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A State-Space Method for Vibration of Double-Beam Systems with Variable Cross-Sections

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Abstract: In this paper, a state-space method for double-beam systems with variable cross-sections is developed, making it possible to calculate the transverse vibration of the double-beams accurately and effectively. Due to the variability of double-beam cross-sections with the viscoelastic interlayer in-between, the governing equations of vibration for the systems become highly coupled partial differential equations, making the problem difficult to solve. A basic double-beam system is introduced to modify the original governing equations to two inhomogeneous differential equations. With the separation of variables, several mode-shape coefficients and a state variable are defined to construct the state-space equations. The coupling terms and variables are transferred into the constant coefficient matrix of the state-space equations, making them decoupled. Numerical procedures are presented to solve the state-space equations to obtain the homogenous and inhomogeneous solutions including the natural frequencies and mode shapes in free vibration and dynamic responses in forced vibration,

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respectively. The method has substantial advantages in decoupling high-order partial differential equations and can be further extended to solve complex structural systems. Numerical results also demonstrate that the method is accurate and efficient. An engineering application with a rail-bridge with floating slab track is finally discussed in detail with the method.

Keywords: Double-beam system; Variable cross-section; Transverse vibration; State-space method.

Introduction

Double-beam systems are a series of idealized structural models, in which there are two parallel beams (upper and lower ones) connected with an interlayer. Numerous engineering applications of them can be found in fields of aerospace, civil, and mechanical engineering, including floating slab tracks (Xin and Gao 2011), sandwich and composite beams (Arikoglu and Ozkol 2010), continuous dynamic vibration absorbers and isolators (Kawazoe et al. 1998; Hu et al. 2023), double-beam cranes (Zhang et al. 2008), and double-walled carbon nanotubes (Murmu and Adhikari 2012). To address issues associated to structural dynamics, vibration control and optimal design, much attention has been drawn to the dynamic characteristics of double-beam systems in the past decades.

Since 1964, three groups of the dynamics of double-beam systems have been developed. In the first group, double-beam systems are without interlayer damping (Oniszczuk 2000; Li and Sun 2015; Mao and Wattanasakulpong 2015) or with viscoelastic interlayer but with restrictions such as two identical beams (Vu et al. 2000; Wu and Gao 2015) and simply supported boundary conditions (Pavlovic et al. 2012; Wu and Gao 2015). In the second group, simple viscoelastic interlayer, two different beams and arbitrary boundary conditions are considered in the systems

(Li et al. 2016). In the third group, the models become more complicated. They consider more specific facts, such as Timoshenko beams (Zhang et al. 2014), realistic interlayer types (Brito et al. 2019; Li et al. 2021a), partially distributed foundation (Liu and Yang 2019), constant axial loads (Liu and Yang 2019; Han et al. 2020), cracked beams (Chen et al. 2021a), and hygrothermal environments (Chen et al. 2021b). Even with numerous research attempts, few have studied the double-beams with variable cross-section, which are widely spread in real beam-type structures.

If the variable cross-section is adopted, the equations of vibration would be changed as the coupled partial differential equations with variable coefficients, which is more difficult to be decoupled. Manconi and Mace (2017) analyze the coupling problems between multiple flexible structures. The perturbation method is used to study both weak and strong couplings for both discrete and continuous systems. However, their investigation is mainly about mode veering. An analytic framework (Zhang et al. 2014) is developed to study vibrations of double Timoshenko beams with variable cross sections and various discontinuities. The dynamical responses are solved by dividing the entire system into a series of distinct components and organizing the compatibility and boundary conditions. Using a modified transfer matrix method, the dynamics of a discretely connected double-beam system are analyzed (Wang et al. 2016), and the variation of cross section is considered. Zhou et al. (2023) propose an approximate discretization method for vibration of a viscoelastic connected Timoshenko double-beam system with variable cross section. The above studies all require discrete structures, which lead to many structural elements. The computation is time-consuming, and the efficiency of them is reduced significantly. Although Li et al. (2021) obtain closed-form solution for vibration of functionally graded beam with variable cross-sections, the varying of cross-section must be continuous and the boundary conditions are simply supported.

State-space approach has proven to solve the complex and coupling problems (Lee et al. 2007; Khdeir and Aldraihem 2016). Based on previous works on state-space approach (Palmeri et al. 2003; Palmeri and Ntotsios 2016), Palmeri and Adhikari (2011) investigated the transverse vibrations of a double-beam system with inhomogeneous beams, arbitrary boundary conditions, and rate-dependent interlayer. The mode shapes used for assumed shapes were buckling modes instead of vibration modes. Further, the damping was not considered in free vibration while simplified Rayleigh's damping model was adopted in forced vibration. Although improved state-space approaches (Li et al. 2021a, 2021b) were proposed to solve vibrations of damped double-beam systems, the beams are both with uniform cross-section.

The investigation presented in this paper is on the double-beam systems with variable cross-sections. For the beams with variable cross-sections, many previous analysis methods require the spatial discretization of the structure, such as finite element method and other similar approaches. In addition, other methods discretize the structure in the modal space, such as the assumed mode method. The total number of the structural elements and the total number of the assumed modes directly affect the solutions of these two types of approaches, respectively. While both total numbers are determined by the frequency range of interest, the conclusions presented in the literature suggest that the required number of modes is smaller than the number of structural elements. The computational efficiency of the assumed mode method is thus more favorable. Based on these approaches and conclusions, an improved state-space method is developed in this paper. A basic double-beam system with uniform cross-section and pure elastic interlayer is first introduced. The original equations of vibration of the to-be-solved system are modified by that basic system. Furthermore, three modal coefficients are defined together with a state variable. Using these constants and state variables, the governing equations in time space are decoupled into a set of first-order state-space equations. The natural frequencies and mode shapes in free vibration and dynamic responses in forced vibration are solved based on the derived state-space equations. Finally, the proposed method is verified with several numerical examples and results from the finite element method. A realistic engineering application, which is a beam bridge with floating slab track, is analyzed to illustrate the practical application value of the proposed research work.

Mechanical model and governing differential equations

The mechanical model of the double-beam system with variable cross-section is shown in Fig. 1(a), in which two slender beams are interconnected with a viscoelastic interlayer. The two slender beams with length L are homogeneous. The assumptions for the studied systems include that (1) two slender beams are Euler-Bernoulli beams, (2) deformation of two beams is in the linear elastic range, (3) forces exerted on the two beams are transverse, (4) the variable cross-sections of two beams are symmetric variables with respect to their central axes, and (5) the change in the cross-sections must be continuous and smooth, without any abrupt changes.

Based on the above fundamental assumptions, the coupled governing equations for the vibration of whole systems are expressed as:

$$
\begin{aligned}
&e_1(x)\cdot W_1''''(x,t) + 2\cdot e_1'(x)\cdot W_1'''(x,t) + e_1''(x)\cdot W_1''(x,t) \\
&+K\left[W_1(x,t) - W_2(x,t)\right] + C\left[\dot{W}_1(x,t) - \dot{W}_2(x,t)\right] + \overline{m}_1(x)\cdot \ddot{W}_1(x,t) = F_1(x,t)
\end{aligned}
$$

(1a)

$$e_2(x)\cdot W_2''''(x,t)+2\cdot e_2'(x)\cdot W_2'''(x,t)+e_2''(x)\cdot W_2''(x,t)$$
$$-K\left[W_1(x,t)-W_2(x,t)\right]-C\left[\dot{W}_1(x,t)-\dot{W}_2(x,t)\right]+\bar{m}_2(x)\cdot\ddot{W}_2(x,t)=F_2(x,t)$$

(1b)

where $W_i(x,t)$ is transverse deflection of either beam (i = 1 or 2 representing the upper beam or lower beam), x and t are the spatial co-ordinate and the time, the prime notations indicate partial derivatives with respect to x, the dot notations indicate partial derivatives with respect to t, $e_i(x)$ and $\bar{m}_i(x)$ are beam flexural rigidity and beam mass per unit length, K and C are the stiffness coefficient and damping coefficient per unit length of the viscoelastic interlayer, and $F_1(x,t)$, $F_2(x,t)$ are the exciting forces acting on the upper and lower beams, respectively.

The boundary conditions at the ends (x=0, L) of two beams are arbitrary. Commonly used ones can be found in literature (Li et al. 2021a). The initial conditions of Eq. (1) of the two beams are

$$W_i(x,0)=W_{i0}(x)$$

(2a)

$$\dot{W}_i(x,0)=V_{i0}(x)$$

(2b)

Referring to the perturbation method, the present study defines a basic double-beam system here. The solutions of the basic double-beam system are used to form a basic solution space. In this space and with the concept of the state-space, the solutions of Eq. (1) will be solved. The mechanical model of such basic double-beam system is shown as Fig. 1(b). The cross-sections of two slender beams are constant. The masses and flexural rigidities of the two beams are uniform. The interlayer is purely elastic. The lengths and boundary conditions of the two beams are the same as the ones in to-be-solved double-beam system. Following the same assumptions, the governing equations for the free vibration of the basic double-beam system are written as follows.

$$e_1 \cdot W_1''''(x,t) + K_0\left[W_1(x,t) - W_2(x,t)\right] + \bar{m}_1 \cdot \ddot{W}_1(x,t) = 0$$

(3a)

$$e_2 \cdot W_2''''(x,t) - K_0\left[W_1(x,t) - W_2(x,t)\right] + \bar{m}_2 \cdot \ddot{W}_2(x,t) = 0$$

(3b)

where e_i and $\bar{m}_i$ are constant beam flexural rigidity and constant beam mass per unit length, and K_0 is the stiffness coefficient per unit length of the pure elastic interlayer.

Free-vibration characteristics

Solution of natural frequencies

The natural frequencies of the to-be-solved system are analyzed in this subsection. In the present study, the solutions of the basic system are used to construct the solution space and solve the dynamic responses of to-be-solved system. First, it requires to introduce these basic solutions into Eq. (1), which are the governing equation of to-be-solved system. Specifically, Eq. (3) is adopted to modify Eq. (1) and make the left side of Eq. (1) to be the same as the left side of Eq. (3). The revised Eq. (1) becomes a kind of governing equations of a forced vibration for the basic double-beam system. Therefore, the basic solutions can be substituted into the modified equation and the orthogonality condition can be applied. Following the above introductions and vanishing the exciting forces $F_1(x,t)$ and $F_2(x,t)$, Eq. (1) can be rewritten for the free vibration as follows:

$$
\begin{aligned}
& e_1 W_1''''(x,t) + K_0\left[W_1(x,t) - W_2(x,t)\right] + \bar{m}_1 \ddot{W}_1(x,t) = \left[e_1 - e_1(x)\right] W_1''''(x,t) \\
& - 2e_1'(x) \cdot W_1'''(x,t) - e_1''(x) \cdot W_1''(x,t) + (K_0 - K)\left[W_1(x,t) - W_2(x,t)\right] \\
& - C\left[\dot{W}_1(x,t) - \dot{W}_2(x,t)\right] + \left[\bar{m}_1 - \bar{m}_1(x)\right] \cdot \ddot{W}_1(x,t)
\end{aligned}
$$

(4a)

$$
\begin{aligned}
& e_2 W_2''''(x,t) - K_0\left[W_1(x,t) - W_2(x,t)\right] + \bar{m}_2 \ddot{W}_2(x,t) = \left[e_2 - e_2(x)\right] W_2''''(x,t) \\
& - 2e_2'(x) \cdot W_2'''(x,t) - e_2''(x) \cdot W_2''(x,t) - (K_0 - K)\left[W_1(x,t) - W_2(x,t)\right] \\
& + C\left[\dot{W}_1(x,t) - \dot{W}_2(x,t)\right] + \left[\bar{m}_2 - \bar{m}_2(x)\right] \cdot \ddot{W}_2(x,t)
\end{aligned}
$$

(4b)

By separating the variables, the assumed solutions of Eq. (4) can be expressed as

$$W_i(x,t) = \sum_{n=1}^{\infty} T_n(t)\,\bar{\phi}_m(x), \qquad (i = 1, 2)$$

(5)

where $T_n(t)$ is the time function and $\bar{\phi}_m(x)$ is the mode shape function of two beams. It is worth noting that $\bar{\phi}_m(x)$ are the mode shapes of the basic double-beam system.

Substituting Eq. (5) into left side of Eq. (4) and applying the orthogonality condition of the basic double-beam system from (Li and Sun 2015), the key equation of to-be-solved double-beam system can be derived out by following the similar derivations developed in (Li and Sun 2015):

$$\ddot{T}_n(t) + \bar{\omega}_n^2 T_n(t) = F_{nW4}(t) + F_{nW3}(t) + F_{nW2}(t) + F_{nK}(t) + F_{nC}(t) + F_{nM}(t) \tag{6}$$

and $\bar{\omega}_n$ is the natural frequencies of basic double-beam system.

Third, three mode-stiffness coefficients, a mode-shape coefficient and a mode-mass coefficient is proposed herein to simplify the coupling terms of the right side of Eq. (6). The assumed solutions to Eq. (4), which could also be denoted as $W_i(x,t) = \sum_{r=1}^{\infty} T_r(t)\,\bar{\phi}_{ri}(x)$, are substituted into the terms of the right side of Eq. (4), and the orthogonality computation is carried out. Then, all the terms of the right side of Eq. (6) and these coefficients are derived out specifically in Appendix A.

Fourth, a state-space approach is proposed to decouple all equations in state space. If there are total *N* modes considered, a state variable $\mathbf{Z}_M(t) = \begin{bmatrix} T_1(t) & \cdots & T_N(t) & \dot{T}_1(t) & \cdots & \dot{T}_n(t) \end{bmatrix}^T$ with dimension 2*N* by 1 is introduced. The key equation Eq. (6) can be written in a state form as:

$$\dot{\boldsymbol{Z}}_M(t) = \boldsymbol{J}_M \cdot \boldsymbol{Z}_M(t) + \boldsymbol{K}_M \cdot \left[\boldsymbol{F}_{MW4}(t) + \boldsymbol{F}_{MW3}(t) + \boldsymbol{F}_{MW2}(t) + \boldsymbol{F}_{MK}(t) + \boldsymbol{F}_{MC}(t) + \boldsymbol{F}_{MM}(t)\right] \tag{7}$$

where

$$\boldsymbol{J}_M = \begin{bmatrix} [\boldsymbol{0}] & [\boldsymbol{I}] \\ [-\bar{\omega}_n^2] & [\boldsymbol{0}] \end{bmatrix}$$

(8a)

$$[-\bar{\omega}_n^2] = diag\left(-\bar{\omega}_1^2 \quad -\bar{\omega}_2^2 \quad \cdots \quad -\bar{\omega}_N^2\right) \tag{8b}$$

$$\boldsymbol{K}_M = \begin{bmatrix} [\boldsymbol{0}] & [\boldsymbol{I}] \end{bmatrix}^T$$

(8c)

and $[\boldsymbol{0}]$, $[\boldsymbol{I}]$, $\left[-\bar{\omega}_n^2\right]$ are all with dimension N by N. $\boldsymbol{F}_{\boldsymbol{MWs}}(t) = [F_{1Ws}(t)\cdots F_{NWs}(t)]^T$ (s=2, 3 or 4), $\boldsymbol{F}_{\boldsymbol{MK}}(t) = [F_{1K}(t)\cdots F_{NK}(t)]^T$, $\boldsymbol{F}_{\boldsymbol{MC}}(t) = [F_{1C}(t)\cdots F_{NC}(t)]^T$ and $\boldsymbol{F}_{\boldsymbol{MM}}(t) = [F_{1M}(t)\cdots F_{NM}(t)]^T$ are three stiffness vectors, an elastic force vector, a damping force vector and an inertial force vector, respectively. All of them are with dimension N by 1. According to Eq. (A.1) to Eq. (A.7) and state-variable $\boldsymbol{Z}_{\boldsymbol{M}}(t)$, those equivalent forces vectors in Eq. (7) are derived out and listed in Appendix B.

With those coefficient matrices in a state form, Eq. (7) is rewritten as:

$$\dot{\boldsymbol{Z}}_M(t) = \boldsymbol{J}_M^* \cdot \boldsymbol{Z}_M(t) \tag{9}$$

where

$$\boldsymbol{J}_M^* = \left[[\boldsymbol{I}] - \boldsymbol{K}_M \left\{\boldsymbol{\Phi}_{MMC}\right\}\right]^{-1} \cdot \left\{\boldsymbol{J}_M + \boldsymbol{K}_M \cdot \left[\left[\boldsymbol{\Phi}_{MW4C}\right] + \left[\boldsymbol{\Phi}_{MW3C}\right] + \left[\boldsymbol{\Phi}_{MW2C}\right] + (K_0 - K)\left\{\boldsymbol{\Phi}_{MKC}\right\} + (-C)\left\{\boldsymbol{\Phi}_{MCC}\right\}\right]\right\}$$

(10)

It is worth noting that $\boldsymbol{J}_M^*$ is a state-space representation with dimension $2N$ by $2N$. As a result, Eq. (9) and Eq. (10) are the final state-space equations for the free vibration of the double-beam system with variable cross-section.

The solutions to free vibration of to-be-solved system with the corresponding equations of vibration Eq. (1) for $F_1(x,t) = F_2(x,t) = 0$, can be written as $W_1(x,t) = \sum_{n=1}^{\infty} e^{j\omega_n t}\phi_{n1}(x)$ and $W_2(x,t) = \sum_{n=1}^{\infty} e^{j\omega_n t}\phi_{n2}(x)$, where $\phi_{n1}(x)$ and $\phi_{n2}(x)$ are the mode shape functions of upper beam and lower beam, respectively. Further, $T_n(t) = e^{j\omega_n t}$ is the time function in which ω_n is the natural frequency and $j=\sqrt{-1}$ is imaginary unit. It is worth noting that $\phi_{n1}(x)$, $\phi_{n2}(x)$ and ω_n in here are all for to-be-solved double-beam system instead of the basic double-beam system.

Substituting $T_n(t)=e^{j\omega_n t}$ into the state variable $\boldsymbol{Z_M}(t)$ yields

$$\dot{\boldsymbol{Z}}_M(t)=\bar{\boldsymbol{J}}_M\boldsymbol{Z}_M(t)$$

(11)

where

$$\bar{\boldsymbol{J}}_M=\begin{bmatrix}[\boldsymbol{0}] & [\boldsymbol{I}]\\ [-\omega_n^2] & [\boldsymbol{0}]\end{bmatrix}$$

(12a)

$$[-\omega_n^2]=diag(-\omega_1^2 \quad -\omega_2^2 \quad \cdots \quad -\omega_N^2) \tag{12b}$$

and submatrices $[\boldsymbol{0}]$, $[\boldsymbol{I}]$ and $[-\omega_n^2]$ in Eq. (12) are all with dimension *N* by *N*.

Equations (9) and (11) are both the state equations of a same double-beam system. The state variables of those two equations are also same. Therefore, the eigenvalues of $\boldsymbol{J}_M^*$ in Eq. (9) must be the same as the eigenvalues of $\bar{\boldsymbol{J}}_M$ in Eq. (11). The eigenvalues of $\bar{\boldsymbol{J}}_M$ are $\lambda_n^{1,2}=\pm\omega_n j$, which is directly related to the natural frequencies of the system. Once the eigenvalues of $\boldsymbol{J}_M^*$ are obtained by analytical or numerical methods, the natural frequencies of to-be-solved double-beam system could be determined. The relationship between the eigenvalues of $\boldsymbol{J}_M^*$ and natural frequencies of to-be-solved double-beam system is $\lambda_n^{1,2}=-\xi\omega_{n,Undamped}\pm\sqrt{1-\xi^2}\,\omega_{n,Undamped}\,j$, in which

$\omega_{n,Undamped}$ is undamped natural frequency and $\omega_{n,Damped} = \sqrt{1-\xi^2}\,\omega_{n,Undamped}$ is damped natural frequency.

Solution of mode shapes

Based on the obtained natural frequencies, the corresponding mode shapes are calculated in this subsection. Similarly, the mode shapes of the basic double-beam system are the basic solutions and they are used to construct the solution space. According to the modal perturbation method (Lou and Chen 2003), the n-th mode shape of to-be-solved double-beam system are assumed as

$$\phi_{ni}(x) = \bar{\phi}_{ni}(x) + \Delta\bar{\phi}_{ni}(x) = \begin{bmatrix} q_{n1i} & \cdots & q_{nNi} \end{bmatrix} \begin{bmatrix} \bar{\phi}_{1i}(x) & \cdots & \bar{\phi}_{Ni}(x) \end{bmatrix}^T = \boldsymbol{Q}_{ni} \cdot \boldsymbol{\Phi}_i(x), \quad (i = 1, 2) \tag{13}$$

in which $\bar{\phi}_{ni}(x)$ are the n-th mode shapes of basic double-beam system, and $\Delta\bar{\phi}_{ni}(x)$ are the n-th additional perturbation increments. q_{nhi} are corresponding Lagrangian coordinates associated with the mode shape $\bar{\phi}_{ni}(x)$, and $\boldsymbol{Q}_{ni}$ = N-dimensional vectors with $q_{nni} = 1$.

The assumed solutions, which represent the n-th mode only, to the free vibration of to-be-solved double-beam system are expressed as

$$W_{ni}(x,t) = e^{j\omega_n t}\phi_{ni}(x), \quad (i=1,2) \tag{14}$$

Substituting Eq. (13) and Eq. (14) into Eq. (1) with $F_1(x,t) = F_2(x,t) = 0$, the equations of vibration to the free vibration in n-th mode could be rewritten as

$$e_1(x)\cdot Q_{n1}\cdot\bar{\Phi}_1^{(4)}(x)+2\cdot e_1'(x)\cdot Q_{n1}\cdot\bar{\Phi}_1'''(x)+e_1''(x)\cdot Q_{n1}\cdot\bar{\Phi}_1''(x)$$
$$+(K+Cj\omega_n)\left[Q_{n1}\cdot\bar{\Phi}_1(x)-Q_{n2}\cdot\bar{\Phi}_2(x)\right]-\bar{m}_1(x)\cdot\omega_n^2\cdot Q_{n1}\cdot\bar{\Phi}_1(x)=0$$

(15a)

$$e_2(x)\cdot Q_{n2}\cdot\bar{\Phi}_2^{(4)}(x)+2\cdot e_2'(x)\cdot Q_{n2}\cdot\bar{\Phi}_2'''(x)+e_2''(x)\cdot Q_{n2}\cdot\bar{\Phi}_2''(x)$$
$$-(K+Cj\omega_n)\left[Q_{n1}\cdot\bar{\Phi}_1(x)-Q_{n2}\cdot\bar{\Phi}_2(x)\right]-\bar{m}_2(x)\cdot\omega_n^2\cdot Q_{n2}\cdot\bar{\Phi}_2(x)=0 \quad (15b)$$

Further, with $\bar{\phi}_{k1}(x)$ ×Eq. (15a) and $\bar{\phi}_{k2}(x)$ ×Eq. (15b) as well as integrating them with respect to *x* from 0 to *L*, each term in Eq. (15) is derived out and listed specifically in Appendix C.

Reorganizing all the terms as shown in Eq. (C.1) to Eq. (C.7), the integral of $\bar{\phi}_{k1}(x)$ ×Eq. (15a) and $\bar{\phi}_{k2}(x)$ ×Eq. (15b) with respect to *x* from 0 to *L* are finally expressed as

$$Q_{n1}\cdot\left[\Phi_{k14}+2\Phi_{k13}+\Phi_{k12}+(K+Cj\omega_n)\Phi_{k110}-\omega_n^2\Phi_{k1M}\right]-(K+Cj\omega_n)Q_{n2}\Phi_{k120}=0$$

(16a)

$$Q_{n2}\cdot\left[\Phi_{k24}+2\Phi_{k23}+\Phi_{k22}+(K+Cj\omega_n)\Phi_{k220}-\omega_n^2\Phi_{k2M}\right]-(K+Cj\omega_n)Q_{n1}\Phi_{k210}=0$$

(16b)

Setting the values of *k* as 1 to *N* and combining all the integral equations together, the final algebra equations for *n*-th mode shapes of to-be-solved double-beam system would be

$$\begin{bmatrix} P_{111} & P_{121} \\ P_{211} & P_{221} \\ \vdots & \vdots \\ P_{11N} & P_{12N} \\ P_{21N} & P_{22N} \end{bmatrix} \begin{Bmatrix} Q_{n1}^T \\ Q_{n2}^T \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{Bmatrix}$$

(17)

in which

$$\boldsymbol{P}_{11k} = \left[\boldsymbol{\Phi}_{k14} + 2\boldsymbol{\Phi}_{k13} + \boldsymbol{\Phi}_{k12} + \left(K + Cj\omega_n \right) \boldsymbol{\Phi}_{k110} - \omega_n^2 \boldsymbol{\Phi}_{k1M} \right]^T$$

(18a)

$$\boldsymbol{P}_{22k} = \left[\boldsymbol{\Phi}_{k24} + 2\boldsymbol{\Phi}_{k23} + \boldsymbol{\Phi}_{k22} + \left(K + Cj\omega_n \right) \boldsymbol{\Phi}_{k220} - \omega_n^2 \boldsymbol{\Phi}_{k2M} \right]^T$$

(18b)

$$\boldsymbol{P}_{12k} = -\left(K + Cj\omega_n \right) \boldsymbol{\Phi}_{k120}^T$$

(18c)

$$\boldsymbol{P}_{21k} = -\left(K + Cj\omega_n \right) \boldsymbol{\Phi}_{k210}^T$$

(18d)

and $\boldsymbol{P}_{11k}$, $\boldsymbol{P}_{22k}$, $\boldsymbol{P}_{21k}$ and $\boldsymbol{P}_{12k}$ are all coefficient vectors with dimension *1* by *N*. The zero vector and unknown variable vector are with dimension 2*N* by 1.

Solving the algebraic Eq. (17) and combing $q_{nn1}=q_{nn2}=1$, the vectors $\boldsymbol{Q}_{n1}$ and $\boldsymbol{Q}_{n2}$ are obtained. By the assumptions as Eq. (13), the *n*-th mode shapes of to-be-solved double-beam system are finally determined. Other mode shapes should be calculated by the same procedure.

Forced-vibration responses

In this section, the forced vibrations of the to-be-solved double-beam system subjected to dynamic loads are analyzed. Based on Eqs. (9) and (10), the general form of state-space governing equations for forced vibrations is

$$\dot{\boldsymbol{Z}}_M(t) = \boldsymbol{J}_M^* \cdot \boldsymbol{Z}_M(t) + \boldsymbol{K}_M^* \cdot \boldsymbol{F}_{MF}(t)$$

(19)

where

$$\boldsymbol{K}_M^* = \left[\left[\boldsymbol{I} \right] - \boldsymbol{K}_M \left[\boldsymbol{\Phi}_{MMC} \right] \right]^{-1} \cdot \boldsymbol{K}_M$$

(20a)

$$\boldsymbol{F}_{MF}(t) = \left[F_{1ff}(t) \quad F_2(t) \quad \cdots \quad F_{Nf}(t) \right]^T$$

(20b)

$$F_{nf}(t) = \left(1/\bar{M}_n\right) \cdot \int_0^L \left[\bar{\phi}_{n1}(x) \cdot F_1(x,t) + \bar{\phi}_{n2}(x) \cdot F_2(x,t) \right] dx$$

(20c)

and matrices $\boldsymbol{J}_M^*$, $\left[\boldsymbol{\Phi}_{MMC}\right]$ and $\boldsymbol{K}_M$ are same as Eq. (10), Eq. (B.4) and Eq. (8c).

Eq. (19) are first-order nonhomogeneous state-space equations with time-invariant coefficient matrix. An alternative incremental solution is adopted herein. Dividing the entire time domain into small intervals of equal length, Δt, and setting the initial time as $t_0 = 0$, the division times can be denoted as t_0, t_1, …, t_k, t_{k+1}, $\cdots$. A piecewise linear force vector is applied in each interval, followed by the incremental solution of Eq. (19) expressed as, similar to those in (Muscolino 1996):

$$\mathbf{Z}_M(t_{k+1}) = \mathbf{\Theta}(\Delta t)\,\mathbf{Z}_M(t_k) + \boldsymbol{\gamma}_0(\Delta t)\,\boldsymbol{F}_{MF}(t_k) + \boldsymbol{\gamma}_1(\Delta t)\,\boldsymbol{F}_{MF}(t_{k+1})$$

(21)

where

$$\mathbf{\Theta}(\Delta t) = \exp\left(\boldsymbol{J}_M^* \Delta t\right) = \mathbf{\Psi} \exp\left(\Delta t \bar{\mathbf{\Lambda}}\right) \mathbf{\Psi}^{-1} \tag{22a}$$

$$\boldsymbol{\gamma}_0(\Delta t) = -\left\{ (1/\Delta t)\left[\mathbf{\Theta}(\Delta t) - \left[\boldsymbol{I}\right]_{2N\times 2N} \right] - \mathbf{\Theta}(\Delta t)\boldsymbol{J}_M^* \right\} \boldsymbol{J}_M^{*-2} \boldsymbol{K}_M^*$$

(22b)

$$\boldsymbol{\gamma}_1(\Delta t) = \left\{ (1/\Delta t)\left[\boldsymbol{\Theta}(\Delta t) - [\boldsymbol{I}]_{2N\times 2N} \right] - \boldsymbol{J}_M^* \right\} \boldsymbol{J}_M^{*-2} \boldsymbol{K}_M^* \tag{22c}$$

and Ψ and $\bar{\Lambda}$ are complex matrices listing the eigenvector and eigenvalues of $\boldsymbol{J}_M^*$, respectively.

Previous studies prove that Eq. (21) supplies an unconditionally stable step-by-step procedure. The numerical errors of Eq. (21) only depends on the modeling of the force vectors as stepwise linear function. If the time interval is small enough and the real forcing function in each time interval is close to the stepwise linear function, the numerical errors can significantly be reduced to a negligible range. After the state variables in Eq. (21) are solved, the dynamic responses of whole double-beam system will be obtained from Eq. (5).

Summarizing the above derivations, the flowchart of the proposed state-space approach is illustrated in Fig. 2. The black flow path represents the solution procedure for the free vibration analysis. The red one, however, is for the forced vibration analysis. Therefore, following the procedure as shown in Fig. 2, the modal information and the dynamical response under dynamical loading of a double-beam system with variable cross-section can be obtained using the derivations in both the main text and the three appendices. The accuracy of the proposed approach depends on the particular solution space constructed from the chosen modes. The specific analysis and conclusions can be found in the authors' previous work (Li et al. 2021b).

Numerical examples and discussion

Numerical examples are applied to validate the proposed state-space approach. Detailed discussion is accomplished to illustrate the influences of structural parameters. Validations and discussions are made in the following three subsections: (Ⅰ) free vibration, (Ⅱ) forced vibration, and (Ⅲ) engineering application. The basic structural parameters are $E = 5\times10^9 Nm^{-2}$, $\rho = 1\times10^3 kgm^{-3}$, $I = 1\times10^{-4} m^4$, $A = 1\times10^{-2} m^2$, $e = EI = 5\times10^5 Nm^2$, $m = \rho A = 10 kgm^{-1}$, and $L = 10m$.

Verification and discussions of free vibration: natural frequencies and corresponding mode shapes

In this example, the accuracy of the proposed method for the free vibration is verified. The natural frequencies and mode shapes are calculated using both the proposed state-space method and the finite element method (FEM). The finite element software applied in this study is MSC Nastran. As for the specific model in the Nastran, CBAR is a typical Euler-Bernoulli beam element, and it is used for two slender beams. The viscoelastic interlayer adopts a simple spring-dash model, and scalar spring connection CELAS1 is chosen to simulate it. The mesh type is CURVE and the dimension of each mesh is 0.05m. 200 elements are built for each beam in the system, which ensures the accuracy of the simulations. Two cases are designed as follows:

Case 1: Simply supported-simply supported upper beam and simply supported-simply supported lower beam, $e_1 = e_2 = e$, $m_1 = m_2 = m$, $e_1(x) = e_1[1\text{-}0.1\sin(\pi x/L)]^3$, $e_2(x) = e_2[1+0.2\sin(\pi x/L)]^3$, $m_1(x) = m_1[1\text{-}0.1\sin(\pi x/L)]$, $m_2(x) = m_2[1+0.2\sin(\pi x/L)]$, $K_0 = 1\times10^5 Nm^{-2}$, $K = 5\times10^4 \sim 2\times10^5 Nm^{-2}$, $C = 0 Nsm^{-1}$.

Case 2: Clamped-free upper beam and simply supported-simply supported lower beam, $e_1 = e_2 = e$, $m_1 = m_2 = m$, $e_1(x) = e_1[1\text{-}0.2x/L]^3$, $e_2(x) = e_2[0.8+0.2x/L]^3$, $m_1(x) = m_1[1\text{-}0.2x/L]$, $m_2(x) = m_2[0.8+0.2x/L]$, $K_0 = 5\times10^3 Nm^{-2}$, $K = 2\times10^3 \sim 1\times10^4 Nm^{-2}$, $C = 0 Nsm^{-1}$.

The first six natural frequencies of the two cases are listed in Table 1 and Table 2, respectively. Compared with the ones from FEM, the results agree to each other very well. The corresponding mode shapes are also shown in Fig. 3 and Fig. 4, respectively. Compared with the ones from FEM, it is shown that they are same. Although the calculation accuracies of the proposed methods and FEM are both excellent, the calculation number in the proposed method is much less than that in FEM. In order to precisely simulate the two beams with variable cross-sections and the continuous elastic interlayer between them, a minimum number of 400 CBAR elements and 201 CELAS1 elements are used in the FEM model. These two numbers of structural elements have been optimized and shown to be the minimum values that make both the free and forced vibrational solutions stable and convergent. In this case, the total number of nodes and nodal degrees are 402 and 1206, respectively. The corresponding dimension of FEM matrix is 1206×1206. However, the dimension of state-space matrix is only 20×20 when ten basic modes are considered. On a computer CPU, the computational space is 1454436 versus 400, indicating that FEM requires more than 3600 times more computational space than the proposed approach. Computation of the proposed method is thus faster with such less occupied space in the CPU.

In these models, the boundary conditions may be the same or different for the two beams. Multiple types of boundary conditions have also been calculated. In contrast to the restrictions in previous studies, the proposed method can handle models with more general boundary conditions. The cross sections of both beams are varied in the longitudinal direction. The changes contain curve type (Case 1) and linear type (Case 2). These two types exist widely in real structures.

In general, the natural frequencies are increased with the increase of interlayer stiffness (as shown in Table 1 and Table 2). However, there are two different phenomena. First, there are some tiny changes in the natural frequencies of synchronous vibration modes in Case 1, while they are usually unchangeable in previous works. It is because the cross-sections of two beams are mutative, and they are not identical. Tiny relative displacements exist in those synchronous modes. Through the relative displacements, the increase of interlayer stiffness produces tiny increase in natural frequencies. Second, the changes of natural frequencies in Case 2 are very small. It is because the relative displacements between two beams are very small. Even with the interlayer stiffness increased significantly, the natural frequencies grew very little.

Finally, a parametric analysis of the accuracy of the state-space approach with respect to the variability of the cross-section is completed herein. The model in Case 1 is adopted. The change rate of cross-section μ is defined as the ratio of area at midspan to area at end of the beam. Parameters of the double-beam system are modified as $e_1(x) = e_1[1-\mu\sin(\pi x/L)]^3$, $e_2(x) = e_2[1+\mu\sin(\pi x/L)]^3$, $m_1(x) = m_1[1-\mu\sin(\pi x/L)]$, and $m_2(x) = m_2[1+\mu\sin(\pi x/L)]$. The first six natural frequencies are calculated and compared with the ones from the FEM in Table 3. It is shown that the increase of μ produces a small yet significant increase in error in each modal natural frequency. More changes in the variational cross-sections would reduce the computational precision of the proposed state-space approach. However, since the maximum value is only 0.4%, the overall error with the present method is within the acceptable range of values. The accuracy of this study is still high, and its application potential is strong.

Verification and discussions of forced vibration: dynamic responses and frequency responses

The proposed state-space approach is validated for the forced vibration of double-beam system with variable cross-section in this subsection. A concentrated harmonic force, $F_1(x,t) = f_0\sin(2\pi f_\omega t)\delta(x-0.5L)$ with the amplitude of force $f_0 = -5000N$, is applied at the midspan of the upper beam. Two cases with same geometrical parameters as the ones in last subsection are applied herein. Boundary conditions and interlayers are listed as follows:

Case 1: Simply supported-simply supported upper beam and simply supported-simply supported lower beam, $K_0 = 2\times10^5 Nm^{-2}$, $K = 2\times10^5 Nm^{-2}$, $C = 0 \sim 1000 Nsm^{-1}$.

Case 2: Clamped-clamped upper beam and clamped-clamped lower beam, $K_0 = 1\times10^4 Nm^{-2}$, $K = 2\times10^4 Nm^{-2}$, $C = 0 \sim 200 Nsm^{-1}$.

The dynamic responses of two beams are calculated by the proposed state-space approach and FEM. Eight modes are adopted in both state-space approach and finite element method to conduct the calculations. The results of two cases are plotted in Fig. 5 and Fig. 6, respectively. It is shown that the dynamic responses of two beams from the proposed state-space approach are in good agreement with the ones from FEM. Both resonance vibration (f_ω = 3.74Hz or 7.17Hz) and ordinary vibration (f_ω = 20Hz or 25Hz) are adopted to demonstrate the good agreement. The precision and reliability of the proposed method are thus validated for the forced vibration analysis.

Furthermore, the frequency responses at midspan of two beams are obtained by the proposed method and are illustrated in Fig. 7 and Fig. 8. The resonance occurs when the exciting frequencies f_ω are close to the system natural frequencies f_n. If the midspan amplitudes of the corresponding mode shapes are not zero, the resonance brings the peak values at natural frequency locations (such as 3.73Hz, 31.56Hz, 32.50Hz and 45.67Hz in Fig. 7; 7.17Hz, 12.30Hz and 38.60Hz in Fig. 8).

As to the damping of interlayer, the dynamic responses of two beams are reduced along with increased damping. However, in the synchronous vibration modes, the dynamic responses are not affected by damping significantly. It is because the relative velocities between two beams in synchronous vibration modes are small so that the damping cannot produce enough damping force. On the contrary, the damping apparently reduces the dynamic responses in asynchronous vibration modes. As shown in Fig. 7 and Fig. 8, the peak values of asynchronous vibration modes (such as 31.56Hz, 32.50Hz and 45.67Hz in Fig. 7; 12.30Hz, 22.08Hz and 39.3Hz in Fig. 8) are cut down. Some peak values (such as 45.67Hz in Fig. 7 and 12.30Hz in Fig. 8) are eliminated due to the damping. The above phenomena are consistent with previous papers about double-beam systems with uniform cross-sections, and it should be considered in future optimal and design works.

Engineering application: a rail-bridge with floating slab track

To further illustrate the practical application potential of the proposed method, an engineering application is analyzed and discussed in detail in this subsection. The designed rail-bridge is in Chengdu-Kangding Railway in Sichuan Province, China, which is a high-speed railway. The bridge is a single span simply supported bridge with variable cross-section. The elevation view of the bridge is shown in Fig. 9(a), and cross-sectional view of three key cross-sections of the bridge are shown in Fig. 9(b) to Fig. 9(d). Since it is a high-speed railway and the location of the bridge is close to a residential area, floating slab track structure is used to build the track system on the bridge. The cross-sectional view of floating slab track (Type: CRTSⅡ) is shown in Fig. 9(e).

To concentrate on the dynamics of the bridge and track system, three simplifications are made: (1) With the shear deformation of bridge ignored, the Euler-Bernoulli beam with variable cross-section is applied to simulate the bridge; (2) The rails and fasteners are ignored so that the whole floating slab track is simplified as one concrete beam; (3) The interactions between wheels and rails are simplified, and the loads of trains are simulated as a series of moving concentrated loads. Based on these simplifications, the designed bridge with floating slab track is modeled as a double-beam system. Due to the steel connectors between track slabs, all track slabs on the bridge behave as one beam in transverse deformation, and thus they are modeled as the upper beam. The bridge is the lower beam with variable cross-section. The rubber mat between floating slab track and bridge is treated as the interlayer. The geometrical and material parameters are listed in Table 4.

The variable cross-section of the rail-bridge can be considered symmetrical, since the vertical locations of the centroids in the three key cross-sections are 1.79m, 1.86m and 1.45m from the top of the bridge beam, respectively. The maximum difference is only 0.41m or 1.03% compared to the total length of the bridge of 39.94m. The changes of the centroid locations are small in terms of the entire length of the bridge. The three centroids lie approximately in a horizontal line, which remains normal to all three key cross-sections before and after deformation. Therefore, the Euler-Bernoulli beam theory is still suitable for the bridge. The proposed method is able to analyze this rail-bridge with floating slab track. Furthermore, the validations in (Martinez-Castro et al. 2006) demonstrate that the Euler-Bernoulli beam theory is accurate to calculate the dynamic responses of the beams with asymmetrical variable cross-sections. Unlike the model in (Yu et al. 2022), the models in previous works (Martinez-Castro et al. 2006) have small differences in the asymmetrical cross-sections. All the centroids are approximately in a horizontal line and all bending planes are treated in one plane. Accordingly, the Euler-Bernoulli beam theory is still useful.

The first six natural frequencies and corresponding mode shapes are solved by the proposed method and FEM for comparison purpose. As shown in Table 5 and Fig. 10, the accuracy of the proposed method in free vibration is verified. Meanwhile, the proposed method has a high precision for the double-beam system with asymmetrical cross-sections when the differences of asymmetrical cross-sections are small. The specific design certifications require that the minimum natural frequency of rail-bridges must be larger than 2.66 Hz in China, 2.88 Hz in Japan, and 2.66 Hz in EUROCODE. The 1st natural frequency of the whole structure is 4.41Hz. Therefore, the design meets the requirements of the above specifications. The first six natural frequencies are all in the range of low frequency, which is less than 20 Hz. The main vibrations of system belong to the low-frequency vibration. It is in line with most previous research conclusions on the structures equipped with floating slab track. The further vibration reduction works on the low-frequency vibration are needed. In each mode shape, the amplitude of floating slab track (W_1) is more than the amplitude of bridge (W_2). It proves that the vibration of floating slab track occupies the main position of entire structural vibration. Therefore, the vibration of rail-bridge is reduced effectively.

In terms of forced vibration, the dynamic responses of whole system under moving high-speed rail trains are analyzed herein. The standard CRH2 high-speed rail train is adopted, and the schematic diagram of the train formation is shown as Fig. 11. There are eight train-cars in one train formation, which consists of motor cars (M) and trailer cars (T). The corresponding static force and dynamic force of each wheelset are calculated and shown in Table 6. Since the interactions between wheels and rails are simplified, the loads of CRH2 trains are finally simulated as a series of moving concentrated loads as shown in Fig. 12. Combining the geometrical parameters of CRH2 train, the details about the moving concentrated loads are listed in Table 7.

According to Eq. (1), the mechanical model of the rail-bridge with floating slab track and the model of the moving loads can be obtained from literature (Museros and Martinez-Rodrigo 2007; Li et al. 2021b). The governing equations for the vibration of the rail-bridge system are written as

$$e_1 \cdot W_1''''(x,t) + K\left[W_1(x,t) - W_2(x,t)\right] + \bar{m}_1 \cdot \ddot{W}_1(x,t) = F_1(x,t)$$

(23a)

$$\begin{aligned} &e_2(x) \cdot W_2''''(x,t) + 2 \cdot e_2'(x) \cdot W_2'''(x,t) + e_2''(x) \cdot W_2''(x,t) \\ &- K\left[W_1(x,t) - W_2(x,t)\right] + \bar{m}_2(x) \cdot \ddot{W}_2(x,t) = 0 \end{aligned}$$

(23b)

in which

$$F_1(x,t) = \sum_{i=1}^{8} \sum_{j=1}^{4} A(P_{ij})\, \delta\left\{x - \left[Vt - L(P_{ij})\right]\right\} \left\{H\left[t - L(P_{ij})/V\right] - H\left[t - \left[L(P_{ij}) + L_1\right]/V\right]\right\}$$

(24)

and i (=1, 2…, 8) is the number of train-cars, j (=1, 2…, 4) is the number of wheels in one train-car, P_{ij} is the simplified moving concentrated loads shown in Table 7, $A(P_{ij})$ is amplitude value of the loads, $L(P_{ij})$ is location of the loads, $H(t-t_{ij})$ is the Heaviside unit function acting at time t_{ij}, $\delta(x-x_{ij})$ is the Dirac delta function acting at location x_{ij}, V is the constant train speed.

Substituting Eq. (24) into Eq. (23), the dynamic responses of the whole system are solved out by the proposed approach. The maximum displacements at midspan of floating slab track and rail-bridge under different speeds of train are drawn in Fig. 13 (a) and (b), respectively. Since the designed maximum speed of Chengdu-Kangding Railway is 250km/h which is also 70m/s, the speed range in Fig. 13 is set between 5m/s and 70m/s. According to the design certifications, the dynamic displacements of rail-bridges must satisfy some restricted values. For the designed rail-bridge in this paper, those restricted values should be 0.04995m in UIC and German, 0.024975m in Japan and 0.02854m in China. In Fig. 13(b), the maximum midspan displacements of bridge under different speeds of trains are all smaller than 0.00444m, which is much smaller than the above restricted values. Therefore, the design meets the requirements of the above specifications.

Furthermore, it is found that there are three peak values in Fig. 13(a) and one peak value in Fig. 13(b) when the speed of trains is between 35m/s and 40m/s. The partial enlarged views of such area are drawn as subfigures in Fig. 13(a) and Fig. 13(b). These peak values were related to resonances and a concept of critical speed was defined. The classical definition of the critical speed is the train speed at which the dynamic response of the railway track and other surrounding structures is intensely amplified, and extraordinary large vibrations occur due to resonances (He et al. 2023). When the speed of train reaches the critical speeds, the frequency produced by the train-cars is close to the natural frequencies of structures, causing the resonance. As shown in Fig. 12 and Table 7, there are three distances between two adjacent wheel forces: 2.5m, 15m and 5m. When 2.5m is considered as the wheel spacing, the exciting frequencies generated by the moving trains at peak values locations are: $1/[2.5m/(35.4m/s)]=14.16Hz$, $1/[2.5m/(36.7m/s)]=14.68Hz$ and $1/[2.5m/(37.68m/s)]=15.07Hz$. Those exciting frequencies are same as the natural frequencies of structure shown in Table 5. It is indeed the resonance that brings the peak values in Fig 13. The midspan displacements-time figures of both floating slab track and rail-bridge under those three critical speeds are plotted in Fig. 14 to Fig. 16. When the train speed is a non-critical speed, the midspan displacement-time figure is shown in Fig. 17.

Comparing Fig. 17 with Fig. 14 to Fig. 16, the dynamic responses of both floating slab track and rail-bridge show stronger periodicity and oscillation when the trains speed is critical speed. The maximum displacements at midspan are also significantly lager in Fig. 14 to Fig. 16.

Comparing the subfigure in Fig. 13(a) with the subfigure in Fig. 13(b), it is observed that there are three peaks in dynamic responses of floating slab track but only one peak in rail-bridge. The resonance of rail-bridge at two critical speeds (35.4m/s and 37.68m/s) disappeared. To better explain this phenomenon, additional numerical experiments based on the same rail-bridge model are performed. The excitation force in these numerical experiments is simplified to be a fixed concentrated harmonic force instead of the moving concentrated force. The excitation frequency of the moving concentrated force is kept and considered in the new fixed concentrated harmonic force, while the amplitude is neglected. Therefore, the new concentrated harmonic force applied at the midspan of the upper beam is $F_1(x,t) = f_0 \sin(2\pi f_\omega t)\delta(x-0.5L_1)$ with $f_0 = -200000N$.

Three excitation frequencies (14.16Hz, 14.68Hz and 15.072Hz), which correspond to three critical speeds (35.4m/s, 36.7m/s, 37.68m/s), are input as the exciting frequencies of the fixed concentrated harmonic force. The midspan displacement-time figures of two beams are drawn in Fig. 18 to Fig. 20. For the floating slab track shown in Fig. 18(a), Fig. 19(a) and Fig. 20(a), all the midspan displacements are increased continually with time. For the bridge, the same phenomenon is only found in Fig. 19(b). They are typical resonance responses. Different from these responses, the dynamic responses of rail-bridge in Fig. 18 (b) and Fig. 20(b) show obvious beat phenomenon instead of resonance phenomenon. The main reason for that difference in the dynamic responses is that these three natural frequencies (14.16Hz, 14.68Hz and 15.072Hz) are too close to each other. When one of these three natural frequencies is applied as exciting frequency, the beat vibration is easy to be generated because the vibration frequency is too close to the other two natural frequencies, which is the sufficient condition to initiate a beat vibration. It is shown with the dynamic responses in Fig. 18(b) and Fig. 20(b). In addition, even in the resonance responses shown in Fig. 18(a), Fig. 19(a) and Fig. 20(a), a tiny beat vibration phenomenon may be found too.

Based on the above interpretation inferred from the dynamical response under a fixed concentrated harmonic force, the vanishing of some resonances in the system under a moving train can be understood. Due to the mutual influence between the three resonances above, the rail-bridge may behave beat vibration at two critical speeds (35.4m/s and 37.68m/s) when the moving trains are considered. The dynamic response of the rail-bridge cannot be continuously increased as in resonance. The two peak values in Fig. 13(b) are eliminated. Although those two peak values are absent, the dynamic response in that speed range is still much larger than that at other speeds. When driving the trains to pass bridges, it still needs to avoid those speeds.

Conclusions

This paper aims to investigate the double-beam systems with variable cross-sections for their dynamic responses. A novel state-space approach is developed to solve the natural frequencies and corresponding mode shapes of the free vibration and obtain the dynamic responses of the forced vibration. The main conclusions could be drawn as follows.

The dynamic problems of the double-beam system with variable cross-section are analyzed and quantified with the proposed method. Due to the variable cross-section, the governing equations of vibration are highly coupled partial differential equations with variable parameters.

The proposed method can decouple those partial differential equations and obtain the homogeneous and non-homogeneous solutions.

Unlike most of previous analysis methods that require a discretization of the structure, the proposed method is an improved state-space approach. A basic double-beam system consisting of two uniform beams and a pure elastic interlayer is initially defined. The modal information of the basic double-beam system is then used to construct a solution space in which both free and forced vibrations of the to-be-solved double-beam system are obtained using the notion of the state-space. Several numerical examples are presented and discussed, while comparisons with FEM results show very good agreement, verifying the proposed methodology and demonstrating its potential use in addressing the study of engineering structure. In particular, the example of a rail-bridge with a floating slab-track has been studied.

The effects of interlayer on the dynamic characteristics of the whole double-beam systems must consider the relative displacement and relative velocity between two beam components. In previous research conclusions, only stiffness and damping of the interlayer are sufficient to analyze the influences. However, the cross-sections of two beams in this paper might be changeable. Even if the vibration modes are synchronous and two beams are identical, some relative displacements and relative velocities may exist due to the non-uniform cross-sections. Unlike previous results, the natural frequencies of those modes are changeable with stiffness and damping of interlayer. As a result, the properties of interlayer itself are not enough. It must combine the relative displacement and relative velocity, which are determined by cross-sections of two beams and mode shapes.

In engineering application part, the usage of floating slab track reduces the vibration of rail-bridge effectively. The shortcoming is that the main vibrations of whole system are in low frequency range, which brings vibration pollutions. Further vibration reduction works are needed. When the high-speed train passes the bridge, the critical speeds may lead to resonance since their exciting frequencies are close to natural frequencies of the structure. If some natural frequencies are too close to each other, the critical speeds may initiate beat vibration. Both resonance and beat vibration generate huge dynamic responses in structures. Therefore, critical speeds must be avoided. The variable cross-sections of beam systems exist in a large number of real engineering structures. It is necessary to study the double-beam systems with variable cross-sections because of its practical values. The proposed method can help engineers design and optimize double-beam systems and other similar vibration reduction systems in engineering practice.

Appendix A

All the terms of the right side of Eq. (7) are derived out specifically as follows:

$$F_{nW4}(t) = (1/\bar{M}_n)\sum_{r=1}^{\infty} T_r(t)\cdot\int_0^L \left[\bar{\varphi}_{r1}^{(4)}(x)\left[e_1 - e_1(x)\right]\bar{\varphi}_{n1}(x) + \bar{\varphi}_{r2}^{(4)}(x)\left[e_2 - e_2(x)\right]\bar{\varphi}_{n2}(x)\right] dx$$
$$= (1/\bar{M}_n)\sum_{r=1}^{\infty} T_r(t)\cdot\bar{\varphi}_{nr-4}$$

(A.1)

$$F_{nW3}(t) = (1/\bar{M}_n)\cdot\sum_{r=1}^{\infty} T_r(t)\cdot\int_0^L \left[\bar{\varphi}_{r1}'''(x)\cdot\left[-2e_1'(x)\right]\cdot\bar{\varphi}_{n1}(x) + \bar{\varphi}_{r2}'''(x)\cdot\left[-2e_2'(x)\right]\cdot\bar{\varphi}_{n2}(x)\right] dx$$
$$= (1/\bar{M}_n)\cdot\sum_{r=1}^{\infty} T_r(t)\cdot\bar{\varphi}_{nr-3}$$

(A.2)

$$F_{nW2}(t) = \left(1/\bar{M}_n\right) \cdot \sum_{r=1}^{\infty} T_r(t) \cdot \int_0^L \left\{ \bar{\phi}''_{r1}(x) \cdot \left[-e_1''(x)\right] \cdot \bar{\phi}_{n1}(x) + \bar{\phi}''_{r2}(x) \cdot \left[-e_2''(x)\right] \cdot \bar{\phi}_{n2}(x) \right\} dx$$

$$= \left(1/\bar{M}_n\right) \cdot \sum_{r=1}^{\infty} T_r(t) \cdot \bar{\phi}_{nr-2}$$

(A.3)

$$F_{nK}(t) = \frac{K_0 - K}{\bar{M}_n} \sum_{r=1}^{\infty} T_r(t) \int_0^L \left\{ \left[\bar{\phi}_{n1}(x) - \bar{\phi}_{n2}(x)\right] \cdot \left[\bar{\phi}_{r1}(x) - \bar{\phi}_{r2}(x)\right] \right\} dx = \frac{K_0 - K}{\bar{M}_n} \sum_{r=1}^{\infty} T_r(t) \cdot \bar{\phi}_{nr} \quad \text{(A.4)}$$

$$F_{nC}(t) = \frac{-C}{\bar{M}_n} \sum_{r=1}^{\infty} \dot{T}_r(t) \cdot \int_0^L \left\{ \left[\bar{\phi}_{n1}(x) - \bar{\phi}_{n2}(x)\right] \cdot \left[\bar{\phi}_{r1}(x) - \bar{\phi}_{r2}(x)\right] \right\} dx = \frac{-C}{\bar{M}_n} \sum_{r=1}^{\infty} \dot{T}_r(t) \cdot \bar{\phi}_{nr}$$

(A.5)

$$F_{nM}(t) = \frac{1}{\bar{M}_n} \sum_{r=1}^{\infty} \ddot{T}_r(t) \cdot \int_0^L \left\{ \bar{\phi}_{r1}(x)\left[\bar{m}_1 - \bar{m}_1(x)\right] \bar{\phi}_{n1}(x) + \bar{\phi}_{r2}(x)\left[\bar{m}_2 - \bar{m}_2(x)\right] \bar{\phi}_{n2}(x) \right\} dx$$

$$= \left(1/\bar{M}_n\right) \sum_{r=1}^{\infty} \ddot{T}_r(t) \cdot \bar{\phi}_{nrM}$$

(A.6)

$$\bar{M}_n = \int_0^L \left\{ \bar{\phi}_{n1}(x) \cdot \bar{m}_1 \cdot \bar{\phi}_{n1}(x) + \bar{\phi}_{n2}(x) \cdot \bar{m}_2 \cdot \bar{\phi}_{n2}(x) \right\} dx \quad \text{(A.7)}$$

Appendix B

The equivalent forces vectors in Eq. (8) are derived out as follows

$$\boldsymbol{F}_{MWs}(t) = \begin{bmatrix} \bar{\phi}_{11-s}/\bar{M}_1 & \cdots & \bar{\phi}_{1N-s}/\bar{M}_1 \\ \vdots & \ddots & \vdots \\ \bar{\phi}_{N1-s}/\bar{M}_N & \cdots & \bar{\phi}_{NN-s}/\bar{M}_N \end{bmatrix} \begin{Bmatrix} T_1(t) \\ \vdots \\ T_N(t) \end{Bmatrix} = \left[\boldsymbol{\Phi}_{MWsC}\right] \boldsymbol{Z}_M(t)$$

(B.1)

$$\boldsymbol{F}_{MK}(t)=(K_0-K)\begin{bmatrix}\bar{\phi}_{11}/\bar{M}_1 & \cdots & \bar{\phi}_{1N}/\bar{M}_1\\ \vdots & \ddots & \vdots\\ \bar{\phi}_{N1}/\bar{M}_N & \cdots & \bar{\phi}_{NN}/\bar{M}_N\end{bmatrix}\begin{Bmatrix}T_1(t)\\ \vdots\\ T_N(t)\end{Bmatrix}=(K_0-K)\left[\boldsymbol{\Phi}_{MKC}\right]\cdot\mathbf{Z}_M(t)$$

(B.2)

$$\boldsymbol{F}_{MC}(t)=(-C)\begin{bmatrix}\bar{\phi}_{11}/\bar{M}_1 & \cdots & \bar{\phi}_{1N}/\bar{M}_1\\ \vdots & \ddots & \vdots\\ \bar{\phi}_{N1}/\bar{M}_N & \cdots & \bar{\phi}_{NN}/\bar{M}_N\end{bmatrix}\begin{Bmatrix}\dot{T}_1(t)\\ \vdots\\ \dot{T}_N(t)\end{Bmatrix}=(-C)\left[\boldsymbol{\Phi}_{MCC}\right]\cdot\mathbf{Z}_M(t)$$ (B.3)

$$\boldsymbol{F}_{MM}(t)=\begin{bmatrix}\bar{\phi}_{1M}/\bar{M}_1 & \cdots & \bar{\phi}_{NM}/\bar{M}_1\\ \vdots & \ddots & \vdots\\ \bar{\phi}_{N1M}/\bar{M}_N & \cdots & \bar{\phi}_{NNM}/\bar{M}_N\end{bmatrix}\begin{Bmatrix}\ddot{T}_1(t)\\ \vdots\\ \ddot{T}_N(t)\end{Bmatrix}=\left[\boldsymbol{\Phi}_{MMC}\right]\cdot\dot{\mathbf{Z}}_M(t)$$

(B.4)

where [$\boldsymbol{\Phi}_{MWsC}$] (s=2, 3 or 4), [$\boldsymbol{\Phi}_{MKC}$], [$\boldsymbol{\Phi}_{MCC}$] and [$\boldsymbol{\Phi}_{MMC}$] are all N by $2N$ coefficient matrices.

Appendix C

The integrations of $\bar{\phi}_{k1}(x)$ × Eq. (19a) and $\bar{\phi}_{k2}(x)$ × Eq. (19b) are derived and listed specifically as follows:

$$\int_0^L e_i(x)\,\boldsymbol{Q}_{ni}\bar{\phi}_{ki}(x)\,\bar{\boldsymbol{\Phi}}_i^{(4)}(x)\,dx$$
$$=\boldsymbol{Q}_{ni}\left[\int_0^L e_i(x)\,\bar{\phi}_{ki}(x)\,\bar{\phi}_{1i}^{(4)}(x)\,dx \quad \cdots \quad \int_0^L e_i(x)\,\bar{\phi}_{ki}(x)\,\bar{\phi}_{Ni}^{(4)}(x)\,dx\right]^T=\boldsymbol{Q}_{ni}\boldsymbol{\Phi}_{ki4}$$

(C.1)

$$\int_0^L 2e_i'(x)\,\boldsymbol{Q}_{ni}\bar{\phi}_{ki}(x)\,\bar{\boldsymbol{\Phi}}_i'''(x)\,dx$$
$$=2\boldsymbol{Q}_{ni}\left[\int_0^L e_i'(x)\,\bar{\phi}_{ki}(x)\,\bar{\phi}_{1i}'''(x)\,dx \quad \cdots \quad \int_0^L e_i'(x)\,\bar{\phi}_{ki}(x)\,\bar{\phi}_{Ni}'''(x)\,dx\right]^T=2\boldsymbol{Q}_{ni}\boldsymbol{\Phi}_{ki3}$$

(C.2)

$$\int_0^L e_i''(x)\, \boldsymbol{Q}_{ni} \bar{\varphi}_{ki}(x)\, \bar{\boldsymbol{\Phi}}_i''(x)\, dx$$
$$=\boldsymbol{Q}_{ni} \left[\int_0^L e_i''(x)\, \bar{\varphi}_{ki}(x)\, \bar{\varphi}_{1i}''(x)\, dx \quad \cdots \quad \int_0^L e_i''(x)\, \bar{\varphi}_{ki}(x)\, \bar{\varphi}_{Ni}''(x)\, dx \right]^T = \boldsymbol{Q}_{ni} \boldsymbol{\Phi}_{ki2}$$

(C.3)

$$\int_0^L \boldsymbol{Q}_{ni} \bar{\varphi}_{ki}(x)\, \bar{\boldsymbol{\Phi}}_i(x)\, dx = \boldsymbol{Q}_{ni} \left[\int_0^L \bar{\varphi}_{ki}(x)\, \bar{\varphi}_{1i}(x)\, dx \quad \cdots \quad \int_0^L \bar{\varphi}_{ki}(x)\, \bar{\varphi}_{Ni}(x)\, dx \right]^T = \boldsymbol{Q}_{ni} \boldsymbol{\Phi}_{kii0}$$

(C.4)

$$\int_0^L \boldsymbol{Q}_{n2} \bar{\varphi}_{k1}(x)\, \bar{\boldsymbol{\Phi}}_2(x)\, dx = \boldsymbol{Q}_{n2} \left[\int_0^L \bar{\varphi}_{k1}(x)\, \bar{\varphi}_{12}(x)\, dx \quad \cdots \quad \int_0^L \bar{\varphi}_{k1}(x)\, \bar{\varphi}_{N2}(x)\, dx \right]^T = \boldsymbol{Q}_{n2} \boldsymbol{\Phi}_{k120} \tag{C.5}$$

$$\int_0^L \boldsymbol{Q}_{n1} \bar{\varphi}_{k2}(x)\, \bar{\boldsymbol{\Phi}}_1(x)\, dx = \boldsymbol{Q}_{n1} \left[\int_0^L \bar{\varphi}_{k2}(x)\, \bar{\varphi}_{11}(x)\, dx \quad \cdots \quad \int_0^L \bar{\varphi}_{k2}(x)\, \bar{\varphi}_{N1}(x)\, dx \right]^T = \boldsymbol{Q}_{n1} \boldsymbol{\Phi}_{k210} \tag{C.6}$$

$$\int_0^L \bar{m}_i(x) \cdot \omega_n^2\, \boldsymbol{Q}_{ni} \cdot \bar{\varphi}_{ki}(x) \cdot \bar{\boldsymbol{\Phi}}_i(x)\, dx$$
$$=\omega_n^2 \boldsymbol{Q}_{ni} \cdot \left[\int_0^L \bar{m}_i(x) \cdot \bar{\varphi}_{ki}(x) \cdot \bar{\varphi}_{1i}(x)\, dx \quad \cdots \quad \int_0^L \bar{m}_i(x) \cdot \bar{\varphi}_{ki}(x) \cdot \bar{\varphi}_{Ni}(x)\, dx \right]^T = \omega_n^2 \boldsymbol{Q}_{ni} \cdot \boldsymbol{\Phi}_{kiM}$$

(C.7)

in which $\boldsymbol{\Phi}_{ki4}$, $\boldsymbol{\Phi}_{ki3}$, $\boldsymbol{\Phi}_{ki2}$, $\boldsymbol{\Phi}_{kii0}$, $\boldsymbol{\Phi}_{k120}$, $\boldsymbol{\Phi}_{k210}$ and $\boldsymbol{\Phi}_{kiM}$ (i=1,2) are all N by 1 constant vectors.

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Conflict of interests

The authors declare that there are no conflicts of interest.

Data Availability Statement

Some or all data, models, or code that support the findings of this study are available from the corresponding author upon reasonable request.

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Table 1. Natural frequencies of double-beam system f_n (Hz); Case 1

Mode	K=5×10^4 N/m^2			K=1×10^5 N/m^2			K=2×10^5 N/m^2		
	Presen	FEM	Error	Present	FEM	Error	Present	FEM	Error (%)
n=1	3.73	3.73	0	3.73	3.73	0	3.74	3.74	0
n=2	14.47	14.47	0	14.59	14.5	0	14.65	14.65	0
n=3	16.12	16.12	0	22.51	22.5	0	31.56	31.56	0
n=4	21.50	21.50	0	26.58	26.5	0	32.50	32.50	0
n=5	31.16	31.16	0	31.89	31.8	0	34.64	34.64	0
n=6	37.73	37.72	0.03	40.32	40.3	0.02	45.67	45.67	0

Table 2. Natural frequencies of double-beam system f_n (Hz); Case 2

Mode	K=2×10^3 N/m^2			K=5×10^3 N/m^2			K=1×10^4 N/m^2		
	Presen	FEM	Error	Present	FEM	Error	Present	FEM	Error (%)
n=1	4.07	4.09	0.49	5.32	5.34	0.37	6.66	6.84	2.63
n=2	7.74	7.74	0	8.39	8.38	0.12	9.53	9.52	0.11
n=3	12.84	12.84	0	13.16	13.1	0	13.66	13.68	0.15
n=4	20.14	20.14	0	20.36	20.3	0	20.75	20.74	0.05
n=5	28.47	28.47	0	28.62	28.6	0	28.86	28.86	0
n=6	38.98	38.97	0.03	39.09	39.0	0.03	39.28	39.27	0.03

Table 3. Natural frequencies of double-beam system f_n (Hz); Case 1

Mode	$\mu = 0$			$\mu = 0.4$			$\mu = 0.8$		
	Presen	FEM	Error	Present	FEM	Error	Present	FEM	Error (%)
n=1	3.51	3.51	0	4.05	4.05	0	5.31	5.31	0
n=2	14.04	14.04	0	15.09	15.0	0.07	18.25	18.20	0.27
n=3	31.59	31.59	0	30.69	30.6	0.16	32.34	32.21	0.40
n=4	32.02	32.02	0	34.42	34.4	0.03	38.29	38.21	0.21
n=5	34.78	34.78	0	35.88	35.8	0.06	46.98	46.86	0.26
n=6	44.86	44.86	0	47.52	47.4	0.25	47.67	47.58	0.19

Table 4. Geometrical parameters and material parameters of rail-bridge with floating slab track

Structure	Parameter	Value	Structure	Parameter	Value
Floating Slab Track	E_1	$3.55\times10^4 Mpa$	Rail-Bridge	E_2	$3.45\times10^4 Mpa$
	$W_1{*}H_1$	$2.55m\times0.2m$		$A_{2\text{-}1}$ (Cross-section 1-1)	17.845 m^2
	A_1	$0.51m^2$		$A_{2\text{-}2}$ (Cross-section 2-2)	11.265 m^2
	I_1	$0.0017m^4$		$A_{2\text{-}3}$ (Cross-section 3-3)	6.6225 m^2
	ρ_1	$2500kg/m^3$		$I_{2\text{-}1}$ (Cross-section 1-1)	23.20202 m^4
	L_1	$39.94m$		$I_{2\text{-}2}$ (Cross-section 2-2)	20.02999 m^4
Rubber Mat	K	$1\times10^7 N/m$		$I_{2\text{-}3}$ (Cross-section 3-3)	10.42134 m^4
				ρ_2	$2500kg/m^3$
				L_2	39.94m

Table 5. Natural frequencies of rail-bridge with floating slab track f_n (Hz)

Mode	n=1	n=2	n=3	n=4	n=5	n=6
Present	4.41	13.5	14.16	14.49	14.68	15.07
Finite element	4.40	13.4	14.16	14.49	14.68	15.07
Error (%)	0.23	0.07	0	0	0	0

Table 6. The main vehicle weight parameters and corresponding wheelset forces

	T_1	M_2	M_3	T_4	T_5	M_6	M_7	T_8
Train Weight (t)	42.8	48.0	46.5	42.0	44.1	48.0	46.8	41.5
Passenger Weight (t)	4.4	8.0	6.8	8.0	4.4	8.0	4.1	5.1
Total Weight (t)	47.2	56.0	53.3	50.0	48.5	56.0	50.9	46.6
Static Force of Each Wheel (kN)	115.6	137.2	130.6	122.5	118.8	137.2	124.7	114.2
Dynamic Force of Each Wheel (kN)	167.7	198.9	189.3	177.6	172.3	198.9	180.8	165.5

Table 7. The locations and amplitudes of the moving concentrated loads

	P_{11}	P_{12}	P_{13}	P_{14}	P_{21}	P_{22}	P_{23}	P_{24}
Location (m)	0	2.5	17.5	20	25	27.5	42.5	45
Amplitude (kN)	167.7	167.7	167.7	167.7	198.9	198.9	198.9	198.9
	P_{31}	P_{32}	P_{33}	P_{34}	P_{41}	P_{42}	P_{43}	P_{44}
Location (m)	50	52.5	67.5	70	75	77.5	92.5	95
Amplitude (kN)	189.3	189.3	189.3	189.3	177.6	177.6	177.6	177.6
	P_{51}	P_{52}	P_{53}	P_{54}	P_{61}	P_{62}	P_{63}	P_{64}
Location (m)	100	102.5	117.5	120	125	127.5	142.5	145
Amplitude (kN)	172.3	172.3	172.3	172.3	198.9	198.9	198.9	198.9
	P_{71}	P_{72}	P_{73}	P_{74}	P_{81}	P_{82}	P_{83}	P_{84}
Location (m)	150	152.5	167.5	170	175	177.5	192.5	195
Amplitude (kN)	180.8	180.8	180.8	180.8	165.5	165.5	165.5	165.5

Figure captions

Fig. 1. The mechanical model of double-beam systems: (a) with variable cross-section; (b) with constant cross-section.

Fig. 2. Flowchart of the proposed state-space method for a double-beam system with variable cross-section.

Fig. 3. The first six normal mode shapes of the double-beam system for Case 1, K=2×10^{5}Nm^{-2}, C=0Nsm^{-1}: (a) Mode 1; (b) Mode 2; (c) Mode 3; (d) Mode 4; (e) Mode 5; (f) Mode 6.

Fig. 4. The first six normal mode shapes of the double-beam system for Case 2, K=5×10^{3}Nm^{-2}, C=0Nsm^{-1}: (a) Mode 1; (b) Mode 2; (c) Mode 3; (d) Mode 4; (e) Mode 5; (f) Mode 6.

Fig. 5. Dynamic responses at midspan point of two beams for Case 1: (a) Upper beam displacement (3.74Hz); (b) Lower beam displacement (3.74Hz); (c) Upper beam displacement (20Hz); (d) Lower beam displacement (20Hz).

Fig. 6. Dynamic responses at midspan point of two beams for Case 2: (a) Upper beam displacement (7.17Hz); (b) Lower beam displacement (7.17Hz); (c) Upper beam displacement (25Hz); (d) Lower beam displacement (25Hz).

Fig. 7. Frequency responses at midspan point of two beams for Case 1: (a) Upper beam; (b) Lower beam.

Fig. 8. Frequency responses at midspan point of two beams for Case 2: (a) Upper beam; (b) Lower beam.

Fig. 9. The design diagram of the rail-bridge with floating slab track (Unit: cm): (a) Elevation view of bridge; (b) Cross-sectional view 1-1; (c) Cross-sectional view 2-2; (d) Cross-sectional view 3-3; (e) Cross-sectional view of floating slab track (Type: CRTSⅡ).

Fig. 10. The first six normal mode shapes of the rail-bridge (W_2) with floating slab track (W_1): (a) Mode 1; (b) Mode 2; (c) Mode 3; (d) Mode 4; (e) Mode 5; (f) Mode 6.

Fig. 11. The schematic diagram of the standard CRH2 train formation (M: motor car; T: trailer car).

Fig. 12. The schematic diagram of a series of moving concentrated loads.

Fig. 13. Maximum displacements at midspan points of two beams under different speeds of trains: (a) Maximum displacements at floating slab track midspan; (b) Maximum displacements at rail-bridge midspan.

Fig. 14. Displacements at midspan points of two beams under 35.4m/s: (a) Displacements at floating slab track midspan; (b) Displacements at rail-bridge midspan.

Fig. 15. Displacements at midspan points of two beams under 36.7m/s: (a) Displacements at floating slab track midspan; (b) Displacements at rail-bridge midspan.

Fig. 16. Displacements at midspan points of two beams under 37.8m/s: (a) Displacements at floating slab track midspan; (b) Displacements at rail-bridge midspan.

Fig. 17. Displacements at midspan points of two beams under 30m/s: (a) Displacements at floating slab track midspan; (b) Displacements at rail-bridge midspan.

Fig. 18. Displacements at midspan points of two beams under harmonic force with exciting frequency 14.16Hz: (a) Displacements at floating slab track midspan; (b) Displacements at rail-bridge midspan.

Fig. 19. Displacements at midspan points of two beams under harmonic force with exciting frequency 14.68Hz: (a) Displacements at floating slab track midspan; (b) Displacements at rail-bridge midspan.

Fig. 20. Displacements at midspan points of two beams under harmonic force with exciting frequency 15.072Hz: (a) Displacements at floating slab track midspan; (b) Displacements at rail-bridge midspan.