REPORT NO.
UCB/SESM-82/06

STRUCTURAL ENGINEERING AND
STRUCTURAL MECHANICS

A CONSISTENT FORMULATION OF NONLINEAR THEORIES OF ELASTIC BEAMS AND PLATES

by
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NOVEMBER 1982

DEPARTMENT OF CIVIL ENGINEERING
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA

Structural Engineering and Structural Mechanics Division
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November 1982

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to professors James M. Kelly and Robert L. Taylor, co-chairmen of the dissertation committee, for their advice and encouragement during the course of this work. Their philosophical attitude which provided me with freedom to pursue my own research interests is equally appreciated. Thanks are also due to professors J. Lubliner and J. E. Marsden for serving on the dissertation committee.

To my friend and colleague Keith Helmstad I give a special word of thanks for many useful and enjoyable discussions.

I will be forever indebted to my wife Constance for her love and constant support during my three years at Berkeley.

Partial support for this research was provided by the Malaysian Rubber Producers Research Association. This support is gratefully acknowledged.

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CHAPTER 0. INTRODUCTION

It is generally agreed that the study of the stability and bifurcation of elastic systems can only be adequately undertaken within the framework of the non-linear field theories of mechanics. However, treatments of the subject in the engineering literature, which are mainly concerned with bifurcation phenomena in special types of bodies such as beams and plates, are often characterized by an over reliance on free body diagrams, non-explicitly stated assumptions and selective truncation of nonlinear terms in establishing the field equations of the theory, procedures which led Truesdell to christen the subject as "unfortunate" [11].

The objective of the present work is a systematic development of the field equations for various types of rods and plates of interest in applications, within the framework of general non-linear elastostatics. The possible appearance of bifurcation phenomena is thus inherent to the formulation. The establishment of approximate kinematic assumptions, as in the plate theory presented in Chapter 5, is based upon the projection method due to Kantorovich [43,47]. The use of a consistent linearization procedure, based upon Taylor's formula in function spaces [12], is employed as an alternative to the selective truncation of non-linear terms. Throughout this work, emphasis is placed on the consideration of the effect of shear deformation and the subsequent transversal warping in the non-linear theories of straight rods and plates. A consistent treatment of the latter effect leads to results which can not be obtained by elementary means.

An Overview

The statement of general nonlinear equilibrium equations in terms of stress resultants for straight rods, is considered in Chapter 1. Motivated by the fact that axial warping of the cross section necessarily occurs as a result of shear deformation, the deformed cross section of the beam is explicitly regarded as a surface in $\mathbb{R}^3$ with associated intrinsic or Gaussian frame. The familiar concepts of axial and shear force then appear naturally as resultants of the components of the first Piola-Kirchhoff stress tensor when the Gaussian or intrinsic frame is chosen as the spatial frame. An interpretation of the second Piola-Kirchhoff tensor, a stress measure without direct physical meaning, is also examined. Our motivation for this interpretation is the extensive use made in the literature of a constitutive model which assumes a linear relationship between the components of this tensor and those of the right Cauchy-Green tensor. Careful derivations of the von-Karman equations, for example [18,61,62], have been based upon the assumption of such a constitutive model. The discussion and numerical results on approximate two-dimensional constitutive models presented in Chapter 2, suggest that assuming this constitutive model should be regarded as a mathematical convenience having questionable physical significance.

The equilibrium equations in terms of stress resultants developed in Chapter 1, are applied in Chapter 2 to derive geometrically exact rod theories. First, the classical Bernoulli-Kirchhoff kinematic assumption is introduced simply by postulating invariance in orientation of the Gaussian frame over the deformed cross section of the beam. Consistent constitutive equations in terms of stress resultants are then obtained by a duality argument employing the stationarity of the total potential energy functional. The resulting theory, capable of modeling finite extension shearing and bending of the beam, is further specialized by introducing the additional assumption that the line of centroids is inextensible. The linearization about the trivial equilibrium configuration of the resulting nonlinear eigenvalue problem leads then to expressions for the critical load well known in the engineering literature [21-24]. By relaxing the Bernoulli-Kirchhoff assumption in a manner which leads to a piece-wise linear deformation pattern of the cross section of the beam, a similar theory is developed for the case of technical interest of a sandwich beam. In both cases, the fully kinematically nonlinear theories developed permit a rigorous post-buckling analysis, presented in appendix I, which employs a modification of the Poincare perturbation method due to Keller [26,55].

In Chapter 3, the straight rod acted on by end forces, a problem which in fact inaugurated the subject of elastic stability, is re-examined and an exact second order solution developed. For this purpose, the exact solution corresponding to Saint-Venant's problem is first recast exclusively in terms of kinematic variables. The resulting displacement field includes a term which exactly accounts for the axial warping of the cross section within the framework of the linear theory. In the context of the projection method due to Kantorovich, the proposed expression for the displacement field represents an optimal choice for the coordinate function of this method for the problem at hand. In addition, this coordinate functions are orthogonal in the L_2 sense over the domain spanned by the cross section of the beam.

Taking the exact linearized kinematics as a point of departure, an exact second order solution is developed by a trivial extension of the results presented in Chapter 1. When the theory is specialized to the situation in which the line of centroids is assumed to be inextensible, the resulting eigenvalue problem leads to a new expression for the critical load which takes into account, up to second order, the effects of shear deformation and subsequent warping of the cross section. Inclusion of these effects leads to values of the critical load which are always lower than those predicted by various proposed modifications of Euler's buckling load based upon the Bernoulli-Kirchhoff assumption.

For columns extremely weak in shear † , the results of Chapter 3 show that the effect of axial warping due to shear deformation, can result in substantial reductions of critical load of the column. An important example of this type of situation is found in the analysis of multilayer elastomeric bearing, a type of column widely used as mounting and isolation device, and recently as key element in the development of effective aseismic base isolation systems in earthquake engineering. Previous analyses of this type of composite column, consisting of thin elastomeric layers bonded to metal plates, have systematically assumed the metal plates perfectly rigid and, consequently, axial warping of the column considered to be completely

† For transversally isotropic elastic solids, the shear modulus is an independent elastic constant.

prevented. Experimental evidence, however, demonstrates that such an assumption is far for being realistic. The analysis of these columns presented in Chapter 4, develops an expression for the displacement field by enforcing compatibility in stresses and displacements between elastomeric pad and metal plate including the flexibility of the plate. The results of Chapter 3 lead to an expression for the buckling load of the column which depends upon the stiffness of the plates.

Finally, the results presented in Chapters 1 to 3 are extended in Chapter 4 to examine the influence of shear deformation and transversal warping in the non-linear theory of elastic plates. For this purpose, the projection method due to Kantorovich is employed to derive an approximate displacement field by enforcing at the outset the shear stress boundary conditions at top and bottom surfaces. Within the framework of classical linear elasticity, the proposed displacement field is shown to exactly reproduce the field equations of the well known plate theory due to Reissner [57,58]. When attention is focussed on the nonlinear theory, however, a new formulation is obtained which, when further simplified by introducing an additional assumption, leads to a plate theory governed by a coupled system of three semi-linear partial differential equations. These equations reproduce the Reissner theory when the effect of the so-called in-plane forces is neglected and reduce to the classical von-Karman model in the limit as the shear stiffness of the plate tends to infinity. Accordingly, the proposed theory furnishes the proper extension of the von-Karman model to thick plates. It is worth noting that our formulation does not make use of the questionable linear relationship between second Piola-Kirchhoff and Cauchy-Green tensors, contrasting previous approaches [61,62]. Such a constitutive model leads to incorrect results for beams subjected to axial loads with shear deformation accounted for.

The formulations developed throughout the present work, are particularly suited for a numerical treatment employing the finite element method. The basis for such numerical implementations are examined in appendix III.

CHAPTER 1.

A FORMAL FRAMEWORK FOR THE NON-LINEAR THEORY OF BEAMS

1.1.- Introduction.

Exact equilibrium equations for beams are derived in the present chapter by integration over the cross section of the beam of the material form of three dimensional equilibrium equations of non-linear elastostatics. A similar approach can be traced back at least to the work of Green [1], although the representation of the deformation map as a formal power series expansion in terms of transversal coordinates assumed in [1], is not introduced in the present work. Starting with the work of Ericksen and Truesdell [2], a considerable body of literature has been devoted to the formulation of rod theories, either by a direct approach, or by regarding the rod as a directed medium or Cosserat Continuum [1],[3-8]. A comprehensive account of both approaches may be found in the important work of Antman [9].

The formulation presented differs from previous ones in the explicit view, adopted throughout this work, of the deformed cross section of the beam as a surface in $\mathbb{R}^3$ with associated Gaussian or intrinsic frame. This point of view is motivated by the fact that warping of the cross section necessarily occurs as a result of shear deformation. Accordingly, the emphasis is placed on a formulation capable of taking into account such an effect and, at the same time, suitable for a systematic consideration of bifurcation phenomena. Furthermore, the explicit introduction of the Gaussian frame allows the expression of the equilibrium equations in terms of normal and tangential stress fields acting on the deformed cross section of the beam; that is vector fields covering the deformation map. A Lagrangian description is employed throughout which leads to a simple form of the final equilibrium equations convenient for practical applications. No use is made, however, of the so-called convected coordinate system briefly discussed at the end of this chapter.

A physical interpretation of the second Piola-Kirchhoff tensor, particularly useful in the formulation of approximate two dimensional constitutive models, often used for computational purposes, is also examined.

1.2.- The Equations of Equilibrium in Terms of Stress Resultants.

This section is concerned with the equilibrium equations for a beam with cross section Ω and length L. The case of a straight beam is considered for convenience, with its longitudinal axis assumed to be a principal axis of inertia and taken to be X^1. Nevertheless, the cross section is allowed to vary smoothly along X^1.

Notation

The reference configuration is then $B := (0,L)\times\Omega$ where $\Omega\subset\mathbb{R}^2$ is a bounded open set with smooth boundary $\partial\Omega$. The deformation map is denoted by Φ: $B\subset\mathbb{R}^3 \rightarrow \mathbb{R}^3$. Points in B are designated by X, whereas $x = \Phi(X)$ designates a point in the deformed configuration $\Phi(B)\subset\mathbb{R}^3$.

The reference configuration B is covered by a coordinate system $\{X^I\}_{I=(1,2,3)}$: $B \rightarrow \mathbb{R}^3$ with associated coordinate vector fields designated by $\{\hat{\mathbf{E}}_I\}_{I=(1,2,3)}$. At each $X\in B$, the coordinate vector fields are maps $\hat{\mathbf{E}}_I(X) : B \rightarrow T_X B$ where $T_X B$ is the tangent space at $X\in B$. The notation $\hat{\mathbf{E}}_I = \dfrac{\partial}{\partial X^I}$ is often employed in the context of manifolds [10]. Similarly, the deformed configuration is covered by a coordinate system $\{x^i\}_{i=(1,2,3)}$ with associated coordinate basis $\hat{\mathbf{e}}_i = \dfrac{\partial}{\partial x^i}$. The metric tensor at a point $X\in B$ is a symmetric, positive definite bilinear form $\mathbf{G}(X) : T_X B\times T_X B \rightarrow \mathbb{R}$ with components

$$G_{IJ}(X) = \hat{\mathbf{E}}_I(X).\hat{\mathbf{E}}_J(X)$$

whereas the metric tensor $\mathbf{g}(x)$ at a point $x \in \Phi(B)$, defined in a similar manner, has components given by

$$g_{ij}(x) = \hat{\mathbf{e}}_i(x).\hat{\mathbf{e}}_j(x)$$

One can then proceed to develop the geometry associated with the deformation of the body within the framework of these general coordinate systems as in [11,12]. A complete account in the general context of manifolds can be found in [12] and need not be repeated here. For the purpose of present work, however, it will be enough to consider the particular case in which $\{X^I\}$ are the standard coordinates in $\mathbb{R}^3$ and $\{\hat{\mathbf{E}}_I\}$ is, therefore, the standard basis in $\mathbb{R}^3$. Although $\{\hat{\mathbf{E}}_I\}$ and $\{\hat{\mathbf{e}}_i\}$ need not coincide, it will be assumed they do so for convenience. By a slight abuse in notation, points $X \in B$ and $x \in \Phi(B)$ will be usually referred to by their position vectors $\mathbf{X}$ and $\mathbf{x}$ respectively.

When attention is restricted to the static case, the balance of linear momentum equation takes the form [11-13]

$$DIV\ \mathbf{P} + \rho_{Ref}(\mathbf{X})\ \mathbf{B}(\mathbf{X}) = 0 \ , \qquad \mathbf{X} \in B \tag{1.1}$$

where $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor, ρ_{Ref} the density in the reference configuration B, and $\mathbf{B}$ the body forces per unit of volume in B. If $\mathbf{F}(\mathbf{X})$ designates the deformation gradient and $J = \det(\mathbf{F})$, the two-point tensor $\mathbf{P}$ is related to the spatial Cauchy stress tensor through the Piola transformation $\mathbf{P} = J\,\sigma\mathbf{F}^{-T}$ [11,12]. The equilibrium equations (1.1a) are supplemented with the balance of angular momentum equation which, in the static case, reduces to the symmetry of the Cauchy stress tensor σ [11-13]. The counterpart of these equilibrium equations, in terms of resultant force and moment acting on the deformed cross section, will be examined next.

Consider an arbitrary cross section Ω, a distance X^1 from the origin, with unit normal $\hat{\mathbf{N}} \equiv \hat{\mathbf{E}}_1 = [1\ 0\ 0]^T$. The unit vector field normal to the lateral contour $\partial\Omega$ is designated by $\hat{\mathbf{N}}_l = [0\ N_{l2}\ N_{l3}]^T$.

(i) *Equilibrium equations for the resultant force* $\mathbf{R}(X^1)$ *acting on the deformed cross section* $\Phi(\Omega)$.

The stress vector acting on $\Phi(\Omega)$, per unit of area in Ω, is defined by

$$\mathbf{T}(\mathbf{X}) = \mathbf{P}\,\hat{\mathbf{N}} = \begin{Bmatrix} P^{11} \\ P^{21} \\ P^{31} \end{Bmatrix} \tag{1.2}$$

and the resultant force $\mathbf{R}(X^1)$ is thus obtained by integration over $\Omega(X^1)$ as

$$\mathbf{R}(X^1) = \int_\Omega [\mathbf{P}\,\hat{\mathbf{N}}]dA \tag{1.3}$$

Integration of (1.1) over the cross sectional area and subsequent application of Green's formula leads, in components, to

$$\frac{d}{dX^1}\int_\Omega P^{i1}\,d\Omega + \int_{\partial\Omega} [P^{i2}\,N_{l2} + P^{i3}\,N_{l3}]\,d\Omega + \int_\Omega \rho_{Ref}\,B^i\,d\Omega = 0\ , \quad (i = 1,2,3) \tag{1.4}$$

Since the applied load on the lateral contour is given by

$$\bar{\mathbf{t}}(\mathbf{X}) = \mathbf{P}|_{\partial\Omega}\hat{\mathbf{N}}_l \quad => \quad \bar{t}^i = P^{i2}\,N_{l2} + P^{i3}\,N_{l3} \tag{1.5}$$

the equilibrium equation for the resultant force $\mathbf{R}(X^1)$ takes the final form

$$\frac{d}{dX^1}\int_\Omega [\mathbf{P}\,\hat{\mathbf{N}}]\,d\Omega + \bar{\mathbf{t}} = 0 \tag{1.6}$$

where $\hat{\mathbf{N}} = [1\ 0\ 0]^T$, and we have assumed, without loss of generality, zero body forces.

(ii) *The equilibrium equation for the resultant moment* $\mathbf{M}(X^1)$ *acting on the deformed section* $\Phi(\Omega)$.

Let $\mathbf{x}_O \in \mathbb{R}^3$ be the point in $\Phi(\Omega)$ image of the centroid $(X^1,0,0)$ of the underformed cross section Ω under the deformation map $\Phi : B \to \mathbb{R}^3$. The vector field $\mathbf{r}(\mathbf{X})$ connecting the point $\mathbf{x}_O$ with an arbitrary point $\mathbf{x}$ in $\Phi(\Omega)$ is then defined by

$$\mathbf{r}(\mathbf{X}) := \mathbf{x} - \mathbf{x}_O = \Phi(\mathbf{X}) - \Phi(\mathbf{X})|_{X^2=X^3=0} \tag{1.7}$$

The notation $\mathbf{x}_O = \Phi_O(X^1)$ will often be used. Thus, with the stress vector $\mathbf{T}$ as given by (1.2), the components of $\mathbf{M}$ may be expressed in the form

$$M_k(X^1) = \epsilon_{ijk}\int_\Omega r^i\,P^{j1}\,d\Omega \tag{1.8}$$

where ϵ_{ijk} are the components of the permutation tensor. Differentiation of (1.8) with respect to X^1 and use of the equilibrium equations (1.1) yields (Summation convention is enforced throughout and the index $D=(1,2)$)

$$\frac{dM_k}{dX^1}(X^1) = \epsilon_{ijk}\left\{\int_\Omega \frac{\partial r^i}{\partial X^1} P^{j1}\, d\Omega - \int_\Omega r^i \frac{\partial P^{jD}}{\partial X^D}\, d\Omega - \int_\Omega \rho_{Ref}\, r^i B^j\, d\Omega\right\}$$

$$= \epsilon_{ijk}\left\{\int_\Omega \frac{\partial r^i}{\partial X^J} P^{jJ}\, d\Omega - \int_\Omega \frac{\partial}{\partial X^D}[r^i P^{jD}]\, d\Omega - \int_\Omega \rho_{Ref} r^i B^j\, d\Omega\right\}$$

$$= \epsilon_{ijk}\int_\Omega \frac{\partial r^i}{\partial X^J} P^{jJ}\, d\Omega - \overline{m}_k$$

where $\overline{m}_k := \epsilon_{ijk}\int_{\partial\Omega} r^i\, \bar{t}^j\, d\Omega$ is the resultant moment of the applied stresses $\bar{\mathbf{t}}$ on the lateral contour $\partial\Omega$. If F^i_J designate the components of the deformation gradient $\mathbf{F}(\mathbf{X})$, from the definition of $\mathbf{r}(\mathbf{X})$ given by (1.7), it follows that

$$\frac{\partial r^i}{\partial X^J} = F^i_J - \frac{dx^i_O}{dX^1}\,\delta^1_J$$

and therefore

$$\frac{dM_k}{dX^1}(X^1) + \overline{m}_k = \epsilon_{ijk}\int_\Omega J\,\sigma^{ij}\, d\Omega - \epsilon_{ijk}\frac{dx^i_O}{dX^1}\int_\Omega P^{j1}\, dA$$

where use has been made of the fact that $J\,\boldsymbol{\sigma} = \mathbf{P}\,\mathbf{F}^T$, $J = \det(\mathbf{F})$ being the jacobian. Since balance of angular momentum is equivalent to the symmetry of the Cauchy stress tensor $\boldsymbol{\sigma}$, $\epsilon_{ijk}\,\sigma_{ij} = 0$ * and one finally gets

$$\frac{d\mathbf{M}(X^1)}{dX^1} + \overline{\mathbf{m}} + \frac{d\mathbf{x}_O(X^1)}{dX^1} \times \int_\Omega [\mathbf{P}\,\hat{\mathbf{N}}]\, d\Omega = 0 \qquad (1.9)$$

where by definition $\mathbf{x}_O := \Phi(\mathbf{X})|_{X^2=X^3=0}$.

Remark

The centroid $\bar{\mathbf{x}} \in \mathbb{R}^3$ of the deformed cross section $\Phi(\Omega)$ is given by

$$\bar{\mathbf{x}} = \overline{\Phi}(X^1) \equiv \frac{1}{\Omega}\int_\Omega \Phi(\mathbf{X})\, d\Omega$$

* Recall that ϵ_{ijk} are the components of a completely anti-symmetric tensor.

and even within the linear theory $\bar{\mathbf{x}} \neq \mathbf{x}_O$, the reason being the axial warping of the cross section Ω. Thus, if equation (1.7) is replaced by $\mathbf{r} = \Phi(\mathbf{X}) - \bar{\mathbf{x}}$ a different definition of the resultant moment acting on $\Phi(\Omega)$ is obtained, although the corresponding equilibrium equation is the same as (1.9) with $\mathbf{x}_O$ replaced by $\bar{\mathbf{x}}$. Clearly, the difference between resultant moments about either $\mathbf{x}_O$ or $\bar{\mathbf{x}}$ occurs only in the non-linear theory, and should become significant only in cases of severe axial warping due to high shear deformation. This point is re-examined in Chap. 3.

Equations (1.6) and (1.9) represent the most general expression of the equations of equilibrium for a beam undergoing finite deformations. They are completely consistent with those of Finite Elasticity. The explicit appearance of the deformation map Φ through $\mathbf{x}_O = \Phi(\mathbf{X})|_{X^2=X^3=0}$ in the equilibrium equation for the resultant moment $\mathbf{M}(X^1)$ should be noted. This explicit dependence does not occur in the linear theory. The important case of a cross section with a plane of symmetry is considered next.

1.3- Cross Section with a Plane of Symmetry

In this section, attention is restricted to the case in which the beam of interest has a plane of symmetry, taken to be $X^1 - X^2$, and bending occurs in this plane. Thus, the external loads are symmetrically applied with respect to $X^1 - X^2$ on $\partial\Omega$, and $\bar{\mathbf{m}} \equiv 0$. The equilibrium equations (1.6) and (1.9) reduce then to

$$
\begin{aligned}
&\frac{d}{dX^1}\int_\Omega P^{11}\, d\Omega + p(X^1) = 0 \\
&\frac{d}{dX^1}\int_\Omega P^{21}\, d\Omega + q(X^1) = 0 \\
&\frac{dM(X^1)}{dX^1} + \frac{dx_O^1}{dX^1}\int_\Omega P^{21}\, d\Omega - \frac{dx_O^2}{dX^1}\int_\Omega P^{11}\, d\Omega = 0
\end{aligned}
\tag{1.10}
$$

where $\bar{\mathbf{t}} = [p(X^1)\ q(X^1)\ 0]^T$ is the applied load on $\partial\Omega$ per unit length. Integration of the first two of (1.10) yields

$$
\int_\Omega P^{11}\, d\Omega = -P - \int_{\xi=0}^{X^1} p(\xi)\, d\xi \ , \quad \int_\Omega P^{21}\, d\Omega = -H - \int_{\xi=0}^{X^1} q(\xi)\, d\xi \tag{1.11}
$$

where P represents the applied horizontal (compressive) load and H the applied vertical load, both at $X^1 = 0$. On the other hand, since $\mathbf{x}_O = \Phi(\mathbf{X})_{X^2=0}$, the most general expression for $\mathbf{x}_O$ is given by

$$x_O^1 = X^1 + u(X^1) \ , \qquad x_O^2 = v(X^1) \tag{1.12}$$

where $u(X^1)$ represents the axial displacement of the line of centroids, and $v(X^1)$ the corresponding vertical displacement. The final equilibrium equation for the bending moment follows by substituting (1.12) into (1.10) and making use of (1.11). The complete system of equilibrium equations is then

$$\begin{aligned} M'(X^1) + v'(X^1)\left[P + \int_{\xi=0}^{X^1} p(\xi)\,d\xi\right] &= [1 + u'(X^1)]\left[H + \int_{\xi=0}^{X^1} q(\xi)\,d\xi\right] \\ \int_\Omega P^{11}\,d\Omega &= -P - \int_{\xi=0}^{X^1} p(\xi)\,d\xi \\ \int_\Omega P^{21}\,d\Omega &= -H - \int_{\xi=0}^{X^1} q(\xi)\,d\xi \end{aligned} \tag{1.13}$$

It is remarked once again that no approximation is involved in equations (1.13); they are consistent with the equilibrium equations of Finite Elasticity. In the section 1.4, these equations will be reformulated in terms of tangential and normal stresses acting on the deformed cross section. An elementary illustration of equations (1.13), is furnished by the classical Euler's elastica.

1.3.1.- Example: Euler's elastica

By assumption, the line of centroids $X^2 = 0$ is regarded as inextensible. Since the stretching of this fiber is given by the Euclidean norm of

$$\boldsymbol{\lambda} = \mathbf{F}|_{X^2=0}\,\hat{\mathbf{E}}_1 = \mathbf{F}|_{X^2=0}\,[1\ 0]^T = \frac{d}{dX^1}\,[x_O^1\ x_O^2]^T$$

from (1.12), the inextensibility condition $||\boldsymbol{\lambda}||^2 = 1$ may be written as

$$\boldsymbol{\lambda} = [(1+u')\ v']^T = [\cos\alpha\ \sin\alpha]^T \tag{1.14}$$

where $\alpha(X^1)$ represents the angle between the line $\mathbf{x} = \Phi(\mathbf{X})|_{X^2=X^3=0}$ and the X^1-axis, for each $X^1 \in (0,L)$.

The second assumption is the usual Bernoulli's kinematic hypothesis. Thus, no shearing occurs and the bending moment is proportional to the change in curvature of the line of centroids. Consequently

$$M(X^1) = EI(X^1)\,\frac{d\alpha(X^1)}{dX^1} \tag{1.15}$$

The substitution of (1.14) and (1.15) into the moment equilibrium equation (1.13) together with the assumption of no distributed loading; i.e: $q(X^1) \equiv 0$ so that $H = 0$, leads to the non-linear eigenvalue problem for the elastica

$$\frac{d}{dX^1}\,[EI(X^1)\,\frac{d\alpha(X^1)}{dX^1}] + P\,\sin\alpha = 0\ , \qquad X^1 \in (0,L) \tag{1.16a}$$

$$u' = \cos\alpha - 1\ , \qquad v' = \sin\alpha \tag{1.16b}$$

An account of this classical problem can be found in [14,15]. In the next chapter, a generalization of problem (1.6) will be presented which takes into account the finite deformation of the beam due to shearing. The nature of the solutions of the resulting non-linear problem will be examined in Appendix I using perturbation methods.

1.4.- Basic Kinematic Relations. Alternative forms of the Equilibrium Equations.

The basic kinematic relationships used throughout this work are presented in this section. With the aid of these relationships, the equilibrium equations (1.13) will be reformulated in terms of stress measures which have direct physical meaning. The main objective is a formulation suitable for a systematic consideration of instability and bifurcation phenomena and, at the same time, capable of taking into account effects such as the warping of the cross section due to shear deformation.

1.4.1.- Geometry of the Deformed Cross Section.

Consider once again, as in section 1.3, an arbitrary cross section Ω, a distance X^1 from the origin of the material frame. The dependence of Ω on X^1 will be often made explicit by setting $\Omega(X^1)$. For a fixed X^1 in (0,L) points in $\Omega(X^1)$ will be denoted simply by

$(X^2,X^3)\in\Omega(X^1)$,

the dependence on X^1 being understood. These points are mapped onto points $\mathbf{x}$ in $\Phi(\Omega)$ through the map

$$\mathbf{x} = \hat{\mathbf{x}}(X^2,X^3) := \Phi(X^1,X^2,X^3)\big|_{X^1=Fixed} \tag{1.17}$$

The lines $X^2 = Constant$ and $X^3 = Constant$ are mapped onto curves in $\Phi(\Omega)$ sometimes referred to as coordinate curves of the surface (1.17). One also says the coordinates $\{X^2,X^3\}$ in Ω are *convected* through the deformation (1.17) [11,12].

The basis vectors $\hat{\mathbf{E}}_2,\hat{\mathbf{E}}_3$ at a point $(X^2,X^3)\in\Omega(X^1)$ are mapped by the deformation gradient $\nabla\hat{\mathbf{x}}$ of (1.17) onto vector fields (convected basis)

$$\mathbf{l}_2 = \nabla\hat{\mathbf{x}}.\hat{\mathbf{E}}_2 = \frac{\partial\hat{\mathbf{x}}}{\partial X^2} \quad \text{and} \quad \mathbf{l}_3 = \nabla\hat{\mathbf{x}}.\hat{\mathbf{E}}_3 = \frac{\partial\hat{\mathbf{x}}}{\partial X^3} \tag{1.18}$$

which are tangent to the coordinate curves and, therefore, to the deformed cross section $\Phi(\Omega)$. In addition, the vector field $\hat{\mathbf{n}}$ normal to $\Phi(\Omega)$ may be defined by

$$\frac{d\omega}{d\Omega}\,\hat{\mathbf{n}} = J\,\mathbf{F}^{-T}\,\hat{\mathbf{E}}_1 = \frac{\partial\hat{\mathbf{x}}}{\partial X^2}\times\frac{\partial\hat{\mathbf{x}}}{\partial X^3} \tag{1.19}$$

where

$$\frac{d\omega}{d\Omega} = \left\|\frac{\partial\hat{\mathbf{x}}}{\partial X^2}\times\frac{\partial\hat{\mathbf{x}}}{\partial X^3}\right\| \tag{1.20}$$

is the relation between deformed and undeformed areas $d\omega$ and $d\Omega$, respectively. The frame composed by $\{\hat{\mathbf{n}},\mathbf{l}_2,\mathbf{l}_3\}$ shown in Fig.1.1, not necessarily orthogonal, is often referred to as the Gaussian frame in the terminology of differential geometry [16].

For simplicity, attention is restricted throughout the rest of this chapter to the cases in which the cross section Ω is either 'narrow', with a plane of symmetry and applied forces contained in this plane, or to a situation of plane strain. The extension to the general case of an arbitrary cross section is straight forward and is considered in chapter 3. Thus, we consider the following situation:

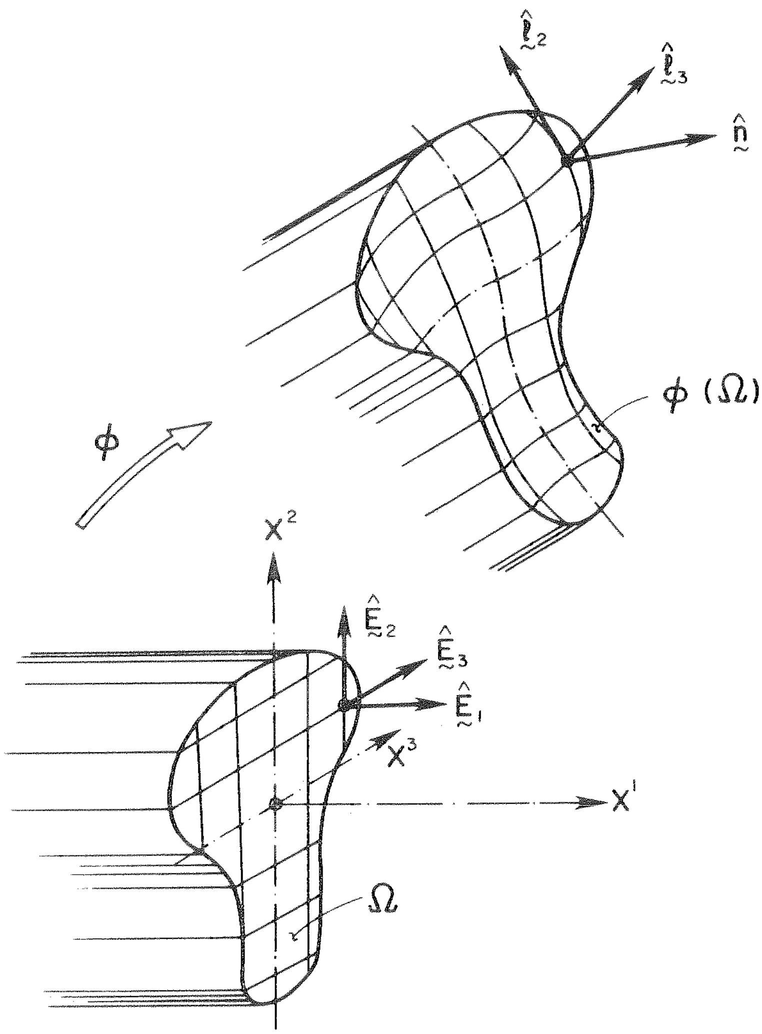


Figure 1.1. Geometry of the deformed cross section. Gaussian frame.

(i) No twisting of the cross section occurs. Therefore

$$\mathbf{l}_2.\mathbf{l}_3 = 0$$

and the Gauss frame is orthogonal.

(ii) Given the type both of cross section and external loading, warping of the cross section occurs only in the direction of the symmetry plane X^1-X^2. Hence, we may assume that

$$\mathbf{l}_3 = \frac{\partial \hat{\mathbf{x}}}{\partial X^3} \equiv \hat{\mathbf{E}}_3 \quad and \quad ||\mathbf{l}_3|| = 1 \quad \#$$

#This is he same as saying that the deformed cross section is a developable surface and, therefore, with zero

In view of these assumptions, further reference to the X^3-coordinate is omitted. Accordingly, $\mathbf{X} = (X^1, X^2)$ is employed to designate points in the reference configuration B, $\hat{\mathbf{E}}_1 = [1\ 0]^T$ and $\hat{\mathbf{E}}_2 = [0\ 1]^T$ for the standard basis at $\mathbf{X}$, and $\mathbf{x} = \Phi(X^1, X^2)$ for the deformation map. With these conventions the relevant part of the deformation gradient is written as

$$\mathbf{F} = \begin{bmatrix} x^1{}_{,1} & x^1{}_{,2} \\ x^2{}_{,1} & x^2{}_{,2} \end{bmatrix} \tag{1.21}$$

where we have set $x^i{}_{,J} = \dfrac{\partial \Phi^i}{\partial X^J}$ for notational convenience.

The coordinate expression for the normal and tangential vector fields in the $\{\hat{\mathbf{e}}_i\}$ basis takes now a particularly simple form. From (1.21) and definitions (1.18) and (1.19), it follows that

$$\hat{\mathbf{n}} = \left[\frac{d\omega}{d\Omega}\right]^{-1} \begin{Bmatrix} x^2{}_{,2} \\ -x^1{}_{,2} \end{Bmatrix} \qquad \hat{\mathbf{l}} = \left[\frac{d\omega}{d\Omega}\right]^{-1} \begin{Bmatrix} x^1{}_{,2} \\ x^2{}_{,2} \end{Bmatrix} \tag{1.22}$$

where $\hat{\mathbf{l}} := \dfrac{\mathbf{l}_2}{||\mathbf{l}_2||}$, and

$$\frac{d\omega}{d\Omega} = \sqrt{(x^1{}_{,2})^2 + (x^2{}_{,2})^2} = ||\mathbf{l}_2|| \tag{1.23}$$

The coordinate matrix of the unit vector fields $\hat{\mathbf{n}}$ and $\hat{\mathbf{l}}$ in the $\{\hat{\mathbf{e}}_i\}$ basis will be denoted by $\Lambda(\mathbf{X})$ and plays a predominant role in subsequent developments. Introducing, for convenience, the angle $\psi(X^1, X^2)$ defined in terms of the components of the deformation gradient by

$$\tan \psi(X^1, X^2) := -\frac{x^1{}_{,2}}{x^2{}_{,2}} \tag{1.24}$$

the matrix $\Lambda(\mathbf{X})$ takes the form

$$\Lambda^T(X^1, X^2) := \frac{1}{\sqrt{(x^1{}_{,2})^2 + (x^2{}_{,2})^2}} \begin{bmatrix} x^2{}_{,2} & -x^1{}_{,2} \\ x^1{}_{,2} & x^2{}_{,2} \end{bmatrix} \equiv \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \tag{1.25a}$$

where, by definition

$$\begin{Bmatrix} \hat{\mathbf{n}} \\ \hat{\mathbf{l}} \end{Bmatrix} = \Lambda^T \begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \end{Bmatrix} \tag{1.25b}$$

Gaussian curvature [16].

Obviously, $\Lambda(\mathbf{X})$ is an orthogonal matrix depending, in general, on both X^1 and X^2.

1.4.2.- Normal and Tangential Stress Fields. Equilibrium Equations.

The equilibrium equation, with the exception of that involving the bending moment, have been so far expressed in terms of the first Piola-Kirchhoff tensor. In applications, an expression in terms of resultant axial and shear forces is often more convenient. For this purpose, we consider the normal and tangential stress vector fields σ and τ covering the deformation map $\Phi : B \rightarrow \mathbb{R}^3$. That is stress vectors acting on the deformed cross section defined per unit of undeformed area, and given by

$$\sigma = \hat{\mathbf{n}}.(\mathbf{P}\,\hat{\mathbf{N}}) \qquad \tau = \hat{\mathbf{l}}.(\mathbf{P}\,\hat{\mathbf{N}}) \tag{1.26}$$

In view of the definition of $\Lambda(\mathbf{X})$ in (1.25), they may be expressed by the relation

$$\begin{Bmatrix} P^{11} \\ P^{21} \end{Bmatrix} = \Lambda(\mathbf{X}) \begin{Bmatrix} \sigma \\ \tau \end{Bmatrix} \tag{1.27}$$

The equations of equilibrium in terms of the normal and tangential stress σ and τ follow then at once by substitution of (1.27) into (1.13). The complete system of equations consists of

$$\frac{d}{dX^1}\int_{\Omega} \Lambda(\mathbf{X}) \begin{Bmatrix} \sigma \\ \tau \end{Bmatrix} d\Omega + \begin{Bmatrix} p(X^1) \\ q(X^1) \end{Bmatrix} = 0 \tag{1.28a}$$

together with the equilibrium equation for the bending moment

$$\frac{dM}{dX^1} + v'(X^1)\left[P + \int_{\xi=0}^{X^1} p(\xi)\,d\xi\right] = [1 + u'(X^1)]\left[\int_{\xi=0}^{X^1} q(\xi)\,d\xi\right] \tag{1.28b}$$

which remains unchanged.

The resultant axial and shear forces are defined by

$$N = \int_{\Omega} \sigma\, d\Omega \qquad V = \int_{\Omega} \tau\, d\Omega \tag{1.30}$$

However, inasmuch as $\Lambda(X^1,X^2)$ is in general a function of X^2, equation (1.27a) shows that an equation involving N and V (and their derivatives) explicitly is, in general, not possible. The warping of the cross section due to shear deformation is the physical reason behind the explicit dependence of Λ on X^2. Due to this warping effect, the angle $\psi(X^1,X^2)$ changes over the deformed cross section and, consequently, so does the frame $(\hat{\mathbf{n}},\hat{\mathbf{l}})$ and the matrix Λ.

Whenever warping is neglected, equations (1.28) reduce to a system of non linear ordinary differential equations involving N,V, and M; such as in the example discussed in 1.3.1. The effect of the warping of the cross section will be discussed at length in the forthcoming chapters.

1.5.- Physical Interpretation of the Second Piola-Kirchhoff stress tensor

One way of characterizing the response of a general non-linear isotropic elastic material is through its strain energy function $\hat{W}(\mathbf{E})$ expressed in terms of the Lagrangian stress tensor $\mathbf{E}$. The second Piola-Kirchhoff stress tensor can then be computed according to [11-13]

$$\mathbf{S} = \frac{\partial \hat{W}(\mathbf{X},\mathbf{E})}{\partial \mathbf{E}} \tag{1.31}$$

In a variety of problems of practical interest, however, one is interested in situations for which the strain is small in some sense and yet the geometry of the problem is such that large displacements and rotations are expected to occur. The question often arises of how to generalize the classical model $\sigma = \lambda \; trace(\epsilon) \; 1 + 2G \; \epsilon$ of linear elasticity to situations of this type. Models of the form

$$\mathbf{S} = \lambda \; trace(\mathbf{E}) \; 1 + 2G \; \mathbf{E} \tag{1.32}$$

have frequently been considered as suitable generalizations [17,18]. However, inasmuch as the tensor $\mathbf{S}$ is defined as the pull-back of $\mathbf{P}$ to the reference configuration [12]; i.e: $\mathbf{S} = \mathbf{F}^{-1}\mathbf{P}$, this object does not have a direct physical meaning, although $\mathbf{P}$ does. One wonders then about the physical significance of the elastic constants λ and G appearing in (1.32), and questions the reasons behind models of this type other than those of mathematical simplicity or computational convenience.

Restricting our attention to the two dimensional case, the components of the tensor $\mathbf{S}$ admit a simple geometric interpretation in terms of the stresses σ and τ which do have direct physical meaning. In fact, substitution of the relation $\mathbf{P} = \mathbf{F}\,\mathbf{S}$ into (1.27) yields

$$\begin{Bmatrix}\sigma\\ \tau\end{Bmatrix} = \left[\frac{d\omega}{d\Omega}\right]^{-1}\begin{bmatrix} J & 0\\ E_{12} & \left[\dfrac{d\omega}{d\Omega}\right]^2\end{bmatrix}\cdot\begin{Bmatrix}S^{11}\\ S^{12}\end{Bmatrix} \tag{1.33}$$

where we have used the fact that the Lagrangian strain tensor is given by $\mathbf{E} = \frac{1}{2}[\mathbf{C} - \mathbf{1}]$, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ being the right Cauchy-Green tensor.

Similarly, planes perpendicular to the X^2-axis are mapped, *locally*, according to $\frac{d\omega_l}{d\Omega_l}\,\hat{\mathbf{n}}_l = J\,\mathbf{F}^{-T}\hat{\mathbf{E}}_2$, where $d\omega_l$ and $d\Omega_l$ are elemental deformed and undeformed areas. An argument analogous to that leading to (1.33) would show that the normal and tangential stress σ_l and τ_l acting on the deformed area $d\omega_l$ are related to the components of $\mathbf{S}$ through

$$\begin{Bmatrix}\sigma_l\\ \tau_l\end{Bmatrix} = \left[\frac{d\omega_l}{d\Omega_l}\right]^{-1}\begin{bmatrix} J & 0\\ E_{12} & \left[\dfrac{d\omega_l}{d\Omega_l}\right]^2\end{bmatrix}\cdot\begin{Bmatrix}S^{22}\\ S^{12}\end{Bmatrix} \tag{1.34}$$

where

$$\frac{d\omega_l}{d\Omega_l} = \sqrt{(x^1{}_{,1})^2 + (x^2{}_{,1})^2} \tag{1.35}$$

Equations (1.33) and (1.34) give an interpretation of the tensor $\mathbf{S}$ highly useful in applications. In the case of beams, for example, these results show that the component S^{11} is exactly proportional to the axial normal stress acting on a deformed cross section of the beam, while S^{22} is proportional to the transversal normal stress. However, this is not the case for the component S^{12}, unless $E_{12} \equiv 0$. In section 1.3.1, It will be shown that the values of the components of $\mathbf{S}$ are in fact exactly proportional to the components of the Cauchy stress tensor with respect to the convected basis.

In view of (1.33), relationships of the form $V = \int_\Omega S^{12}\, d\Omega$, sometimes used in the analysis of beams and plates subjected to large displacements [18], should be regarded as merely formal with questionable physical significance.

The results contained in equations (1.33) and (1.34) are illustrated in Fig.1.2, and may be summarized in the following proposition.

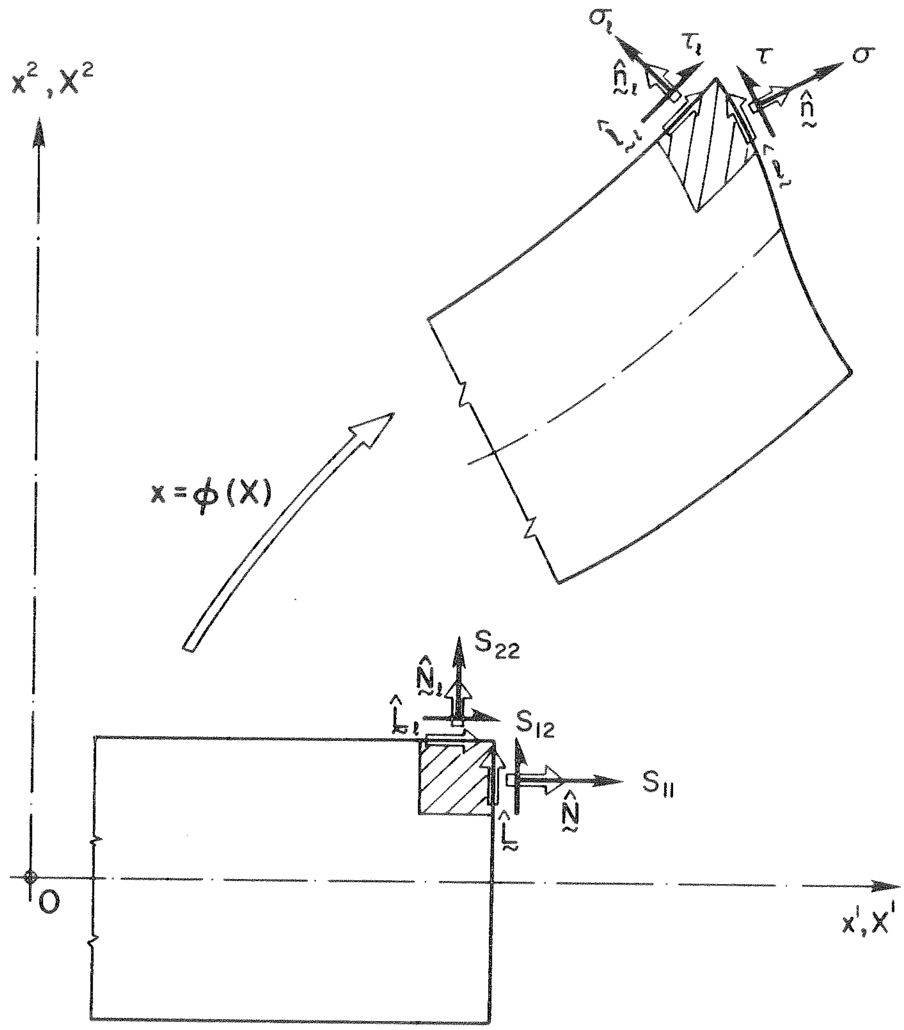


Figure 1.2. Interpretation of the second Piola-Kirchoff stress tensor **S**.

Proposition

Consider a squared neighborhood with sides $d\Omega$ and $d\Omega_l$ parallel to the the coordinate axis X^1, X^2; and deformed onto a neighborhood with sides $d\omega, d\omega_l$, as shown in Fig.1.2.

The components of the second Piola-Kirchhoff stress tensor S are given, in terms of the normal and tangential stresses (σ, τ) and (σ_l, τ_l) acting on $d\omega$ and $d\omega_l$ respectively, by

$$S^{11} = \frac{d\omega}{d\Omega}\,\frac{\sigma}{J}$$
$$S^{22} = \frac{d\omega_l}{d\Omega_l}\,\frac{\sigma_l}{J} \tag{1.36}$$
$$S^{12} = \left[\frac{d\omega}{d\Omega}\right]^{-1}\left[\tau - 2E_{12}\,\frac{\sigma}{J}\right] = \left[\frac{d\omega_l}{d\Omega_l}\right]^{-1}\left[\tau_l - 2E_{12}\,\frac{\sigma_l}{J}\right]$$

Remark

A direct coordinate free proof of this proposition is extremely simple and generalizes immediately to the three dimensional case. In fact, we have

$$\begin{aligned}\sigma &= \hat{\mathbf{n}}.(\mathbf{P}\,\hat{\mathbf{E}}_1)\\ &= \left[\frac{d\omega}{d\Omega}\right]^{-1} J\,\hat{\mathbf{E}}_1.[\mathbf{F}^{-1}\mathbf{P}\,\hat{\mathbf{E}}_1]\\ &= \left[\frac{d\omega}{d\Omega}\right]^{-1} J\,\hat{\mathbf{E}}_1.[\mathbf{S}\,\hat{\mathbf{E}}_1]\\ &= \left[\frac{d\omega}{d\Omega}\right]^{-1} S^{11}\end{aligned}$$

which proves the first of (1.36). A similar proof holds for the second relation. For the shearing stress we have

$$\begin{aligned}\tau &= \hat{\mathbf{l}}.(\mathbf{P}\,\hat{\mathbf{E}}_1)\\ &= \frac{1}{||\mathbf{l}||}[\mathbf{F}\,\hat{\mathbf{E}}_2].[\mathbf{F}\,\mathbf{S}\,\hat{\mathbf{E}}_1]\\ &= \frac{1}{||\mathbf{l}||}\hat{\mathbf{E}}_2.[(\mathbf{F}^T\mathbf{F})\,\mathbf{S}\,\hat{\mathbf{E}}_1]\\ &= \frac{1}{||\mathbf{l}||}\hat{\mathbf{E}}_2.[\mathbf{C}\,\mathbf{S}\,\hat{\mathbf{E}}_1]\end{aligned}$$

since $C_{22} = ||\mathbf{l}||^2 = \left[\frac{d\omega}{d\Omega}\right]^2$, it follows that

$$\tau = E_{12}\,\frac{\sigma}{J} + \left[\frac{d\omega}{d\Omega}\right] S^{12}$$

which proves the third of (1.36).

1.5.1.- Convected Basis. Alternative Interpretation.

The so-called convected coordinate system is often used in the formulation of rod theories. Its introduction provides yet another interpretation of the second Piola-Kirchhoff tensor.

Let the reference configuration B, a three dimensional manifold over R^3, be covered by a system of coordinates $\{X^I\}$ with associated basis $\{\hat{\mathbf{E}}_I\}$. The convected coordinate system on $\Phi(B)$ is then defined as

$$\theta^I(x) = \theta^I(\Phi^{-1}(x)) = \theta^I o \Phi^{-1}(x) \tag{1.37}$$

for any $x \in B$ such that $x = \Phi(X)$.

The usefulness of the convected system lies in the simple coordinate representation taken by the deformation map $\Phi : B \rightarrow R^3$. Denoting the latter by $\Phi_\Theta^I = X^I o \Phi$, from (1.37) follows that

$$\Phi_\Theta^I(X) = X^I o \Phi^{-1} o \Phi(X) = X^I(X) \tag{1.38}$$

Thus, the convected coordinates θ^I of any point $x = \Phi(X)$ in $\Phi(B)$ are numerically equal to the coordinates X^I of the point $X \in B$. Furthermore from (1.37) it immediately follows that the basis $\{\hat{\mathbf{I}}_I\}$ associated with the convected system is given by

$$\hat{\mathbf{I}}_I(X) = \mathbf{F}(X).\hat{\mathbf{E}}_I \tag{1.39}$$

Hence, the metric tensor $G_{\Theta_{IJ}} = \hat{\mathbf{I}}_I . \hat{\mathbf{I}}_J$ in the convected basis reduces to the right Cauchy-Green tensor $\mathbf{C}$; i.e:

$$\mathbf{G}_\Theta(x) = \mathbf{C} o \Phi^{-1}(x) \;, \qquad \textit{where} \;\; x = \Phi(X) \tag{1.40}$$

To obtain an alternative interpretation of the second Piola-Kirchhoff stress tensor let σ_Θ^{IJ} be the components of the Cauchy stress tensor $\boldsymbol{\sigma}$ with respect to the convected basis. Accordingly

$$\begin{aligned} \boldsymbol{\sigma} &= \sigma^{ij}\, \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \\ &= \sigma_\Theta^{IJ}\, \hat{\mathbf{I}}_I \times \hat{\mathbf{I}}_J = \sigma_\Theta^{IJ}\, F_I^i\, F_J^j\, \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \end{aligned} \tag{1.41}$$

however, by the Piola transformation

$$\sigma^{ij} = \frac{1}{J} P^{iJ} F^j_J = \frac{1}{J} S^{IJ} F^i_I F^j_J \tag{1.42}$$

a comparison between (1.41) and (1.42) shows that the numerical values of the components S^{IJ} of the tensor $\mathbf{S}$ with respect to the material coordinates $\{X^I\}$ are given by

$$S^{IJ}(\mathbf{X}) = J \, \sigma^{IJ}_{\theta} \circ \Phi(\mathbf{X}) \tag{1.43}$$

The spatial tensor $\boldsymbol{\tau} = J \, \boldsymbol{\sigma}$ is often referred to as Kirchhoff tensor. Thus, equation (1.43) shows that *the components of the Kirchhoff stress tensor* $\boldsymbol{\tau}(\mathbf{x})$ *with respect to the convected basis* $\{\theta^I\}$ *coincide with the components of the second Piola-Kirchhoff stress tensor* $\mathbf{S}(\mathbf{X})$ *with respect to the material basis* $\{X^I\}$.

Although the convected coordinate system leads to more compact expressions of the field equations, the deformation gradient is implicitly contained in these expressions. In practical applications, we find a Lagrangian formulation in terms of the material coordinates $\{X^I\}$ more useful.

CHAPTER 2.

FINITE DEFORMATION AND BUCKLING OF BEAMS BASED UPON BERNOULLI'S GENERALIZED KINEMATIC ASSUMPTION

2.1.- Introduction.

Formulations of non-linear, geometrically exact beam theories in terms of stress resultants require the introduction of some kinematic constraint. The Bernoulli-Kirchhoff hypothesis is undoubtly the most widely enforced kinematic constraint. In the context of Chapter 1, this hypothesis amounts simply to enforcing invariance in orientation of the Gaussian frame over the deformed cross section $\Phi(\Omega)\subset\mathbb{R}^3$ of the beam. Under this restriction, the Gaussian frame plays a role entirely equivalent to that of the *directors* in beam theories which regard the rod as a Cosserat Continuum. A comprehensive account of formulations based upon the director approach may be found in [8].

In the present chapter, the transversally homogeneous beam is first considered under the aforementioned kinematic hypothesis. Subsequently, the Bernoulli-Kirchhoff assumption is relaxed to reflect a piece-wise linear deformation pattern of the cross section. This generalization allows the rigorous treatment of a case of technical interest: the sandwich beam. The emphasis is placed on bifurcation phenomena and attention is thus confined to the static case and hyperelastic material. The results obtained in this chapter may be summarized as follows:

(i) The results of chapter 1 allow a simple derivation of a geometrically exact theory capable of modeling finite compression bending and shear of the beam. For the transversally homogeneous beam, similar results appear to be implicit in Antman's work [19]. For the sandwich beam, however, the fully non-linear theory presented seems to be entirely new. Finite element implementations of our formulation for the transversally homogeneous beam can be found in [20].

(ii) The elastic stability of both types of beams is examined by introducing the customary assumption of inextensibility of the line of centroids. The result is then a non-linear eigenvalue problem for the critical axial load in which finite shear of the beam remains accounted for. Remarkably, the linear problem obtained by consistent linearization about the reference configuration yields, for both types of beams, expressions for the buckling loads well known in the engineering literature [21,22,23,24,25] which exhibit the important reduction experienced by Euler's buckling load as a result of shear deformation.

(iii) The fully non-linear formulations presented, allow a rigorous study of the post-buckling behavior for both the transversally homogeneous and the sandwich beam. Making use of the Poincare-Keller [26] perturbation method, it is shown in Appendix I that the adjacent equilibrium configurations are locally stable, the bifurcation diagram being the familiar pitchfork. Similar bifurcation analysis, employing the Poincare-Keller method or the Liapunov-Schmidt procedure, have been restricted in the past to either Euler's elastica [14,15] or to an extensible although unshearable rod [27,28]. Related results for the elastica with no-convex strain energy are given in [29,30]. No results appear to have been reported regarding the influence of shear in the post-buckling behavior except for those presented in [18]. The case of the sandwich has been overlooked.

(iv) Finally, two dimensional constitutive models convenient for computational purposes, are examined in the context of approximate theories obtained by consistent linearization. Such model have often been use within the framework of finite element method [31,32,33] and, to some extent, to justify models such as the Von-Karman theory of thin plates [18].

2.2.- Transversally Homogeneous Beam. Geometrically "Exact" Theory.

2.2.1.- Kinematic Assumption.

The central assumption in the theory discussed in this section, is furnished by Bernoulli's "plane sections remain plane" hypothesis. When attention is confined to the case of bending in a plane of symmetry of the beam, the expression for the deformation map $\Phi : B \rightarrow \mathbb{R}^3$ consistent with this assumption is given by

$$\mathbf{x} = \mathbf{x}_O + X^2 \begin{Bmatrix} -\sin\psi(X^1) \\ \cos\psi(X^1) \end{Bmatrix} \tag{2.1}$$

where, as in the previous chapter

$$\mathbf{x}_O = \Phi(\mathbf{X})\big|_{X^2=0} = \begin{Bmatrix} X^1 + u(X^1) \\ v(X^1) \end{Bmatrix} \tag{2.2}$$

Expressed in geometric terms, (2.1) states that an arbitrary cross section $\Omega(X^1)$ is translated according to (2.2) and rotated an angle $\psi(X^1)$ in plane of symmetry. Any warping effect of the cross section is ignored by (2.1).

From (2.1) the explicit components of the deformation gradient are

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} 1 + u' - X^2\psi'\cos\psi & -\sin\psi \\ v' - X^2\psi'\sin\psi & \cos\psi \end{bmatrix} \tag{2.3}$$

where $(')$ denotes differentiation with respect to X^1. In addition the jacobian $J = \det(\mathbf{F})$ is given by

$$J = (1 + u')\cos\psi + v'\sin\psi - X^2\,\psi' \tag{2.4}$$

Finally, the unit vector fields $\hat{\mathbf{N}} \equiv \hat{\mathbf{E}}_1 = [1\ 0]^T$ and $\hat{\mathbf{E}}_2 = [0\ 1]^T$ are mapped onto $\hat{\mathbf{n}}$ and $\hat{\mathbf{l}}$ according to (1.22); i.e

$$\hat{\mathbf{n}} = \left[\frac{d\omega}{d\Omega}\right]^{-1} J\, \mathbf{F}^{-T}\, \hat{\mathbf{N}} = [\cos\psi(X^1)\ \sin\psi(X^1)]^T \tag{2.5a}$$

$$\hat{\mathbf{l}} = \mathbf{F}\, \hat{\mathbf{E}}_2 = [-\sin\psi(X^1)\ \cos\psi(X^1)]^T \tag{2.5b}$$

The kinematics of the beam is summarized in Fig.2.1.

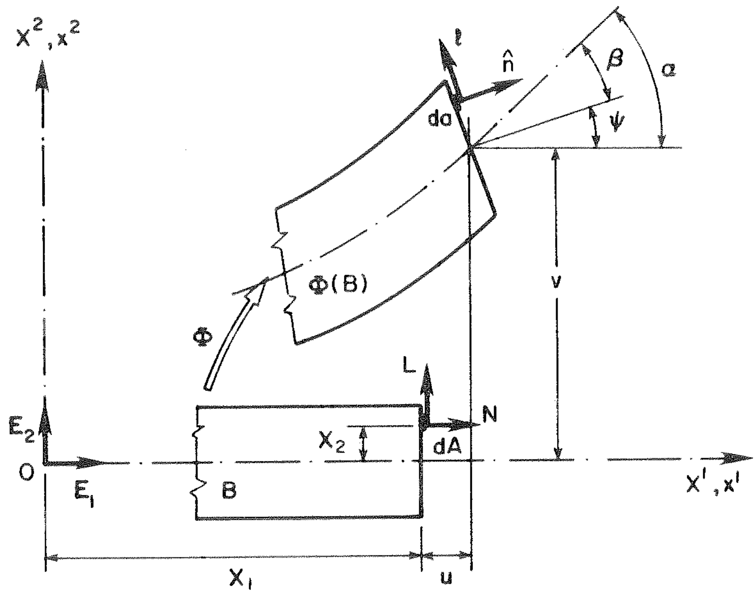


Figure 2.1. Kinematics of a transversally homogeneous beam.

Clearly, the assumption (2.1)-(2.2) represents, in the context of Chap.1, the simplest possible expression for the general kinematics discussed in section 1.4. Explicitly $\frac{d\omega}{d\Omega} \equiv 1$ and the $\psi(X^1,X^2)$, as defined by (1.24), is independent of X^2. Consequently, the matrix $\Lambda(X^1,X^2)$ is independent of X^2 and , according to (1.25), takes the simple form

$$\Lambda(X^1,X^2) \equiv \begin{bmatrix} \cos\psi(X^1) & -\sin\psi(X^1) \\ \sin\psi(X^1) & \cos\psi(X^1) \end{bmatrix} \tag{2.6}$$

2.2.2.- Equilibrium Equations.

Once an explicit expression for the orthogonal matrix Λ is known, the equilibrium equations of the theory can be immediately established. To express them in terms of the axial and shear forces acting on the deformed cross section, recall that these stress resultants are related to the normal and tangential stress fields σ and τ by

$$N = \int_{\Omega} \sigma \, d\Omega \qquad V = \int_{\Omega} \tau \, d\Omega \tag{2.7}$$

The introduction of these expressions together with (2.6) into equation (1.28a), and use of the fact that $\Lambda(X^1,X^2)$ is independent of X^2, leads to

$$\frac{d}{dX^1}\left[\begin{bmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{bmatrix}\cdot\begin{Bmatrix}N\\V\end{Bmatrix}\right] + \begin{Bmatrix}p(X^1)\\q(X^1)\end{Bmatrix} = 0 \tag{2.8a}$$

together with the equilibrium equation (1.28b) for the bending moment M; i.e:

$$M' + \left[P + \int_{\xi=0}^{X^1} p(\xi)\, d\xi\right] v' = (1+u')\left[H + \int_{\xi=0}^{X^1} q(\xi)\, d\xi\right] \tag{2.8b}$$

which, remarkably enough, always remains unchanged.

It is possible, however, to recast the equilibrium equation (2.8b) for M in a form including explicitly the forces N and V. In fact, substitution of (2.8a) into (2.8b) yields

$$M' + [(1+u')\cos\psi + v'\sin\psi]\, V - [v'\cos\psi - (1+u')\cos\psi]\, N = 0 \tag{2.9}$$

From (2.4) one notices that the term multiplying V is precisely the value of the jacobian J at $X^2 \equiv 0$. Similarly, it is easy to see that the term multiplying N is the component E_{12} of the Lagrangian strain tensor **E**. Consequently, equation (2.9) may be written as

$$M' + J_O\, V - E_{12}\, N = 0 \qquad X^1 \in (0,L) \tag{2.10}$$

where

$$J_O := \det(\mathbf{F})\big|_{X^2=0} \qquad \text{so that} \qquad J = J_O - X^2\, \psi' \tag{2.11}$$

The use of the equilibrium equation (2.10) rather than (2.8b) is particularly useful in numerical formulations employing a finite element technique. Next attention is focussed on the development of appropriate constitutive equations in terms of the stress resultants N, V and M under the assumption of hyperelasticity. Only the isothermal case will be considered.

2.2.3.- Constitutive equations for Hyperelasticity.

It will be assumed in what follows, that the material is hyperelastic with strain energy $W(X,\mathbf{F})$. The first Piola-Kirchhoff tensor **P** may then be computed according to

$$\mathbb{P} = \frac{\partial W(X,\mathbb{F})}{\partial \mathbb{F}}$$

To obtain consistent constitutive equations in terms of stress resultants, it is first necessary to derive the conjugate strain measures to the generalized stresses N, V, and M. A duality argument is then necessary. The derivation will be carried out in some detail.

Consider the total potential energy functional defined, for a hyperelastic material in any configuration $\mathbf{x} = \Phi(\mathbf{X})$, by

$$\Pi(\Phi) := \int_B W(X,\mathbb{F})\, dV - \int_{\partial B_t} \bar{\mathbf{t}}.\mathbf{x}\, dS - \int_B \rho_{Ref}\, \mathbf{B}.\mathbf{x}\, dV \tag{2.12}$$

where $\mathbb{F} = \nabla\Phi$, ∂B_t denotes the part of the boundary where the stresses $\bar{\mathbf{t}}$ are prescribed, and ∂B_u that part of the boundary where displacements are prescribed. $\partial B_t \bigcap \partial B_u = \phi$. In addition, the linear space of kinematically admissible variations is defined in the usual manner; i.e

$$V = \{\ \delta\mathbf{u} : B \to \mathbb{R}^3 \ \mid \delta\mathbf{u}|_{\partial B_u} = 0\ \} \tag{2.13}$$

In order to find the dual strain measures to N, V and M, it is sufficient to focus on the strain energy part, herein denoted by $\Pi^*(\Phi)$, of the total potential energy functional. The Frechet differential of Π^* at an equilibrium configuration $\mathbf{x} = \Phi(\mathbf{X})$ is then

$$\delta\Pi(\Phi) = \int_0^L \left[\int_\Omega \mathbb{P}(\nabla\Phi).\delta\mathbb{F}\, d\Omega \right] dx \tag{2.14}$$

where $\delta\mathbb{F}$ may be easily computed from the explicit expression (2.3) of the deformation gradient, by

$$\delta\mathbb{F} = \frac{d}{d\alpha} \nabla[\Phi + \alpha\, \delta\mathbf{u}]\big|_{\alpha=0} \tag{2.15}$$

Performing the indicated computation one finds

$$\delta\Pi^*(\Phi) = \int_0^L \Bigg\{ \delta u' \int_\Omega P^{11} d\Omega + \delta v' \int_\Omega P^{21}\, d\Omega + \delta\psi' \left[-\hat{\mathbf{n}}.\int_\Omega X^2 \begin{Bmatrix} P^{11} \\ P^{21} \end{Bmatrix} d\Omega \right]$$
$$- \delta\psi \left[\psi'\, \hat{\mathbf{i}}.\int_\Omega X^2 \begin{Bmatrix} P^{11} \\ P^{21} \end{Bmatrix} d\Omega + \hat{\mathbf{n}}.\int_\Omega \begin{Bmatrix} P^{12} \\ P^{22} \end{Bmatrix} d\Omega \right] \Bigg\} dx \tag{2.16}$$

To interpret the terms appearing in (2.16) recall first that according to (1.7), the bending moment M may be expressed in terms of components of $\mathbf{P}$ as

$$M = -[\cos\psi \ \sin\psi] . \int_{\Omega} X^2 \begin{Bmatrix} P^{11} \\ P^{21} \end{Bmatrix} d\Omega \equiv -\hat{\mathbf{n}} . \int_{\Omega} X^2 \begin{Bmatrix} P^{11} \\ P^{21} \end{Bmatrix} d\Omega \tag{2.17}$$

Next, the two integrals appearing in the coefficient multiplying $\delta\psi$ in (2.17) may be identified in terms of the stresses σ and τ as follows. For the first integral one has

$$\hat{\mathbf{l}} . \int_{\Omega} X^2 \begin{Bmatrix} P^{11} \\ P^{21} \end{Bmatrix} d\Omega = [-\sin\psi \ \cos\psi] . \int_{\Omega} \Lambda . \begin{Bmatrix} \sigma \\ \tau \end{Bmatrix} d\Omega$$

$$= \int_{\Omega} X^2 \, [0 \ 1] . \begin{Bmatrix} \sigma \\ \tau \end{Bmatrix} d\Omega = \int_{\Omega} X^2 \, \tau \, d\Omega \tag{2.19}$$

In addition, since $\mathbf{P} = \mathbf{F}^{-1}\mathbf{S}$ and the components of $\mathbf{S}$ are related to σ and τ through (1.36), the second integral may be written as

$$\hat{\mathbf{n}} . \int_{\Omega} X^2 \begin{Bmatrix} P^{12} \\ P^{22} \end{Bmatrix} d\Omega = \int_{\Omega} J \, S^{12} \, d\Omega$$

$$= \int_{\Omega} [J \, \tau - 2E_{12} \, \sigma] \, d\Omega \tag{2.20}$$

Thus, substitution of (2.18)-(2.20) into (2.17) yields

$$\delta\Pi^*(\Phi) = \int_0^L \{\delta u'[N\cos\psi - V\sin\psi] + \delta v'[N\sin\psi + V\cos\psi]\} \, dx$$

$$+ \int_0^L \{\delta\psi' M - \delta\psi[J_O V - 2E_{12}N]\} \, dx \tag{2.21}$$

It should be noticed that integration by parts and standard arguments of variational calculus [33] would yield the equations of equilibrium previously derived. Instead, collecting terms one finds

$$\delta\Pi(\Phi) = \int_0^L \Bigg\{ M \, \delta\psi' + N \Big[[v'\cos\psi - (1+u')\sin\psi]\delta\psi + \cos\psi\delta u' + \sin\psi\delta v' \Big]$$

$$+ V \Big[[-(1+u')\cos\psi - v'\sin\psi]\delta\psi - \sin\psi\delta u' + \cos\psi\delta v' \Big] \Bigg\} \, dx \tag{2.22}$$

Observe finally that the stretching of the line of centroids is given by the norm of the vector field

$$\boldsymbol{\lambda} = \mathbf{F}|_{X^2=0}\cdot\begin{Bmatrix}1\\0\end{Bmatrix} \equiv \begin{Bmatrix}(1+u')\\ v'\end{Bmatrix} \tag{2.23}$$

Therefore, the strain measures λ_n and λ_l defined by

$$\lambda_n := \boldsymbol{\lambda}.\hat{\mathbf{n}} = (1 + u')\cos\psi + v'\sin\psi \equiv J_O \tag{2.24a}$$

$$\lambda_l := \boldsymbol{\lambda}.\hat{\mathbf{l}} = v'\cos\psi - (1 + u')\sin\psi \equiv E_{12} \tag{2.24b}$$

represent the stretching in the direction $\hat{\mathbf{n}}$ normal to the deformed cross section and the shearing of the cross section (in the direction $\hat{\mathbf{l}}$), respectively. In view of (2.24), the coefficients multiplying N and V in (2.21) are identified as the variations (Frechet differentials) of λ_n and λ_l respectively. In conclusion, (2.21) may be finally written as

$$\delta\Pi(\Phi) = \int_0^L [M\ \delta\psi' + N\ \delta\lambda_n + V\delta\lambda_l]\ d\Omega \tag{2.25}$$

which shows that ψ', $[\lambda_n - 1]$ and λ_l are the strain measures conjugate to M, N and V, respectively. Accordingly, the strain energy function of the rod is of the form $\overline{W}(X^1,\psi',\lambda_n,\lambda_l)$. However, the question regarding which suitable restrictions are to be imposed on the strain energy function in order to obtain physically meaningful and, at the same time, well posed problems is far from settled, even in the simple one dimensional case ([12,13,19] and references therein). In the sequel, the following constitutive equations are considered

$$\begin{aligned} N &= E\Omega(X^1)\ [\lambda_n - 1] \equiv E\Omega(X^1)\ [J_O - 1] \\ V &= G\Omega(X^1)\ \lambda_l \equiv G\Omega(X^1)\ E_{12} \\ M &= EI(X^1)\ \psi' \end{aligned} \tag{2.26}$$

Equations (2.26) furnish the simplest constitutive model in terms of the generalized strains ψ', $[\lambda_n - 1]$, and λ_l. In addition, its linearization about the reference configuration yields the classical constitutive equations of Timoshenko's beam theory. However, an estimate of the so-called shear coefficient, requires a more elaborate kinematic assumption. This question will be considered in some detail in the next chapter. Clearly, the model (2.26) derives from the strain energy potential $\overline{W}(X^1,\psi',\lambda_n,\lambda_l)$ defined by

$$\overline{W}(X^1,\psi',\lambda_n,\lambda_l) := \frac{1}{2}\ EI(X^1)\ \psi'^2 + \frac{1}{2}\ G\Omega(X^1)\ \lambda_l{}^2 + \frac{1}{2}\ E\Omega(X^1)\ [\lambda_n - 1]^2 \tag{2.27}$$

by means of the general relations

$$M = \frac{\partial \overline{W}}{\partial \psi'} , \quad V = \frac{\partial \overline{W}}{\partial \lambda_t} , \quad N = \frac{\partial \overline{W}}{\partial \lambda_n} \tag{2.28}$$

The equilibrium equation (2.8) together with the constitutive equations (2.26), comprise the complete system of equations governing the response of the beam under the kinematic assumption (2.1)-(2.2). They can be reduced to a system of semi-linear ordinary differential equations † involving ψ, v and u. However, explicit solutions in terms of quadratures are not possible in general. Nevertheless, these equations can be easily treated by a numerical procedure such as the finite element method. All that is needed is a standard weak formulation of the equations and subsequent discretization by means of interpolation functions as shown in AppendixIII. Numerical examples employing our formulation can be found in [19].

In applications, large axial displacements are seldom encountered. Accordingly, it is often assumed that the axial displacement is "infinitesimal" so that $1 + u' \approx 1$. Approximations of this type will be more systematically treated in section 2.4. The limiting case in which the line of centroids is inextensible is, therefore, relevant to practical applications and is considered next. Remarkably enough, the consistent linearization of this problem yields the equations of a theory widely used in the treatment of multilayer elastomeric bearings.

2.2.4.- The Elastica with Shear Deformation.

The stretching of the line of centroids of the beam is given by the norm of the vector field $\boldsymbol{\lambda}$ defined in (2.23). Hence, the inextensibility condition $||\boldsymbol{\lambda}|| = 1$ of the line of centroids implies

$$[(1+u') \;\; v'] \equiv [\cos\alpha \;\; \sin\alpha] \tag{2.29}$$

where, as in Chap.1, α represents the angle the deformed line of centroids forms with the x_1-axis. The constitutive equation for the shear force takes then the simple form

$$V = G\Omega(X^1) \sin[\alpha(X^1) - \psi(X^1)] \tag{2.30}$$

The substitution of (2.30) and the constitutive equation for the bending moment into the

† It is noted that for semi-linear equations the classical energy criterion can be rigorously proved; i.e: stable equilibrium configurations correspond to a minimum of the total potential energy [12].

equilibrium equations (2.8) yields, after use is made of the inextensibility condition (2.29), the system of non-linear ordinary differential equations

$$[EI(X^1)\ \psi']' + P\sin\alpha = \cos\alpha \left[H + \int_{\xi=0}^{X^1} q(\xi)\ d\xi\right]$$

$$\sin(\alpha-\psi) = \frac{P}{G\Omega}\sin\psi - \frac{\cos\psi}{G\Omega}\left[H + \int_{\xi=0}^{X^1} q(\xi)\ d\xi\right] \qquad X^1\in(0,L) \tag{2.31}$$

$$v' = \sin\alpha \qquad u' = \cos\alpha - 1$$

where it has been assumed for simplicity that the applied distributed (horizontal) force $p(X^1) \equiv 0$. It is noted that the axial force N should be regarded as a reaction to be determined by the corresponding equilibrium equation.

The linearization of the equations (2.31) about the reference configuration $\psi \equiv \alpha \equiv 0$, gives a linearized deformation characterized by $\bar{\psi}$, $\bar{\alpha}$ which satisfy

$$[EI(X^1)\ \bar{\psi}']' + P\,\bar{\alpha} = H + \int_{\xi=0}^{X^1} q(\xi)\ d\xi$$

$$\bar{\alpha} = \left[1 + \frac{P}{G\Omega}\right]\bar{\psi} - \frac{1}{G\Omega}\left[H + \int_{\xi=0}^{X^1} q(\xi)\ d\xi\right] \qquad X^1\in(0,L) \tag{2.32}$$

$$\bar{v}' = \bar{\alpha} \qquad \bar{u}' = 0$$

It follows immediately from these equations, that the linear part $\bar{\psi}$ of ψ is given by the linear operator L defined by

$$L\,\bar{\psi} \equiv \frac{[EI(X^1)\bar{\psi}']'}{1 + \dfrac{P}{G\Omega}} + P\,\bar{\psi} = H + \int_{\xi=0}^{X^1} q(\xi)\ d\xi \ , \quad X^1\in(0,L) \tag{2.33}$$

When proper boundary conditions are appended, the values of P for which the problem $L\,\bar{\psi} = H$ admits non-trivial solutions, provides the critical axial loads for which bifurcation occurs. See Appendix I. It is noted that equation (2.33) differs from the classical Euler's bifurcation equation (See example 1.3.1, Chap.1) by the fact that the bending stiffness $EI(X^1)$ of the beam experiences a reduction by a factor $\dfrac{1}{1 + \dfrac{P}{G\Omega}}$ due to the shear deformation effect. Such a result appears to be first derived by Haringx [21,22,23], although in the limited context of a linearized approach. No reference in this work is made to the fully non-linear equations (2.32).

In applications, a variational formulation of the linearized problem is often more convenient; i.e: the so called energy method [24,35]. By multiplying the first of equations (2.32) by $\bar{\psi}'$, the second by $[\bar{v}'-\bar{\psi}]$ adding and integrating by parts, it immediately follows that

$$P = \frac{\frac{1}{2}\int_0^L EI(\xi)[\bar{\psi}'(\xi)]^2 d\xi + \frac{1}{2}\int_0^L G\Omega(\xi)[\bar{v}'(\xi)-\bar{\psi}(\xi)]^2 d\xi + \textit{Boundary Terms}}{\frac{1}{2}\int_0^L [2\bar{v}'(\xi) - \bar{\psi}(\xi)]\psi(\xi)\,d\xi} \tag{2.34}$$

Clearly, (2.34) is simply the Rayleigh quotient for the linear problem (2.31); in physical terms, the ratio of the total potential energy to the axial end displacement of the rod. The minimization of (2.34) over the class of functions with "finite energy" † yields the critical loads.

The critical values of the axial load P can be related to the classical Euler's bifurcation load P_E by the expression (Appendix I)

$$P_{crit} = \frac{2P_E}{1 + \sqrt{1 + \frac{4P_E}{G\Omega}}} \tag{2.35}$$

It is interesting to examine the asymptotic behavior of the critical load, as given by (2.35), in the limiting cases when the shear stiffness tends either to zero or infinity. From (3.35) it is easily found that

$$P_{crit} \approx P_E \sqrt{\frac{G\Omega}{P_E}} \to 0\,, \quad as \quad \frac{G\Omega}{P_E} \to 0 \tag{2.36}$$

while

$$P_{crit} \approx P_E \left[1 - \frac{P_E}{G\Omega}\right] \to P_E\,, \quad as \quad \frac{G\Omega}{P_E} \to \infty \tag{2.37}$$

It can be shown (see Appendix I) that the critical loads obtained from the linearized problem (2.33) correspond to values of the axial load for which bifurcation from the trivial equilibrium configuration actually occurs. In applications, the question often arises regarding the

† The classical example of Sobolev space [36,44,45].

stability of these adjacent equilibrium configurations. It is shown in Appendix I, using perturbation methods, that these adjacent equilibrium positions are locally stable. In fact, they behave as parabolas in a neighborhood of the trivial equilibrium configuration.

It should be emphasised that the stability at bifurcation points can not be studied with only the information provided by the linearized problem. The knowledge of the equations governing the fully non-linear problem is necessary for this purpose. In contrast with previous formulations [21,22,23], equation (2.33) arises naturally as the consistent linearization of the non-linear problem (2.31), making possible the analysis of Appendix I.

2.3.- The Sandwich Beam.

It is a well known result in classical linear elasticity that due to shear deformation a warping of the cross section of the beam always occurs. One might then ask what effect the warping of the cross section has, if any, on the overall buckling load of the beam when shear deformation is taken into account. Clearly, the answer to this question involves the replacement of the "cross section remain plane" assumption by an alternative deformation pattern. This subject will be discussed at length in the forthcoming chapters.

Nevertheless, it is possible for composite beams of the sandwich type, widely used in certain applications requiring lightweight members, to extend the kinematic assumption (2.1)-(2.2) as to approximately account for the warping effect of the cross section. For simplicity, the case of a rectangular cross section is assumed throughout and attention is confined to the case in which the beam is composed of three layers: a "soft" core of thickness h and top and bottom flanges each with thickness $\frac{\epsilon}{2}$. For this type of beams, most of the axial load is carried by the top and bottom layers while the core is mainly responsible for resisting the shear force. It is then reasonable to assume a piece-wise linear deformation pattern of the cross section, as illustrated in Fig.2.2. This assumption turns out to be particularly accurate [25].

Remarkably enough, the inclusion of the described warping pattern has a significant effect

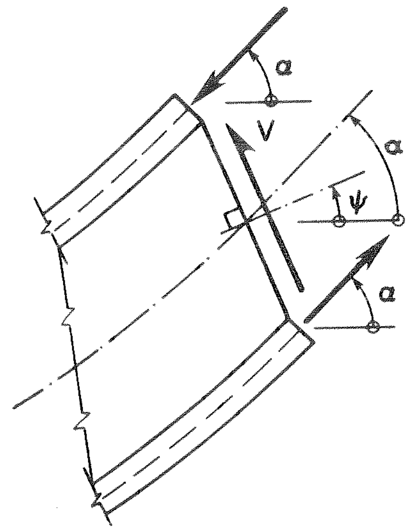


Figure 2.2. The kinematics of a sandwich beam.

on the overall buckling load, particularly important in the case of low shear stiffness. As opposed to previous analyses [25], the formulation presented is not restricted to a small angle approximation and should, therefore, be regarded as a fully non-linear theory.

2.3.1.- The Kinematic Assumption.

Under the assumption of a piece-wise deformation pattern for the cross section of the beam, the deformation map $\mathbf{x} = \Phi(\mathbf{X})$ takes the form

$$
\begin{aligned}
\mathbf{x} &= \mathbf{x}_O + \frac{h}{2}\begin{Bmatrix}-\sin\psi(X^1)\\ \cos\psi(X^1)\end{Bmatrix} + (X^2 - \frac{h}{2})\begin{Bmatrix}-\sin\alpha(X^1)\\ \cos\alpha(X^1)\end{Bmatrix} && \text{if } \frac{h}{2}\leqslant X^2 \leqslant \frac{h}{2}+\frac{\epsilon}{2}\\
\mathbf{x} &= \mathbf{x}_O + X^2\begin{Bmatrix}-\sin\psi(X^1)\\ \cos\psi(X^1)\end{Bmatrix} && \text{if } 0\leqslant |X^2| \leqslant \frac{h}{2} \qquad (2.38)\\
\mathbf{x} &= \mathbf{x}_O - \frac{h}{2}\begin{Bmatrix}-\sin\psi(X^1)\\ \cos\psi(X^1)\end{Bmatrix} + (X^2 + \frac{h}{2})\begin{Bmatrix}-\sin\alpha(X^1)\\ \cos\alpha(X^1)\end{Bmatrix} && \text{if } -\frac{h}{2}-\frac{\epsilon}{2}\leqslant X^2 \leqslant -\frac{h}{2}
\end{aligned}
$$

where, as in the previous section, α denotes the angle the deformed line of centroids forms with the x_1 axis, and $\mathbf{x}_o$ is given by (2.2). Introducing the Heaviside step function $H(x-\xi)$ †

† A locally integrable function defined by $H(x-\xi) = \begin{cases}1 & \text{if } x>\xi\\ 0 & \text{if } x<\xi\end{cases}$

the kinematic assumption (2.38a)-(2.38b) can be recast in the following more convenient form

$$\mathbf{x} = \mathbf{x}_O + X^2 \begin{Bmatrix} -\sin\psi \\ \cos\psi \end{Bmatrix}$$
$$+ \left\{ (X^2-\frac{h}{2})H(X^2-\frac{h}{2})+(X^2+\frac{h}{2})[1-H(X^2+\frac{h}{2})] \right\} \left[\begin{Bmatrix} -\sin\alpha \\ \cos\alpha \end{Bmatrix} - \begin{Bmatrix} -\sin\psi \\ \cos\psi \end{Bmatrix} \right] \quad (2.39)$$

It is noted that (3.39) differs from (2.1) only by the term containing the Heaviside functions, which accounts for the warping of the cross section.

From the Kinematics (2.39) the components of the deformation gradient and the normal and tangential stress fields $\hat{\mathbf{n}}$ and $\hat{\mathbf{l}}$, can be immediately computed from (1.21)-(1.23). The matrix Λ containing the components of $\hat{\mathbf{n}}$ and $\hat{\mathbf{l}}$ takes, according to (1.25), the form

$$\Lambda^T(X^1,X^2) = [H(X^2+\frac{h}{2})-H(X^2-\frac{h}{2})] \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix}$$
$$+ [1-H(X^2+\frac{h}{2})+H(X^2-\frac{h}{2})] \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \quad (2.40)$$

Introducing the definitions

$$N_{core} := a \int_{0 \leqslant |\xi| \leqslant \frac{h}{2}} \sigma(\xi)\, d\xi \qquad V_{core} := a \int_{0 \leqslant |\xi| \leqslant \frac{h}{2}} \tau(\xi)\, d\xi \quad (2.41a)$$

and

$$N_{flange} := a \int_{\frac{h}{2} \leqslant |\xi| \leqslant \frac{h}{2}+\frac{\epsilon}{2}} \sigma(\xi)\, d\xi \quad V_{flange} := a \int_{\frac{h}{2} \leqslant |\xi| \leqslant \frac{h}{2}+\frac{\epsilon}{2}} \tau(\xi)\, d\xi \quad (2.41b)$$

where a is the width of the cross section, and making use of (2.40), the equilibrium equations (1.28a) can be written as

$$\begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \begin{Bmatrix} N_{core} \\ V_{core} \end{Bmatrix} + \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} N_{flange} \\ V_{flange} \end{Bmatrix} = \begin{Bmatrix} P \\ H \end{Bmatrix} - \int_{\xi=0}^{X^1} \begin{Bmatrix} p(\xi) \\ q(\xi) \end{Bmatrix} d\xi \quad (2.42)$$

Equation (2.42) together with the bending moment equilibrium equation (2.8b) comprise the complete system of equilibrium equations for the problem at hand. It has been mentioned earlier, that the case in which the thickness ratio $\frac{\epsilon}{h} << 1$ and the core is composed by a 'soft' material is the most frequently encountered in applications [25]. Accordingly, it is often assumed that $N_{core} \approx 0$ and $V_{flange} \approx 0$, so that the total axial and shear forces N and V are defined as

$$N = N_{flange} \qquad V = V_{core} \tag{2.43}$$

In this event, the equilibrium equation (2.42) takes the following simple form

$$\begin{bmatrix} \cos\alpha & -\sin\psi \\ \sin\alpha & \cos\psi \end{bmatrix} \begin{Bmatrix} N \\ V \end{Bmatrix} \equiv \begin{Bmatrix} P \\ H \end{Bmatrix} - \int_{\xi=0}^{X^1} \begin{Bmatrix} p(\xi) \\ q(\xi) \end{Bmatrix} d\xi \tag{2.44}$$

In what follows, attention is restricted to this simplified case. The limiting situation in which the line of centroids is regarded as inextensible is examined next.

2.3.2.- Inextensible Sandwich Beam.

Since the inextensibility condition and constitutive equation for the shear force are given by (2.29) and (2.30), the system of ordinary differential equations governing the problem differs from (2.31) only in the equilibrium equation for the shear force, now given by (2.44). Explicitly

$$\begin{aligned} &[EI(X^1)\psi']' + P\sin\alpha = \cos\alpha \left[H + \int_{\xi=0}^{X^1} q(\xi)\, d\xi \right] \\ &\sin(\alpha-\psi) = \frac{P}{G\Omega}\sin\alpha - \frac{1}{G\Omega}\left[H + \int_{\xi=0}^{X^1} q(\xi)\, d\xi \right] , \quad X^1 \in (0,L) \\ &u' = \cos\alpha - 1 \qquad v' = \sin\alpha \end{aligned} \tag{2.45}$$

where Ω designates now the area of the core. The consistent linearization of equations (3.45) yields a linear problem

$$\begin{aligned} &[EI(X^1)\, \bar{\psi}']' + P\bar{\alpha} = H + \int_{\xi=0}^{X^1} q(\xi)\, d\xi \\ &\bar{\psi} = \left[1 - \frac{P}{G\Omega}\right]\bar{\alpha} + \frac{1}{G\Omega}\left[H + \int_{\xi=0}^{X^1} q(\xi)\, d\xi \right] , \quad X^1 \in (0,L) \\ &\bar{u}' = 0 \qquad \bar{v}' = \alpha \end{aligned} \tag{2.46}$$

from which easily follows that the linear part $\bar{\psi}$ of ψ is given by

$$L\,\bar{\psi} \equiv \left[1 - \frac{P}{G\Omega}\right][EI(X^1)\, \bar{\psi}']' + P\bar{\psi} = H + \int_{\xi=0}^{X^1} q(\xi)\, d\xi \qquad X^1 \in (0,L) \tag{2.47}$$

It is noted that due to the effect of shear deformation, the bending stiffness appears in (2.47) reduced by a factor of $1 - \frac{P}{G\Omega}$, in contrast with that of $\frac{1}{1+\frac{P}{G\Omega}}$ found for the homogeneous beam. In fact, the critical loads obtained from the associated eigenvalue problem are now related to Euler's buckling load P_E according to (see Appendix I)

$$\frac{1}{P_{crit}} = \frac{1}{G\Omega} + \frac{1}{P_E} \tag{2.48}$$

The asymptotic behavior of P_{crit} is then quite different from that found for the critical load in the case of a homogeneous beam. Even though one has

$$P_{crit} \approx P_E \left[1 - \frac{P_E}{G\Omega}\right] \rightarrow P_E \qquad as \quad G\Omega \rightarrow \infty \tag{2.49}$$

as in (2.37), from (2.48) it follows that

$$P_{crit} \approx G\Omega \rightarrow 0 \qquad as \quad G\Omega \rightarrow 0 \tag{2.50}$$

which for low shear stiffness exhibits a linear behavior of the critical load as a function of $G\Omega$, quite different from that corresponding to a homogeneous beam predicted by (2.36). For low values of shear stiffness, The later predicts values of the buckling load substantially greater than those given by (2.48). It is noted that (2.48) is, in any case, bounded by $G\Omega$. Clearly, the distinction between both theories becomes irrelevant for high values of $G\Omega$. For an homogeneous isotropic material for example, $\frac{G\Omega}{P_E} \approx O(\frac{L^2}{h^2})$ and, except for extremely short beams, both (2.35) and (2.48) coincide for all practical purposes.

Since the fully non-linear problem (2.45) is at our disposal, It is possible to study the stability of the adjacent equilibrium configurations at the critical values given by (2.48). Using perturbation methods it is shown in Appendix I that such adjacent equilibrium positions are in fact locally stable equilibrium configurations.

It is finally noted, that the linearized eigenvalue problem (2.46) can be recast, as in the previous section, in a variational form. The Rayleigh quotient associated with (2.46) takes the form

$$P = \frac{\frac{1}{2}\int_0^L EI(\xi)[\bar{\psi}'(\xi)]^2 d\xi + \frac{1}{2}\int_0^L G\Omega(\xi)[\bar{v}'(\xi)-\bar{\psi}(\xi)]^2 d\xi + Boundary\ Terms}{\frac{1}{2}\int_0^L [\bar{v}'(\xi)]^2 d\xi} \tag{2.51}$$

Equation (2.51) differs from its counterpart (2.37) valid for a homogeneous beam, in the different expression for "external work" done by the applied end load P, now given as $\frac{P}{2}\int_0^L \bar{v}'dx$. Remarkably enough, the same expression for the external work is found for Euler's elastica, the difference in Rayleigh quotients being the appearance of the shear strain energy term in (2.51). This similarity might lead to the erroneous conclusion that the linearized equations (2.46) furnish the appropriate extension of Euler's elastica for the case in which shear deformation is taken into account. This has been in fact the case in some alternative derivations [24].

The source of confusion appears to be the angle the resultant axial force forms with the X^1-axis. The equilibrium equation (2.44) shows that a sandwich beam may be viewed as a homogeneous beam in which the axial force is no longer normal to the deformed cross section, but forms an angle $\bar{\alpha}$ with the X^1-axis. Such an assumption leads to the linearized equations (2.45). The derivation presented shows that this conclusion stems from the assumed warping pattern leading to the kinematics (2.39).

2.4.- Consistent Approximate Theories.

The consistent formulation of n-order approximate theories relies on the observation that the classical Taylor's formula from elementary calculus can be immediately extented to the general setting of functional (Banach) spaces. All that is needed for practical purposes are the two following facts [12,36]

Let X and Y be Banach spaces †, S $\subset$ X an open set, and $f : S \subset X \rightarrow$ Y a given (non-

† By a slight generalization, the same conclusions apply to Banach manifolds [37]. This is in fact the case of interest for general three dimensional non-linear elasticity [12].

linear) map. Then

(i) If f is Fréchet differentiable on S then f is continuous on S, Gateaux differentiable, and the Gateaux differential coincides with the Fréchet differential. Conversely, Gateaux differentiability and continuity imply Fréchet differentiability. Consequently, the Fréchet differential may be computed by the Gateaux differential formula

$$\delta f(\mathbf{x}_O;\bar{\mathbf{x}}) = \frac{d}{d\alpha}[f(\mathbf{x}_O+\alpha\bar{\mathbf{x}})]\big|_{\alpha=0} \tag{2.52}$$

Higher differentials are computed in the same manner; for example

$$\delta^2 f(\mathbf{x}_O,\bar{\mathbf{x}};\hat{\mathbf{x}}) = \frac{d}{d\alpha}[\delta f(\mathbf{x}_O+\hat{\mathbf{x}},\bar{\mathbf{x}})]\big|_{\alpha=0} \tag{2.53}$$

(ii) Any smooth (C^k) map may be approximated at $\mathbf{x}_O+\bar{\mathbf{x}}\in S$ by Taylor's formula; i.e:

$$f(\mathbf{x}_O + \bar{\mathbf{x}}) = f(\mathbf{x}_O) + \sum_{n=1}^{k-1} \frac{1}{k!}\delta^k f(\mathbf{x}_O;\bar{\mathbf{x}}^k) + \omega^k(\mathbf{x}_O,\bar{\mathbf{x}}) \tag{2.54}$$

where

$$\frac{||\omega^k(\mathbf{x}_O,\bar{\mathbf{x}})||_Y}{||\bar{\mathbf{x}}||_X} \to 0 \quad as \quad ||\bar{\mathbf{x}}||_X \to 0$$

One often writes $\delta f(\mathbf{x}_O;\bar{\mathbf{x}}) = Df(\mathbf{x}_O).\bar{\mathbf{x}}$, where the linear map $Df(\mathbf{x}_O) : X \to Y$ is the Frechet derivative at $\mathbf{x}_O \in S$. $Lf := f(\mathbf{x}_O) + Df(\mathbf{x}_O).\bar{\mathbf{x}}$ is then referred to as the linear part of f at $\mathbf{x}_O \in S$.

These two facts can be effectively exploited in mechanics of solids [38] to derive consistent approximations to within arbitrary order to the field equations †. If the expansion (2.54) is carried out about the stress free reference configuration, one is led to a sequence of approximate problems often referred to as the method of successive approximations [11].

It is shown below, that consistent approximate beam theories, of interest in engineering applications, can be obtained in a simple manner from the non-linear field equations by a

† Typically, f is taken to be a vector field over the configuration $\Phi = \Phi_O$, a stress tensor or a strain measure of interest. X is taken as the product space of the configuration manifold and an appropriate tangent bundle and $\hat{\mathbf{x}}$, the super-imposed infinitesimal deformation, a vector field over Φ. See [12] for a complete account of a covariant formulation not restricted to linear spaces.

straightforward application of (i) and (ii). Furthermore, the consistent second order approximation to the equilibrium equations requires only the knowledge of the kinematics of the linear theory. This fact plays a key role in the developments of the forthcoming chapters.

Second Order Approximate Theories.

Recall that, under the kinematic assumption (2.1)-(2.2), the equilibrium equations (2.8a) and (2.10) together with the constitutive equations (2.26) comprise the field equations governing the response of a homogeneous rod. Making use of (2.52) and (2.53) the strain measures defined by (2.24) takes, up to second order, the form

$$\begin{aligned} J_O &= 1 + \bar{u}' + \frac{1}{2}[2\bar{v}' - \bar{\psi}] + \cdots \\ E_{12} &= 0 + [\bar{v}' - \bar{\psi}] + \frac{1}{2}[-2\bar{u}'\,\bar{\psi}] + \cdots \\ E_{11} &= 0 + [\bar{u}' - X^2\bar{\psi}'] + \frac{1}{2}[\bar{u}'^2 + \bar{v}'^2 - 2X^2\bar{u}'\,\bar{\psi}'] + \cdots \end{aligned} \tag{2.55}$$

If the values of the variables at a configuration Φ_o, characterized by (u_o, v_o, ψ_o) are designated by subscript "o", the linearization of the equilibrium equations (2.8a) and (2.10) about the configuration Φ_o, take the form

$$\begin{aligned} &\left[\frac{dM}{d\psi'}\Big|_o \bar{\psi}'\right]' + J_O\big|_o \frac{dV}{dE_{12}}\Big|_o(\bar{v}'-\bar{\psi}) + V\big|_o\,\bar{u}' + E_{12}\big|_o \frac{dN}{dJ_O}\Big|_o + N\big|_o\,(\bar{v}'-\bar{\psi}) \\ &\qquad + \left\{M' + J_O V + E_{12}N\right\}\Big|_o = 0 \end{aligned} \tag{2.56}$$

$$\frac{d}{dX^1}\left(\Lambda\big|_o \begin{bmatrix} \dfrac{dN}{dJ_O}\Big|_o\,\bar{u}' \\ \dfrac{dV}{dE_{12}}\Big|_o\,(\bar{v}'-\bar{\psi}) \end{bmatrix} + \bar{\psi}\,\frac{d\Lambda}{d\psi}\Big|_o \begin{Bmatrix} N|_o \\ V|_o \end{Bmatrix}\right) + \left\{\frac{d}{dX^1}\,\Lambda\big|_o \begin{Bmatrix} N|_o \\ V|_o \end{Bmatrix} + \begin{Bmatrix} p \\ q \end{Bmatrix}\right\} = 0$$

The terms within braces in (2.56), the so-called residual, give the out-of-balance forces at the configuration Φ_o and appear naturally when the non-linear problem is solved by an iterative solution procedure [32,38]. If Φ_o is chosen to be the reference stress free configuration, then $N|_o = V|_o = M|_o = E_{12}|_o \equiv 0$, and $J_O|_o \equiv 1$. Therefore, since $\Lambda|_o \equiv 1$, the first approximation to the field equations takes the form

$$
\begin{aligned}
M' + V &= 0 \qquad & M &= EI\bar{\psi}' \\
V' + q &= 0 \qquad & V &= G\Omega[\bar{v}' - \psi'] \\
N' + p &= 0 \qquad & N &= E\Omega\bar{u}'
\end{aligned}
\tag{2.57}
$$

Thus, the first order approximation yield the well known equations of Timoshenko's beam theory. Equations (2.57) could have been derived directly from the three dimensional equations of linear elasticity [39] without resorting to the non-linear theory and subsequent consistent linearization process.

Applying again the Gateaux differential formula (2.52) to (2.56), making use of (2.55) and substituting in Taylor's expansion formula (2.53); one arrives at the second order approximation expressed as

$$
\begin{aligned}
M' + [1+\bar{u}']\, V - [\bar{v}' - \bar{\psi}]\, N &= 0 \qquad & M &= EI\,\bar{\psi}' \\
V' + [\bar{\psi}\, N]' + q &= 0 \qquad & V &= G\Omega\ [\bar{v}' - (1+\bar{u}')\bar{\psi}] \\
N' - [\bar{\psi}\, V]' + p &= 0 \qquad & N &= E\Omega\ [\bar{u}' + \frac{1}{2}(2\bar{v}'-\bar{\psi})\bar{\psi}]
\end{aligned}
\tag{2.58}
$$

Higher order approximations may be systematically obtained in the same manner by computing successive terms in the expansion (2.53).

It is noted that once the equilibrium equations are established to within the desired order of approximation a duality argument, entirely analogous to that of 2.2.3, yields the consistent conjugate measures of deformation. Thus, appropriate constitutive equations can be immediately stated by this procedure. In the case of a sandwich beam, for example, the bending moment equilibrium equation (2.8b) may be written with the aid of (2.44) as

$$
M' + J_O V + [v'\cos\alpha - (1+u')\sin\alpha]N = 0 \tag{2.59}
$$

and since

$$
\sin(\alpha) = \frac{v'}{\sqrt{(1+u')^2 + v'^2}} \quad \textit{so that} \quad \bar{\alpha} = \bar{v}'
$$

by (2.52), a computation analogous to that leading to (2.57) and (2.58) shows that the second order approximation to the equilibrium equations (2.44) and (2.59) is given by

$$M' + [1 + \bar{u}']V = 0$$
$$[V' + \bar{v}'N] + q = 0 \qquad (2.60a)$$
$$[N' - \bar{\psi}V] + p = 0$$

In view of (2.60a), the aforementioned duality argument shows that the constitutive equations consistent with (2.60a) are

$$M = EI(X^1)\bar{\psi}'(X^1)$$
$$V = G\Omega(X^1)[\bar{v}'(X^1) - (1+\bar{u}')\,\bar{\psi}(X^1)] \qquad (2.60b)$$
$$N = E\Omega(X^1)[\bar{u}' + \frac{1}{2}\bar{v}'^2 + \frac{1}{2}\bar{u}'^2]$$

where Ω stands for the area of the core. The physical difference between the axial measures of deformation appearing in (2.58) and (2.60b) is illustrated in Fig.2.3.

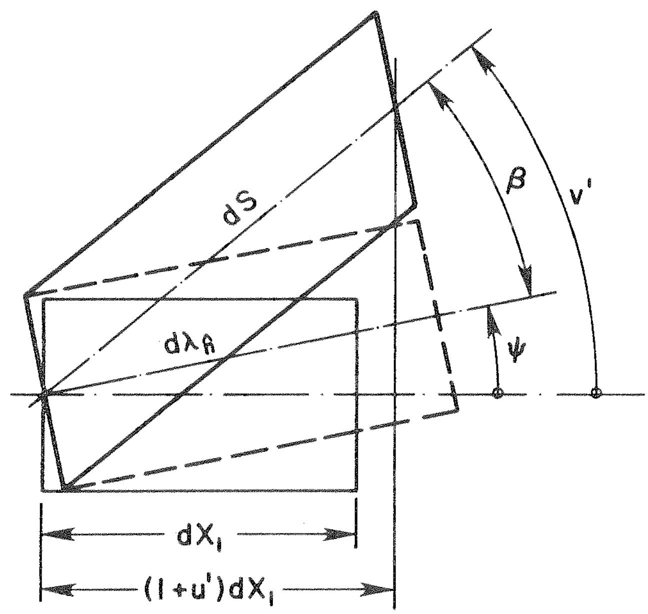


Figure 2.3. Illustration of the axial measures of deformation.

The approximations (2.57) and (2.58) or (2.60) illustrate the following facts

(a) The second order approximation to the equilibrium equations involves only linear measures of deformation; i.e: $\boldsymbol{\epsilon}$, LJ_O. Accordingly, they can be exactly established from the knowledge of the linearized kinematics. In the case of the kinematic assumption (2.1)-

(2.2), the linear part is given by

$$x^1 = X^1 + \bar{u}(X^1) - X^2\bar{\psi}(X^1) \qquad x^2 = X^2 + v(X^1)$$

and the systematic application of the equations of linear elasticity yield then (2.57) while those of non-linear elasticity lead to (2.58). This remark equally applies to the general case of three dimensional elasticity.

(b) Equations (2.58) and (2.60) have been derived by a consistent linearization process based upon Taylor's formula, without resorting to a selective truncation of non-linear terms, a procedure often invoked. In engineering applications, however, the axial displacements are usually small in comparison with actual rotations and lateral displacements. Thus, the additional assumption

$$||u'(x)|| := \max_{0 \leqslant x \leqslant L} \{|u'(x)|\} = O(||\psi||^2, ||v'||^2) \tag{2.61}$$

is quite realistic in most situations, and leads to the simplification $1 + \bar{u}' \approx 1$, $\bar{u}'\bar{\psi} \approx 0$ in equations (2.58) and (2.60). In situations for which the additional assumption (2.61) is not applicable, equations (2.58) or (2.60) can always be employed, although a numerical treatment may be necessary.

2.4.2.- Remarks on Approximate two-dimensional Constitutive Models.

It has been pointed out in section 1.5 that for computational purposes, particularly in the context of the finite element method [31,32,33], as well as for the derivation of approximate theories [18], the ad-hoc constitutive model

$$\mathbf{S} = \lambda^{\#}\ trace(\mathbf{E}) + 2G^{\#}\ \mathbf{E} \tag{2.62}$$

is sometimes employed, often in conjunction with the kinematic assumption (2.1)-(2.2) [31,33]. In this section, it will be examined to what extent convenient models of this type can be used in the context of the approximate theories discussed in the previous section.

First, recall that according to (1.31) and (2.17) the resultant forces N, V, and M can be expressed in terms of the components of the stress tensor S as follows

$$N = \int_\Omega \sigma \, d\Omega = \int_\Omega J \, S^{11} \, d\Omega$$
$$V = \int_\Omega \tau \, d\Omega = \int_\Omega S^{12} \, d\Omega + E_{12} \int_\Omega S^{11} \, d\Omega \tag{2.63}$$
$$M = -\hat{\mathbf{n}} . \int_\Omega \begin{Bmatrix} P^{11} \\ P^{21} \end{Bmatrix} X^2 \, d\Omega = -\int_\Omega J X^2 \, S^{11} \, d\Omega$$

Next, attention is confined to a second order approximation. Since the linear part of $\dfrac{E_{12}}{J}$ is , according to (2.52), given by $L\left[\dfrac{E_{12}}{J}\right] = \epsilon_{12}$ the second order approximation to $(2.63)_2$ takes the form

$$V = \int_\Omega S^{12} \, d\Omega + \epsilon_{12} \int_\Omega \sigma \, d\Omega \tag{2.64}$$

Let us consider again the two cases of the homogeneous and the sandwich beam in the situation for which assumption (2.61) holds so that it may be assume $1 + \bar{u}' \approx 1$ and $\bar{u}' \bar{\psi} \approx 0$.

(a) Homogeneous Beam: Since $\epsilon_{12} = \bar{v}' - \bar{\psi}$, equation (2.64) takes the form

$$V = \int_\Omega S^{12} \, d\Omega + [\bar{v}' - \bar{\psi}] \, N \tag{2.65}$$

Equation (2.65) shows, for example, that the formal definition $V = \int_\Omega S^{12} \, d\Omega$ is meaningless unless $N = 0$ (no axial load) or $\epsilon_{12} = 0$ (negligible shear deformation). Furthermore, when the constitutive model (2.62) is assumed $S^{12} = G^{\#} E_{12}$ and if P designates the applied axial (compressive) load applied at the ends of the beam, (2.65) is estimated to within second order by

$$V = \left[G^{\#} - \frac{P}{\Omega}\right] \Omega \, [\bar{v}' - \bar{\psi}] \tag{2.66}$$

Therefore, since $V = G \, \Omega \, [\bar{v}' - \bar{\psi}]$, it follows from (2.66) that the shear modulus $G^{\#}$ in the constitutive model (2.62) is given, in the presence of axial load and to within a second order approximation, by

$$G^{\#} = G + \frac{P}{\Omega} \tag{2.67}$$

(b) Sandwich Beam: According to the kinematics given by (2.39), ϵ_{12} has the expression

$$\epsilon_{12} = [\bar{v}' - \bar{\psi}][H(X^2 + \frac{h}{2}) - H(X^2 - \frac{h}{2})] \tag{2.68}$$

and the substitution of (2.68) into (2.64) yields

$$V = \int_\Omega S^{12}\, d\Omega + [\bar{v}' - \bar{\psi}] N_{core} \tag{2.69}$$

For the common case encountered in practice in which assumption (2.43) holds, $N_{core} \approx 0$ and (2.69) yields

$$V = \int_\Omega S^{12}\, d\Omega \qquad \textit{so that} \qquad G^{\#} = G \tag{2.70}$$

Thus, one arrives to the somewhat surprising result that for a sandwich beam the average of the stress component S^{12} over the cross section gives the total shear force and, furthermore, $G^{\#} \equiv G$ in the constitutive model (2.62). This result may be interpreted physically as follows.

Physical Interpretation: The equilibrium equation (2.44) shows that a sandwich beam may be viewed as an homogeneous beam in which the resultant axial force is no longer normal to the cross section, but forms an angle $\alpha \approx v'$ with the X^1-axis. See Fig.2.4. When this approach is taken, instead of the definitions (2.7) for N and V in terms of σ and τ, by equilibrium considerations one has

$$\int_\Omega \sigma\, d\Omega = N\cos(\alpha - \psi) \approx N$$
$$\int_\Omega \tau\, d\Omega = N\sin(\alpha - \psi) + V \approx N[\bar{v}' - \bar{\psi}] + V$$

However, the relationship (1.36) always holds and, therefore, so does its second order approximation; i.e:

$$\int_\Omega \tau\, d\Omega = \int_\Omega S^{12}\, d\Omega + \epsilon_{12} \int_\Omega \sigma\, d\Omega$$

Clearly, from these equations (2.70) immediately follows.

2.4.3.- A Numerical Assessment.

The two-dimensional non-linear finite element developed in reference [32] for the

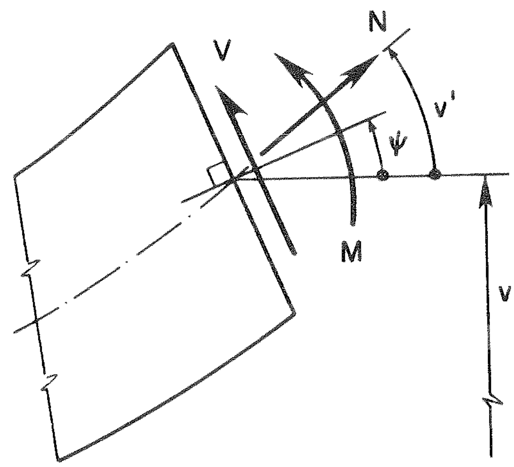


Figure 2.4. Simplified model for the sandwich beam.

general purpose finite element computer program FEAP [46] was used in order to assess numerically the accuracy of the estimates given in (2.67) and (2.69). A slender beam with ratio $\frac{length}{width} = 10$ and left end clamped, was subjected to a constant vertical load and progressively increasing axial load, both applied at its right end. The finite element discretization consisted of five 9-node isoparametric elements of equal length. The values of the elastic constants in the constitutive model (2.62) are shown in Fig.2.5 together with the finite element mesh.

Figure 2.6 shows the computed lateral stiffness as a function of the axial load for the case in which $G^{\#} = G + \frac{P}{\Omega}$ as in (2.67), and for the case $G^{\#} \equiv G$ corresponding to (2.69). The agreement between the computed values and those predicted by the respective theories is excellent.

Material Properties	Model Properties
$E = 10^6$ $G = 500.$ $\nu = 0.25$	$\lambda^{\#} = 26666.$ $\mu^{\#} = 400000.$ $\nu^{\#} = 0.25$

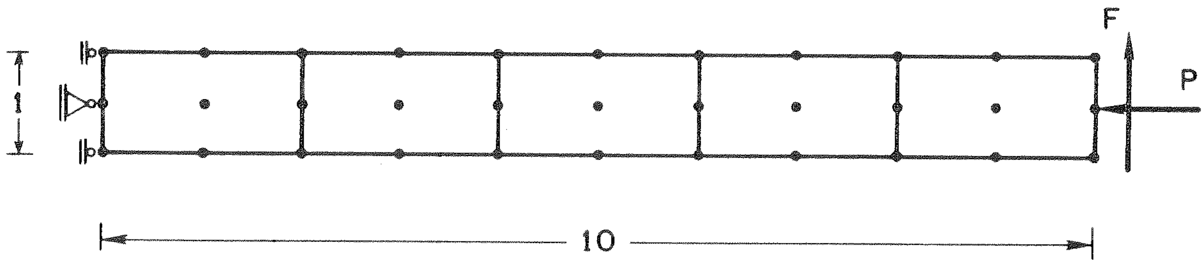


Figure 2.5. Two-dimensional Finite Element mesh and material properties.

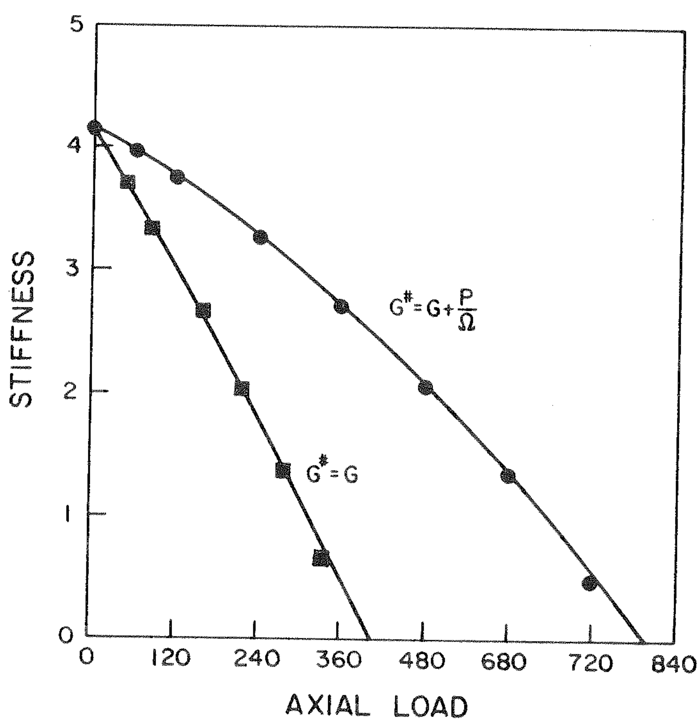


Figure 2.6. Numerical results. Two-dimensional Finite Elasticity Finite Element model.

CHAPTER 3.

CONSISTENT FORMULATION OF A LINEARIZED THEORY OF BUCKLING FOR TRANSVERSALLY HOMOGENEOUS BEAMS

3.1.- Introduction

The formulation presented in this chapter hinges on the following two key facts

(i) The method of successive approximations, based upon Taylor's formula, shows that only linear measures of deformation are involved in an exact second order approximation to the non-linear equilibrium equations. Accordingly, a second order approximation can be exactly established provided an exact solution to the displacement field in the context of the linear theory is available.

(ii) The exact three dimensional solution for the displacement field corresponding to a beam acted upon by end loads (Saint Venant's problem) is known, not only for a linear elastic isotropic material [40,41], but in the more general situation of transversally isotropic solid [42]. Furthermore, the conclusions obtained in the former situation carry over with no essential modification to the latter.

Taking this result as a point of departure, the proposed formulation can be summarized as follows

(1) As a first step, the exact displacement field for a beam acted by end loads is recast exclusively in terms of kinematic variables, which are taken as the average displacement and average rotation of the cross section and are designated by $\bar{u}$ and $\bar{\psi}$ respectively. The

resulting kinematics includes an axial warping of the cross section which is shown to be proportional to the average shear angle of deformation $\bar{\beta}$.

Based upon this exact kinematics, one can derive a set of ordinary differential equations for the variables $\{\bar{u}, \bar{\psi}\}$ which are exact in the sense that their integration yields the exact solution for $\{\bar{u}, \bar{\psi}\}$. Remarkably enough these equations correspond to the well known theory due to Timoshenko and lead to the expression for the shear coefficient first derived by Cowper [39].

(2) Next, the physical description of the body is restricted to the knowledge of the kinematic variables $\{\bar{u}, \bar{\psi}\}$ and attention is focussed on the non-linear theory. A trivial extension of the results presented in Chap.1 allow then the derivation of an exact second order approximation in terms of the variables $\{\bar{u}, \bar{\psi}\}$ to the non-linear equilibrium equations. In this approach the deformed cross section is viewed as a two dimensional surface and the introduction of its Gaussian frame allows the expression of the final equilibrium equations in terms of axial and shear forces defined as stress resultants of normal and tangential stresses.

Once the equilibrium equations are known, consistent measures of deformation dual to the resultant stress measures can be derived for a hyperelastic material. Thus, no assumption is involved in the derivation of the field equations summarized in Table 3.1.

(3) This formulation is recast into an eigenvalue problem for the critical load by introducing the customary assumption of quasi-inextensibility. The result is then a new expression for the critical load which takes into account, to within a second order approximation, the effect of axial warping due to shear deformation. This expression should be regarded as an exact second order approximation to the buckling load corresponding to a straight beam acted on by end loads.

The formulation presented is re-examined in the context of the projection method originally proposed by Kantorovich ([43] and references therein). Its shown that the exact kinematics derived in step (1) corresponds to an optimal choice for the coordinate functions of this method, for the problem at hand. The physical motivation for the choice of the kinematic variables is also examined. It is shown that the plane defined by the average angle $\bar{\psi}$ not only corresponds to the average plane of bending but, furthermore, defines the plane in which the resultant of tangential stresses over the deformed cross section, the shear force, is contained. However, due to the effect of axial warping the resultant of normal stresses no longer remains normal to the plane of bending.

3.2.- The Exact Linearized Kinematics.

The exact solution for the displacement field corresponding to a straight beam with axial axis x_1, cross section $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$, symmetry plane $x_1\!-\!x_2$ and acted upon by end loads contained in $x_1\!-\!x_2$, may be expressed as [40,41]

$$
\begin{aligned}
u_1(\mathbf{x}) &= -\frac{M}{EI_2}x_1x_2 - \frac{V}{EI_2}[\chi(x_2,x_3) + x_2x_3^2 + \frac{1}{2}x_1^2] + \frac{N}{E\Omega}x_1 \\
u_2(\mathbf{x}) &= \frac{M}{2EI_2}x_1^2 + \frac{V}{3EI_2}x_1^3 + \nu\left[\frac{M}{2EI_2}[x_2^2 - x_3^2] - \frac{N}{E\Omega}x_2\right], \quad \mathbf{x}\in(0,L)\times\Omega \qquad (3.1)\\
u_3(\mathbf{x}) &= \nu\left[\frac{M}{EI_2}x_2x_3 - \frac{N}{E\Omega}x_3\right]
\end{aligned}
$$

where N, V and M are the resultant axial and shear force and bending moment contained in the symmetry plane $x_1\!-\!x_2$, $I_2 = \int_\Omega [x_2]^2 d\Omega$ the moment of inertia, and $\chi : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ a harmonic function satisfying the Neumann problem

$$
\begin{aligned}
&\Delta\chi(x_2,x_3) = 0 \ , \quad (x_2,x_3)\in\Omega \\
&\frac{\partial\chi}{\partial n}\Big|_{\partial\Omega} = -n_2\left[\frac{1}{2}\nu x_2^2 + (1-\frac{\nu}{2})x_3^2\right] - n_3(2+\nu)x_2x_3 \qquad (3.2)
\end{aligned}
$$

Next, the displacement field (3.1) is reformulated fully in terms of kinematic variables. These variables will, in turn, be related to the forces N, V and M (generalized stresses) through appropriate constitutive equations. This approach will be referred to as reduction of the dimensionality of the problem † . We select as a set of generalized kinematic variables, the mean

† Since the equilibrium equations, a system of partial differential equations, are reduced to a set of ordinary differential equation, the denomination is justified. A systematic procedure of reduction of dimensionality is

displacement $\bar{\mathbf{u}}(x_1)$ and the average rotation $\bar{\psi}(x_1)$ about the x_3-axis of an arbitrary cross section $\Omega(x_1)$. Alternative sets will be discussed later. Since the axes $\{x_i\}$ are assumed to be principal axis of inertia, $\bar{\mathbf{u}}(x_1)$ and $\bar{\psi}(x_1)$ may be defined by

$$\bar{\mathbf{u}}(x_1) := \frac{1}{\Omega} \int_{\Omega} \mathbf{u}(x_1,x_2,x_3)\, d\Omega \tag{3.3}$$

and

$$\bar{\psi}(x_1) := -\frac{1}{I_2}\int_{\Omega} x_2 u_1(x_1,x_2,x_3)\, d\Omega \tag{3.4}$$

The physical significance of the angle $\bar{\psi}$ is illustrated in Fig.3.1.

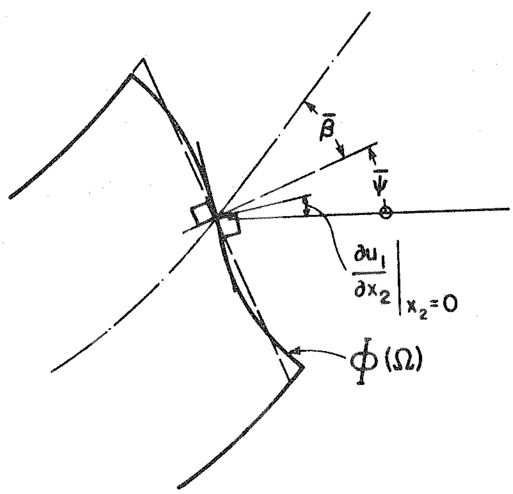


Figure 3.1. The average rotation $\bar{\psi}(x_1)$.

In terms of these kinematic variables, the displacement field (3.1) can be recast in the form

$$\begin{aligned} u_1(\mathbf{x}) &= \bar{u}_1(x_1) - x_2\,\bar{\psi}(x_1) + \hat{u}_1(\mathbf{x}) \\ u_\alpha(\mathbf{x}) &= \bar{u}_\alpha(x_1) + \hat{u}_\alpha(\mathbf{x}) \ , \quad (\alpha=2,3) \end{aligned} \tag{3.5}$$

where the components $\hat{u}_i(\mathbf{x})$,$(i=1,2,3)$ of the vector field $\hat{\mathbf{u}}(\mathbf{x})$, referred to as *residual displacement* in the sequel, satisfy the conditions

$$\int_{\Omega} \hat{u}_i(\mathbf{x})\, d\Omega = \int_{\Omega} x_2\, \hat{u}_1(\mathbf{x})\, d\Omega \equiv 0 \qquad (i=1,2,3) \tag{3.6}$$

differential equation, the denomination is justified. A systematic procedure of reduction of dimensionality is furnished by the Kantorovich method. See section 2.5.

A physical interpretation of the residual displacement $\hat{\mathbf{u}}(\mathbf{x})$ will be given shortly. First, it is noted from (3.5) and the definitions (3.3) and (3.4), that the axial component $\hat{u}_1(\mathbf{x})$ is given by

$$\hat{u}_1(\mathbf{x}) = u_1(\mathbf{x}) - \frac{1}{\Omega}\int_\Omega u_1(\mathbf{x})\,d\Omega - \frac{x_2}{I_2}\int_\Omega x_2\, u_1(\mathbf{x})\,d\Omega$$
$$= -\frac{V}{G\Omega}\,\frac{G\Omega}{EI_2}\left[\chi + x_2x_3^2 - \frac{1}{\Omega}\int_\Omega [\chi+x_2x_3^2]\,d\Omega - \frac{x_2}{I_2}\int_\Omega [\chi+x_2x_3^2]\,x_2\,d\Omega\right]$$

Hence, introducing the function $\phi:\Omega \rightarrow \mathbb{R}$ defined by

$$\phi(x_2,x_3) := \frac{G\Omega}{EI_2}\left[\chi(x_2,x_3) + x_2x_3^2 - \frac{1}{\Omega}\int_\Omega [\chi+x_2x_3^2]\,d\Omega - \frac{x_2}{I_2}\int_\Omega x_2[\chi+x_2x_3^2]\,d\Omega\right] \quad (3.7)$$

which depends solely upon the geometry of the cross section Ω and the Poisson ratio ν, the component $\hat{u}_1(\mathbf{x})$ may be expressed as

$$\hat{u}_1(\mathbf{x}) = -\frac{V}{G\Omega}\,\phi(x_2,x_3) \quad (3.8)$$

Proceeding in the same manner with the components $\hat{u}_2(\mathbf{x})$ and $\hat{u}_3(\mathbf{x})$, we arrive at

$$\hat{u}_2(\mathbf{x}) = \nu\left[\frac{M}{2EI_2}\left(x_2^2 - x_3^2 - \frac{I_2-I_3}{\Omega}\right) - \frac{N}{E\Omega}x_2\right]$$
$$\hat{u}_3(\mathbf{x}) = \nu\left[\frac{M}{EI_2}x_2x_3 - \frac{N}{E\Omega}x_3\right] \quad (3.9)$$

together with the result

$$\bar{u}_3(x_1) \equiv 0 \quad (3.10)$$

To express the residual displacement $\hat{\mathbf{u}}(\mathbf{x})$ in terms of the kinematic variables $\bar{u}(x_1)$ and $\bar{\psi}(x_1)$, appropriate constitutive equations for M , V and N will be first derived.

The exact solution of the problem at hand [40,41] shows that $\sigma_{22} \equiv \sigma_{33} = \sigma_{23} \equiv 0$. Thus, integration of the constitutive equation $\sigma_{11} = E\epsilon_{11} + \nu(\sigma_{22}+\sigma_{33})$ over Ω yields

$$M = -\int_\Omega x_2\sigma_{11}\,d\Omega = EI_2\,\bar{\psi}'(x_1) \quad (3.11a)$$
$$N = \int_\Omega \sigma_{11}\,d\Omega = E\Omega\,\bar{u}_1'(x_1) \quad (3.11b)$$

Similarly, the integration of $\sigma_{12} = G\epsilon_{12}$ over Ω together with condition (3.6) gives

$$V = \int_\Omega \sigma_{12} d\Omega = G \int_\Omega \epsilon_{12} d\Omega$$
$$= G\Omega [\bar{u}_2' - \bar{\psi}] - V \frac{1}{\Omega} \int_\Omega \frac{\partial \phi}{\partial x_2} d\Omega$$

Thus, introducing the average shearing angle

$$\bar{\beta}(x_1) = \bar{u}_2'(x_1) - \bar{\psi}(x_1) \tag{3.12}$$

the constitutive equation for the shear force V takes the final form

$$V = G\Omega \ \kappa \ \bar{\beta}(x_1) \tag{3.13a}$$

where

$$\kappa = \frac{1}{1 + \dfrac{1}{\Omega} \displaystyle\int_\Omega \frac{\partial \phi}{\partial x_2} d\Omega} \tag{3.13b}$$

is the Timoshenko's celebrated shear coefficient. The use of the definition of $\phi(x_2, x_3)$ given by (3.7) together with a well known identity for harmonic functions † and the boundary condition for χ, yields the alternative expression of the shear coefficient

$$\kappa = \frac{2(1+\nu)\ I_2}{\nu \dfrac{I_3 - I_2}{2} - \dfrac{\Omega}{I_2} \displaystyle\int_\Omega x_2 [\chi + x_2 x_3^2] d\Omega} \tag{3.14}$$

first derived by Cowper [39].

It should be noted that for the problem at hand, equations (3.11) through (3.14) are exact. Summarizing our results, the exact kinematics for a straight beam with axial axis x_1 and symmetry plane $x_1 - x_2$ may be expressed in the form

$$\begin{aligned} u_1(\mathrm{x}) &= \bar{u}(x_1) - x_2 \bar{\psi}(x_1) - \phi(x_2, x_3) \ \kappa \ \bar{\beta}(x_1) \\ u_2(\mathrm{x}) &= \bar{v}(x_1) + \nu \ [g(x_2, x_3) \ \bar{\psi}'(x_1) - x_2 \bar{u}'(x_1)] \\ u_3(\mathrm{x}) &= \qquad + \nu \ [x_2 x_3 \bar{\psi}'(x_1) - x_3 \bar{u}'(x_1)] \end{aligned} \tag{3.15}$$

where we have written for short

$$\bar{u}(x_1) \equiv \bar{u}_1(x_1) \ , \qquad \bar{v}(x_1) = \bar{u}_2(x_1) \ , \qquad \bar{\beta}(x_1) = \bar{v}'(x_1) - \bar{\psi}(x_1) \tag{3.16}$$

The function $\phi : \Omega \to \mathbb{R}$ and the shear coefficient κ are given by (3.7) and (3.14) respec-

† $\displaystyle\int_\Omega \frac{\partial \chi(x_2, x_3)}{\partial x_\alpha} d\Omega = \int_{\partial\Omega} x_\alpha \frac{\partial \chi}{\partial n} dS$, $(\alpha = 1,2)$, provided $\Delta\chi = 0$ in Ω.

tively, while $g : \Omega \rightarrow \mathbb{R}$ is defined by

$$g(x_2, x_3) = \frac{1}{2}\left[x_2^2 - x_3^2 - \frac{I_2 - I_3}{\Omega}\right] \tag{3.17}$$

It is of some interest to compare (3.15) with the classical Bernoulli's kinematic assumption. First, (3.15) includes an axial warping of the cross section due to the shear deformation which is proportional to the average shear angle $\overline{\beta}(x_1)$. Second, the lateral displacements $u_2(\mathrm{x})$ and $u_3(\mathrm{x})$ in (3.15) depend upon x_2 and x_3 through the Poisson ratio. The part proportional to $\overline{\psi}'(x_1)$, therefore proportional to the bending moment, gives rise to the so-called *anticlastic surfaces* [41].

3.3.- The Second Order Approximation

With the exact expression (3.15) for the kinematics of the linearized theory at our disposal, we proceed to develop an exact second order approximation to the equilibrium equations of the non-linear theory by making use of the method of successive approximations.

3.3.1.- Kinematic Relations.

Consider the situation in which the externally applied end loads are controlled by a small parameter ϵ. Accordingly, let the components with respect to the $\{\hat{\mathrm{e}}_i\}$ of the applied force $\mathbf{R}|_{X_1=0}$ at $X_1=0$ be expressed as

$$P_\epsilon = \epsilon\overline{P} + \epsilon^2\overline{\overline{P}} + \cdots, \qquad H_\epsilon = \epsilon\overline{H} + \epsilon^2\overline{\overline{H}} + \cdots \tag{3.18}$$

In accord with the method of successive approximations, we consider a family of configurations $\Phi_\epsilon : \mathbb{R}\times B \rightarrow \mathbb{R}^3$ defined by

$$\Phi_\epsilon(\mathrm{X}) \equiv \mathrm{X} + \overline{\mathrm{u}}_\epsilon(\mathrm{X}) \equiv \mathrm{X} + \epsilon\overline{\mathrm{u}}(\mathrm{X}) + \epsilon^2\overline{\overline{\mathrm{u}}}(\mathrm{X}) + O(\epsilon^3) \tag{3.19}$$

and such that $\overline{\mathrm{u}}(\mathrm{X}) = \frac{d}{d\epsilon}[\Phi_\epsilon(\mathrm{X})]|_{\epsilon=0}$ is the exact linearized displacement field given by (3.15). From (3.19) it immediately follows that

$$\begin{aligned} \mathbf{F}_\epsilon(\mathrm{X}) &= 1 + \epsilon\, GRAD\, \overline{\mathrm{u}}(\mathrm{X}) + O(\epsilon^2) \\ J_\epsilon(\mathrm{X}) &:= det(\mathbf{F}_\epsilon) = 1 + DIV\, \overline{\mathrm{u}}(\mathrm{X}) + O(\epsilon^2) \end{aligned} \tag{3.20}$$

$$[J\mathbf{F}^{-1}(\mathbf{X})]^{\epsilon} = 1 + \epsilon[DIV\,\mathbf{u}(\mathbf{X}) - GRAD\,\mathbf{u}(\mathbf{X})] + O(\epsilon^2)$$

The explicit expression for the components of $GRAD\,\mathbf{u}$ with respect to the basis $\{\hat{\mathbf{e}}_i \times \hat{\mathbf{E}}_j\}$ can be computed from (3.15) as

$$GRAD\,\mathbf{u}(\mathbf{X}) = \begin{bmatrix} \bar{u}' - X_2\bar{\psi}' - \phi\kappa\bar{\beta}' & -\bar{\psi} - \phi,_2\kappa\bar{\beta} & -\phi,_3\kappa\bar{\beta} \\ \nu' + \nu(g\bar{\psi}'' - X_2\bar{u}'') & \nu(g,_2\bar{\psi}' - \bar{u}') & \nu g,_3\bar{\psi}' \\ \nu X_3(X_2\bar{\psi}'' - \bar{u}'') & \nu X_3\bar{\psi}' & \nu(X_2\bar{\psi}' - \bar{u}') \end{bmatrix} \tag{3.21}$$

Next, recall from Chap.1 that the Gaussian frame at a point $\mathbf{x}$ in the deformed cross section $\Phi(\Omega)$ is composed by $\{\hat{\mathbf{n}}, \hat{\mathbf{l}}_2, \hat{\mathbf{l}}_3\}$, where $\hat{\mathbf{n}} = \left[\frac{d\omega}{d\Omega}\right]^{-1} J\mathbf{F}^{-1}\hat{\mathbf{E}}_1$ is the unit vector field normal to $\Phi(\Omega)$ and $\hat{\mathbf{l}}_\alpha = \mathbf{l}_\alpha/||\mathbf{l}_\alpha||$, $(\alpha=2,3)$; being $\mathbf{l}_\alpha = \mathbf{F}\hat{\mathbf{E}}_\alpha$ the convected vector fields tangent to $\Phi(\Omega)$. Thus, from (3.20) and (3.21) the components of the Gaussian frame with respect to the basis $\{\hat{\mathbf{e}}_i\}$ are given, in matrix notation, by

$$\left[\frac{d\omega}{d\Omega}\right]\hat{\mathbf{n}} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} + \epsilon\left(\begin{Bmatrix} 0 \\ \bar{\psi} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \phi,_2 \\ \phi,_3 \end{Bmatrix}\kappa\bar{\beta} + \begin{Bmatrix} 2X_2 \\ 0 \\ 0 \end{Bmatrix}\nu\bar{\psi}' + \begin{Bmatrix} 2 \\ 0 \\ 0 \end{Bmatrix}\nu\bar{u}'\right) + O(\epsilon^2)$$

$$||\mathbf{l}_2||\,\hat{\mathbf{l}}_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} + \epsilon\left(\begin{Bmatrix} -\bar{\psi} \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -\phi,_2 \\ 0 \\ 0 \end{Bmatrix}\kappa\bar{\beta} + \begin{Bmatrix} 0 \\ X_2 \\ X_3 \end{Bmatrix}\nu\bar{\psi}' + \begin{Bmatrix} 0 \\ -1 \\ 0 \end{Bmatrix}\nu\bar{u}'\right) + O(\epsilon^2) \tag{3.22}$$

$$||\mathbf{l}_3||\,\hat{\mathbf{l}}_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} + \epsilon\left(\begin{Bmatrix} -\phi,_3 \\ 0 \\ 0 \end{Bmatrix}\kappa\bar{\beta} + \begin{Bmatrix} 0 \\ -X_3 \\ X_2 \end{Bmatrix}\nu\bar{\psi}' + \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix}\nu\bar{u}'\right) + O(\epsilon^2)$$

where use has been made of the expression for $g(X_2, X_3)$ given by (3.7). Clearly, the Gaussian frame is orthogonal to within second order, since $\hat{\mathbf{l}}_2.\hat{\mathbf{l}}_3 = \epsilon_{23} + O(\epsilon^2) = 0 + O(\epsilon^2)$. Furthermore, in view of (3.22), one has the estimates

$$\begin{aligned} ||\mathbf{l}_2|| &= 1 + \epsilon\,\epsilon_{22} + O(\epsilon^2) = 1 + \epsilon(\nu X_2\bar{\psi}' - \nu\bar{u}') + O(\epsilon^2) \\ ||\mathbf{l}_3|| &= 1 + \epsilon\,\epsilon_{33} + O(\epsilon^2) = 1 + \epsilon(\nu X_2\bar{\psi}' - \nu\bar{u}') + O(\epsilon^2) \\ \left[\frac{d\omega}{d\Omega}\right] &= 1 + \epsilon(2X_2\bar{\psi}' + 2\nu\bar{u}') + O(\epsilon^2) \end{aligned} \tag{3.23}$$

As in Chap.1, the relationship between Gaussian frame and the spatial basis $\{\hat{\mathbf{e}}_i\}$ will be expressed as $\{\hat{\mathbf{n}}, \hat{\mathbf{l}}_2, \hat{\mathbf{l}}_3\}^T = \mathbf{\Lambda}(\mathbf{X})^T\{\hat{\mathbf{e}}_i\}^T$. Thus, in view of (3.22) and (3.23) the components of the matrix $\mathbf{\Lambda}_\epsilon(\mathbf{X})$ consistent with the configurations (3.19) may be written in the form

$$\mathbf{\Lambda}_\epsilon(\mathbf{X}) = 1 + \epsilon\left[\mathbf{\Omega}\ \bar{\psi}(X_1) + \mathbf{\Xi}(X_2,X_3)\ \kappa\bar{\beta}(X_1) + \mathbf{\Sigma}(X_2,X_3)\ \nu\bar{\psi}'(X_1)\right] + O(\epsilon^2) \quad (3.24a)$$

where

$$\mathbf{\Omega} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad \mathbf{\Xi} = \begin{bmatrix} 0 & -\phi,_2 & -\phi,_3 \\ \phi,_2 & 0 & 0 \\ \phi,_3 & 0 & 0 \end{bmatrix},\quad \mathbf{\Sigma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -X_3 \\ 0 & X_3 & 0 \end{bmatrix} \quad (3.24b)$$

Equation (3.24) admits an extremely simple geometrical interpretation. $\mathbf{\Omega}\bar{\psi}(X_1)$ represents a rotation of magnitude $\bar{\psi}$ about the X_3-axis, and this is the only effect the kinematics based upon Bernoulli's assumption considers. $\mathbf{\Xi}(X_2,X_3)$ reflects the change in orientation the Gaussian frame experiences over the deformed cross section $\Phi(\Omega)$ as a consequence of the axial warping of $\Phi(\Omega)$ due to shear deformation. Finally, $\mathbf{\Sigma}(X_2,X_3)$ takes into account the deformation of the cross section due to Poisson's effect.

3.3.2.- Equilibrium Equations.

Consistent with the expression (3.18) for the configurations $\Phi_\epsilon : \mathbb{R}\times B \rightarrow \mathbb{R}^3$, we consider the following representation of the first Piola-Kirchhoff stress tensor

$$\mathbf{P}_\epsilon(\mathbf{X}) = \epsilon\ \sigma + \epsilon^2\ \bar{\bar{\mathbf{P}}} + O(\epsilon^3) \quad (3.25)$$

where $\sigma = \frac{d}{d\epsilon}[\mathbf{P}_\epsilon]|_{\epsilon=0}$ is the (symmetric) stress tensor of the linear theory, and $\bar{\bar{\mathbf{P}}} = \frac{1}{2}\frac{d^2}{d\epsilon^2}[\mathbf{P}_\epsilon]|_{\epsilon=0}$ a (non-symmetric) two-point tensor. Let us denote by $\mathbf{P}_\epsilon^G$ the components of of the first Piola-Kirchhoff tensor with respect to the Gaussian and material frames. For simplicity, the same symbol $\mathbf{P}_\epsilon$ is employed to designate the tensor and its components with respect to the basis $\{\hat{\mathbf{e}}_i \times \hat{\mathbf{E}}_I\}$. Both set of components are related according to $\mathbf{P}_\epsilon = \mathbf{\Lambda}_\epsilon \mathbf{P}_\epsilon^G$. Hence, in view of (3.24) the linear part of $\mathbf{P}_\epsilon^G$ is given by $\sigma = \frac{d}{d\epsilon}[\mathbf{P}_\epsilon^G]|_{\epsilon=0}$ and $\mathbf{P}_\epsilon^G$ admits the representation

$$\mathbf{P}_\epsilon^G(\mathbf{X}) = \epsilon\ \sigma + \epsilon\ \bar{\bar{\mathbf{P}}}_\epsilon^G + O(\epsilon^3) \quad (3.26)$$

The stress vector $\mathbf{T}_\epsilon = \mathbf{P}_\epsilon \hat{\mathbf{E}}_1$ acting on the deformed cross section $\Phi_\epsilon(\Omega)$ is then given by

$$\mathbf{T}_\epsilon = P_{\epsilon i1}\,\hat{\mathbf{e}}_i = P^G_{\epsilon\,1 1}\hat{\mathbf{n}} + P^G_{\epsilon\,2 1}\hat{\mathbf{I}}_2 + P^G_{\epsilon\,3 1}\hat{\mathbf{I}}_3 \tag{3.27}$$

and the normal and tangential stress resultants over $\Phi_\epsilon(\Omega)$ are defined as

$$\begin{aligned} N_\epsilon(X_1) &= \epsilon\int_\Omega \sigma_{11} d\Omega + \epsilon^2 \int_\Omega P^G_{\epsilon\,11} d\Omega + O(\epsilon^3) \equiv \epsilon\,\overline{N} + \epsilon^2 \overline{\overline{N}} + O(\epsilon^3) \\ V_{\epsilon_\alpha}(X_1) &= \epsilon\int_\Omega \sigma_{\alpha 1} d\Omega + \epsilon^2 \int_\Omega P^G_{\epsilon\,\alpha 1}\, d\Omega + O(\epsilon^3) \equiv \epsilon\,\overline{V}_\alpha + \epsilon^2\,\overline{\overline{V}}_\alpha + O(\epsilon^3) \end{aligned} \tag{3.28}$$

The equilibrium equations (1.6) for the resultant force then leads to the equalities

$$\begin{aligned} -\epsilon\begin{Bmatrix}\overline{P}\\ \overline{H}\\ 0\end{Bmatrix} + \epsilon^2\begin{Bmatrix}\overline{\overline{P}}\\ \overline{\overline{H}}\\ 0\end{Bmatrix} &= \int_\Omega \mathbf{P}_\epsilon(\mathbf{X})\hat{\mathbf{E}}_1 d\Omega \\ &= \int_\Omega \boldsymbol{\Lambda}_\epsilon(\mathbf{X})\mathbf{P}^G_\epsilon(\mathbf{X})\hat{\mathbf{E}}_1 d\Omega \\ &= \epsilon\int_\Omega \sigma\hat{\mathbf{E}}_1 d\Omega + \epsilon^2\int_\Omega [\overline{\overline{\mathbf{P}}}^G_\epsilon + (\boldsymbol{\Omega}\overline{\psi} + \boldsymbol{\Xi}\kappa\overline{\beta} + \boldsymbol{\Sigma}\nu\overline{\psi}')\sigma]\hat{\mathbf{E}}_1 d\Omega + O(\epsilon^3) \end{aligned}$$

and the use of equation (3.28) yields the following second order approximation to the equilibrium equation (1.6) for the resultant force

$$\begin{aligned} &\epsilon\left\{\begin{Bmatrix}\overline{N}\\ \overline{V}_2\\ \overline{V}_3\end{Bmatrix} + \begin{Bmatrix}\overline{P}\\ \overline{H}\\ 0\end{Bmatrix}\right\} \\ &+ \epsilon^2\left\{\begin{Bmatrix}\overline{\overline{N}}\\ \overline{\overline{V}}_2\\ \overline{\overline{V}}_3\end{Bmatrix} + \begin{Bmatrix}-\overline{V}_2\overline{\psi}\\ \overline{N}\overline{\psi}\\ 0\end{Bmatrix} + \kappa\overline{\beta}\int_\Omega \boldsymbol{\Xi}\begin{Bmatrix}\sigma_{11}\\ \sigma_{12}\\ \sigma_{13}\end{Bmatrix} d\Omega + \nu\overline{\psi}'\int_\Omega \boldsymbol{\Sigma}\begin{Bmatrix}\sigma_{11}\\ \sigma_{12}\\ \sigma_{13}\end{Bmatrix} d\Omega + \begin{Bmatrix}\overline{\overline{P}}\\ \overline{\overline{H}}\\ 0\end{Bmatrix}\right\} + O(\epsilon^3) = 0 \end{aligned} \tag{3.29a}$$

The second order approximation to the equilibrium equation for the bending moment $\mathbf{M}_\epsilon = M_\epsilon \hat{\mathbf{e}}_3 = [\epsilon\,\overline{M} + \epsilon^2\,\overline{\overline{M}}]\hat{\mathbf{e}}_3 + O(\epsilon^3)$ may be written in a similar form as

$$\epsilon[\overline{M} - \overline{H}] + \epsilon^2[\overline{\overline{M}} - \overline{\overline{H}} - \overline{u}'\overline{H} + \overline{v}'\overline{P}] + O(\epsilon^3) = 0 \tag{3.29b}$$

Equations (3.29) represents the first two terms of a formal expansion of the nonlinear equilibrium equations in terms of the small parameter ϵ which controls the applied loads. By equating to zero the coefficient of the first power of ϵ one obtains the equilibrium equations of the linear theory in terms of stress resultants, in agreement with the principle of successive approximations. The coefficient of the second power ϵ^2 yields, when set equal to zero, the second order correction within the nonlinear theory. The second order terms in (3.29a) containing $\Xi(X_2, X_3)$ and $\Sigma(X_2, X_3)$ arise as a result of the transverse warping of the cross section and the Poisson's effect, respectively, and need to be estimated in terms of resultant forces. For this purpose, use is made of the following key result:

Lemma

Regardless the shape of the cross section Ω, the following relation always holds

$$\int_\Omega \left(\phi_{,2}\,\sigma_{12} + \phi_{,3}\sigma_{13}\right) d\Omega = 0$$

Proof

The proof makes use of the definition (3.7) of the warping function $\phi(X_2, X_3)$ in terms of the harmonic function $\chi(X_2, X_3)$ given by the Neumann problem (3.2). The computation is somewhat lengthy and details can be found in Appendix II.

Second order estimates of the warping and Poisson's effects.

Since $\Xi(X_2, X_3)$ and $\Sigma(X_2, X_3)$ are given by (3.24b), in view of the previous lemma equations (3.29) reduce to

$$\begin{aligned}
&\epsilon[\overline{N} - \overline{P}] + \epsilon^2[\overline{\overline{N}} - \overline{\psi}\,\overline{V}_2 + \overline{P}] + O(\epsilon^3) = 0 \\
&\epsilon[\overline{V}_2 - \overline{H}] + \epsilon^2[\overline{\overline{V}}_2 + \overline{\psi}\overline{N} + \kappa\overline{\beta}\int_\Omega \phi_{,2}\sigma_{11} d\Omega - \nu\overline{\psi}'\int_\Omega X_3\sigma_{12} d\Omega + \overline{\overline{H}}] + O(\epsilon^3) = 0 \\
&\epsilon[\overline{M} - \overline{H}] + \epsilon^2[\overline{\overline{M}} - \overline{\overline{H}} - \overline{u}'\overline{H} + \overline{v}'\overline{P}] + O(\epsilon^3) = 0
\end{aligned} \tag{3.30a}$$

To complete the estimate, the two integrals appearing in (3.30a) will be expressed in terms of stress resultants as follows. Proceeding as in Appendix II

$$\begin{aligned}
\nu\overline{\psi}'\int_\Omega X_3\sigma_{12} d\Omega &= -\nu G\overline{\psi}'\,\frac{\overline{V}_2}{EI_2}\int_\Omega \left[X_3\frac{\partial\chi}{\partial X_3} + (2+\nu)X_2(X_3)^2\right] d\Omega \\
&= -\nu G\overline{\psi}'\,\frac{\overline{V}_2}{EI_2}\left[(2+\nu)\int_\Omega X_2(X_3)^2 d\Omega + \int_{\partial\Omega}\frac{1}{2}(X_3)^2\frac{\partial\chi}{\partial n} d\Gamma\right] \\
&= \overline{V}_2\nu\overline{\psi}'\frac{1}{2I_2}\int_\Omega X_2(X_3)^2 d\Omega
\end{aligned} \tag{3.30b}$$

Clearly, this term vanishes for cross sections with two axes of symmetry. A similar computation and the definition (3.13b) of the shear coefficient κ yields

$$\kappa\bar{\beta}\int_\Omega \phi,_2\sigma_{11}d\Omega = \bar{N}\bar{\beta}\kappa\,\frac{1}{\Omega}\int_\Omega \phi,_2 d\Omega - \frac{\bar{V}_2}{I_2}\bar{\psi}'\int_\Omega \frac{1}{2}(X_2)^2\frac{\partial\chi}{\partial \mathbf{n}}d\Omega$$

$$= (1-\kappa)\beta\,\bar{N} + \bar{V}_2\,\frac{\bar{\psi}'}{I_2}\int_\Omega \left[(1+\frac{3}{2}\nu)(X_2)^3 + (1-\frac{1}{2}\nu)X_2(X_3)^2\right]d\Omega \quad (3.30c)$$

Therefore, if attention is confined to cross sections with two axes of symmetry, equations (3.30a)-(3.30c) lead to following expression for the second order approximation to the nonlinear equilibrium equations

$$\begin{aligned} N_\epsilon - (\epsilon\bar{\psi})\;V_\epsilon &= -\epsilon\bar{P} - \epsilon^2\bar{\bar{P}} - O(\epsilon^3) \\ [(\epsilon\bar{\psi}) + (1-\kappa)(\epsilon\bar{\beta})]\;N_\epsilon + V_\epsilon &= -\epsilon\bar{H} - \epsilon^2\bar{\bar{H}} - O(\epsilon^3) \\ M'_\epsilon + [1 + (\epsilon\bar{u}')]\;V_\epsilon - \kappa(\epsilon\bar{\beta})\;N_\epsilon &= O(\epsilon^3) \end{aligned} \quad (3.31)$$

where N_ϵ and V_ϵ are defined by (3.28) and M_ϵ in an analogous manner. Since $\bar{u}_\epsilon = \epsilon\bar{u} + O(\epsilon^2)$ and $\bar{\psi}_\epsilon = \epsilon\bar{\psi} + O(\epsilon^2)$, the subindex "$\epsilon$" may be omited and the second order approximation tates the final form

$$\begin{aligned} N - \bar{\psi}\;V &= -P \\ V + [\bar{\psi} + (1-\kappa)\bar{\beta}]\;N &= -H \\ M' - \kappa\bar{\beta}N + [1 + \bar{u}']\;V &= 0 \end{aligned} \quad (3.32)$$

Equations (3.32) should be compared with the linearized system of equilibrium equations (2.58) derived in the previous chapter. The former show the following two results

(a) Due to warping of the cross section the resultant force N is rotated an extra angle $(1-\kappa)\bar{\beta}$ and therefore no longer remains normal to the average plane of bending defined by the angle $\bar{\psi}(X_1)$

(b) However, the shear force V always remains contained in the average plane of bending regardless the shape of the cross section.

These two conclusions are illustrated in Fig.3.2 and Fig.3.3.

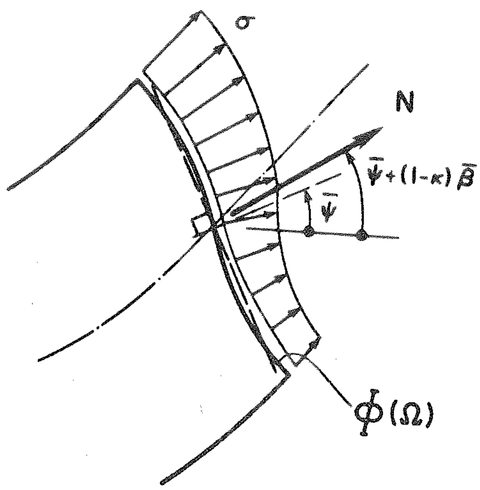


Figure 3.2. Resultant N of axial stresses over $\Phi(\Omega)$.

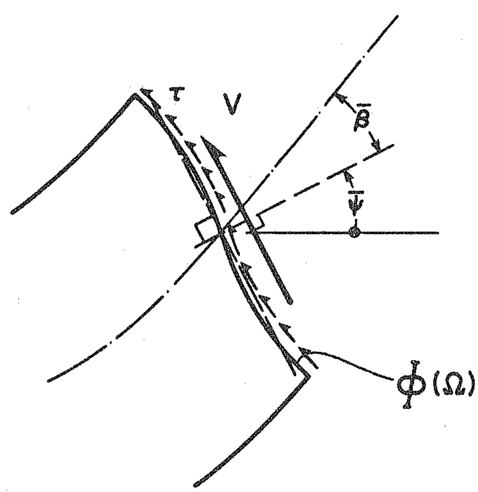


Figure 3.3. Resultant V of shearing stresses over $\Phi(\Omega)$

3.3.3.- Constitutive Equations.

It is again assumed that the material is hyperelastic with strain energy function $W(X,\nabla\Phi)$ and attention is restricted to isothermal processes. Let V be the linear space of kinematic admissible variations defined by (2.13). An argument similar to that presented in section 2.2.3 then shows that the Frechet differential of $\Pi^*(\Phi) = \int_B W\, dV$ at the configuration $\Phi : B \rightarrow \mathbb{R}^3$ may be written as

$$\delta\Pi^*(\Phi) = \int_{(-\frac{h}{2},\frac{h}{2})\times\Omega} \frac{\partial W(X,\nabla\Phi)}{\partial \mathbf{F}}:\delta\mathbf{F}\,dV$$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\int_{\Omega} \mathbf{P}(\nabla\Phi):\delta\mathbf{F}\,d\Omega \right] dX_3$$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} [M\,\delta\psi' + V\,\delta\lambda_{\nu} + N\,\delta\lambda_{n}]\,dX_3$$

where

$$\lambda_{\nu} = \bar{v}' - [1+\bar{u}']\,\bar{\psi}$$

$$\lambda_{n} = \bar{u}' + \frac{1}{2}\,[\bar{v}']^2 - \frac{1}{2}\,\kappa\bar{\beta}^2 \tag{3.33}$$

Therefore, in view of (3.33) the simplest possible hyperelastic constitutive model consistent with the non-linear equilibrium equations (3.32) is furnished by that of the linear theory with the axial and shearing measures of deformation $\bar{u}'$ and $\bar{\beta}'$ replaced by λ_n and λ_{ν}, respectively. The complete system of equations of the second approximation discussed in this section has been summarized in Table 3.1.

TABLE 3.1

The second order Approximation.

Equilibrium Equations

$$N - \bar{\psi}\,V = -P$$

$$V + [\bar{\psi} + (1-\kappa)\,\bar{\beta}]\,N = -H$$

$$M' + \bar{v}'\,P + [1+\bar{u}']\,H = 0$$

Constitutive Equations

$$M(X_1) = EI(X_1)\,\bar{\psi}'$$

$$V(X_1) = G\Omega(X_1)\kappa\,[\bar{v}' - (1+\bar{u}')\bar{\psi}]$$

$$N(X_1) = E\Omega(X_1)\,[\bar{u}' + \frac{1}{2}\,(\bar{v}')^2 - \frac{1}{2}\,\kappa(\bar{\beta}')^2]$$

In order to obtain a closed form expression for the buckling load of the rod relevant to problems of technical interest, the field equations summarized in Table 3.1 are further restricted by introducing the customary assumption of inextensibility of the line of centroids.

3.4.- The Linearized Eigenvalue Problem for the Elastica.

If the the line of centroids is assumed to be inextensible, the axial force N should then be regarded as a reaction to be determined from the equilibrium equation $(3.32)_1$ once the problem is solved. Thus, under this assumption $\bar{u}'=0$ and the constitutive equations in Table 3.1 take the form

$$M = EI_2\,\bar{\psi}'(X_1) \qquad V = G\Omega\ \kappa\,\bar{\beta}(X_1) \tag{3.34}$$

These equations together with the equilibrium equations (3.32) and the inextensibility constraint lead to the linear problem

$$\begin{gathered}\left[1 - [1-\kappa]\frac{P}{G\Omega\kappa}\right]\bar{v}' + \frac{H}{G\Omega\kappa} = \left[1 + \frac{P}{G\Omega}\right]\bar{\psi} \\ EI_2\,\bar{\psi}'' + P\,\bar{v}' = H \qquad\qquad x_1 \in (0,L) \\ \bar{u}' = 0\end{gathered} \tag{3.35}$$

from which immediately follows that both the average rotation $\bar{\psi}(x_1)$ and the lateral deflection $\bar{v}(x_1)$ are governed by

$$L\bar{\psi} \equiv \frac{EI_2\left[1-(1-\kappa)\dfrac{P}{G\Omega\kappa}\right]}{1 + \dfrac{P}{G\Omega}}\bar{\psi}'' + P\,\bar{\psi} = H \tag{3.36a}$$

$$L\,\bar{v}' = H \tag{3.36b}$$

When proper homogeneous boundary conditions are appended, (3.36) leads to an eigenvalue problem from which the critical values of P may be determined. Alternatively, a variational formulation may be employed. In this event, from (3.35) easily follows that the lowest eigenvalue of (3.36), the critical load, is characterized as the minimum value of the functional

$$P = \min_{\bar{v},\bar{\psi}\in H^1} \frac{\dfrac{1}{2}\displaystyle\int_0^L EI_2[\bar{\psi}']^2 dx + \dfrac{1}{2}\displaystyle\int_0^L G\Omega\kappa[\bar{\beta}]^2 dx + \textit{Boundary Terms}}{\dfrac{1}{2}\displaystyle\int_0^L [(\bar{v}')^2 - \kappa\bar{\beta}^2]\,dx} \tag{3.37}$$

over the class of functions with finite energy; i.e $\bar{\psi}$, $\bar{v} \in H^1(0,L)$ and satisfying the essential boundary conditions [44,45]. It is easily checked that the critical values of the axial load derived either from (3.35) or (3.36) can be related to the values P_E of the Euler's critical load by the expression

$$P_{crit} = \frac{2P_E}{1 + [1-\kappa]\dfrac{P_E}{G\Omega\kappa} + \sqrt{1 + [1-\kappa]\dfrac{P_E}{G\Omega\kappa} + \dfrac{4P_E}{G\Omega\kappa}}} \qquad (3.37)$$

The derivation presented heretofore, does not provide any information about the fully non-linear problem. Therefore, a stability analysis at bifurcation points, similar to that carried out in Appendix I for the simplified theories discussed in the previous chapter is not possible. However, by comparison with these simplified theories and under physical grounds it will be reasonable to regard such bifurcation positions as leading to locally stable configurations.

3.5.- Comparison with other Formulations.

Formulations based upon the 'plane sections remain plane' kinematic assumption have been discussed at length in the previous chapter. This simplifying assumption allowed the derivation of fully non-linear theories which, when consistently linearized, yielded well known engineering approximations to the bending stiffness and buckling loads. These formulations take into account the so-called effect of shear deformation in an approximate manner, in the sense that warping of the cross section which necessarily appears as a result of shear deformation is, except for the case of the sandwich beam, systematically neglected. On the other hand, the warping effect as well as the Poisson ratio effect, are inherently built into the formulation presented in this chapter. Furthermore, it is again emphasised that this formulation represents an exact second order approximation to the three dimensional non-linear theory whenever only end loads are considered. In the light of these results some comparisons may be drawn.

Bending Stiffness.

In view of equations $(3.34)_1$ and $(3.35)_1$, the constitutive equation for the bending moment can be written in in the form

$$M = EI_2\,\bar{\psi}' = \frac{EI_2\left[1 - (1-\kappa)\dfrac{P}{G\Omega\kappa}\right]}{1 + \dfrac{P}{G\Omega}}\,\bar{v}'' \tag{3.38}$$

Clearly, if the deformation due to shear is neglected, $G\Omega \rightarrow \infty$ and (3.38) reduces to the classical relation $M = EI_2\bar{v}''$ between bending moment and linearized curvature of elementary beam theory. However, when shear deformation is taken into account, equation (3.38) shows the following effects

(a) The bending stiffness of the elementary theory experiences a reduction by the factor $1 + \dfrac{P}{G\Omega}$ present in the denominator of (3.38). An analogous reduction factor is predicted by the elementary theory, based upon Bernouilli's assumption. However to account for the unrealistic uniform shear stress distribution predicted by this simplified approach, $G\Omega$ is simply relaced by $G\Omega\kappa$ and the reduction factor takes the form $1 + \dfrac{P}{G\Omega\kappa}$.

(b) An extra reduction in bending stiffness by the factor $1 - (1-\kappa)P/G\Omega\kappa$ appearing in the numerator of (3.38). Such a reduction appears as a result of the axial warping of the cross section. It is noted that the total effective stiffness predicted by (3.38) results in a value lower than that of $\dfrac{EI}{1 + P/G\Omega\kappa}$ obtained under Bernouilli's assumption.

(c) For a sandwich beam, the constitutive equation for the bending moment takes the form

$$M = EI_2\left[1 - \frac{P}{G\Omega\kappa}\right]\bar{v}'' \tag{3.39}$$

where, in the case of a "soft core", Ω is the total area of the cross section and $\kappa = \dfrac{\Omega_{core}}{\Omega}$.

The same type of expression was incorrectly proposed by Timoshenko [24], for the case of a homogeneous beam. Nevertheless, when $\dfrac{P}{G\Omega}$ is *small* (3.38) reduces for all practical purposes to (3.39).

Buckling load.

Similar remarks are applicable to the buckling load, and the situation is summarized in the Table 3.2. It is noted that the formulation presented yields, in addition, an explicit expression for the shear coefficient κ. Remarkably enough, when $\frac{G\Omega}{EI_2} >> 1$ all the proposed expressions including the exact formula (3.37) reduce for all practical purposes to that corresponding to the sandwich beam. For a linear isotropic material, $2 \leqslant \frac{E}{G} \leqslant 3$ and except for extremely short beams this will always be the case.

TABLE 3.2.

Proposed expressions for the buckling load.

CRITICAL LOAD	General Expression	$\frac{G\Omega}{EI_2} \rightarrow 0$	$\frac{G\Omega}{EI_2} \rightarrow \infty$
Homogeneous Beam (*Haringx, et al*)	$\dfrac{2P_E}{1+\left(1+\dfrac{4P_E}{G\Omega\kappa}\right)^{1/2}}$	$\left(\dfrac{G\Omega\kappa}{P_E}\right)^{1/2} P_E$	$\left(1-\dfrac{P_E}{G\Omega\kappa}\right) P_E$
Sandwich Beam (*Plantema, et al*)	$\dfrac{P_E}{1+\dfrac{P_E}{G\Omega\kappa}}$	$G\Omega\kappa$	$\left(1-\dfrac{P_E}{G\Omega\kappa}\right) P_E$
Homogeneous Beam (*Present Study*)	$\dfrac{2P_E}{1+\dfrac{(1-\kappa)P_E}{G\Omega\kappa}+\left(1+\dfrac{(5-\kappa)P_E}{G\Omega\kappa}\right)^{1/2}}$	$\dfrac{G\Omega\kappa}{1-\kappa}$	$\left(1-\dfrac{P_E}{G\Omega\kappa}\right) P_E$

However, for a transversally isotropic material the elastic constants G and E are independent [42], and the ratio $\frac{G\Omega}{EI_2}$ can take extremely low values, particularly in the case of short beams. In such an event, since the expressions of the buckling load summarized in Table 3.2 show quite different asymptotic behavior as $\frac{G\Omega}{EI_2} \rightarrow 0$, the buckling load can be severely under estimated unless (3.37) is employed. See Fig.3.4.

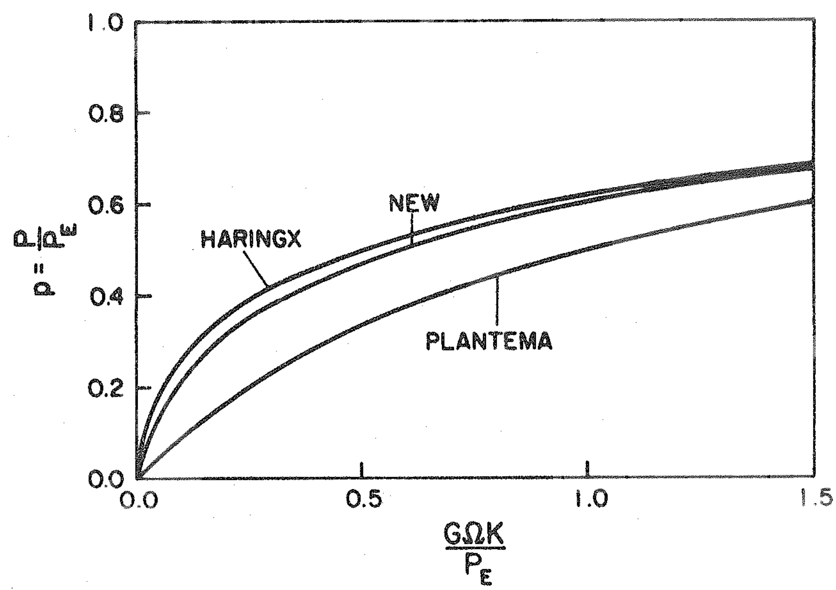


Figure 3.4. Comparison between different expressions of the buckling load.

An important example of this type of beam is furnished by a multilayer elastomeric bearing, widely used in base isolation systems. A complete study of this type of device is undertaken in the next chapter.

3.6.- Remarks on the Derivation of the Proposed Theory.

The derivation of the first two equilibrium equations in (3.32) made explicit use of the following two facts

(i) *The knowledge of the exact solution, in the context of the linear theory of elasticity, for the displacement field of a beam acted upon by end loads (Saint Venant's problem).*

The consideration of more general types of loading would require, strictly speaking, the exact solution of the corresponding displacement field. However, the estimates of the terms appearing in (3.28) show that only the warping function $\phi : \Omega \rightarrow \mathbb{R}$ gives an additional non-zero contribution to the final equilibrium equations (3.32) through the shear factor κ. Accordingly, the explicit expression of the warping function ϕ or, alternatively the value of the shear coefficient κ, is all that is required for the consideration of more general loading situations. The explicit form taken by the residual lateral displacements $\hat{u}_2(\mathbf{x})$ and $\hat{u}_3(\mathbf{x})$ is irrelevant for this purpose.

Remarkably enough, the warping function $\phi(x_2, x_3)$ takes the same form (3.7) for the case of a constant load q distributed over the span of the beam [40]. Thus, as pointed out by Cowper [39], the shear coefficient κ computed from (3.14) remains as an excellent approximation provided the transversally applied load does not vary wildly over the span of the beam.

(ii) *A particular choice of kinematic variables was adopted to model the response of the beam. Namely the average displacement vector* $\hat{\mathbf{u}}(x_1)$ *and the average rotation* $\bar{\psi}(x_1)$ *of an arbitrary cross section* $\Omega(x_1)$.

Clearly, the question can be raised as to whether a different choice of kinematic variables might lead to a complete different description of the response of the beam. This point will be examined in the more general context of Kantorovich's method discussed next.

3.6.1.- Reduction of Dimension in the Theory of Elasticity.

The method proposed by L. V. Kantorovich [43], is essentially a generalization of the Ritz method and falls in the class of the so-called energy methods of variational calculus. The method is treated at length in the Russian literature [43,44] and has been extensively applied to a variety of problems in the linear theory of elasticity [41,43,47]. Applications to the solution of time dependent problems, particularly in the context of the Finite Element Method are also well known [45,46].

In this section, attention will be focussed on the use of Kantorovich method as a systematic procedure to accomplish the reduction of dimension in boundary value problems of the theory of Elasticity. We shall restrict ourselves to an outline of the relevant facts. A complete account of the procedure can be found in [47] and references therein. Applications to non-linear rod theories can be found in [9].

Outline of the Method

For an elastic beam with axial axis x_1 and cross section Ω, the displacement field is expressed in the form

$$\mathbf{u}(\mathbf{x}) = \sum_{k=0}^{N} \boldsymbol{\phi}_{\mathbf{k}}{}^{T}(x_2,x_3)\,.\,\bar{\mathbf{u}}_k(x_1) \tag{3.39}$$

where the functions $\boldsymbol{\phi}_{\mathbf{k}}{}^{T} = [\phi_1,\phi_2,\phi_3]$ are given *a−priori* and satisfy the boundary conditions on the lateral contour $\partial\Omega$ of the beam. The objective is to determine the generalized displacements $\{\bar{\mathbf{u}}_k(x_1)\}$. For this purpose, the linear space of kinematically admissible displacements

$$V = \{\ \bar{\mathbf{u}} : (0,L)\times\Omega \rightarrow I\!R^3\ \mid \mathbf{u}|_{\partial\Omega_u} = prescribed\}$$

is endowed with the energy norm

$$||\mathbf{u}||^2 = \int_0^L \left\{ \frac{\lambda}{2} \int_\Omega [div\ \mathbf{u}]^2\, d\Omega + \frac{2G}{2} \int_\Omega [\nabla\mathbf{u}:\nabla\mathbf{u}]\ d\Omega \right\} dx \tag{3.40}$$

The total potential energy of the beam loaded in the plane $x_1 - x_2$ by a distributed force q, say, is then given by the quadratic functional

$$\mathbf{u}(\mathbf{x}) \rightarrow \Pi(\mathbf{u}) = ||\mathbf{u}||^2 - \int_0^L q(x) \left[\int_\Omega u_2\, d\Omega \right] dx \tag{3.41}$$

and the functions $\{\bar{\mathbf{u}}_k(x_1)\}$ are determined by substituting (3.39) into (3.41) and enforcing the variational condition of minimum potential energy over V. It is easily shown, making use of standard methods of variational calculus, that this condition leads to a system of ordinary differential equations for the displacements $\{\bar{\mathbf{u}}_k\}$. Thus, the solution of a system of partial differential equations is reduced by Kantorovich method, to the integration of a set of ordinary differential equations.

Clearly, the success of the method depends upon the choice of the coordinate functions $\{\phi_k\}$ in the expansion (3.39). Remarkably, it can be shown that an optimal choice for the functions $\{\phi_k\}$ is possible which is independent of the type of loading and leads to a minimum of the error between approximate and exact solutions in the sense of the energy norm (3.40) † . Furthermore, when the expansion (3.39) is limited to N=1, This optimal choice yields the classical Euler-Bernouilli kinematics of the elementary beam theory.

We are now in the position of re-examine the objection raised in (ii): Does the choice of kinematic variables affect substantially the response of the beam?

(iia) Clearly, the kinematics (3.15) corresponds to an expansion of the form (3.39) in which, in view of condition (3.6), the coordinate functions $\{1, x_2, \phi\}$ and $\{1, g\}$ in (3.15) are orthogonal over Ω with respect to the L^2 inner product $<\phi_1, \phi_2> = \int_\Omega \phi_1 \phi_2 \, d\Omega$.

Further, the results of section 3.2 show that these coordinate functions, as given by (3.7) and (3.17), are exact for the problem at hand of a beam acted upon by end loads. It is noted that we could have proceeded in reverse order and from the displacement field (3.15) determine the explicit form if the coordinate functions by enforcing at the outset the boundary conditions

$$\sigma_{12}|_{\partial\Omega} = \sigma_{13}|_{\partial\Omega} = 0$$

This approach will be pursued in the analysis of plates presented in chapter 5.

(iib) Once the functions $\{\phi_k\}$ in (3.39), or ϕ and g in (3.15), have been established it is irrelevant, from a formal stand point, to consider instead of the variables $\{\bar{\mathbf{u}}_k\}$ an alternative set of kinematic variables $\{\bar{\bar{\mathbf{u}}}_k\}$, say, the later being in a one-to-one, smooth enough correspondence with the former. The minimization process or the procedure described in section 2.3 would lead to an equally consistent, in general different, set of ordinary differential equations. Nevertheless, $\{\bar{\mathbf{u}}_k\}$ and $\{\bar{\bar{\mathbf{u}}}_k\}$ will generally have different physical meanings and care must be exercised when solving specific problems, particularly in

† A fairly complete account of this fact including explicit expressions for the functions $\{\phi_k\}$ can be found in [47]. We have been unable to locate the references made in [47] to the original work of Babuska and Prager.

regard to the enforcement of the physically meaningful boundary conditions. This point, as well as the effect in the overall buckling load of a small change in boundary conditions is illustrated in the following example.

3.6.2.- Example: Narrow Rectangular Cross Section.

Consider a narrow beam with rectangular section acted upon by end loads. The exact displacement field can be found in [40] and is given in terms of polynomials. The following two sets of generalized kinematic variables are considered.

(a) *First choice.*

Let us first consider the set $\{\bar{u},\bar{v},\bar{\psi}\}$ where, as in section 3.2, $\bar{u}(x_1)$ and $\bar{v}(x_1)$ are the components of the mean displacement given by (3.3) and $\bar{\psi}(x_1)$ is the average rotation defined by (3.4).

The displacement field then takes the form (3.15) with $u_3 \equiv 0$ and the functions ϕ and g given by

$$\phi = \frac{1}{6I}\left[\Omega x_2^3 - \frac{9}{5}Ix_2\right]\left(1 - \frac{\nu}{2(1+\nu)}\right), \qquad g = \frac{1}{2}\left[x_2^2 - \frac{I}{\Omega}\right] \tag{3.43}$$

In view of (3.43), the shear coefficient κ given by (1.13b) takes the value

$$\kappa = \frac{10(1+\nu)}{12+11\nu} \tag{3.44}$$

(b) *Second choice.*

Let us consider next the alternative set $\{u_o, v_o, \psi_o\}$ illustrated in Fig.3.5 and defined by

$$u_o := u_1|_{x_2=0}\,, \quad v_o := u_2|_{x_2=0}\,, \quad \psi_o := -\frac{\partial u_1}{\partial x_2}|_{x_2=0} \tag{3.45}$$

Making use of these definitions, we find that $\{\bar{u},\bar{v},\bar{\psi}\}$ and $\{u_o, v_o, \psi_o\}$ are related through

$$
\begin{aligned}
u_o &= \bar{u} \\
v_o &= \bar{v} - \frac{1}{2}\,\frac{I}{\Omega}\,\bar{\psi}' \\
\psi_o &= \bar{\psi} - \frac{3}{10}\,\kappa\left[1 - \frac{\nu}{2(1+\nu)}\right]\bar{\beta}
\end{aligned}
\tag{3.46}
$$

The shearing angle $\beta_o(x_1)$ is defined in a manner similar to (3.12) as

$$\beta_o(x_1) = v_o'(x_1) - \psi_o(x_1) \tag{3.47}$$

and can be related to $\bar{\beta}(x_1)$ as follows. Differentiating $(3.46)_2$ and making use of the constitutive equations for M and V together with the equilibrium relation $M' = -V$, one arrives at

$$v_o' = \bar{v}' + \frac{\nu G}{2E}\,\bar{\beta} = \bar{v}' + \frac{\nu}{4(1+\nu)}\,\bar{\beta} \tag{3.48}$$

The substitution of this relation together with $(3.46)_3$ in the definition (3.47) leads to

$$\beta_o(x_1) = \frac{3}{2}\,\kappa\,\bar{\beta}(x_1) \tag{3.49}$$

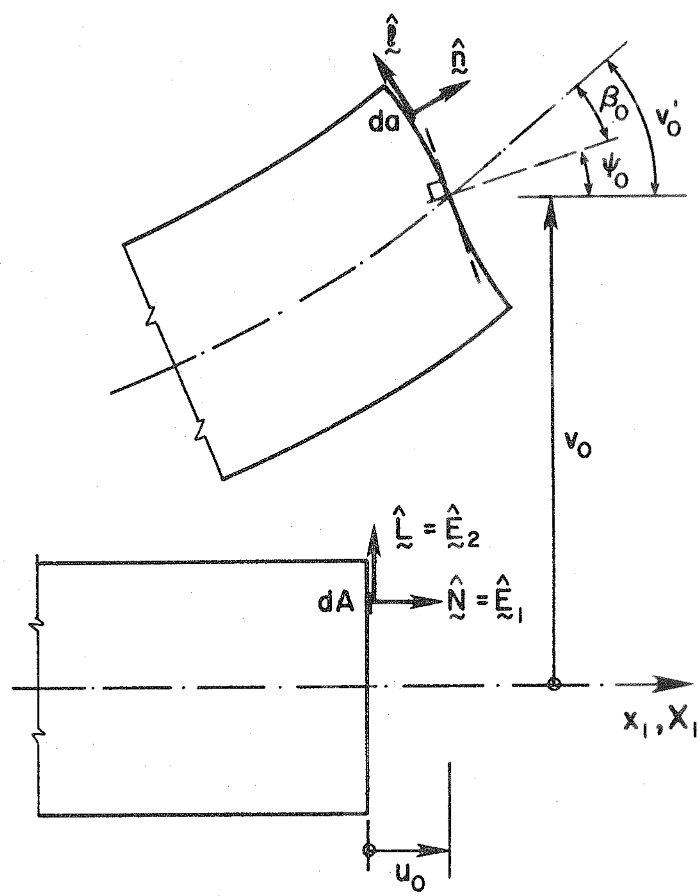


Figure 3.5. The kinematic variables u_o, v_o and ψ_o.

To summarize the situation, either from the solution in [40] or by substitution of (3.46) and (3.49) in the kinematics (3.15), we find that the exact displacement field may be expressed in terms of the variables (3.45) as

$$u_1(\mathrm{x}) = u_o(x_1) - x_2\,\psi_o(x_1) - \frac{4}{3h^2}\,[x_2]^3\left(1 - \frac{\nu}{2(1+\nu)}\right)\beta_0$$

$$u_2(\mathrm{x}) = v_o(x_1) + \frac{1}{2}\,\nu\,x_2^2\psi_o{}'(x_1) - \nu\,x_2\,\bar{u}'(x_1) \tag{3.50}$$

and the constitutive equations (3.11a) and (3.13a) for the bending moment and shear force take, in view of (3.46) and (3.49), the form

$$M = EI\left[\psi_o{}' + \frac{1}{5}\left(1 - \frac{\nu}{2(1+\nu)}\right)\beta_o{}'\right], \qquad V = G\Omega\,\frac{2}{3}\,\beta_o \tag{3.51}$$

Equation (2.51) shows that when the shear force is expressed in terms of β_o the shear coefficient takes the value $\frac{2}{3}$. Finally, the second order approximation to the equilibrium equations takes the form of equation (3.32) where the angle $\bar{\psi} + (1-\kappa)\bar{\beta}$ is replaced by

$$\bar{\psi} + (1-\kappa)\bar{\beta} = \psi_o + \frac{2+\nu}{6(1+\nu)}\,\beta_o$$

As remarked in the previous section, the formulations based upon either choices (a) or (b) of kinematic variables, are entirely equivalent, although the sets $\{\bar{u},\bar{v},\bar{\psi}\}$ and $\{u_o, v_o,\psi_o\}$ have different physical meaning. The distinction between both sets becomes important in the actual modeling of physical boundary conditions. To illustrate the point, let us first assume for simplicity, as in elementary beam theory, that the Poisson ratio $\nu = 0$. Then

$$\bar{v} = v_o\,, \quad \bar{\beta} = \frac{4}{5}\,\beta_o\,, \quad \bar{\psi} = \psi_o + \frac{1}{5}\,\beta_o \tag{3.52}$$

and the linearized problem for the elastica (3.35) takes, in terms of β_o the form

$$\left[1 - \frac{P}{2G\Omega}\right]\bar{v}' + \frac{H}{\frac{2}{3}G\Omega} = \left[1 + \frac{P}{G\Omega}\right]\psi_o$$

$$EI\left[\frac{4}{5}\,\psi_o{}' + \frac{1}{5}\,\bar{v}''\right] = H\,, \qquad X_1 \in (0,L) \tag{3.53}$$

$$\bar{u}' = u_o = 0$$

It follows from these equations that the rotation ψ_o at x_2=0 is governed by exactly the same differential equation (3.36) that $\bar{\psi}$ and $\bar{v}'$ satisfy; i.e:

$$L\,\bar{\psi} = L\,\psi_o = L\,\bar{v}' \equiv H \qquad (3.54a)$$

where the explicit form of the linear operator L is

$$L \equiv \frac{EI\left[1 - \dfrac{P}{5G\Omega}\right]}{1 + \dfrac{P}{G\Omega}}\,\frac{d^2}{dX^2} + P \qquad (3.54b)$$

Example

As a concrete example, consider a cantilever beam subjected to axial and vertical forces P and F applied at the right end $x=L$, respectively. To model the clamped boundary condition at the left end we may assumed that either the average rotation $\bar{\psi}(x)$ or the rotation $\psi_o(x)$ vanishes at $x=0$. For these two possibilities, it is easily found that

(a) the condition $\bar{\psi}(0) = \bar{v}(0) \equiv 0$ leads to a value of the tip deflection

$$\bar{v}(L) = \frac{FL}{P}\left[\frac{1 + \dfrac{P}{G\Omega}}{1 - \dfrac{P}{5G\Omega}}\,\frac{\tan(\lambda L)}{\lambda L} - 1\right]$$

(b) while the condition $\psi_o(0) = \bar{v}(0) \equiv 0$ yields the value

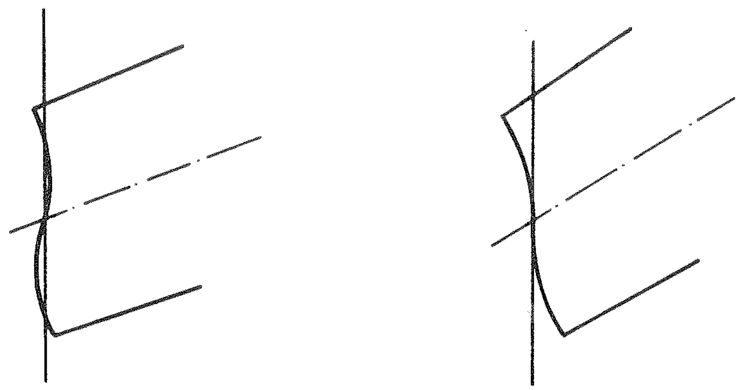
$$\bar{v}(L) = \frac{FL}{P}\left[\frac{1 + \dfrac{P}{G\Omega}}{1 - \dfrac{P}{2G\Omega}}\,\frac{\tan(\lambda L)}{\lambda L} - 1\right]$$

where in both cases

$$\lambda^2 = \frac{\dfrac{P}{EI}\left[1 + \dfrac{P}{G\Omega}\right]}{1 - \dfrac{P}{5G\Omega}} < \left[\frac{\pi}{2L}\right]^2$$

Clearly, the boundary condition in (a) leads to a lower value for the tip deflection. Furthermore, since we always have the bound $P_{crit} < 5G\Omega$, the beam subjected to the boundary condition in (a) can never buckle unless the critical load (3.37) is reached and $\lambda L = \frac{\pi}{2}$. However, for the boundary condition in (b) we obtain the surprising result that, if the shear stiffness $G\Omega$ is small enough so that the critical load (3.37) is greater than $2G\Omega$, the beam may buckle for the lower value $P = 2G\Omega$ of the axial load.

It should be noted that there is no contradiction between the different values for the tip deflection of the cantilever found in (a) and (b). Physically, the boundary condition at x=0 in case (a) is essentially different from that in (b), as shown in Fig.3.6. In fact, the difference in the values found for the tip deflection illustrates the importance a small change in boundary conditions has in the response of beams extremely weak in shear † .



(a) $\bar{\psi}(0) = 0$ (b) $\psi_0(0) = 0$

Figure 3.6. Boundary conditions for a cantilever.

A numerical treatment of this effect in the context of the Finite Element Method can be found in [32].

Nevertheless the question remains regarding which measure of rotation is physically more appropriate for the modeling of the response of a beam. Beam theories are the result of the assumption that resultant stress measures such as N, V and M are acceptable for the physical description of the response of the body. It appears to be physically more appropriate, to relate this average stress measures to average measures of deformation rather than to the deformation at a point. The choice of the set $\{\bar{u}, \bar{v}, \bar{\psi}\}$ of kinematic variables then appears to be physically more compelling.

† Experimental research conducted in the E.E.R.C. University of California, Berkeley, with multilayer elastomeric bearings confirms this result.

CHAPTER 4.

A CONSISTENT THEORY FOR THE ANALYSIS OF MULTILAYER ELASTOMERIC BEARINGS

Introduction

A multilayer elastomeric bearing is a type of composite column consisting of alternate layers of thin rubber perfectly bonded to metal plates. Due to the extremely high values of the ratio $\frac{\textit{Bulk Modulus (K)}}{\textit{Shear modulus (G)}}$ characteristic of natural rubber, such an arrangement prevents the lateral expansion of the rubber layers and, therefore, results in a column capable of withstanding high compressive loads with only a small axial deflection while, at the same time, preserving the low shear resistance of rubber to shearing [48,49].

The two characteristics of high compressive stiffness and low shear stiffness make this type of composite column widely used in applications ranging from their traditional use as vibration mounts, shock absorbers or bridge pads, to their recent application in earthquake engineering as base isolation devices for the protection of buildings against strong ground motions [50]. Although the axial dimension of a typical elastomeric bearing is of the same order of magnitude as its plane dimensions, as shown in Figures 4.1 and 4.3, the low value of its shear stiffness makes the column prone to buckling under compressive load [48].

In the past, the stability analysis of elastomeric bearings has been based upon the approximate linearized theory of buckling developed Haringx [21,22,23], in which the effective compression bending and shear stiffness of the composite column are determined from the so-called apparent compressive bending and shear stiffness of a single column unit [48,49].

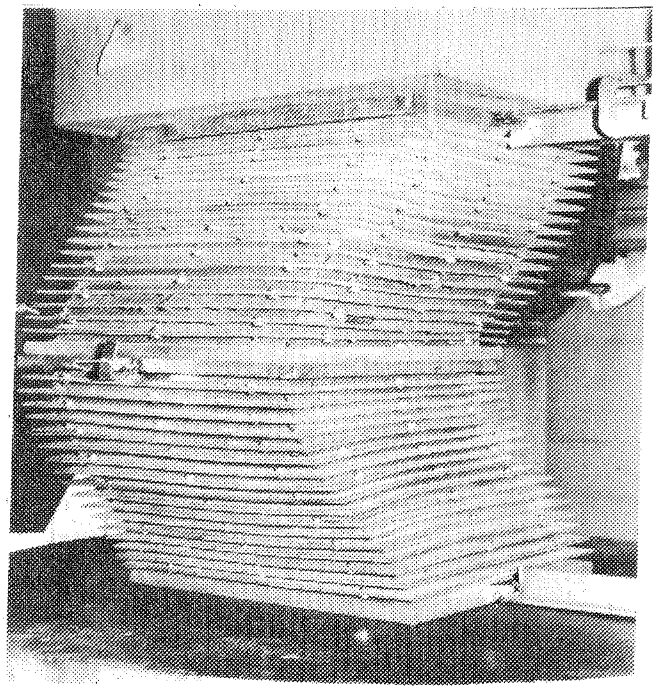


Figure 4.1. Experimental test of a bearing used in base-isolation systems.

Expressions for these elastic constants can be found in the literature [51,52,53]. As shown in [32], Haringx's treatment may be derived as a second order approximation of the non-linear equilibrium equations of finite elasticity under the "plane sections remain plane" assumption. For laminated bearings, this kinematic assumption amounts to considering the plates perfectly rigid so that the axial warping is completely restraint. An extension of Haringx's formulation to the case of plates of an arbitrary shape, not necessarily flat although perfectly rigid, has been reported in [54]. The fully non-linear theory, not restricted to a small angle or small axial displacement approximations, has been derived in Chapter 2 of this work.

The systematic neglection of the finite stiffness of the plate is, therefore, the central assumption in previous analysis of the stability of multilayer elastomeric bearings. In certain applications, particularly in the context of earthquake engineering, this assumption might be quite unrealistic. Fig.4.1 shows an elastomeric bearing typically used in base isolation systems, under test in the E.E.R.C of the University of California, Berkeley. The severe warping experienced by the plates is apparent and a physical explanation for this effect is illustrated in Fig.4.2. Due to the shearing of the column, the top and bottom surfaces of any plate are subjected to a shear stress distribution, *a−priori* unknown, which causes bending of the thin plate. In this chapter, a theory which consistently includes the effect of the finite stiffness of the plate will be developed. Attention is confined to the case of interest in earthquake engineering in which the plates are flat. Our formulation can be outlined as follows.

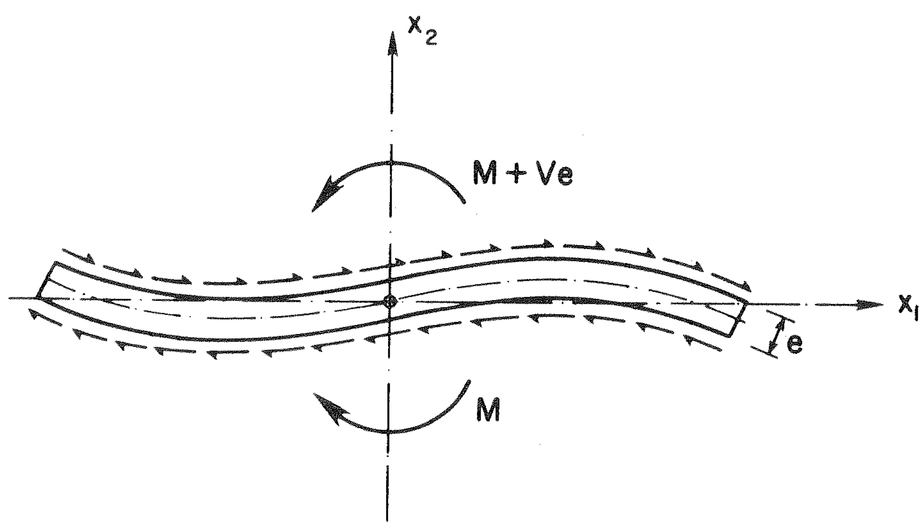


Figure 4.2. Physical motivation for the warping of the plate.

(i) The distribution of shearing stress acting on an arbitrary plate is first obtained by enforcing compatibility of displacements and shear stress between metal plate and rubber layer. The exclusive dependence of the lateral deflection of the bearing on the axial coordinate is the only assumption made beyond those of the linear theory of elasticity. Such an assumption amounts to neglecting the in-plane extension of the plate.

(ii) Integration of the strain-displacement relations leads to an expression of the displacement field which includes a warping function depending upon stiffness of the plate and proportional the amount of shear. Thus, the "plane sections remain plane" assumption no longer holds in the present approach. When attention is confined to the linear theory, the resulting field equations are shown to correspond to a Timoshenko type of beam theory in which the explicit expression of the shear coefficient depends upon the stiffness of the plate. The limiting cases in which the stiffness of the plate tend to either zero or infinity are also examined.

(iii) According to the formulation presented in Chapter 3, a consistent second order approximation to the non-linear equilibrium equations in terms of resultant forces can be developed from the knowledge of the linearized kinematics. The results presented in Chapter 3 are then entirely applicable and lead to an expression for the buckling load which depends on the stiffness of the plate through the shear coefficient. This expression yields values of the buckling load always lower than those predicted by Haringx's formulation. In the limit as the plate becames infinitely stiff, both formulations are shown to coincide.

4.1.- Basic assumptions.

A composite beam in the form of a typical multilayer elastomeric bearing is illustrated in Fig.4.3. The x_1-axis is taken to be the axial axis of the beam. In the analysis of the composite system consisting of the rubber layers and steel plates, we shall introduce the following assumptions:

(a) State of plane strain (or stress). Thus, further reference to the x_3 coordinate will be omitted.

(b) The lateral displacement of the composite system $u_2(x_1,x_2)$ depends solely on the axial coordinate x_1. That is, Poisson's effect is neglected.

With these two assumptions, the displacement field of the composite system takes the form

$$\begin{aligned} u_1(x_1,x_2) &= u(x_1,x_2) \\ u_2(x_1,x_2) &= v(x_1) \end{aligned} \tag{4.1}$$

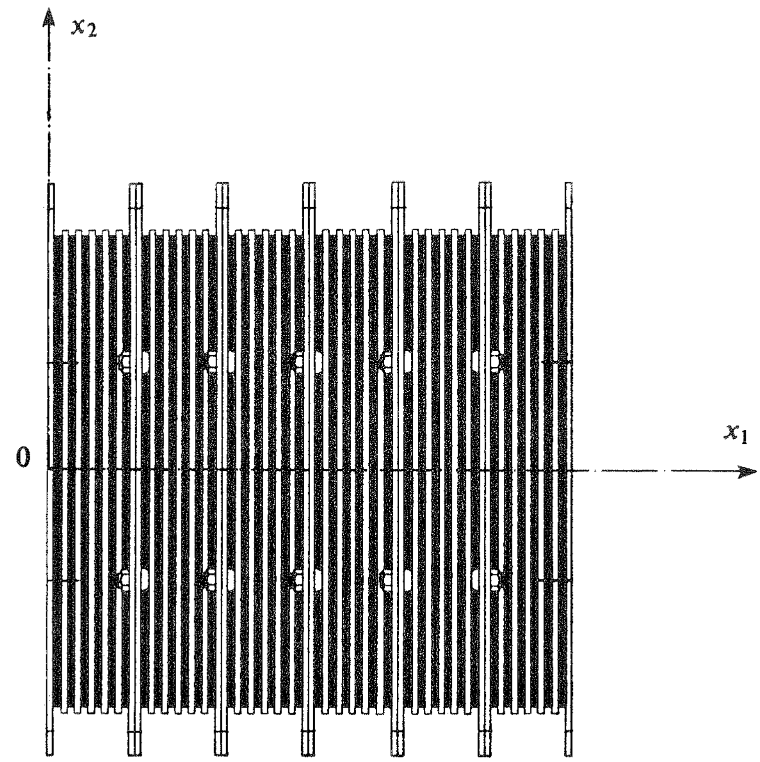


Figure 4.3. A typical multilayer elastomeric bearing.

and the components of the infinitesimal strain tensor ϵ are then

$$\epsilon_{11} = \frac{\partial u}{\partial x_1}$$

$$\epsilon_{12} = \frac{1}{2}[v'(x_1) + \frac{\partial u}{\partial x_2}] \tag{4.2}$$

Let G denote the shear modulus of the rubber. We shall assume the metal plates to be thin enough so that the shearing stress is determined exclusively by the deformation of the rubber. Hence, from (4.2) the constitutive equation for the shear stress is

$$\sigma_{12} = 2G\,\epsilon_{12} = G\,[v'(x_1) + \frac{\partial u}{\partial x_2}] \tag{4.3}$$

By differentiating both sides of equation (4.3) with respect to x_2 it follows that

$$\frac{\partial^2 u}{\partial x_2^2} = \frac{1}{G}\frac{\partial \sigma_{12}}{\partial x_2} \tag{4.4}$$

It will be shown herein that the form of the axial displacement $u(x_1,x_2)$ follows from the compatibility between the rubber pad and metal plate. Notice that no constitutive assumption for the axial stress σ_{11} is introduced for this purpose.

4.2.- Analysis of the plates.

Consider a typical plate of thickness e located at $x_1=\zeta$ as illustrated in Fig.4.4. It will be assumed that the thickness e is small enough so that the usual Kirchhoff's assumption is applicable.

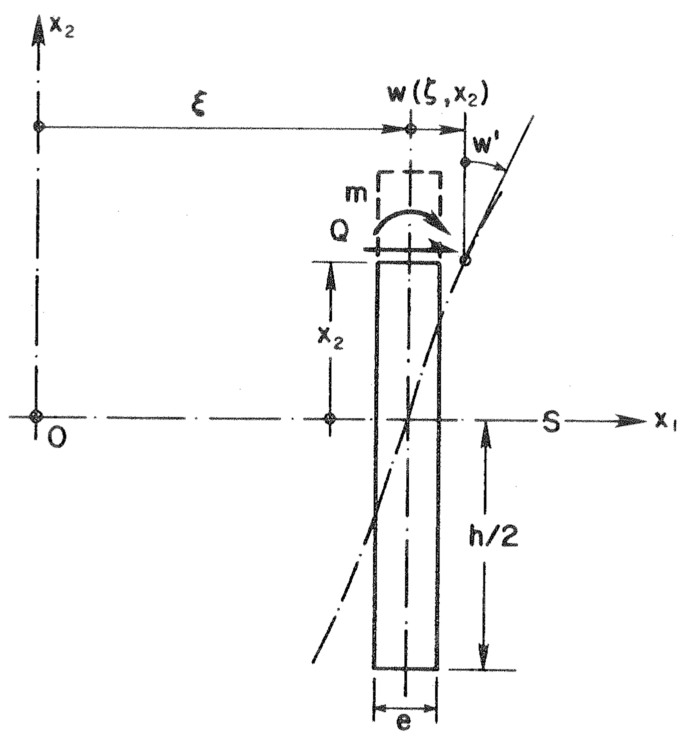


Figure 4.4. Geometry of a typical plate.

No assumption, however, is made regarding the distribution of stresses acting on the top and bottom surfaces. Denoting by $w(\zeta,x_2)$ the deflection of the middle surface, the displacement field is then given by

$$u_1(\zeta,x_2) = w(\zeta,x_2)$$
$$u_2(\zeta,x_2) = -x_1 \frac{dw(\zeta,x_2)}{dx_2} \tag{4.5}$$

and the components of the infinitesimal strain tensor ϵ by

$$\epsilon_{22} = -x_1 \frac{d^2 w(\zeta, x_2)}{dx_2^2}$$

$$\epsilon_{12} = 0 \tag{4.6}$$

Let M and Q be the bending moment and shear force respectively, acting on an arbitrary cross section a distance x_2 from the axial axis x_1. These forces are given by

$$M = -\int_{-\frac{e}{2}}^{\frac{e}{2}} x_1 \, \sigma_{22} \, dx_1 \qquad Q = \int_{-\frac{e}{2}}^{\frac{e}{2}} \sigma_{12} \, dx_1 \tag{4.7}$$

If the Young's modulus and Poisson's ratio of the plate are denoted by E_p and ν_p respectively, the constitutive equation for the bending moment M follows at once from (4.6) and (4.7); i.e:

$$M = D_p \frac{d^2 w(\zeta, x_2)}{dx_2^2} \tag{4.8}$$

where $D_p = \dfrac{E_p \, e^3}{12 \, (1-\nu_p^2)}$ is the bending stiffness of the plate.

Assuming zero body forces, the equations of equilibrium of the plate are given by

$$\sigma_{11,1} + \sigma_{12,2} = 0$$

$$\sigma_{12,1} + \sigma_{22,2} = 0 \tag{4.9}$$

Denote by τ^-,σ^- and τ^+,σ^+ the tangential and normal stresses acting on the surfaces $x_1 = \zeta - \frac{e}{2}$ and $x_1 = \zeta + \frac{e}{2}$, respectively. Integration of the second of eqs. (4.9) through the thickness e of the plate gives

$$\frac{d}{dx_2} \left[\int_{-\frac{e}{2}}^{\frac{e}{2}} \sigma_{22} dx_1 \right] + [\tau^+ - \tau^-] = 0 \tag{4.10}$$

however, by equation $(4.6)_1$

$$\int_{-\frac{e}{2}}^{\frac{e}{2}} \sigma_{22} dx_1 = -E_p \frac{d^2 w(\zeta, x_2)}{dx_2} \int_{-\frac{e}{2}}^{\frac{e}{2}} x_1 \, dx_1 = 0 \tag{4.11}$$

and thus equation (4.10) yields

$$\tau^- = \tau^+ \equiv \tau \tag{4.12}$$

Introducing $\Delta\sigma \equiv \sigma^+ - \sigma^-$, and integration of equations (4.9) through the thickness gives

$$\begin{aligned} &\frac{dM}{dx_2} + Q - e\tau = 0 \\ &\frac{dQ}{dx_2} + \Delta\sigma = 0 \end{aligned} \tag{4.13}$$

from which we obtain

$$\frac{d^2M}{dx_2^2} - e\frac{d\tau}{dx_2} = \Delta\sigma \tag{4.14}$$

Let us consider next the compatibility conditions to be satisfied by the plate. They can be formulated as follows:

(i) $w(\zeta,x_2) = u(\zeta-\frac{e}{2},x_2) = u(\zeta+\frac{e}{2},x_2)$

(ii) $\tau^+ = \tau = \sigma_{12}(\zeta+\frac{e}{2},x_2)$

$\tau^- = \tau = \sigma_{12}(\zeta-\frac{e}{2},x_2)$

(iii) $\sigma^+ = \sigma_{11}(\zeta+\frac{e}{2},x_2)$

$\sigma^- = \sigma_{11}(\zeta-\frac{e}{2},x_2)$

Conditions (i) and (ii) together with equation (4.4) imply

$$\frac{d^2w(\zeta,x_2)}{dx_2^2} = \frac{1}{G}\frac{d\tau}{dx_2} \tag{4.15}$$

Thus, with the aid of the constitutive equation (4.8) and the equilibrium equation $(4.13)_1$, the bending moment M and the shear force Q may be expressed in terms of the tangential stress τ as

$$M = \frac{D_p}{G}\frac{d\tau}{dx_2}, \quad Q = -\frac{D_p}{G}\frac{d^2\tau}{dx_2^2} + e\tau \tag{4.16}$$

and by substitution into equation (4.14) we arrive at

$$L\,\tau := \frac{d^3\tau}{dx_2^3} - [\frac{Ge}{D_p}]\frac{d\tau}{dx_2} = [\frac{Ge}{D_p}]\,\frac{\Delta\sigma}{e} \qquad x_2 \in (-\frac{h}{2},\frac{h}{2}) \tag{4.17}$$

which is the differential equation of equilibrium for the shearing stress

$$\tau = \sigma_{12}(\zeta-\frac{e}{2},x_2) = \sigma_{12}(\zeta+\frac{e}{2},x_2)$$

The key point in determining the shear stress τ is the establishment of proper boundary conditions for equation (4.17). This point will be considered next.

4.2.1.- The boundary value problem.

The distribution of shear stresses on the plate must be such that the resultant moment and shear force at the end of the plate vanish. Thus, we have the boundary conditions

$$M\,\Big|_{x_2=\pm\frac{h}{2}} = 0 \qquad Q\,\Big|_{x_2=\pm\frac{h}{2}} = 0 \tag{4.18}$$

Furthermore, the symmetry of the problem demands the shear stress σ_{12} to be an even function of x_2. This condition together with (4.18) determine, in view of (4.16), the set of boundary conditions

$$\begin{aligned} &\tau(+x_2) = \tau(-x_2) \qquad x_2 \in (-\frac{h}{2},\frac{h}{2}) \\ &\frac{d\tau}{dx_2}\,\Big|_{x_2=\pm\frac{h}{2}} = 0 \\ &[\frac{d^2\tau}{dx_2^2} - \lambda^2\tau]\,\Big|_{x_2=\pm\frac{h}{2}} = 0 \end{aligned} \tag{4.19}$$

where the parameter $\lambda^2 = \dfrac{Ge}{D_p}$ has been introduced for convenience.

However, the boundary value problem posed by equation (4.17) together with the boundary conditions (4.19) does not have unique solution. Moreover, it might even have no solution for an arbitrary function $f(x_2) = \lambda^2\dfrac{\Delta\sigma}{e}$. To see this, let us note that the completely homogeneous problem

$$L\tau_H \equiv \frac{d^3\tau_H}{dx_2^3} - \lambda^2\frac{d\tau_H}{dx_2} \equiv 0 \qquad x_2 \in (-\frac{h}{2},\frac{h}{2}) \tag{4.20}$$

with the homogeneous boundary conditions (4.19), has the solution τ_H = *Constant.* Thus, by Fredholm alternative theorem [55], the problem posed by equation (4.17) with boundary conditions given by (4.19) is undetermined up to an arbitrary constant. Furthermore, for the problem to have solution, the forcing function $f(x_2) = \lambda^2 \frac{\Delta\sigma}{e}$ has to be orthogonal to the solutions of the completely homogeneous problem and, therefore, meet the condition

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \Delta\sigma \; dx_2 = 0 \tag{4.21}$$

which shows that the resultant of the normal stresses acting on the plate must be a moment ΔM, say.

Nevertheless, a unique solution for the problem posed by equation (4.17) with boundary conditions (4.19) can be singled out by requiring that

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau \; dx_2 = V \tag{4.22}$$

where $V = V(\zeta - \frac{e}{2}) = V(\zeta + \frac{e}{2})$ is the resultant shear force acting on both sides $x_1 = \zeta - \frac{e}{2}$ and $x_1 = \zeta + \frac{e}{2}$ of the plate.

Finally, by multiplying equation (4.17) by x_2, integration by parts and enforcement of conditions (4.19) it is found that the resultant moment ΔM is related to the shear force V through the overall equilibrium condition

$$\Delta M \equiv -\int_{-\frac{h}{2}}^{\frac{h}{2}} \Delta\sigma \; x_2 \; dx_2 = -Ve \tag{4.23}$$

In conclusion, we can estate that: *the boundary value problem posed by equation (4.17) with boundary conditions given by (4.19) together with condition (4.22) has a unique solution, provided the normal stresses* $\Delta\sigma$ *satisfy conditions (4.21) and (4.23).*

4.2.2.- Determination of the shear stress τ

A explicit solution for the boundary value problem posed by equation (4.17) with boundary conditions (4.19) and condition (4.22) requires a explicit expression for the resultant normal stresses $\Delta\sigma$ acting on the plate. It will be assumed that they are distributed linearly along the plate. The consistency of this assumption will be assessed later. Conditions (4.21) and (4.23) imply then

$$\Delta\sigma = -\frac{\Delta M}{I}x_2 = \frac{Ve}{I}x_2 \tag{4.24}$$

where $I = \frac{1}{12}h^3$.

The solution of the boundary value problem is then easily found to be

$$\tau = \frac{V}{h}\left\{\frac{3}{2}[1-\frac{4x_2^2}{h^2}] - \frac{3}{\gamma^2}[1-\frac{\gamma}{Sinh\gamma}Cosh(\frac{2\gamma x_2}{h})]\right\} \tag{4.25}$$

where $\gamma \equiv \frac{h}{e}\sqrt{\frac{3G}{E_p}(1-\nu_p^2)}$.

The shear stress τ given by equation (4.25) is plotted versus the parameter γ in Fig.4.5.

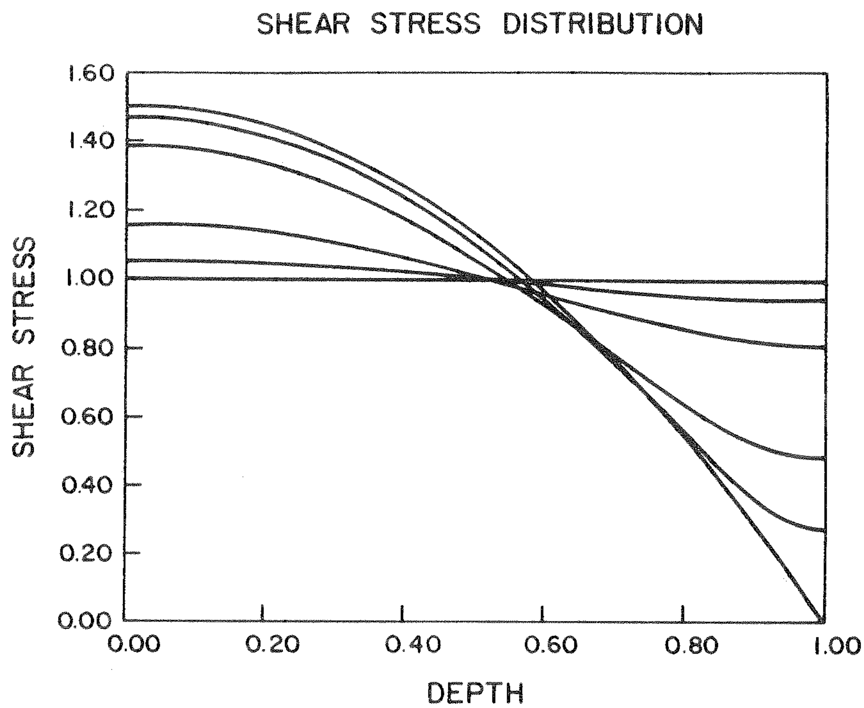


Figure 4.5. Shear stress distribution.

Notice that as the plate becomes more flexible; i.e: $D_p \rightarrow 0$ or $e \rightarrow 0$, the parameter $\gamma \rightarrow \infty$ and the shear stress τ tends to the parabolic distribution $\tau \rightarrow \frac{3V}{2h}[1-\frac{4x_2^2}{h^2}]$. On the other hand, as the plate becomes stiffer $\gamma \to 0$ and the shear stress $\tau \to \frac{V}{h}$; i.e: takes a uniform distribution over the plate.

4.3.- The composite system.

Kinematics

Once the distribution of the shear stress τ is known, the expression for the axial displacement $u_1(x_1,x_2)$ follows at once. In fact, equation (4.3) and the compatibility conditions (ii) lead to

$$\frac{\partial u(x_1,x_2)}{\partial x_2} = -v'(x_1) + \frac{1}{G}\tau \tag{4.26}$$

and the substitution of the expression for τ given by (4.25) yields

$$u_1(x_1,x_2) = \bar{u}(x_1) - x_2 v'(x_1) + \frac{V(x_1)}{Gh}\left\{\frac{3}{2}x_2 - \frac{2}{h^2}x_2^3 - \frac{3}{\gamma^2}[x_2 - \frac{h}{2Sinh\gamma}Sinh(\frac{2\gamma x_2}{h})]\right\} \tag{4.27}$$

where the arbitrary function $\bar{u}(x_1)$ represents the axial displacement of the neutral axis x_1.

To express the displacement field fully in terms of kinematic variables we introduce, as in the previous chapter, the average rotation $\bar{\psi}(x_1)$ of a cross section defined by

$$\bar{\psi}(x_1) = -\frac{1}{I}\int_{-\frac{h}{2}}^{\frac{h}{2}} x_2\, u_1(x_1,x_2)\, dx_2 \tag{4.28}$$

where $I = \frac{1}{12}h^3$ is the moment of inertia of the cross section. The substitution of (4.27) into (4.28) shows that the angle $\bar{\psi}(x_1)$ is related to the resultant shear force $V(x_1)$ by

$$\bar{\psi}(x_1) = v'(x_1) - \frac{V(x_1)}{G\Omega\kappa} \tag{4.29}$$

where the constant κ has the expression

$$\kappa = \frac{\frac{5}{6}}{1 - \frac{5}{2\gamma^2}\left[1 - 3\left(\frac{1}{\gamma\, Tanh\gamma} - \frac{1}{\gamma^2}\right)\right]} \tag{4.30}$$

Equation (4.29) relates, through $G\Omega\kappa$, the resultant shear force $V(x_1)$ acting on a cross section of the composite system to the angle $\bar{\beta}(x_1) \equiv v'(x_1) - \bar{\psi}(x_1)$ difference between the slope of the deformed neutral axis and the average angle rotated by the cross section. Hence, the angle $\bar{\beta}(x_1)$ gives a measure of the average distortion of a cross section due to shear, and the constant $\Omega\kappa$ represents an "effective" shear area. The parameter κ, plotted in Fig.4.6 versus γ, has then an analogous significance to that of the shear coefficient in Timoshenko's beam theory. In fact, in the limit as $\gamma \rightarrow \infty$ and the plate becomes infinitely flexible, it follows from (4.30) that $\kappa \rightarrow \frac{5}{6}$ which is in agreement with the expression (3.45) for the shear coefficient of a rectangular section when $\nu = 0$ [39].

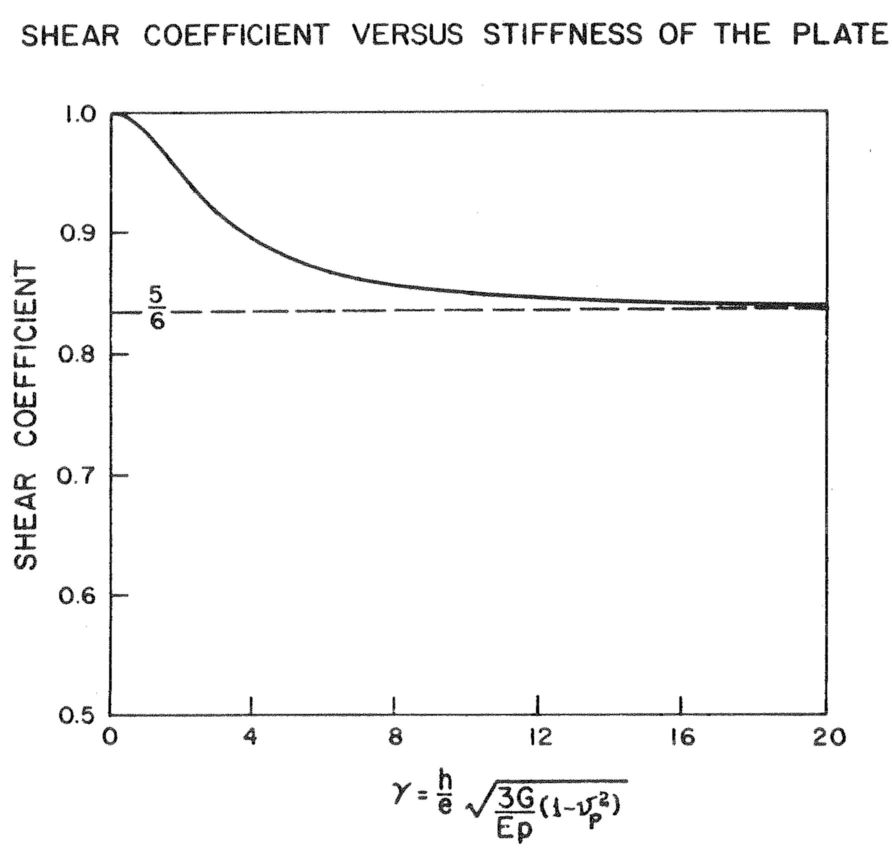


Figure 4.6. Variation of the shear coefficient κ with the stiffness of the plate.

The kinematics of the composite system takes then the following final form

$$u_1(x_2,x_1) = \bar{u}(x_1) - x_2\,\bar{\psi}'(x_1) - \phi(x_2)\,\kappa\,\bar{\beta}(x_1)$$
$$u_2(x_2,x_1) = v(x_1) \tag{4.31}$$

where the shear coefficient κ is given by (4.30), and in addition

$$\phi(x_1) = \left[\frac{1}{\kappa} - \frac{3}{2} + \frac{3}{\gamma^2}\right] x_2 + \frac{2}{h^2}\,x_2^3 - \frac{3h}{2\gamma^2 Sinh(\gamma)}\,Sinh(\frac{2\gamma x_2}{h})$$
$$\gamma = \frac{h}{e}\sqrt{\frac{3G}{E_p}(1-\nu_p^2)} \tag{4.32}$$
$$\bar{\beta}(x_1) = v'(x_1) - \bar{\psi}(x_1)$$

Equation (4.31) has exactly the same form as the displacement field (3.43) with the Poisson ratio $\nu = 0$. The dependence on the stiffness of the plate of the warping function $\phi : (-\frac{h}{2}, \frac{h}{2}) \rightarrow I\!R$ given in the present case by (4.32), is noted.

Constitutive equations

In addition to the constitutive assumption for the shear stress σ_{12} given by equation (4.3), let us consider for the axial stress σ_{11} a relationship of the form

$$\sigma_{11} = E_a\,\epsilon_{11} = E_a\,\frac{\partial u_1(x_1,x_2)}{\partial x_1} \tag{4.33}$$

Since the bending moment $M(x_1)$ acting on an arbitrary cross section is given by

$$M(x_1) = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{11}\,x_2\,dx_2 \tag{4.34}$$

from equations (4.33) and (4.32) and the definition of $\bar{\psi}(x_1)$ given by (4.29) it follows that

$$M(x_1) = E_a I\,\bar{\psi}(x_1) \tag{4.35}$$

Therefore, the elastic constant of the composite system E_a can be chosen as $E_a = \frac{K_b}{I}$, where K_b is the so-called apparent bending stiffness of the system [52,53]. The constitutive equation for the shear force follows immediately from equation (4.29), namely

$$V(x_1) = G\,\Omega\,\kappa\,[v'(x_1) - \bar{\psi}(x_1)] \tag{4.36}$$

where the shear coefficient κ is given by (4.30). The shear modulus G should be replaced by $\frac{G\,e_R}{e_T}$ where e_T is the total height of a single column unit and e_R the height of the rubber pad and one plate [53].

Equilibrium equations. Linear Theory.

The equilibrium equations in terms of the resultant shear force and bending moment, follow by integration of the equilibrium equations of linear elasticity over the cross section in the standard manner. The result is

$$\begin{aligned} &M'(x_1) + V(x_1) = 0 \\ &V'(x_1) + q(x_1) = 0 \end{aligned} \tag{4.37}$$

being $q(x_1)$ the applied transversal load.

The set of equations governing the behavior of the composite system consists of the constitutive equations (4.35) and (4.36) together with the equilibrium equation (4.37). Formally, they correspond to a Timoshenko type of beam theory. However, the displacement field given by (4.31), obtained by enforcing compatibility of stresses and displacements between the steel plate and rubber pad, includes a warping of the cross section which depends upon the stiffness of the plate. The expression for the effective shear coefficient given by (4.30) is then consistently derived from this displacement pattern.

Two limiting cases of particular interest in the theory presented heretofore are possible. They are summarized next.

4.4.- Limiting cases.

(a) *The stiffness of the plate* $D_p \rightarrow \infty$.

Typically, this assumption is always made in the analysis of multilayer elastomeric bearings [52,54] where the plate is assumed to be so stiff as to prevent any possible warping of the cross section of the bearing. However, in some practical applications, such an assumption might be quite unrealistic as illustrated in Fig.4.1.

From equations (4.25), (4.30) and (4.32), it easily follows that the asymptotic expansions as $\gamma \rightarrow 0$ for the shear stress τ, the shear coefficient κ and the warping function $\phi(x_2)$ are

$$\tau(x_1,x_2) = \frac{V(x_1)}{\Omega}\left[1 + \left(2\frac{x_2^4}{h^4} - \frac{x_2^2}{h^2} + \frac{7}{120}\right)\gamma^2\right] + O(\gamma^4)$$

$$\kappa = 1 - \frac{2}{105}\gamma^2 + O(\gamma^4) \qquad (4.38)$$

$$\phi(x_2) = 0 - \left[\frac{2}{5}\frac{x_2^5}{h^4} - \frac{1}{3}\frac{x_2^3}{h^2} + \frac{7}{120}x_2\right]\gamma^2 + O(\gamma^4)$$

Since $\phi(x_2) \to 0$ uniformly in $(-\frac{h}{2},\frac{h}{2})$, in the limit as the plates become infinitely stiff we recover the classical Bernouilli's kinematic assumption. Furthermore, since the shear coefficient takes the value $\kappa = 1$ in the limit as $\gamma \to 0$, our derivation provides a rational justification for the usual choice of $G\Omega$ as the effective shear stiffness of columns with extremely stiff plates. This has been the only case considered in previous formulations, in which the flexibility of the plate is systematically neglected in the analysis of the bearing.

(b) *The case in which either the stiffness D_p of the plate of its thickness tend to zero.*

This case corresponds to that of an homogeneous beam with elastic constants G and $E = E_a$. Again, from equations (4.25), (4.30), and (4.32) we find that in the limit as $\gamma \to 0$ the distribution of shear stresses, the shear coefficient κ and the warping function $\phi(x_1)$ reduce to

$$\tau = \frac{3V(x_1)}{2\Omega}\,[1 - \frac{4x_2^2}{h^2}] \equiv \frac{5}{4}\left[1 - \frac{4x_2^2}{h^2}\right]\bar{\beta}(x_1)$$

$$\kappa = \frac{5}{6} \qquad (4.39)$$

$$\phi(x_2) = \frac{2}{h^2}\,x_2^3 - \frac{3}{10}\,x_2$$

The constitutive equations (4.35) and (4.36) together with the equilibrium equations (4.37) show that in the limit as the thickness of the plate or its stiffness tend to zero, we recover the classical Timoshenko's beam theory with shear coefficient $\kappa = \frac{5}{6}$. Notice however, that the usual "plane sections remain plane" deformation pattern no longer holds in the present approach due to the presence of the warping function $\phi(x_1)$, given by $(4.39)_3$, in the kinematics (4.31). The consideration of the warping of the cross section is precisely what leads to an expression for the shear coefficient. The agreement between the warping function given by $(4.39)_3$ when $\gamma \to \infty$, and equation (3.43) with $\nu = 0$, should be noted.

Remark

An elementary derivation of a beam theory that includes warping of the cross section has been recently reported [56]. Although this theory is claimed to be new, it is in fact completely equivalent to the classical Timoshenko's theory. The source of confusion lies in the erroneous identification made in [56] between the average angle $\bar{\psi}(x_1)$ defined by (4.28) and the rotation $\psi_o(x_1)$ at the neutral axis. Both angles are related by

$$\bar{\psi}(x_1) = \frac{4}{5}\,\psi_o(x_1) + \frac{1}{5}\,v'(x_1) \tag{4.40}$$

and the analysis contained in sections 2.4 and 2.5, show that this distinction is only relevant to the enforcement of boundary conditions.

4.5.- The Non-Linear Theory.

4.5.1.- The second order approximation.

Inasmuch as the displacement field derived in section 4.3 is exactly of the same form as equation (3.15) considered in the previous chapter, the results of that chapter entirely apply to the large displacement analysis of multilayer elastomeric bearings. The Kinematics of the formulation heretofore presented, as well as the second order approximation to the non-linear equilibrium equations, are then summarized in Table 4.1 at the end of this Chapter.

4.5.2.- The Eigenvalue Problem for the Critical Load.

Again the formulation presented in section 3.4 is entirely applicable. Nevertheless, for the case under consideration of an elastomeric bearing some remarks are in order.

(a) The inextensibility assumption expressed by $\bar{u}'(X_1) \approx 0$, is particularly accurate for the case of an elastomeric bearing. This type of column is regarded as incompressible in the axial direction for all practical purposes.

(b) The exclusive dependence of the lateral deflection of the bearing on the axial coordinate, was the only assumption made to derive the linearized displacement field. This assumption is equivalent, for closely spaced metal plates, to neglecting the in-plane extension of the plate. Due to the high values the ratio $\frac{E_p}{E_a}$ takes in most applications, such an assumption is quite accurate.

Accordingly, the eigenvalue problem arising from

$$L\,\bar{\psi}(X_1) = H\ , \qquad L\,v'(X_1) = H \qquad X_1 \in (0,L) \tag{4.45}$$

where the operator L is defined by

$$L \equiv \frac{K_b\left[1 - (1-\kappa)\dfrac{P}{G\Omega\kappa}\right]}{1 + \dfrac{P}{G\Omega}}\,\frac{d^2}{dX_1^2} + P \tag{4.46}$$

with K_b being the apparent stiffness of the column, is expected to yield values of the critical load accurate enough for all practical purposes. The expression of our critical load in terms of Euler's buckling load is given by equation (3.37) and has been compared in Table 3.1 with other proposed formulations. For multilayer elastomeric bearings, the bending stiffness appearing in the expression of Euler's buckling load is the apparent stiffness K_b of the column which can be determined either from experimental testing or from analytical expressions [53].

As pointed out before, the proposed expression for the buckling load is in agreement with that due to Haringx, which neglects the the flexibility of the metal plates, in the event of extremely stiff plates for which $\kappa = 1$.

TABLE 4.1: Summary of the Formulation.

Kinematics

Displacement field

$$x_1 = X_1 + \bar{u}(X_1) - X_2\,\bar{\psi}(X_1) - \phi(X_2)\,\kappa\,\bar{\beta}(X_1)$$

$$x_2 = X_2 + v(X_1)$$

$$\gamma = \frac{h}{e}\sqrt{\frac{3G}{E_p}(1-\nu_p^2)}$$

$$\kappa = \frac{\frac{5}{6}}{1 - \frac{5}{2\gamma^2}\left[1 - \frac{3}{\gamma}\left(\frac{1}{Tanh(\gamma)} - \frac{1}{\gamma}\right)\right]}$$

$$\phi(X_2) = \left[\frac{1}{\kappa} - \frac{3}{2} + \frac{3}{\gamma^2}\right] X_2 + \frac{2}{h^2}\,X_2^3 - 3\frac{h}{2\gamma^2 Sinh(\gamma)} Sinh\left(\frac{2\gamma X_2}{h}\right)$$

Gaussian frame

$$\begin{Bmatrix}\hat{\mathbf{n}}\\ \hat{\mathbf{i}}\end{Bmatrix} = \Lambda^T(\mathbf{X})\begin{Bmatrix}\hat{\mathbf{e}}_1\\ \hat{\mathbf{e}}_2\end{Bmatrix} \qquad \Lambda^T = \Lambda^{-1}$$

$$\Lambda^T(\mathbf{X}) = 1 + \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\bar{\psi}(X_1) + \begin{bmatrix}0 & \phi'(X_2)\\ -\phi'(X_2) & 0\end{bmatrix}\kappa\,\bar{\beta}(X_1)$$

Equilibrium Equations.

Linear Momentum

$$-\begin{Bmatrix}P\\ H\end{Bmatrix} - \int_{\xi=0}^{X_1}\begin{Bmatrix}0\\ q(\xi)\end{Bmatrix} d\xi = \begin{Bmatrix}N\\ V\end{Bmatrix} + \begin{Bmatrix}-V\\ N\end{Bmatrix}\bar{\psi}(X_1) + \begin{Bmatrix}0\\ N\end{Bmatrix}(1-\kappa)\,\bar{\beta}(X_1)$$

Angular Momentum

$$0 = M' + v'\,P - [1+\bar{u}']\left[H + \int_0^L q(\xi)\,d\xi\right]$$

or, alternatively

$$0 = M' + V + [1+\bar{u}']\,V - \kappa\bar{\beta}'\,N$$

CHAPTER 5.

THE EFFECT OF WARPING IN THE LINEAR AND NON-LINEAR THEORIES OF ELASTIC PLATES.

5.1.- Introduction.

In this Chapter, the formulations presented in Chapters one and three are extended to examine the influence of the transversal warping, which necessarily appears as a result of shear deformation, in the linear and no-linear elastic response of plates. The linear theory and a second order approximation to the non-linear theory will in turn be considered. Our approach essentially follows that presented in Chap. 3, namely:

(i) An expression for the displacement field of the plate is first derived in the context of the linear theory. Since exact three-dimensional solutions are not available, as opposed to the situation encountered in the analysis of Chap.3, use is made of Kantorovich's method to derive an approximate displacement field for which plane sections parallel to the coordinate planes x_3-x_α, $(\alpha=1,2)$ no longer remain plane.

(ii) With the explicit expression for the displacement field at our disposal, the consistent linear theory and a second order approximation to the non-linear theory are systematically derived. To develop the latter, the general non-linear equilibrium equations for plates are first examined. These equations are derived without introducing any kinematic assumption from the three dimensional non-linear equilibrium equations.

The conclusions obtained in the present analysis are to a large extent analogous to those found in Chap.3. For beams, the consideration of the axial warping of the cross section led, in the context of the linear theory, to a consistent derivation of the well-known Timoshenko's beam theory together with an explicit expression for the shear coefficient. It will be seen that the analogous result for plates is a very simple derivation of the theory due to Reissner [57,58]. In the present approach, the assumptions made in [57,58] regarding the stress distribution across the thickness appear naturally as a result of the proposed displacement field.

However, a formulation which appear to be new is obtained, as in Chap.3, when attention is focussed on the non-linear theory. The formulation presented leads to a non-linear plate theory for which, due to the effect of shear deformation and subsequent transversal warping, the normal stress resultants over the thickness of the plate are no longer normal to the planes of bending. This formulation is further simplified by introducing an additional assumption which allows the definition of an Airy stress function as potential for the in-plane forces. As a result, the response of the plate is governed by a coupled system of three non-linear partial differential equations which, as opposed to the classical Von-Karman model, includes the effects of shear deformation and transversal warping of the plate. Furthermore, the proposed formulation reduces to the Von-Karman model in the limit as the shear stiffness of the plate tends to infinity, and reproduces Reissner theory if the non-linear effect of the in-plane forces is neglected.

Derivation of non-linear plate theories from the general three dimensional non-linear theory have been considered in [59,60,61]. However, a theory analogous to the Von-Karman model, thorough treated in [62], which consistently includes the effect of shear deformation and subsequent transversal warping of the plate has, to the knowledge of the author, not been yet proposed.

5.2.- The Displacement Field.

Consider a plate with thickness h, whose middle plane coincides with the $x_1 - x_2$ coordinate plane. Let us introduce the kinematic variables $\{\bar{u}_\alpha(x_1,x_2);\ \tilde{\psi}_\alpha(x_1,x_2)\}_{\alpha=1,2}$, defined by

$$\bar{u}_\alpha(x_1,x_2) := \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} u_\alpha(\mathbf{x})\, dx_3 \;, \quad \bar{\psi}_\alpha(x_1,x_2) := -\frac{1}{I}\int_{-\frac{h}{2}}^{\frac{h}{2}} x_3\, u_1(\mathbf{x})\, dx_3 \tag{5.1}$$

where $I = \frac{1}{12}h^3$. The physical meaning of these variables is analogous to that of the kinematic variables introduced in Chap.3, section 3.1, for the analysis of beams.

According to Kantorovich's method, the displacement field is to be expressed as an expansion of the type (3.39). With the experience gained from our previous analysis of beams, we consider a truncated expansion of the form

$$\begin{aligned} u_\alpha(\mathbf{x}) &= \bar{u}_\alpha(x_1,x_2) - x_3\,\bar{\psi}_\alpha(x_1,x_2) - \phi(x_3)\,\kappa\,\bar{\beta}_\alpha(x_1,x_2) \quad (\alpha{=}1,2) \\ u_3(\mathbf{x}) &= w(x_1,x_2) \end{aligned} \tag{5.2}$$

where $\kappa\,\phi(x_3)$ is a polynomial of third degree and $\bar{\beta}_\alpha(x_1,x_2)$, $(\alpha{=}1,2)$ are as yet unknown functions. In view of definitions (5.1), the polynomial $\kappa\,\phi(x_3)$ must satisfy the conditions

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \phi(x_3)\, dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3\,\phi(x_3)\, dx_3 \equiv 0 \tag{5.3}$$

which suffice to determine $\kappa\,\phi(x_3)$ up to the arbitrary constant κ; i.e:

$$\kappa\,\phi(x_3) = \kappa\left[\frac{2}{h^2}x_3^3 - \frac{3}{10}x_3\right] \tag{5.4}$$

Conditions (5.3) show that the coordinate functions $\{1, -x_3, -\phi(x_3)\}$ in the expansion (5.2) are orthogonal in $(-\frac{h}{2}, \frac{h}{2})$ in the L_2 sense. Thus the choice of kinematic variables defined by (5.1) is not only physically compelling but mathematically convenient.

To determine the functions $\bar{\beta}_\alpha(x_1,x_2)$ we can resort to the minimization process described in sec.3.5. However, as noted by Kantorovich in [43], it is often advantageous to enforce at the outset the boundary conditions at $x_3 = \pm\frac{h}{2}$ so that the functions $\bar{\beta}_\alpha$ are further restricted. For the expansion (5.2), the enforcement of the stress boundary conditions at $x_3 = \pm\frac{h}{2}$ suffices to completely determine $\bar{\beta}_\alpha(x_1,x_2)$ and the constant κ. In fact assuming, without lost of generality, the conditions

$$\sigma_{3\alpha}\Big|_{x_3=\pm\frac{h}{2}}=0 \quad => \quad \epsilon_{3\alpha}\Big|_{x_3=\pm\frac{h}{2}}=0 \qquad (\alpha=1,2) \tag{5.5}$$

it follows, in view of (5.2), the explicit expressions

$$\bar{\beta}_\alpha(x_1,x_2) = w_{,\alpha}(x_1,x_2) - \bar{\psi}_\alpha(x_1,x_2)$$
$$\kappa = \frac{5}{6} \tag{5.6}$$

Therefore, the introduction of (5.2) reduces the problem to the determination of the functions $\{\bar{u}_\alpha; \bar{\psi}_\alpha\}_{\alpha=1,2}$ and $w(x_1,x_2)$, the functions $\bar{\beta}_\alpha$, $(\alpha=1,2)$ being linearly dependent by virtue of equation (5.6). We next show that the displacement field (5.2) leads, in the context of the linear theory of elasticity, to what we believe is the simplest derivation of Reissner theory yet proposed.

5.3.- The Linear Theory.

5.3.1.- Equilibrium Equations.

The integration of the local form of the linearized equations of equilibrium

$$\sigma_{\alpha\beta,\beta} + \sigma_{\alpha 3,3} + b_\alpha = 0 \qquad (\alpha,\beta=1,2)$$
$$\sigma_{3\alpha,\alpha} + \sigma_{33,3} + b_3 = 0 \tag{5.7}$$

leads, when the body forces are assumed to vanish, to the usual global form of the equilibrium equations, namely

$$N_{\alpha\beta,\beta} = 0$$
$$V_{\alpha,\alpha} + q = 0 \quad (\alpha,\beta=1,2), \quad (x_1,x_2)\in\Omega \tag{5.8}$$
$$M_{\alpha\beta,\beta} + V_\alpha = 0$$

where, by definition $q(x_1,x_2) = \sigma_{33}\Big|_{x_3=-\frac{h}{2}} + \sigma_{33}\Big|_{x_3=+\frac{h}{2}}$, and $M_{\alpha\beta}$, V_α, $N_{\alpha\beta}$ have the usual meaning, that is

$$N_{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta}dx_3\,, \quad M_{\alpha\beta} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} x_3\,\sigma_{\alpha\beta}dx_3\,, \quad V_\alpha = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{3\alpha}dx_3 \quad (\alpha,\beta=1,2) \tag{5.9}$$

5.3.2.- Constitutive Equations.

The Constitutive equations for a linear isotropic material may be expressed as

$$\sigma_{\alpha\beta} = \left[\frac{E\nu}{1-\nu^2}\,\epsilon_{\gamma\gamma} + \frac{\nu}{1-\nu}\,\sigma_{33} \right] \delta_{\alpha\beta} + 2G\,\epsilon_{\alpha\beta}$$

$$\sigma_{3\alpha} = 2G\,\epsilon_{3\alpha} \qquad (\alpha,\beta=1,2) \qquad (5.10)$$

Equation $(5.10)_2$ together with the kinematics (5.2) determine the distribution of shear stresses across the thickness h of the plate as

$$\sigma_{\alpha 3} = G\,\frac{5}{4}\left[1 - \left(\frac{2x_3}{h}\right)^2\right] \bar{\beta}_\alpha(x_1,x_2) \qquad -\frac{h}{2} \leqslant x_3 \leqslant \frac{h}{2} \qquad (5.11)$$

and the integration the shear stress distribution (5.11) over the thickness leads to the constitutive equation for the shear forces

$$V_\alpha = G\,h\frac{5}{6}\,\bar{\beta}_\alpha \equiv G\,h\frac{5}{6}\,[w,_\alpha - \bar{\psi}_\alpha] \qquad (5.12)$$

Before proceeding further, it is noted that the distribution of the stress component σ_{33} over the thickness is completely determined exclusively from equilibrium considerations. In fact, the local equilibrium equation $(5.7)_2$ together with (5.11) and (5.12) imply that

$$\frac{\partial \sigma_{33}}{\partial x_3} = -\sigma_{3\alpha,\alpha} = -\left[1 - \left(\frac{2x_3}{h}\right)^2\right] \frac{V_{\alpha,\alpha}}{\frac{2h}{3}}$$

and integration of this equation over the thickness leads, after use is made of the global equilibrium equation $(5.8)_2$, to

$$\sigma_{33}(x_1,x_2) = q(x_1,x_2)\,\frac{3}{4}\left[\left(\frac{2x_3}{h}\right) - \frac{1}{3}\left(\frac{2x_3}{h}\right)^3\right] \qquad -\frac{h}{2} \leqslant x_3 \leqslant \frac{h}{2} \qquad (5.13)$$

The distribution of the stress components $\sigma_{3\alpha}$ and σ_{33} over the thickness of the plate is depicted in Fig.5.1

The resultant moments over the thickness of the plate are defined by equation $(5.9)_2$. Therefore, the corresponding equilibrium equations follow at once from the local constitutive equation $(5.10)_1$, the kinematics (5.2) and the derived expression (5.13) for the stress component σ_{33} in terms of the applied load $q(x_1,x_2)$. The result may be written as

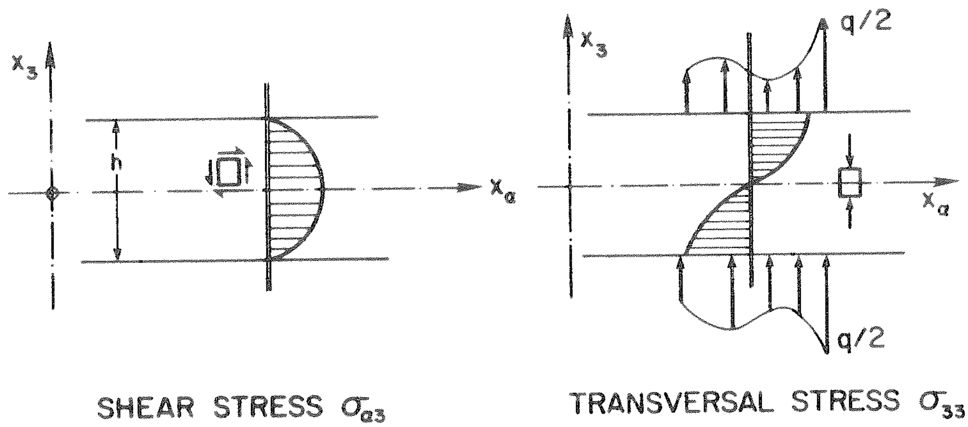


Figure 5.1. Shear and transversal stress distributions.

$$M_{\alpha\beta} = D\left[\nu\bar{\psi}_{\gamma,\gamma}\delta_{\alpha\beta} + (1-\nu)\ \frac{1}{2}\left(\bar{\psi}_{\alpha,\beta}+\bar{\psi}_{\beta,\alpha}\right)\right]$$

$$- D\frac{\nu}{2}\ \frac{q}{G\frac{5}{6}\ h}\ \delta_{\alpha\beta} \qquad (\alpha,\beta=1,2) \qquad (5.14)$$

where $D = \dfrac{E\ h^3}{12(1-\nu^2)}$.

Constituitive equations for the in-plane forces $N_{\alpha\beta}$ defined by $(5.9)_1$, may be established by the same procedure of integration over the thickness of the local form $(5.10)_1$ of the constitutive equations. For future reference we state the final result

$$N_{\alpha\beta} = \frac{Eh\nu}{1-\nu^2}\bar{\epsilon}_{\gamma,\gamma}\ \delta_{\alpha\beta} + 2Gh\ \bar{\epsilon}_{\alpha\beta} \qquad (5.15a)$$

where we have set

$$\bar{\epsilon}_{\alpha\beta} = \frac{1}{2}\ [\bar{u}_{\alpha,\beta} + \bar{u}_{\beta,\alpha}] \qquad (5.15b)$$

5.3.3.- The Equations of the Linear Theory.

(a) *First Approach*

Combining the constitutive equations (5.12) and (5.14) for shear forces and resultant moments, respectively, and making use of the equilibrium equation $(5.8)_2$, the variables $\bar{\psi}_\alpha$ can be eliminated to obtain

$$M_{\alpha\beta} = D\left[\nu\, w_{,\gamma\gamma}\delta_{\alpha\beta} + (1-\nu)\, w_{,\alpha\beta}\right] - \frac{h^2}{5}\,\frac{V_{\alpha,\beta}+V_{\beta,\alpha}}{2} + \frac{\nu h^2}{10(1-\nu)}\, q\, \delta_{\alpha\beta}$$

The substitution of this equation into the moment equilibrium equation $(5.8)_1$ yields the system of partial differential equations

$$V_\alpha - \frac{h^2}{10}\Delta V_\alpha + \frac{\nu h^2}{10(1-\nu)}\, q_{,\alpha} = -D\,\Delta w_{,\alpha} \quad (\alpha=1,2) \tag{5.16}$$

where use has been made of the equilibrium equation $(5.8)_2$.

Equations (5.16) are exactly the same as those first derived by Reissner, employing a variational procedure, in the original reference [58]. These equations can be further simplified by introducing, as noted in [57,58], a stress function $\chi : \Omega \rightarrow \mathbb{R}$ so that the equilibrium equation $(5.8)_2$ be satisfied for $q \equiv 0$.

(b) *Second Approach.*

Alternatively, the following procedure is often useful and plays a key role in the non-linear theory discussed in section 5.4.6. Let us introduce the function $w^S(x_1,x_2) : \Omega \rightarrow \mathbb{R}$ satisfying the conditions

$$\frac{\partial w^S(x_1,x_2)}{\partial x_\alpha} = \bar{\beta}_\alpha(x_1,x_2)\,, \qquad (\alpha=1,2) \tag{5.17a}$$

In addition, we define $w^B(x_1,x_2) = w - w^S$. Thus, in view of $(5.6)_2$ it follows that

$$\frac{\partial w^B(x_1,x_2)}{\partial x_\alpha} = \bar{\psi}_\alpha(x_1,x_2)\,, \qquad (\alpha=1,2) \tag{5.17b}$$

The physical meaning of the functions $w^B(x_1,x_2)$ and $w^S(x_1,x_2)$ is clear. The former represents the partial deflection of the plate due to bending, while the latter gives the deflection due to shear. In terms of these functions, the constitutive equations (5.12) and (5.14) for the shear force and bending moment take the form

$$M_{\alpha\beta} = D\left[\nu\, w^B{}_{,\gamma\gamma}\,\delta_{\alpha\beta} + (1-\nu)\, w^B{}_{,\alpha\beta}\right] - D\,\frac{q}{Gh\kappa}\,\frac{\nu}{2}\,\delta_{\alpha\beta}$$

$$V_\alpha = Gh\kappa\, w^S{}_{,\alpha}\,, \qquad \kappa = \frac{5}{6} \tag{5.18a}$$

and the substitution into the equilibrium equations (5.8) yields the uncoupled system of equations

$$\Delta^2 w^B = \frac{q}{D} + \frac{\Delta q}{Gh\kappa}\,\frac{\nu}{2}$$

$$\Delta w^S = -\frac{q}{Gh\kappa}\,, \qquad (x_1, x_2) \in \Omega \tag{5.18b}$$

The solution procedure based upon equations (5.18) is often referred to as "method of split rigidities". The first of equations (5.18b) differs from the usual Kirchhoff equations for thin plates by the term $\nu\psi/2Gh\kappa$ which appears as a result of fact that σ_{33} is not zero but given by (5.13).

The results presented in this section show that the inclusion of the transversal warping of the plate does not lead to essentially new results, except for the explicit expression (5.2) for the displacement field, when the analysis is restricted to the framework of the classical linear theory. Rather, it furnishes a simple and at the same time consistent derivation of a well known plate theory due to Reissner. However, essentially new results can be obtained in the context of the non-linear theory considered next.

5.4.- The Non-Linear Theory.

We shall examine first a form of the general non-linear equilibrium equations particularly convenient for the analysis of plates. These equations are developed essentially by an extension of the procedure employed in Chapter 1 to establish analogous equilibrium equations for the analysis of beams.

5.4.1.- The Non-Linear Equilibrium Equations.

The point of departure, as in Chap.1, is furnished by the three dimensional equilibrium equations

$$DIV\ \mathbf{P}(\mathbf{X}) + \rho_{Ref} B(\mathbf{X}) = 0$$
$$\mathbf{P}\,\mathbf{F}^T - \mathbf{F}\,\mathbf{P}^T = 0 \qquad \mathbf{X} \in \Omega \times (-\frac{h}{2}, \frac{h}{2}) \tag{5.19}$$

where $\Omega \subset \mathbb{R}^2$ is the domain expanded by the middle plane of the undeformed plate, a bounded open set with smooth boundary $\partial\Omega$. The material coordinates are again designated by $\{X^I\}$ with the X^3-axis directed along the thickness h of the plate; $-\frac{h}{2} \leqslant X^3 \leqslant \frac{h}{2}$. The corresponding basis $\{\hat{E}_I\}$ is the standard basis in $\mathbb{R}^3$.

(i) *Balance of Linear Momentum*

The integration of the equilibrium equations $(5.19)_1$ over the thickness h over plate together with Green's formula leads to

$$\frac{\partial}{\partial X^A} \int_{-\frac{h}{2}}^{\frac{h}{2}} P^{aA}(\mathbf{X})\, dX^3 = 0 \qquad (a, A{=}1,2)$$
$$\frac{\partial}{\partial X^A} \int_{-\frac{h}{2}}^{\frac{h}{2}} P^{3A}(\mathbf{X})\, dX^3 + q(X^1, X^2) = 0 \tag{5.20}$$

(ii) *Balance of Angular Momentum.*

The components of the average displacement $\bar{\mathbf{u}}(\mathbf{X})$ over the thickness h of the plate are given by

$$\bar{u}^a(X^1, X^2) := \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} u^a(\mathbf{X})\, dX^3, \qquad w(X^1, X^2) := \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} u^3(\mathbf{X})\, dX^3 \tag{5.21}$$

Therefore, the position of the average middle surface of the deformed plate is determined by the map $\bar{\mathbf{x}} = \bar{\Phi}(X^1, X^2)$ defined by

$$\bar{x}^1 = X^1 + \bar{u}^1(X^1, X^2), \quad \bar{x}^2 = X^2 + \bar{u}^2(X^1, X^2), \quad \bar{x}^3 = X^3 + w(X^1, X^2) \quad (X^1, X^2) \in \Omega \tag{5.22}$$

By regarding $X^A = Constant$, $(A=1,2)$, in (5.22) we obtain the position of the centroid of deformed surfaces which correspond to planes perpendicular to the X^A-axis in the undeformed configuration $\Omega \times (-\frac{h}{2}, \frac{h}{2})$ of the plate. The moment $\mathbf{M}^A$ acting on such surfaces is then given by

$$\mathbf{M}^A = \int_{-\frac{h}{2}}^{\frac{h}{2}} [\Phi(\mathbf{X}) - \bar{\mathbf{x}}]|_{X^A=Const.} \times (\mathbf{P}\,\hat{\mathbf{E}}_A)\, dX^3 \tag{5.23}$$

For simplicity in the notation, the subscript $X^A = Const.$ will be understood when appropriate and is dropped in the sequel. Equation (5.23) may be written in components as

$$\hat{M}^A_{a.} = \epsilon_{aij} \int_{-\frac{h}{2}}^{\frac{h}{2}} [x^i - \bar{x}^i]\; P^{jA} dX^3 \tag{5.24}$$

where the notation $\hat{M}^A_{a.}$ is used to distinguish the components of $\mathbf{M}^A$ as given by (5.24) from those usually employed in plate theory which are designated by M^{aA}. Figure 5.2 illustrates the difference between both sets of components which are related by

$$M^{aA} = -\epsilon^{ab3} \hat{M}^A_{b.} \tag{5.25}$$

The balance of angular momentum equation can then be easily established from equation (5.24). Proceeding as in Chap.1, differentiation of (5.24) and application of the equilibrium equations (5.20) leads to (summation convention is enforced throughout and A=1,2; I=1,2,3)

$$\frac{\partial \hat{M}^A_{a.}}{\partial X^A} = \epsilon_{aij} \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{\partial x^i}{\partial X^A} - \frac{\partial \bar{x}^i}{\partial X^A} \right) P^{jA} dX^3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} [x^i - \bar{x}^i]\, \frac{\partial P^{jA}}{\partial X^A} dX^3 \right]$$

$$= \epsilon_{aij} \frac{\partial x^i}{\partial X^A} \int_{-\frac{h}{2}}^{\frac{h}{2}} P^{jA} dX^3 + \epsilon_{aij} \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} F^i{}_I\, P^{jI} dX^3 - [x^i - \bar{x}^i]\; P^{j3}\Big|_{\pm\frac{h}{2}} - \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho_{Ref} B^j dX^3 \right]$$

Since $[\mathbf{x} - \bar{\mathbf{x}}] \times (\mathbf{P}\,\hat{\mathbf{E}}_1)|_{X^3=\pm\frac{h}{2}} = 0$, and we may assume without lost of generality that $\mathbf{B} = 0$, the substitution of the balance of angular momentum equation $(5.19)_2$ yields

$$\frac{\partial \hat{M}^A_{a.}}{\partial X^A} + \epsilon_{aij} \frac{\partial \bar{x}^i}{\partial X^A} \int_{-\frac{h}{2}}^{\frac{h}{2}} P^{jA} dX^3 = 0 \tag{5.26}$$

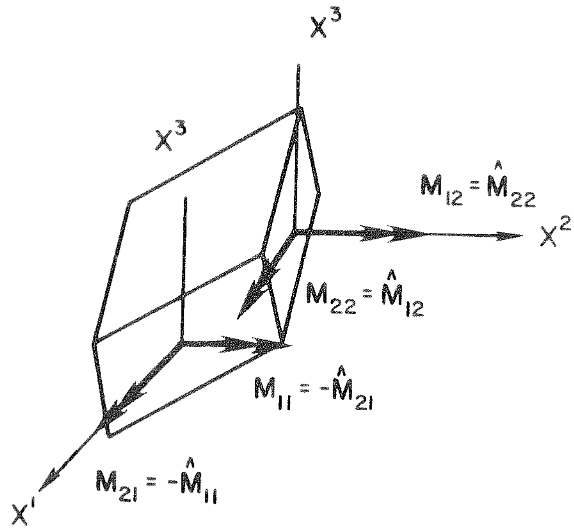


Figure 5.2. Components of the moment $\mathbf{M}^A$.

Making use of (5.25), equation (5.26) can be recast in terms of the components M^{aA} shown in Fig.5.2, which are those typically employed in plate theory. If, in addition, use is made of the definition (5.22) for the components $\bar{x}^i$, (i=1,2,3), equation (5.26) takes the final form

$$\frac{\partial M^{aA}}{\partial X^A} + \frac{\partial w}{\partial X^A} \int_{-\frac{h}{2}}^{\frac{h}{2}} P^{aA} \, dX^3 - \left[\delta^a{}_A + \frac{\partial \bar{u}^a}{\partial X^A} \right] \int_{-\frac{h}{2}}^{\frac{h}{2}} P^{3A} \, dX^3 = 0 \,, \quad (a, A{=}1,2) \tag{5.27}$$

Equations (5.20) and (5.27) comprise the complete system of equilibrium equations for a plate of thickness h undergoing finite deformation. It is noted that no kinematic assumption is needed to obtain these equations, as the derivation presented shows. For a general non-linear elastic material with constitutive equations $\mathbf{P} = \dfrac{\partial W(X, \mathbf{F})}{\partial \mathbf{F}}$, these equations could be used in a numerical treatment by a Finite Element technique. However, if an approximate kinematic assumption such as our proposed expression (5.2) which includes the effect of transversal warping of the plate is introduced, a second order approximation to these non-linear equilibrium equations can be systematically derived. To explicitly state this second order approximation in terms of resultant axial and shear forces we need some geometric preliminaries which will be considered next. Our approach will be similar to that discussed in chapter 3 in the analysis of beams.

5.4.2.- Geometry of the Deformed Plate.

Consider again sections of the plate in the undeformed configuration $\Omega \times (-\frac{h}{2}, \frac{h}{2})$ parallel to the coordinate planes $X^A - X^3$, $(A=1,2)$. These sections are mapped onto surfaces $\mathbf{x}^A = \Phi(\mathbf{X})|_{X^A = Constant}$ in the deformed configuration, with unit vector fields given by

$$\hat{\mathbf{n}}_1 \left[\frac{d\omega_1}{d\Omega_1} \right] = J\, \mathbf{F}^{-T}|_{X^1 = Const.} \hat{\mathbf{E}}_1 \qquad \hat{\mathbf{n}}_2 \left[\frac{d\omega_2}{d\Omega_2} \right] = J\, \mathbf{F}^{-T}|_{X^2 = Const.} \hat{\mathbf{E}}_2 \tag{5.28}$$

where $d\Omega_A$, $(A=1,2)$, designates the undeformed element of area normal to the X^A-axis, and $d\omega_a$, $(a=1,2)$, the corresponding element of are in the deformed configuration. Let $\mathbf{l}_A = \mathbf{F}\,\hat{\mathbf{E}}_A$ be the convected basis. The frame composed by the unit vector fields

$$\{\hat{\mathbf{n}}_1(\mathbf{X}), \hat{\mathbf{n}}_2(\mathbf{X}), \hat{\mathbf{l}}_3(\mathbf{X})\} \tag{5.29}$$

furnishes the natural extension for plates of the Gaussian frame considered in our previous analysis of beams and plays a fundamental role in the formulation that follows. It is again emphasised that $\hat{\mathbf{n}}_A(\mathbf{X})$, $(A=1,2)$, and $\mathbf{l}_I(\mathbf{X})$, $(I=1,2,3)$ are vector fields over the deformation map $\Phi : B \rightarrow \mathbb{R}^3$; that is, vectors at points $\mathbf{x} = \Phi(\mathbf{X})$ parametrized by material coordinates † .

The relationship between the frame (5.29), henceforth referred to as *moving frame* , and the standard basis $\{\hat{\mathbf{e}}_i\}$ in $\Phi(B)$ will be written as

$$\begin{Bmatrix} \hat{\mathbf{n}}_1 \\ \hat{\mathbf{n}}_2 \\ \hat{\mathbf{l}}_3 \end{Bmatrix} = \Lambda^T(\mathbf{X}) \begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{Bmatrix} \tag{5.30}$$

It is noted the vector $\hat{\mathbf{l}}_3$ of the frame (5.29) is orthogonal to the plane defined by $\hat{\mathbf{n}}_1$ and

† more precisely, $\hat{\mathbf{n}}_A$ and $\mathbf{l}_I$ are maps from B onto the tangent bundle $T\Phi(B)$ of the deformed configuration $\Phi(B)$.

$\hat{\mathbf{n}}_2$. However, $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ are not orthogonal and, as a result, the "moving frame" (5.29) is not orthogonal. To prove this, note first that $||\hat{\mathbf{l}}_3||^2 = \hat{\mathbf{E}}_3.(\mathbf{F}^T\mathbf{F}\,\hat{\mathbf{E}}_3) = C_{33}$, and from (5.28) and the definition of convected basis it follows that

$$\sqrt{C_{33}}\left[\frac{d\omega_a}{d\Omega_A}\right]\hat{\mathbf{n}}_A.\hat{\mathbf{l}}_3 = J\,\hat{\mathbf{E}}_3.\left[(\mathbf{F}^T\mathbf{F}^{-T})\,\hat{\mathbf{E}}_3\right]$$
$$= J\,\hat{\mathbf{E}}_3.\hat{\mathbf{E}}_A \equiv 0$$

A similar computation shows that

$$\left[\frac{d\omega_1}{d\Omega_1}\right]\left[\frac{d\omega_2}{d\Omega_2}\right]\hat{\mathbf{n}}_1.\hat{\mathbf{n}}_2 = J^2\,\hat{\mathbf{E}}_1.[(\mathbf{F}^{-1}\mathbf{F}^{-T})\,\hat{\mathbf{E}}_2]$$
$$= J^2\,[E_{12}]^{-1}$$

and even within a first order approximation one has

$$\hat{\mathbf{n}}_1.\hat{\mathbf{n}}_2 = 0 - \epsilon_{12} + \textit{Higher order Terms} \neq 0 \tag{5.31}$$

Therefore, the matrix $\mathbf{\Lambda}(\mathbf{X})$ in (5.28) is not orthogonal except in the linear theory where no distinction is made between undeformed and deformed configurations and, consequently, $\mathbf{\Lambda} \equiv 1$.

It is shown next that the introduction of the moving frame (5.27), defined through the matrix $\mathbf{\Lambda}(\mathbf{X})$ by (5.30), makes possible a simple definition of the resultant axial and shear forces in terms of the components of the first Piola-Kirchhoff tensor, even when transversal warping is taken into account.

5.4.3.- Resultant Forces over the thickness of the deformed plate.

Let $\mathbf{T}^A(\mathbf{X})$ be the stress vector field acting on the deformed surface $\mathbf{x}^A = \Phi(\mathbf{X})|_{X^A=Const}$ which corresponds to a plane parallel to $X^A - X^2$, (A=1,2), in the undeformed configuration. The components of $\mathbf{T}^A$ with respect to the standard basis $\{\hat{\mathbf{e}}_i\}$ in $\Phi(B)$ are given by $\{P^{jA}|_{X^A=Const.}\}_{(j=1,2,3)}$; and its components with respect to the moving frame $\{\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{l}}_3\}$ will be designated by $\{\overline{\tau}^{jA}\}_{(j=1,2,3)}$. As in the previous section, the subscript X^A=*Constant* will be understood when appropriate and is omitted in the sequel for notational simplicity. The stress vector $\mathbf{T}^A$ can be expressed as

$$\mathbf{T}^A = P^{jA}\,\hat{\mathbf{e}}_j = \bar{\tau}^{1A}\,\hat{\mathbf{n}}_1 + \bar{\tau}^{2A}\hat{\mathbf{n}}_2 + \bar{\tau}^{3A}\hat{\mathbf{1}}_3 \quad (A{=}1,2) \tag{5.32}$$

Clearly, $\bar{\tau}^{jA}$ are simply the components of the first Piola-Kirchhoff tensor $\mathbf{P}$ with respect to the basis $\{\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{1}}_3\}$ and $\{\hat{\mathbf{E}}_I\}$. Thus, in view of (5.30), the relationship between P^{jA} and $\bar{\tau}^{jA}$ is given by

$$\begin{Bmatrix} P^{1A} \\ P^{2A} \\ P^{3A} \end{Bmatrix} = \mathbf{\Lambda}(\mathbf{X}) \begin{Bmatrix} \bar{\tau}^{1A} \\ \bar{\tau}^{2A} \\ \bar{\tau}^{3A} \end{Bmatrix} \tag{5.33}$$

which is an immediate consequence of the fact that $\mathbf{P}(\mathbf{X})$ is a two-point tensor.

It is clear from equation (5.32), that the component $\bar{\tau}^{3A}$ represents the transversal shearing stress over the thickness of the plate while $\bar{\tau}^{11}$ and $\bar{\tau}^{22}$ give the corresponding normal stresses. Thus, the resultant shear forces V^A and in-plane forces N^{aA} are defined according to

$$V^A = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{\tau}^{3A}\, dX^3 \;, \qquad N^{aA} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{\tau}^{aA}\, dX^3 \;, \quad (a, A{=}1,2) \tag{5.34}$$

With the aid of equations (5.33) and (5.34), the equilibrium equations (5.20) and (5.27) can be recast in terms of resultant forces, provided an explicit expression for the matrix $\mathbf{\Lambda}(\mathbf{X})$ is at our disposal. Restricting ourselves to a second order approximation, an explicit expression for $\mathbf{\Lambda}(\mathbf{X})$ can be derived from the linearized displacement field (5.2). As in Chap.3, this fact is a consequence of the method of successive approximations.

5.4.4.- The Second Order Approximation

To consistently develop a second order approximation to the non-linear equilibrium equations, recall from Chap.3 that to within a first order approximation we may write

$$\begin{aligned} &\mathbf{F} = \mathbf{1} + GRAD\ \mathbf{u}(\mathbf{X}) \\ &J \equiv \det(\mathbf{F}) = 1 + DIV\ \mathbf{u}(\mathbf{X}) + \cdots \\ &J\,\mathbf{F}^{-T} = [1 + DIV\ \mathbf{u}(\mathbf{X})]\,\mathbf{1} - [GRAD\ \mathbf{u}(\mathbf{X})]^{-T} + \cdots \end{aligned} \tag{5.35}$$

where the $\mathbf{u}(\mathbf{X})$ is taken as the linearized displacement field (5.2) with κ and $\phi(X^3)$ as defined by (5.6). Since the components of $GRAD\ u$ are, from (5.2), given by

$$GRAD\ u(\mathbf{X}) = \begin{bmatrix} u^1{}_{,1} & u^2{}_{,2} & -\psi_1 - \kappa\phi'\beta_1 \\ u^2{}_{,1} & u^2{}_{,2} & -\psi_2 - \kappa\phi'\beta_2 \\ w_{,1} & w_{,2} & 0 \end{bmatrix} \tag{5.36}$$

the explicit expression for $\mathbf{\Lambda}(\mathbf{X})$ follows at once from (5.28) and (5.30) together with (5.35) and (5.36). The result can be written as

$$\mathbf{\Lambda}(\mathbf{X}) = \mathbf{1} + \begin{bmatrix} 0 & -u^2{}_{,1} & 0 \\ -u^2{}_{,2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\psi_1 \\ 0 & 0 & -\psi_2 \\ \psi_1 & \psi_2 & 0 \end{bmatrix} + \kappa\,\phi(X^3) \begin{bmatrix} 0 & 0 & -\beta_1 \\ 0 & 0 & -\beta_2 \\ \beta_1 & \beta_2 & 0 \end{bmatrix}$$
$$\equiv \mathbf{1} + \mathbf{\Sigma}(\mathbf{X}) + \mathbf{\Omega}(X^1,X^2) + \phi(X^3)\,\kappa\,\mathbf{\Xi}(X^1,X^2) \tag{5.37}$$

It follows from this expression and (5.30), that $\hat{\mathbf{n}}_1 . \hat{\mathbf{n}}_2 = -\epsilon_{12}$ in agreement with the result of (5.31). Equation (5.37) admits the following physical interpretation. The first term is the only one that appears in the linear theory in which no distinction is made between undeformed and deformed configurations. The terms $\mathbf{\Omega}$ and $\kappa\phi\mathbf{\Xi}$ represent additional contributions arising from the rotation of an element of plate and the transversal warping of the plate due to shear deformation, respectively. Finally, the term $\mathbf{\Sigma}$ is due to the in-plane deformation of the plate.

The substitution of (5.37) into (5.33) leads, after use is made of the definition of resultant forces given by (5.34), to

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} P^{1A} \\ P^{2A} \\ P^{3A} \end{Bmatrix} dX^3 = \begin{Bmatrix} N^{1A} \\ N^{2A} \\ V^A \end{Bmatrix} + \begin{Bmatrix} -\bar{\psi}_1\, V^A \\ -\bar{\psi}_2\, V^A \\ \bar{\psi}_1 N^{1A} + \bar{\psi}_2 N^{2A} \end{Bmatrix}$$
$$+ \int_{-\frac{h}{2}}^{\frac{h}{2}} \kappa\phi(X^3) \begin{Bmatrix} -\bar{\beta}_1\, \bar{\tau}^{3A} \\ -\bar{\beta}_2\, \bar{\tau}^{3A} \\ \bar{\beta}_1 \bar{\tau}^{1A} + \bar{\beta}_2 \bar{\tau}^{2A} \end{Bmatrix} + \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} -u^2{}_{,1}\, \bar{\tau}^{2A} \\ -u^2{}_{,2}\, \bar{\tau}^{2A} \\ 0 \end{Bmatrix} dX^3 \tag{5.38}$$

Equation (5.38) admits again a simple physical interpretation. The first term corresponds to the linear theory, the second appears as a result of the rotation of the plate and the last two are the consequence of the transversal warping and in-plane deformation of plate, respectively. These last two terms can be estimated two within a second order approximation from the observation that the infinitesimal symmetric stress tensor $\boldsymbol{\sigma}$, is the linear part of the non-symmetric Piola-Kirchhoff tensor $\mathbf{P}$. Accordingly, the components $\bar{\tau}^{jA}$ of $\mathbf{P}$ with respect to the basis

$\{\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{l}}_3\}$ and $\{\hat{\mathbf{E}}_I\}$ may be written as

$$\bar{\tau}^{jA} = \sigma^{jA} + \bar{\bar{\tau}}^{jA} + \cdots \tag{5.39}$$

where σ^{jA} are the components $\boldsymbol{\sigma}$ and $\bar{\bar{\tau}}^{jA}$ those of a non-symmetric tensor. Given the form of the last two terms in (5.38), it is clear that only the linear term $\boldsymbol{\sigma}$ in (5.39) needs to be considered in a second order approximation. Thus, making use of the constitutive equations (5.11)-(5.12) we obtain

$$\kappa \bar{\beta}_a \int_{-\frac{h}{2}}^{\frac{h}{2}} \psi'(X^3)\, \bar{\tau}^{3A} dX^3 = \kappa \bar{\beta}_a \int_{-\frac{h}{2}}^{\frac{h}{2}} \phi'(X^3)\, \sigma^{3A} dX^3 + \cdots$$

$$= \frac{5}{6} \frac{V^A}{\frac{2}{3}h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[1 - \left(\frac{4X^3}{h}\right)^2\right] \left[\left(\frac{4X^3}{h}\right)^2 - \frac{1}{5}\right] dX^3 + \cdots \equiv 0 \tag{5.40}$$

Similarly, when use is made of (5.10) and (5.14), we obtain the estimate

$$\kappa \bar{\beta}_a \int_{-\frac{h}{2}}^{\frac{h}{2}} \phi'(X^3) \bar{\tau}^{aA} dX^3 = \kappa \bar{\beta}_a \int_{-\frac{h}{2}}^{\frac{h}{2}} \phi'(X^3) \sigma^{aA} dX^3$$

$$= \frac{1}{6} N^{aA} \tag{5.41}$$

Equations (5.40) and (5.41) complete the estimate of the third term of (5.38). Similar estimates could be carried out for the terms appearing in the last summand of (5.38). However, we shall not do so and assume instead, on physical grounds, that this term can be neglected.

If the estimates (5.40) and (5.41) are substituted into (5.38) the equilibrium equation (5.20) and (5.27) take in terms of resultant force over the thickness of the deformed plate the final form

$$\begin{aligned} &\frac{\partial}{\partial X^A} [N^{aA} - \bar{\psi}_b \delta^b{}_A\, V^A] = 0 \\ &\frac{\partial}{\partial X^A} [V^A + (\bar{\psi}_a + \frac{1}{6}\bar{\beta}_a)\, N^{aA}] + q = 0 \\ &\frac{\partial M^{aA}}{\partial X^A} + \left[\delta^a{}_A + \frac{\partial \bar{u}^a}{\partial X^A}\right] V^A - \frac{5}{6}\bar{\beta}_b \delta^b{}_A\, N^{aA} = 0 \end{aligned} \tag{5.42}$$

It is again noted that the linear terms appearing in the equilibrium equations (5.42) furnish the equilibrium equations (5.8) of the linear theory, in accord with the method of successive approximations. The additional terms in the first two of (5.42) arise as a result of the rotation and transversal warping of the plate. Due to the effect of transversal warping, the resultant axial forces N^{aA} are shifted by an angle $\frac{1}{6}\bar{\beta}_a$. The resultant shear forces V^A, however, remain in the (average) planes of bending as $(5.42)_1$ shows. The factor of $\frac{5}{6}$ appearing in the moment equilibrium equation $(5.42)_3$ should also be attributed to the effect of the transversal warping.

5.4.5.- Constitutive Equations

To establish constitutive equations for the resultant forces N^{aA}, V^A and M^{aA} the corresponding measures of deformation need first to be derived. An argument analogous to that presented in section 2.2.3 is employed for this purpose.

Assume the material is hyperelastic with strain energy $W(X,\nabla\Phi)$. Let V be the linear space of kinematic admissible variations defined by (2.13) and $\Pi^*(\Phi) = \int_B W\, dV$. We can then show that the Frechet differential of Π^* at a configuration Φ may be written as

$$\begin{aligned}\delta\Pi^*(\Phi) &= \int_{\Omega\times(-\frac{h}{2},\frac{h}{2})} \frac{\partial W(X,\nabla\Phi)}{\partial \mathbf{F}} : \delta\mathbf{F}\, dV \\ &= \int_\Omega \left\{ \mathbf{P}(\nabla\Phi):\delta\mathbf{F}\, dX^3 \right\} d\Omega \\ &= \int_\Omega [M^{aA}\,\delta\bar{\psi}_{aA} + N^{aA}\,\delta\lambda_{aA} + V^A\,\delta\gamma_A]\, d\Omega \end{aligned} \tag{5.43}$$

where

$$\begin{aligned}\lambda_{aA} &= \delta_a^B[\bar{\epsilon}_{BA} + \frac{1}{2} w,_B w,_A - \frac{1}{2}\kappa\bar{\beta}_B\bar{\beta}_A] \\ \gamma_A &= w,_A - (\delta^a{}_A + \bar{u}^a{}_{,A})\,\bar{\psi}_a \end{aligned} \tag{5.44}$$

Equation (5.43) shows that $\bar{\psi}_{aA}$, λ_{aA} and γ_A, the latter two being given by (5.44), are the measures of deformation dual to M^{aA}, N^{aA} and V^A, respectively. It is important to note that the symmetry of M^{aA} and N^{aA} holds only for this second approximation and does not carry over to the general non-linear theory.

In view of (5.44), the simplest possible constitutive model consistent with the equilibrium equations (5.42) is furnished by the constitutive equations (5.12), (5.14) and (5.15) of the linear theory, with $\bar{\epsilon}_{aA}$ and $\bar{\beta}_A$ replaced by λ_{aA} and γ_A respectively. The complete system of equations of the non-linear theory presented herein is summarized in Table 5.1. Since material and spatial coordinates are taken as coincident, no notational distinction between both sets of coordinates has been made. We shall adhere to this convention in the sequel.

TABLE 5.1.

The second order approximation to the non-linear theory.

Equilibrium Equations.

$$N_{\alpha\beta,\beta} - [\bar{\psi}_\alpha \; V_\beta]_{,\beta} = 0$$

$$V_{\alpha,\alpha} + \left\{ [w_{,\alpha} - \frac{5}{6}(w_{,\alpha} - \bar{\psi}_\beta] \; N_{\alpha\beta} \right\}_{,\beta} + q = 0$$

$$M_{\alpha\beta,\beta} + [\delta_{\alpha\beta} + \bar{u}_{\alpha,\beta}] \; V_\beta - \frac{5}{6}[w_{,\beta} - \bar{\psi}_\beta] \; N_{\alpha\beta} = 0$$

Constitutive Equations

$$N_{\alpha\beta} = \frac{Eh\nu}{1-\nu^2} \lambda_{\gamma,\gamma} \, \delta_{\alpha\beta} + 2Gh \, \lambda_{\alpha\beta}$$

$$M_{\alpha\beta} = D \left[\nu \, \bar{\psi}_{\gamma,\gamma} \, \delta_{\alpha\beta} + (1-\nu) \, \frac{1}{2}(\bar{\psi}_{\alpha,\beta} + \bar{\psi}_{\beta,\alpha}) \right] - D\frac{1+\nu}{E} \, \frac{q}{\frac{5}{6}h} \, \delta_{\alpha\beta}$$

$$V_\alpha = Gh\frac{5}{6} \, [w_{,\alpha} - (1+\bar{u}_{\alpha,\alpha}) \, \bar{\psi}_\alpha] \qquad (\textit{no summ on } \alpha)$$

$$N_{\alpha\beta} = N_{\beta\alpha} \qquad M_{\alpha\beta} = M_{\beta\alpha}$$

Kinematic Variables

$$\beta_\alpha = w_{,\alpha} - \bar{\psi}_\alpha$$

$$\bar{\epsilon}_{\alpha\beta} = \frac{1}{2}[\bar{u}_{\alpha,\beta} + \bar{u}_{\beta,\alpha}]$$

$$\lambda_{\alpha\beta} = \bar{\epsilon}_{\alpha\beta} + \frac{1}{2} \, w_{,\alpha} w_{,\beta} - \frac{1}{2} \, \frac{5}{6} \bar{\beta}_\alpha \bar{\beta}_\beta$$

The formulation summarized in Table 5.1 could be used as a rigorous basis for a numerical treatment employing finite differences or the finite element method. It is noted that this formulation reduces to equations (3.33) with $\kappa = \frac{5}{6}$ in the one-dimensional case. This value for the constant κ is the result of the inclusion of the transversal warping.

5.4.6.- Simplified Theory.

We present in this section a simplified theory restricted, as the Von-Karman model, to 'moderate' rotations and infinitesimal axial displacements which, nevertheless, retains the basic features of accounting both for shear deformation and axial displacement of the plate. This formulation exactly reproduces the Von-Karman equations in the limit as the shear stiffness of the plate tends to infinity. On the other hand, it reduces to Reissner theory when the in-plane forces are zero.

The formulation summarized in Table 5.1 is simplified by introducing, based on physical grounds, the following two additional assumptions

Additional Assumptions.

(i) The in-plane displacements $\bar{u}_\alpha$ are assumed to be 'infinitesimal'. Accordingly, when establishing the equilibrium in the deformed configuration $\Phi(B)$ we may assume

$$\delta_{\alpha\beta} + \bar{u}_{\alpha,\beta} \approx \delta_{\alpha\beta}$$

(ii) The contribution to the horizontal equilibrium equations of the rotated shear forces is neglected†. Therefore

$$\bar{\psi}_{\alpha,\beta}\, V_{\alpha,\beta} \approx 0 \quad => \quad N_{\alpha\beta,\beta} = 0$$

It is noted that these two assumptions, although sometimes not explicitly recognized, are equally present in the Von-Karman theory of plates. Assumption (i) is quite realistic in most applications. The formal reason for assumption (ii) is that it allows the introduction of the potential $\Upsilon : \Omega \rightarrow \mathbb{R}$ for the forces $N_{\alpha\beta}$ such that

$$N_{11} = \Upsilon_{,22}\,, \quad N_{22} = \Upsilon_{,11}\,, \quad N_{12} = -\Upsilon_{,12} \tag{5.45}$$

† Actually, it can be shown by making use of the principle of virtual work that this assumption is redundant with (i). We omit the details.

so that the axial equilibrium equation, which reduces to $N_{\alpha\beta,\beta} = 0$ by assumption (ii), is identically satisfied.

Making use of assumptions (i) and (ii) the equations of equilibrium and constitutive equations summarized in Table 5.1 can be combined to yield the following system of partial differential equations

$$M_{\alpha\beta,\alpha\beta} = D\left[\bar{\psi}_{\alpha,\alpha\beta\beta} - \frac{1+\nu}{E}\frac{q_{,\alpha\alpha}}{\frac{5}{6}h}\right] \equiv N_{\alpha\beta}\, w_{,\alpha\beta} + q$$
$$\left[\frac{N_{\alpha\beta}}{Gh} + \delta_{\alpha\beta}\right] = -\frac{1}{G\,\frac{5}{6}h}\left[q + N_{\alpha\beta}\, w_{,\alpha\beta}\right] \qquad (5.46)$$
$$N_{\alpha\beta,\beta} = 0$$

It should be noted that for the one-dimensional case with transversally applied load $q \equiv 0$, equations (5.46) lead to the eigenvalue problem

$$\frac{D\left[1 - \dfrac{P}{5Gh}\right]}{1 + \dfrac{P}{Gh}}\,\frac{d^4 w}{dX^4} + P\,\frac{d^2 w}{dX^2} = 0$$

which is in complete agreement for $\kappa = \frac{5}{6}$ with the general expression (3.36) derived in chapter 3.

Yet, equations (5.46) can be further simplified by introducing the deflection due to shear $w^S(X^1, X^2) : \Omega \to I\!R$ which satisfies (5.17a). The reason for this is that from (5.45) the constitutive equations for the in-plane forces $N_{\alpha\beta}$ may be reduced to the following simple relation between the potential $\Upsilon(X^1, X^2)$ for the in-plane forces and Gaussian curvatures

$$\Delta^2 \Upsilon = Eh\,\frac{1}{2}\left[\{w;\, w\} - \{w^S;\, w^S\}\right] \qquad (5.47)$$

where $\{\,.\,;\,.\,\}$ denotes the differential operator

$$\{f;\, g\} \equiv f_{,11} g_{,22} + f_{,22} g_{,11} - 2\, f_{,12} g_{,12}$$
$$\equiv \Delta f\, \Delta g - \nabla(\nabla f) : \nabla(\nabla g) \qquad (5.48)$$

and, therefore, $\frac{1}{2}\{f;\, f\}$ gives the Gaussian curvature of the map $f : \Omega \subset I\!R^2 \to I\!R$.

Thus, with the aid of (5.17a) and (5.45), equation $(5.46)_1$ may be written in the following form

$$\Delta^2 w - \Delta^2 w^S = \frac{1}{D} N_{\alpha\beta}\, w,_{\alpha\beta} + [\frac{q}{D} + \frac{\Delta q}{G\frac{5}{6}h}\frac{\nu}{2}]$$

$$\equiv \frac{1}{D}\{\Upsilon;\ w\} + [\frac{q}{D} + \frac{\Delta q}{G\frac{5}{6}h}\frac{\nu}{2}] \tag{5.49}$$

In conclusion, when the deflection due to shear $w^S(X^1,X^2)$ satisfying (5.17a) and the potential $\Upsilon(X^1,X^2)$ satisfying conditions (5.45) are introduced, the present theory yields the coupled system of non-linear partial differential equations shown in Table 5.2. It is again noted that these equations reduce to the classical Von-Karman equations of plates when the shear stiffness of the plate $Gh \rightarrow \infty$. Finally, if the in-plane forces vanish, since $w^B = w - w^S$, one obtains equations (5.18b) of the linear plate theory due to Reissner. The consistency of the formulation presented and the clear statement of the assumptions required at each step of the derivation, is noted.

TABLE 5.2

Large deflection of Plates.
Effects of warping and shear deformation included.

$$\Delta^2 w - \Delta^2 w^S = [\frac{q}{D} + \frac{\Delta q}{Gh\kappa}\frac{\nu}{2}] + \frac{1}{D}\{\Upsilon;\ w\}$$

$$\nabla(\nabla w^S) = -\frac{q}{Gh\kappa} - \frac{1}{Gh\kappa}[\{\Upsilon;\ w\} - \kappa\{\Upsilon;\ w^S\}]$$

$$\Delta^2\Upsilon = Eh\,\frac{1}{2}\left[\{w;\ w\} - \{w^S;\ w^S\}\right]$$

where

$$\mathbf{M} = \mathbf{M}^T = D\left[\nu\,\Delta(w-w^S)\mathbf{1} + (1-\nu)\,\nabla(\nabla w - \nabla w^S)\right] - D\frac{\nu}{2}\frac{q}{Gh\kappa}$$

$$\mathbf{N} = \mathbf{N}^T = \Delta\Upsilon\ \mathbf{1} - \nabla(\nabla\Upsilon)$$

$$\mathbf{V} = Gh\kappa\,\nabla w^S \qquad \kappa = \frac{5}{6}$$

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APPENDIX I

STABILITY AT BIFURCATION POINTS FOR THE ELASTICA WITH SHEAR DEFORMATION

In this appendix, the stability of the adjacent equilibrium configurations for both the transversally homogeneous and the sandwich beam, is examined making use of perturbation methods. The line of centroids is assumed to be perfectly inextensible, and finite shear deformation is taken into account.

The case of the transversally homogeneous beam is considered first, and taken as a model problem to present the methodology in some detail. The stability analysis for the sandwich beam is entirely analogous and will be briefly examined later.

I.- THE TRANSVERSALLY HOMOGENEOUS BEAM

1.- Statement of the Problem.

The system of semi-linear ordinary differential equations (2.31), derived in chapter 2., can be recast into the following non-linear eigenvalue problem for the applied axial load P

$$\psi''(x) + F(\psi(x),P) = 0 \ ; \qquad x \in (0,L) \tag{I.1a}$$

$$\psi'(0) = 0 \ ; \quad \psi(L) = 0 \tag{I.1b}$$

where

$$F(\psi,P) = \frac{P}{EI}\left[\sqrt{1 - \left[\frac{P}{G\Omega}\right]^2 \sin^2\psi} + \frac{P}{G\Omega}\cos\psi\right]\sin\psi \tag{I.2}$$

Remarks

(i) Boundary conditions corresponding to a cantilever beam fixed at x = 1 have been assumed for simplicity.

(ii) It is noted that $F(\psi,P)$ is an odd function of ψ; i.e:

$$F(\psi,P) = -F(-\psi,P)$$

This remark plays a key role in the subsequent analysis.

(iii) Problem (I.1) always has the solution $\psi_O(x) \equiv 0$. The objective is to study possible bifurcations from this trivial solution for certain values of the parameter P representing the applied axial load. For this purpose, the consistently linearized problem about $\psi_O \equiv 0$, considered next, plays a key role.

1.1.- Linearized Problem.

Consider first the linearized problem about the configuration $\psi_O(x) \equiv 0$. The Frechet differential of F at ψ_O is given by

$$D_1 F(\psi_O,P).\bar{\psi} = \frac{d}{d\alpha} F(\psi_O+\alpha\bar{\psi},P)|_{\alpha=0} = \frac{P}{EI}\left[\frac{P}{1+G\Omega}\right]\bar{\psi}$$

Introducing the notation

$$\lambda^2(P) = \frac{P}{EI}\left[1 + \frac{P}{G\Omega}\right] \tag{I.3}$$

the following linear problem is obtained

$$\bar{\psi}'' + \lambda^2(P)\,\bar{\psi} = 0\,; \qquad x \in (o,L) \tag{I.4a}$$

$$\bar{\psi}(0) = 0\,; \qquad \bar{\psi}(L) = 0 \tag{I.4b}$$

Problem (I.4) defines a self-adjoint linear operator (from the Hilbert space $H^2(0,L) \to L^2(0,L)$) whose inverse is compact. Hence, its spectrum is discrete and consists of the eigenvalues and associated eigenfunctions

$$\bar{\psi}_n(x) = \cos[\lambda_n x]\,; \qquad P_n = \frac{2\,P_{En}}{1+\sqrt{1+\dfrac{4P_{En}}{G\Omega}}} \tag{I.5}$$

where

$$\lambda_n^2 = \frac{(2n-1)^2\pi^2}{4L^2} \qquad and \qquad P_{E\,n} = EI\,\lambda_n^2 \tag{I.6}$$

Notice that $P_{E_n} = \frac{EI\pi^2}{4L^2}$ is just Euler's buckling load. Since $\psi(0)\neq 0$ for any non trivial solution of problem (I.4), the following normalizing condition is chosen

$$\bar{\psi}(0) = 1 \tag{I.7}$$

Using the definition of branching point and the Frechet differentiability condition, one can prove the following key fact [4],[5]

Proposition

Bifurcation for the non linear problem (I.1) from the trivial solution $\psi_O \equiv 0$ can only occur at points which are in the spectrum of the linearized problem (I.4); i.e: at the eigenvalues P_n given by (I.5)-(I.6).

To study the stability of these adjacent equilibrium configurations, departing from $\psi_O \equiv 0$ at points P_n, the local behavior of the corresponding branch in a neigborhood of $(P_n,\psi\equiv 0)$ needs to be examined. The most convenient way of treating this problem is to make use of perturbation theory of operators in Hilbert space. Relevant to the problem at hand are the Liapunov-Schmidt procedure, which translates problem (I.1) into a non-linear integral equation [4]; and a modification of Poincare's classical method, developed by Keller [1]-[2], which exploits the connection between problem (I.1) and the associated initial value problem [3],[4]. Use will be made of the latter in the developments that follow.

2.- Perturbation Analysis for the Elastica with Shear Deformation.

We start the perturbation analysis, by setting

$$\psi(x) = \psi_O + \epsilon\, z(x) \tag{I.8}$$

where the perturbation parameter ϵ is chosen so that

$$z(x,P,\epsilon)|_{x=0} = 1 \tag{I.9}$$

Thus, since $\epsilon = \psi|_{x=0}$, this perturbation parameter represents physically the rotation of the cantilever at $x = 0$, clearly non zero for any non-trivial solution $\psi \neq \psi_O$ of (I.1).

2.1.- Poincare-Keller's Method.

Let us consider the non-linear initial value problem for z, obtained by substituting (I.8) into (I.1) and replacing the boundary condition at x = L by the normalizing condition (I.9); i.e: the problem

$$z'' + \frac{1}{\epsilon}F(\epsilon z,P) = 0\ , \qquad x \in (0,L) \tag{I.10a}$$

$$z(0) = 1\ , \qquad z'(0) = 0 \tag{I.10b}$$

Let $z(x,P,\epsilon)$ be the solution of this problem depending parametrically (and continuously) on P and ϵ. We first note that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}F(\epsilon z,P) = \frac{P}{EI}\left[1 + \frac{P}{G\Omega}\right] z = \lambda^2(P)\ z$$

so that $z(x,P,0)$ satisfies

$$z''(x,P,0) + \lambda^2(P)\ z(x,P,0) = 0\ , \qquad x \in (O,L) \tag{I.11a}$$

$$z(0,P,0) = 1\ ; \qquad z'(0,P,0) = 0 \tag{I.11b}$$

and admits, therefore, the solution

$$z(x,P,0) = \cos\,[\lambda(P)x] \tag{I.12}$$

When $P = P_n$, the solution (I.12) satisfies the boundary condition $z(L,P_n,0) = 0$, and problem (I.11) reduces identically to (I.4) with $z(0,P_n,0) = 1$ as normalizing condition. The objective now is to obtain solutions of the non-linear problem (I.1) for P near P_n and ϵ near zero (i.e: $\psi(x)$ near $\psi_O \equiv 0$). These solutions can be constructed from those of the initial value problem (I.10) in the following manner.

Let $z(x,P,\epsilon)$ be a solution of (I.10). $z(x,P,\epsilon)$ will be a solution of the eigenvalue problem (I.1) if it satisfies the boundary condition at x = L; that is, provided there is a pair (P,ϵ) such that

$$b(P,\epsilon) := z(L,P,\epsilon) = 0 \tag{I.13}$$

This equation becomes then the "bifurcation equation", and is always satisfied by $(P_n,0)$. By the Implicit Function theorem, it will be possible to obtain a solution (P,ϵ) near $(P_n,0)$, provided one of the derivatives of $b(P,\epsilon)$ with respect to either P or ϵ does not vanish at $P = P_n$ and $\epsilon = 0$.

(a) Let us examine first $\frac{\partial b}{\partial \epsilon}(P_n,0)$. From (I.13) it follows that

$$\frac{\partial b}{\partial \epsilon}(P_n,0) = \frac{\partial z}{\partial \epsilon}(L,P_n,\epsilon)$$

since $F(\psi,P)$ is an odd function of ψ, $\frac{1}{\epsilon}F(\epsilon z,P)$ is an even function of ϵ and its derivative with respect to ϵ is, therefore, zero at $\epsilon = 0$. Differentiation of (I.10) with respect to ϵ leads then to

$$\frac{\partial z}{\partial \epsilon}(x,P,0) = 0 \quad => \quad \frac{\partial b}{\partial \epsilon}(P_n,0) \equiv 0 \tag{I.14}$$

Equation $(I.14)_1$ indicates that $z(x,P,\epsilon)$ is an *even function of* ϵ. In addition, $(I.14)_2$ shows that it is not possible to solve (I.13) for ϵ as a function of P.

(b) Consider next $\frac{\partial b}{\partial P}(P_n,0)$. From (I.12) and (I.13) it follows that

$$\frac{\partial b}{\partial P}(P_n,0) = \frac{\partial z}{\partial P}(1,P_n,0)$$

$$= -\lambda'(P_n)\sin[\lambda(P_n)]$$

$$= (-1)^n \frac{1}{EI}\sqrt{1+\frac{(2n-1)^2\pi^2 EI}{L^2}}\sin\left[\frac{(2n-1)\pi x}{2L}\right] \neq 0 \tag{I.15}$$

It is, therefore, possible to solve (I.13) in a neighborhood of $(P_n,0)$ to obtain a function

$$P = \hat{P}_n(\epsilon) \quad \textit{with} \quad P_n = \hat{P}_n(0) \tag{I.16}$$

satisfying (I.13); i.e:

$$B(\epsilon) := b(\hat{P}_n(\epsilon),\epsilon) = 0 \tag{I.17}$$

In addition, by differentiating (I.17) with respect to ϵ and applying the chain rule it is found that

$$0 \equiv \frac{dB}{d\epsilon}(0) = \frac{\partial b}{\partial P}(P_n,0).\hat{P}_n{}'(0) + \frac{\partial b}{\partial \epsilon}(P_n,0)$$

which by (I.14) and (I.15) yields the result

$$\frac{d}{d\epsilon}\hat{P}_n(\epsilon)\big|_{\epsilon=0} = 0 \tag{I.18}$$

Therefore, according to (I.18), $\hat{P}_n(\epsilon)$ is an *even function of* ϵ.

Once the function $P = \hat{P}(\epsilon)$ is obtained, the behavior of the branch near $P = P_n$ and $\psi_O(x) \equiv 0$ is completely characterized by $\psi(x) = \epsilon\, z(x,\hat{P}_n(\epsilon),\epsilon)$. The explicit determination of the function $P = \hat{P}_n(\epsilon)$ is considered next.

2.2.- Determination of $P = \hat{P}(\epsilon)$

First, it is noted that $\frac{1}{\epsilon}\, F(\epsilon z,P)$, where the function F is given by (I.2), admits an expansion in powers of ϵ of the following form

$$\frac{1}{\epsilon}\, F(\epsilon z,P) = \lambda^2(P)\left[z - \left(\frac{1}{3} + \frac{P}{G\Omega}\right) z^3 \frac{\epsilon^2}{2}\right] + O(\epsilon^4) \tag{I.19}$$

Next, the function $Z(x,\epsilon)$ is defined by the relation

$$Z(x,\epsilon) := z(x,\hat{P}(\epsilon),\epsilon) \tag{I.20}$$

Clearly, $Z(x,\epsilon)$ is an even function of ϵ since by (I.14) and (I.19) $z(x,P,\epsilon)$ and $\hat{P}_n(\epsilon)$ are even functions of ϵ too. Therefore, both $Z(x,\epsilon)$ and $\hat{P}_n(\epsilon)$ admit a power series expansion of the form

$$\hat{P}_n(\epsilon) = P_n + \hat{P}_{n\epsilon\epsilon}(0)\,\frac{\epsilon^2}{2} + O(\epsilon^4) \tag{I.21a}$$

$$Z(x,\epsilon) = Z(x,0) + Z_{\epsilon\epsilon}\,\frac{\epsilon^2}{2} + O(\epsilon^4) \tag{I.21b}$$

where the sub index ϵ indicates differentiation with respect to ϵ. Substitution of (I.21) into (I.19) gives

$$\frac{1}{\epsilon}\, F[\epsilon Z(x,\epsilon),\hat{P}_n(\epsilon)] = \lambda_n^2 Z(x,0) + \left\{\lambda_n^2 Z_{\epsilon\epsilon}(x,0) - \lambda_n^2\left(\frac{1}{3} + \frac{P_n}{G\Omega}\right) Z^3(x,0) + \frac{\hat{P}_{\epsilon\epsilon}(0)}{EI}\left(1 + \frac{2P_n}{G\Omega}\right) Z(x,0)\right\}\frac{\epsilon^2}{2} + O(\epsilon^4) \tag{I.22}$$

The function $Z(x,\epsilon)$, defined by (I.20), satisfies the non-linear initial value problem (I.10), since z does. Furthermore, it also satisfies the boundary condition at x = L, since $Z(1,\epsilon) = z(1,\hat{P}(\epsilon),\epsilon) = B(\epsilon) = 0$, by (I.17). Hence, $Z(x,\epsilon)$ satisfies the non-linear eigenvalue problem (I.1) with the normalizing condition

$$Z(0,\epsilon) = 1 \tag{I.23}$$

The substitution of (I.21) and (I.22) into (I.1) and subsequent equating to zero of the coefficients in the resulting expansion in powers of ϵ, leads then to a sequence of problems the first of which is a linear problem for $Z(x,0)$ identical to (I.4) and, therefore, with solution

$$Z(x,O) = \cos[\lambda_n\, x] \tag{I.24}$$

whereas the second one, the coefficient of the term ϵ^2, gives rise a linear inhomogeneous problem for $Z_{\epsilon\epsilon}(x,0)$ of the form

$$Z''_{\epsilon\epsilon}(x,0) + \lambda_n{}^2 Z_{\epsilon\epsilon}(x,0) = \lambda_n{}^2\left[\frac{1}{3} + \frac{P_n}{G\Omega}\right] Z^3(x,0) - \frac{\hat{P}_{\epsilon\epsilon}(0)}{EI}\left[1 + \frac{2P_n}{G\Omega}\right] Z(x,0)$$

$$Z'_{\epsilon\epsilon}(0,\epsilon) = 0 \ , \qquad Z_{\epsilon\epsilon}(L,\epsilon) = 0 \tag{I.25}$$

The homogeneous problem associated with (I.25) has the non-trivial solution given by (I.24). By Freldholm alternative theorem, problem (I.25) will have a solution only if the right hand side is orthogonal to (I.24). The application of this solvability condition yields

$$\frac{\hat{P}_{\epsilon\epsilon}}{EI} = \frac{\lambda_n{}^2}{3}\;\frac{1 + \dfrac{3P_n}{G\Omega}}{1 + \dfrac{2P_n}{G\Omega}}\;\frac{\int_0^L \cos^4\,[\lambda_n x]\; dx}{\int_0^L \cos^2\,[\lambda_n x]\; dx} \tag{I.26}$$

Therefore, up to terms of order $O(\epsilon^4)$, at each bifurcation point P_n the following estimate holds

$$\hat{P}_n(\epsilon) - P_n = \hat{P}_{\epsilon\epsilon}(0)\,\frac{\epsilon^2}{2} + O(\epsilon^4)$$

$$= \frac{EI\,\lambda_n{}^2}{8}\;\frac{1 + \dfrac{3P_n}{G\Omega}}{1 + \dfrac{2P_n}{G\Omega}}\;\epsilon^2 \tag{I.27}$$

with λ_n and P_n given by (I.6).

Equation (I.27) leads to the following conclusion:

At each bifurcation point P_n, the adjacent equilibrium positions behave locally as parabolas. These adjacent equilibrium configurations are locally stable and the bifurcation diagram is, therefore, the pitchfork.

II.- THE SANDWICH BEAM.

For the sandwich beam, equations (2.54) derived in Chapter 2, lead to a non-linear eigenvalue problem entirely analogous to (I.1a)-(I.1b) with the function $F(\psi,P)$ defined now by

$$F(\psi,P) = \frac{P}{EI}\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = \frac{P}{EI}\frac{\dfrac{\sin\psi}{\cos\psi - \dfrac{P}{G\Omega}}}{\left[1 + \dfrac{\sin^2\psi}{\left(\cos\psi - \dfrac{P}{G\Omega}\right)^2}\right]^{1/2}} \tag{I.29}$$

The linearization about $\psi_O \equiv 0$ yields

$$D_1F(\psi_O,P).\bar{\psi} = \frac{d}{d\alpha}F(\psi_O+\alpha\bar{\psi})|_{\alpha=0}$$

$$= \frac{P}{EI\left[1 - \dfrac{P}{G\Omega}\right]}.\bar{\psi} \tag{I.30}$$

Hence, one has a linearized eigenvalue problem identical to (I.4a)-(I.4b) with $\lambda(P)$ defined now, in view of (I.30), by

$$\lambda^2(P) = \frac{P}{EI\left[1 - \dfrac{P}{G\Omega}\right]} \tag{I.31}$$

The same procedure discussed in detail in sec.2.1 of this appendix shows that the two following facts again hold

(i) $\dfrac{1}{\epsilon}F(\epsilon z,P)$ is an even function of ϵ.

(ii) $\hat{P}_n(\epsilon)$ is an even function of ϵ.

The results (i) and (ii) are all that is needed to determine the local behavior of the adjacent equilibrium configuration at P_n. Proceeding as in section 2.2, one is led to the following power series expansion

$$\frac{1}{\epsilon}F[\epsilon Z(x,\epsilon),P_n(\epsilon)] = \lambda^2{}_n Z(x,0) +$$

$$+\left\{ \lambda^2{}_n Z_{\epsilon\epsilon}(x,0) - \lambda^2{}_n\left[\frac{1}{3}+\frac{2-\dfrac{P_n}{G\Omega}}{\left(1-\dfrac{P_n}{G\Omega}\right)^2}\right]Z^3(x,0)\right.$$

$$\left. + \frac{P_{\epsilon\epsilon}(0)}{EI\left(1-\dfrac{P_n}{G\Omega}\right)^2}Z(x,0)\right\}\frac{\epsilon^2}{2} + O(\epsilon^4) \qquad \text{(I.32)}$$

Since $Z(x,\epsilon)$ satisfies the non-linear eigenvalue problem (I.1a)-(I.1b), $Z_{\epsilon\epsilon}(x,0)$ satisfies again an inhomogeneous problem analogous to (I.25a)-(I.25b) with the right hand side now given by the coefficient of ϵ^2 in the expansion (I.32). The solvability condition yields then

$$\frac{P_{\epsilon\epsilon}}{EI} = \lambda^2{}_n\left[\frac{1}{3}\left(1-\frac{P_n}{G\Omega}\right)^2 + 2 - \frac{P_n}{G\Omega}\right]\frac{1}{8} > 0 \qquad \text{(I.33)}$$

where P_n are is the critical load, given by

$$P_n = \frac{P_{En}}{1+\dfrac{P_{En}}{G\Omega}} \qquad \text{(I.34)}$$

being P_{En} and $\lambda^2{}_n$ as in (I.6).

Therefore, identical conclusion to that found for the transversally homogeneous beam holds for the sandwich beam; namely:

The adjacent equilibrium configurations behave locally as parabolas and are locally stable, the bifurcation diagram being the pitchfork.

References.

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APPENDIX II

In this appendix, we give a detailed proof of the following identity employed in chapter 3.

$$\int_{\Omega} \left[\sigma_{12}\, \phi_{,2} + \sigma_{13}\, \phi_{,3} \right] d\Omega \equiv 0$$

Proof

It is first noted that the exact shear stress distribution over $\Omega \subset \mathbb{R}^2$ can be obtained directly from the displacement field (3.15). From equations (3.11a), (3.12) and (3.13b) it follows that

$$\bar{\psi}''(x_1) = \frac{M'(x_1)}{EI_2} = -\frac{V(x_1)}{EI_2} = -\left[\frac{G\Omega}{EI_2}\right] \kappa\, \bar{\beta}(x_1) \tag{II.1}$$

and this relation, together with the expression (3.7) for the warping function $\phi(x_2, x_3) : \Omega \rightarrow \mathbb{R}$, the definition (3.13b) of the shear coefficient κ and the displacement field (3.15) leads to

$$\sigma_{12} = -\left[\frac{G\Omega}{EI_2}\right] \frac{V}{\Omega} \left[\chi_{,2} + \frac{\nu}{2}(x_2)^2 + (1-\frac{\nu}{2})(x_3)^2 \right]$$

$$\sigma_{13} = -\left[\frac{G\Omega}{EI_2}\right] \frac{V}{\Omega} \left[\chi_{,3} + (2+\nu)\, x_2 x_3 \right] \tag{II.2}$$

which is, of course, in complete agreement with the exact solution of the flexure problem.

Next, we proceed to prove the proposed identity by a direct computation. Since $\chi : \Omega \rightarrow \mathbb{R}$ is a harmonic function which satisfies the Neumman problem (3.2), the following identities hold

$$\int_\Omega |\nabla\chi|^2\, d\Omega = \int_{\partial\Omega} \chi \frac{\partial\chi}{\partial n}\, d\Gamma$$
$$= -\int_\Omega \left[\chi_{,2}\left\{\frac{\nu}{2}(x_2)^2 + (1-\frac{\nu}{2})(x_3)^2\right\} + \chi_{,3}(2+\nu)x_2x_3\right] dA$$
$$- 2(1+\nu)\int_\Omega x_2\chi\, d\Omega \qquad \text{(II.3)}$$

and

$$\int_\Omega \chi_{,2}\, d\Omega = \int_{\partial\Omega} x_2 \frac{\partial\chi}{\partial n}\, dS$$
$$= -(2+\frac{5}{2}\nu)I_2 - (1-\frac{1}{2}\nu)I_3 \qquad \text{(II.4)}$$

where $I_\alpha \equiv \int_\Omega (x_\alpha)^2\, d\Omega$, $(\alpha=2,3)$ are the principal moments of inertia of Ω.

When use is made of the identities (II.3) and (II.4), the stress distribution (II.2) together with (3.7) leads to

$$\int_\Omega \left(\phi_{,2}\sigma_{12} + \phi_{,3}\sigma_{13}\right) d\Omega = -\left(\frac{G\Omega}{EI_2}\right)^2 \frac{V}{\Omega}\left\{\int_\Omega \left((x_3)^2\chi_{,2} + 2x_2x_3\chi_{,3}\right) d\Omega\right.$$
$$\left. + \int_\Omega \left((6+\frac{9}{2}\nu)(x_2x_3)^2 + (1-\frac{\nu}{2})(x_3)^4\right) d\Omega\right\} \qquad \text{(II.5)}$$

However, equation (II.5) vanishes identically due to the fact that

$$\int_\Omega \left((x_3)^2\chi_{,2} + 2x_2x\chi_{,3}\right) d\Omega = \int_\Omega \left[\frac{\partial}{\partial x_2}\left(x_2(x_3)^2\chi_{,2}\right) + \frac{\partial}{\partial x_3}\left(x_2(x_3)^2\chi_{,3}\right)\right] d\Omega$$
$$= \int_{\partial\Omega} x_2(x_3)^2 \frac{\partial\chi}{\partial n}\, dS$$
$$= -\int_\Omega \left((6+\frac{9}{2}\nu)(x_2x_3)^2 + (1-\frac{1}{2}\nu)(x_3)^4\right) d\Omega$$

which completes the proof.

APPENDIX III

FORMAL VARIATIONAL STRUCTURE

In this appendix, we examine the formal variational structure of the formulations presented in Chapter 2 through 5. For simplicity, the kinematically exact rod theory presented in Chapter 2, sections 2.1 and 2.2, is taken as a model problem. The formulations developed in Chapters 3 and 4 follow exactly the same pattern, and the extension to the plate theory presented in Chapter 5 and summarized in table 5.1, is straightforward.

1.- Notation.

The following notation is introduced for convenience and employed throughout this appendix. The axial coordinate $X^1 \in (0,\ L)$ is designated simply by X.

$$
\begin{aligned}
\mathbf{x}_o &= [(X+u(X))\quad v(X)]^T \\
\mathbf{w} &= [\mathbf{x}_o\quad \psi]^T \\
\mathbf{R}_x &= [R^1\quad R^2\quad M]^T \\
\mathbf{R}_g &= [N\quad V\quad M]^T \\
\boldsymbol{\Lambda}(\psi) &= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
\tag{III.1}
$$

The components R^1 and R^2 of the resultant force $\mathbf{R} = R^i\hat{\mathbf{e}}_i = N\hat{\mathbf{n}} + V\hat{\mathbf{l}}$ and the resultant moment M are defined as in Chapter 1, namely

$$
R^i(X) = \int_\Omega P^{i1}(\mathrm{X})\ d\Omega\ ,\quad (i=1,2),\qquad M(X) = \epsilon_{ij3}\int_\Omega [\Phi(\mathrm{X}) - \mathrm{x}_o]^i\ P^{j1}(\mathrm{X})\ d\Omega \tag{III.2}
$$

Thus, the set of components $\mathbf{R}_x$ and $\mathbf{R}_g$ with respect to the spatial and Gaussian frames, defined in (III.1), are related as

$$\mathbf{R}_x = \mathbf{\Lambda}(\psi)\ \mathbf{R}_g \tag{III.3}$$

In addition, we introduce the linear operator $\mathbb{F}:[H^1(0, L)]^3 \to [L_2(0, L)]^3$ defined by

$$\mathbf{w} \to \mathbb{F}(\mathbf{w}) = \begin{Bmatrix} \mathbf{x}_o' \\ \psi' \end{Bmatrix} \tag{III.4}$$

where a "prime" designates $\frac{d}{dX}$. Therefore, the strain measures $[\lambda_n-1]$, λ_l and ψ' associated with N , V and M, and defined by (2.24), may be written as

$$\begin{Bmatrix} \lambda_n-1 \\ \lambda_l \\ \psi' \end{Bmatrix} = \mathbf{\Lambda}^T(\psi)\begin{Bmatrix} 1+u'-\cos\psi \\ v'-\sin\psi \\ \psi' \end{Bmatrix}$$
$$= \mathbf{\Lambda}^T(\psi)[\mathbb{F}(\mathbf{w}) - \hat{\mathbf{n}}] \tag{III.5}$$

where we have set $\hat{\mathbf{n}} = [\cos\psi \ \ -\sin\psi \ \ 0]^T$. The notation $\boldsymbol{\lambda}_g = [(\lambda_n-1) \ \ \lambda_l \ \ \psi']^T$ will be often employed.

Remarks

(i) Since $\boldsymbol{\lambda}_g = [(\lambda_n-1) \ \ \lambda_l \ \ \psi']^T$ are the strain measures associated with $\mathbf{R}_g$, and both $\mathbf{R}_x$ and $\mathbf{R}_g$ are related through (III.3), equation (III.5) shows that $[\mathbb{F}(\mathbf{w}) - \hat{\mathbf{n}}]$ are the strain measures associated with the components $\mathbf{R}_x$ Obviously, $[\mathbb{F}(\mathbf{w}) - \hat{\mathbf{n}}]$ and $\boldsymbol{\lambda}_g$ define the same object in different coordinate systems.

(ii) If the rod undergoes pure bending

$$1 + u' - \cos\psi \equiv v' - \sin\psi \equiv 0 \quad <=> \quad \lambda_n \equiv \lambda_l \equiv 0$$

and, therefore, $N \equiv V \equiv 0$ in agreement with our intuition.

(iii) Clearly, the strain measures $\boldsymbol{\lambda}_g$ are invariant under superimposed rigid body motions.

2.- Weak form of the equilibrium equations.

To further simplify the notation, let us designate by ∂_w the directions at either $X = 0$ or $X = L$ along which some or all the components of $\mathbf{w} = [\mathbf{x}_o \ \ \psi]^T$ are prescribed to values designated by $\bar{\mathbf{w}}$. Similarly, $\partial_{\mathbf{R}}$ will designate the directions at $X = 0$, L along which entries in $\mathbf{R}_x$

are prescribed to have values $\tilde{\mathbf{R}}_x$. We require $\partial_w \cap \partial_\mathbf{R} = \emptyset$. The linear space of kinematically admissible variations may then be defined as

$$W = \left\{ \eta \in [H^1(0,\ L)]^3 \ |\ \ \eta|_{\partial_w} = 0 \right\} \tag{III.6}$$

Consider next the equilibrium equation (2.10) for the bending moment M. Upon noting that

$$\begin{Bmatrix} 2E_{12} \\ -J_0 \end{Bmatrix} \equiv \begin{Bmatrix} \lambda_l \\ \lambda_n \end{Bmatrix} = \begin{bmatrix} -\sin\psi & \cos\psi \\ -\cos\psi & -\sin\psi \end{bmatrix} \begin{Bmatrix} (1+u') \\ v' \end{Bmatrix} \tag{III.7}$$

equation (2.10) may be written as

$$-\frac{dM}{dX} + \mathbb{F}^T(\mathbf{w}) . \frac{d\mathbf{\Lambda}^T}{d\psi}(\psi) . \mathbf{R}_g = 0 \tag{III.8}$$

Combining (III.8) with the equilibrium equation (2.8a) for the components N and V of the resultant force $\mathbf{R}$, and making use of the (III.3), the weak form of the equilibrium equations takes the form

$$\begin{aligned} G(\boldsymbol{\eta}, \mathbf{w}) = \int_0^L \left[\mathbb{F}^T(\boldsymbol{\eta}) . \mathbf{\Lambda}(\psi) \quad \eta_3 \mathbb{F}^T(\mathbf{w}) . \frac{d\mathbf{\Lambda}}{d\psi}(\psi) \right] . \mathbf{R}_g(\boldsymbol{\lambda}_g)\ dX \\ - \int_0^L q(X)\, v(X)\ dX - \left[\boldsymbol{\eta} . \tilde{\mathbf{R}}_x \right]_{\partial_\mathbf{R}} = 0\ , \quad \text{for } any \quad \boldsymbol{\eta} \in V \end{aligned} \tag{III.9}$$

As shown in section 2.2.3, the forces $\mathbf{R}_g(\boldsymbol{\lambda}_g)$ derive from the strain energy potential $W(\lambda_n, \lambda_l, \psi')$ according to

$$\mathbf{R}_g = \frac{\partial W(\boldsymbol{\lambda}_g)}{\partial \boldsymbol{\lambda}_g} \tag{III.10}$$

Thus, for the simple constitutive model (2.26) proposed in Chapter 2, equation (III.10) can be written in the form

$$\mathbf{R}_g(\boldsymbol{\lambda}_g) = \mathbf{D} . \boldsymbol{\lambda}_g = \mathbf{D} . \mathbf{\Lambda}^T(\psi) . [\mathbb{F}(\mathbf{w}) - \hat{\mathbf{n}}] \tag{III.11a}$$

where

$$\mathbf{D} = diag[E\Omega \quad G\Omega \quad EI] \tag{III.11b}$$

3.- Total potential energy functional

From equations (III.9) and (III.11) it immediately follows that

$$G(\eta, \mathbf{w}) = \frac{d}{d\alpha}\Big[\Pi(\mathbf{w}+\alpha\eta)\Big]_{\alpha=0} \tag{III.12}$$

where $\Pi(\mathbf{w})$ is the total potential energy defined by

$$\Pi(\mathbf{w}) = \int_0^L [\mathbb{F}(\mathbf{w}) - \hat{\mathbf{n}}]^T.\mathbf{\Lambda}(\psi)\mathbf{D}\mathbf{\Lambda}^T(\psi).\{\mathbb{F}(\mathbf{w}) - \hat{\mathbf{n}}\}\, dX - \int_0^L q(X)\,v(X)\,dX \tag{III.13}$$

Equation (III.13) is perfectly consistent with the fact that the material is hyperelastic with strain energy given by (2.27).

4.- Linearization of the weak form.

The linearization of the weak form of the equilibrium equations about a intermediate configuration plays a key role in numerical implementations employing an iterative solution procedure. A complete account of linearization procedures in the general context of infinite dimensional manifolds can be found in [1]. For the development that follows the relevant results have been outlined in section 2.4.1.

Consider an intermediate configuration $\bar{\mathbf{w}} : \mathbb{R} \to \mathbb{R}^3$. Let $\Delta\mathbf{w} : \mathbb{R} \to \mathbb{R}^3$ be a superimposed infinitesimal deformation; that is, a vector field covering $\bar{\mathbf{w}}(X)$. The linear part of $G(\eta, \mathbf{w})$ at $\bar{\mathbf{w}}(X)$, denoted by $L[G]_{\bar{\mathbf{w}}}$, can be computed as

$$L[G]_{\bar{\mathbf{w}}} = \frac{d}{d\alpha}\Big[G(\eta,\, \bar{\mathbf{w}} + \alpha\eta)\Big]_{\alpha=0} \tag{III.14}$$

Thus, since $G(\eta, \mathbf{w})$ is defined by (III.9), equation (III.14) leads to

$$L[G]_{\bar{\mathbf{w}}} = \int_0^L \left\{\mathbb{F}^T(\eta).\frac{d\mathbf{\Lambda}}{d\psi}(\bar{\psi})\Delta\psi + \mathbb{F}^T(\Delta\mathbf{w}).\frac{d\mathbf{\Lambda}}{d\psi}(\bar{\psi})\eta_3 + \mathbb{F}^T(\bar{\mathbf{w}}).\frac{d^2\mathbf{\Lambda}}{d\psi^2}(\bar{\psi})\Delta\psi\eta_3\right\}.\bar{\mathbf{R}}_g\, dX$$
$$+ \int_0^L \left\{\mathbb{F}^T(\eta)\,\mathbf{\Lambda}(\bar{\psi}) + \mathbb{F}^T(\bar{\mathbf{w}}).\frac{d\mathbf{\Lambda}}{d\psi}(\bar{\psi})\eta_3\right\}.L[\mathbf{R}_g]_{\bar{\mathbf{w}}}\, dX \tag{III.15}$$

If (III.14) is again applied to the constitutive relation (III.11), the linear part $L[\mathbf{R}_g]_{\bar{\mathbf{w}}}$ of $\mathbf{R}_g$ at $\bar{\mathbf{w}}(X)$ can be written as

$$L[\mathbf{R}_g]_{\bar{\mathbf{w}}} = \bar{\mathbf{R}}_g + \mathbf{D}\left[\Delta\psi \frac{d\mathbf{\Lambda}^T}{d\psi}(\bar{\psi}).\mathbb{F}(\bar{\mathbf{w}}) + \mathbf{\Lambda}^T(\bar{\psi}).\mathbb{F}(\Delta\mathbf{w})\right]$$

$$= \bar{\mathbf{R}}_g + \mathbf{D}\left[\mathbf{\Lambda}^T(\bar{\psi}) \quad \frac{d\mathbf{\Lambda}^T}{d\psi}(\bar{\psi}).\mathbb{F}(\bar{\mathbf{w}})\right]\begin{Bmatrix}\mathbb{F}^T(\Delta\mathbf{w}) \\ \Delta\psi\end{Bmatrix} \tag{III.16}$$

The linearized weak form of the field equations for the problem at hand is obtained by substituting the expression for $L[\mathbf{R}_g]_{\bar{\mathbf{w}}}$ given by (III.16) into (III.15). Employing matrix notation, the final result can be conveniently written as

$$L[G]_{\bar{\mathbf{w}}} = \int_\Omega [\mathbb{F}^T(\eta) \quad \eta_3]\left[\mathbf{K} + \mathbf{K}_G\right]\begin{Bmatrix}\mathbb{F}^T(\Delta\mathbf{w}) \\ \Delta\psi\end{Bmatrix} dX + G(\eta, \Delta\mathbf{w}) \tag{III.17}$$

Where $\mathbf{K}_G$ and $\mathbf{K}$ have the expressions

$$\mathbf{K}_G = \begin{bmatrix} \mathbf{O} & \frac{d\mathbf{\Lambda}}{d\psi}(\bar{\psi}).\bar{\mathbf{R}}_g \\ \bar{\mathbf{R}}_g^T.\frac{d\mathbf{\Lambda}^T}{d\psi}(\bar{\psi}) & \mathbb{F}^T(\bar{\mathbf{w}}).\frac{d^2\mathbf{\Lambda}}{d\psi^2}(\bar{\psi}).\bar{\mathbf{R}}_g \end{bmatrix} \tag{III.18}$$

and

$$\mathbf{K} = \begin{Bmatrix} \mathbf{\Lambda}(\bar{\psi}) \\ \mathbb{F}^T(\bar{\mathbf{w}}).\frac{d\mathbf{\Lambda}}{d\psi}(\bar{\psi}) \end{Bmatrix} \mathbf{D}\left[\mathbf{\Lambda}^T(\bar{\psi}) \quad \frac{d\mathbf{\Lambda}^T}{d\psi}(\bar{\psi}).\mathbb{F}(\bar{\mathbf{w}})\right] \tag{III.19}$$

$\mathbf{K}_G$ is often referred to as the *geometric stiffness* and $\mathbf{K}_T = \mathbf{K}_G + \mathbf{K}$ represents the *tangent stiffness* at the intermediate configuration $\bar{\mathbf{w}} : \mathbb{R} \rightarrow \mathbb{R}^3$. The *residual* or out-of-balance forces at the configuration $\bar{\mathbf{w}}(X)$ is given by

$$-\,G(\eta, \bar{\mathbf{w}}) = [\eta.\tilde{\mathbf{R}}_x]_{\partial_R} - \int_0^L [\mathbb{F}^T\eta \quad \eta_3]\begin{Bmatrix} \mathbf{\Lambda}(\bar{\psi}) \\ \mathbb{F}^T\bar{\mathbf{w}}.\frac{d\mathbf{\Lambda}}{d\psi}(\bar{\psi}) \end{Bmatrix}.\bar{\mathbf{R}}_g \, dX \tag{III.20}$$

and vanishes identically if $\bar{\mathbf{w}}(X)$ is an equilibrium configuration. By setting $L[G]_{\bar{\mathbf{w}}} = 0$ for any $\eta \in W$ in (III.17) one obtains a classical variational problem from which the incremental deformation $\Delta\mathbf{w} : \mathbb{R} \rightarrow \mathbb{R}^3$, such that $\Delta\mathbf{w}(X) - \tilde{\mathbf{w}}(X) \in W$, may be obtained. The numerical treatment of this problem by a finite element technique involves the discretization of the open interval $(0, L)$ and the interpolation of $\eta \in W$ and $\Delta\mathbf{w}(X)$ by means of shape functions. The procedure is standard and details, which may be found in the literature [2,3,4], are omited.

References

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