal{E}(T)} \mathrm{length}(e) > 0$$

and fix a $T_0 \in \mathcal{T}$ that obtains this minimum. Note that there exists an $N \in \mathrm{Nor}(\mathcal{B}(\mathcal{H}))$ with spectrum equal to T_0. Since T_0 is connected and contains zero, $N \in \overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}$ by Theorem 1.8.3. By the Spectral Theorem for Normal Operators there exists a unitary $U \in \mathcal{B}(\mathcal{H})$ such that $\|D - UNU^*\| \leq \frac{1}{2}\delta$. Hence, as $N \in \overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}$, the result follows. $\square$

Note the following interesting result (which is in the spirit of [32, Example 1.5]) implies the inequality in Lemma 4.1.2 is an equality when D is positive.

Lemma 4.1.3. *Let $\mathfrak{A}$ be a C^*-algebra and let $A \in \mathfrak{A}_+$. Suppose*

$$\sigma(A) = \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k\}.$$

Then

$$\mathrm{dist}(A, \mathrm{QuasiNil}(\mathfrak{A})) \geq \frac{1}{2} \max_{1 \leq i \leq k} |\lambda_i - \lambda_{i-1}|.$$

Proof. Choose $x_0, y_0 \in \sigma(A)$ such that $x_0 < y_0$ and

$$|x_0 - y_0| = \max_{1 \leq i \leq k} |\lambda_i - \lambda_{i-1}|.$$

Let $\sigma_0 := \{z \in \sigma(A) \mid z \leq x_0\}$ and let $\sigma_1 := \{z \in \sigma(A) \mid z \geq y_0\}$. Thus σ_0 and σ_1 are non-empty, disjoint, compact subsets of $\sigma(A)$ such that $\sigma(A) = \sigma_0 \cup \sigma_1$. Let

$$\Omega := \left\{ z \in \mathbb{C} \mid |z - \lambda_k| < \lambda_k - y_0 + \frac{1}{2}|x_0 - y_0| \right\}.$$

It is clear that $0 \notin \Omega$, $\sigma_1 \subseteq \Omega$, $\sigma_0 \cap \overline{\Omega} = \emptyset$, and

$$\inf\{|\lambda_j - z| \mid z \in \partial\Omega, j \in \{0, 1, \ldots, k\}\} = \frac{1}{2}|x_0 - y_0|.$$

By [32, Theorem 1.1] (which is an application of the Analytic Functional Calculus for Banach Algebras) if $M \in \mathfrak{A}$ is such that

$$\|A - M\| < \inf\left\{ \left\|(\lambda I - A)^{-1}\right\|^{-1} \mid \lambda \in \partial\Omega \right\}$$

then $\sigma(M) \cap \Omega \neq \emptyset$. Since A is self-adjoint

$$\inf\left\{ \left\|(\lambda I - A)^{-1}\right\|^{-1} \mid \lambda \in \partial\Omega \right\} = \inf\{|\lambda_j - z| \mid z \in \partial\Omega, j \in \{0, 1, \ldots, k\}\}.$$

Hence, if $M \in \mathfrak{A}$ is such that $\|A - M\| < \frac{1}{2}|x_0 - y_0|$ then $\sigma(M) \cap \Omega \neq \emptyset$. As $0 \notin \Omega$ and the spectrum of any quasinilpotent operator is $\{0\}$, the result follows. $\square$

Using Lemma 4.1.2 and an idea motivated by [26], we obtain the following result that will enable us to bypass the problem of unbounded nilpotency degrees.

Lemma 4.1.4. *Let $\{T_n\}_{n\geq 1} \subseteq \mathrm{Nor}(\mathcal{B}(\mathcal{H}))$ be a bounded set such that $\sigma(T_n)$ is connected and contains zero for all $n \in \mathbb{N}$. Then for every $\epsilon > 0$ there exists a $q \in \mathbb{N}$ and $\{M_n\}_{n\geq 1} \subseteq \mathrm{Nil}(\mathcal{B}(\mathcal{H}))$ such that $\|T_n - M_n\| \leq \epsilon$ and $M_n^q = 0$ for all $n \in \mathbb{N}$.*

Proof. Without loss of generality $\|T_n\| \leq 1$ for all $n \in \mathbb{N}$ and $\epsilon = \frac{1}{2^m}$ for some $m \in \mathbb{N}$. Since

$\|T_n\| \leq 1$, $\sigma(T_n) \subseteq \overline{\mathbb{D}}$ for all $n \in \mathbb{N}$. Define

$$C_{k,\ell} := \left[\frac{2k-1}{2^{m+1}}, \frac{2k+1}{2^{m+1}}\right) + i\left[\frac{2\ell-1}{2^{m+1}}, \frac{2\ell+1}{2^{m+1}}\right) \subseteq \mathbb{C}$$

for all $k, \ell \in \{-2^m, -2^m+1, \ldots, 2^m\}$ (note $\bigcup_{k,\ell=-2^m}^{2^m} C_{k,\ell}$ contains the closed unit square) and for each

$$Y \subseteq \{-2^m, -2^m+1, \ldots, 2^m\} \times \{-2^m, -2^m+1, \ldots, 2^m\}$$

define

$$X_Y := \{T_n \mid \chi_{C_{k,\ell}}(T_n) \neq 0 \text{ if and only if } (k,\ell) \in Y\}$$

where $\chi_Z(T_n)$ is the spectral projection of T_n onto the subset Z. Note that $\bigcup_Y X_Y = \{T_n\}_{n\geq 1}$ and $X_Y = \emptyset$ if $(0,0) \notin Y$ or $\bigcup_{(k,\ell)\in Y} C_{k,\ell}$ is disconnected.

Since the number of possible sets Y is finite, it suffices to show that for each Y there exists a $q \in \mathbb{N}$ such that for every $T_n \in X_Y$ there exists an $M_n \in \mathrm{Nil}(\mathcal{B}(\mathcal{H}))$ such that $\|T_n - M_n\| \leq 3\epsilon$ and $M_n^q = 0$. Fix Y such that $(0,0) \in Y$ and $\bigcup_{(k,\ell)\in Y} C_{k,\ell}$ is connected. For each $(k,\ell) \in Y$ let $z_{k,\ell} \in C_{k,\ell}$ to be the centre of $C_{k,\ell}$ (so $z_{0,0} = 0$). Let D_Y be a diagonal operator whose spectrum and essential spectrum is $\{z_{k,\ell} \mid (k,\ell) \in Y\}$. By the Spectral Theorem for Normal Operators, for each $T_n \in X_Y$ there exists a unitary $U_n \in \mathcal{B}(\mathcal{H})$ such that $\|T_n - U_n D_Y U_n^*\| \leq 2\epsilon$. Since D_Y is within ϵ of an element of $\mathrm{Nil}(\mathcal{B}(\mathcal{H}))$ by Lemma 4.1.2, the result follows. □

Theorem 4.1.5. *Let $\mathfrak{M} := L_\infty(X, \mathcal{B}(\mathcal{H}))$ where (X,μ) is a Radon measure space. If $f \in \mathrm{Nor}(\mathfrak{M})$ then the following are equivalent:*

1. $f \in \overline{\mathrm{Nil}(\mathfrak{M})}$.
2. $f \in \overline{\mathrm{QuasiNil}(\mathfrak{M})}$.
3. $f(x) \in \overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))}$ *μ-almost everywhere.*
4. $f(x) \in \overline{\mathrm{QuasiNil}(\mathcal{B}(\mathcal{H}))}$ *μ-almost everywhere.*

5. *$\sigma(f(x))$ is connected and contains zero μ-almost everywhere.*

Proof. The equivalence of (3), (4), and (5) is clear from Theorem 1.8.3. Clearly (1) implies (2). To see that (2) implies (5), suppose that $f \in \overline{\mathrm{QuasiNil}(\mathfrak{M})}$. If $M \in \mathrm{QuasiNil}(\mathfrak{M})$ then $M(x) \in \mathrm{QuasiNil}(\mathcal{B}(\mathcal{H}))$ for almost every $x \in X$ by the spectral radius formula. Therefore f is almost everywhere the pointwise limit of elements of $\mathrm{QuasiNil}(\mathcal{B}(\mathcal{H}))$ and thus $\sigma(f(x))$ is connected and contains zero for almost every $x \in X$ by Lemma 1.8.4.

To see that (5) implies (1), suppose $\sigma(f(x))$ is connected and contains zero for almost every $x \in X$. Let $\epsilon > 0$. Note we may assume without loss of generality that $f(x)$ is normal for every $x \in X$, $\sup_{x\in X} \|f(x)\| < \infty$, and $\sigma(f(x))$ is connected and contains zero for every $x \in X$. Since f is measurable, the range of f is separable and $x \mapsto \|f(x)\|$ is a measurable function. Thus there exist $\{T_n\}_{n\geq 1} \subseteq f(X)$ and disjoint measurable subsets $\{E_n\}_{n\geq 1} \subseteq X$ such that if

$$h := \sum_{n\geq 1} T_n \chi_{E_n} \in \mathfrak{M}$$

then $\|h - f\| \leq \epsilon$. Since $T_n \in f(X)$ for all $n \in \mathbb{N}$, $\{T_n\}_{n\geq 1}$ is a bounded set of normal operators such that $\sigma(T_n)$ is connected and contains zero for all $n \in \mathbb{N}$. By Lemma 4.1.4 there exist $\{M_n\}_{n\geq 1} \subseteq \mathrm{Nil}(\mathcal{B}(\mathcal{H}))$ and a $q \in \mathbb{N}$ such that $\|T_n - M_n\| \leq \epsilon$ and $M_n^q = 0$ for all $n \in \mathbb{N}$. Let

$$g := \sum_{n\geq 1} M_n \chi_{E_n}.$$

Then $g \in \mathfrak{M}$, $\|f - g\| \leq 2\epsilon$, and $g^q = 0$ so $g \in \mathrm{Nil}(\mathfrak{M})$. Hence $f \in \overline{\mathrm{Nil}(\mathfrak{M})}$. □

Thus we have completely characterized when a normal operator is a limit of nilpotent operators in a type I von Neumann algebra with separable predual:

Corollary 4.1.6. *Suppose*

$$\mathfrak{M} = L_\infty(X, \mathcal{B}(\mathcal{H})) \oplus \left(\prod_{n\geq 1} \mathcal{M}_n(\mathbb{C}) \overline{\otimes} L_\infty(X_n) \right)$$

where (X, μ) and (X_n, μ_n) are Radon measure spaces. Let $P \in \mathfrak{M}$ be the (central) projection onto $L_\infty(X, \mathcal{B}(\mathcal{H}))$ and let $N \in \mathrm{Nor}(\mathfrak{M})$. Then the following are equivalent:

1. *$N \in \overline{\mathrm{Nil}(\mathfrak{M})}$.*
2. *$N \in \overline{\mathrm{QuasiNil}(\mathfrak{M})}$.*
3. *$PN = N$ and $\sigma(N(x))$ is connected and contains zero for almost every $x \in X$.*

4.2 Type III von Neumann Algebras

In this section we will determine when a normal operator in a type III von Neumann algebra with separable predual is a norm limit of nilpotent operators. Our first result is the following generalization of Theorem 1.8.3 to type III factors.

Proposition 4.2.1. *Let $\mathfrak{M}$ be a type III factor with separable predual and let $N \in \mathrm{Nor}(\mathfrak{M})$. Then the following are equivalent:*

1. *$N \in \overline{\mathrm{Nil}(\mathfrak{M})}$.*
2. *$N \in \overline{\mathrm{QuasiNil}(\mathfrak{M})}$.*
3. *$\sigma(N)$ is connected and contains zero.*

Proof. Clearly (1) implies (2) and (2) implies (3) is trivial by Lemma 1.8.4. Suppose $N \in \mathrm{Nor}(\mathfrak{M})$ is such that $\sigma(N)$ is connected and contains zero. Since $\mathfrak{M}$ is a type III factor with separable predual, there exists a unital copy of $\mathcal{B}(\mathcal{H})$ inside $\mathfrak{M}$. Choose a normal operator N_0 inside this copy of $\mathcal{B}(\mathcal{H})$ such that $\sigma(N_0) = \sigma(N)$. Therefore $N_0 \in \overline{\mathrm{Nil}(\mathcal{B}(\mathcal{H}))} \subseteq \overline{\mathrm{Nil}(\mathfrak{M})}$ by Theorem 1.8.3.

Since $\sigma(N) = \sigma(N_0)$ and since $\mathfrak{M}$ is a type III factor, N and N_0 are approximately unitarily equivalent in $\mathfrak{M}$ (see [70]). Thus $N \in \overline{\mathrm{Nil}(\mathfrak{M})}$ since $N_0 \in \overline{\mathrm{Nil}(\mathfrak{M})}$. □

To determine when a normal operator in a type III von Neumann algebra is a limit of nilpotent operators, we will use the fact that every type III von Neumann algebra is a direct integral of type III factors (for a reminder on direct integrals of von Neumann algebras, we refer the reader to [35]). This causes greater difficulty than seen in the proof of Theorem 4.1.5 as the type III factors are allowed to vary over the direct integral. The idea of the proof is similar to the proof of Proposition 4.2.1 except that the copy of $\mathcal{B}(\mathcal{H})$ inside our von Neumann algebra must be done 'in a measurable way'.

Theorem 4.2.2. *Let $\mathfrak{M}$ be a type III von Neumann algebra with separable predual. Choose a locally compact, complete, separable, metrizable measure space (X, μ) and a collection of type III factors $(\mathfrak{M}_x)_{x\in X}$ with separable predual such that $\mathfrak{M}$ is a direct integral of $(\mathfrak{M}_x)_{x\in X}$. If $N \in \mathrm{Nor}(\mathfrak{M})$ we may write $N = \int_X^\oplus N_x \, d\mu(x)$ where $N_x \in \mathfrak{M}_x$ is a normal operator μ-almost everywhere. Then the following are equivalent:*

1. *$N \in \overline{\mathrm{Nil}(\mathfrak{M})}$.*
2. *$N \in \overline{\mathrm{QuasiNil}(\mathfrak{M})}$.*
3. *$N_x \in \overline{\mathrm{Nil}(\mathfrak{M}_x)}$ μ-almost everywhere.*
4. *$N_x \in \overline{\mathrm{QuasiNil}(\mathfrak{M}_x)}$ μ-almost everywhere.*
5. *$\sigma(N_x)$ is connected and contains zero μ-almost everywhere.*

Proof. Clearly (3), (4), and (5) are equivalent by Proposition 4.2.1 and clearly (1) implies (2). To see that (2) implies (5), suppose that $N \in \overline{\mathrm{QuasiNil}(\mathfrak{M})}$. If $M \in \mathrm{QuasiNil}(\mathfrak{M})$ then $M_x \in \mathrm{QuasiNil}(\mathfrak{M}_x)$ for almost every $x \in X$ by the spectral radius formula. Therefore N_x is almost everywhere the pointwise limit of elements of $\mathrm{QuasiNil}(\mathfrak{M}_x)$ and thus $\sigma(N_x)$ is connected and contains zero for almost every $x \in X$ by Lemma 1.8.4.

To see that (5) implies (1), suppose $N \in \mathrm{Nor}(\mathfrak{M})$ is such that $\sigma(N_x)$ is connected and contains zero μ-almost everywhere. Thus we can assume that N_x is normal, $\|N_x\| \leq \|N\|$, $0 \in \sigma(N_x)$, and $\sigma(N_x)$ is connected for all $x \in X$.

Unfortunately greater difficult arises in the following proof than in the proof of Theorem 4.1.5 as we need to deal with direct integrals and the fact that the type III factors $\{\mathfrak{M}_x \mid x \in$

$X\}$ may differ. Our hope is to show for any $\epsilon > 0$ there exists a $q \in \mathbb{N}$ and $M_x \in \mathrm{Nil}(\mathfrak{M}_x)$ for all $x \in X$ such that $\|N_x - M_x\| \leq 2\epsilon$ for all $x \in X$ and $M_x^q = 0$ for all $x \in X$. If $(x \mapsto M_x)$ is measurable then $\int_X^{\oplus} M_x \, d\mu(x)$ will be a nilpotent element of $\mathfrak{M}$ within 2ϵ of N.

To get $(x \mapsto M_x)$ to be measurable we need to modify the proof of Lemma 4.1.4. Without loss of generality we may assume $\|N_x\| \leq 1$ for all $x \in X$ and $\epsilon = \frac{1}{2^m}$ for some $m \in \mathbb{N}$. Let $C_{k,\ell}$ be as in Lemma 4.1.4 and, for each subset $Y \subseteq \{-2^m, -2^m+1, \ldots, 2^m\} \times \{-2^m, -2^m+1, \ldots, 2^m\}$ define

$$X_Y := \{x \in X \mid \chi_{C_{k,\ell}}(N_x) \neq 0 \text{ if and only if } (k,\ell) \in Y\} \subseteq X$$

where $\chi_Z(N_x)$ is the spectral projection of N_x onto the subset Z.

Since $f(N) = \int_X^{\oplus} f(N_x) \, d\mu(x)$ for all bounded Borel functions f on the spectrum of N, each X_Y is a measurable subset of X. Therefore, since the number of possible sets Y is finite and the sets X_Y are disjoint, it suffices to show that for each Y there exists a nilpotent operator M_Y in $\mathfrak{M}$ such that the support of M_Y is X_Y and N is within 2ϵ of M_Y when restricted to X_Y. Fix a potential Y. Note that $\bigcup_Y X_Y = X$ and $X_Y = \emptyset$ if $(0,0) \notin Y$ or $\bigcup_{(k,\ell)\in Y} C_{k,\ell}$ is disconnected. Thus we may assume that $\bigcup_{(k,\ell)\in Y} C_{k,\ell}$ is connected, $(0,0) \in Y$, and $X = X_Y$ when performing our approximations.

For each $x \in X$ and $(k,\ell) \in Y$ let $P_{x,k,\ell} := \chi_{C_{k,\ell}}(N_x)$. Note the maps $(x \mapsto P_{x,k,\ell}) = \chi_{C_{k,\ell}}(N)$ are elements of $\mathfrak{M}$ for all $(k,\ell) \in Y$. We claim that $\{(x \mapsto P_{x,k,\ell})\}_{(k,\ell)\in Y}$ are equivalent in $\mathfrak{M}$. To see this, we notice by construction that $\{P_{x,k,\ell}\}_{(k,\ell)\in Y}$ are orthogonal equivalent projections in $\mathfrak{M}_x$ almost everywhere. However, in a type III von Neumann algebra, two projections are equivalent if and only if they have the same central support. By [35, Lemma 14.1.20.v], the central support in $\mathfrak{M}$ is the direct integral of the central supports in $\mathfrak{M}_x$ and thus the claim is complete.

Recall $(0,0) \in Y$. Since $\mathfrak{M}$ is a type III von Neumann algebra, every non-zero projection of $\mathfrak{M}$ is properly infinite. Thus, as $(x \mapsto P_{x,0,0})$ is non-zero almost everywhere, $(x \mapsto P_{x,0,0})$ is a properly infinite projection. Thus there exist equivalent, pairwise orthogonal, measurable projections

$$\{(x \mapsto P_{x,0,0,w})\}_{w\geq 1}$$

such that

$$(x \mapsto P_{x,0,0}) = \sum_{w\geq 1}(x \mapsto P_{x,0,0,w}).$$

Since $\{(x \mapsto P_{x,k,\ell})\}_{(k,\ell)\in Y}$ are equivalent in $\mathfrak{M}$, by using $\{(x \mapsto P_{x,0,0,w})\}_{w\geq 1}$ there exist equivalent, pairwise orthogonal, measurable projections

$$\{\{(x \mapsto P_{x,k,\ell,w})\}_{(k,\ell)\in Y}\}_{w\geq 1}$$

such that

$$(x \mapsto P_{x,k,\ell}) = \sum_{w\geq 1}(x \mapsto P_{x,k,\ell,w})$$

for all $(k, \ell) \in Y$.

For each $(k, \ell) \in Y$ let $z_{k,\ell} \in C_{k,\ell}$ be the centre of $C_{k,\ell}$ (so $z_{0,0} = 0$). Let

$$T := \left(x \mapsto \sum_{(k,\ell)\in Y}\sum_{w\geq 1} z_{k,\ell}P_{x,k,\ell,w}\right)$$

which is a measurable and decomposable operator in $\mathfrak{M}$. Clearly $\|T - N\| < \epsilon$ by construction.

To construct our nilpotent operator, let D be the diagonal operator on a separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{\{e_{k,\ell,w}\}_{(k,\ell)\in Y}\}_{w\geq 1}$ such that $D(e_{k,\ell,w}) = z_{k,\ell}e_{k,\ell,w}$ for all $w \in \mathbb{N}$ and $(k, \ell) \in Y$. By Lemma 4.1.2 there exists an $M' \in \mathrm{Nil}(\mathcal{B}(\mathcal{H}))$ such that $\|D - M'\| \leq \epsilon$. For each $w_1, w_2 \in \mathbb{N}$ and $(k_1, \ell_1), (k_2, \ell_2) \in Y$ let

$$a_{(k_1,\ell_1,w_1),(k_2,\ell_2,w_2)} := \langle M'e_{k_2,\ell_2,w_2}, e_{k_1,\ell_1,w_1}\rangle \in \mathbb{C}$$

and let $\left(x \mapsto V_{x,(k_1,\ell_1,w_1),(k_2,\ell_2,w_2)}\right) \in \mathfrak{M}$ be the partial isometry such that

$$\left(x \mapsto V_{x,(k_1,\ell_1,w_1),(k_2,\ell_2,w_2)}\right)\left(x \mapsto V_{x,(k_1,\ell_1,w_1),(k_2,\ell_2,w_2)}\right)^* = (x \mapsto P_{x,k_1,\ell_1,w_1})$$

and

$$\left(x \mapsto V_{x,(k_1,\ell_1,w_1),(k_2,\ell_2,w_2)}\right)^*\left(x \mapsto V_{x,(k_1,\ell_1,w_1),(k_2,\ell_2,w_2)}\right) = (x \mapsto P_{x,k_2,\ell_2,w_2}).$$

Finally let

$$M := \left(x \mapsto \sum_{w_1,w_2\geq 1}\sum_{(k_1,\ell_1),(k_2,\ell_2)\in Y} a_{(k_1,\ell_1,w_1),(k_2,\ell_2,w_2)} V_{x,(k_1,\ell_1,w_1),(k_2,\ell_2,w_2)}\right)$$

which is a measurable and decomposable operator in $\mathfrak{M}$. Moreover M is also a nilpotent operator as, for each $x \in X$, M_x is a copy of M'. Since $\|D - M'\| \leq \epsilon$, $\|(T)_x - (M)_x\| \leq \epsilon$ for all $x \in X$. Whence $\|T - M\| \leq \epsilon$ so $\|N - M\| \leq 2\epsilon$ thus completing the proof. □

4.3 Restrictions by Tracial States

This section will demonstrate how tracial states on C*-algebras provide restrictions to the spectra of normal operators which may be in the closure of the quasinilpotent operators. In particular, these restrictions directly apply to type II_1 von Neumann algebras and thus prevent an elegant classification of which normal operators are norm limits of nilpotent or quasinilpotent operators. Note that the following result that enables us to create additional elements of $\overline{\mathrm{QuasiNil}(\mathfrak{A})}$.

Lemma 4.3.1. *Let $\mathfrak{A}$ be a C*-algebra and let $T \in \overline{\mathrm{QuasiNil}(\mathfrak{A})}$. Then*

$$\overline{alg(T)}, \overline{alg(T^*)} \subseteq \overline{\mathrm{QuasiNil}(\mathfrak{A})}.$$

Similarly if $T \in \overline{\mathrm{Nil}(\mathfrak{A})}$ then $\overline{alg(T)}, \overline{alg(T^*)} \subseteq \overline{\mathrm{Nil}(\mathfrak{A})}$.

Proof. It is clear that the adjoint of an element of $\mathrm{QuasiNil}(\mathfrak{A})$ (respectively $\mathrm{Nil}(\mathfrak{A})$) is an element of QuasiNil($\mathfrak{A}$) (respectively Nil($\mathfrak{A}$)). Moreover, if p is a polynomial such that $p(0) = 0$ then if $M \in \mathrm{QuasiNil}(\mathfrak{A})$ (respectively $M \in \mathrm{Nil}(\mathfrak{A})$) then $p(M) \in \mathrm{QuasiNil}(\mathfrak{A})$ (respectively $p(M) \in \mathrm{Nil}(\mathfrak{A})$). □

Now we shall introduce the main tool for the results of this section.

Definition 4.3.2. Let $\mathfrak{A}$ be a C*-algebra. A tracial state τ on $\mathfrak{A}$ is a positive linear functional of norm one such that $\tau(AB) = \tau(BA)$ for all $A, B \in \mathfrak{A}$.

There are several examples of C*-algebras with tracial states. For example, finite dimensional C*-algebras, the reduced group C*-algebra of a countable discrete group, abelian C*-algebras, type II_1 von Neumann algebras, and uniformly hyperfinite C*-algebras all have tracial states. The reason for examining C*-algebras with tracial states is the following.

Lemma 4.3.3. *Let $\mathfrak{A}$ be a C*-algebra and let τ be a tracial state on $\mathfrak{A}$. Then $\tau(M) = 0$ whenever $M \in \overline{\mathrm{QuasiNil}(\mathfrak{A})}$.*

Proof. By the continuity of τ we may assume that $M \in \mathrm{QuasiNil}(\mathfrak{A})$. If $\mathfrak{A}$ is not unital, the linear map $\tilde{\tau} : \tilde{\mathfrak{A}} \to \mathbb{C}$ on the unitization $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$ defined by

$$\tilde{\tau}(\lambda I_{\tilde{\mathfrak{A}}} + A) = \lambda + \tau(A)$$

for all $A \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$ is easily seen to be a tracial state on $\tilde{\mathfrak{A}}$ that extends τ. Hence we may assume that $\mathfrak{A}$ is unital.

By Rota's Theorem (see [63] and note the proof holds in a general C*-algebra; alternatively see [49, Proposition 4] or [51, Exercise 9.15] for another proof) and the fact that $\sigma(M) = \{0\}$, for all $\epsilon > 0$ there exists a $B_\epsilon \in \mathfrak{A}^{-1}$ such that $\|B_\epsilon^{-1} M B_\epsilon\| < \epsilon$. Therefore

$$|\tau(M)| = |\tau(B_\epsilon^{-1} M B_\epsilon)| \leq \left\|B_\epsilon^{-1} M B_\epsilon\right\| < \epsilon.$$

Thus, as this holds for all $\epsilon > 0$, $\tau(M) = 0$. □

Corollary 4.3.4. *Let $\mathfrak{A}$ be a C*-algebra and let τ be a tracial state on $\mathfrak{A}$. Then*

$$\mathrm{dist}(T, \mathrm{QuasiNil}(\mathfrak{A})) \geq |\tau(T)|$$

for all $T \in \mathfrak{A}$.

As Lemma 4.3.3 proves that the closure of the quasinilpotent operators in a C*-algebra are in the kernel of every trace on the C*-algebra, it is useful to examine C*-algebras with several tracial states.

Definition 4.3.5. Let $\mathfrak{A}$ be a C*-algebra. A tracial state τ on $\mathfrak{A}$ is said to be faithful if $\tau(A) > 0$ for all $A \in \mathfrak{A}_+ \setminus \{0\}$.

A C*-algebra $\mathfrak{A}$ is said to have a separating family of tracial states if for every $A \in \mathfrak{A}_+ \setminus \{0\}$ there exists a tracial state on $\mathfrak{A}$ such that $\tau(A) > 0$.

For example, finite dimensional C*-algebras, the reduced group C*-algebra of a countable discrete group, abelian C*-algebras, type II_1 factors, and uniformly hyperfinite C*-algebras all have faithful tracial states. Every type II_1 von Neumann algebra has a separating family of tracial states.

Using Lemma 4.3.3 we easily obtain the following restriction.

Proposition 4.3.6. *Let $\mathfrak{A}$ be a C*-algebra with a separating family of tracial states and let $N \in \mathrm{Nor}(\mathfrak{A})$ be such that there exists a polynomial p with $p(0) = 0$, $p(N) \neq 0$, and $p(\sigma(N)) \subseteq [0, \infty)$. Then $N \notin \overline{\mathrm{QuasiNil}(\mathfrak{A})}$. Thus $\mathfrak{A}_{\mathrm{sa}} \cap \overline{\mathrm{QuasiNil}(\mathfrak{A})} = \{0\}$.*

Proof. Suppose there exists an $N \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{QuasiNil}(\mathfrak{A})}$ and a polynomial p such that $p(0) = 0$, $p(N) \neq 0$, and $p(\sigma(N)) \subseteq [0, \infty)$. Then $p(N) \in \mathfrak{A}_+ \cap \overline{\mathrm{QuasiNil}(\mathfrak{A})}$ by Lemma 4.3.1. Since $p(N) \neq 0$, the assumptions on $\mathfrak{A}$ imply that there exists a tracial state τ on $\mathfrak{A}$ such that $\tau(p(N)) > 0$ which contradicts Lemma 4.3.3. □

Proposition 4.3.6 easily extends using the following well-known result.

Theorem 4.3.7 (Mergelyan's Theorem; see [64, Theorem 20.5])**.** *Let K be a compact set in the complex plane such that $\mathbb{C} \setminus K$ is connected. If f is a continuous function on K which is holomorphic on the interior of K then f can be uniformly approximated by polynomials on K.*

Corollary 4.3.8. *Let $\mathfrak{A}$ be a C*-algebra and let $N \in \mathrm{Nor}(\mathfrak{A})\setminus\{0\}$ be such that $int(\sigma(N)) = \emptyset$ and $\mathbb{C} \setminus \sigma(N)$ is connected. Then the following are true:*

1. *If $\mathfrak{A}_+ \cap \overline{\mathrm{QuasiNil}(\mathfrak{A})} = \{0\}$ then $N \notin \overline{\mathrm{QuasiNil}(\mathfrak{A})}$.*
2. *If $\mathfrak{A}_+ \cap \overline{\mathrm{Nil}(\mathfrak{A})} = \{0\}$ then $N \notin \overline{\mathrm{Nil}(\mathfrak{A})}$.*

Consequently, if $\mathfrak{A}$ is a C-algebra with a separating family of tracial states then $N \notin \overline{\mathrm{QuasiNil}(\mathfrak{A})}$.*

Proof. Suppose $\mathfrak{A}_+ \cap \overline{\mathrm{QuasiNil}(\mathfrak{A})} = \{0\}$ and $N \in \overline{\mathrm{QuasiNil}(\mathfrak{A})}$. Then $0 \in \sigma(N)$ by Lemma 1.8.4. Define $f \in C(\sigma(N))$ by $f(z) = |z|$ for all $z \in \sigma(N)$. Since $f(0) = 0$, Mergelyan's Theorem implies f is the uniform limit on $\sigma(N)$ of polynomials that vanish at zero and thus $f(N) \in \overline{\mathrm{QuasiNil}(\mathfrak{A})}$ by Lemma 4.3.1. Since $f(N) \in \mathfrak{A}_+$, $f(N) = 0$ so $N = 0$ as claimed.

The proof of the second claim is nearly identical to the first and the final claim follows from Proposition 4.3.6. ☐

Corollary 4.3.8 is the strongest restriction that has been obtained on the spectrum of a normal operator in the closure of the quasinilpotent operators of a C*-algebra with a separating family of tracial states. To obtain stronger restrictions, we turn our attention to C*-algebras with faithful tracial states.

Remarks 4.3.9. Let $\mathfrak{A}$ be a unital C*-algebra, let τ be a tracial state on $\mathfrak{A}$, and let $N \in \mathrm{Nor}(\mathfrak{A})$. Consider the C*-algebra $\mathfrak{C} := C^*(1, N, N^*)$ and $\tau|_{\mathfrak{C}}$. Then $\tau|_{\mathfrak{C}}$ is an element of the dual space of $\mathfrak{C}$ and thus can be associated with a complex, regular, Borel measure μ on $\sigma(N)$. Thus we view $\tau|_{\mathbb{C}}(f(N)) = \int_{\sigma(N)} f\, d\mu$ for $f \in C(\sigma(N))$. Moreover, since τ is positive and unital, μ is a probability measure on $\sigma(N)$. If τ is faithful then $\mu(U) > 0$ for all non-empty relatively open sets U in $\sigma(N)$.

If $N \in \overline{\mathrm{QuasiNil}(\mathfrak{A})}$ then $\tau(p(N)) = \tau(p(N^*)) = 0$ for all polynomials p that vanish at zero by Lemma 4.3.1 and Lemma 4.3.3. Therefore, since $N \simeq z$ and $N^* \simeq \overline{z}$, $\int_{\sigma(N)} z^n \, d\mu = 0$ and $\int_{\sigma(N)} \overline{z}^n \, d\mu = 0$ for all $n \in \mathbb{N}$.

It is therefore of interest to our main problem to determine the supports of all probability measures μ with compact support such that $\int z^n \, d\mu = 0$ and $\int \overline{z}^n \, d\mu = 0$ for all $n \in \mathbb{N}$. Unfortunately we have not been able to classify the supports of such measures. However, some progress has been made that enables us to improve Corollary 4.3.8 in the case our C*-algebra has a faithful tracial state.

To begin our discussion of normal limits of quasinilpotent operators in C*-algebras with faithful tracial states, we make the following definition.

Definition 4.3.10. A subset $X \subseteq \mathbb{C}$ is said to be a non-quasinilpotent spectrum if for every C*-algebra $\mathfrak{A}$ with a faithful tracial state, $N \notin \overline{\mathrm{QuasiNil}(\mathfrak{A})}$ whenever $N \in \mathrm{Nor}(\mathfrak{A}) \setminus \{0\}$ is such that $\sigma(N) \subseteq X$.

It is clear if $X \subseteq \mathbb{C} \setminus \{0\}$ then X is a non-quasinilpotent spectrum by Lemma 1.8.4. Moreover a subset of a non-quasinilpotent spectrum is a non-quasinilpotent spectrum and Corollary 4.3.8 provides some examples of non-quasinilpotent spectra. In addition, we have the following.

Lemma 4.3.11. *If X is a non-quasinilpotent spectrum then for every $r, \theta \in \mathbb{R}$ the set $re^{i\theta}X$ is a non-quasinilpotent spectrum. Furthermore every closed half-plane with zero on the boundary is a non-quasinilpotent spectrum.*

Proof. The first claim is trivial and thus it suffices to prove that the closed upper half-plane is a non-quasinilpotent spectrum.

Let $\mathfrak{A}$ be a C*-algebra with a faithful tracial state τ and let $N \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{QuasiNil}(\mathfrak{A})}$ be such that $\sigma(N)$ is contained in the closed upper half-plane. Let μ be the measure on

$\sigma(N)$ from Remarks 4.3.9. Then

$$\int_{\sigma(N)} \mathrm{Im}(z)\, d\mu = \frac{1}{2i} \int_{\sigma(N)} z - \overline{z}\, d\mu = 0.$$

However, since relatively open subsets of $\sigma(N)$ have positive μ-measure and $\mathrm{Im}(z) > 0$ above the x-axis, the above integral implies $\sigma(N)$ must lie on the x-axis. This implies that

$$N \in \mathfrak{A}_{\mathrm{sa}} \cap \overline{\mathrm{QuasiNil}(\mathfrak{A})}.$$

Thus Proposition 4.3.6 implies $N = 0$. □

Lemma 4.3.12. *For each $\alpha \in [0, 2\pi)$ let*

$$X_\alpha := \{\lambda \in \mathbb{C} \mid \lambda = re^{i\theta}, r \geq 0, \theta \in [0, 2\pi) \setminus \{\alpha\}\}.$$

Then each X_α is a non-quasinilpotent spectrum.

Proof. It suffices to prove the result for $\alpha = \pi$ by Lemma 4.3.11. Let $\mathfrak{A}$ be a C*-algebra with a faithful tracial state τ and let $N \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{QuasiNil}(\mathfrak{A})}$ be such that $\sigma(N) \subseteq X_\pi$. Thus $0 \in \sigma(N)$ by Lemma 1.8.4.

Recall $\sigma(N)$ is compact and bounded. Let K' be the union of $\sigma(N)$ with the bounded components of the complement of $\sigma(N)$. Then K' is a compact set such that $0 \notin int(K')$, $\mathbb{C} \setminus K'$ is connected, and $K' \subseteq X_\pi$.

Consider the function $f(z) = z^{\frac{1}{2}}$ on X_π (where the principal branch has been selected). Then f is a continuous function on X_π and holomorphic on the interior of K'. Consequently, as $f(0) = 0$, f is the uniform limit on K' of polynomials that vanish at zero by Mergelyan's Theorem. Therefore, since $N \in \overline{\mathrm{QuasiNil}(\mathfrak{A})}$, Lemma 4.3.1 implies

$$f(N) \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{QuasiNil}(\mathfrak{A})}.$$

However $\sigma(f(N)) = f(\sigma(N))$ is contained in the closed right half plane through the origin.

Hence $f(N) = 0$ by Lemma 4.3.11. Thus $N = f(N)^2 = 0$ as desired. $\square$

Theorem 4.3.13. *Suppose that $X \subseteq \mathbb{C}$ is such that $0 \in X$ and $\mathbb{C} \setminus X$ is connected. Suppose further that there exists an element $y \in \mathbb{C} \setminus X$ such that the line segment $f(t) = ty$ for $t \in (0, 1]$ is contained in $\mathbb{C} \setminus X$. Then X is a non-quasinilpotent spectrum.*

Proof. By Lemma 4.3.11 we can assume that $y = 1$. Let $\mathfrak{A}$ be a C*-algebra with a faithful tracial state τ and let $N \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{QuasiNil}(\mathfrak{A})}$ be such that $\sigma(N) \subseteq X$. Thus $0 \in \sigma(N)$ by Lemma 1.8.4.

Consider the function $g(z) = \frac{1}{z-1} + 1$ on X. Then, since $1 \in \mathbb{C} \setminus X \subseteq \mathbb{C} \setminus \sigma(N)$ (which is open), g is analytic on a neighbourhood of $\sigma(N)$. Since $g(0) = 0$, Mergelyan's Theorem implies g is the uniform limit on $\sigma(N)$ of polynomials that vanish at zero. Hence

$$g(N) \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{QuasiNil}(\mathfrak{A})}.$$

However, since $(0, 1] \notin X$, $g((0, 1]) = (-\infty, 0)$, and g is injective, $(-\infty, 0) \notin \sigma(g(N))$. Thus $g(N) = 0$ by Lemma 4.3.12. Since g is a fractional linear transformation, g is invertible and thus $N = g^{-1}(g(N)) = g^{-1}(0) = 0$. $\square$

It would be pleasant if the assumptions of Theorem 4.3.13 could be reduced to supposing zero is in the boundary of the unbounded connected component of $\mathbb{C} \setminus X$.

Theorem 4.4.6 will demonstrate that we cannot expect $\mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{Nil}(\mathfrak{A})} = \{0\}$ for an arbitrary C*-algebra $\mathfrak{A}$ with a faithful tracial state.

4.4 Type II_1 Factors

In this section we will examine when a normal operator in a type II_1 von Neumann algebra is the limit of nilpotent operators. We begin by applying the results of Section 4.3 to type II_1 von Neumann algebras and type II_1 factors.

Remarks 4.4.1. If $\mathfrak{M}$ is a type II_1 von Neumann algebra then $\mathfrak{M}$ has a separating family of tracial states. Therefore Corollary 4.3.8 applies. Moreover Theorem 4.3.13 applies in the case $\mathfrak{M}$ is a type II_1 factor. Thus, as every type II_1 von Neumann algebra is the direct integral of type II_1 factors, if $N = \int_X^\oplus N_x \, d\mu$ is a normal operator in a type II_1 von Neumann algebra that is a norm limit of nilpotent operators then $\sigma(N_x)$ cannot satisfy the assumptions of Theorem 4.3.13 on a set of positive μ-measure.

Remarks 4.4.2. Let $\mathfrak{M}$ be a type II_1 factor and let τ be the faithful, normal, tracial state on $\mathfrak{M}$. Remarks 4.3.9 imply that for each $N \in \mathrm{Nor}(\mathfrak{M})$ there exists a probability measure μ_N with support $\sigma(N)$ defined by τ. We will call μ_N the spectral distribution of N. Note two normal operators $N_1, N_2 \in \mathfrak{M}$ are approximately unitarily equivalent in $\mathfrak{M}$ if and only if $\mu_{N_1} = \mu_{N_2}$ (see [70]). Since the question of when a normal operator is in $\overline{\mathrm{Nil}(\mathfrak{M})}$ is clearly invariant under approximate unitary equivalence, the elements N of $\mathrm{Nor}(\mathfrak{M}) \cap \overline{\mathrm{Nil}(\mathfrak{M})}$ can be completely classified based on μ_N.

Our next goal is to demonstrate several measures μ_N as described in Remarks 4.4.2 such that $N \in \mathrm{Nor}(\mathfrak{M}) \cap \overline{\mathrm{Nil}(\mathfrak{M})}$. The main tool in this construction is Lemma 4.4.4 which is based on [32, Section 2.3.3]. For completeness we include the statement of the following well-known result.

Theorem 4.4.3 (Berg's Technique, see [21, Theorem VI.4.1] for a proof)**.** *Let $\{e_j\}_{j=0}^n \cup \{f_j\}_{j=0}^n$ be an orthonormal set in $\mathcal{H}$. Suppose $T \in \mathcal{B}(\mathcal{H})$ has the property that*

$$Te_j = e_{j+1} \quad \text{and} \quad Tf_j = f_{j+1}$$

for all $j \in \{0, \ldots, n-1\}$. Then there exists an $S \in \mathcal{B}(\mathcal{H})$ such that

1. *$S\xi = T\xi$ for all $\xi \in \left(\{e_j\}_{j=0}^{n-1} \cup \{f_j\}_{j=0}^{n-1}\right)^\perp$,*
2. *$S(\mathrm{span}\{e_j, f_j\}) = \mathrm{span}\{e_{j+1}, f_{j+1}\}$ for all $j \in \{0, \ldots, n-1\}$,*
3. *S is an isometry on $\mathrm{span}\{e_j, f_j\}$ for all $j \in \{0, \ldots, n-1\}$,*

4. $S^n e_0 = f_n$*,*

5. $S^n f_0 = e_n$*, and*

6. $\|S - T\| < \frac{\pi}{n}$*.*

Lemma 4.4.4 (see [32, Section 2.3.3])**.** *Let* $n, m \in \mathbb{N}$ *with* $m \geq 2$ *and choose*

$$0 = a_0 < a_1 < a_2 < \ldots < a_m = 1.$$

Then there exists

$$M \in \mathrm{Nil}(\mathcal{M}_{(2m+1)n+1}(\mathbb{C})) \quad \textit{and} \quad N \in \mathrm{Nor}(\mathcal{M}_{(2m+1)n+1}(\mathbb{C}))$$

such that

$$\|M - N\| \leq \frac{\pi}{n} + \max_{0 \leq k \leq m-1} |a_{k+1} - a_k|$$

and

$$\sigma(N) = \left\{ a_k e^{\frac{\pi i}{n} j} \;\middle|\; j \in \{1, \ldots, 2n\}, k \in \{0, \ldots, m\} \right\}$$

where the multiplicity of each non-zero eigenvalue is one.

Proof. Let $\{e_k\}_{k=0}^{(2m+1)n}$ be the standard orthonormal basis of $\mathbb{C}^{(2m+1)n+1}$ and define $M \in \mathrm{Nil}(\mathcal{M}_{(2m+1)n+1}(\mathbb{C}))$ by $Me_{(2m+1)n} = 0$,

$$M(e_{kn+j}) = a_{k+1} e_{kn+j+1}$$

for all $k \in \{0, 1, \ldots, m-1\}$ and $j \in \{0, 1, \ldots, n-1\}$,

$$M(e_{mn+j}) = a_m e_{mn+j+1}$$

for all $j \in \{0, 1, \ldots, n-1\}$,

$$M(e_{kn+j}) = a_{2m+1-k} e_{kn+j+1}$$

for all $k \in \{m+1, m+2, \ldots, 2m\}$ and $j \in \{0, 1, \ldots, n-1\}$, and by extending M by linearity. Thus M is a nilpotent weighted forward shift on $\mathbb{C}^{(2m+1)n+1}$ with weights

$$a_1, a_2, \ldots, a_{m-1}, a_m, a_m, a_m, a_{m-1}, \ldots, a_2, a_1$$

for consecutive blocks of length n. It is clear that $\|M\| = 1$.

For an arbitrary Hilbert space $\mathcal{K}$ with orthonormal basis $\{f_j\}_{j=1}^{2n}$ let $U_{2n} : \mathcal{K} \to \mathcal{K}$ be defined by $U_{2n}(f_j) = f_{j+1}$ for all $j \in \{1, 2 \ldots, 2n-1\}$, $U_{2n}(f_{2n}) = f_1$, and by extending U by linearity. It is clear that U_{2n} is a unitary operator with

$$\sigma(U_{2n}) = \left\{ e^{\frac{\pi i}{n} j} \mid j \in \{1, \ldots, 2n\} \right\}$$

with the multiplicity of each eigenvalue being one.

Our goal is to use Berg's Technique to approximate M with a direct sum of multiples of U_{2n}. For each $k \in \{0, 1, \ldots, 2m-1\}$ let

$$\mathcal{H}_k := \operatorname{span}\{e_{nk+j} \mid j \in \{0, 1, \ldots, n-1\}\}.$$

Let $\mathcal{K}_{m-1,m+1} := \mathcal{H}_{m-1} \oplus \mathcal{H}_m \oplus \mathcal{H}_{m+1}$. By Berg's Technique on

$$\{e_{nm-n}, e_{nm-n+1}, \ldots, e_{nm}\} \quad \text{and} \quad \{e_{nm+n}, e_{nm+n+1}, \ldots, e_{nm+2n}\},$$

there exists an $S_1 \in \mathcal{M}_{(2m+1)n+1}(\mathbb{C})$ such that $\|S_1 - M\| < \frac{\pi}{n}$, $S_1(f) = M(f)$ for all $f \in (\mathcal{H}_{m-1} \oplus \mathcal{H}_{m+1})^\perp$,

$$S_1(\operatorname{span}\{e_{nm-n+j}, e_{nm+n+j}\}) \subseteq \operatorname{span}\{e_{nm-n+j+1}, e_{nm+n+j+1}\}$$

for all $j \in \{0, 1, \ldots, n-1\}$, S_1 is an isometry on

$$\operatorname{span}\{e_{nm-n}, e_{nm-n+1}, \ldots, e_{nm-1}, e_{nm+n}, e_{nm+n+1}, \ldots, e_{nm+2n-1}\},$$

$S_1^n(e_{nm-n}) = e_{nm+2n}$, and $S_1^n(e_{nm+n}) = e_{nm}$. Therefore

$$\mathcal{K}'_{m-1,m+1} := \operatorname{span}\{e_{nm+n}, S_1(e_{nm+n}), S_1^2(e_{nm+n}), \ldots, S_1^{n-1}(e_{nm+n})\} \oplus \mathcal{H}_m$$

is an S_1-reducing subspace and S_1 is unitarily equivalent to U_{2n} when restricted to $\mathcal{K}'_{m-1,m+1}$.

Let

$$\mathcal{K}''_{m-1,m+1} := \mathcal{K}_{m-1,m+1} \ominus \mathcal{K}'_{m-1,m+1}.$$

By construction, $e_{nm-n} \in \mathcal{K}''_{m-1,m+1}$, S_1 is a forward shift with weights one on

$$\operatorname{span}\{e_{nm-n}, S_1(e_{nm-n}), S_1^2(e_{nm-n}), \ldots, S_1^{n-1}(e_{nm-n})\} = \mathcal{K}''_{m-1,m+1},$$

and $S_1(e_{nm+2n}) = M(e_{nm+2n}) = a_{m-1}e_{nm+2n+1}$. Let M_1 be the operator obtained from S_1 by reducing the weights on $\mathcal{K}''_{m-1,m+1}$ from $1 = a_m$ to a_{m-1} (so $M_1(S_1^{n-1}(e_{nm-n})) = a_{m-1}S_1(S_1^{n-1}(e_{nm-n})) = a_{m-1}e_{nm+2n}$). Hence

$$\|M - M_1\| \leq \frac{\pi}{n} + |a_m - a_{m-1}|.$$

Moreover, by construction, $\mathcal{K}'_{m-1,m+1}$ is a reducing subspace for M_1 such that $M_1|_{\mathcal{K}'_{m-1,m+1}} = U_{2n}$ and $M_1|_{(\mathcal{K}'_{m-1,m+1})^\perp}$ is an $((2m-1)n+1)$ by $((2m-1)n+1)$ matrix that is a nilpotent, weighted forward shift with weights

$$a_1, a_2, \ldots, a_{m-2}, a_{m-1}, a_{m-1}, a_{m-1}, a_{m-2}, \ldots, a_2, a_1$$

for consecutive blocks of length n.

For our next approximation, we will apply Berg's Technique on M_1 in 'an orthogonal way' in order not to disturb the above approximation. Let

$$\mathcal{K}_{m-2,m+2} := \mathcal{H}_{m-2} \oplus \mathcal{K}''_{m-1,m+1} \oplus \mathcal{H}_{m+2}.$$

By Berg's Technique on

$$\{e_{nm-2n}, e_{nm-2n+1}, \ldots, e_{nm-n}\} \quad \text{and} \quad \{e_{nm+2n}, e_{nm+2n+1}, \ldots, e_{nm+3n}\},$$

there exists an $S_2 \in \mathcal{M}_{(2m+1)n+1}(\mathbb{C})$ such that $\|S_2 - M_1\| < \frac{\pi}{n}$, $S_2(f) = M_1(f)$ for all $f \in (\mathcal{H}_{m-2} \oplus \mathcal{H}_{m+2})^{\perp}$,

$$S_2(\operatorname{span}\{e_{nm-2n+j}, e_{nm+2n+j}\}) \subseteq \operatorname{span}\{e_{nm-2n+j+1}, e_{nm+2n+j+1}\}$$

for all $j \in \{0, 1, \ldots, n-1\}$, S_2 is a_{m-1} times an isometry on

$$\operatorname{span}\{e_{nm-2n}, e_{nm-2n+1}, \ldots, e_{nm-n-1}, e_{nm+2n}, e_{nm+2n+1}, \ldots, e_{nm+3n-1}\},$$

$S_2^n(e_{nm-2n}) = a_{m-1}^n e_{nm+3n}$, and $S_2^n(e_{nm+2n}) = a_{m-1}^n e_{nm-n}$. Therefore the above implies that

$$\mathcal{K}'_{m-2,m+2} := \mathcal{K}''_{m-1,m+1} \oplus \operatorname{span}\{e_{nm+2n}, S_2(e_{nm+2n}), \ldots, S_2^{n-1}(e_{nm+2n})\}$$

is a reducing subspace of S_2 such that the restriction of S_2 to this subspace is unitarily equivalent to $a_{m-1}U_{2n}$ and on

$$\mathcal{K}''_{m-2,m+2} := \mathcal{K}_{m-2,m+2} \ominus \mathcal{K}'_{m-2,m+2} \subseteq \mathcal{H}_{m-2} \oplus \mathcal{H}_{m+2}$$

S_2 is a forward shift with weights a_{m-1}. By dropping these weights to a_{m-2}, we obtain a matrix M_2 such that

$$\|M_2 - M_1\| \leq \frac{\pi}{n} + |a_{m-1} - a_{m-2}|,$$

$\mathcal{K}'_{m-1,m+1}$ and $\mathcal{K}'_{m-2,m+2}$ are a reducing subspace for M_2 such that

$$M_2|_{\mathcal{K}'_{m-1,m+1}} = a_m U_{2n}, \ \ M_2|_{\mathcal{K}'_{m-2,m+2}} = a_{m-1} U_{2n},$$

and $M_2|_{(\mathcal{K}'_{m-1,m+1} \oplus \mathcal{K}'_{m-2,m+2})^{\perp}}$ is a $((2m-3)n+1)$ by $((2m-3)n+1)$ matrix that is a nilpotent, weighted forward shift with weights

$$a_1, a_2, \ldots, a_{m-3}, a_{m-2}, a_{m-2}, a_{m-2}, a_{m-3}, \ldots, a_2, a_1$$

for consecutive blocks of length n (except in the case that $(2m-3)n+1 = n+1$ in which case we have the $(n+1)\times(n+1)$ zero matrix). Moreover

$$\begin{aligned} \|M - M_2\| &\leq \max\{\|M - M_1\|, \|M_1 - M_2\|\} \\ &\leq \max\left\{\tfrac{\pi}{n} + |a_m - a_{m-1}|, \tfrac{\pi}{n} + |a_{m-1} - a_{m-2}|\right\} \end{aligned}$$

since M and M_1 only differ on $\mathcal{K}_{m-1,m+1}$ and M_1 and M_2 differ only on $\mathcal{H}_{m-2} \oplus \mathcal{H}_{m+2}$ which are orthogonal spaces.

By continuing this process ad nauseum, we eventually obtain that M is within

$$\frac{\pi}{n} + \max_{0 \leq k \leq m-1} |a_{k+1} - a_k|$$

of an operator unitarily equivalent to $\left(\bigoplus_{1 \leq k \leq m} a_k U_{2n}\right) \oplus 0_{n+1}$ where 0_{n+1} is the $(n+1)\times(n+1)$ zero matrix. Since $N_0 := \left(\bigoplus_{1 \leq k \leq m} a_k U_{2n}\right) \oplus 0_{n+1}$ is a normal operator with the desired spectrum, the result trivially follows. $\square$

Lemma 4.4.5. *Let $\mathfrak{M}$ be a II$_1$ factor, let τ be the faithful, normal, tracial state on $\mathfrak{M}$, and let $N \in \mathrm{Nor}(\mathfrak{M})$ be such that $\sigma(N) = \overline{\mathbb{D}}$. Suppose there exists an increasing, unbounded sequence of natural numbers $(n_k)_{k\geq 1}$ and real numbers $0 = a_{0,k} < a_{1,k} < a_{2,k} < \cdots < a_{n_k+1,k} = 1$ such that*

$$\lim_{k\to\infty} \max_{0 \leq p \leq n_k} |a_{p+1,k} - a_{p,k}| = 0$$

and if $E_{k,p,q}$ is the spectral projection of N onto

$$\left\{ z \in \overline{\mathbb{D}} \mid z = re^{i\theta}, a_{p,k} < r \leq a_{p+1,k}, \frac{q\pi}{n_k} < \theta \leq \pi\frac{(q+1)\pi}{n_k} \right\}$$

for all $q \in \{1, \ldots, 2n_k\}$ and $p \in \{1, \ldots, n_k\}$ and $E_{k,0}$ is the spectral projection onto the closed *disk of radius $a_{1,k}$ centred at zero then*

$$\tau(E_{k,0}) = \frac{n_k+1}{2n_k^2+n_k+1} \qquad \textit{and} \qquad \tau\left(E_{k,p,q}\right) = \frac{1}{2n_k^2+n_k+1}$$

for all $q \in \{1, \ldots, 2n_k\}$ and for all $p \in \{1, \ldots, n_k\}$. Then N is a norm limit of nilpotent operators from $\mathfrak{M}$.

Proof. Suppose N has the above conditions. For each $k \in \mathbb{N}$ let

$$N_k := 0E_{k,0} + \sum_{q=1}^{2n_k}\sum_{p=1}^{n_k} a_{p,k} e^{\frac{\pi i}{n_k}q} E_{k,p,q}.$$

Then $N_k \in \mathrm{Nor}(\mathfrak{M})$ is such that the norm of $N_k - N$ is at most the maximum of $a_{1,k}$ and

$$\max_{1\leq p\leq n_k} \textit{diameter}\left(\text{wedge of radii } a_{p,k} \text{ and } a_{p+1,k} \text{ with an angle of } \frac{\pi}{n_k}\right)$$

by the Spectral Theorem for Normal Operators. Since $\lim_{k\to\infty} n_k = \infty$ and

$$\lim_{k\to\infty} \max_{0\leq p\leq n_k} |a_{p+1,k} - a_{p,k}| = 0,$$

$N = \lim_{k\to\infty} N_k$. Thus it suffices to show $\lim_{k\to\infty} \mathrm{dist}(N_k, \mathrm{Nil}(\mathfrak{M})) = 0$.

Since

$$\tau(E_{k,0}) = \frac{n_k+1}{2n_k^2+n_k+1},$$

by the theory of II_1 factors there exists a collection $\{E_{k,0,p} \mid p \in \{0, \ldots, n_k\}\}$ of mutually orthogonal, equivalent projections of trace $\frac{1}{2n_k^2+n_k+1}$ that sum to $E_{k,0}$. Hence, for each $k \in \mathbb{N}$, the tracial conditions given in the hypotheses and the properties of II_1 factors imply that

$$\{E_{k,0,p} \mid p \in \{0, \ldots, n_k\}\} \cup \{E_{k,p,q} \mid q \in \{1, \ldots, 2n_k\}, p \in \{1, \ldots, n_k\}\}$$

are mutually orthogonal, equivalent projections that sum to $I_{\mathfrak{M}}$. Thus, using the partial isometries between these equivalent projections as matrix units, we can construct a copy of

$\mathcal{M}_{2n_k^2+n_k+1}(\mathbb{C})$ such that N_k can be viewed as a normal operator of $\mathcal{M}_{2n_k^2+n_k+1}(\mathbb{C})$ with

$$\sigma(N_k) = \left\{ a_{p,k} e^{\frac{\pi i}{n_k} q} \mid q \in \{1, \ldots, 2n_k\}, p \in \{0, 1, \ldots, n_k\} \right\}$$

where the multiplicity of each non-zero eigenvalue is one. Thus Lemma 4.4.4 implies N_k is within $\max_{0 \leq p \leq n_k - 1} |a_{p+1,k} - a_{p,k}|$ of an element of $\mathrm{Nil}(\mathfrak{M})$. By the assumption that $\lim_{k\to\infty} \max_{0\leq p\leq n_k} |a_{p+1,k} - a_{p,k}| = 0$, the result follows. $\square$

Theorem 4.4.6. *Let $\mathfrak{M}$ be a II_1 factor, let τ be the faithful, normal, tracial state on $\mathfrak{M}$, and let $N \in \mathrm{Nor}(\mathfrak{M})$ be such that $\sigma(N) = \overline{\mathbb{D}}$. Let μ_N be the spectral distribution of N. Suppose μ_N is absolutely continuous with respect to the two-dimensional Lebesgue measure and invariant under rotations of the disk. Then $N \in \overline{\mathrm{Nil}(\mathfrak{M})}$.*

Proof. By the assumptions on μ_N there exists a function f such that $rf(r) \in L_1([0,1])$, $f > 0$ almost everywhere with respect to the Lebesgue measure, and

$$\mu_N(X) = \int_X f(r)(r dr d\theta)$$

for all $X \subseteq \overline{\mathbb{D}}$ where $r dr d\theta$ is the two-dimensional Lebesgue measure. The construction of the $0 = a_{0,n} < a_{1,n} < \cdots < a_{n+1,n} = 1$ necessary to apply Lemma 4.4.5 at the n^{th} step is done by choosing $a_{j+1,n}$ such that

$$\frac{n+1}{2n^2+n+1} + \frac{2jn}{2n^2+n+1} = 2\pi \int_0^{a_{j+1,n}} rf(r) dr.$$

Since $F(x) = 2\pi \int_0^x rf(r)dr$ is an absolutely continuous, strictly increasing bijection from $[0,1]$ to $[0,1]$ by assumption, F^{-1} exists and is a strictly increasing continuous bijection from $[0,1]$ to $[0,1]$ such that

$$a_{j+1,n} = F^{-1}\left(\frac{n+1}{2n^2+n+1} + \frac{2jn}{2n^2+n+1} \right).$$

Thus $\lim_{n\to\infty} \max_{1\leq j\leq n-1} |a_{j+1,n} - a_{j,n}| = 0$ as required. $\square$

Next we will demonstrate how complex analysis may be used to construct additional non-zero normal operators in the closure of the nilpotent operators in type II_1 factors. We begin with the following observation that is trivial by Mergelyan's Theorem and Lemma 4.3.1.

Lemma 4.4.7. *Let $\mathfrak{A}$ be a C*-algebra and let $N \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{Nil}(\mathfrak{A})}$ be such that $\sigma(N) = \overline{\mathbb{D}}$. If $f : \overline{\mathbb{D}} \to \mathbb{C}$ is continuous on $\overline{\mathbb{D}}$, holomorphic on $\mathbb{D}$, and $f(0) = 0$, then $f(N) \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{Nil}(\mathfrak{A})}$ and $\sigma(f(N)) = f(\overline{\mathbb{D}})$.*

The same holds with $\overline{\mathrm{Nil}(\mathfrak{A})}$ replaced with $\overline{\mathrm{QuasiNil}(\mathfrak{A})}$.

Theorem 4.4.8. *Let Ω be a non-empty, open, connected and simply connected subset of $\mathbb{C}$ containing zero such that $\partial\Omega$ contains at least two points and is a Jordan curve. Let $\mathfrak{A}$ be a C*-algebra and let $N \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{Nil}(\mathfrak{A})}$ be such that $\sigma(N) = \overline{\mathbb{D}}$. Then there exists an operator $N_0 \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{Nil}(\mathfrak{A})}$ with $\sigma(N_0) = \overline{\Omega}$.*

The same holds with $\overline{\mathrm{Nil}(\mathfrak{A})}$ replaced with $\overline{\mathrm{QuasiNil}(\mathfrak{A})}$.

Proof. Let $f : \mathbb{D} \to \Omega$ be the biholomorphism given by the Riemann Mapping Theorem. By Carathéodory's Theorem f extends to a function $g : \overline{\mathbb{D}} \to \overline{\Omega}$ such that g is continuous on $\overline{\mathbb{D}}$, g is holomorphic on $\mathbb{D}$, and g is a bijection. Since $0 \in \Omega$, $0 \notin \partial\Omega$ and thus there exists an $a \in \mathbb{D}$ such that $g(a) = 0$. Let $h(z) = \frac{z+a}{\overline{a}z+1}$. Then h is a homeomorphism of the closed unit disk and is a biholomorphism of the open unit disk as $|a| < 1$. Let $F : \overline{\mathbb{D}} \to \overline{\Omega}$ be defined by $F(z) = g(h(z))$. Then F is well-defined, F is continuous on $\overline{\mathbb{D}}$, F is holomorphic on $\mathbb{D}$, F is surjective, and $F(0) = g(h(0)) = g(a) = 0$. Hence, by Lemma 4.4.7, $N_0 = F(N) \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{Nil}(\mathfrak{A})}$ is such that $\sigma(N_0) = F(\overline{\mathbb{D}}) = \overline{\Omega}$. $\square$

Unfortunately the solution to the question of when a normal operator in a type II_1 factor is a norm limit of nilpotent operators remains open. In particular, the results of Section 4.3 raise the following the following question.

Question 4.4.9. *If $\mathfrak{M}$ is a type II_1 factor and $N \in \mathfrak{M}$ is a normal operator with connected* *spectrum containing zero such that the spectral distribution, μ_N, of N has the property that*

$$\int_{\sigma(N)} p(z)\, d\mu_N(z) = 0$$

for every polynomial p that vanishes at zero, is $N \in \overline{\mathrm{Nil}(\mathfrak{M})}$?

Theorem 4.4.8 can be used to modify the spectral distributions in Theorem 4.4.6 to obtain more elements of $\mathrm{Nor}(\mathfrak{M}) \cap \overline{\mathrm{Nil}(\mathfrak{M})}$ for a type II_1 factor $\mathfrak{M}$. However, an answer to Question 4.4.9 seems to be directly related to the following finite dimensional question.

Question 4.4.10. *Given $N \in \mathrm{Nor}(\mathcal{M}_n(\mathbb{C}))$, can $\mathrm{dist}(N, \mathrm{Nil}(\mathcal{M}_n(\mathbb{C}))$ be computed using only knowledge of the spectrum (including multiplicity) of N?*

Very little is known in regards to the above question. Lemma 4.4.4 does provide some information and was used to derive the positive results of this section. In addition, [2, Section A1.2] provides a good summary of what is known. We will see at the end of Section 4.5 that this question reappears in the discussion of $\mathrm{Nor}(\mathfrak{M}) \cap \overline{\mathrm{Nil}(\mathfrak{M})}$ for a type II_∞ factor $\mathfrak{M}$ in a slightly simpler form.

On the other side of things, Theorem 4.3.13 provides examples of normal operators in a type II_1 factor that are not limits of nilpotent (or even quasinilpotent) operators. In particular, unlike the results of Section 4.1 and Section 4.2, it is unclear if $\mathrm{Nor}(\mathfrak{M}) \cap \overline{\mathrm{Nil}(\mathfrak{M})}$ and $\mathrm{Nor}(\mathfrak{M}) \cap \overline{\mathrm{QuasiNil}(\mathfrak{M})}$ agree for a type II_1 factor $\mathfrak{M}$. This raises the following question.

Question 4.4.11. *Is $\overline{\mathrm{Nil}(\mathfrak{M})} = \overline{\mathrm{QuasiNil}(\mathfrak{M})}$ for an arbitrary von Neumann algebra $\mathfrak{M}$?*

It is not difficult to see that Question 4.4.11 has a positive answer in the case of a type I_∞ von Neumann algebra.

Theorem 4.4.12. *Let $\mathfrak{M} := L_\infty(X, \mathcal{B}(\mathcal{H}))$ where (X, μ) is a Radon measure space. Then $\overline{\mathrm{Nil}(\mathfrak{M})} = \overline{\mathrm{QuasiNil}(\mathfrak{M})}$.*

Proof. It is clear that $\overline{\mathrm{Nil}(\mathfrak{M})} \subseteq \overline{\mathrm{QuasiNil}(\mathfrak{M})}$. To complete the proof it suffices to show that $\mathrm{QuasiNil}(\mathfrak{M}) \subseteq \overline{\mathrm{Nil}(\mathfrak{M})}$. Let $f \in \mathrm{QuasiNil}(\mathfrak{M}) \setminus \{0\}$ and let $\epsilon > 0$. Since

$$\left\{x \in X \mid \left\|f(x)^k\right\| > \left\|f^k\right\|\right\}$$

has measure zero for each $k \in \mathbb{N}$, we may assume without loss of generality that $\left\|f(x)^k\right\| \leq \left\|f^k\right\|$ for each $x \in X$ and $k \in \mathbb{N}$.

Since f is measurable, the range of f is separable and $x \mapsto \|f(x)\|$ is a measurable function. Thus there exist $\{T_n\}_{n\geq 1} \subseteq f(X)$ and disjoint measurable subsets $\{E_n\}_{n\geq 1} \subseteq X$ such that if

$$h := \sum_{n\geq 1} T_n \chi_{E_n} \in \mathfrak{M}$$

then $\|h - f\| \leq \epsilon$.

By [6, Theorem 2.2] (or [32, Theorem 5.18]) for every $\alpha, \beta > 0$ and $k \in \mathbb{N}$ there exist $\{M_{k,n}\}_{n\geq 1} \subseteq \mathrm{Nil}(\mathcal{B}(\mathcal{H}))$ such that $M_{k,n}^{2k} = 0$ for all $n \in \mathbb{N}$ and

$$\|T_n - M_{k,n}\| \leq 2\left(\alpha \|T_n\| + \beta + \frac{\left\|T_n^k\right\|}{\alpha\beta^{k-1}}\right) \leq 2\left(\alpha \|f\| + \beta + \frac{\left\|f^k\right\|}{\alpha\beta^{k-1}}\right)$$

for all $n \in \mathbb{N}$. By choosing $\alpha = \delta^{-k}$ and $\beta = \delta \left\|f^k\right\|^{\frac{1}{k}}$ for some fixed $\delta > 1$, for each $k \in \mathbb{N}$ and $\delta > 1$ there exist $\{M_{k,n}\}_{n\geq 1} \subseteq \mathrm{Nil}(\mathcal{B}(\mathcal{H}))$ such that $M_{k,n}^{2k} = 0$ for all $n \in \mathbb{N}$ and

$$\|T_n - M_{k,n}\| \leq 2\left(\delta^{-k} \|f\| + 2\delta \left\|f^k\right\|^{\frac{1}{k}}\right)$$

for all $n \in \mathbb{N}$. Since f is quasinilpotent, $\lim_{k\to\infty} \left\|f^k\right\|^{\frac{1}{k}} = 0$ so there exists a $k_0 \in \mathbb{N}$ and a $\delta_0 > 1$ such that

$$2\left(\delta_0^{-k} \|f\| + 2\delta_0 \left\|f^{k_0}\right\|^{\frac{1}{k_0}}\right) < \epsilon.$$

Hence there exist $\{M_n\}_{n\geq 1} \subseteq \mathrm{Nil}(\mathcal{B}(\mathcal{H}))$ such that $M_n^{2k_0} = 0$ for all $n \in \mathbb{N}$ and

$$\|T_n - M_n\| \leq \epsilon$$

for all $n \in \mathbb{N}$.

Let

$$g := \sum_{n\geq 1} M_n \chi_{E_n}.$$

Then $g \in \mathfrak{M}$, $\|f - g\| \leq 2\epsilon$, and $g^{2k_0} = 0$ so $g \in \mathrm{Nil}(\mathfrak{M})$. Hence $f \in \overline{\mathrm{Nil}(\mathfrak{M})}$. $\square$

Unfortunately the results of [6] do little to solve Question 4.4.11 for a general von Neumann algebra. For example, if

$$(M_n)_{n\geq 1} \in \prod_{n\geq 1} \mathcal{M}_n(\mathbb{C}) := \left\{ (T_n)_{n\geq 1} \mid T_n \in \mathcal{M}_n(\mathbb{C}), \sup_{n\geq 1} \|T_n\| < \infty \right\}$$

is quasinilpotent then $\lim_{k\to\infty} \sup_{n\geq 1} \left\|M_n^k\right\|^{\frac{1}{k}} = 0$. Clearly this implies each M_n is a nilpotent matrix. However, an element $(T_n)_{n\geq 1} \in \prod_{n\geq 1} \mathcal{M}_n(\mathbb{C})$ is nilpotent if and only if there exists a $k \in \mathbb{N}$ such that $T_n^k = 0$ for all $n \in \mathbb{N}$ and it is unclear that $(M_n)_{n\geq 1}$ can be approximated by elements of this form. The answer to Question 4.4.11 appears even more elusive for von Neumann algebras of other type since, unlike with normal operators, it is not apparent that quasinilpotent operators in factors of other types may be approximated with elements from $\mathcal{M}_n(\mathbb{C})$ or $\mathcal{B}(\mathcal{H})$.

4.5 Type II$_\infty$ Factors

In this section we will study when normal operators are norm limits of nilpotent and quasinilpotent operators in type II$_\infty$ factors with separable predual. Although the tracial restrictions of Section 4.3 do not apply, the finite projections do pose another restriction. This additional restriction is similar to a restriction that appears in Theorem 1.8.2 but not in Theorem 1.8.3.

Remarks 4.5.1. Let $\mathfrak{M}$ be a type II_∞ factor with separable predual. Then there exists a type II_1 factor $\mathfrak{N}$ such that $\mathfrak{M} = \mathfrak{N}\overline{\otimes}\mathcal{B}(\mathcal{H})$. Let $\mathfrak{M}_0 := \mathfrak{N} \otimes_{\min} \mathfrak{K}$ where $\mathfrak{K} \subseteq \mathcal{B}(\mathcal{H})$ is the C*-algebra of all compact operators. Then $\mathfrak{M}_0$ can be viewed as an ideal of $\mathfrak{M}$ (that is not weak*-closed). Let $q : \mathfrak{M} \to \mathfrak{M}/\mathfrak{M}_0$ be the canonical quotient map. For each $T \in \mathfrak{M}$ let $\sigma_e(T) := \sigma(q(T))$. We will call $\sigma_e(T)$ the essential spectrum of $T \in \mathfrak{M}$. Alternatively $\mathfrak{M}_0$ can be shown to be the ideal generated by operators supported on finite projections and thus the essential spectrum does not depend on the decomposition of $\mathfrak{M}$ chosen.

If $T \in \overline{\mathrm{QuasiNil}(\mathfrak{M})}$ then $q(T) \in \overline{\mathrm{QuasiNil}(\mathfrak{M}/\mathfrak{M}_0)}$. Hence $\sigma_e(T)$ must be connected and contain zero by Lemma 1.8.4. This additional condition is unnecessary for $\mathcal{B}(\mathcal{H})$ as the spectrum and essential spectrum of an $N \in \mathrm{Nor}(\mathcal{B}(\mathcal{H}))$ agree when $\sigma(N)$ is connected.

Let τ be an unbounded tracial state on $\mathfrak{M}$ such that $\tau(T \otimes P) = \tau'(T)$ for all $T \in \mathfrak{N}$ where τ' is the faithful, normal, tracial state on $\mathfrak{N}$ and $P \in \mathcal{B}(\mathcal{H})$ is a rank one projection. As in Remarks 4.4.2, for each $N \in \mathfrak{M}$, τ gives rise to a positive measure μ_N with support $\sigma(N)$ and $N_1, N_2 \in \mathfrak{M}$ are approximately unitarily equivalent in $\mathfrak{M}$ if and only if $\mu_{N_1} = \mu_{N_2}$ (see [70]). Thus the elements N of $\mathrm{Nor}(\mathfrak{M}) \cap \overline{\mathrm{Nil}(\mathfrak{M})}$ can be completely classified based on μ_N. Moreover, note $\lambda \in \sigma_e(N)$ if and only if

$$\mu_N(\{z \in \mathbb{C} \mid |z - \lambda| < \epsilon\}) = \infty$$

for all $\epsilon > 0$. Thus the measure μ_N captures the information about $\sigma_e(N)$.

Since every type II_∞ factor has infinite projections, we easily obtain (as in Section 4.1 and Section 4.2) that there are several normal operators in the closure of the nilpotent operators.

Theorem 4.5.2. *Let $\mathfrak{M}$ be a von Neumann algebra and let $N \in \mathrm{Nor}(\mathfrak{M})$ be such that $\sigma(N)$ is connected and contains zero. Suppose further that for every $\epsilon > 0$ there exist a finite number of disjoint Borel sets $\{E_{k,\epsilon}\}_{k=1}^{n_\epsilon}$ such that $\sigma(N) = \bigcup_{k=1}^{n_\epsilon} E_{k,\epsilon}$, diam$(E_{k,\epsilon}) < \epsilon$, and if $P_{k,\epsilon} := \chi_{E_{k,\epsilon}}(N)$ then $\{P_{k,\epsilon}\}_{k=1}^{n_\epsilon}$ are equivalent, properly infinite projections. Then $N \in \overline{\mathrm{Nil}(\mathfrak{M})}$.*

Proof. Let N be as described above and fix $\epsilon > 0$. Fix $a_k \in E_{k,\epsilon}$ such that $a_k = 0$ for the unique $k \in \{1, \ldots, n_\epsilon\}$ where $0 \in E_{k,\epsilon}$ and let $T_\epsilon := \sum_{k=1}^{n_\epsilon} a_k P_{k,\epsilon}$. Then $\|T_\epsilon - N\| \leq \epsilon$ and $T_\epsilon \in \mathrm{Nor}(\mathfrak{M})$.

Since $P_{1,\epsilon}$ is properly infinite, there exist mutually orthogonal, equivalent projections $\{P_{1,\epsilon,\ell}\}_{\ell \geq 1}$ such that

$$P_{1,\epsilon} = \sum_{\ell \geq 1} P_{1,\epsilon,\ell}.$$

Since $\{P_{k,\epsilon}\}_{k=1}^{n_\epsilon}$ are equivalent, mutually orthogonal projections, there exist mutually orthogonal, equivalent projections $\{\{P_{k,\epsilon,\ell}\}_{\ell \geq 1}\}_{k=1}^{n_\epsilon}$ such that

$$P_{k,\epsilon} = \sum_{\ell \geq 1} P_{k,\epsilon,\ell}$$

for all $k \in \{1, \ldots, n_\epsilon\}$.

Let $\mathcal{B}(\mathcal{H}) \subseteq \mathfrak{M}$ be a copy of the bounded linear operators on a separable Hilbert space generated by the partial isometries implementing the equivalences of $\{\{P_{k,\epsilon,\ell}\}_{\ell \geq 1}\}_{k=1}^{n_\epsilon}$ inside of $\mathfrak{M}$. Thus T_ϵ can be viewed as normal element of $\mathcal{B}(\mathcal{H}) \subseteq \mathfrak{M}$ with spectrum and essential spectrum equal to $\{a_k\}_{k=1}^{n_\epsilon}$. Since $\sigma(N)$ is connected and $diam(E_{k,\epsilon}) < \epsilon$ for all $k \in \{1, \ldots, n_\epsilon\}$, Lemma 4.1.2 implies T_ϵ is within 3ϵ of an element of $\mathrm{Nil}(\mathcal{B}(\mathcal{H})) \subseteq \mathrm{Nil}(\mathfrak{M})$. Hence $\mathrm{dist}(N, \mathrm{Nil}(\mathfrak{M})) \leq 4\epsilon$. □

Corollary 4.5.3. *Let $\mathfrak{M}$ be a type II_∞ factor with separable predual and let $N \in \mathrm{Nor}(\mathfrak{M})$ be such that $\sigma(N)$ is connected and contains zero. If $\sigma_e(N) = \sigma(N)$ then $N \in \overline{\mathrm{Nil}(\mathfrak{M})}$.*

Proof. Since $\sigma_e(N) = \sigma(N)$, every non-zero spectral projection of N is an infinite projection in $\mathfrak{M}$. Since $\mathfrak{M}$ is a type II_∞ factor, every infinite projection is properly infinite and any two infinite projections are equivalent. Thus the result follows from Theorem 4.5.2. □

Combining our results from Section 4.4, the following provides examples of normal operators N in type II_∞ factors that are limits of nilpotent operators yet $\sigma(N) \neq \sigma_e(N)$.

Proposition 4.5.4. *Let $\mathfrak{M}$ be a II_∞ factor with separable predual and write $\mathfrak{M} \simeq \mathfrak{N}\overline{\otimes}\mathcal{B}(\mathcal{H})$ where $\mathfrak{N}$ is a II_1 factor. Let τ be an unbounded tracial state on $\mathfrak{M}$ such that $\tau(T \otimes P) = \tau'(T)$ for all $T \in \mathfrak{N}$ where τ' is the faithful, normal, tracial state on $\mathfrak{N}$ and $P \in \mathcal{B}(\mathcal{H})$ is a rank one projection.*

Let $N \in \mathrm{Nor}(\mathfrak{M})$ be such that $\sigma(N)$ and $\sigma_e(N)$ are both connected and contain zero. Suppose further there exists an $N_0 \in \mathrm{Nor}(\mathfrak{N}) \cap \overline{\mathrm{Nil}(\mathfrak{N})}$ and a $k \in \mathbb{N}$ such that $\sigma(N_0) = \sigma(N)$ and $k\tau'(\chi_X(N_0)) = \tau(\chi_X(N))$ whenever $X \subseteq \sigma(N) \setminus \sigma_e(N)$ is Borel. Then $N \in \overline{\mathrm{Nil}(\mathfrak{M})}$.

Proof. Choose $N_1 \in \mathrm{Nor}(\mathfrak{N})$ such that $\sigma(N_1) = \sigma_e(N)$. Let $Q \in \mathcal{B}(\mathcal{H})$ be a rank k projection and consider $T := N_0 \otimes Q + N_1 \otimes (I_\mathcal{H} - Q)$. Then $T \in \mathrm{Nor}(\mathfrak{M})$ has the same spectral distribution as N so T and N are approximately unitarily equivalent in $\mathfrak{M}$.

Note $\mathfrak{N}\overline{\otimes}(I_\mathcal{H} - Q)\mathcal{B}(\mathcal{H})(I_\mathcal{H} - Q)$ is a type II_∞ factor and

$$N_1 \otimes (I_\mathcal{H} - Q) \in \mathrm{Nor}(\mathfrak{N}\overline{\otimes}(I_\mathcal{H} - Q)\mathcal{B}(\mathcal{H})(I_\mathcal{H} - Q))$$

satisfies the hypotheses of Corollary 4.5.3. Therefore

$$N_1 \otimes (I_\mathcal{H} - Q) \in \overline{\mathrm{Nil}(\mathfrak{N}\overline{\otimes}(I_\mathcal{H} - Q)\mathcal{B}(\mathcal{H})(I_\mathcal{H} - Q))}.$$

Since $N_0 \in \overline{\mathrm{Nil}(\mathfrak{N})}$ by assumption and the direct sum of two nilpotent operators is a nilpotent operator, $T \in \overline{\mathrm{Nil}(\mathfrak{M})}$. Hence $N \in \overline{\mathrm{Nil}(\mathfrak{M})}$. $\square$

Unfortunately Proposition 4.5.4 requires the normal operator N_0 to be a limit of nilpotent operators from $\mathfrak{N}$. Since type II_1 factors have no self-adjoint operators in the closure of the nilpotent operators, Proposition 4.5.4 does not enable us to classify $\mathfrak{M}_{\mathrm{sa}} \cap \overline{\mathrm{Nil}(\mathfrak{M})}$ for a type II_∞ factor. For example, using the notation of Proposition 4.5.4, let $N_0 \in \mathfrak{N}_{\mathrm{sa}}$ have the Lebesgue measure on $[0, 1]$ as its spectral distribution. Then $N_0 \notin \overline{\mathrm{Nil}(\mathfrak{N})}$ yet, if $P \in \mathcal{B}(\mathcal{H})$ is a rank one projection, it might be possible that $N_0 \otimes P \in \overline{\mathrm{Nil}(\mathfrak{M})}$.

One way to view this problem for this particular N_0 is as follows. From Corollary 4.3.4

we know

$$\liminf_{n\to\infty} \operatorname{dist}\left(\operatorname{diag}\left(\frac{1}{2^n}, \frac{2}{2^n}, \dots, \frac{2^n-1}{2^n}, 1\right), \operatorname{Nil}(\mathcal{M}_n(\mathbb{C}))\right) \geq \frac{1}{2}$$

so it is not possible to use spectral projections in $\mathfrak{N}$ to approximate N_0 with nilpotent matrices of this form. However, in $\mathfrak{M}$, the spectral projection of $N_0 \otimes P$ corresponding to $\{0\}$ is infinite. This allows us to add an infinite number of zero eigenvalues to any matrix in a matrix approximation of $N_0 \otimes P$ inside $\mathfrak{M}$. In particular, if

$$\liminf_{n\to\infty} \liminf_{k\to\infty} \operatorname{dist}\left(\operatorname{diag}\left(\frac{1}{2^n}, \frac{2}{2^n}, \dots, \frac{2^n-1}{2^n}, 1\right) \oplus 0_k, \operatorname{Nil}(\mathcal{M}_{n+k}(\mathbb{C}))\right) = 0$$

where 0_k is the $k \times k$ zero matrix, it would be easy to conclude that $N_0 \otimes P \in \overline{\operatorname{Nil}(\mathfrak{M})}$.

In fact, this limiting question is intrinsically related to the distance from a normal operator in $\mathcal{B}(\mathcal{H})$ with finite spectrum to $\operatorname{Nil}(\mathcal{B}(\mathcal{H}))$.

Proposition 4.5.5. *Let $n \in \mathbb{N}$ and let $a_1, \dots, a_n \in \mathbb{C}$. Let $\{e_n\}_{n\geq 1}$ be an orthonormal basis for $\mathcal{H}$ and define $D \in \operatorname{Nor}(\mathcal{B}(\mathcal{H}))$ by $De_j = a_j e_j$ if $j \in \{1, \dots, n\}$ and $De_j = 0$ otherwise. Then*

$$\lim_{k\to\infty} \operatorname{dist}(\operatorname{diag}(a_1, \dots, a_n) \oplus 0_k, \operatorname{Nil}(\mathcal{M}_{n+k}(\mathbb{C}))) = \operatorname{dist}(D, \mathcal{B}(\mathcal{H})).$$

Proof. It is clear that $\operatorname{dist}(\operatorname{diag}(a_1, \dots, a_n) \oplus 0_k, \operatorname{Nil}(\mathcal{M}_{n+k}(\mathbb{C})))$ decreases as k increases so the limit exists. Moreover

$$\lim_{k\to\infty} \operatorname{dist}(\operatorname{diag}(a_1, \dots, a_n) \oplus 0_k, \operatorname{Nil}(\mathcal{M}_{n+k}(\mathbb{C}))) \geq \operatorname{dist}(D, \mathcal{B}(\mathcal{H}))$$

by considering the direct sum of nilpotent matrices with a zero operator.

Let $M \in \operatorname{Nil}(\mathcal{B}(\mathcal{H}))$ be arbitrary. Let $m \in \mathbb{N}$ be such that $M^m = 0$, let $\mathcal{L} := \operatorname{span}\{e_j \mid j \in \{1, \dots, n\}\}$, and let

$$\mathcal{K} := \operatorname{span}\{\mathcal{L}, M(\mathcal{L}), \dots, M^{m-1}(\mathcal{L})\}.$$

Clearly $\mathcal{K}$ is a finite dimensional Hilbert space containing $\mathcal{L}$ that is invariant under M.

Choose $k \in \mathbb{N} \cup \{0\}$ such that $\dim(\mathcal{K}) = k + n = k + \dim(\mathcal{L})$. With respect to the decomposition $\mathcal{K} \oplus \mathcal{K}^{\perp}$ of $\mathcal{H}$, write

$$D = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \text{ and } M = \begin{bmatrix} M_1 & M_2 \\ 0 & M_3 \end{bmatrix}.$$

By construction T can be viewed as an $(k+n) \times (k+n)$ matrix that is unitarily equivalent to $\mathrm{diag}\,(a_1, \ldots, a_n) \oplus 0_k$. Moreover M_1 can be viewed as an $(k+n) \times (k+n)$ nilpotent matrix as M is nilpotent. Therefore

$$\mathrm{dist}\,(\mathrm{diag}\,(a_1, \ldots, a_n) \oplus 0_k, \mathrm{Nil}(\mathcal{M}_{n+k}(\mathbb{C}))) \leq \|T - M_1\| \leq \|D - M\|$$

so the result follows. $\square$

Some work towards evaluating the distance between a normal operator N of $\mathcal{B}(\mathcal{H})$ with finite spectrum and $\mathrm{Nil}(\mathcal{B}(\mathcal{H}))$ has been performed. One example of this is Lemma 4.1.2 which investigates the above distance when the spectrum and essential spectrum of N agree. Another example is [65, Theorem 2.3a] which gives a bound for $\mathrm{dist}(N, \mathrm{Nil}(\mathcal{B}(\mathcal{H})))$ based on the spectrum and essential spectrum of N. Unfortunately the bound from [65] does not appear to be tight in this setting.

Alternatively, looking at the limiting property, [32, Theorem 2.12] (originally in [30, Section 7]) shows

$$\lim_{n \to \infty} \mathrm{dist}(P_n, \mathrm{Nil}(\mathcal{M}_n(\mathbb{C}))) = \frac{1}{2}$$

where $P_n \in \mathcal{M}_n(\mathbb{C})$ is an arbitrary rank one projection. Moreover, in [45] and [46], a tight upper bound for $\mathrm{dist}(P_n, \mathrm{Nil}(\mathcal{M}_n(\mathbb{C})))$ has been obtained and examples have been given that obtain this bound for small n.

Note the only two obvious lower bounds for $\mathrm{dist}(A, \mathrm{Nil}(\mathcal{M}_n(\mathbb{C})))$ for a positive matrix $A \in \mathcal{M}_n(\mathbb{C})$ of norm one are Lemma 4.1.3 and Corollary 4.3.4. This provides some support to the following conjecture.

Conjecture 4.5.6. *There exists a continuous function $f : [0,1]^2 \to [0,1]$ such that $f(0,0) = 0$ and, if $A \in \mathcal{M}_{k+1}(\mathbb{C})$ is a positive matrix of norm one with $\sigma(A) = \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k = 1\}$, then*

$$\mathrm{dist}(A, \mathrm{Nil}(\mathcal{M}_{k+1}(\mathbb{C}))) \leq f\left(\tau_{k+1}(A), \max_{1 \leq i \leq k} |\lambda_i - \lambda_{i-1}|\right)$$

where τ_{k+1} is the tracial state on $\mathcal{M}_{k+1}(\mathbb{C})$.

If the above conjecture holds, it would be a simple argument to show that if $\mathfrak{M}$ is a type II_∞ factor and if $A \in \mathfrak{M}_{\mathrm{sa}}$ is such that $\sigma_e(A) = \{0\}$ and $\sigma(A) = [0,1]$ with the spectral distribution being a multiple of the Lebesgue measure, then $A \in \overline{\mathrm{Nil}(\mathfrak{M})}$. It would then be possible to use some elementary mapping arguments and approximations by normal operators with finite spectrum to show that if $\mathfrak{M}$ is a type II_∞ factor and if $N \in \mathrm{Nor}(\mathfrak{M})$, then $N \in \overline{\mathrm{Nil}(\mathfrak{M})}$ if and only if $\sigma(N)$ and $\sigma_e(N)$ are connected and contain zero.

4.6 Normal Limits of Sums of Nilpotent Operators in Von Neumann Algebras

In this section we will investigate $\mathrm{Nor}(\mathfrak{M}) \cap \overline{\mathrm{span}(\mathrm{Nil}(\mathfrak{M}))}$ for an arbitrary von Neumann algebra $\mathfrak{M}$. In [29, Corollary 5] Herrero showed that the unit of $\mathcal{B}(\mathcal{H})$ is a limit of sums of two nilpotent operators. Since certain von Neumann algebras contain unital copies of $\mathcal{B}(\mathcal{H})$, the following result is trivial.

Proposition 4.6.1. *Let $\mathfrak{M}$ be a type I_∞, II_∞, or type III von Neumann algebra. Then there exist sequences of nilpotent operators $(M_{i,n})_{n\geq 1}$ in $\mathfrak{M}$ such that $I_{\mathfrak{M}} = \lim_{n\to\infty} M_{1,n} + M_{2,n}$.*

Our next goal is to generalize [29, Corollary 6] to type I and type III von Neumann algebras with separable predual. The arguments used in these two results are a modification of the arguments in [29] using the theory developed in Sections 4.1 and 4.2.

Theorem 4.6.2. *Let $\mathfrak{M} := L_\infty(X, \mathcal{B}(\mathcal{H}))$ where (X, μ) a Radon measure space and let $f \in \mathrm{Nor}(\mathfrak{M})$. Then there exists two sequences $(M_n)_{n\geq 1}$ and $(M'_n)_{n\geq 1}$ of nilpotent operators* *in $\mathfrak{M}$ such that*

$$\lim_{n\to\infty} \|f - (M_n + M_n')\| = 0.$$

Furthermore these sequences can be chosen such that $\max\{\|M_n\|, \|M_n'\|\} \leq \frac{3}{2}\|f\|$.

Proof. Fix $\epsilon > 0$ and choose a representation of f such that $\sup_{x\in X} \|f(x)\| < \infty$ and $f(x)$ is normal for every $x \in X$. Since f is measurable, the range of f is separable and $x \mapsto \|f(x)\|$ is a measurable function. As such there exist $\{T_n\}_{n\geq 1} \subseteq f(X)$ and disjoint measurable subsets $\{E_n\}_{n\geq 1} \subseteq X$ such that if $h := \sum_{n\geq 1} T_n\chi_{E_n}$ (so $h \in \mathfrak{M}$) then $\|h - f\| \leq \epsilon$. Since $T_n \in f(X)$ for all $n \in \mathbb{N}$, T_n is normal and $\|T_n\| \leq \|f\|$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$ choose λ_n in the essential spectrum of T_n and fix $R_n \in \mathrm{Nor}(\mathcal{B}(\mathcal{H}))$ such that the spectrum and essential spectrum of R_n is the closed ball of radius $\|T_n\|$ around zero. For each $n \in \mathbb{N}$ let

$$S_n := \frac{\lambda_n}{2} I_{\mathcal{H}} + R_n \in \mathcal{B}(\mathcal{H}) \quad \text{and} \quad S_n' := \frac{\lambda_n}{2} I_{\mathcal{H}} - R_n \in \mathcal{B}(\mathcal{H}).$$

Thus for each $n \in \mathbb{N}$ the spectrum and essential spectrum of S_n and S_n' are the closed unit ball of radius $\|T_n\|$ centred at $\frac{\lambda_n}{2}$.

Recall that $\mathcal{M}_2(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H})$. For each $n \in \mathbb{N}$ let

$$L_n := \left(\frac{1}{2}T_n\right) \oplus S_n \in \mathcal{M}_2(\mathcal{B}(\mathcal{H})) \quad \text{and} \quad L_n' := \left(\frac{1}{2}T_n\right) \oplus S_n' \in \mathcal{M}_2(\mathcal{B}(\mathcal{H})).$$

Therefore, for each $n \in \mathbb{N}$, $L_n, L_n' \in \mathrm{Nor}(\mathcal{M}_2(\mathcal{B}(\mathcal{H})))$, $\sigma(L_n)$ and $\sigma(L_n')$ are connected and contain zero, and $\{L_n, L_n'\}_{n\geq 1}$ is bounded in norm by $\frac{3}{2}\|f\|$. Thus Lemma 4.1.4 implies there exists a $q \in \mathbb{N}$ and $\{Q_n, Q_n'\}_{n\geq 1} \subseteq \mathrm{Nil}(\mathcal{M}_2(\mathcal{B}(\mathcal{H})))$ such that $\|Q_n\|, \|Q_n'\| \leq \frac{3}{2}\|f\|$, $\|L_n - Q_n\|, \|L_n' - Q_n'\| \leq \epsilon$, and $Q_n^q = 0 = (Q_n')^q$ for all $n \in \mathbb{N}$.

Notice

$$L_n + L_n' = T_n \oplus (S_n + S_n') = T_n \oplus (\lambda_n I_{\mathcal{H}}).$$

Since λ_n was chosen to be in the essential spectrum of T_n, it is clear that T_n and $L_n + L_n'$ are approximately unitarily equivalent (viewing $\mathcal{M}_2(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H})$). Thus the above implies there exist $\{M_n, M_n'\}_{n\geq 1} \subseteq \mathrm{Nil}(\mathcal{B}(\mathcal{H}))$ such that $\|M_n\|, \|M_n'\| \leq \frac{3}{2}\|f\|$, $M_n^q = 0 = (M_n')^q$, and $\|T_n - (M_n + M_n')\| \leq 3\epsilon$ for all $n \in \mathbb{N}$. Therefore, if we define

$$g_1 := \sum_{n\geq 1} M_n \chi_{E_n} \quad \text{and} \quad g_2 := \sum_{n\geq 1} M_n' \chi_{E_n},$$

then $g_1, g_2 \in \mathrm{Nil}(\mathfrak{M})$ are such that $\|g_1\|, \|g_2\| \leq \frac{3}{2}\|f\|$ and

$$\|h - (g_1 + g_2)\| \leq \sup_{n\geq 1} \|T_n - (M_n + M_n')\| \leq 3\epsilon.$$

Thus $\|f - (g_1 + g_2)\| \leq 4\epsilon$ completing the proof. $\square$

Theorem 4.6.3. *Let $\mathfrak{M}$ be a type III von Neumann algebra with separable predual and let $N \in \mathrm{Nor}(\mathfrak{M})$. Then there exists two sequences $(M_n)_{n\geq 1}$ and $(M_n')_{n\geq 1}$ of nilpotent operators in $\mathfrak{M}$ such that*

$$\lim_{n\to\infty} \|N - (M_n + M_n')\| = 0.$$

Furthermore these sequences can be chosen such that $\max\{\|Q_n\|, \|Q_n'\|\} \leq \frac{3}{2}\|N\|$ for all $n \in \mathbb{N}$.

Proof. Most of the arguments of this result are similar to those used in Theorem 4.2.2 and thus will be omitted. Since $\mathfrak{M}$ is a type III von Neumann algebra with separable predual, there exists a locally compact, complete, separable, metrizable, measure space (X, μ) and a collection of type III factors $(\mathfrak{M}_x)_{x\in X}$ with separable predual such that $\mathfrak{M}$ is a direct integral of $(\mathfrak{M}_x)_{x\in X}$. Thus we can write $N = \int_X^{\oplus} N_x \, d\mu(x)$ where $N_x \in \mathfrak{M}_x$ is a normal operator and $\|N_x\| \leq \|N\|$ for all $x \in X$. Without loss of generality, $\|N\| = 1$.

Let $\epsilon := \frac{1}{2^m}$ for some fixed $m \in \mathbb{N}$ and let $C_{k,\ell}$ and X_Y be as in Theorem 4.2.2. Since the number of possible sets Y is finite and the sets X_Y are disjoint, it suffices to show that for each Y there exists two nilpotent operators M_Y and M_Y' in $\mathfrak{M}$ such that the supports of M_Y and M_Y' are X_Y and N is within 3ϵ of $M_Y + M_Y'$ when restricted to X_Y.

Fix a potential Y and for each $(k,\ell) \in Y$ let $z_{k,\ell} \in C_{k,\ell}$ be any element within the closed unit ball. As in Theorem 4.2.2 there exist equivalent, pairwise orthogonal, measurable projections $\{\{(x \mapsto P_{x,k,\ell,w})\}_{(k,\ell)\in Y}\}_{w\geq 1}$ such that

$$T_Y := \left(x \mapsto \sum_{(k,\ell)\in Y} \sum_{w\geq 1} z_{k,\ell} P_{x,k,\ell,w} \right)$$

is a measurable and decomposable operator in $\mathfrak{M}$ that has X_Y as support and is within 2ϵ of the restriction of N to X_Y.

To construct our nilpotent operators, let D_Y be the diagonal operator on a separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{\{e_{k,\ell,w}\}_{(k,\ell)\in Y}\}_{w\geq 1}$ such that $D_Y(e_{k,\ell,w}) = z_{k,\ell} e_{k,\ell,w}$ for all $w \in \mathbb{N}$ and $(k,\ell) \in Y$. By [29, Corollary 6] (or by Theorem 4.6.2) there exist $Q_Y^{(1)}, Q_Y^{(2)} \in \mathrm{Nil}(\mathcal{B}(\mathcal{H}))$ such that $\left\| Q_Y^{(j)} \right\| \leq \frac{3}{2}\|D_Y\| \leq \frac{3}{2}\|N\|$ for $j \in \{1,2\}$ and $\left\| D_Y - \left(Q_Y^{(1)} + Q_Y^{(2)} \right) \right\| \leq \epsilon$. For each $w_1, w_2 \in \mathbb{N}$, $j \in \{1,2\}$, and $(k_1,\ell_1),(k_2,\ell_2) \in Y$ let

$$a^{(j)}_{(k_1,\ell_1,w_1),(k_2,\ell_2,w_2)} := \left\langle Q_Y^{(j)} e_{k_2,\ell_2,w_2}, e_{k_1,\ell_1,w_1} \right\rangle \in \mathbb{C}$$

and let $\left(x \mapsto V_{x,(k_1,\ell_1,w_1),(k_2,\ell_2,w_2)}\right) \in \mathfrak{M}$ be the partial isometry such that

$$\left(x \mapsto V_{x,(k_1,\ell_1,w_1),(k_2,\ell_2,w_2)}\right)\left(x \mapsto V_{x,(k_1,\ell_1,w_1),(k_2,\ell_2,w_2)}\right)^* = (x \mapsto P_{x,k_1,\ell_1,w_1})$$

and

$$\left(x \mapsto V_{x,(k_1,\ell_1,w_1),(k_2,\ell_2,w_2)}\right)^*\left(x \mapsto V_{x,(k_1,\ell_1,w_1),(k_2,\ell_2,w_2)}\right) = (x \mapsto P_{x,k_2,\ell_2,w_2}).$$

Finally for $j \in \{1,2\}$ let $M_Y^{(j)}$ be the operator

$$\left(x \mapsto \sum_{w_1,w_2\geq 1} \sum_{(k_1,\ell_1),(k_2,\ell_2)\in Y} a^{(j)}_{(k_1,\ell_1,w_1),(k_2,\ell_2,w_2)} V_{x,(k_1,\ell_1,w_1),(k_2,\ell_2,w_2)} \right)$$

which are measurable, decomposable, nilpotent operators in $\mathfrak{M}$ whose sum is within ϵ of T_Y. Thus the result follows. $\square$

In the case of a C*-algebra $\mathfrak{A}$ with a tracial state (in particular, finite von Neumann algebras), the following corollary of Lemma 4.3.3 demonstrates that $I_{\mathfrak{A}} \notin \overline{\mathrm{span}(\mathrm{QuasiNil}(\mathfrak{A}))}$.

Proposition 4.6.4. *Let $\mathfrak{A}$ be a unital C*-algebra with a tracial state τ. Then*

$$\overline{\mathrm{span}(\mathrm{QuasiNil}(\mathfrak{A}))} \subseteq \ker(\tau).$$

Thus $I_{\mathfrak{A}} \notin \overline{\mathrm{span}(\mathrm{QuasiNil}(\mathfrak{A}))}$.

We end this section by generalizing [29, Corollary 6] to type II_∞ factors with separable predual.

Theorem 4.6.5. *Let $\mathfrak{M}$ be a type II_∞ factor with separable predual and let $N \in \mathrm{Nor}(\mathfrak{M})$. Then there exists two sequences $(M_n)_{n\geq 1}$ and $(M_n')_{n\geq 1}$ of nilpotent operators in $\mathfrak{M}$ such that*

$$\lim_{n\to\infty} \|N - (M_n + M_n')\| = 0.$$

Furthermore these sequences can be chosen such that

$$\|M_n\|, \|M_n'\| \leq \frac{1}{2}\|N\| + \mathrm{dist}(0, \sigma_e(N))$$

for all $n \in \mathbb{N}$ (where $\sigma_e(N)$ was defined in Remarks 4.5.1).

Proof. Let $\lambda \in \sigma_e(N)$ be any point such that $|\lambda| = \mathrm{dist}(0, \sigma_e(N))$ and let $R := \frac{1}{2}(\|N\| + |\lambda|)$. Let M_0 be a normal operator in $\mathfrak{M}$ such that $\sigma(M_0)$ and $\sigma_e(M_0)$ are both the closed ball of radius R around 0. Let $M_1 := \frac{\lambda}{2} I_{\mathfrak{M}} + M_0$ and $M_2 := \frac{\lambda}{2} I_{\mathfrak{M}} - M_0$. Thus $\sigma(M_1)$, $\sigma_e(M_1)$, $\sigma(M_2)$, and $\sigma_e(M_2)$ are all the closed ball of radius R around $\frac{\lambda}{2}$.

Since $\mathfrak{M}$ is a II_∞ factor, there exists a unital embedding of $\mathcal{M}_2(\mathfrak{M})$ into $\mathfrak{M}$ such that $N \oplus (\lambda I_{\mathfrak{M}})$ and N are approximately unitarily equivalent (as $\lambda \in \sigma_e(N)$). Let $L_1 := \frac{1}{2}N \oplus M_1 \in \mathcal{M}_2(\mathfrak{M})$ and $L_2 := \frac{1}{2}N \oplus M_2 \in \mathcal{M}_2(\mathfrak{M})$. By construction it is clear that $L_1 + L_2 = N \oplus (\lambda I_{\mathfrak{M}})$,

$$\|L_1\| = \|L_2\| = R + \frac{|\lambda|}{2} = \frac{1}{2}\|N\| + \mathrm{dist}(0, \sigma_e(N)),$$

and $\sigma(L_1) = \sigma_e(L_1)$ and $\sigma(L_2) = \sigma_e(L_2)$ are both connected sets containing zero. Since $\mathcal{M}_2(\mathfrak{M})$ is also a II_∞ factor, Corollary 4.5.3 implies that L_1 and L_2 are norm limits of nilpotent operators from $\mathcal{M}_2(\mathfrak{M})$. Hence there exists two sequences $(M_n)_{n\geq 1}$ and $(M'_n)_{n\geq 1}$ of nilpotent operators in $\mathcal{M}_2(\mathfrak{M}) \subseteq \mathfrak{M}$ such that

$$\lim_{n\to\infty} \|N \oplus (\lambda I) - (M_n + M'_n)\| = 0$$

and $\|Q_n\|, \|Q'_n\| \leq \frac{1}{2}\|N\| + \mathrm{dist}(0, \sigma_e(N))$ for all $n \in \mathbb{N}$. Hence $N \oplus \lambda I$ is a norm limit of sums of nilpotent operators from $\mathcal{M}_2(\mathfrak{M}) \subseteq \mathfrak{M}$ with the desired properties. Since $N \oplus \lambda I$ and N are approximately unitarily equivalent in $\mathfrak{M}$ (see [70]), the result follows. □

4.7 Distance from Projections to Nilpotent Operators in Von Neumann Algebras

In this section we will investigate the distance from an arbitrary fixed projection to the nilpotent and quasinilpotent operators in von Neumann algebras. We begin with the following simple result.

Lemma 4.7.1. *Let $\mathfrak{A}$ be a unital C^*-algebra. Then*

$$\mathrm{dist}(I_{\mathfrak{A}}, \mathrm{Nil}(\mathfrak{A})) = \mathrm{dist}(I_{\mathfrak{A}}, \mathrm{QuasiNil}(\mathfrak{A})) = 1.$$

Furthermore if $P \in \mathfrak{A}$ is a non-trivial projection then

$$\frac{1}{2} \leq \mathrm{dist}(P, QuasNil(\mathfrak{A})) \leq \mathrm{dist}(P, \mathrm{Nil}(\mathfrak{A})) \leq 1.$$

Proof. The first claim follows since nilpotent and quasinilpotent operators are not invertible and the open unit ball around $I_{\mathfrak{A}}$ contains invertible operators. The second result follows trivially from Lemma 4.1.3. □

Our first goal is to generalize [29, Corollary 9] to type I and type III von Neumann algebras with separable predual. These arguments are simple generalizations of [29, Corollary 9] based on the techniques used in Sections 4.1 and 4.2.

Theorem 4.7.2. *Let (X, μ) a Radon measure space and let $\mathfrak{M} := L_\infty(X, \mathcal{B}(\mathcal{H}))$. If $P \in \mathfrak{M}$ is a projection then*

1. $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{M})) = \mathrm{dist}(P, \mathrm{QuasiNil}(\mathfrak{M})) = 0$ *if $P = 0$,*
2. $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{M})) = \mathrm{dist}(P, \mathrm{QuasiNil}(\mathfrak{M})) = 1$ *if $P(x)$ has finite dimensional kernel on a set of positive μ-measure, and*
3. $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{M})) = \mathrm{dist}(P, \mathrm{QuasiNil}(\mathfrak{M})) = \frac{1}{2}$ *otherwise.*

Proof. Clearly (1) holds. To see that (2) holds, note if $M \in \mathrm{Nil}(\mathfrak{M})$ then $M(x) \in \mathrm{Nil}(\mathcal{B}(\mathcal{H}))$ for almost every $x \in X$ and if $M \in \mathrm{QuasiNil}(\mathfrak{M})$ then $M(x) \in \mathrm{QuasiNil}(\mathcal{B}(\mathcal{H}))$ for almost every $x \in X$. If $P \in \mathfrak{M}$ is a projection where $P(x)$ is a projection with finite dimensional kernel on a set of positive μ-measure then since every projection in $\mathcal{B}(\mathcal{H})$ with finite dimensional kernel is distance one from the nilpotent and quasinilpotent operators (by [29, Corollary 9]) the above description of the nilpotent and quasinilpotent operators of $\mathfrak{M}$ implies $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{M})) = \mathrm{dist}(P, \mathrm{QuasiNil}(\mathfrak{M})) = 1$.

To see that (3) holds, it suffices to show $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{M})) \leq \frac{1}{2}$ by Lemma 4.7.1. Fix $\epsilon > 0$. Note we can choose a representation of P such that $P(x)$ is a projection with infinite dimensional kernel for every $x \in X$. Since P is measurable, the range of P is separable and $x \mapsto \|P(x)\|$ is a measurable function. Thus there exist $\{P_n\}_{n\geq 1} \subseteq P(X)$ and disjoint measurable subsets $\{E_n\}_{n\geq 1} \subseteq X$ such that if

$$Q := \sum_{n\geq 1} P_n \chi_{E_n}$$

(so $Q \in \mathfrak{M}$) then $\|Q - P\| \leq \epsilon$.

Since $P_n \in f(X)$ for all $n \in \mathbb{N}$, each P_n is rather the zero projection, a projection with infinite range and kernel, or a projection with finite range. Since each projection with finite range can be viewed as the direct sum of rank one projections, [29, Corollary 9] implies there exists a $q \in \mathbb{N}$ and $\{M_n\}_{n\geq 1} \subseteq \mathrm{Nil}(\mathcal{B}(\mathcal{H}))$ such that $\|P_n - M_n\| \leq \frac{1}{2} + \epsilon$ and $M_n^q = 0$ for all $n \in \mathbb{N}$.

Let

$$M := \sum_{n\geq 1} M_n \chi_{E_n}.$$

Then $M \in \mathfrak{M}$, $\|P - M\| \leq \frac{1}{2} + 2\epsilon$, and $M^q = 0$ so $M \in \mathrm{Nil}(\mathfrak{M})$. Hence $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{M})) \leq \frac{1}{2}$. □

Theorem 4.7.3. *Let $\mathfrak{M}$ be a type III von Neumann algebra with separable predual. Choose a locally compact, complete, separable, metrizable measure space (X, μ) and a collection of type III factors $(\mathfrak{M}_x)_{x\in X}$ with separable predual such that $\mathfrak{M}$ is a direct integral of $(\mathfrak{M}_x)_{x\in X}$. If $P \in \mathfrak{M}$ is a projection, we may write $P = \int_X^{\oplus} P_x \, d\mu(x)$ where $P_x \in \mathfrak{M}_x$ is a projection for all $x \in X$. Then*

1. $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{M})) = \mathrm{dist}(P, \mathrm{QuasiNil}(\mathfrak{M})) = 0$ *if* $P = 0$,
2. $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{M})) = \mathrm{dist}(P, \mathrm{QuasiNil}(\mathfrak{M})) = 1$ *if* $P(x) = I_{\mathfrak{M}_x}$ *on a set of positive* μ*-measure, and*
3. $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{M})) = \mathrm{dist}(P, \mathrm{QuasiNil}(\mathfrak{M})) = \frac{1}{2}$ *otherwise.*

Proof. Clearly (1) holds and (2) follows in a similar fashion as in Theorem 4.7.2. To see that (3) holds, it suffices to show $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{M})) \leq \frac{1}{2}$ by Lemma 4.7.1. Since $\mathfrak{M}$ is a type III von Neumann algebra, every non-zero projection of $\mathfrak{M}$ is properly infinite. Thus, as $P = (x \mapsto P_x)$ is non-zero, $(x \mapsto P_x)$ is a properly infinite projection. Thus there exist equivalent, pairwise orthogonal, measurable projections $\{(x \mapsto P_{x,w,1})\}_{w\geq 1}$ such that

$$(x \mapsto P_x) = \sum_{w\geq 1} (x \mapsto P_{x,w,1}).$$

Let $Z \in \mathfrak{M}$ be the central support of P and let $Q := (I_{\mathfrak{M}} - P)Z$. Thus P and Q have the same central support Z as $I_{\mathfrak{M}_x} - P_x \neq 0$ for all x in the support of Z. Hence P and Q are equivalent projections in $\mathfrak{M}$. Therefore, using the projections $\{(x \mapsto P_{x,w,1})\}_{w\geq 1}$ and the fact that $QP = 0$, there exist measurable projections $\{(x \mapsto P_{x,w,2})\}_{w\geq 1}$ such that

$$(x \mapsto Q_x) = \sum_{w\geq 1}(x \mapsto P_{x,w,2})$$

and

$$\{\{(x \mapsto P_{x,w,j})\}_{w\geq 1}\}_{j=1,2}$$

is a collection of equivalent, pairwise orthogonal projections.

Let $\epsilon > 0$. To construct our nilpotent operator, let D be the diagonal operator on an infinite dimensional, separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{\{e_{w,j}\}_{w\geq 1}\}_{j,=1,2}$ such that $D(e_{w,1}) = e_{w,1}$ and $D(e_{w,2}) = 0$. By [29, Corollary 9] there exists an $M' \in \mathrm{Nil}(\mathcal{B}(\mathcal{H}))$ such that $\|D - M'\| \leq \frac{1}{2} + \epsilon$. For each $w_1, w_2 \in \mathbb{N}$ and $j_1, j_2 \in \{1, 2\}$ let

$$a_{(w_1,j_1),(w_2,j_2)} := \langle M' e_{w_2,j_2}, e_{w_1,j_1}\rangle \in \mathbb{C}$$

and let $\left(x \mapsto V_{x,(w_1,j_1),(w_2,j_2)}\right) \in \mathfrak{M}$ be the partial isometry such that

$$\left(x \mapsto V_{x,(w_1,j_1),(w_2,j_2)}\right)\left(x \mapsto V_{x,(w_1,j_1),(w_2,j_2)}\right)^* = (x \mapsto P_{x,w_1,j_1})$$

and

$$\left(x \mapsto V_{x,(w_1,j_1),(w_2,j_2)}\right)^*\left(x \mapsto V_{x,(w_1,j_1),(w_2,j_2)}\right) = (x \mapsto P_{x,w_2,j_2}).$$

Finally let

$$M := \left(x \mapsto \sum_{w_1,w_2\geq 1}\ \sum_{j_1,j_2\in\{1,2\}} a_{(w_1,j_1),(w_2,j_2)} V_{x,(w_1,j_1),(w_2,j_2)}\right)$$

which is a measurable and decomposable operator in $\mathfrak{M}$. Moreover M is also nilpotent as, for each $x \in X$, M_x is a copy of M'. Since $\|D - M'\| \leq \frac{1}{2}+\epsilon$, it is clear that $\|P - M\| \leq \frac{1}{2}+\epsilon$. $\square$

In the case of a C*-algebra $\mathfrak{A}$ with a tracial state τ, Corollary 4.3.4 clearly provides an additional restriction. Thus it is unlikely to for us to generalize the above results to finite von Neumann algebras. However [29, Corollary 9] generalizes to type II_∞ factors.

Theorem 4.7.4. *Let $\mathfrak{M}$ be a type II_∞ von Neumann algebra with separable predual. Choose a locally compact, complete, separable, metrizable, measure space (X, μ) and a collection of type II_∞ factors $(\mathfrak{M}_x)_{x\in X}$ with separable predual such that $\mathfrak{M}$ is a direct integral of $(\mathfrak{M}_x)_{x\in X}$. If $P \in \mathfrak{M}$ is a projection, we may write $P = \int_X^\oplus P_x \, d\mu(x)$ where $P_x \in \mathfrak{M}_x$ is a projection for all $x \in X$.*

1. $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{M})) = \mathrm{dist}(P, \mathrm{QuasiNil}(\mathfrak{M})) = 0$ *if $P = 0$,*
2. $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{M})) = \mathrm{dist}(P, \mathrm{QuasiNil}(\mathfrak{M})) = 1$ *if $I_{\mathfrak{M}_x} - P(x)$ is finite on a set of positive μ-measure, and*
3. $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{M})) = \mathrm{dist}(P, \mathrm{QuasiNil}(\mathfrak{M})) = \frac{1}{2}$ *otherwise.*

Proof. Clearly (1) holds. To see that (2) holds, first suppose $\mathfrak{M}$ is a factor (that is μ is a point-mass measure) let $\mathfrak{M}_0$ be the ideal of $\mathfrak{M}$ given in Remarks 4.5.1 and let $q : \mathfrak{M} \to \mathfrak{M}/\mathfrak{M}_0$ be the canonical quotient map. If $I_\mathfrak{M} - P$ is finite then $q(P) = I_{\mathfrak{M}/\mathfrak{M}_0}$. Thus

$$1 = \mathrm{dist}(q(P), \mathrm{Nil}(\mathfrak{M}/\mathfrak{M}_0)) \leq \mathrm{dist}(P, \mathrm{Nil}(\mathfrak{M})) \leq 1$$

and

$$1 = \mathrm{dist}(q(P), \mathrm{QuasiNil}(\mathfrak{M}/\mathfrak{M}_0)) \leq \mathrm{dist}(P, \mathrm{QuasiNil}(\mathfrak{M})) \leq 1.$$

Therefore (2) follows for general $\mathfrak{M}$ as in Theorem 4.7.3.

To see that (3) holds, it suffices to show $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{M})) \leq \frac{1}{2}$ by Lemma 4.7.1. By assumption $I_\mathfrak{M} - P$ is an properly infinite projection so there exists a collection of projections $\{Q_j\}_{j\geq 1}$ such that $\{P\} \cup \{Q_j\}_{j\geq 1}$ is a set of mutually equivalent, orthogonal projections that sum to the identity. Using the partial isometries implementing the equivalence of these projections, a copy of $\mathcal{B}(\mathcal{H})$ can be constructed inside $\mathfrak{M}$ such that P can be viewed as a rank one projection inside $\mathcal{B}(\mathcal{H})$. Thus [29, Corollary 9] implies $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{M})) \leq \mathrm{dist}(P, \mathrm{Nil}(\mathcal{B}(\mathcal{H}))) \leq \frac{1}{2}$ as desired. $\square$

4.8 Purely Infinite C*-Algebras

In this section we will prove Theorem 4.8.6, which completely classifies when a normal operator in a unital, simple, purely infinite C*-algebra is a norm limit of nilpotent and quasinilpotent operators. The main tools of the proof are the existence and equivalence of projections in unital, simple, purely infinite C*-algebras and Lemma 4.8.1 which gives positive matrices of norm one that are asymptotically approximated by nilpotent matrices as we allow the size of the matrices to increase. In fact, in the case that $\mathfrak{A}_0^{-1} = \mathfrak{A}^{-1}$, the conditions of Theorem 4.8.6 are identical to the conditions of Theorem 1.8.3. This is not a surprise as the proof of Theorem 4.8.6 relies only on Lemma 4.8.1 and the structure of the projections in a unital, simple, purely infinite C*-algebra. In fact, the proof of Theorem 4.8.6 can be adapted to prove Theorem 1.8.3. When the proof of Theorem 4.8.6 has been completed, we will apply similar techniques to obtain information about the closed span of nilpotent operators and the distance from a fixed projection to the nilpotent operators in unital, simple, purely infinite C*-algebras.

For completeness we include an outline of the following previously known result.

Lemma 4.8.1 (See [2, A1.14])**.** *For each $n \in \mathbb{N}$ there exists a positive matrix $A_n \in \mathcal{M}_n(\mathbb{C})$ with norm one such that* $\lim_{n\to\infty} \mathrm{dist}(A_n, \mathrm{Nil}(\mathcal{M}_n(\mathbb{C}))) = 0$.

Proof. Let

$$Q_n' := \sum_{j=1}^{n-1} \frac{1}{j} q_n^j \in \mathcal{M}_n(\mathbb{C})$$

where $q_n \in \mathcal{M}_n(\mathbb{C})$ is the nilpotent Jordan block of order n. It was shown in [36] that $\|\mathrm{Re}(Q_n')\| \leq \frac{\pi}{2}$. If $Q_n := \frac{\mathrm{i}}{\ln(n)} Q_n' \in \mathcal{M}_n(\mathbb{C})$ and $H_n := \mathrm{Re}(Q_n) \in \mathcal{M}_n(\mathbb{C})$, then, by [36],

$-I_n \leq H_n \leq I_n$, $Q_n \in \mathrm{Nil}(\mathcal{M}_n(\mathbb{C}))$,

$$| \, \|H_n\| - 1| \leq \frac{\ln(2)}{2\ln(n)}, \text{ and } \|H_n - Q_n\| \leq \frac{\pi}{2\ln(n)}.$$

By normalizing each H_n, we obtain self-adjoint matrices $B_n \in \mathcal{M}_n(\mathbb{C})$ with norm one such that

$$\lim_{n\to\infty} \mathrm{dist}(B_n, \mathrm{Nil}(\mathcal{M}_n(\mathbb{C}))) = 0.$$

For each $n \in \mathbb{N}$ let $A_n := B_n^2$. Hence $A_n \in \mathcal{M}_n(\mathbb{C})$ is a positive matrix with norm one. Since the square of any nilpotent matrix is a nilpotent matrix, it is easy to obtain

$$\lim_{n\to\infty} \mathrm{dist}(A_n, \mathrm{Nil}(\mathcal{M}_n(\mathbb{C}))) = 0$$

as desired. □

Although, in general, little can be said about the spectrum of the matrices A_n in Lemma 4.8.1, the following does provide some information.

Corollary 4.8.2. *Let $\{A_n\}_{n\geq 1}$ be the positive matrices of norm one from Lemma 4.8.1. For every $m \in \mathbb{N}$ there exists an $N_m \in \mathbb{N}$ such that*

$$\sigma(A_n) \cap \left[\frac{k}{m}, \frac{k+1}{m}\right) \neq \emptyset$$

for all $k \in \{0, 1, \ldots, m-1\}$ and for all $n \geq N_m$. That is, the spectrum of the matrices A_n are asymptotically dense in $[0, 1]$.

Proof. Since $\lim_{n\to\infty} \mathrm{dist}(A_n, \mathrm{Nil}(\mathcal{M}_n(\mathbb{C}))) = 0$ by Lemma 4.8.1, $\lim_{n\to\infty} \mathrm{dist}(\sigma(A_n), 0) = 0$ and the distance between adjacent eigenvalues (when arranged in increasing order) of each A_n tends to zero as n tends to infinity by Lemma 4.1.3. Hence the result easily follows. □

We will require the use of the following trivial result in the proof of Theorem 4.8.6.

Lemma 4.8.3. *Let $\mathfrak{A}$ be a C*-algebra, let $N \in \mathrm{Nor}(\mathfrak{A})$, let $(N_n)_{n\geq 1}$ be a sequence of normal operators of $\mathfrak{A}$ such that $N = \lim_{n\to\infty} N_n$, and let U be an open subset of $\mathbb{C}$ such that $U \cap \sigma(N) \neq \emptyset$. Then there exists an $k \in \mathbb{N}$ such that $\sigma(N_n) \cap U \neq \emptyset$ for all $n \geq k$.*

Proof. Fix $\lambda \in U \cap \sigma(N)$. By Urysohn's Lemma there exists a continuous function f on $\mathbb{C}$ such that $f|_{U^c} = 0$ yet $f(\lambda) = 1$. Note $f(N) = \lim_{n\to\infty} f(N_n)$ by standard functional calculus results. If $\sigma(N_n) \cap U = \emptyset$ for infinitely many n, then $f(N_n) = 0$ for infinitely many n yet $f(N) \neq 0$ by construction. This is clearly a contradiction. $\square$

Now we will prove Theorem 4.8.6 for positive operators. Although Proposition 4.8.4 is not required in the proof of Theorem 4.8.6, the proof of Proposition 4.8.4 contains all the conceptual difficulties and technical approximations thus easing in the comprehension of Theorem 4.8.6.

Proposition 4.8.4. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra and let $A \in \mathfrak{A}_+$. Then the following are equivalent:*

1. *$A \in \overline{\mathrm{Nil}(\mathfrak{A})}$.*
2. *$A \in \overline{\mathrm{QuasiNil}(\mathfrak{A})}$.*
3. *The spectrum of A is connected and contains zero.*

Proof. Clearly (1) implies (2) and (2) implies (3) is trivial by Lemma 1.8.4. We shall demonstrate that (3) implies (1).

Suppose the spectrum of A is connected and contains zero and let $\epsilon > 0$. Since $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra, $\mathfrak{A}$ has real rank zero (see [84] or [21, Theorem V.7.4]). Thus there exists scalars $0 = a_n < a_{n-1} < \ldots < a_1 = \|A\|$ and non-zero pairwise orthogonal projections $P_1^{(1)}, \ldots, P_n^{(1)} \in \mathfrak{A}$ such that $\|A - A_1\| \leq \epsilon$ where

$$A_1 := \sum_{k=1}^{n} a_k P_k^{(1)}.$$

Moreover, since the spectrum of A is connected, we may also assume that

$$\max_{1\leq k\leq n-1} |a_{k+1} - a_k| \leq \epsilon$$

by Lemma 4.8.3. The idea behind the remainder of the proof is to systematically remove the largest eigenvalue of A_1 by approximating with a nilpotent operator.

By Lemma 4.8.1 there exists an $\ell \in \mathbb{N}$, a positive matrix $T_1 \in \mathcal{M}_\ell(\mathbb{C})$ with $\|T_1\| = a_1$, and a nilpotent matrix $M_1 \in \mathcal{M}_\ell(\mathbb{C})$ such that $\|T_1 - M_1\| \leq \epsilon$. In addition, by a small perturbation, we may assume that the geometric multiplicity of the eigenvalue a_1 of T_1 is one. For each $k \in \{2, \ldots, n\}$ let

$$\left\{\lambda_{1,k}, \lambda_{2,k}, \ldots, \lambda_{m_k^{(1)},k}\right\}$$

be the spectrum of $\sigma(T_1)$ contained in $[a_k, a_{k-1})$ counting multiplicity (where zero intersection is possible). Since $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra, for each $k \in \{2, \ldots, n\}$ there exists pairwise orthogonal projections

$$Q_{1,k}^{(1)}, Q_{2,k}^{(1)}, \ldots, Q_{m_k^{(1)},k}^{(1)}$$

such that $P_1^{(1)}$ is equivalent $Q_{j,k}^{(1)}$ for each $j \in \left\{1, \ldots, m_k^{(1)}\right\}$ and $\sum_{j=1}^{m_k} Q_{j,k}^{(1)} < P_k^{(1)}$.

For $k \in \{2, \ldots, n\}$ let

$$P_k^{(2)} := P_k^{(1)} - \sum_{j=1}^{m_k^{(1)}} Q_{j,k}^{(1)} > 0$$

(where the empty sum is the zero projection). Therefore, if

$$A_1' := a_1 P_1^{(1)} + \sum_{k=2}^{n} a_k \left(\sum_{j=1}^{m_k^{(1)}} Q_{j,k}^{(1)}\right) \qquad \text{and} \qquad A_2 := \sum_{k=2}^{n} a_k P_k^{(2)}$$

then A_1' and A_2 are self-adjoint operators such that $A_1 = A_1' + A_2$. Notice if $P^{(2)} := \sum_{k=2}^{n} P_k^{(2)}$

then $P^{(2)}$ is a non-trivial projection such that

$$A_2 \in P^{(2)}\mathfrak{A}P^{(2)} \qquad \text{and} \qquad A_1' \in \left(I_{\mathfrak{A}} - P^{(2)}\right)\mathfrak{A}\left(I_{\mathfrak{A}} - P^{(2)}\right).$$

Thus the proof will be complete if we can demonstrate that A_1' is within 2ϵ of a nilpotent operator from $\left(I_{\mathfrak{A}} - P^{(2)}\right)\mathfrak{A}\left(I_{\mathfrak{A}} - P^{(2)}\right)$ and A_2 is within 2ϵ of a nilpotent operator from $P^{(2)}\mathfrak{A}P^{(2)}$.

Recall $P^{(2)}\mathfrak{A}P^{(2)}$ and $\left(I_{\mathfrak{A}} - P^{(2)}\right)\mathfrak{A}\left(I_{\mathfrak{A}} - P^{(2)}\right)$ are unital, simple, purely infinite C*-algebras. Moreover, if

$$A_1'' := a_1 P_1^{(1)} + \sum_{k=2}^{n}\sum_{j=1}^{m_k^{(1)}} \lambda_{j,k} Q_{j,k}^{(1)} \in \left(I_{\mathfrak{A}} - P^{(2)}\right)\mathfrak{A}\left(I_{\mathfrak{A}} - P^{(2)}\right)$$

then $\|A_1'' - A_1'\| \leq \epsilon$ by the assumption that $\max_{1\leq k\leq n-1} |a_{k+1} - a_k| \leq \epsilon$. Since

$$\left\{P_1^{(1)}\right\} \cup \left\{\left\{Q_{j,k}^{(1)}\right\}_{j=1}^{m_k^{(1)}}\right\}_{k=2}^{n}$$

are pairwise orthogonal, equivalent projections in $\left(I_{\mathfrak{A}} - P^{(2)}\right)\mathfrak{A}\left(I_{\mathfrak{A}} - P^{(2)}\right)$, we can use the partial isometries implementing the equivalence to construct a matrix algebra with these projections as the orthogonal minimal projections. Moreover, by construction, inside this matrix algebra A_1'' has the same spectrum as T_1 (including multiplicity) so A_1'' can be approximated with the analog of M_1 inside $\left(I_{\mathfrak{A}} - P^{(2)}\right)\mathfrak{A}\left(I_{\mathfrak{A}} - P^{(2)}\right)$. Hence A_1' is within 2ϵ of a nilpotent operator from $\left(I_{\mathfrak{A}} - P^{(2)}\right)\mathfrak{A}\left(I_{\mathfrak{A}} - P^{(2)}\right)$.

To approximate A_2 with a nilpotent operator from $P^{(2)}\mathfrak{A}P^{(2)}$, we repeat the same argument with a positive matrix T_2 of norm a_2. Due to the nature of the above approximations, the above process gives a non-trivial projection $P^{(3)} < P^{(2)}$ and a positive operator A_3 of $P^{(3)}\mathfrak{A}P^{(3)}$ with spectrum $\{a_3, a_4, \ldots, a_n\}$ such that $A_2 - A_3 \in \left(P^{(2)} - P^{(3)}\right)\mathfrak{A}\left(P^{(2)} - P^{(3)}\right)$ can be approximated within 2ϵ of a nilpotent operator from $\left(P^{(2)} - P^{(3)}\right)\mathfrak{A}\left(P^{(2)} - P^{(3)}\right)$. By repeating this process a finite number of times (eventually ending with a zero operator), we can write A_1 as a finite direct sum of positive matrices each within 2ϵ of a nilpotent matrix from the respective matrix algebra. Hence A_1 is within 2ϵ of a nilpotent operator from $\mathfrak{A}$ and thus A is within 3ϵ of a nilpotent operator from $\mathfrak{A}$. $\square$

In order to adapt the proof of Proposition 4.8.4 to normal operators, it is necessary to be able to approximate said operators with normal operators with finite spectra. This difficult work has already been completed by Lin in Theorem 1.3.19. It turns out that the condition '$\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \in \mathbb{C} \setminus \sigma(N)$' is a necessary condition for an operator to be a limit of nilpotent operators.

Lemma 4.8.5. *Let $\mathfrak{A}$ be a unital C^*-algebra and let $T \in \overline{\mathrm{QuasiNil}(\mathfrak{A})}$. Then $\lambda I_{\mathfrak{A}} - T \in \mathfrak{A}_0^{-1}$ for all $\lambda \in \mathbb{C} \setminus \sigma(T)$.*

Proof. If $M \in \mathrm{QuasiNil}(\mathfrak{A})$ then $\lambda I_{\mathfrak{A}} - tM$ is invertible for all $\lambda \in \mathbb{C} \setminus \{0\}$ and for all $t \in \mathbb{C}$. Therefore $\lambda I_{\mathfrak{A}} - M \in \mathfrak{A}_0^{-1}$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.

If $T \in \overline{\mathrm{QuasiNil}(\mathfrak{A})}$ then $0 \in \sigma(T)$ by Lemma 1.8.4. As $\mathfrak{A}_0^{-1}$ is closed in the relative topology on $\mathfrak{A}^{-1}$, $\lambda I_{\mathfrak{A}} - T \in \mathfrak{A}_0^{-1}$ for all $\lambda \in \mathbb{C} \setminus \sigma(T)$. $\square$

With Lemma 4.8.5 giving another necessary condition for a normal operator to be a limit of nilpotent operators, we can now address our main theorem.

Theorem 4.8.6. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N \in \mathrm{Nor}(\mathfrak{A})$. Then the following are equivalent:*

1. *$N \in \overline{\mathrm{Nil}(\mathfrak{A})}$.*
2. *$N \in \overline{\mathrm{QuasiNil}(\mathfrak{A})}$.*
3. *$0 \in \sigma(N)$, $\sigma(N)$ is connected, and $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \in \mathbb{C} \setminus \sigma(N)$.*

Proof. Clearly (1) implies (2) and (2) implies (3) is trivial by Lemma 1.8.4 and Lemma 4.8.5. We shall demonstrate that (3) implies (1). As the approximations contained in the proof are identical to those used in Proposition 4.8.4, we will only outline the main technique and omit the approximations.

Suppose $0 \in \sigma(N)$, $\sigma(N)$ is connected, and $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \in \mathbb{C} \setminus \sigma(N)$. Fix $\epsilon > 0$ and for each $(n, m) \in \mathbb{Z}^2$ let

$$B_{n,m} := \left(\epsilon n - \frac{\epsilon}{2}, \epsilon n + \frac{\epsilon}{2}\right] + \mathrm{i}\left(\epsilon m - \frac{\epsilon}{2}, \epsilon m + \frac{\epsilon}{2}\right] \subseteq \mathbb{C}.$$

By Theorem 1.3.19 there exists a normal operator N_ϵ with finite spectrum such that $\|N - N_\epsilon\| \leq \epsilon$. For each $(n, m) \in \mathbb{Z}^2$ we label the box $B_{n,m}$ relevant if $\sigma(N_\epsilon) \cap B_{n,m} \neq \emptyset$ and we label the box $B_{n,m}$ irrelevant if $\sigma(N_\epsilon) \cap B_{n,m} = \emptyset$. Since $\sigma(N)$ is connected, we may assume (via Lemma 4.8.3) that the union of all relevant $B_{n,m}$ is a connected set and $B_{0,0}$ is relevant. By a perturbation of at most ϵ, we can assume that $\sigma(N_\epsilon)$ is precisely the centres of all relevant boxes and $\|N - N_\epsilon\| \leq 2\epsilon$.

The remainder of the proof is similar in nature to the proof of Proposition 4.8.4 in that we will use a recursive algorithm to write N_ϵ as a finite direct sum of matrices inside of $\mathfrak{A}$ each of which is within 5ϵ of the set of nilpotent matrices. If the only relevant box is $B_{0,0}$, the algorithm may stop as N_ϵ is the zero operator and thus nilpotent. Otherwise we label a relevant box bad if its removal disconnects the union of the relevant boxes or it is $B_{0,0}$ and we label a relevant box good if it is not bad. Elementary graph theory implies that at least one box is good.

Let B_{n_0,m_0} be a good, relevant box. Since the union of the relevant boxes is connected, there exists a continuous path $\gamma : [0, 1] \to \mathbb{C}$ that connects 0 to $\epsilon n_0 + \mathrm{i}\epsilon m_0$ whose image lies in the union of the relevant boxes. By Lemma 4.8.1 and since γ can be approximated uniformly by a polynomial that vanishes at zero, there exists an $\ell \in \mathbb{N}$, a normal operator $N_\ell \in \mathcal{M}_\ell(\mathbb{C})$, and a nilpotent $M_\ell \in \mathcal{M}_\ell(\mathbb{C})$ such that the spectrum of N_ℓ is contained within an ϵ-neighbourhood of the union of relevant boxes and $\|N_\ell - M_\ell\| \leq \epsilon$. By perturbing the eigenvalues of N_ℓ by at most 4ϵ, we can assume that the spectrum of N_ℓ is precisely a subset of the centres of relevant boxes, the multiplicity of $\epsilon n_0 + \mathrm{i}\epsilon m_0$ is precisely one, and

$\|M_\ell - N_\ell\| \leq 5\epsilon$.

For each $(n, m) \in \mathbb{Z}^2$ let $P_{n,m}$ be the spectral projection of N_ϵ for the box $B_{n,m}$. Using P_{n_0,m_0} as a main projection, for each other $(n, m) \in \mathbb{Z}^2$ such that $\epsilon n + i\epsilon m$ is in the spectrum of N_ℓ we can find the algebraic multiplicity of the eigenvalue $\epsilon n + i\epsilon m$ of N_ℓ many orthogonal subprojections of $P_{n,m}$ whose sum is strictly less then $P_{n,m}$ and each of which is equivalent to P_{n_0,m_0}. Thus, as in the proof of Proposition 4.8.4, we can find a projection $P_1 \in \mathfrak{A}$ such that P_1 commutes with N_ϵ, $P_1 N_\epsilon P_1$ can be approximated by a nilpotent operator from $P_1 \mathfrak{A} P_1$ within 5ϵ, and $(I - P_1)N_\epsilon(I - P_1)$ has the same spectrum as N_ϵ minus $\epsilon n_0 + i\epsilon m_0$.

By our selection of (n_0, m_0) and choice of projection P_1, the number of relevant $B_{n,m}$ for $(I - P_1)N_\epsilon(I - P_1)$ is one less than the number of relevant $B_{n,m}$ for N_ϵ and the union of the relevant $B_{n,m}$ for $(I - P_1)N_\epsilon(I - P_1)$ is connected and contains $B_{0,0}$. Thus, by repeating the above process a finite number of times, we obtain a nilpotent operator $M \in \mathfrak{A}$ such that $\|N - M\| \leq 7\epsilon$. Hence the result follows. □

In the case of our C*-algebra is not a purely infinite C*-algebra, we note that the following can easily be proved using the techniques illustrated above.

Lemma 4.8.7. *Let $\mathfrak{A}$ be a unital, simple C*-algebra and let $N \in \mathrm{Nor}(\mathfrak{A})$ be such that $\sigma(N)$ is connected and contains zero. If $N = \lim_{n\to\infty} \sum_{k=1}^{m_n} a_{k,n} P_{k,n}$ where $a_{k,n} \in \mathbb{C}$ and $P_{k,n}$ are infinite projections with $\sum_{k=1}^{m_n} P_{k,n} = I_{\mathfrak{A}}$ then $N \in \overline{\mathrm{Nil}(\mathfrak{A})}$.*

Proof. The conditions that $\mathfrak{A}$ is simple and the projections are infinite imply that the projections are properly infinite (see [21, Theorem V.5.1]) and every projection is equivalent to a subprojection of any infinite projection (see [21, Lemma V.5.4]). Thus the process used above works (where we note the small technical detail that, when removing one projection from the sum, we can still take the differences containing the other projections to be infinite by showing that they containing a strict subprojection equivalent to the original $P_{k,n}$ by [21, Theorem V.5.1]). □

With the proof of Theorem 4.8.6 complete, we turn our attention to other interesting questions pertaining to limits of nilpotent operators in unital, simple, purely infinite C*-algebras. To begin, we recall that [29, Corollary 6] shows that the closure of $\mathrm{Nil}(\mathcal{B}(\mathcal{H})) + \mathrm{Nil}(\mathcal{B}(\mathcal{H}))$ contained every normal operator. We now demonstrate a similar result for unital, simple, purely infinite C*-algebras.

Theorem 4.8.8. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra. Then*

$$\mathfrak{A}_{\mathrm{sa}} \subseteq \overline{\{M_1 + M_2 \mid M_1, M_2 \in \mathrm{Nil}(\mathfrak{A})\}}$$

and

$$\mathfrak{A} \subseteq \overline{\{M_1 + M_2 + M_3 + M_4 \mid M_1, M_2, M_3, M_4 \in \mathrm{Nil}(\mathfrak{A})\}}.$$

Proof. Clearly the second result follows from the first by considering real and imaginary parts. To prove the first result, we will first demonstrate that

$$I_{\mathfrak{A}} \in \overline{\{M_1 + M_2 \mid M_1, M_2 \in \mathrm{Nil}(\mathfrak{A})\}}.$$

Note that there exists a positive operator $A \in \mathfrak{A}$ such that $\sigma(A) = [0, 1]$. Thus A and $I_{\mathfrak{A}} - A$ are limits of nilpotent operators by Theorem 4.8.6 (or Proposition 4.8.4) which completes the claim.

Let $A \in \mathfrak{A}_{\mathrm{sa}}$ be arbitrary and fix $\epsilon > 0$. Since $\mathfrak{A}$ has real rank zero (see [21, Theorem V.7.4]), there exists non-zero pairwise orthogonal projections $\{P_k\}_{k=1}^n \subseteq \mathfrak{A}$ and scalars $\{a_k\}_{k=1}^n$ such that $\|\sum_{k=1}^n a_k P_k - A\| < \epsilon$. Since each $P_k \mathfrak{A} P_k$ is a unital, simple, purely infinite C*-algebra with unit P_k, P_k is a limit of the sum of two nilpotent operators from $P_k \mathfrak{A} P_k$. Since the finite direct sum of nilpotent operators is a nilpotent operator, $\sum_{k=1}^n a_k P_k$ is a limit of sums of two nilpotent operators from $\mathfrak{A}$ and thus the result follows. ∎

Corollary 4.8.9. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra and let $N \in \mathrm{Nor}(\mathfrak{A})$ be such that $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \in \mathbb{C} \setminus \sigma(N)$. Then*

$$N \in \overline{\{M_1 + M_2 \mid M_1, M_2 \in \mathrm{Nil}(\mathfrak{A})\}}$$

Proof. The result follows from the same argument in Theorem 4.8.8 where N can be approximated by normal operators with finite spectrum by Theorem 1.3.19. $\square$

We note that if $\mathfrak{A} := \mathcal{O}_n$ is the Cuntz algebra generated by n isometries then $\mathfrak{A}_0^{-1} = \mathfrak{A}^{-1}$ by [16]. Thus $\mathrm{Nor}(\mathcal{O}_n) \subseteq \overline{\{M_1 + M_2 \mid M_1, M_2 \in \mathrm{Nil}(\mathcal{O}_n)\}}$ for all $n \in \mathbb{N}$.

In [29, Corollary 9], Herrero determined the distance from a fixed projection in $\mathcal{B}(\mathcal{H})$ to the nilpotent and quasinilpotent operators was either 0, 1, or $\frac{1}{2}$ and gave necessary and sufficient conditions for each distance. Using the structure of projections in unital, simple, purely infinite C*-algebras, it is possible to imitate Herrero's work.

Theorem 4.8.10. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra and let $P \in \mathfrak{A}$ be a projection. Then*

1. $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{A})) = \mathrm{dist}(P, \mathrm{QuasiNil}(\mathfrak{A})) = 0$ *if* $P = 0$,
2. $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{A})) = \mathrm{dist}(P, \mathrm{QuasiNil}(\mathfrak{A})) = 1$ *if* $P = I_{\mathfrak{A}}$, *and*
3. $\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{A})) = \mathrm{dist}(P, \mathrm{QuasiNil}(\mathfrak{A})) = \frac{1}{2}$ *otherwise.*

Proof. Clearly (1) and (2) hold by Lemma 4.7.1. To see that (3) holds, it suffices to show

$$\mathrm{dist}(P, \mathrm{Nil}(\mathfrak{A})) \leq \frac{1}{2}$$

by Lemma 4.7.1. Since $I_{\mathfrak{A}} - P$ is a properly infinite projection, for each $k \in \mathbb{N}$ there exists pairwise orthogonal projections $Q_{1,k}, Q_{2,k}, \ldots, Q_{k,k}$ such that P is equivalent to $Q_{j,k}$ for each $j \in \{1, \ldots, k\}$ and $\sum_{j=1}^{k} Q_{j,k} < I_{\mathfrak{A}} - P$.

Let

$$Q_k := P + \sum_{j=1}^{k} Q_{j,k}.$$

Then $P \in Q_k \mathfrak{A} Q_k$ for all $k \in \mathbb{N}$. Thus it suffices to show that

$$\inf_{k \geq 1} \mathrm{dist}(P, \mathrm{Nil}(Q_k \mathfrak{A} Q_k)) \leq \frac{1}{2}.$$

Since

$$\{P\} \cup \{Q_{j,k}\}_{j=1}^{k}$$

is a set of equivalent, pairwise orthogonal projections in $\mathfrak{A}$, we can use the partial isometries implementing the equivalence to construct a copy of $\mathcal{M}_{k+1}(\mathbb{C})$ with these projections as the orthogonal minimal projections. Moreover, by construction, inside this matrix algebra P is a rank one projection. Thus, by [32, Theorem 2.12], P is within $\frac{1}{2} + \sin\left(\frac{\pi}{m_k+1}\right)$ (where m_k is the integer part of $\frac{k}{2}$) of a nilpotent matrix. Thus

$$\mathrm{dist}(P, \mathrm{Nil}(Q_k\mathfrak{A}Q_k)) \leq \frac{1}{2} + \sin\left(\frac{\pi}{m_k+1}\right)$$

so the result follows. ☐

4.9 AFD C*-Algebras

In this section we will investigate when a normal operator in a AFD C*-algebra is a norm limit of nilpotent operators. The study of such operators is intrinsically related to how normal matrices can be asymptotically approximated by nilpotent matrices as we allow the dimension of our matrices to increase. Proposition 4.9.7 will provide conditions on an AFD C*-algebra that guarantee the intersection of the normal operators and the quasinilpotent operators is trivial whereas Theorem 4.9.5 exhibits an AFD C*-algebra $\mathfrak{A}$ where $\mathfrak{A}_{\mathrm{sa}} \cap \overline{\mathrm{Nil}(\mathfrak{A})} \neq \{0\}$. Moreover, in Theorem 4.9.8 which is the main result of this section, we will demonstrate that every UHF C*-algebra has a normal operator with spectrum equal to the closed unit disk that is a norm limit of nilpotent operators. All of this together (along with Proposition 4.10.10) implies that the study of $\mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{Nil}(\mathfrak{A})}$ for AFD C*-algebras $\mathfrak{A}$ is incredibly complex.

We begin with the following important result.

Proposition 4.9.1. *Let $\mathfrak{A}$ be an AFD C*-algebra and write $\mathfrak{A} = \overline{\bigcup_{k\geq 1} \mathfrak{A}_k}$ where each $\mathfrak{A}_k$ is a finite dimensional C*-algebra. For each $T \in \mathfrak{A}$ following are equivalent:*

1. $T \in \overline{\mathrm{QuasiNil}(\mathfrak{A})}$.

2. $T \in \overline{\mathrm{Nil}(\mathfrak{A})}$.

3. $T \in \overline{\bigcup_{k\geq 1} \mathrm{Nil}(\mathfrak{A}_k)}$.

Proof. Clearly (3) implies (2) and (2) implies (1). Suppose $T \in \overline{\mathrm{QuasiNil}(\mathfrak{A})}$. Let $\epsilon > 0$ and choose $M \in \mathrm{QuasiNil}(\mathfrak{A})$ such that $\|T - M\| < \epsilon$. Since $M \in \overline{\bigcup_{k\geq 1} \mathfrak{A}_k}$ and by the semicontinuity of the spectrum, there exist an $k \in \mathbb{N}$ and an operator $M_0 \in \mathfrak{A}_k$ such that $\|M_0 - M\| < \epsilon$ and

$$\sigma(M_0) \subseteq \{z \in \mathbb{C} \mid \mathrm{dist}(z, \sigma(M)) < \epsilon\} = \{z \in \mathbb{C} \mid |z| < \epsilon\}.$$

Since $\mathfrak{A}_k$ is a finite dimensional C*-algebra, $\mathfrak{A}_k$ is a direct sum of matrix algebras. Thus M_0 is unitarily equivalent to a direct sum of upper triangular matrices. Each of these upper triangular matrices is the sum of a nilpotent matrix and a diagonal matrix whose diagonal entries are in $\sigma(M_0)$. Since the equivalence is via a unitary, by subtracting the diagonal part we obtain an $M' \in \mathrm{Nil}(\mathfrak{A}_k)$ such that

$$\|M_0 - M'\| \leq \sup\{|z| \mid z \in \sigma(M_0)\} < \epsilon.$$

Therefore $\|T - M'\| < 3\epsilon$ completing the proof. $\square$

Remarks 4.9.2. The study of which normal operators of an AFD C*-algebra are in the closure of the nilpotent operators is intrinsically connected to the distribution of eigenvalues of normal matrices that are asymptotically approximated by nilpotent matrices as we allow the dimension of the matrices to increase.

Indeed if $\mathfrak{A}$ is an AFD C*-algebra with $\mathfrak{A} = \overline{\bigcup_{k\geq 1} \mathfrak{A}_k}$ where $\mathfrak{A}_1 \overset{\alpha_1}{\to} \mathfrak{A}_2 \overset{\alpha_2}{\to} \mathfrak{A}_3 \overset{\alpha_3}{\to} \cdots$ is a direct limit of finite dimensional C*-algebras with α_k injective for all $k \in \mathbb{N}$, then it is easy to see by Proposition 4.9.1 and by [38] that $N \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{Nil}(\mathfrak{A})}$ if and only if for each $k \in \mathbb{N}$ there exists an $N_k \in \mathrm{Nor}(\mathfrak{A}_k)$ such that $N = \lim_{k\to\infty} N_k$ and $\lim_{k\to\infty} \mathrm{dist}(N_k, \mathrm{Nil}(\mathfrak{A}_k)) = 0$.

Moreover, since $N = \lim_{k\to\infty} N_k$, $\lim_{k\to\infty} \|\alpha_k(N_k) - N_{k+1}\| = 0$. This is possible only if for each $k \in \mathbb{N}$ the eigenvalues of $\alpha_k(N_k)$ and N_{k+1} (including multiplicities) can be paired together in a manner such that the maximum of the absolute values of the differences tends to zero as k tends to infinity.

Similarly, if $N_k \in \mathrm{Nor}(\mathfrak{A}_k)$ can be chosen such that for each $k \in \mathbb{N}$ the eigenvalues of $\alpha_k(N_k)$ and N_{k+1} (including multiplicities) can be paired together inside the appropriate direct summand of $\mathfrak{A}_{k+1}$ in a manner such that the maximum of the absolute values of the differences tends to zero as k tends to infinity and $\lim_{k\to\infty} \mathrm{dist}(N_k, \mathrm{Nil}(\mathfrak{A}_k)) = 0$, then, by taking unitary conjugates of the matrices N_k, it is possible to construct a Cauchy sequence in $\mathfrak{A}$ that converges to a normal operator N in the closure of the nilpotent operators.

Example 4.9.3. For each $n \in \mathbb{N}$ let $A_n \in \mathcal{M}_{2^n}(\mathbb{C})$ be a diagonal matrix with spectrum $\left\{\frac{1}{2^n}, \frac{2}{2^n}, \ldots, 1\right\}$. Then

$$\liminf_{n\to\infty} \mathrm{dist}(A_n, \mathrm{Nil}(\mathcal{M}_{2^n}(\mathbb{C}))) > 0.$$

To see this, we note that the sequence $(A_n)_{n\geq 1}$ can be used to construct a Cauchy sequence in the 2^∞-UHF C*-algebra $\mathfrak{A}$ that converges to a non-zero positive operator A. If $\liminf_{n\to\infty} \mathrm{dist}(A_n, \mathrm{Nil}(\mathcal{M}_{2^n}(\mathbb{C}))) = 0$ then A would be the limit of elements of $\mathrm{Nil}(\mathfrak{A})$ which would contradict Proposition 4.3.6 as $\mathfrak{A}$ has a faithful tracial state.

Alternatively

$$\liminf_{n\to\infty} \mathrm{dist}(A_n, \mathrm{Nil}(\mathcal{M}_{2^n}(\mathbb{C}))) \geq \frac{1}{2}$$

since the normalized trace on $\mathcal{M}_{2^n}(\mathbb{C})$ has norm one, the normalized traces of A_n tend to $\frac{1}{2}$ as n tends to infinity, and the trace of any nilpotent matrix is zero.

Note, in the above example, we can view each A_n as a positive operator whose spectrum is the first 2^n entries of the sequence $\{1, \frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{7}{8}, \ldots\}$. Thus, by Remarks 4.9.2, we are interested in the following question: "Given a sequence $(a_n)_{n\geq 1} \in \ell_\infty(\mathbb{N})$ does $\liminf_{n\to\infty} \mathrm{dist}(\mathrm{diag}(a_1, \ldots, a_n), \mathrm{Nil}(\mathcal{M}_n(\mathbb{C}))) = 0$?" An application of Lemma 4.1.3 implies $\overline{\{a_n\}_{n\geq 1}}$ must be a connected set containing zero in order for an affirmative answer to this question. Thus the following is of particular interest.

Proposition 4.9.4. *There exists a sequence* $(a_n)_{n\geq 1} \in \ell_\infty(\mathbb{C})_+$ *with* $\overline{\{a_n\}_{n\geq 1}} = [0,1]$ *such that*

$$\liminf_{n\to\infty} \mathrm{dist}(\mathrm{diag}(a_1,\ldots,a_n), \mathrm{Nil}(\mathcal{M}_n(\mathbb{C}))) = 0.$$

Proof. By Lemma 4.8.1 for each $n \in \mathbb{N}$ there exists a positive matrix $A_n \in \mathcal{M}_n(\mathbb{C})$ of norm one such that $\lim_{n\to\infty} \mathrm{dist}(A_n, \mathrm{Nil}(\mathcal{M}_n(\mathbb{C}))) = 0$. Choose $n_1 \in \mathbb{N}$ such that

$$\mathrm{dist}(A_{n_1}, \mathrm{Nil}(\mathcal{M}_{n_1}(\mathbb{C}))) \leq 1.$$

Let the first n_1 of the scalars a_j be the eigenvalues of A_{n_1} (including multiplicity).

Let $R_1 := A_{n_1}$. By Corollary 4.8.2 $\sigma(A_n)$ progressively gets dense in $[0,1]$ as n increases. Therefore there exists an $n_2 \in \mathbb{N}$ such that

$$\sigma(A_{n_2}) \cap \left[\frac{k}{2^2}, \frac{k+1}{2^2}\right) \neq \emptyset$$

for all $k \in \{0,1,2,3\}$ and

$$\mathrm{dist}(A_{n_2}, \mathrm{Nil}(\mathcal{M}_{n_2}(\mathbb{C}))) \leq \frac{1}{2^2}.$$

By comparing the eigenvalues of R_1 and A_{n_2} there exists an $m_1 \in \mathbb{N}$ and an injective map f_1 from the eigenvalues of R_1 (including multiplicity) to the eigenvalues of $A_{n_2}^{\oplus m_1}$ (including multiplicity) such that $|\lambda - f_1(\lambda)| \leq \frac{1}{4}$ for all eigenvalues λ of R_1 (including multiplicity). Therefore, if $A_{n_2}^{\oplus m_1} \ominus R_1$ denotes the $(m_1n_2 - n_1) \times (m_1n_2 - n_1)$ diagonal matrix whose diagonal entries are the eigenvalues of $A_{n_2}^{\oplus m_1}$ (including multiplicities) excluding $f_1(\lambda)$ for all eigenvalues λ of R_1 (including multiplicity), then

$$R_2 := R_1 \oplus (A_{n_2}^{\oplus m_1} \ominus R_1)$$

is within $\frac{1}{4}$ of a unitary conjugate of $A_{n_2}^{\oplus m_1}$ and thus

$$\begin{aligned}\operatorname{dist}(R_2, \operatorname{Nil}(\mathcal{M}_{n_2m_1}(\mathbb{C})) &\leq \tfrac{1}{4} + \operatorname{dist}(A_{n_2}^{\oplus m_1}, \operatorname{Nil}(\mathcal{M}_{n_2m_1}(\mathbb{C})) \\ &\leq \tfrac{1}{4} + \operatorname{dist}(A_{n_2}, \operatorname{Nil}(\mathcal{M}_{n_2}(\mathbb{C})) \leq \tfrac{1}{2}.\end{aligned}$$

Thus define the next $m_1n_2 - n_1$ of the scalars a_j to be the eigenvalues of $A_{n_2}^{\oplus m_1} \ominus R_1$ (including multiplicity).

By continuing this process ad infinitum, the desired sequence $(a_n)_{n\geq 1}$ is obtained. □

Of course the existence of the above sequence does not imply that there exists an AFD C*-algebra with a non-zero positive operator in the closure of the nilpotent operators as the structure required for such an operator is more complex (see Remarks 4.9.2). However, an example of such a AFD C*-algebra is an easy application of the theory developed in Section 4.8.

Theorem 4.9.5. *There exists an AFD C*-algebra* $\mathfrak{A}$ *such that* $\mathfrak{A}_+ \cap \overline{\operatorname{Nil}(\mathfrak{A})} \neq \{0\}$.

Proof. Let $\mathcal{O}_2$ be the Cuntz algebra generated by two isometries. Since $\mathcal{O}_2$ is a separable, nuclear C*-algebra, the cone of $\mathcal{O}_2$, $\mathfrak{C} := C_0((0,1], \mathcal{O}_2)$, is AF-embeddable (see [50, Proposition 2] or [14, Theorem 8.3.5]). Hence there exists an AFD C*-algebra $\mathfrak{A}$ such that $\mathfrak{C} \subseteq \mathfrak{A}$. Thus it suffices to show $\mathfrak{C}_+ \cap \overline{\operatorname{Nil}(\mathfrak{C})} \neq \{0\}$.

Let $A \in (\mathcal{O}_2)_+ \setminus \{0\}$ be such that $\sigma(A) = [0,1]$ and let $A' \in \mathfrak{C}_+$ be defined by $A'(x) = Ax$ for all $x \in (0,1]$. Since $A \neq 0$, $A' \neq 0$. Since $A \in \overline{\operatorname{Nil}(\mathcal{O}_2)}$ by Theorem 4.8.6 (or simply Proposition 4.8.4), it is trivial to verify that $A' \in \overline{\operatorname{Nil}(\mathfrak{C})}$ as desired. □

Using Theorem 4.9.5 and Proposition 4.9.1, it is easy to obtain the following that enables us to improve Lemma 4.8.1 by bounding the nilpotency degrees of the approximating nilpotent matrices. Theorem 4.9.5, Proposition 4.9.1, and Remarks 4.9.2 together also imply that Lemma 4.8.1 holds with the additional property that the distribution of eigenvalues of the sequence A_n is 'not too poorly behaved'.

Corollary 4.9.6. *There exists an increasing sequence of natural numbers $(k_n)_{n\geq 1}$ and a sequence of positive matrices $A_n \in \mathcal{M}_{k_n}(\mathbb{C})$ of norm one such that for every $\epsilon > 0$ there exists an index $m \in \mathbb{N}$ and a $\ell \in \mathbb{N}$ such that*

$$\mathrm{dist}(A_n, \mathrm{Nil}_\ell(\mathcal{M}_{k_n}(\mathbb{C}))) < \epsilon$$

for all $n \geq m$ (where $\mathrm{Nil}_\ell(\mathcal{M}_{k_n}(\mathbb{C}))$ is the set of nilpotent $k_n \times k_n$-matrices of nilpotency index at most ℓ).

Next we have the following trivial observation that demonstrates several AFD C*-algebras where no non-zero normal operators are limits of quasinilpotent operators.

Proposition 4.9.7. *Suppose $\mathfrak{A} = \overline{\bigcup_{k\geq 1} \mathfrak{A}_k}$ where $\mathfrak{A}_1 \overset{\alpha_1}{\to} \mathfrak{A}_2 \overset{\alpha_2}{\to} \mathfrak{A}_3 \overset{\alpha_3}{\to} \cdots$ is a direct limit of finite dimensional C*-algebras with α_k injective for all $k \in \mathbb{N}$. If $\mathfrak{A}_k = \oplus_{j=1}^{m_k} \mathcal{M}_{n_{j,k}}(\mathbb{C})$ and $\{n_{j,k}\}_{j,k\geq 1}$ is a bounded set, then $\mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{QuasiNil}(\mathfrak{A})} = \{0\}$.*

Proof. Suppose $N \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{QuasiNil}(\mathfrak{A})}$ and let $\ell := \sup_{j,k\geq 1} n_{j,k} < \infty$. Therefore $M^\ell = 0$ for all $M \in \bigcup_{k\geq 1} \mathrm{Nil}(\mathfrak{A}_k)$ so $N^\ell = 0$ by Proposition 4.9.1. Hence $N = 0$. □

The main result of this section is Theorem 4.9.8 which gives examples of normal operators in each UHF C*-algebra that are limits of nilpotent operators. This result is slightly surprising since every UHF C*-algebra has a faithful tracial state yet Section 4.3 demonstrated that faithful tracial states impose restrictions on when normal operators can be limits of nilpotent operators. In particular, Proposition 4.3.6 shows that $\mathfrak{A}_{\mathrm{sa}} \cap \overline{\mathrm{QuasiNil}(\mathfrak{A})} = \{0\}$ for every UHF C*-algebra $\mathfrak{A}$ (also see Corollary 4.3.8, Lemma 4.3.12, and Theorem 4.3.13).

The main tool in this construction is Lemma 4.4.4 which is based on Section 2.3.3 of [32]. The following result was known to Marcoux and was communicated to the author.

Theorem 4.9.8 (Marcoux)**.** *Let $\mathfrak{A}$ be a nonelementary UHF C*-algebra. There exists an $N \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{Nil}(\mathfrak{A})}$ such that $\sigma(N)$ is the closed unit disk.*

Proof. Write $\mathfrak{A} = \overline{\bigcup_{k\geq 1} \mathcal{M}_{\ell_k}(\mathbb{C})}$ where $\mathcal{M}_{\ell_1}(\mathbb{C}) \overset{\alpha_1}{\to} \mathcal{M}_{\ell_2}(\mathbb{C}) \overset{\alpha_2}{\to} \mathcal{M}_{\ell_3}(\mathbb{C}) \overset{\alpha_3}{\to} \cdots$ is a direct limit of full matrix algebras with α_k injective for all $k \in \mathbb{N}$. Moreover we can assume that $\frac{\ell_{k+1}}{\ell_k}$ is composite for all $k \in \mathbb{N}$ and $\ell_1 \geq 11$.

For each $k \in \mathbb{N}$ we will construct $n_k, m_k \in \mathbb{N}$ and $q_k \in \mathbb{N} \cup \{0\}$ such that $m_1, n_1 \geq 2$, $(2m_k + 1)n_k + 1 + q_k = \ell_k$ for all $k \in \mathbb{N}$, $2m_k \leq m_{k+1}$ for all $k \in \mathbb{N}$, $2n_k \leq n_{k+1}$ for all $k \in \mathbb{N}$, and, if $N_k \in \mathrm{Nor}(\mathcal{M}_{(2m_k+1)n_k+1+q_k}(\mathbb{C}))$ is a specific unitary conjugate of the normal matrix obtain by taking the direct sum of the $q_k \times q_k$ zero matrix with the normal matrix from Lemma 4.4.4 with $n = n_k$, $m = m_k$, and $a_j = \frac{j}{m_k}$ for all $j \in \{0, 1, \ldots, m_k\}$ then $(N_k)_{k\geq 1}$ is a Cauchy sequence in $\mathfrak{A}$.

If such a sequence exists then, since $\lim_{k\to\infty} m_k = \infty$ and $\lim_{k\to\infty} n_k = \infty$ and since adding a zero direct summand at most decreases the distance to the nilpotent operators, Lemma 4.4.4 implies

$$\lim_{k\to\infty} \mathrm{dist}(N_k, \mathrm{Nil}(\mathcal{M}_{\ell_k}(\mathbb{C}))) = 0.$$

Thus, if $N = \lim_{k\to\infty} N_k$ then $N \in \mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{Nil}(\mathfrak{A})}$ by construction. Since $\|N_k\| \leq 1$, $\|N\| \leq 1$. Since $\lim_{k\to\infty} m_k = \infty$ and $\lim_{k\to\infty} n_k = \infty$, Lemma 4.4.4 implies the intersection of $\sigma(N_k)$ with any open subset of the closed unit ball is non-empty for sufficiently large k. This implies $\sigma(N)$ is the closed unit disk by the semicontinuity of the spectrum.

To show that the claim is true, let $m_1 = 2$ and select $n_1 \in \mathbb{N}$ with $n_1 \geq 2$ and $q_1 \in \{0, 1, 2, 3, 4\}$ such that $\ell_1 = (2m_1 + 1)n_1 + 1 + q_1$. Let N_1 be as described above.

Suppose we have performed the construction for some fixed $k \in \mathbb{N}$. Since $\frac{\ell_{k+1}}{\ell_k}$ is composite, we may write $\frac{\ell_{k+1}}{\ell_k} = pz$ where $p, z \geq 2$. Then, when we view N_k as an element of $\mathcal{M}_{\ell_{k+1}}(\mathbb{C})$, each eigenvalue of N_k has pz times the multiplicity it did in $\mathcal{M}_{\ell_k}(\mathbb{C})$. Let $n_{k+1} := pn_k \geq 2n_k$ and $m_{k+1} := zm_k \geq 2m_k$. Then

$$(2m_{k+1} + 1)n_{k+1} + 1 = \ell_{k+1} - ((z-1)pn_k + pz + pzq_k - 1).$$

Thus let $q_{k+1} := ((z-1)pn_k + pz + pzq_k - 1) \geq 0$ so

$$(2m_{k+1}+1)n_{k+1} + 1 + q_{k+1} = \ell_{k+1}.$$

If N'_{k+1} is the normal matrix obtain by taking the direct sum of the $q_k \times q_k$ zero matrix with the normal matrix from Lemma 4.4.4 with $n = n_{k+1}$, $m = m_{k+1}$, and $a_j = \frac{j}{m_{k+1}}$ for all $j \in \{0, 1, \ldots, m_{k+1}\}$, then, by construction, we can pair the eigenvalues of N_k (including multiplicity) when viewed an element of $\mathcal{M}_{\ell_{k+1}}(\mathbb{C})$ with the eigenvalues of N'_{k+1} in a bijective way such that the difference of any pair is at most $\frac{\pi}{n_k} + \frac{1}{m_k}$ by our knowledge of the eigenvalues from Lemma 4.4.4. Thus there exists a unitary conjugate N_{k+1} of N'_{k+1} that is within $\frac{\pi}{n_k} + \frac{1}{m_k}$ of the image of N_k in $\mathcal{M}_{\ell_{k+1}}(\mathbb{C})$. Since $2m_k \leq m_{k+1}$ for all $k \in \mathbb{N}$ and $2n_k \leq n_{k+1}$ for all $k \in \mathbb{N}$, this implies $(N_k)_{k\geq 1}$ is a Cauchy sequence in $\mathfrak{A}$ as desired. $\square$

Note that Theorem 4.4.8 can now be applied to every UHF C*-algebra by Theorem 4.9.8.

To conclude this section, we will demonstrate that [29, Corollary 6] cannot be generalized to AFD C*-algebras (it was demonstrated in Section 4.7 that [29, Corollary 9] cannot be generalized to C*-algebras with faithful tracial states). It is the existence of faithful tracial states on finite dimensional C*-algebras that prevent the generalization of Herrero's result.

Lemma 4.9.9. *Let $\mathfrak{A}$ be a unital AFD C*-algebra and let $T \in \mathfrak{A}$. Then each of the following sets is either the empty set or a singleton:*

1. $\left\{\lambda \in \mathbb{C} \mid \lambda I_{\mathfrak{A}} + T \in \overline{\mathrm{Nil}(\mathfrak{A})}\right\}$.
2. $\left\{\lambda \in \mathbb{C} \mid \lambda I_{\mathfrak{A}} + T \in \overline{\mathrm{span}(\mathrm{Nil}(\mathfrak{A}))}\right\}$.

Proof. We shall only prove the first claim since the proof of the second claim is exactly the same. Suppose

$$\lambda_0 \in \left\{\lambda \in \mathbb{C} \mid \lambda I_{\mathfrak{A}} + T \in \overline{\mathrm{Nil}(\mathfrak{A})}\right\}$$

and let $R := \lambda_0 I_{\mathfrak{A}} + T$. Thus to show that $\lambda I_{\mathfrak{A}} + T \notin \overline{\mathrm{Nil}(\mathfrak{A})}$ for all $\lambda \in \mathbb{C} \setminus \{\lambda_0\}$ it suffices to show that $\mu I_{\mathfrak{A}} + R \notin \overline{\mathrm{Nil}(\mathfrak{A})}$ for all $\mu \in \mathbb{C} \setminus \{0\}$.

Since $\mathfrak{A}$ is a unital AFD C*-algebra, $\mathfrak{A} = \overline{\bigcup_{k\geq 1}\mathfrak{A}_k}$ where $\mathfrak{A}_1 \xrightarrow{\alpha_1} \mathfrak{A}_2 \xrightarrow{\alpha_2} \mathfrak{A}_3 \xrightarrow{\alpha_3} \cdots$ is a direct limit of finite dimensional C*-algebras with α_k unital and injective for all $k \in \mathbb{N}$. Therefore there exists $R_k \in \mathfrak{A}_k$ such that $R = \lim_{k\to\infty} R_k$. However, since $R \in \overline{\mathrm{Nil}(\mathfrak{A})}$, Proposition 4.9.1 implies that $R = \lim_{k\to\infty} M_k$ where $M_k \in \mathrm{Nil}(\mathfrak{A}_k)$ for all $k \in \mathbb{N}$. Hence $\lim_{k\to\infty} \|R_k - M_k\| = 0$. Thus $\lim_{k\to\infty} \mathrm{tr}_{\mathfrak{A}_k}(R_k) = 0$ (where $\mathrm{tr}_{\mathfrak{A}_k}$ is any faithful tracial state on $\mathfrak{A}_k$) as every nilpotent matrix has zero trace.

Fix $\mu \in \mathbb{C} \setminus \{0\}$. Then $\mu I_{\mathfrak{A}} + R = \lim_{k\to\infty} \mu I_{\mathfrak{A}_k} + R_k$. If $\mu I_{\mathfrak{A}} + R \in \overline{\mathrm{Nil}(\mathfrak{A})}$ then the above argument implies that $\lim_{k\to\infty} \mathrm{tr}_{\mathfrak{A}_k}(\mu I_{\mathfrak{A}_k} + R_k) = 0$ which is impossible as $\mu \neq 0$ and $\lim_{k\to\infty} \mathrm{tr}_{\mathfrak{A}_k}(R_k) = 0$. ☐

Corollary 4.9.10. *Let $\mathfrak{A}$ be a unital AFD C*-algebra. Then*

$$I_{\mathfrak{A}} \notin \overline{\mathrm{span}(\mathrm{Nil}(\mathfrak{A}))}.$$

Proof. Note $0 \in \overline{\mathrm{span}(\mathrm{Nil}(\mathfrak{A}))}$ and apply Lemma 4.9.9. ☐

4.10 C*-Algebras with Dense Subalgebras of Nilpotent Operators

In [57], Read gave an example of a separable C*-algebra that contains a dense subalgebra consisting entirely of nilpotent operators. In this section we will use Lemma 4.8.1 and the construction in [57] to construct an approximately homogeneous (and thus separable, nuclear, and quasidiagonal) C*-algebra that contains dense subalgebra consisting entirely of nilpotent operators. It will also be demonstrated that there exists an AFD C*-algebra with a C*-subalgebra $\mathfrak{D}$ where $\mathfrak{D} = \overline{\mathrm{Nil}(\mathfrak{D})}$. Thus the study of the closure of nilpotent operators in AFD C*-algebras is incredibly complex.

Construction 4.10.1. Note by Lemma 4.8.1 there exists finite dimensional Hilbert spaces $\{\mathcal{H}_n\}_{n\geq 1}$, positive matrices $A_n \in \mathcal{B}(\mathcal{H}_n)$ of norm one, and nilpotent matrices $M_n \in \mathcal{B}(\mathcal{H}_n)$ such that $\sum_{n\geq 1} \|A_n - M_n\| < \infty$. Since each A_n is a positive matrix with norm one, there exists unit vectors $\xi_n \in \mathcal{H}_n$ such that $A_n\xi_n = \xi_n$ for all $n \in \mathbb{N}$.

We will use $\{\mathcal{H}_n\}_{n\geq 1}$ and $\{\xi_n\}_{n\geq 1}$ to generalize Read's construction. Consider the sequence of pointed Hilbert spaces $(\mathcal{H}_n, \xi_n)$. For each $n < m$ define $\phi_{n,m} : \otimes_{k=1}^n \mathcal{H}_k \to \otimes_{k=1}^m \mathcal{H}_k$ such that

$$\phi_{n,m}(\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n) = \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n \otimes \xi_{n+1} \otimes \xi_{n+2} \otimes \cdots \otimes \xi_m.$$

Let $\mathcal{K} := \otimes_{k=1}^\infty \mathcal{H}_k$ be the completion of the direct limit of the nested sequence of Hilbert spaces $\otimes_{k=1}^n \mathcal{H}_k$ with the connecting maps $\phi_{n,m}$. Since each $\mathcal{H}_k$ is separable, each $\otimes_{k=1}^n \mathcal{H}_k$ is separable and thus $\mathcal{K}$ is separable. Let $\phi_n : \otimes_{k=1}^n \mathcal{H}_k \to \mathcal{K}$ be the natural inclusion.

We will maintain the above notation throughout the rest of this section.

The following are new versions of [57, Lemma 0.3] and [57, Corollary 0.4] respectively that will serve our purposes. We omit the proofs as they follow as in [57].

Lemma 4.10.2. *Let $(S_n)_{n\geq 1}$ be a sequence of operators with $S_n \in \mathcal{B}(\mathcal{H}_n)$ such that*

$$C := \prod_{n\geq 1} \max\{\|S_n\|, 1\} < \infty \quad \text{and} \quad \sum_{n\geq 1} \|S_n\xi_n - \xi_n\| < \infty.$$

Then there exists a unique operator $S' \in \mathcal{B}(\mathcal{K})$ such that $S'(\phi_n\zeta) = S'_n\zeta$ for each $\zeta \in \otimes_{k=1}^n \mathcal{H}_k$ where $S'_n := \lim_{m\to\infty} S'_{n,m}$ where, for each $m > n$, $S'_{n,m} : \otimes_{k=1}^n \mathcal{H}_k \to \mathcal{K}$ is defined by

$$S'_{n,m} = \phi_m \circ (\otimes_{i=1}^m S_i) \circ \phi_{n,m}.$$

We will use $\otimes_{n=1}^\infty S_n$ to denote S'.

Corollary 4.10.3. *Let $S' = \otimes_{n=1}^\infty S_n$ and $R' = \otimes_{n=1}^\infty R_n$ be elements of $\mathcal{B}(\mathcal{K})$ as constructed in Lemma 4.10.2. Then*

$$\|S' - R'\| \leq C_S C_R \sum_{n\geq 1} \|S_n - R_n\|$$

where

$$C_S := \prod_{n\geq 1} \max\{1, \|S_n\|\} \quad \text{and} \quad C_R := \prod_{n\geq 1} \max\{1, \|R_n\|\}.$$

Construction 4.10.4. Let $\mathfrak{B}$ be the C*-subalgebra of $\mathcal{B}(\mathcal{K})$ generated by all operators of the form $\otimes_{n=1}^{\infty} S_n$ given by Lemma 4.10.2. Let $\mathcal{E}_A$ be subset of $\mathfrak{B}$ containing all operators of the form $\otimes_{n=1}^{\infty} S_n$ from Lemma 4.10.2 such that there exist a $k \in \mathbb{N}$ such that $S_n = A_n$ for each $n \geq k$. Since $\sum_{n\geq 1} \|A_n\xi_n - \xi_n\| = 0$ and $\|A_n\| = 1$ for all $n \in \mathbb{N}$, $\mathcal{E}_A$ is non-empty. Let $\mathfrak{C}$ be the C*-algebra generated by $\mathcal{E}_A$. Note that $\mathcal{E}_A$ is a self-adjoint set so $\mathfrak{C}$ is the closure of the algebra generated by $\mathcal{E}_A$.

Lemma 4.10.5. *The C*-algebra $\mathfrak{C}$ from Construction 4.10.4 is nuclear, quasidiagonal, approximately homogeneous, and separable.*

Proof. For each $k \in \mathbb{N}$ let $\mathfrak{C}_k$ be the C*-subalgebra of $\mathfrak{C}$ generated by all operators of the form $\otimes_{n=1}^{\infty} S_n$ from Lemma 4.10.2 such that $S_n = A_n$ for all $n > k$. Then $\mathfrak{C}_k$ is isomorphic to $\mathcal{B}(\mathcal{H}_1) \otimes_{\min} \cdots \otimes_{\min} \mathcal{B}(\mathcal{H}_k) \otimes_{\min} \mathfrak{A}_{k+1}$ where $\mathfrak{A}_{k+1}$ is the abelian C*-algebra generated by the infinite tensor $\otimes_{n=k+1}^{\infty} S_n$ where $S_n = A_n$ for all $n > k$. Since $\mathfrak{C}$ is the inductive limit of the C*-algebras $\mathfrak{C}_k$, the result follows. $\square$

Theorem 4.10.6. *The C*-algebra $\mathfrak{C}$ from Construction 4.10.4 has a dense subalgebra $\mathfrak{N}$ such that every operator of $\mathfrak{N}$ is nilpotent.*

Proof. This proof is nearly identical to that of [57, Theorem 1.2] where the only changes are our simple modifications. Let $\mathcal{E}_N$ be the subset of $\mathfrak{B}$ consisting of all operators of the form $\otimes_{n=1}^{\infty} S_n$ from Lemma 4.10.2 such that there exist a $k \in \mathbb{N}$ such that $S_n = M_n$ for all $n \geq k$. Let $\mathfrak{N}$ be the (not necessarily closed nor self-adjoint) subalgebra of $\mathfrak{B}$ generated by $\mathcal{E}_N$. It suffices to show three things: (1) $\mathfrak{N} \subseteq \mathfrak{C}$; (2) $\mathfrak{N}$ is dense in $\mathfrak{C}$; (3) every operator of $\mathfrak{N}$ is nilpotent.

Proof of (1): It suffices to show that $\mathcal{E}_N \subseteq \mathfrak{C}$. To begin we will show that $\mathfrak{N}$ is not empty. Suppose that $(S_n)_{n\geq 1}$ is a sequence of operators where $S_n \in \mathcal{B}(\mathcal{H}_n)$ for all $n \in \mathbb{N}$ and $S_n = M_n$ for all $n \geq k$ for some fixed $k \in \mathbb{N}$. Since $\|A_n\| = 1$ for all $n \in \mathbb{N}$ and $\sum_{n\geq 1} \|M_n - A_n\| < \infty$, $\sum_{n\geq 1} | \|S_n\| - 1| < \infty$ and so $\prod_{n\geq 1} \max\{1, \|S_n\|\} < \infty$. Moreover, since $\sum_{n\geq 1}\|A_n\xi_n - \xi_n\| = 0$,

$$\sum_{n\geq k}\|S_n\xi_n - \xi_n\| \leq \sum_{n\geq k}\|M_n - A_n\| + \sum_{n\geq k}\|A_n\xi_n - \xi_n\| < \infty.$$

Hence Lemma 4.10.2 implies that the operator $\otimes_{n=1}^{\infty}S_n$ exists. Hence $\mathfrak{N}$ is not empty.

Fix a sequence $(S_n)_{n\geq 1}$ of operators where $S_n \in \mathcal{B}(\mathcal{H}_n)$ for all $n \in \mathbb{N}$ and $S_n = M_n$ for all $n \geq k$. For each $m \geq k$ define $R_m := (\otimes_{n=1}^{m}S_n) \otimes \left(\otimes_{n=m+1}^{\infty}A_n\right)$. Then $\{R_m\}_{m\geq k} \subseteq \mathcal{E}_A$ by construction and, by Corollary 4.10.3,

$$\|\otimes_{n=1}^{\infty}S_n - R_m\| \leq \left(\prod_{n\geq 1}\max\{\|S_n\|,1\}\right)^2 \sum_{n\geq m+1}\|A_n - M_n\|.$$

Therefore, since $\lim_{m\to\infty}\sum_{n\geq m+1}\|A_n - M_n\| = 0$, $\otimes_{n=1}^{\infty}S_n$ is in the closure of $\{R_m\}_{m\geq k}$ and thus $\otimes_{n=1}^{\infty}S_n \in \mathfrak{C}$. Hence $\mathfrak{N} \subseteq \mathfrak{C}$ as desired.

Proof of (2): It suffices to show that $\mathcal{E}_A$ is in the closure of $\mathfrak{N}$ since $\mathfrak{C}$ is the closure of the algebra (and not *-algebra) generated by $\mathcal{E}_A$. Fix an operator $T := \left(\otimes_{n=1}^{k}S_n\right)\otimes\left(\otimes_{n=k+1}^{\infty}A_n\right) \in \mathcal{E}_A$. For each $m \geq k$ let $R_m := (\otimes_{n=1}^{m}S_n) \otimes \left(\otimes_{n=m+1}^{\infty}M_n\right)$. Then $\{R_m\}_{m\geq k} \subseteq \mathfrak{N}$ and, by Corollary 4.10.3, $\|T - R_m\|$ is at most

$$\left(\prod_{n=1}^{k}\max\{\|S_n\|,1\}\right)^2 \left(\prod_{n\geq 1}\max\{\|M_n\|,1\}\right) \sum_{n\geq m+1}\|A_n - M_n\|.$$

Therefore, since $\lim_{m\to\infty}\sum_{n\geq m+1}\|A_n - M_n\| = 0$, $T \in \overline{\mathfrak{N}}$. Hence $\mathcal{E}_A$ is in the closure of $\mathfrak{N}$ so $\mathfrak{N}$ is dense in $\mathfrak{C}$.

Proof of (3): Notice that every operator N of $\mathfrak{N}$ can be written in the form

$$N = \sum_{k=1}^{\ell} S_k \otimes \left(\otimes_{i=n+1}^{\infty}M_i^k\right)$$

for some $n, \ell \in \mathbb{N}$ and $S_1, \ldots, S_\ell \in \mathcal{B}\left(\otimes_{k=1}^{n}\mathcal{H}_k\right)$. Therefore, since there exists an $m_{n+1} \in \mathbb{N}$ such that $M_{n+1}^{m_{n+1}} = 0$, $N^{m_{n+1}} = 0$ (by the trivial computation that $(\otimes_{n=1}^{\infty}R_n)(\otimes_{n=1}^{\infty}R_n') =$

$\otimes_{n=1}^{\infty} R_n R_n'$). Hence N is nilpotent so every operator of $\mathfrak{N}$ is nilpotent. $\square$

One interesting consequence is the following which is quite surprising since every other C*-algebra $\mathfrak{A}$ with $\mathrm{Nor}(\mathfrak{A}) \cap \overline{\mathrm{Nil}(\mathfrak{A})} \neq \{0\}$ studied in this dissertation has had a plethora of projections.

Corollary 4.10.7. *Let $\mathfrak{C}$ be the C*-algebra from Construction 4.10.4. Then $\sigma(T)$ is connected and contains zero for all $T \in \mathfrak{C}$. Thus $\mathfrak{C}$ is projectionless.*

Proof. The result is trivial by Theorem 4.10.6 and Lemma 1.8.4. $\square$

To conclude this section we will demonstrate that there exists an AFD C*-algebra that contains a C*-subalgebra $\mathfrak{D}$ such that $\mathfrak{D} = \overline{\mathrm{Nil}(\mathfrak{D})}$. This demonstrates that the study of the closure of the nilpotent operators in an AFD C*-algebra is incredibly complex. To begin we note the following trivial observation from the proof of Theorem 4.10.6.

Lemma 4.10.8. *Let $\mathfrak{C}$ be the C*-algebra from Construction 4.10.4, let $\mathfrak{N}$ be the subalgebra of $\mathfrak{C}$ from Theorem 4.10.6, and $N_1, \ldots, N_m \in \mathfrak{N}$. Then there exists an $\ell \in \mathbb{N}$ (depending on $N_1, \ldots, N_m$) such that $N_{n_1} N_{n_2} \cdots N_{n_\ell} = 0$ for any selection of $n_j \in \{1, \ldots, m\}$.*

Proof. This result is trivial by the structure of elements of $\mathfrak{N}$ from the third part of the proof of Theorem 4.10.6. $\square$

Lemma 4.10.9. *Let $\mathfrak{C}$ be the C*-algebra from Construction 4.10.4 and let $\mathfrak{N}$ be the subalgebra of $\mathfrak{C}$ from Theorem 4.10.6. The subalgebra*

$$C_0(0,1] \odot \mathfrak{N} := \left\{ \sum_{j=1}^{m} f_j \otimes N_j \;\middle|\; m \in \mathbb{N}, N_j \in \mathfrak{N}, f_j \in C_0(0,1] \right\}$$

of $C_0(0,1] \otimes_{\min} \mathfrak{C}$ is dense and consists entirely of nilpotent operators.

Proof. Clearly $C_0(0,1] \odot \mathfrak{N}$ is a dense subalgebra of $C_0(0,1] \otimes_{\min} \mathfrak{C}$ as $\mathfrak{N}$ is a dense subalgebra of $\mathfrak{C}$. Let $\sum_{j=1}^{m} f_j \otimes N_j \in C_0(0,1] \odot \mathfrak{N}$ be arbitrary. By Lemma 4.10.8 there exists an $\ell \in \mathbb{N}$

such that $N_{n_1}N_{n_2}\cdots N_{n_\ell}=0$ for any $n_j\in\{1,\ldots,m\}$. Thus $\left(\sum_{j=1}^m f_j\otimes N_j\right)^\ell=0$ so every element of $C_0(0,1]\odot\mathfrak{N}$ is nilpotent. $\square$

Proposition 4.10.10. *There exists an AFD C*-algebra $\mathfrak{A}$ and a C*-subalgebra $\mathfrak{D}$ of $\mathfrak{A}$ such that $\mathfrak{D}$ has a dense subalgebra consisting entirely of nilpotent operators.*

Proof. Let $\mathfrak{C}$ be the C*-algebra from Construction 4.10.4 and let

$$\mathfrak{D}:=C_0(0,1]\otimes_{\min}\mathfrak{C}.$$

Then $\mathfrak{D}$ is AF-embeddable by Lemma 4.10.5 and by [50, Proposition 2]. Thus the result follows from Lemma 4.10.9. $\square$

CHAPTER 5

Closed Unitary and Similarity Orbits in Purely Infinite C*-Algebras

In this chapter, which is based on the author's work in [76], we will investigate the norm closure of the unitary and similarity orbits of normal operators in unital, simple, purely infinite C*-algebras. In Section 5.2 we shall use previously known techniques based on [40, Theorem 4.4] to provide a simple proof of the classification of when two normal operators are approximately unitarily equivalent in a unital, simple, purely infinite C*-algebra with trivial K_1-group. Although this proof is less powerful than [17, Theorem 1.7], the techniques used enables the study of additional operator theoretic problems on these C*-algebras. Section 5.2 will also develop the necessary technical results and techniques needed in later sections.

One particularly interesting problem is the study of the distance between unitary orbits of operators. Significant progress has been made in determining the distance between two unitary orbits of bounded operators on a complex, infinite dimensional Hilbert space (see [19] and [20]). In terms of determining the distance between unitary orbits of normal operators inside other C*-algebras, [18] makes significant progress for the Calkin algebra (which is a unital, simple, purely infinite C*-algebra) and [33] makes significant progress for semifinite factors.

In Section 5.3 we will make use of the approach of Section 5.2 to compute some bounds on the distance between unitary orbits of normal operators in unital, simple, purely infinite C*-algebras with trivial K_1-group. Using [17, Theorem 1.7] along with some additional K-theory arguments, we will extend these results to unital, simple, purely infinite C*-algebras without any constraints on the K_1-groups.

Before [5, Theorem 1] a classification of when one normal operator on a complex, infinite dimensional, separable Hilbert space was in the closed similarity orbit of another operator with minor additional constraints was given in [9, Theorem 1]. Thus it appears natural when tackling the problem of computing the norm closure of the similarity orbit of an operator in a unital C*-algebra to first consider the normal operators. Using the results from Section 5.3 along with Theorem 4.9.8 and ideas from [32], a classification of when one normal operator is in the closed unitary orbit of another normal operator in unital, simple, purely infinite C*-algebras and type III factors with separable predual will be given in Section 5.4.

5.1 Dadarlat's Result

Given a normal operator N in a unital C*-algebra $\mathfrak{A}$, the Continuous Functional Calculus for Normal Operators provides a unital, injective *-homomorphism from the continuous functions on the spectrum of N into $\mathfrak{A}$ sending the identity function to N. It is easy to see that two normal operators are approximately unitarily equivalent in $\mathfrak{A}$ if and only if the corresponding unital, injective *-homomorphism are approximately unitarily equivalent. Thus it is of interest to determine when two unital, injective *-homomorphisms from an abelian C*-algebra to a fixed unital C*-algebra are approximately unitarily equivalent. In particular, when $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra, several preliminary results were developed in [39], [23], [40], and [41] (to name a few) and a complete classification was given in [17].

Theorem 5.1.1 ([17, Theorem 1.7])**.** *Let X be a compact metric space, let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra, and let $\varphi, \psi : C(X) \to \mathfrak{A}$ be two unital, injective *-homomorphisms. Then φ and ψ are approximately unitarily equivalent if and only if $[[\varphi]] = [[\psi]]$ in $KL(C(X), \mathfrak{A})$ (see* [60] *for the definition of KL).*

As a specific case of [17, Theorem 1.7], if $X \subseteq \mathbb{C}$ is compact it is a corollary of the Universal Coefficient Theorem for C*-algebras (see [62]), the definition of $KL(C(X), \mathfrak{A})$, and the fact that $K_*(C(X))$ is a free abelian group that

$$KL(C(X), \mathfrak{A}) = KK(C(X), \mathfrak{A}) = \mathrm{Hom}(K_*(C(X)), K_*(\mathfrak{A}))$$

where $\mathrm{Hom}(K_*(C(X)), K_*(\mathfrak{A}))$ is the set of all homomorphisms from $K_*(C(X))$ to $K_*(\mathfrak{A})$. Thus [17, Theorem 1.7] implies that for a unital, simple, purely infinite C*-algebra $\mathfrak{A}$ and a compact subset X of $\mathbb{C}$, two unital, injective *-homomorphisms $\varphi, \psi : C(X) \to \mathfrak{A}$ are approximately unitarily equivalent if and only if $\varphi_* = \psi_*$ where φ_* and ψ_* are the group homomorphisms from $K_*(C(X))$ to $K_*(\mathfrak{A})$ induced by φ and ψ respectively. Thus a complete classification of when two normal operator with the same spectrum in a unital, simple, purely infinite C*-algebra is obtained.

The proof of Dadarlat's result greatly varies from the traditional proof of when two normal operators on a complex, infinite dimensional, separable Hilbert space are approximately unitarily equivalent. Thus, in Section 5.2, we will develop an alternate proof.

5.2 Closed Unitary Orbits of Normal Operators

In this section we will use [40, Theorem 4.4] and previously known techniques of manipulating projections in unital, simple, purely infinite C*-algebras (which are present in [39], [17], [40], and [23] to name but a few) to provide a simple proof of when two normal operators in a unital, simple, purely infinite C*-algebra with trivial K_1-group are approximately unitarily equivalent (see Corollary 5.2.14). Along the way we shall develop the notation and several technical results that will necessary in later sections and develop analogous results for other C*-algebras.

It is useful for discussions in this dissertation to recall the generalized index function introduced in [40].

Definition 5.2.1. Let $\mathfrak{A}$ be a unital C*-algebra and let $N \in \mathfrak{A}$ be a normal operator. By the Continuous Functional Calculus for Normal Operators, there exists a canonical unital, injective *-homomorphism $\varphi_N : C(\sigma(N)) \to \mathfrak{A}$ such that $\varphi_N(z) = N$. As φ_N is unital and injective, this induces a group homomorphism $\Gamma(N) : K_1(C(\sigma(N))) \to K_1(\mathfrak{A})$. The group homomorphism $\Gamma(N)$ is called the index function of N. To simplify notation, we will write $\Gamma(N)(\lambda)$ to denote $[\lambda I_{\mathfrak{A}} - N]_1$ in $\mathfrak{A}$.

In the case that $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra, $K_1(\mathfrak{A})$ is canonically isomorphic to $\mathfrak{A}^{-1}/\mathfrak{A}_0^{-1}$ by [16, Theorem 1.9]. Thus if $N \in \mathfrak{A}$ is a normal operator such that $\Gamma(N)$ is trivial then $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$. Furthermore if $N \in \mathfrak{A}$ is a normal operator and $\lambda \notin \sigma(N)$ then $\Gamma(N)(\lambda)$ describes the connected component of $\lambda I_{\mathfrak{A}} - N$ in $\mathfrak{A}^{-1}$.

The reason for examining the index function in the context of approximately unitarily equivalent normal operators is seen by the following necessary condition.

Lemma 5.2.2. *Let $\mathfrak{A}$ be a unital and let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that $N_1 \in \overline{\mathcal{S}(N_2)}$. Then*

1. *if $\lambda I_{\mathfrak{A}} - N_2 \in \mathfrak{A}_0^{-1}$ for some $\lambda \notin \sigma(N_1)$ then $\lambda I_{\mathfrak{A}} - N_1 \in \mathfrak{A}_0^{-1}$, and*
2. *if $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra then $\Gamma(N_1)(\lambda) = \Gamma(N_2)(\lambda)$ for all $\lambda \notin \sigma(N_1)$.*

Proof. Suppose $N_1 \in \overline{\mathcal{S}(N_2)}$ and $\lambda \notin \sigma(N_1)$. Then $\sigma(N_2) \subseteq \sigma(N_1)$ and there exists a sequence of invertible elements $V_n \in \mathfrak{A}$ such that

$$\lim_{n\to\infty} \left\| N_1 - V_n N_2 V_n^{-1} \right\| = 0.$$

Thus it is clear that

$$\lim_{n\to\infty} \left\| (\lambda I_{\mathfrak{A}} - N_1) - V_n(\lambda I_{\mathfrak{A}} - N_2)V_n^{-1} \right\| = 0.$$

Therefore, if $\lambda I_{\mathfrak{A}} - N_2 \in \mathfrak{A}_0^{-1}$ then $V_n(\lambda I_{\mathfrak{A}} - N_2)V_n^{-1} \in \mathfrak{A}_0^{-1}$ for all $n \in \mathbb{N}$ and thus first result trivially follows.

In the case $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra, the above implies that $\lambda I_{\mathfrak{A}} - N_1$ and $V_n(\lambda I_{\mathfrak{A}} - N_2)V_n^{-1}$ are in the same connected component of $\mathfrak{A}^{-1}$ for sufficiently large n. Therefore

$$\begin{aligned} [\lambda I_{\mathfrak{A}} - N_1]_1 &= [V_n(\lambda I_{\mathfrak{A}} - N_2)V_n^{-1}]_1 \\ &= [V_n]_1[\lambda I_{\mathfrak{A}} - N_2]_1[V_n^{-1}]_1 \\ &= [\lambda I_{\mathfrak{A}} - N_2]_1. \end{aligned}$$

Hence $\Gamma(N_1)(\lambda) = \Gamma(N_2)(\lambda)$. □

The main tools for our alternate proof of [17, Theorem 1.7] are the K-theory of unital, simple, purely infinite C*-algebras along with Theorem 1.3.19 due to Lin (see [40, Theorem 4.4]). Using Lin's result and Lemma 4.8.3, we can easily provide a simple proof of [17, Theorem 1.7] for unital, simple, purely infinite C*-algebras with trivial K_0-group and normal operators with trivial index function.

Proposition 5.2.3. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra such that $K_0(\mathfrak{A})$ is trivial. Let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial. Then $N_1 \sim_{au} N_2$ if and only if $\sigma(N_1) = \sigma(N_2)$.*

Proof. By previous discussions it is clear that $\sigma(N_1) = \sigma(N_2)$ if $N_1 \sim_{au} N_2$. Suppose $\sigma(N_1) = \sigma(N_2)$. Since $K_0(\mathfrak{A}) = \{0\}$, all non-trivial projections are Murray-von Neumann equivalent by [16, Theorem 1.4]. Thus any two normal operators with the same finite spectrum are unitarily equivalent.

By the assumption that $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial, N_1 and N_2 can be approximated by normal operators with finite spectrum by [40, Theorem 4.4]. By small perturbations using Lemma 4.8.3 and the semicontinuity of the spectrum, we can assume that N_1 and N_2 can be approximated arbitrarily well by normal operators with the same finite spectrum. Thus the result follows. □

Note the condition '$\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial' holds when $\mathfrak{A}_0^{-1} = \mathfrak{A}^{-1}$ or equivalently when $K_1(\mathfrak{A})$ is trivial (see [16, Theorem 1.9]).

If $\mathcal{O}_2$ is the Cuntz algebra generated by two isometries, $K_0(\mathcal{O}_2)$ and $K_1(\mathcal{O}_2)$ are trivial by [16, Theorem 3.7] and [16, Theorem 3.8] respectively. Thus Proposition 5.2.3 completely classifies when two normal operators in $\mathcal{O}_2$ are approximately unitarily equivalent.

Corollary 5.2.4. *Let $N, M \in \mathcal{O}_2$ be normal operators. Then $N \sim_{au} M$ if and only if $\sigma(N) = \sigma(M)$.*

Note that the proof of Proposition 5.2.3 is easily modified to a more general setting.

Corollary 5.2.5. *Let $\mathfrak{A}$ be a unital C*-algebra such that $\mathfrak{A}$ has property weak (FN) and any two non-zero projections in $\mathfrak{A}$ are Murray-von Neumann equivalent. If $N_1, N_2 \in \mathfrak{A}$ are two normal operators such that $\lambda I_{\mathfrak{A}} - N_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1, 2\}$ then $N_1 \sim_{au} N_2$ if and only if $\sigma(N_1) = \sigma(N_2)$.*

Corollary 5.2.6. *Let $\mathfrak{A}$ be a unital C*-algebra such that $\mathfrak{A}$ has property (FN) and any two non-zero projections in $\mathfrak{A}$ are Murray-von Neumann equivalent. If $N_1, N_2 \in \mathfrak{A}$ are two normal operators then $N_1 \sim_{au} N_2$ if and only if $\sigma(N_1) = \sigma(N_2)$.*

Corollary 5.2.7. *Let $\mathfrak{M}$ be a type III factor with separable predual and let $N_1, N_2 \in \mathfrak{M}$ be normal operators. Then $N_1 \sim_{au} N_2$ if and only if $\sigma(N_1) = \sigma(N_2)$.*

Our next task is to provide a simple proof of [17, Theorem 1.7] when $K_0(\mathfrak{A})$ is non-trivial yet $K_1(\mathfrak{A})$ is trivial. The Cuntz algebras, $\mathcal{O}_n$, generated by $n \in \mathbb{N} \cup \{\infty\}$ isometries (where $K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1}$ and $K_1(\mathcal{O}_n)$ is trivial by [16, Theorem 3.7] and [16, Theorem 3.8] respectively) are excellent examples of such algebras. We begin with the case that our two normal operators have the same connected spectrum. The following lemma is motivated by the proof of Theorem 4.8.6 and contains the essential ideas used in main result of this section (Theorem 5.2.13) and in Section 5.3.

Lemma 5.2.8. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators. Suppose that $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial, $\sigma(N_1) = \sigma(N_2)$, and $\sigma(N_1)$ is connected. Then $N_1 \sim_{au} N_2$.*

Proof. We shall begin with the case that $\sigma(N_1) = \sigma(N_2) = [0,1]$ and then modify the proof for the general case.

Suppose $\sigma(N_1) = [0,1] = \sigma(N_2)$. Let $\epsilon > 0$ and choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. By [40, Theorem 4.4] (or the fact that unital, simple, purely infinite C*-algebras have real rank zero (see [21, Theorem V.7.4])), by Lemma 4.8.3, by the semicontinuity of the spectrum, and by perturbing eigenvalues, there exists two collections of non-zero, pairwise orthogonal projections

$$\left\{P_j^{(1)}\right\}_{j=0}^n \text{ and } \left\{P_j^{(2)}\right\}_{j=0}^n$$

in $\mathfrak{A}$ such that

$$\sum_{j=0}^n P_j^{(q)} = I_{\mathfrak{A}} \text{ and } \left\|N_q - \sum_{j=0}^n \frac{j}{n} P_j^{(q)}\right\| < 2\epsilon$$

for all $q \in \{1,2\}$. The idea of the proof is to apply a 'back and forth' argument to produce a unitary that intertwines the approximations of N_1 and N_2.

Since $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra, $P_0^{(1)}$ is Murray-von Neumann equivalent to a proper subprojection of $P_0^{(2)}$ (see [21, Lemma V.5.4]). Thus we can write $P_0^{(2)} = Q_0^{(2)} + R_0^{(2)}$ where $Q_0^{(2)}$ and $R_0^{(2)}$ are non-zero orthogonal projections in $\mathfrak{A}$ such that $Q_0^{(2)}$ and $P_0^{(1)}$ are Murray-von Neumann equivalent. Furthermore $R_0^{(2)}$ is Murray-von Neumann equivalent to a proper subprojection of $P_1^{(1)}$. Thus we can write $P_1^{(1)} = Q_1^{(1)} + R_1^{(1)}$ where $Q_1^{(1)}$ and $R_1^{(1)}$ are non-zero orthogonal projections in $\mathfrak{A}$ such that $Q_1^{(1)}$ and $R_0^{(2)}$ are Murray-von Neumann equivalent.

For notional purposes, let $Q_0^{(1)} := 0$, $R_0^{(1)} := P_0^{(1)}$, $Q_n^{(2)} := P_n^{(2)}$, and $R_n^{(2)} := 0$. By repeating this procedure (using $R_1^{(1)}$ in place of $P_0^{(1)}$), we obtain sets of non-zero, pairwise orthogonal projections

$$\left\{Q_j^{(1)}, R_j^{(1)}\right\}_{j=1}^n \text{ and } \left\{Q_j^{(2)}, R_j^{(2)}\right\}_{j=0}^{n-1}$$

such that $P_j^{(q)} = Q_j^{(q)} + R_j^{(q)}$ for all $j \in \{0,\ldots,n\}$ and $q \in \{1,2\}$, $R_j^{(2)}$ is Murray-von Neumann equivalent to $Q_{j+1}^{(1)}$ for all $j \in \{0,\ldots,n-1\}$, and $R_j^{(1)}$ is Murray-von Neumann equivalent to

$Q_j^{(2)}$ for all $j \in \{0, \ldots, n-1\}$. Since

$$I_{\mathfrak{A}} = \sum_{j=0}^{n} Q_j^{(1)} + R_j^{(1)} = \sum_{j=0}^{n} Q_j^{(2)} + R_j^{(2)}, \qquad (*)$$

we note that

$$\begin{aligned} \left[R_n^{(1)}\right]_0 &= [I_{\mathfrak{A}}]_0 - \textstyle\sum_{j=1}^{n} \left[Q_j^{(1)}\right]_0 - \sum_{j=0}^{n-1} \left[R_j^{(1)}\right]_0 \\ &= [I_{\mathfrak{A}}]_0 - \textstyle\sum_{j=1}^{n} \left[R_{j-1}^{(2)}\right]_0 - \sum_{j=0}^{n-1} \left[Q_j^{(2)}\right]_0 \\ &= \left[Q_n^{(2)}\right]_0 . \end{aligned}$$

Hence $R_n^{(1)}$ and $Q_n^{(2)}$ are Murray-von Neumann equivalent by [16, Theorem 1.4].

Let $\{V_j\}_{j=0}^{n} \cup \{W_j\}_{j=0}^{n-1}$ be partial isometries in $\mathfrak{A}$ such that $V_j^*V_j = R_j^{(1)}$ and $V_jV_j^* = Q_j^{(2)}$ for all $j \in \{0, \ldots, n\}$, and $W_j^*W_j = Q_{j+1}^{(1)}$ and $W_jW_j^* = R_j^{(2)}$ for all $j \in \{0, \ldots, n-1\}$. Hence $(*)$ implies that

$$U := \sum_{j=0}^{n} V_j + \sum_{j=0}^{n-1} W_j$$

is a unitary operator in $\mathfrak{A}$. Moreover

$$\begin{aligned} U^* \left(\textstyle\sum_{j=0}^{n} \frac{j}{n} P_j^{(2)}\right) U &= U^* \left(\textstyle\sum_{j=0}^{n} \frac{j}{n} Q_j^{(2)} + \sum_{j=0}^{n} \frac{j}{n} R_j^{(2)}\right) U \\ &= \textstyle\sum_{j=0}^{n} \frac{j}{n} R_j^{(1)} + \sum_{j=0}^{n-1} \frac{j}{n} Q_{j+1}^{(1)}. \end{aligned}$$

Hence, since

$$\sum_{j=0}^{n} \frac{j}{n} P_j^{(1)} = \sum_{j=0}^{n} \frac{j}{n} Q_j^{(1)} + \sum_{j=0}^{n} \frac{j}{n} R_j^{(1)},$$

we obtain that

$$\|N_1 - U^*N_2U\| \leq 5\epsilon.$$

Since $\epsilon > 0$ was arbitrary, $N_1 \sim_{au} N_2$.

To complete the general case, we will use a technique similar to that used in the proof of Theorem 4.8.6. To begin, let N_1 and N_2 be as in the statement of the lemma. Fix $\epsilon > 0$

and for each $(n, m) \in \mathbb{Z}^2$ let

$$B_{n,m} := \left(\epsilon n - \frac{\epsilon}{2}, \epsilon n + \frac{\epsilon}{2}\right] + i\left(\epsilon m - \frac{\epsilon}{2}, \epsilon m + \frac{\epsilon}{2}\right] \subseteq \mathbb{C}.$$

Thus the sets $B_{n,m}$ partition the complex plane into a grid with side-lengths ϵ.

For each $(n, m) \in \mathbb{Z}^2$ we label the box $B_{n,m}$ relevant if $\sigma(N_1) \cap B_{n,m} \neq \emptyset$ and we will say two boxes are adjacent if their union is connected. Since $\sigma(N_1)$ is connected, the union of the relevant boxes is connected.

By [40, Theorem 4.4] we can approximate N_1 and N_2 within ϵ by normal operators M_1 and M_2 in $\mathfrak{A}$ with finite spectrum. By Lemma 4.8.3, by the semicontinuity of the spectrum, and by perturbing eigenvalues, we can assume that $\sigma(M_q)$ is precisely the centres of the relevant boxes and $\|N_q - M_q\| \leq 2\epsilon$ for all $q \in \{1, 2\}$.

We claim that there exists a unitary $U \in \mathfrak{A}$ such that $\|M_1 - U^*M_2U\| \leq \sqrt{2}\epsilon$. Consider a tree $\mathcal{T}$ in $\mathbb{C}$ whose vertices are the centres of the relevant boxes and whose edges are straight lines that connect vertices in adjacent relevant boxes. Consider a leaf of $\mathcal{T}$. We can identify this leaf with the spectral projections of M_1 and M_2 corresponding to the eigenvalue defined by the vertex. We can then apply the 'back and forth' technique illustrated above to embed the spectral projection of M_1 under the corresponding spectral projection of M_2 and the remaining spectral projection of M_2 under a spectral projection of M_1 corresponding to the adjacent vertex of the leaf (which is within $\sqrt{2}\epsilon$). By considering $\mathcal{T}$ with the above leaf removed, we then have a smaller tree. By continually repeating this 'back and forth'-crossing technique, we are eventually left with the trivial tree. As before, K-theory implies the remaining projections are Murray-von Neumann equivalent. It is then possible to use the partial isometries from the 'back and forth' construction to create a unitary with the desired properties. $\square$

Our next goal is to remove the condition '$\sigma(N_1)$ is connected' from Lemma 5.2.8. Unfortunately, two normal operators having equal spectrum is not enough to guarantee that the normal operators are approximately unitarily equivalent (even in the case that $K_1(\mathfrak{A})$

is trivial). The technicality is the same as why two projections in $\mathcal{B}(\mathcal{H})$ are not always approximately unitarily equivalent. To see this, we note the following lemmas.

Lemma 5.2.9. *Let $\mathfrak{A}$ be a unital C^*-algebra and let $P, Q \in \mathfrak{A}$ be projections. If there exists an element $V \in \mathfrak{A}^{-1}$ such that*

$$\left\|Q - VPV^{-1}\right\| < \frac{1}{2}$$

then P and Q are Murray-von Neumann equivalent.

Proof. Let $P_0 := VPV^{-1} \in \mathfrak{A}$ and let $Z := P_0Q + (I_{\mathfrak{A}} - P_0)(I_{\mathfrak{A}} - Q) \in \mathfrak{A}$. Hence P_0 is an idempotent and it is clear that

$$\begin{aligned}
\|Z - I_{\mathfrak{A}}\| &= \|(P_0Q + (I_{\mathfrak{A}} - P_0)(I_{\mathfrak{A}} - Q)) - (Q + (I_{\mathfrak{A}} - Q))\| \\
&\leq \|(P_0 - I_{\mathfrak{A}})Q\| + \|((I_{\mathfrak{A}} - P_0) - I_{\mathfrak{A}})(I_{\mathfrak{A}} - Q)\| \\
&= \|(P_0 - Q)Q\| + \|((I_{\mathfrak{A}} - P_0) - (I_{\mathfrak{A}} - Q))(I_{\mathfrak{A}} - Q)\| \\
&\leq \|P_0 - Q\| + \|Q - P_0\| < 1.
\end{aligned}$$

Hence $Z \in \mathfrak{A}^{-1}$. Therefore, if U is the partial isometry in the polar decomposition of Z, $Z = U|Z|$ and U is a unitary element of $\mathfrak{A}$.

We claim that $UQU^* = P_0$. To see this, we notice that $U = Z|Z|^{-1}$, $ZQ = P_0Q = P_0Z$, and

$$Z^*Z = QP_0Q + (I_{\mathfrak{A}} - Q)(I_{\mathfrak{A}} - P_0)(I_{\mathfrak{A}} - Q).$$

Thus $QZ^*Z = QP_0Q = Z^*ZQ$ so Q commutes with Z^*Z. Hence Q commutes with $C^*(Z^*Z)$ and thus Q commutes with $|Z|^{-1}$. Thus

$$\begin{aligned}
UQU^* &= Z|Z|^{-1}Q|Z|^{-1}Z^* \\
&= ZQ|Z|^{-2}Z^* \\
&= P_0Z|Z|^{-2}Z^* = P_0
\end{aligned}$$

as claimed.

Therefore $Q = (U^*V)P(U^*V)^{-1}$ where $U^*V \in \mathfrak{A}^{-1}$. It is standard to verify that if

W is the partial isometry in the polar decomposition of U^*V then W is a unitary such that $Q = WPW^*$ (see [61, Proposition 2.2.5]). Therefore $P \sim_u Q$ and thus P and Q are Murray-von Neumann equivalent. $\square$

Lemma 5.2.10. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra and let P and Q be projections in $\mathfrak{A}$. Then $P \sim_u Q$ if and only if $P \sim_{au} Q$ if and only if $Q \in \overline{\mathcal{S}(P)}$ only if P and Q are Murray-von Neumann equivalent. If $P \neq I_{\mathfrak{A}}$ and $Q \neq I_{\mathfrak{A}}$, then $P \sim_u Q$ whenever P and Q are Murray-von Neumann equivalent.*

Proof. The result follows from Lemma 5.2.9 and standard K-theory arguments. $\square$

The above shows that if $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra with $K_0(\mathfrak{A})$ being non-trivial, there exists two projections $P, Q \in \mathfrak{A}$ with $\sigma(P) = \sigma(Q) = \{0, 1\}$ that are not approximately unitarily equivalent. Thus knowledge of the spectrum is not enough to complete our classification.

To avoid the above technicality, we will describe an additional condition for two normal operators to be approximately unitarily equivalent in a unital C*-algebra. The construction of this conditions makes use of the analytical functional calculus.

Lemma 5.2.11. *Let $\mathfrak{A}$ be a unital C*-algebra, let $A, B \in \mathfrak{A}$, and let $f : \mathbb{C} \to \mathbb{C}$ be a function that is analytic on an open neighbourhood U of $\sigma(A) \cup \sigma(B)$. If $A \in \overline{\mathcal{S}(B)}$ then $f(A) \in \overline{\mathcal{S}(f(B))}$. Similarly if $A \sim_{au} B$ then $f(A) \sim_{au} f(B)$.*

Proof. Let $(V_n)_{n\geq 1}$ be a sequence of invertible elements in $\mathfrak{A}$ such that

$$\lim_{n\to\infty} \left\|A - V_n B V_n^{-1}\right\| = 0.$$

Let γ be any compact, rectifiable curve inside U such that $(\sigma(A) \cup \sigma(B)) \cap \gamma = \emptyset$, $\mathrm{Ind}_\gamma(z) \in \{0, 1\}$ for all $z \in \mathbb{C} \setminus \gamma$, $\mathrm{Ind}_\gamma(z) = 1$ for all $z \in \sigma(A) \cup \sigma(B)$, and $\{z \in \mathbb{C} \mid \mathrm{Ind}_\gamma(z) \neq 0\} \subseteq U$.

Then

$$\begin{aligned}
& f(A) - V_n f(B) V_n^{-1} \\
= \; & \tfrac{1}{2\pi i} \int_\gamma f(z) \left((zI_{\mathfrak{A}} - A)^{-1} - V_n (zI_{\mathfrak{A}} - B)^{-1} V_n^{-1} \right) dz \\
= \; & \tfrac{1}{2\pi i} \int_\gamma f(z) \left((zI_{\mathfrak{A}} - A)^{-1} - (zI_{\mathfrak{A}} - V_n B V_n^{-1})^{-1} \right) dz \\
= \; & \tfrac{1}{2\pi i} \int_\gamma f(z) (zI_{\mathfrak{A}} - A)^{-1} (A - V_n B V_n^{-1}) (zI_{\mathfrak{A}} - V_n B V_n^{-1})^{-1} dz.
\end{aligned}$$

Hence $\|f(A) - V_n f(B) V_n^{-1}\|$ is at most

$$\frac{\operatorname{length}(\gamma) \, \|A - V_n B V_n^{-1}\|}{2\pi} \sup_{z \in \gamma} |f(z)| \left\|(zI_{\mathfrak{A}} - A)^{-1}\right\| \left\|(zI_{\mathfrak{A}} - V_n B V_n^{-1})^{-1}\right\|.$$

Provided $\|A - V_n B V_n^{-1}\| \, \|(zI_{\mathfrak{A}} - A)^{-1}\| < 1$ for all $z \in \gamma$, the second resolvent equation can be used to show that

$$\left\|(zI_{\mathfrak{A}} - V_n B V_n^{-1})^{-1}\right\| \leq \frac{\|(zI_{\mathfrak{A}} - A)^{-1}\|}{1 - \|A - V_n B V_n^{-1}\| \, \|(zI_{\mathfrak{A}} - A)^{-1}\|}$$

for all $z \in \gamma$. Since $\lim_{n \to \infty} \|A - V_n B V_n^{-1}\| = 0$, γ is compact, and the resolvent function of an operator is continuous on the resolvent, $\|f(A) - V_n f(B) V_n^{-1}\|$ is at most

$$\frac{\operatorname{length}(\gamma) \, \|A - V_n B V_n^{-1}\|}{2\pi} \sup_{z \in \gamma} |f(z)| \frac{\left\|(zI_{\mathfrak{A}} - A)^{-1}\right\|^2}{1 - \|A - V_n B V_n^{-1}\| \, \|(zI_{\mathfrak{A}} - A)^{-1}\|}$$

for sufficiently large n. Since the resolvent function is a continuous function on the resolvent of an operator and γ is compact, the above supremum is finite and tends to

$$\sup_{z \in \gamma} |f(z)| \left\|(zI_{\mathfrak{A}} - A)^{-1}\right\|^2$$

as $n \to \infty$. Thus, as

$$\lim_{n \to \infty} \left\|A - V_n B V_n^{-1}\right\| = 0$$

and $\operatorname{length}(\gamma)$ is finite, $f(A) \in \overline{\mathcal{S}(f(B))}$.

The proof that $A \sim_{au} B$ implies $f(A) \sim_{au} f(B)$ follows directly by replacing the invertible elements V_n with unitary operators. $\square$

If $\mathfrak{A}$ in Lemma 5.2.11 were a unital, simple, purely infinite C*-algebra, if A and B were normal operators, and if f took values in $\{0, 1\}$ with $f(A)$ and $f(B)$ being non-trivial, then Lemma 5.2.10 would imply that the projections $f(A)$ and $f(B)$ are Murray-von Neumann equivalent in $\mathfrak{A}$. Thus, to simplify notation, we make the following definition.

Definition 5.2.12. Let $\mathfrak{A}$ be a unital C*-algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators. We say that N_1 and N_2 have equivalent common spectral projections if for every function $f : \mathbb{C} \to \mathbb{C}$ that is analytic on an open neighbourhood U of $\sigma(N_1) \cup \sigma(N_2)$ with $f(U) \subseteq \{0, 1\}$, the projections $f(N_1)$ and $f(N_2)$ are Murray-von Neumann equivalent.

If $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra and $\sigma(N_1) = \sigma(N_2)$, it is elementary to show that using [16, Theorem 1.4] that N_1 and N_2 have equivalent spectral projections if and only if they induce the same group homomorphisms from $K_0(\sigma(N_1))$ to $K_0(\mathfrak{A})$ via the Continuous Functional Calculus of Normal Operators.

Finally, with the above and the arguments used in Lemma 5.2.8, we a simple proof of [17, Theorem 1.7] based on [40, Theorem 4.4] for planar compact sets in the case that $K_1(\mathfrak{A})$ is trivial.

Theorem 5.2.13. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators. Suppose*

1. *$\sigma(N_1) = \sigma(N_2)$,*
2. *$\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial, and*
3. *N_1 and N_2 have equivalent common spectral projections.*

Then $N_1 \sim_{au} N_2$.

Proof. Fix $\epsilon > 0$ and consider the ϵ-grid used in Lemma 5.2.8. We label the box $B_{n,m}$ relevant if $B_{n,m} \cap \sigma(N_1) \neq \emptyset$. Let K be the union of the relevant boxes. Since $\sigma(N_1)$ is compact, K has finitely many connected components. Let $L_1, \ldots, L_k$ be the connected components of K. By construction $\mathrm{dist}(L_i, L_j) \geq \epsilon$ for all $i \neq j$. Therefore, if f_i is the characteristic function of L_i, the third assumptions of the theorem implies $f_i(N_1)$ and $f_i(N_2)$ are Murray-von Neumann equivalent for each $i \in \{1, \ldots, k\}$.

Note the second assumption of the theorem implies that there exists normal operators M_1 and M_2 in $\mathfrak{A}$ with finite spectrum such that $\|N_q - M_q\| < \epsilon$ for all $q \in \{1, 2\}$. By an application of Lemma 4.8.3, by the semicontinuity of the spectrum, and by small perturbations, we can assume that M_q has spectrum contained in K and $\sigma(M_q) \cap B_{n,m} \neq \emptyset$ for all relevant boxes $B_{n,m}$ and $q \in \{1, 2\}$. Furthermore, since each f_i extends to a continuous function on an open neighbourhood of K, we can assume that $\|f_i(N_q) - f_i(M_q)\| < \frac{1}{2}$ for all $i \in \{1, \ldots, k\}$ and $q \in \{1, 2\}$ by properties of the continuous functional calculus. Therefore, for each $i \in \{1, \ldots, k\}$ and $q \in \{1, 2\}$, $f_i(N_q)$ and $f_i(M_q)$ can be assumed to be Murray-von Neumann equivalent by Lemma 5.2.9. Since $f_i(N_1)$ and $f_i(N_2)$ are Murray-von Neumann equivalent for each $i \in \{1, \ldots, k\}$, $f_i(M_1)$ and $f_i(M_2)$ are Murray-von Neumann equivalent for each $i \in \{1, \ldots, k\}$. By perturbing the spectrum of M_1 and M_2 inside each L_i, we can assume that $\sigma(M_q)$ is precisely the centres of the relevant boxes for all $q \in \{1, 2\}$, $f_i(M_1)$ and $f_i(M_2)$ are Murray-von Neumann equivalent for each $i \in \{1, \ldots, k\}$, and $\|N_q - M_q\| < 2\epsilon$ for all $q \in \{1, 2\}$.

Next we apply the 'back and forth' argument of Lemma 5.2.8 to the spectrum of M_1 and M_2 in each L_i separately. This process can be applied to each L_i separately as in Lemma 5.2.8 due to the fact that $f_i(M_1)$ and $f_i(M_2)$ are Murray-von Neumann equivalent so the final step of the construction (that is, $R_n^{(1)}$ and $Q_n^{(2)}$ are Murray-von Neumann equivalent) can be completed. Thus, for each $i \in \{1, \ldots, k\}$, the 'back and forth' process produces a partial isometry $V_i \in \mathfrak{A}$ such that $V_i^* V_i = f_i(M_1)$, $V_i V_i^* = f_i(M_2)$, and $\|M_1 f_i(M_1) - V_i^* M_2 f_i(M_2) V_i\| \leq \sqrt{2}\epsilon$. Therefore, if $U := \sum_{i=1}^k V_i$ then $U \in \mathfrak{A}$ is a unitary as

$$\sum_{i=1}^{k} f_i(M_1) = I_{\mathfrak{A}} = \sum_{i=1}^{k} f_i(M_2)$$

are sums of orthogonal projections. Moreover, a trivial computation shows

$$\|M_1 - U^*M_2U\| \leq \sqrt{2}\epsilon$$

so

$$\|N_1 - U^*N_2U\| \leq (4 + \sqrt{2})\epsilon$$

completing the proof. □

Corollary 5.2.14. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra such that $K_1(\mathfrak{A})$ is trivial and let $N_1, N_2 \in \mathfrak{A}$ be normal operators. Then $N_1 \sim_{au} N_2$ if and only if*

1. *$\sigma(N_1) = \sigma(N_2)$ and*
2. *N_1 and N_2 have equivalent common spectral projections.*

Proof. One direction is follows from Theorem 5.2.13 and the fact that $K_1(\mathfrak{A})$ is trivial implies $\mathfrak{A}^{-1} = \mathfrak{A}_0^{-1}$ by [16, Theorem 1.9]. The other direction follows from Lemma 5.2.11 and Lemma 5.2.10. □

5.3 Distance Between Unitary Orbits of Normal Operators

In this section we will make use of the techniques of Section 5.2 to provide some bounds for the distance between the unitary orbits of two normal operator in unital, simple, purely infinite C*-algebras. In particular, Corollary 5.3.7 can be used to deduce Theorem 5.2.13. These results along with [17, Theorem 1.7] will provide information about the distance between unitary orbits of normal operators with non-trivial index function.

We begin with the following definition that is common in the discussion of the distance between unitary orbits.

Definition 5.3.1. Let X and Y be subsets of $\mathbb{C}$. The Hausdorff distance between X and

Y, denoted $d_H(X, Y)$, is

$$d_H(X, Y) := \max\left\{\sup_{x\in X} \mathrm{dist}(x, Y), \sup_{y\in Y} \mathrm{dist}(y, X)\right\}.$$

In [18], Davidson developed the following notation for the Calkin algebra that will be of particular use to us.

Definition 5.3.2. Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra. For normal operators $N_1, N_2 \in \mathfrak{A}$ let $\rho(N_1, N_2)$ denote the maximum of $d_H(\sigma(N_1), \sigma(N_2))$ and

$$\sup\{\mathrm{dist}(\lambda, \sigma(N_1)) + \mathrm{dist}(\lambda, \sigma(N_2)) \mid \lambda \notin \sigma(N_1) \cup \sigma(N_2), \Gamma(N_1)(\lambda) \neq \Gamma(N_2)(\lambda)\}.$$

We begin by noting the following adaptation of [18, Proposition 1.2].

Proposition 5.3.3. *Let $\mathfrak{A}$ be a unital C*-algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators. Then*

$$\mathrm{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \geq d_H(\sigma(N_1), \sigma(N_2)).$$

If $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra then

$$\mathrm{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \geq \rho(N_1, N_2).$$

Proof. The proof of the first statement follows from [19, Proposition 2.1] and the proof of the second statement follows from the proof of [18, Proposition 1.2] where the index function Γ is substituted for the traditional index function. $\square$

For our discussions of the distance between unitary orbits of normal operators in unital, simple, purely infinite C*-algebras, we shall begin with the case our normal operators have trivial index function so that $\rho(N_1, N_2) = d_H(\sigma(N_1), \sigma(N_2))$ and we may apply the techniques from Section 5.2.

We first turn our attention to the Cuntz algebra $\mathcal{O}_2$. As $K_0(\mathcal{O}_2)$ and $K_1(\mathcal{O}_2)$ are trivial, we are led to the following generalization of [33, Theorem 1.5] whose proof is identical to the one given below.

Proposition 5.3.4 (see [33, Theorem 1.5])**.** *Let $\mathfrak{A}$ be a unital C^*-algebra such that $\mathfrak{A}$ has property weak (FN), any two non-zero projections in $\mathfrak{A}$ are Murray-von Neumann equivalent, and every non-zero projection in $\mathfrak{A}$ is properly infinite. Let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial. Then*

$$\mathrm{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) = d_H(\sigma(N_1), \sigma(N_2)).$$

Proof. One inequality follows from Proposition 5.3.3. Let $\epsilon > 0$. Since $\mathfrak{A}$ has weak (FN), the conditions on N_1 and N_2 imply that there exists two normal operators $M_1, M_2 \in \mathfrak{A}$ with finite spectrum such that $\|N_q - M_q\| < \epsilon$ for all $q \in \{1, 2\}$. By Lemma 4.8.3, by the semicontinuity of the spectrum, and by applying small perturbations, we may assume that $\sigma(M_q) \subseteq \sigma(N_q)$ and $\sigma(M_q)$ is an ϵ-net for $\sigma(N_q)$ for all $q \in \{1, 2\}$.

Let X be the set of all ordered pairs $(\lambda, \mu) \in \sigma(M_1) \times \sigma(M_2)$ such that either

$$|\lambda - \mu| = \mathrm{dist}(\lambda, \sigma(M_2)) \text{ or } |\lambda - \mu| = \mathrm{dist}(\mu, \sigma(M_1)).$$

For each $\lambda \in \sigma(M_1)$ and $\mu \in \sigma(M_2)$, let $n_\lambda := |\{(\lambda, \zeta) \in X\}|$ and $m_\mu := |\{(\zeta, \mu) \in X\}|$. Clearly $n_\lambda \geq 1$ for all $\lambda \in \sigma(M_1)$, $m_\mu \geq 1$ for all $\mu \in \sigma(M_2)$, and $\sum_{\lambda\in\sigma(M_1)} n_\lambda = \sum_{\mu\in\sigma(M_2)} m_\mu$.

Since every projection in $\mathfrak{A}$ is properly infinite, we can write

$$M_1 = \sum_{\lambda\in\sigma(M_1)} \sum_{k=1}^{n_\lambda} \lambda P_{\lambda,k} \quad \text{and} \quad M_2 = \sum_{\mu\in\sigma(M_2)} \sum_{k=1}^{m_\mu} \mu Q_{\mu,k}$$

where $\{\{P_{\lambda,k}\}_{k=1}^{n_\lambda}\}_{\lambda\in\sigma(M_1)}$ and $\left\{\{Q_{\mu,k}\}_{k=1}^{m_\mu}\right\}_{\mu\in\sigma(M_2)}$ are sets of non-zero orthogonal projections in $\mathfrak{A}$ each of which sums to the identity. Since all projections in $\mathfrak{A}$ are Murray-von Neumann equivalent, using X we can pair off the projections in these finite sums to obtain a unitary

$U \in \mathfrak{A}$ (that is a sum of partial isometries) such that

$$\|M_1 - UM_2U^*\| \leq \sup\{|\lambda - \mu| \mid (\lambda, \mu) \in X\} = d_H(\sigma(M_1), \sigma(M_2)).$$

Hence

$$\mathrm{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \leq 2\epsilon + d_H(\sigma(M_1), \sigma(M_2)).$$

Since $\sigma(M_1)$ is an ϵ-net for $\sigma(N_1)$, and $\sigma(M_2)$ is an ϵ-net for $\sigma(N_2)$,

$$d_H(\sigma(M_1), \sigma(M_2)) \leq d_H(\sigma(N_1), \sigma(N_2)) + \epsilon$$

completing the proof. □

Unfortunately Proposition 5.3.4 does not completely generalize to unital, simple, purely infinite C*-algebras with non-trivial K_0-group. The following uses the ideas of Section 5.2 to obtain a preliminary result.

Lemma 5.3.5. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial. If $\sigma(N_1)$ is connected then*

$$\mathrm{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) = d_H(\sigma(N_1), \sigma(N_2)).$$

Proof. One inequality follows from Proposition 5.3.3. The proof of the other inequality is a more complicated 'back and forth' argument. Fix $\epsilon > 0$ and let $B_{n,m}$ be as in Lemma 5.2.8. For each $q \in \{1, 2\}$, we will say that $B_{n,m}$ is N_q-relevant if $B_{n,m} \cap \sigma(N_q) \neq \emptyset$. By [40, Theorem 4.4] there exists normal operators $M_1, M_2 \in \mathfrak{A}$ with finite spectrum such that $\|N_q - M_q\| < \epsilon$ for all $q = \{1, 2\}$. By Lemma 4.8.3, by the semicontinuity of the spectrum, and by a small perturbation, we can assume that $\sigma(M_q)$ is precisely the centres of the N_q-relevant boxes and $\|N_q - M_q\| \leq 2\epsilon$. For each $q \in \{1, 2\}$ and $\lambda \in \sigma(M_q)$ let $P_\lambda^{(q)}$ be the non-zero spectral projection of M_q corresponding to λ.

To begin our 'back and forth' argument, we will construct a bipartite graph, $\mathcal{G}$, using

$\sigma(M_1)$ and $\sigma(M_2)$ as vertices (where we have two vertices for λ if $\lambda \in \sigma(M_1) \cap \sigma(M_2)$). The process for constructing the edges in $\mathcal{G}$ is as follows: for each $i, j \in \{1, 2\}$ with $i \neq j$ and each $\lambda \in \sigma(M_i)$, for every $\mu \in \sigma(M_j)$ such that

$$|\lambda - \mu| \leq 2\sqrt{2}\epsilon + d_H(\sigma(N_1), \sigma(N_2))$$

(note that at least one such μ exists) add edges to $\mathcal{G}$ from μ to λ and the centre of any N_i-relevant box adjacent (including diagonally adjacent) to the N_i-relevant box λ describes.

Clearly $\mathcal{G}$ is a bipartite graph and, by construction, if $\lambda \in \sigma(M_1)$ and $\mu \in \sigma(M_2)$ are connected by an edge of $\mathcal{G}$ then $|\lambda - \mu| \leq 2\sqrt{2}\epsilon + d_H(\sigma(N_1), \sigma(N_2))$. We claim that $\mathcal{G}$ is connected. To see this, we note that since $\mathcal{G}$ is bipartite and every vertex is the endpoint of at least one edge, it suffices to show that for each pair $\lambda, \mu \in \sigma(M_1)$ there exists a path from λ to μ. Fix a pair $\lambda, \mu \in \sigma(M_1)$. Since $\sigma(N_1)$ is connected, the union of the N_1-relevant boxes is connected so there exists a finite sequence $\lambda = \lambda_0, \lambda_1, \ldots, \lambda_k = \mu$ where $\lambda_{\ell-1}$ and λ_ℓ are centres of adjacent N_1-relevant boxes for all $\ell \in \{1, \ldots, k\}$. However $\lambda_{\ell-1}$ and λ_ℓ are connected in $\mathcal{G}$ (via an element of $\sigma(M_2)$) by construction. Hence the claim follows.

Now that $\mathcal{G}$ is constructed, we will progressively remove vertices and edges from $\mathcal{G}$ and modify the non-zero projections $\left\{\left\{P_\lambda^{(q)}\right\}_{\lambda \in \sigma(M_j)}\right\}_{q \in \{1,2\}}$ in a specific manner to construct partial isometries in $\mathfrak{A}$ that will enable us to create a unitary $U \in \mathfrak{A}$ such that

$$\|M_1 - U^* M_2 U\| \leq 2\sqrt{2}\epsilon + d_H(\sigma(N_1), \sigma(N_2)).$$

Since $\mathcal{G}$ is a connected graph, there exists a $j \in \{1, 2\}$ and a vertex $\lambda \in \sigma(M_j)$ in $\mathcal{G}$ whose removal (along with all edges with λ as an endpoint) does not disconnect $\mathcal{G}$. Choose any vertex μ in $\mathcal{G}$ connected to λ by an edge. By the construction of $\mathcal{G}$ $|\lambda - \mu| \leq 2\sqrt{2}\epsilon + d_H(\sigma(N_1), \sigma(N_2))$ and $\mu \in \sigma(M_i)$ where $i \in \{1, 2\} \setminus \{j\}$. Since $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra and $P_\mu^{(i)}$ is non-zero, there exists non-zero projections $Q_\mu^{(i)}$ and $R_\mu^{(i)}$ in $\mathfrak{A}$ such that $P_\lambda^{(j)}$ and $Q_\mu^{(i)}$ are Murray-von Neumann equivalent and $P_\mu^{(i)} = Q_\mu^{(i)} + R_\mu^{(i)}$ by [21, Lemma V.5.4]. To complete our recursive step, remove λ from $\mathcal{G}$ (so $\mathcal{G}$ will still be a connected, bipartite graph), remove $P_\lambda^{(j)}$ from our list of projections, and replace $P_\mu^{(i)}$ with $R_\mu^{(i)}$ in our list of projections.

Continue the recursive process in the above paragraph until two vertices are left in $\mathcal{G}$ that must be connected by an edge. Since $\mathcal{G}$ is bipartite, one of these two remaining vertices is a non-zero subprojection of a spectral projection of M_1 and the other is a non-zero subprojection of a spectral projection of M_2. These two projections are Murray-von Neumann equivalent by the same K-theory argument used in Lemma 5.2.8.

By the same arguments as Lemma 5.2.8, the Murray-von Neumann equivalence of the projections created in the above process allows us to create partial isometries and thus, by taking a sum, a unitary $U \in \mathfrak{A}$ with the claimed property. Hence

$$\|N_1 - U^*N_2U\| \leq (4 + 2\sqrt{2})\epsilon + d_H(\sigma(N_1), \sigma(N_2)).$$

As $\epsilon > 0$ was arbitrary, the result follows. □

The above proof can be modified to show the following results.

Corollary 5.3.6. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial. Suppose for each $q \in \{1, 2\}$ that $\sigma(N_q) = \bigcup_{i=1}^n K_i^{(q)}$ is a disjoint union of compact sets with $K_i^{(1)}$ connected for all $i \in \{1, \ldots, n\}$. Let $\chi_i^{(q)}$ be the characteristic function of $K_i^{(q)}$ for all $q \in \{1, 2\}$ and $i \in \{1, \ldots, n\}$. If $\chi_i^{(1)}(N_1)$ and $\chi_i^{(2)}(N_2)$ are Murray-von Neumann equivalent for all $i \in \{1, \ldots, n\}$ then*

$$\mathrm{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \leq \max_{i \in \{1, \ldots, n\}} d_H\left(K_i^{(1)}, K_i^{(2)}\right).$$

Proof. Fix $\epsilon > 0$. The condition that '$\chi_i^{(1)}(N_1)$ and $\chi_i^{(2)}(N_2)$ are Murray-von Neumann equivalent' allows the arguments of Lemma 5.3.5 to be applied on each pair $\left(K_i^{(1)}, K_i^{(2)}\right)$ to produce a partial isometry $V_i \in \mathfrak{A}$ such that $V_i^*V_i = \chi_i^{(1)}(N_1)$, $V_iV_i^* = \chi_i^{(2)}(N_2)$, and

$$\left\|N_1\chi_i^{(1)}(N_1) - V_i^*N_2\chi_i^{(2)}(N_2)V_i\right\| < \epsilon + d_H\left(K_i^{(1)}, K_i^{(2)}\right).$$

If $U := \sum_{i=1}^k V_i \in \mathfrak{A}$ then U is a unitary operator such that

$$\|N_1 - U^* N_2 U\| < \epsilon + \max_{i \in \{1, \ldots, n\}} d_H\left(K_i^{(1)}, K_i^{(2)}\right).$$

Hence the result follows. □

Corollary 5.3.7. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that $\Gamma(N_1)$ and $\Gamma(N_2)$ are trivial. If N_1 and N_2 have equivalent common spectral projections then*

$$\operatorname{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) = d_H(\sigma(N_1), \sigma(N_2)).$$

Proof. Let $\epsilon > 0$ and let M_1 and M_2 be the normal operators as constructed in Lemma 5.3.5. Notice we can apply the same technique as in Theorem 5.2.13 to assume for each $q \in \{1, 2\}$ that $\chi_K(N_q)$ and $\chi_K(M_q)$ are Murray-von Neumann equivalent whenever K is a connected component of the union of the N_q-relevant boxes.

Construct the bipartite graph $\mathcal{G}$ as in the proof of Lemma 5.3.5. The only caveat remaining in the proof of Lemma 5.3.5 is that we required $\mathcal{G}$ to be connected. Let $\mathcal{G}_0$ be a connected component of $\mathcal{G}$. If K is the union of the N_1- and N_2-relevant boxes with vertices in $\mathcal{G}_0$ then the distance from K to any other N_q-relevant box is at least ϵ. Hence the characteristic function χ_K of K is a continuous function on $\sigma(N_1)$ and $\sigma(N_2)$. Since N_1 and N_2 have equivalent common spectral projections, $\chi_K(N_1)$ and $\chi_K(N_2)$ are Murray-von Neumann equivalent and thus, by our additional assumptions on M_1 and M_2, $\chi_K(M_1)$ and $\chi_K(M_2)$ are Murray-von Neumann equivalent. Hence we can apply the proof of Lemma 5.3.5 to each of the finite number of connected component of $\mathcal{G}$ separately and combine the resulting partial isometries as in Corollary 5.3.6 to obtain a unitary U such that

$$\|N_1 - U N_2 U^*\| \leq (4 + 2\sqrt{2})\epsilon + d_H(\sigma(N_1), \sigma(N_2)).$$

Hence the result follows. □

We have made use of the equivalence of certain spectral projections in the creation of all of the above bounds. To illustrate the necessity of these assumptions, we note the following example.

Example 5.3.8. Let P and Q be non-trivial projections in $\mathcal{O}_3$ with $[P]_0 \neq [Q]_0$. Then $\sigma(P) = \sigma(Q)$ yet $\mathrm{dist}(\mathcal{U}(P), \mathcal{U}(Q)) \geq 1$ or else P and Q would be Murray-von Neumann equivalent (see [61, Proposition 2.2.4] and [61, Proposition 2.2.7]).

In particular we have the following quantitative version of the above example.

Proposition 5.3.9. *Let $\mathfrak{A}$ be a unital C*-algebra, let $N_1, N_2 \in \mathfrak{A}$ be normal operators, and let $f : \mathbb{C} \to \mathbb{C}$ be a function that is analytic on an open neighbourhood U of $\sigma(N_1) \cup \sigma(N_2)$ with $f(U) \subseteq \{0, 1\}$. Let γ be a compact, rectifiable curve inside U with $(\sigma(N_1) \cup \sigma(N_2)) \cap \gamma = \emptyset$, $\mathrm{Ind}_\gamma(z) \in \{0, 1\}$ for all $z \in \mathbb{C} \setminus \gamma$, $\mathrm{Ind}_\gamma(z) = 1$ for all $z \in \sigma(N_1) \cup \sigma(N_2)$, and $\{z \in \mathbb{C} \mid \mathrm{Ind}_\gamma(z) \neq 0\} \subseteq U$. If $f(N_1)$ and $f(N_2)$ are not Murray-von Neumann equivalent then*

$$\mathrm{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \geq \frac{2\pi}{l_0(\gamma) \sup_{z \in \gamma} \|(zI_{\mathfrak{A}} - N_1)^{-1}\| \, \|(zI_{\mathfrak{A}} - N_2)^{-1}\|}$$

where $l_0(\gamma)$ is the length of γ in the regions where $f(z) = 1$.

Proof. By the proof of Lemma 5.2.11, we know that $\|f(N_1) - Uf(N_2)U^*\|$ is at most

$$\frac{l_0(\gamma) \, \|N_1 - UN_2U^*\|}{2\pi} \sup_{z \in \gamma} \left\|(zI_{\mathfrak{A}} - N_1)^{-1}\right\| \left\|(zI_{\mathfrak{A}} - N_2)^{-1}\right\|$$

for all unitaries U in $\mathfrak{A}$. Since $f(N_1)$ and $f(N_2)$ are not Murray-von Neumann equivalent, $f(N_1)$ and $Uf(N_2)U^*$ are not Murray-von Neumann equivalent so

$$1 \leq \|f(N_1) - Uf(N_2)U^*\|$$

by [61, Proposition 2.2.5] and [61, Proposition 2.2.7]. Hence the result follows. □

Next we desire to examine the distance between unitary orbits of normal operators with non-trivial index function. Unfortunately, as this problem is not complete even for the Calkin algebra and due to the technical restraints illustrated above, a complete description of the distance between unitary orbits will not be given. In particular our goal is to generalize Corollary 5.3.7 to a sufficient degree to be used in Section 5.4. We shall proceed with this goal by attempting to adapt the proof of [18, Theorem 1.4] via an application of [17, Theorem 1.7].

As in the proof of [18, Theorem 1.4], we will need a notion of direct sums inside unital, simple, purely infinite C*-algebras. This leads us to the following construction.

Lemma 5.3.10. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra, let $V \in \mathfrak{A}$ be a non-unitary isometry, and let $P := VV^*$. Then there exists a unital embedding of the 2^∞-UHF C*-algebra $\mathfrak{B} := \overline{\bigcup_{\ell \geq 1} \mathcal{M}_{2^\ell}(\mathbb{C})}$ into $(I_\mathfrak{A} - P)\mathfrak{A}(I_\mathfrak{A} - P)$ such that $[Q]_0 = 0$ in $\mathfrak{A}$ for every projection $Q \in \mathfrak{B}$.*

Proof. Let $P_0 := I_\mathfrak{A} - P$. Since $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra, there exists a projection $P_1 \in \mathfrak{A}$ such that P_0 and P_1 are Murray-von Neumann equivalent and $0 < P_1 < P_0$ (see [21, Lemma V.5.4]). Let $P_2 := P_0 - P_1$ which is a non-trivial projection. Note $[P_0]_0 = 0$ in $\mathfrak{A}$ by [16, Theorem 1.4]. Hence

$$[P_1]_0 = [P_0]_0 = 0 = [P_1 + P_2]_0 = [P_1]_0 + [P_2]_0 = [P_2]_0.$$

Thus P_1 and P_2 are Murray-von Neumann equivalent in $\mathfrak{A}$ by [16, Theorem 1.4]. Thus, since $P_1, P_2 \leq P_0$, P_1 and P_2 are Murray-von Neumann equivalent in $P_0\mathfrak{A}P_0$.

For $q \in \{1, 2\}$ let $V_q \in P_0\mathfrak{A}P_0$ be an isometry such that $V_qV_q^* = P_q$. Then it is not difficult to see for each $\ell \in \mathbb{N}$ that

$$\mathfrak{B}_\ell := *\text{-alg}\left(\{V_{i_1}V_{i_2}\cdots V_{i_\ell}V_{j_\ell}^*\cdots V_{j_2}^*V_{j_1}^* \mid i_1, i_2, \ldots, i_\ell, j_1, j_2\ldots, j_\ell \in \{1, 2\}\}\right)$$

is a C*-subalgebra of $P_0\mathfrak{A}P_0$ containing P_0 that is isomorphic to $\mathcal{M}_{2^\ell}(\mathbb{C})$. Moreover, it is clear that $\mathfrak{B}_\ell \subseteq \mathfrak{B}_{\ell+1}$ for all $\ell \in \mathbb{N}$ and

$$\{V_{i_1}V_{i_2}\cdots V_{i_\ell}V_{j_\ell}^*\cdots V_{j_2}^*V_{j_1}^* \mid i_1, i_2, \ldots, i_\ell, j_1, j_2 \ldots, j_\ell \in \{1, 2\}\}$$

are matrix units for $\mathfrak{B}_\ell$ in such a way that $\mathfrak{B} := \overline{\bigcup_{\ell\geq 1} \mathfrak{B}_\ell}$ is the 2^∞-UHF C*-algebra. Notice every rank one projection in $\mathfrak{B}_\ell$ is Murray-von Neumann equivalent in $\mathfrak{B}_\ell$ (and thus in $P_0\mathfrak{A}P_0$) to the rank one matrix unit $(V_1)^\ell(V_1^*)^\ell$ which is Murray-von Neumann equivalent in $\mathfrak{A}$ to P_0. Therefore $[Q]_0 = [P_0]_0 = 0$ in $\mathfrak{A}$ for every rank one projection $Q \in \mathfrak{B}_\ell$. Hence $[Q]_0 = 0$ in $\mathfrak{A}$ for every non-zero projection $Q \in \mathfrak{B}_\ell$. However, if $Q \in \mathfrak{B}$ is a non-zero projection, it is easy to see that there exists an $\ell \in \mathbb{N}$ and a non-zero projection $Q_0 \in \mathfrak{B}_\ell$ such that $\|Q - Q_0\| < \frac{1}{2}$. Hence Q and Q_0 are Murray-von Neumann equivalent in $\mathfrak{A}$ by Lemma 5.2.9. Thus $[Q]_0 = [Q_0]_0 = 0$ as desired. □

We will need the following two well-known results to adapt the proof of [18, Theorem 1.4] to our desired context.

Lemma 5.3.11. *Let $\mathfrak{B} := \overline{\bigcup_{\ell\geq 1} \mathcal{M}_{2^\ell}(\mathbb{C})}$ be the 2^∞-UHF C*-algebra. If $X \subseteq \mathbb{C}$ is compact, there exists a normal operator $N \in \mathfrak{B}$ such that $\sigma(N) = X$.*

Lemma 5.3.12 (see [16, Lemma 1.2])**.** *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra, let $V \in \mathfrak{A}$ be an isometry, and let $U \in \mathfrak{A}$ be a unitary. Then $[U]_1 = [VUV^* + (I_\mathfrak{A} - VV^*)]_1$.*

Using the above lemmas we obtain the following extension of Corollary 5.3.7 to a normal operators with non-trivial index functions provided certain assumptions apply. The techniques used in this lemma will be essential for the remainder of the chapter.

Lemma 5.3.13. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra and let $N, M \in \mathfrak{A}$ be normal operators such that*

1. *$\sigma(M) \subseteq \sigma(N)$,*
2. *$\Gamma(M)(\lambda) = \Gamma(N)(\lambda)$ for all $\lambda \notin \sigma(N)$, and*

3. N and M have equivalent common spectral projections.

Then

$$\mathrm{dist}(\mathcal{U}(N),\mathcal{U}(M)) = d_H(\sigma(N), \sigma(M)).$$

Proof. One inequality follows from Proposition 5.3.3. Since $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra, there exists a non-unitary isometry $V \in \mathfrak{A}$. Let $P := VV^*$, let $\mathfrak{C} := (I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$, and let $\mathfrak{B}$ be the unital copy of the 2^∞-UHF C*-algebra in $\mathfrak{C}$ given by Lemma 5.3.10. By Lemma 5.3.11 there exists normal operators $N_0, M_0 \in \mathfrak{B}$ such that $\sigma(N_0) = \sigma(N)$ and $\sigma(M_0) = \sigma(M)$.

Let $N' := VMV^* + N_0$ and let $M' := VMV^* + M_0$ which are clearly normal operators as V is an isometry. We will demonstrate that $N' \in \overline{\mathcal{U}(N)}$ and $M' \in \overline{\mathcal{U}(M)}$ by appealing to [17, Theorem 1.7]. Notice that $\sigma(N') = \sigma(M) \cup \sigma(N_0) = \sigma(N)$ as V is an isometry. Furthermore if $f : \mathbb{C} \to \mathbb{C}$ is a function that is analytic on an open neighbourhood U of $\sigma(N)$ with $f(U) \subseteq \{0, 1\}$ then

$$f(N') = f(VMV^*) + f(N_0) = Vf(M)V^* + f(N_0).$$

If $f(M) = 0$ then $f(N) = 0$ as $f(M)$ and $f(N)$ are Murray-von Neumann equivalent. This implies f is zero on $\sigma(N)$ and thus $f(N') = f(N_0) = 0 = f(N)$. If $f(M) \neq 0$ then $f(N') \neq 0$ and

$$[f(N')]_0 = [Vf(M)V^*]_0 + [f(N_0)]_0 = [f(M)]_0 = [f(N)]_0$$

as $f(N_0) \in \mathfrak{B}$ and as every projection in $\mathfrak{B}$ is trivial in the K_0-group of $\mathfrak{A}$ by Lemma 5.3.10. In any case $f(N')$ and $f(N)$ are Murray-von Neumann equivalent. Furthermore, since $\mathfrak{B}_0^{-1} = \mathfrak{B}^{-1}$ as $\mathfrak{B}$ is a UHF C*-algebra, we notice for any $\lambda \notin \sigma(N)$ that $\lambda I_{\mathfrak{A}} - N'$ is in the same component of $\mathfrak{A}^{-1}$ as

$$V(\lambda I_{\mathfrak{A}} - M)V^* + (\lambda I_{\mathfrak{A}} - P)$$

which is in the same connected component of $\mathfrak{A}^{-1}$ as $\lambda I_{\mathfrak{A}} - M$ by Lemma 5.3.12. Therefore, since $\Gamma(M)(\lambda) = \Gamma(N)(\lambda)$ for all $\lambda \notin \sigma(N)$ by assumption, we obtain that $\Gamma(N') = \Gamma(N)$. Therefore N and N' are approximately unitarily equivalent in $\mathfrak{A}$ by [17, Theorem 1.7]. Similarly M and M' are approximately unitarily equivalent in $\mathfrak{A}$ by [17, Theorem 1.7].

Hence it is easy to see for any unitary $U \in \mathfrak{C}$ that

$$\mathrm{dist}(\mathcal{U}(N),\mathcal{U}(M)) \leq \|(P+U)N'(P+U)^* - M'\| = \|UN_0U^* - M_0\|.$$

However, since $\mathfrak{C}$ is a unital, simple, purely infinite C*-algebra and $N_0, M_0 \in \mathfrak{C}$ are in the unital inclusion of the UHF C*-algebra $\mathfrak{B}$ in $\mathfrak{C}$, it is easy to see that $\Gamma(N_0)$ and $\Gamma(M_0)$ are trivial (when viewed as elements of $\mathfrak{C}$). Since any two non-zero projections in $\mathfrak{B} \subseteq \mathfrak{C}$ are Murray-von Neumann equivalent, the hypotheses of Corollary 5.3.7 are satisfied for N_0 and M_0 in $\mathfrak{C}$. Hence for any $\epsilon > 0$ there exists a unitary $U \in \mathfrak{C}$ such that

$$\|UN_0U^* - M_0\| \leq \epsilon + d_H(\sigma(N_0), \sigma(M_0)) = \epsilon + d_H(\sigma(N), \sigma(M)).$$

Hence

$$\mathrm{dist}(\mathcal{U}(N),\mathcal{U}(M)) \leq d_H(\sigma(N), \sigma(M))$$

as desired. ☐

Lemma 5.3.13 is enough to proceed with the results of Section 5.4. However it is possible to remove a hypothesis from Lemma 5.3.13. The following lemma is a specific case of [42, Theorem 10.6] that we prove using elementary techniques developed in this dissertation.

Lemma 5.3.14. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra and let $X \subseteq \mathbb{C}$ be a compact subset. Suppose X is a union of finitely many compact, connected components $\{K_i\}_{i=1}^n$ and $\mathbb{C} \setminus X$ is the union of finitely many connected components $\{\Omega_j\}_{j=0}^m$ where Ω_0 is the unbounded component. Let $\{g_i\}_{i=1}^n \subseteq K_0(\mathfrak{A})$ be such that $\sum_{i=1}^n g_i = [I_{\mathfrak{A}}]_0$ and let $\{h_j\}_{j=1}^m \subseteq K_1(\mathfrak{A})$. Then there exists a normal operator $N \in \mathfrak{A}$ such that $\sigma(N) = X$, $[\chi_{K_i}(N)]_0 = g_i$ for all $i \in \{1, \ldots, n\}$ (where χ_{K_i} is the characteristic function of K_i), and $[\lambda I_{\mathfrak{A}} - N]_1 = h_j$ whenever $\lambda \in U_j$ for all $j \in \{1, \ldots, m\}$. That is, for any element*

$\gamma \in \mathrm{Hom}(K_*(C(X)), K_*(\mathfrak{A})) \simeq KK(C(X), \mathfrak{A})$ *there exists a normal operator in* $\mathfrak{A}$ *whose continuous functional calculus realizes* γ*.*

Proof. We may assume without loss of generality that if $1 \leq j_1 < j_2 \leq m$ then Ω_{j_1} is contained in the unbounded component of $\mathbb{C} \setminus \Omega_{j_2}$. Since $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra, $K_1(\mathfrak{A})$ is canonically isomorphic to $\mathfrak{A}^{-1}/\mathfrak{A}_0^{-1}$ by [16, Theorem 1.9]. Choose a unitary $U_1 \in \mathfrak{A}$ such that $[U_1]_1 = h_1$. By the Continuous Functional Calculus for Normal Operators there exists a normal operator $T_1 \in \mathfrak{A}$ such that $\sigma(T_1)$ is a simple closed curve contained in X such that $[\lambda I_{\mathfrak{A}} - T_1]_1 = h_1$ for all $\lambda \in \Omega_1$. If Ω_2 is contained the unbounded component of $\mathbb{C} \setminus \sigma(T_1)$, we can repeat the above procedure to obtain a normal operator $T_2 \in \mathfrak{A}$ such that $\sigma(T_2)$ is a simple closed curve contained in X and in the unbounded component of $\mathbb{C} \setminus \sigma(T_1)$ such that $[\lambda I_{\mathfrak{A}} - T_2]_1 = h_2$ for all $\lambda \in \Omega_2$. If Ω_2 is contained the bounded component of $\mathbb{C} \setminus \sigma(T_1)$, we can repeat the above procedure to obtain a normal operator $T_2 \in \mathfrak{A}$ such that $\sigma(T_2)$ is a simple closed curve contained in X and in the bounded component of $\mathbb{C} \setminus \sigma(T_1)$ such that $[\lambda I_{\mathfrak{A}} - T_2]_1 = h_2 - h_1$ for all $\lambda \in \Omega_2$. Due to the ordering of $\{\Omega_j\}_{j=1}^m$, we can find normal operators $\{T_j\}_{j=1}^m$ such that each $\sigma(T_j)$ is a simple closed curve contained in X with the property that if $\mathcal{J}_j \subseteq \{1, \ldots, m\}$ is the set of all indices $\ell \in \{1, \ldots, m\}$ such that Ω_j is contained in the bounded component of $\mathbb{C} \setminus \sigma(T_\ell)$ then $\sum_{\ell \in \mathcal{J}_j} [\lambda I_{\mathfrak{A}} - T_j]_1 = h_j$ for all $\lambda \in \Omega_j$ and $j \in \{1, \ldots, m\}$. Hence

$$\sum_{j=1}^{m} [\lambda I_{\mathfrak{A}} - T_j]_1 = h_j$$

for all $\lambda \in \Omega_\ell$ and all $\ell \in \{1, \ldots, m\}$.

Since $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra, [21, Theorem V.5.1] implies there exists m isometries $\{V_j\}_{j=1}^m$ such that $Q := \sum_{j=1}^m V_j V_j^* < I_{\mathfrak{A}}$. Furthermore [21, Lemma V.5.4] and [16, Theorem 1.4] imply that there exists orthogonal projections $\{Q_i\}_{i=1}^{n-1}$ such that $\sum_{i=1}^{n-1} Q_i < I_{\mathfrak{A}} - Q$ and $[Q_i]_0 + \sum_{j=1}^m [\chi_{K_i}(T_j)]_0 = g_i$ for all $i \in \{1, \ldots, n-1\}$ (where χ_{K_i}

is the characteristic function of K_i). Let

$$Q_n := I_{\mathfrak{A}} - Q - \sum_{i=1}^{n-1} Q_i.$$

For each $i \in \{1, \ldots, n\}$ choose $\mu_i \in K_i$ and let

$$M := \sum_{j=1}^{m} V_j T_j V_j^* + \sum_{i=1}^{n} \mu_i Q_i.$$

Clearly M is a normal operator with $\sigma(M) \subseteq X$. Suppose $\lambda \in \Omega_{j_0}$ for some $j_0 \in \{1, \ldots, m\}$. Then

$$\lambda I_{\mathfrak{A}} - M = \sum_{j=1}^{m} V_j(\lambda I_{\mathfrak{A}} - T_j)V_j^* + \sum_{i=1}^{n} (\lambda - \mu_i) Q_i.$$

Since clearly $[Q + \sum_{i=1}^{n}(\lambda - \mu_i)Q_i]_1 = 0$, by writing $\lambda I_{\mathfrak{A}} - M$ as a product of unitaries and by applying Lemma 5.3.12 we clearly obtain that

$$[\lambda I_{\mathfrak{A}} - M]_1 = \sum_{j=1}^{m} [\lambda I_{\mathfrak{A}} - T_j]_1 = h_j.$$

Furthermore

$$\chi_{K_{i_0}}(M) = \sum_{j=1}^{m} V_j \chi_{K_{i_0}}(T_j) V_j^* + \sum_{i=1}^{n} \chi_{K_{i_0}}(\mu_i) Q_i$$

for all $i_0 \in \{1, \ldots, n\}$. Hence

$$[\chi_{K_{i_0}}(M)]_0 = \sum_{j=1}^{m} [\chi_{K_{i_0}}(T_j)]_0 + [Q_{i_0}]_0 = g_{i_0}$$

for all $i_0 \in \{1, \ldots, n-1\}$. Since $\sum_{i=1}^{n} [\chi_{K_i}(M)]_0 = [I_{\mathfrak{A}}]_0$, by our assumption that $\sum_{i=1}^{n} g_i = [I_{\mathfrak{A}}]_0$ we clearly obtain $[\chi_{K_n}(M)]_0 = g_n$. Thus M satisfies the conclusions of the lemma except for the fact that $\sigma(M)$ may be strictly contained in X.

Since $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra, there exists a non-unitary isometry $V \in \mathfrak{A}$. Let $P := VV^*$, let $\mathfrak{C} := (I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$, and let $\mathfrak{B}$ be the unital copy of the

2^∞-UHF C*-algebra in $\mathfrak{C}$ given by Lemma 5.3.10. By Lemma 5.3.11 there exists normal operator $N_0 \in \mathfrak{B}$ such that $\sigma(N_0) = X$. Let $N := VMV^* + N_0 \in \mathfrak{A}$. Then it is clear that N is a normal operator with $\sigma(N) = X$. Furthermore the proof of Lemma 5.3.13 implies that N has the desired properties. □

Before generalizing Lemma 5.3.13, we note we may use Lemma 5.3.13 and Lemma 5.3.14 to prove the following corollary that is a specific case of [42, Theorem 10.6].

Corollary 5.3.15. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra and let $X \subseteq \mathbb{C}$ be compact. For each bounded, connected component Ω of $\mathbb{C} \setminus X$ let $h_\Omega \in K_1(\mathfrak{A})$. Let $\mathcal{I}$ be the set of closed subsets K of X such that the characteristic function χ_K of K is a continuous function on X. Suppose there exists $\{g_K\}_{K \in \mathcal{I}} \subseteq K_0(\mathfrak{A})$ such that $g_X = [I_\mathfrak{A}]$ and $g_{K_1} + g_{K_2} = g_{K_1 \cup K_2}$ whenever $K_1, K_2 \in \mathcal{I}$ are disjoint. Then there exists a normal operator $N \in \mathfrak{A}$ such that $[\chi_K(N)]_0 = g_K$ for all $K \in \mathcal{I}$ and $[\lambda I_\mathfrak{A} - N]_1 = h_\Omega$ whenever $\lambda \in \Omega$ and Ω is a bounded component of $\mathbb{C} \setminus X$. That is, for any element $\gamma \in \mathrm{Hom}(K_*(C(X)), K_*(\mathfrak{A})) \simeq KK(C(X), \mathfrak{A})$ there exists a normal operator in $\mathfrak{A}$ whose continuous functional calculus realizes γ.*

Proof. For each $n \in \mathbb{N}$ let

$$X_n := \left\{ z \in \mathbb{C} \mid \mathrm{dist}(z, X) \leq \frac{1}{2^n} \right\}.$$

Note X_n satisfies the conditions of the compact subset in Lemma 5.3.14 and if K is a connected component of X_n then $K \cap X \in \mathcal{I}$. Thus Lemma 5.3.14 implies there exists normal elements $\{M_n\}_{n \geq 1} \subseteq \mathfrak{A}$ such that $\sigma(M_n) = X_n$, if K is a connected component of X_n then $[\chi_K(M_n)]_0 = g_K$, and if $\lambda \in (\mathbb{C} \setminus X_n) \cap \Omega$ where $\Omega \subseteq \mathbb{C} \setminus X$ is a bounded, connected component then $[\lambda I_\mathfrak{A} - M]_1 = h_\Omega$.

Let $N_1 := M_1$. Since $\sigma(M_2) \subseteq \sigma(N_1)$, since M_2 and N_1 have equivalent common projections by the assumptions on the set $\{g_K\}_{K \in \mathcal{I}}$, and since $\Gamma(M_2)(\lambda) = \Gamma(N_1)(\lambda)$ whenever $\lambda \notin \sigma(N_1)$, Lemma 5.3.13 implies there exists a unitary $U_2 \in \mathfrak{A}$ such that $\|N_1 - U_2 M_2 U_2^*\| \leq \frac{1}{2}$. Let $N_2 := U_2 M_2 U_2^*$. By repeating this process there exists a sequence $(N_n)_{n \geq 1} \subseteq \mathfrak{A}$ such that each N_n is a normal operator with the same conditions as M_n listed in the above paragraph and such that $\|N_n - N_{n+1}\| \leq \frac{1}{2^n}$. Hence $(N_n)_{n\geq 1}$ is a Cauchy sequence and thus converges to a normal operator $N \in \mathfrak{A}$. Clearly $\sigma(N) = X$ by the semicontinuity of the spectrum and by Lemma 4.8.3. Furthermore N has the desired properties by Lemma 5.2.9 and since the connected components of $\mathfrak{A}^{-1}$ are open and completely determine the K_1-group element. $\square$

With the above complete, we remove an assumption from 5.3.13.

Theorem 5.3.16. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that*

1. $\Gamma(N_1)(\lambda) = \Gamma(N_2)(\lambda)$ *for all* $\lambda \notin \sigma(N_1) \cup \sigma(N_2)$, *and*

2. N_1 *and* N_2 *have equivalent common spectral projections.*

Then

$$\mathrm{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) = d_H(\sigma(N_1), \sigma(N_2)).$$

Proof. Let $\epsilon > 0$. For each $q \in \{1, 2\}$ Lemma 5.3.14 implies there exists a normal operator M_q such that

$$\sigma(M_q) = \{z \in \mathbb{C} \mid \mathrm{dist}(z, \sigma(N_q)) \leq \epsilon\},$$

$\Gamma(M_q)(\lambda) = \Gamma(N_q)(\lambda)$ for all $\lambda \notin \sigma(M_q)$, and M_q and N_q have equivalent common spectral projections. Hence Lemma 5.3.13 implies that $\mathrm{dist}(\mathcal{U}(N_q), \mathcal{U}(M_q)) \leq \epsilon$ for all $q \in \{1, 2\}$.

We claim there exists a normal operator $M \in \mathfrak{A}$ such that $\sigma(M) = \sigma(M_1) \cap \sigma(M_2)$, M and M_q have equivalent common spectral projections for all $q \in \{1, 2\}$, and $\Gamma(M)(\lambda) = \Gamma(M_q)(\lambda)$ for all $\lambda \notin \sigma(M_q)$ and $q \in \{1, 2\}$. The claim will follow from Lemma 5.3.14 provided $\sigma(M_1) \cap \sigma(M_2)$ is non-empty, we can choose the correct K_1-elements for the bounded, connected components of $\mathbb{C} \setminus \sigma(M)$, and we can construct the correct K_0-elements for the connected components of $\sigma(M)$. Since N_1 and N_2 have equivalent common spectral projections, it is clear that $\sigma(M_1) \cap \sigma(M_2)$ is non-empty.

If Ω is a bounded, connected component of the complement of $\mathbb{C} \setminus \sigma(M)$ then either Ω intersects both or exactly one of $\mathbb{C} \setminus \sigma(M_1)$ and $\mathbb{C} \setminus \sigma(M_2)$. If Ω intersects both $\mathbb{C} \setminus \sigma(M_1)$ and $\mathbb{C} \setminus \sigma(M_2)$, the condition that $\Gamma(N_1)(\lambda) = \Gamma(N_2)(\lambda)$ for all $\lambda \notin \sigma(N_1) \cup \sigma(N_2)$ implies we can select a single element of $K_1(\mathfrak{A})$ for $\Gamma(M)(\lambda)$ to take for all $\lambda \in \Omega$ such that $\Gamma(M)(\lambda) = \Gamma(M_q)(\lambda)$ for all $\lambda \in \Omega \setminus \sigma(M_q)$ for $q \in \{1, 2\}$. If Ω intersects $\mathbb{C} \setminus \sigma(M_q)$ but not the other complement, we define $\Gamma(M)(\lambda) = \Gamma(M_q)(\lambda)$ for all $\lambda \in \Omega \subseteq \mathbb{C} \setminus \sigma(M_q)$.

To construct M such that M and M_q have equivalent common spectral projections for all $q \in \{1, 2\}$, we need to define the K_0-elements that should be taken by the spectral projections of the finite number of connected components of $\sigma(M)$ in such a way that if K is a connected component of $\sigma(M_q)$, the sum of K_0-element of the spectral projections of $\sigma(M)$ corresponding to components contained in K is the same as the K_0-element of the spectral projection of M_q corresponding to K. Since, by construction, M_1 and M_2 have equivalent common spectral projections and $\sigma(M_1) \cup \sigma(M_2)$ has a finite number of connected components, we may assume for the purposes of this argument that $\sigma(M_1) \cup \sigma(M_2)$ is connected. Construct a connected, bipartite graph $\mathcal{G}$ whose vertices correspond to the connected components of $\sigma(M_1)$ and $\sigma(M_2)$ and where we connect two vertices with n edges provided the intersection of the corresponding connected components has n connected components. Thus we can view the edges of $\mathcal{G}$ as the connected components of $\sigma(M_1) \cap \sigma(M_2)$. Thinking of each vertex being labelled with the K_0-element of the spectral projection of the corresponding connected component, it suffices to label the edges of $\mathcal{G}$ with K_0-elements in such a way that the K_0-element at any vertex is the sum of the K_0-elements of the adjacent edges. This can be done by selecting a subgraph $\mathcal{T}$ of $\mathcal{G}$ that is a tree, selecting a root for $\mathcal{T}$, labelling all edges not in $\mathcal{T}$ to have the trivial K_0-element, starting at the vertices farthest from the root (which must be leaves) and labelling the one adjacent edge to each vertex to be the correct K_0-element, and by recursively labelling the remaining edges of the vertices farthest from the root that have a unlabelled edges to be such that the K_0-element of the vertex is the sum of the K_0-elements of the adjacent vertices. This process is well-defined (that is, we will always have an edge remaining to label so we can have the correct

K_0-element at each vertex we consider), will terminate, and give such a labelling since M_1 and M_2 have equivalent common spectral projections so the same K-theory using in Lemma 5.2.8 will imply the last step (which is labelling a single edge between the root and another vertex) is correct. Hence the claim is complete.

Since $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra, there exists a non-unitary isometry $V \in \mathfrak{A}$. Let $P := VV^*$, let $\mathfrak{C} := (I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$, and let $\mathfrak{B}$ be the unital copy of the 2^∞-UHF C*-algebra in $\mathfrak{C}$ given by Lemma 5.3.10. By Lemma 5.3.11 there exists normal operators $M_{q,0} \in \mathfrak{B}$ such that $\sigma(M_{q,0}) = \sigma(M_q)$ for all $q \in \{1, 2\}$. For each $q \in \{1, 2\}$ let $M_q' := VMV^* + M_{q,0}$. The proof of Lemma 5.3.13 then demonstrates that $M_q' \in \overline{\mathcal{U}(M_q)}$ for all $q \in \{1, 2\}$,

$$\mathrm{dist}(\mathcal{U}(M_1), \mathcal{U}(M_2)) \leq \inf_{U \in \mathcal{U}(\mathfrak{C})} \|UM_{1,0}U^* - M_{2,0}\|,$$

and thus

$$\mathrm{dist}(\mathcal{U}(M_1), \mathcal{U}(M_2)) = d_H(\sigma(M_1), \sigma(M_2)) \leq 2\epsilon + d_H(\sigma(N_1), \sigma(N_2))$$

by Corollary 5.3.7. Hence $\mathrm{dist}(\mathcal{U}(N_q), \mathcal{U}(M_q)) \leq \epsilon$ for $q \in \{1, 2\}$ implies that

$$\mathrm{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \leq d_H(\sigma(N_1), \sigma(N_2)) + 4\epsilon.$$

As $\epsilon > 0$, the result follows. □

To complete this section we note that the proof of Theorem 5.3.16 can be adapted to obtain additional results provided there is a method for matching spectral projections. In particular [18, Theorem 1.4] clearly generalizes to the following results.

Proposition 5.3.17. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra with trivial K_0-group. If $N_1, N_2 \in \mathfrak{A}$ are normal operators then*

$$\mathrm{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \leq 2\rho(N_1, N_2)$$

where $\rho(N_1, N_2)$ is as defined in Definition 5.3.2.

Proof. Since $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra, there exists a non-unitary isometry $V \in \mathfrak{A}$. Let $P := VV^*$, let $\mathfrak{C} := (I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$, and let $\mathfrak{B}$ be the unital copy of the 2^∞-UHF C*-algebra in $\mathfrak{C}$ given by Lemma 5.3.10.

Let

$$X := \sigma(N_1) \cup \sigma(N_2) \cup \{\lambda \in \mathbb{C} \mid \lambda \notin \sigma(N_1) \cup \sigma(N_2), \Gamma(N_1)(\lambda) \neq \Gamma(N_2)(\lambda)\}.$$

By Lemma 5.3.11 there exists a normal operator $N' \in \mathfrak{B}$ such that $\sigma(N') = X$. Therefore, if

$$M := VN_1V^* + N'$$

then M is a normal operator in $\mathfrak{A}$ such that $\sigma(M) = X$ and $\Gamma(M)(\lambda) = \Gamma(N_1)(\lambda) = \Gamma(N_2)(\lambda)$ for all $\lambda \notin X$ (alternatively we could have used Lemma 5.3.14 to construct M). Therefore it suffices to show for any $q \in \{1, 2\}$ that

$$\mathrm{dist}(\mathcal{U}(N_q), \mathcal{U}(M)) \leq \rho(N_1, N_2).$$

By the definition of ρ we see that

$$\rho(N_q, M) = d_H(\sigma(N_q), \sigma(M)) \leq \rho(N_1, N_2).$$

Furthermore, by applying Lemma 5.3.11, there exists normal operators $N_0, M_0 \in \mathfrak{B}$ such that $\sigma(N_0) = \sigma(N_q)$ and $\sigma(M_0) = \sigma(M)$. As in the proof of Lemma 5.3.13, we see that $VN_qV^* + N_0 \in \overline{\mathcal{U}(N_q)}$ and $VN_qV^* + M_0 \in \overline{\mathcal{U}(M)}$. Hence it is easy to see that for any unitary $U \in \mathfrak{C}$ that

$$\begin{aligned}
\mathrm{dist}(\mathcal{U}(N_q), \mathcal{U}(M)) &\leq \|(P + U)(VN_qV^* + N_0)(P + U)^* - (VN_qV^* + M_0)\| \\
&= \|UN_0U^* - M_0\|.
\end{aligned}$$

Thus, as in the proof of Lemma 5.3.13, for any $\epsilon > 0$ there exists a $U \in \mathfrak{C}$ such that

$$\|UN_0U^* - M_0\| \leq \epsilon + d_H(\sigma(N_1), \sigma(M)) \leq \epsilon + \rho(N_1, N_2).$$

Hence the result follows. $\square$

Proposition 5.3.18. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra. If $N_1, N_2 \in \mathfrak{A}$ are normal operators with equivalent common spectral projections then*

$$\mathrm{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \leq 2\rho(N_1, N_2).$$

Proof. The proof of this result follows the proof of Proposition 5.3.17 where we note N_1 and N_2 having common spectral projections implies that N_1 and M have common spectral projections and N_2 and M have common spectral projections. This facilitates the proof that $VN_qV^* + N_0 \in \overline{\mathcal{U}(N_q)}$ and $VN_qV^* + M_0 \in \overline{\mathcal{U}(M)}$ and thus the rest of the proof follows. $\square$

5.4 Closed Similarity Orbits of Normal Operators

As the Calkin algebra is a unital, simple, purely infinite C*-algebra, in this section we endeavour to use the results of Section 5.3 and Theorem 4.9.8 to generalize Theorem 1.8.15. In addition, we will obtain a generalization of Theorem 1.8.15 to type III factors with separable predual. The two main results of this section are similar in proof but pose slight technical differences and thus are listed separately.

Theorem 5.4.1. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra and let $N, M \in \mathfrak{A}$ be normal operators. Then $N \in \overline{\mathcal{S}(M)}$ if and only if*

1. $\sigma(M) \subseteq \sigma(N)$,
2. *each component of* $\sigma(N)$ *intersects* $\sigma(M)$,
3. $\Gamma(N)(\lambda) = \Gamma(M)(\lambda)$ *for all* $\lambda \notin \sigma(N)$,

4. *if $\lambda \in \sigma(N)$ is not isolated in $\sigma(N)$, the component of λ in $\sigma(N)$ contains some non-isolated point of $\sigma(M)$, and*

5. *N and M have equivalent common spectral projections.*

Theorem 5.4.2. *Let $\mathfrak{A}$ be a unital C^*-algebra with the following properties:*

1. *$\mathfrak{A}$ has property weak (FN),*

2. *every non-zero projection in $\mathfrak{A}$ is properly infinite, and*

3. *any two non-zero projections in $\mathfrak{A}$ are Murray-von Neumann equivalent.*

(For example, $\mathcal{O}_2$ and every type III factor with separable predual.)

Let $N, M \in \mathfrak{A}$ be normal operators such that $\lambda I_{\mathfrak{A}} - M \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M)$. Then $N \in \overline{\mathcal{S}(M)}$ if and only if

1. *$\sigma(M) \subseteq \sigma(N)$,*

2. *each component of $\sigma(N)$ intersects $\sigma(M)$,*

3. *$\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$, and*

4. *if $\lambda \in \sigma(N)$ is not isolated in $\sigma(N)$, the component of λ in $\sigma(N)$ contains some non-isolated point of $\sigma(M)$.*

Note if $N \in \overline{\mathcal{S}(M)}$ then the first two conditions must hold by discussions from the beginning of Section 5.2 and the third condition follows from Lemma 5.2.2. The fifth condition of Theorem 5.4.1 is necessary by Lemma 5.2.11 and Lemma 5.2.9.

To see that the fourth conclusion is necessary, let K_λ be the connected component of $\sigma(N)$ containing λ. We note that if K_λ is not isolated in $\sigma(N)$ (that is, every open neighbourhood of K_λ intersects a different connected component of $\sigma(N)$) then the first two conditions imply that $\sigma(M) \cap K_\lambda$ contains a cluster point of $\sigma(M)$. Otherwise if K_λ is isolated in $\sigma(N)$, the characteristic function χ_{K_λ} of K_λ can be extended to an analytic function on a neighbourhood of $\sigma(N)$. Thus Lemma 5.2.11 implies $\chi_{K_\lambda}(N) \in \overline{\mathcal{S}(\chi_{K_\lambda}(M))}$. If $\sigma(M) \cap K_\lambda$ does not contain a cluster point of $\sigma(M)$ then $\chi_{K_\lambda}(M)$ must have finite spectrum. Hence there exists a non-zero polynomial p such that $p(\chi_{K_\lambda}(M)) = 0$. Clearly this implies $p(T) = 0$ for all $T \in \overline{\mathcal{S}(\chi_{K_\lambda}(M))}$ so $p(\chi_{K_\lambda}(N)) = 0$. Since K_λ is a connected, compact subset of $\sigma(N)$ that is not a singleton, this is impossible. Hence the fourth condition is necessary. An alternative proof of the necessity of the fourth condition may be obtained by considering the separable C*-algebra generated by N, M, and a countable number of invertible elements, by taking an infinite direct sum of a faithful representation of this C*-algebra on a separable Hilbert space, and by appealing to property (e) of [9, Theorem 1].

By applying Theorem 5.4.1 in conjunction with [17, Theorem 1.7], the following result is easily obtained.

Corollary 5.4.3. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators. If $N_1 \in \overline{\mathcal{S}(N_2)}$ and $N_2 \in \overline{\mathcal{S}(N_1)}$ then $N_1 \sim_{au} N_2$.*

To begin the proofs of Theorem 5.4.1 and Theorem 5.4.2 we note the following trivial result about similarity of operators in C*-algebras.

Lemma 5.4.4. *Let $\mathfrak{A}$ be a unital C*-algebra, let $P \in \mathfrak{A}$ be a non-trivial projection, let $Z \in (I_\mathfrak{A} - P)\mathfrak{A}(I_\mathfrak{A} - P)$, and let $X \in \mathfrak{A}$ be such that $PX(I_\mathfrak{A} - P) = X$. If $\lambda \notin \sigma_{(I_\mathfrak{A}-P)\mathfrak{A}(I_\mathfrak{A}-P)}(Z)$ then*

$$\lambda P + X + Z \sim \lambda P + Z.$$

Proof. Note that if $Y := X(\lambda(I_\mathfrak{A} - P) - Z)^{-1}$ then

$$T := I_\mathfrak{A} + Y$$

is invertible with

$$T^{-1} = I_\mathfrak{A} - Y.$$

A trivial computation shows

$$T(\lambda P + X + Z)T^{-1} = \lambda P + Z.$$

$\square$

Corollary 5.4.5. *Let $\mathfrak{A}$ be a unital C*-algebra, let $n \in \mathbb{N}$, let $\lambda_1, \ldots, \lambda_n$ be distinct complex scalars, let $\{P_j\}_{j=1}^n \subseteq \mathfrak{A}$ be a set of non-trivial orthogonal projections with $\sum_{j=1}^n P_j = I_{\mathfrak{A}}$, and let $\{A_{i,j}\}_{i,j=1}^n \subseteq \mathfrak{A}$ be such that $A_{i,j} = 0$ if $i \geq j$ and $P_i A_{i,j} P_j = A_{i,j}$ for all $i < j$. Then*

$$\sum_{j=1}^n \lambda_j P_j + \sum_{i,j=1}^n A_{i,j} \sim \sum_{j=1}^n \lambda_j P_j.$$

Proof. By applying Lemma 5.4.4 with $P := P_1$, $Z := \sum_{j=1}^n \lambda_j P_j + \sum_{i,j=2}^n A_{i,j}$ (it is elementary to show that $\sigma_{(I_{\mathfrak{A}}-P)\mathfrak{A}(I_{\mathfrak{A}}-P)}(Z) = \{\lambda_2, \ldots, \lambda_n\}$ so $\lambda_1 \notin \sigma(Z)$ by assumption), and $X := \sum_{j=1}^n A_{1,j}$, we obtain that

$$\sum_{j=1}^n \lambda_j P_j + \sum_{i,j=1}^n A_{i,j} \sim \sum_{j=1}^n \lambda_j P_j + \sum_{i,j=2}^n A_{i,j}.$$

The result then proceeds by recursion by considering the unital C*-algebra $(I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$. $\square$

To begin the proof of Theorem 5.4.1 we first show that a 'direct sum' of a normal operator and a nilpotent operator is in the similarity orbit of the normal operator. The idea of this result is based on [32, Lemma 5.3].

Lemma 5.4.6. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra, let $M \in \mathfrak{A}$ be a normal operator, let $V \in \mathfrak{A}$ be a non-unitary isometry, let $P := VV^*$, and let $\mathfrak{B} := \overline{\bigcup_{\ell \geq 1} \mathcal{M}_{2^\ell}(\mathbb{C})}$ be the unital copy of the 2^∞-UHF C*-algebra in $\mathfrak{C}$ given by Lemma 5.3.10. Suppose μ is a cluster point of $\sigma(M)$ and $Q \in \mathcal{M}_{2^\ell}(\mathbb{C}) \subseteq \mathfrak{B}$ is a nilpotent matrix for some $\ell \in \mathbb{N}$. Then $VMV^* + \mu(I_{\mathfrak{A}} - P) + Q \in \overline{\mathcal{S}(M)}$.*

Proof. Since $Q \in \mathcal{M}_{2^\ell}(\mathbb{C}) \subseteq \mathfrak{B}$ is a nilpotent matrix, Q is unitarily equivalent to a strictly upper triangular matrix. Thus we can assume Q is strictly upper triangular. By our assumptions on μ there exists a sequence $(\mu_j)_{j\geq 1}$ of distinct scalars contained in $\sigma(M)$ that converges to μ. For each $q \in \mathbb{N}$ let

$$T_q := \mathrm{diag}(\mu_q, \mu_{q+1}, \ldots \mu_{q+2^\ell-1}) \in \mathcal{M}_{2^\ell}(\mathbb{C}) \subseteq \mathfrak{B}$$

be the diagonal matrix with μ_q, ..., $\mu_{q+2^\ell-1}$ along the diagonal.

Let $M_q := VMV^* + T_q \in \mathfrak{A}$. As in the proof of Lemma 5.3.13, it is easy to see by [17, Theorem 1.7] that M_q is approximately unitarily equivalent to M for each $q \in \mathbb{N}$. Hence

$$M \sim_{au} M_q \sim VMV^* + (T_q + Q)$$

by Lemma 5.4.5. Since $\lim_{q\to\infty} T_q + Q = \mu(I_{\mathfrak{A}} - P) + Q$, the result follows. □

Subsequently we have our next stepping-stone which based on [32, Corollary 5.5].

Lemma 5.4.7. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra. Let $N, M \in \mathfrak{A}$ be normal operators and write $\sigma(N) = K_1 \cup K_2$ where K_1 and K_2 are disjoint compact sets with K_1 connected. Suppose*

1. *$\sigma(M) = K_1' \cup K_2$ where $K_1' \subseteq K_1$,*
2. *$\Gamma(N)(\lambda) = \Gamma(M)(\lambda)$ for all $\lambda \notin \sigma(N)$, and*
3. *N and M have equivalent common spectral projections.*

If K_1' contains a cluster point of $\sigma(M)$ then $N \in \overline{\mathcal{S}(M)}$.

Proof. If K_1 is a singleton, $K_1' = K_1$ as K_1' is non-empty. Thus $\sigma(M) = \sigma(N)$ so Theorem 5.2.13 implies N and M are approximately unitarily equivalent.

Otherwise K_1' is not a singleton. Fix a non-unitary isometry $V \in \mathfrak{A}$ and $\epsilon > 0$. Let $P := VV^*$ and let $\mathfrak{B} := \overline{\bigcup_{\ell\geq 1} \mathcal{M}_{2^\ell}(\mathbb{C})}$ be the unital copy of the 2^∞-UHF C*-algebra in

$(I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$ given by Lemma 5.3.10. By Theorem 4.9.8 there exists a normal operator $T \in \mathfrak{B}$ with

$$\sigma(T) = \{z \in \mathbb{C} \mid |z| \leq \epsilon\}$$

such that T is a norm limit of nilpotent matrices from $\bigcup_{\ell \geq 1} \mathcal{M}_{2^\ell}(\mathbb{C}) \subseteq \mathfrak{B} \subseteq \mathfrak{A}$. Let $\mu \in K_1'$ be any cluster point of $\sigma(M)$. Lemma 5.4.6 implies that

$$VMV^* + \mu(I_{\mathfrak{A}} - P) + Q \in \overline{\mathcal{S}(M)}$$

for every nilpotent matrix $Q \in \bigcup_{\ell \geq 1} \mathcal{M}_{2^\ell}(\mathbb{C}) \subseteq \mathfrak{B}$. Since T is a norm limit of nilpotent matrices from $\bigcup_{\ell \geq 1} \mathcal{M}_{2^\ell}(\mathbb{C})$, we obtain that

$$VMV^* + \mu(I_{\mathfrak{A}} - P) + T \in \overline{\mathcal{S}(M)}.$$

Let $M_1 := VMV^* + \mu(I_{\mathfrak{A}} - P) + T$. As in the proof of Lemma 5.3.13, it is easy to see that M_1 is a normal operator such that $\Gamma(M_1)(\lambda) = \Gamma(M)(\lambda) = \Gamma(N)(\lambda)$ for all $\lambda \notin \sigma(M_1) \cup \sigma(N)$ and M_1 and N have equivalent common spectral projections.

Since K_1 is connected and $\sigma(M_1)$ contains an open neighbourhood around $\mu \in K_1$, we can repeat the above argument a finite number of times to obtain a normal operator $M_0 \in \overline{\mathcal{S}(M)}$ such that $\sigma(M_0) = K_1'' \cup K_2$ where K_1'' is connected, $K_1 \subseteq K_1''$,

$$K_1'' \subseteq \{z \in \mathbb{C} \mid \operatorname{dist}(z, K_1) \leq \epsilon\},$$

$\Gamma(M_0)(\lambda) = \Gamma(N)(\lambda)$ for all $\lambda \notin \sigma(M_1) \cup \sigma(N)$, and M_0 and N have equivalent common spectral projections. Therefore Lemma 5.3.13 implies

$$\operatorname{dist}(\mathcal{U}(N), \mathcal{U}(M_0)) = d_H(\sigma(N), \sigma(M_0)) \leq \epsilon$$

so $\operatorname{dist}(N, \mathcal{S}(M)) \leq \epsilon$. Thus, as $\epsilon > 0$ was arbitrary, the result follows. $\square$

We can now complete the proof of Theorem 5.4.1 using the above result.

Proof of Theorem 5.4.1. Let N and M satisfy the five conditions of Theorem 5.4.1. By applying Lemma 5.4.7 recursively a finite number of times, we can find a normal operator M' such that $M' \in \overline{\mathcal{S}(M)}$, $\sigma(M')$ is $\sigma(M)$ unioned with a finite number of connected components of $\sigma(N)$, and N and M' satisfy the five conditions of Theorem 5.4.1.

Fix $\epsilon > 0$. Since $\sigma(N)$ is compact, $\sigma(N)$ has a finite ϵ-net. Thus the normal operator M' in the above paragraph can be selected with the additional requirement that $\mathrm{dist}(\lambda, \sigma(M')) \leq \epsilon$ for all $\lambda \in \sigma(N)$. By Lemma 5.3.13 $\mathrm{dist}(\mathcal{U}(N), \mathcal{U}(M')) \leq \epsilon$ so $\mathrm{dist}(N, \mathcal{S}(M)) \leq \epsilon$ as desired. $\square$

Note that by using Corollary 5.2.14 instead of [17, Theorem 1.7] and Corollary 5.3.7 instead of Lemma 5.3.13, a proof of Theorem 5.4.1 that is independent of [17, Theorem 1.7] may be obtained for any unital, simple, purely infinite C*-algebra with trivial K_1-group. Similarly, using [12, Theorem 11.1] and [18, Theorem 1.4], the proof of Theorem 5.4.1 is greatly simplified for the Calkin algebra and provides an alternate proof of Theorem 1.8.15.

With the proof of Theorem 5.4.1 complete, we endeavour to prove Theorem 5.4.2. As the proof of Theorem 5.4.1 relies on an embedding of the scalar matrices inside the C*-algebra under consideration, we make the following definition.

Definition 5.4.8. Let $\mathfrak{A}$ be a unital C*-algebra. An operator $A \in \mathfrak{A}$ is said to be a scalar matrix in $\mathfrak{A}$ if there exists a finite dimensional C*-algebra $\mathfrak{B}$ and a unital, injective *-homomorphism $\pi : \mathfrak{B} \to \mathfrak{A}$ such that $A \in \pi(\mathfrak{B})$.

The point of considering scalar matrices in the context of Theorem 5.4.2 is the following.

Proposition 5.4.9. *Let $\mathfrak{A}$ be a unital C*-algebra with the three properties listed in Theorem 5.4.2. If $N \in \mathfrak{A}$ is a normal operator with the closed unit disk as spectrum then N is a norm limit of nilpotent scalar matrices from $\mathfrak{A}$.*

Proof. It is easy to see the second and third assumptions in Theorem 5.4.2 imply that the 2^∞-UHF C*-algebra has a unital, faithful embedding into $\mathfrak{A}$. Therefore, by Theorem 4.9.8,

$\mathfrak{A}$ has a normal operator N_0 with the closed unit disk as spectrum that is a norm limit of nilpotent scalar matrices from $\mathfrak{A}$. Since every two normal operators with spectrum equal to the closed unit disk are approximately unitarily equivalent by Corollary 5.2.5 the result follows. $\square$

Using the ideas contained in the proof of Lemma 5.4.6, it is possible to prove the following.

Lemma 5.4.10. *Let $\mathfrak{A}$ be a unital C*-algebra such that*

1. *there exits a unital, injective *-homomorphism $\pi : \mathfrak{A} \oplus \mathfrak{A} \to \mathfrak{A}$, and*
2. *if $N_1, N_2 \in \mathfrak{A}$ are normal operators with $\lambda I_{\mathfrak{A}} - N_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1, 2\}$, $N_1 \sim_{au} N_2$ if and only if $\sigma(N_1) = \sigma(N_2)$.*

Let $M \in \mathfrak{A}$ be a normal operator with $\lambda I_{\mathfrak{A}} - M \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M)$, let $\mu \in \sigma(M)$ be a cluster point of $\sigma(M)$, and let $Q \in \mathfrak{A}$ be a nilpotent scalar matrix. Then $\pi(M \oplus (\mu I + Q)) \in \overline{\mathcal{S}(M)}$.

By using similar ideas to the proof of Theorem 5.4.1 and by using the following lemma, the proof of Theorem 5.4.2 is also complete.

Lemma 5.4.11. *Let $\mathfrak{A}$ be a unital C*-algebra with the three properties listed in Theorem 5.4.2. Let $N, M \in \mathfrak{A}$ be normal operators with $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$ and $\lambda I_{\mathfrak{A}} - M \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M)$. Let $\{K_\lambda\}_\Lambda$ be the connected components of $\sigma(N)$. Suppose*

$$\sigma(M) = \left(\bigcup_{\lambda \in \Lambda \setminus \{\lambda_0\}} K_\lambda \right) \cup K_0$$

where $K_0 \subseteq K_{\lambda_0}$. If K_0 contains a cluster point of $\sigma(M)$ then $N \in \overline{\mathcal{S}(M)}$.

Proof. The proof of this lemma follows the proof of Lemma 5.4.7 by using direct sums instead of non-unitary isometries and an application of Proposition 5.3.4 provided that Lemma 5.4.10 applies. Note that the second and third assumptions of Theorem 5.4.2 imply that the first assumption of Lemma 5.4.10 holds and Corollary 5.2.5 implies that the second assumption of Lemma 5.4.10 holds. $\square$

With the proofs of Theorem 5.4.1 and Theorem 5.4.2 complete, we will use said theorems to classify when a normal operator is a limit of nilpotents in these C*-algebras. Thus Corollary 5.4.12 provides another proof (although a more complicated proof) of Theorem 4.8.6. Moreover Corollary 5.4.13 has slightly weaker conditions to any result of Chapter 4 (that is, there should exists C*-algebras satisfying the assumptions of the following theorem that are not studied in Chapter 4 although the author is not aware of them). However, we note the proof of Theorem 4.8.6 can be adapted to this setting. These proofs are based on the proof of [32, Proposition 5.6].

Corollary 5.4.12. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra. A normal operator $N \in \mathfrak{A}$ is a norm limits of nilpotent operators from $\mathfrak{A}$ if and only if $0 \in \sigma(N)$, $\sigma(N)$ is connected, and $\Gamma(N)$ is trivial.*

Proof. The requirements that $\sigma(N)$ is connected and contains zero follows by Lemma 1.8.4. The condition that $\Gamma(N)$ is trivial follows from Lemma 4.8.5.

Suppose $N \in \mathfrak{A}$ is a normal operator such that $0 \in \sigma(N)$, $\sigma(N)$ is connected, and $\Gamma(N)$ is trivial. Let $\epsilon > 0$ and fix a non-unitary isometry $V \in \mathfrak{A}$. Let $P := VV^*$ and let $\mathfrak{B} := \overline{\bigcup_{\ell \geq 1} \mathcal{M}_{2^\ell}(\mathbb{C})}$ be the unital copy of the 2^∞-UHF C*-algebra in $(I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$ given by Lemma 5.3.10. By Theorem 4.9.8 there exists a normal operator $T \in \mathfrak{B}$ with

$$\sigma(T) = \{z \in \mathbb{C} \mid |z| \leq \epsilon\}$$

such that T is a norm limit of nilpotent matrices from $\bigcup_{\ell \geq 1} \mathcal{M}_{2^\ell}(\mathbb{C}) \subseteq \mathfrak{B} \subseteq \mathfrak{A}$.

Let $M := VNV^* + T \in \mathfrak{A}$. Clearly M is a normal operator such that $\sigma(M) = \sigma(N) \cup \sigma(T)$, M and N have equivalent common spectral projections, and $\Gamma(M)$ is trivial as in the proof of Lemma 5.3.13. Therefore Corollary 5.3.7 implies that

$$\mathrm{dist}(\mathcal{U}(N), \mathcal{U}(M)) \leq \epsilon.$$

However, we note that $\Gamma(T)$ is trivial when we view T as a normal element in $\mathfrak{A}$. Moreover, as $\sigma(N)$ is connected and contains zero, $\sigma(M)$ is connected and contains $\sigma(T)$. Thus Theorem 5.4.1 (where conditions (4) and (5) are easily satisfied) implies that $M \in \overline{\mathcal{S}(T)}$ so

$$\mathrm{dist}(N, \mathcal{S}(T)) \leq \epsilon.$$

However, as T is a norm limit of nilpotent operators from $\mathfrak{B} \subseteq \mathfrak{A}$, the above inequality implies N is within 2ϵ of a nilpotent operator from $\mathfrak{A}$. Thus the proof is complete. □

Corollary 5.4.13. *Let $\mathfrak{A}$ be a unital, separable C^*-algebra with the three properties listed in Theorem 5.4.2. A normal operator $N \in \mathfrak{A}$ is a norm limits of nilpotent operators from $\mathfrak{A}$ if and only if $0 \in \sigma(N)$, $\sigma(N)$ is connected, and $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$.*

Proof. The proof of this result follows the proof of Corollary 5.4.12 by using direct sums instead of non-unitary isometries (as in Lemma 5.4.10), Proposition 5.3.4 instead of Corollary 5.3.7, Theorem 5.4.2 instead of Theorem 5.4.1, and Proposition 5.4.9. □

To conclude this dissertation, we will briefly discuss closed similarity orbits of normal operators in von Neumann algebras. We recall that [70] completely classifies when two normal operators are approximately unitarily equivalent in von Neumann algebras. Furthermore Theorem 5.4.2 completely determines when one normal operator is in the closed similarity orbit of another normal operator in type III factors with separable predual. Thus it is natural to ask whether a generalization of Theorem 5.4.2 to type II factors may be obtained.

Unfortunately the existence of a faithful, normal, tracial state on type II_1 factors inhibits when a normal operator can be in the closed similarity orbit of another normal operator. Indeed suppose $\mathfrak{M}$ is a type II_1 factor and let τ be the faithful, normal, tracial state on $\mathfrak{M}$. If $N, M \in \mathfrak{M}$ are such that $N \in \overline{\mathcal{S}(M)}$, it is trivial to verify that $\tau(p(N)) = \tau(p(M))$ for all polynomials p in one variable. In particular if $N, M \in \mathfrak{M}$ are self-adjoint and $N \in \overline{\mathcal{S}(M)}$ we obtain that $\tau(f(N)) = \tau(f(M))$ for all continuous functions on $\sigma(N) \cup \sigma(N)$ and, as τ is faithful and normal, this implies that N and M must have the same spectral distribution. Therefore, if $N, M \in \mathfrak{M}$ are self-adjoint operators, $\sigma(M) = [0, \frac{1}{2}]$, and $\sigma(N) = [0, 1]$, then, unlike in $\mathcal{B}(\mathcal{H})$, $N \notin \overline{\mathcal{S}(M)}$. Combining the above arguments and [70, Theorem 1.3] we have the following result.

Proposition 5.4.14. *Let $\mathfrak{M}$ be a type II_1 factor. If $N, M \in \mathfrak{M}$ are self-adjoint operators and $N \in \overline{\mathcal{S}(M)}$, then $N \sim_{au} M$.*

Bibliography

[1] G. Anderson and O. Zeitouni, *A law of large numbers for finite-range dependent random matrices*, Communications on Pure and Applied Mathematics **61** (2008), no. 8, 1118-1154.

[2] C. Apostol, L. Fialkow, D. A. Herrero, and D. Voiculescu, *Approximation of Hilbert Space Operators, Volume II*, Research Notes in Mathematics, vol. 102, Pitman Books Limited, 1984.

[3] C. Apostol, C. Foiaş, and C. Pearcy, *That Quasinilpotent Operators are Norm-Limits of Nilpotent Operators Revisited*, Proc. Amer. Math. Soc. **73** (1979), no. 1, 61-64.

[4] C. Apostol, C. Foiaş, and D. Voiculescu, *On the Norm-Closure of Nilpotents, II*, Rev. Roum. Math. Pures et Appl. **19** (1974), 549-577.

[5] C. Apostol, D. A. Herrero, and D. Voiculescu, *The Closure of the Similarity Orbit of a Hilbert Space Operator*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), no. 3, 421-426.

[6] C. Apostol and N. Salinas, *Nilpotent Approximations and Quasinilpotent Operators*, Pacific J. Math. **61** (1975), no. 2, 327-337.

[7] C. Apostol and D. Voiculescu, *On a problem of Halmos*, Rev. Roumaine Math. Pures Appl. **19** (1974), 283-284.

[8] M. F. Atiyah, *Elliptic operators and compact groups*, Lecture Notes in Math., vol. 401, Springer, 1974.

[9] J. Barría and D. A. Herrero, *Closure of Similarity Orbits of Hilbert Space Operators, IV: Normal Operators*, J. London Math. Soc. (2) **17** (1978), 525-536.

[10] S. T. Belinschi, T. Mai, and R. Speicher, *Analytic subordination theory of operator-valued free additive convolution and the solution of a general random matrix problem*, 24 pp., available at `arXiv:1303.3196`.

[11] H. Bercovici and D. Voiculescu, *Regularity questions for free convolution*, Oper. Theory Adv. Appl. **104** (1998), 37-47.

[12] L. G. Brown, R. G. Douglas, and P. A. Fillmore, *Unitary Equivalence Modulo the Compact Operators and Extensions of C^*-Algebras*, Proc. Conf. on Operator Theory, Lecture Notes in Mathematics **345** (1973), 58-128.

[13] N. Brown, K. Dykema, and D. Shlyakhtenko, *Topological entropy of free product automorphisms*, Acta Math. **189** (2002), 1-35.

[14] N. Brown and N. Ozawz, *C^*-algebras and Finite Dimensional Approximations*, Graduate Studies in Mathematics, American Mathematical Society, 2008.

[15] A. Connes, *Noncommutative geometry*, Academic Press Inc., San Diego, CA, 1994.

[16] J. Cuntz, *K-theory for certain C^*-algebras*, Ann. Math. **131** (1981), 181-197.

[17] M. Dadarlat, *Approximately Unitarily Equivalent Morphisms and Inductive Limit C^*-Algebras*, K-Theory **9** (1995), 117-137.

[18] K. Davidson, *The distance between unitary orbits of normal elements in the Calkin algebra*, P. Roy. Soc. Edinb. **99A** (1984), 35-43.

[19] ———, *The distance between unitary orbits of normal operators*, Acta Sci. Math. **50** (1986), 213-223.

[20] ———, *Estimating the Distance Between Unitary Orbits*, J. Operator Theory **20** (1988), 21-40.

[21] ———, *C^*-Algebras by Example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996.

[22] K. Dykema and D. Shlyakhtenko, *Exactness of Cuntz-Pimsner C^*-algebras*, Proc. Edinburgh Math. Soc. **44** (2001), 425-444.

[23] G. A. Elliot, G. Gong, H. Lin, and C. Pasnicu, *Abelian C^*-subalgebras of C^*-algebras of real rank zero and inductive limit C^*-algebras*, Duke Math. J. **85** (1996), no. 3, 511-554.

[24] R. I. Grigorchuk and A. Żuk, *The lamplighter group as a group generated by a 2-state automaton, and its spectrum*, Geom. Dedicata **87** (2001), no. 1, 209-244.

[25] U. Haagerup, H. Schultz, and S. Thorbjørnsen, *A random matrix approach to the lack of projections in* $C^*_{\mathrm{red}}(F_2)$, Advances in Mathematics **204** (2006), no. 1, 1-83.

[26] D. Hadwin, *Triangular Truncation and Normal Limits of Nilpotent Operators*, Proc. Amer. Math. Soc. **123** (1995), no. 6, 1741-1745.

[27] P. R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **33** (1970), no. 5, 887-933.

[28] J. H. Hedlund, *Limits of Nilpotent and Quasinilpotent Operators*, Michigan Math. J. **19** (1972), no. 3, 249-255.

[29] D. A. Herrero, *Normal Limits of Nilpotent Operators*, Indiana Univ. Math. J. **23** (1974), 1097-1108.

[30] ———, *Toward a Spectral Characterization of the set of Norm Limits of Nilpotent Operators*, Indiana Univ. Math. J. **24** (1975), 847-864.

[31] ———, *Quasidiagonality, Similarity and Approximation by Nilpotent Operators*, Indiana Univ. Math. J. **30** (1981), 199-232.

[32] ———, *Approximation of Hilbert Space Operators, Volume 1*, Research Notes in Mathematics, vol. 72, Pitman Books Limited, 1982.

[33] F. Hiai and Y. Nakamura, *Distance Between Unitary Orbits in Von Neumann Algebras*, Pacific J. Math. **138** (1989), no. 2, 259-294.

[34] F. Hiai and D. Petz, *The Semicircle Law, Free Random Variables and Entropy*, Mathematical Surveys and Monographs, vol. 77, American Mathematical Society, 2000.

[35] R. Kadison and J. Ringrose, *Fundamentals of the Theory of Operator Algebra: Advance Theory, Volume II*, Graduate Studies in Mathematics, The American Mathematical Society, 1986.

[36] W. Kahan, *Every $n \times n$ Symmetric Matrix Z with Real Spectrum Satisfies $\|Z - Z^*\| \leq \|Z + Z^*\| (\log_2(n) + 0.038)$*, Proc. Amer. Math. Soc. **39** (1973), no. 2, 235-241.

[37] E. Kirchberg, *Exact C^*-algebras, tensor products and the classification of purely infinite algebras*, Proceedings of the International Congress of Mathematicians (1995), 943-954.

[38] H. Lin, *Almost Commuting Self-Adjoint Matrices and Applications*, Operator Algebras and their Applications, Fields Institute Communications **13** (1995), no. 13, 193-233.

[39] ———, *Almost commuting unitary elements in purely infinite simple C^*-algebras*, Math. Ann. **303** (1995), no. 1, 599-616.

[40] ———, *Approximation by Normal Elements with Finite Spectra in C^*-Algebras of Real Rank Zero*, Pacific J. Math. **173** (1996), no. 2, 413-487.

[41] ———, *Homomorphisms from $C(X)$ into C^*-Algebras*, Can. J. Math. **49** (1997), no. 5, 963-1009.

[42] ———, *Approximate Homotopy of Homomorphisms from $C(X)$ Into a Simple C^*-Algebra*, The American Mathematical Society, 2010.

[43] P. A. Linnell, *Division rings and group von Neumann algebras*, Forum Math. **5** (1993), no. 5, 561-576.

[44] W. Lück, *L^2-Invariants: Theory and Applications to Geometry and K-Theory*, A Series of Modern Surveys in Mathematics, vol. 44, Springer, 2002.

[45] G. W. MacDonald, *Distance from Projections to Nilpotents*, Canad. J. Math. **47** (1995), no. 4, 841-851.

[46] ———, *Distance from Idempotents to Nilpotents*, Canad. J. Math. **59** (2008), no. 3, 638-657.

[47] C. Male, *The norm of polynomials in large random and deterministic matrices* (2011), 41 pp., available at `arXiv:1004.4155v5`.

[48] L. W. Marcoux, *Projections, commutators and Lie ideals in C^*-algebras*, Proc. Roy. Irish Acad. **110A** (2010), 31-55.

[49] G. J. Murphy and T. T. West, *Spectral radius formulae*, Proc. Edinburgh Math. Soc.(2) **22** (1979), no. 3, 271-275.

[50] N. Ozawa, *Homotopy invariance of AF-embeddability*, Geom. Funct. Anal. **13** (2003), 216-222.

[51] V. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge Studies in Advanced Mathematics, vol. 58, Cambridge University Press, 2002.

[52] M. Pimsner, *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$*, Fields Institute Communications **12** (1997), 189-212.

[53] M. Pimsner and D. Voiculescu, *Exact sequences for K-groups and Ext-groups of certain cross-product C^*-algebras*, J. Operator Theory. **4** (1980), no. 1, 93-118.

[54] ______, *K-groups of reduced crossed products by free groups*, J. Operator Theory. **8** (1982), no. 1, 131-156.

[55] G. Pisier, *Remarks on a recent result by Paul Skoufranis* (2012), 10 pp., available at `arXiv:1203.3186v1`.

[56] N. R. Rao and A. Edelman, *The polynomial method for random matrices*, Foundations of Computational Mathematics **8** (2008), no. 6, 649-702.

[57] C. J. Read, *C^*-Algebras with Dense Nilpotent Subalgebras*, Bull. Lond. Math. Soc. **35** (2003), no. 35, 362-366.

[58] H. Reich, *Group von Neumann algebras and related algebras*, Ph.D. Thesis, Universität Göttingen, `http://www.mi.fu-berlin.de/math/groups/top/members/publ/diss.pdf`.

[59] E. Ricard and Q. Xu, *Khintchine type inequalities for reduced free products and applications*, J reine angew. Math. **599** (2006), 27-59.

[60] M. Rørdam, *Classification of Certain Infinite Simple C^*-Algebras*, J. Funct. Anal. **131** (1995), 415-458.

[61] M. Rørdam, F. Larsen, and N. J. Lausten, *An Introduction to K-Theory for C^*-Algebras*, London Mathematics Society, Student Texts, vol. 49, Cambridge University Press, 2000.

[62] J. Rosenberg and C. Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized functor*, Duke Math. J. **55** (1987), no. 2, 431-474.

[63] G. C. Rota, *On models for linear operators*, Comm. Pure Appl. Math. **13** (1960), no. 3, 469-472.

[64] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, 1987.

[65] N. Salinas, *On the Distance to the Set of Compact Perturbations of Nilpotent Operators*, J. Operator Theory **3** (1980), no. 2, 179-194.

[66] R. Sauer, *Power series over the group ring of a free group and applications to Novikov-Shubin invariants*, High-dimensional manifold topology (2003), 449-468.

[67] T. Schick, *Integrality of L^2-Betti numbers*, Math. Ann. **317** (2000), no. 4, 727-750.

[68] ———, *Integrality of L^2-Betti numbers*, 24 pp., available at `arXiv:math/0001101v4`.

[69] M. P. Schützenberger, *On a theorem of R. Jungen*, Proc. Amer. Math. Soc. **13** (1962), no. 6, 885-890.

[70] D. Sherman, *Unitary orbits of normal operators in von Neumann algebras*, J. Reine Angew. Math. **605** (2007), 95-132.

[71] D. Shlyakhtenko, *Some applications of freeness with amalgamation*, J. Reine angew. Math. **500** (1998), 191-212.

[72] D. Shlyakhtenko and P. Skoufranis, *Freely Independent Random Variables with Non-Atomic Distributions*, to appear in Trans. Amer. Math. Soc. (2013), 25, available at `arXiv:1305.1920`.

[73] P. Skoufranis, *On a Notion of Exactness for Reduced Free Products of C^*-Algebras*, to appear in J reine angew. Math. (2013), 25 pp., available at `arXiv:1202.5455`.

[74] ———, *Normal Limits of Nilpotent Operators in von Neumann Algebras*, Integral Equations and Operator Theory **77** (2013), no. 3, 407-439.

[75] ———, *Normal Limits of Nilpotent Operators in C^*-Algebras*, to appear in J. Operator Theory (2013), 25 pp., available at `arXiv:1205.3537`.

[76] ———, *Closed Unitary and Similarity Orbits of Normal Operators in Purely Infinite C^*-Algebras*, J. Funct. Anal. **265** (2013), no. 3, 474-506.

[77] D. Voiculescu, *Symmetries of some reduced free product C^*-algebras*, Operator Algebras and Their Connections with Topology and Ergodic Theory (Busteni, 1983), Lecture Notes in Mathematics **1132** (1985), 556-588.

[78] ———, *Addition of certain non-commuting random variables*, J. Funct. Anal. **66** (1986), no. 3, 323-346.

[79] ———, *Multiplication of certain non-commuting random variables*, J. Operator Theory **18** (1987), no. 2, 223-235.

[80] ———, *The analogues of entropy and of Fisher's information measure in free probability theory, I*, Communications in mathematical physics **155** (1993), no. 1, 71-92.

[81] ———, *The analogues of entropy and of Fisher's information measure in free probability theory V. Noncommutative Hilbert Transforms*, Inventiones mathematicae **132** (1998), no. 1, 189-227.

[82] D. V. Voiculescu, K. Dykema, and A. Nica, *Free Random Variables*, CRM Monograph Series, vol. 1, American Mathematical Society, 1992.

[83] S. Wassermann, *Exact C^*-algebras and related topics*, Lecture Notes Series, vol. 19, Seoul National University, 1994.

[84] S. Zhang, *A property of purely infinite simple C^*-algebras*, Proc. Amer. Math. Soc. **109** (1990), no. 3, 717-720.