Title
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Publication Date
1967-01-26

UCRL-17311

University of California

Ernest O. Lawrence Radiation Laboratory

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Berkeley, California

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Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

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Jerome Finkelstein

(Ph. D. Thesis)

January 26, 1967

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ABSTRACT

The equivalent-potential method is generalized so as to permit calculation both of multichannel scattering and of certain scattering amplitudes of particles of nonzero spin. It is pointed out that amplitudes calculated by this method have correct dependence upon momentum transfer; it is argued that this feature may make the equivalent-potential method a better approximation to the unitarity iteration than is the N/D method. As examples, the $\pi\pi$, πK, and πN amplitudes are considered. For the first two of these cases it is found that the force due to vector meson exchange is far too small (it is suggested that inelastic effects may be quite important), but that if the input couplings are increased so as to reproduce the vector mesons, then 2^+ mesons are also produced, at roughly the masses of the $f^0(1250)$, $f^{0\prime}(1500)$, and $K^{**}(1405)$. In the πN case, with the potential given by nucleon and $N^*(1238)$ exchange, there are no free parameters in the calculation. The N^* is then "predicted" at a mass of 1100 MeV. The nonresonant phase shifts also agree in

a general way with the results of phase-shift analysis; in particular, the S_{11} scattering length has the correct sign, while N/D calculations produce the wrong sign. It is argued that this result indicates that the force due to iteration of the potential, included in this method but not in N/D, can be important.

I. INTRODUCTION

An important problem in strong interaction physics has been the calculation of scattering amplitudes from input "forces" which are assumed known. One common way of calculating two-particle amplitudes is the N/D approximation.[1] In this paper I discuss and generalize an alternative procedure suggested several years ago by Charap and Fubini.[2] With this procedure I am able to calculate simply and without the introduction of adjustable parameters the scattering amplitude implied by a given choice of input "force." Also, I can investigate whether certain discrepancies between N/D calculations and experiment are due to a failure of the N/D approximation or of more basic assumptions.

The analysis of the scattering problem into "forces" and their "effects" is of course based on a classical analogue. Even in the classical case, difficulties arise if this distinction is taken too seriously;[3] since forces are always accompanied by their effects, it is not surprising that sooner or later one must get into trouble if one insists on discussing them separately. Nevertheless, this way of thinking about problems is certainly extremely useful in classical physics. In strong interaction physics it may also be useful, but the difficulties will appear much sooner. Loosely speaking, since the "effect" in one channel can be the "force" for another, this distinction can be expected to be useful only in the approximation of neglecting exact crossing symmetry.

In the framework of the Mandelstam representation, which shall be assumed throughout this paper, the notion of force can be defined as follows: Let $A(s,t)$ be the (matrix) scattering amplitude connecting a specified, finite number of two-body channels, where s and t are the usual kinematic variables. Let $A_t(s,t)$ be the t discontinuity of $A(s,t)$. (All amplitudes in this paper have definite exchange parity). Then the elastic double spectral function is defined by the usual unitarity formula

$$\rho_{el}(s,t) = \frac{1}{\pi q s^{\frac{1}{2}}} \int_{t_0}^{\infty} dt'\, dt'' \; \frac{A_t(s,t')\, A_t^*(s,t'')}{K^{\frac{1}{2}}(s,t,t',t'')} \; ; \qquad \text{(I-1)}$$

t_0 is the position of the lowest singularity in t. The generalized potential V, which corresponds to the "force," is then defined by the relation

$$A_t(s,t) = V(s,t) + \frac{1}{\pi} \int ds' \; \frac{\rho_{el}(s',t)}{s' - s} \; , \qquad \text{(I-2)}$$

where, if necessary, a cutoff is to be understood on this last integral. Clearly this definition of the force depends on which channels are explicitly included in the matrix A. If V is real, the amplitude satisfies "elastic" unitarity, by which we mean that the S matrix of the channels explicitly included is unitary.

These definitions permit the scattering problem to be attacked in two stages. The first stage is the construction of V, the second the calculation of the amplitude in terms of V. For the examples

which I shall discuss I will take V to be real, and approximate it by the most obvious single particle exchange terms. I shall not discuss here the possible justifications or limitations of these approximations, and in fact shall have essentially nothing new to say about the first stage, except that my numerical results might be thought to provide some measure of the reliability of these approximations. The problem that I will discuss is, given an approximation to V, to find the corresponding amplitude A.

A popular approach to this problem consists in approximating the left-hand cuts of partial wave amplitudes by the projections of V, and then unitarizing the amplitude by means of the N/D equations (In the following, whenever I refer to the N/D method this approximation to the left-hand cuts should be understood). Such a procedure assures that the nearest part of the left-hand cut is treated correctly;[5] one hopes that this feature, together with the correct treatment of the elastic cut, assures an adequate approximation at low energies. Since Eqs. (I-1) and (I-2) can be solved for A_t by iteration, in principle the entire left-hand cut could be obtained, in the approximation that we somehow know V. However, since at best we can only know the long range parts of V, it might be thought that if the problem can be handled at all, only the nearby left-hand cut is important, and hence it is legitimate to neglect effects due to iterations of V.

A hint that this argument may not be correct can be seen in the fact that, for large ℓ, amplitudes calculated with the N/D

approximation can not have both their asymptotic and their threshold behavior correct. Thus iteration of long-range components in the potential is not dynamically equivalent to the effect of short-range components. I will suggest below, furthermore, that there are examples for low ℓ, such as scattering in the $\pi N S_{11}$ state, in which including the iterations of the parts of the potential that we do know gives a significantly better answer than does the N/D approximation. The equivalent-potential method suggested by the work of Charap and Fubini,[2] which shares with N/D the feature of treating the nearby singularities correctly, but which does not neglect the force due to iterations of the potential, can be used to provide a basis for comparison with N/D results, as well as to produce amplitudes which are interesting for more general purposes.

Briefly, this method is as follows: We imagine that we know $V(s,t)$ and will settle for an amplitude A which fulfills these requirements:

a) $A(s,t)$ satisfies elastic unitarity. (of the channels explicitly considered).

b) For small t, $A_t(s,t) = V(s,t)$. (I-3)

These requirements demand that we include the nearest singularities correctly. If we define

$$V(r,s) = \lambda(s) \int_{t_0}^{\infty} dt\, V(s,t)\, e^{-r\sqrt{t}}/r\,, \qquad \text{(I-4)}$$

and if the normalization $\lambda(s)$ is judiciously chosen, the scattering amplitude implied by the Schrödinger equation of which $V(r,s)$ is the potential can be shown to fulfill requirements (I-3). It is the common hope of this method and of N/D that these requirements will ensure a reasonable approximation at low energies; at high energies the short-range part of V are surely important.

The equivalent-potential method is presented in more detail and generality in Section II below. It is applied to examples of scattering problems in the next two sections: Section III contains calculations of $\pi\pi$ and πK scattering, and Section IV a calculation of πN scattering. The results of these calculations are summarized and discussed in Section V.

Some of the material presented in this paper is contained in two preliminary reports of these calculations.[6,7]

II. THE EQUIVALENT-POTENTIAL METHOD

The equivalent-potential method was first suggested by Charap and Fubini[2] for the special case of spinless external particles and zero kinetic energy. They were able to argue that the energy-independent Schrödinger equation potential they obtained would be a good approximation when the kinetic energy was small compared to the rest mass. A few years later Balázs[8] generalized this method by letting the potential be energy-dependent, and suggested its application to $\pi\pi$ scattering.

Requirements (I-3) determine the normalization $\lambda(s)$ appearing in (I-4), and this completely determines the method. Before discussing this, I will, essentially follow the outline of Balázs, exhibit a potential $U(r,s)$ which in principle would produce an amplitude satisfying (I-1) and (I-2) exactly. We will then be able to see in what sense our $V(r,s)$ is an approximation to this U.

A. Construction of Exact Potential

Let us assume that we have solved (I-1) and (I-2) by iteration, and that we now know $A_t(s,t)$ exactly. For application to a two channel calculation, I will for a time display the matrix indices of A_t, although a single channel would have been sufficient to illustrate this construction. Knowing $A_t(s,t)$, define matrices $\alpha_{ij}(s,E,t)$ and $[f_t(s,E,t)]_{ij}$ by

$$\alpha_{ij}(s,E,t) = \sum_k \frac{1}{2\pi p_k} \int dt'\, dt'' \, \frac{[f_t(s,E,t')]_{ik}[f_t^*(s,E,t'')]_{kj}}{K_{ijk}^{\frac{1}{2}}(E,t,t't'')} \qquad \text{(II-1)}$$

$$[f_t(s,E,t)]_{ij} = \frac{2[A_t(s,t)]_{ij}}{s^{\frac{1}{2}}} + \frac{1}{\pi}\int^{\infty} dE'\ \alpha_{ij}(s,E',t)\left[\frac{1}{E'-E} - \frac{1}{E'-s}\right] . \tag{II-2}$$

Here i,j,k refer to the channels; p_k is the same function of E that the center-of-mass momentum q_k is of s, and K is the standard unitarity kernel, which is a function of the three angle cosines. Equations (II-1) and (II-2) can be solved for α and f_t by iteration. Having obtained α, define $U(s,t)$ by

$$U_{ij}(s,t) = 2s^{-\frac{1}{2}}[A_t(s,t)]_{ij} - \frac{1}{\pi}\int^{\infty} dE'\ \frac{\alpha_{ij}(s,E',t)}{E'-s} . \tag{II-3}$$

By combining (II-2) and (II-3), we see

$$[f_t(s,E,t)]_{ij} = U_{ij}(s,t) + \frac{1}{\pi}\int^{\infty} \frac{dE'\ \alpha_{ij}(s,E',t)}{E'-E} . \tag{II-4}$$

Now Eqs. (II-1) and (II-4) can be solved very simply. Consider the set of coupled Schrödinger equations, parameterized by s :

$$(\nabla^2 + p_i^2)\,\psi_i(r) = \sum_j U_{ij}(r,s)\,\psi_j(r) , \tag{II-5}$$

where

$$U_{ij}(r,s) = -\frac{1}{\pi}\int_{t_0}^{\infty} dt\ U_{ij}(s,t)\ e^{-r\sqrt{t}}/r . \tag{II-6}$$

With certain restrictions on $U(s,t)$, to be discussed below, we know that the amplitude implied by these Schrödinger equations, call it $f_{ij}(s,E,t)$, will have its t discontinuity equal to $[f_t(s,E,t)]_{ij}$, thus justifying the notation. This is so because the t discontinuity of f will satisfy Eqs. (II-1) and (II-4),[9] and the solution to these equations is unique.

The restrictions are, first, that $U_{ij}(r,s)$ be less singular than r^{-2} at the origin, which from (II-6) means $U_{ij}(s,t)$ goes to zero faster than $t^{-\frac{1}{2}}$ at large t. This will have to be checked in individual cases, and in fact will give us trouble when we try to generalize to include scattering of particles with non-zero spin. The second restriction is that $U_{ij}(s,t)$ be real. From (II-3),

$$\text{Im } U_{ij}(s,t) = 2s^{-\frac{1}{2}} \text{ Im } [A_t(s,t)]_{ij} - \alpha_{ij}(s,s,t) , \tag{II-7}$$

and then using (II-1) and (II-2),

$$\frac{\text{Im } U_{ij}(s,t)}{2s^{-\frac{1}{2}}} = \text{Im } [A_t(s,t)]_{ij} - \sum_k \frac{1}{\pi p_k s^{\frac{1}{2}}} \int dt'\, dt'' \, \frac{[A_t(s,t')]_{ik}[A_t^*(s,t'')]_{kj}}{K^{\frac{1}{2}}(s,t,t',t'')} . \tag{II-8}$$

That this is zero follows from the assumed unitarity of A_t, that is from the matrix analogue of (I-1) and (I-2) and the fact that the input $V(s,t)$ is real.

Let me now suppress the channel indices. So far we have seen that the t discontinuity of $f(s,E,t)$ is indeed $f_t(s,E,t)$. If we define

$$A(s,t) = \tfrac{1}{2}\, s^{\frac{1}{2}}\, f(s,s,t) \qquad \text{(II-9)}$$

we see from (II-2) that the t discontinuity of $A(s,t)$ is just $A_t(s,t)$, which by assumption satisfies (I-1) and (I-2). Having started with $V(s,t)$, we have constructed an amplitude whose t discontinuity does satisfy (I-1) and (I-2). Of course there are many functions with the same t discontinuity, but by using the Schrödinger equation we have picked out that unique one which is analytic in angular momentum. This will be another assumption we shall have to make: that the low partial waves of the amplitude we are seeking are the continuation of the high partial waves. A similar assumption is made in N/D calculations in ignoring the possibility of CDD poles.

While it might be tempting to try to interprete r as some kind of position coordinate, and $f(s,E,t)$ as an off-mass-shell amplitude, this is not necessary, and will not be done here. It is sufficient for our purposes to regard the introduction of these new variables as a purely formal device to generate a mass-shell amplitude $A(s,t)$ which satisfies the mass-shell conditions (I-1) and (I-2).

B. The Approximation

The potential $U(s,t)$ defined by (II-3) would in principle be obtained from $V(s,t)$ by iterating (I-1) and (I-2) an infinite number of times, and then iterating (II-1) and (II-2) an infinite number of times. If we were to iterate each of these equations n times (keeping only the first n powers of V), and call the resulting potential $U^n(s,t)$, then it could be seen that U^n is real, and that $U^n(s,t) = U(s,t)$ for $t < (n+1)^2 t_0$. If we put $U^n(r,s) \equiv -(1/\pi) \int dt\, U^n(s,t)\, e^{-r\sqrt{t}}/r$, the resulting amplitude would still be unitary, and would have its t discontinuity correct[5] for $t < (n+1)^2 t_0$. Comparing with Eq. (I-4), if we set the normalization $\lambda(s) = -2/(\pi s^{\frac{1}{2}})$, then $V(r,s) = U^1(r,s)$; once again with channel indices, that is

$$V_{ij}(r,s) = \frac{-2}{\pi s^{\frac{1}{2}}} \int_{t_0}^{\infty} dt\, V_{ij}(s,t)\, e^{-r\sqrt{t}}/r \,. \qquad \text{(II-10)}$$

Equation (II-10) is the basis for the equivalent-potential calculation for spinless external particles. We have just seen in what sense this $V(r,s)$ is an approximation to the $U(r,s)$ defined above. However, the properties of the amplitude implied by $V(r,s)$ can be deduced from (II-10) directly. Clearly $f(s,E,t)$ will satisfy the non-relativistically normalized unitarity relation, and for $t < 4t_0$ its t discontinuity will be $2s^{-\frac{1}{2}} V(s,t)$. Hence $A(s,t)$ defined in (II-9) will fulfill requirements (I-3). Moreover, since

the amplitude does come from a Schrödinger equation, its full t discontinuity will have the structure implied by the (non-relativistic) Mandelstam unitarity iteration. Thus the contribution to the force coming from higher terms of the iteration, while not included correctly, are not ignored either. This is the advantage that the equivalent-potential method has over the N/D method. One way to appreciate the advantage of getting the structure in t correctly is to observe that the amplitudes that I will calculate have correct threshold behavior, while N/D amplitudes, just because the higher terms of the iteration are neglected, do not automatically have correct threshold behavior.

Blankenbecler and Sugar[10] have recently proposed a method of making dynamical calculations which shares with the equivalent-potential the feature of including the force due to iterated exchange, but which requires the off-shell generalized potential.

C. Behavior in s

So far we have examined our amplitude at a fixed value of s, and found that as a function of t it has the structure we expect; we have speculated that this feature might make it a better approximation than is the N/D amplitude. When we look at our amplitude as a function of s, things do not look quite so good. In the first place, at large energies the short-range parts of the force are important, and even if we were to know $V(s,t)$ exactly, our approximate treatment of the short-range contributions of the iterations would not be justified. The N/D method also suffers from this defect, perhaps even more severely. Also,

the analytic structure in the s plane is not correct. We started by considering the unitarity expression in the physical region, but having defined the potential in (II-10), we can continue it anywhere we like, and in particular notice that $V(r,s)$ will become infinite as $s \to 0$, since in general $V(s,t)$ will be finite there. It will still be true as $s \to 0$ that the t discontinuity of our amplitude will be correct for $t < 4t_0$, but for $t > 4t_0$ it will become more and more incorrect as $s \to 0$. Moreover, this is a defect not shared by N/D; it was introduced when we accepted for the higher unitarity iterations expressions which differ from the correct ones in not always carefully distinguishing between m^{-1} and $2s^{-\frac{1}{2}}$.

This defect would be fatal if we ever had to write a dispersion relation in s. But we do not; Eq. (II-10) can be justified, and the Schrödinger equation solved, at fixed s. It is entirely possible for an amplitude to be a good approximation in the scattering region and to have bad analytic structure somewhere else. We will certainly not be able to continue our solution far out of the scattering region.

Below I will want to suggest that closed coupled channels may be important constituents of the ρ meson. If this be so, then in a one-channel calculation, requirements (I-3) are not sufficient to assure a good approximation. In N/D language we would explain this by saying that there is an important singularity, the inelastic threshold, which must be taken into account. In our language we say that the t discontinuity for $t > 4t_0$ is affected by the closed channel. It will

perhaps seem strange that we can discuss the neglect of a closed channel (or the introduction of a spurious singularity at $s = 0$) in terms of the t discontinuity at fixed s, when the more usual way to think about this is in terms of a dispersion relation in s. That we can so discuss this can be seen from the following observation: If our amplitude had the correct t discontinuity for a particular s, then (under the assumption of analyticity in angular momentum) it would be correct[5] for that value of s.

D. Modification for πN

If the external particles do have spin, there may be several coupled amplitudes, in which case the unitarity equations become more complicated. It is possible, however, to generalize the results of the preceding sections to some amplitudes involving particles with spin. For the case of πN scattering, I have been able to calculate an amplitude for the states with $J = \ell + \frac{1}{2}$, but not for those with $J = \ell - \frac{1}{2}$.

The kinematics of πN scattering have been summarized by, for example, Frautschi and Walecka,[11] whose notation I shall use. The invariant amplitudes A and B are not unitary in the sense of Eq. (I-1), and so could not possibly be reproduced by a Schrödinger equation. Balázs had tried to write a Schrödinger equation for the πN amplitude which is a matrix in spin space.[12] He found that the attractive potential corresponding to nucleon exchange behaved like r^{-3} at the origin, which he proposed to handle with a cutoff. By

using amplitudes which obey simple unitarity relations, we shall be able partially to overcome this difficulty.

Consider the partial-wave amplitudes $f_{\ell^{\pm}}(s)$ for orbital angular momentum ℓ and $J = \ell \pm \frac{1}{2}$, which are normalized to $e^{i\delta} \sin \delta/q$ under the assumption of elastic unitarity. Then let $F_{\pm}$ be defined by

$$F_{\pm}(s,t) = \sum (2\ell + 1)\, f_{\ell^{\pm}}(s)\, P_{\ell}(1 + t/2q^2) \qquad \text{(II-11)}$$

in the physical region, and by analytic continuation wherever the sum in (II-11) does not converge. The sum defining F_+ begins at $\ell = 0$, the sum for F_- at $\ell = 1$. Since the $f_{\ell^{\pm}}$ satisfy the unitarity of spinless particles, so will F_+ and F_-. That is, we have the familiar relations

$$\text{Im}\, F_{\pm}(s,t) = (q/4\pi) \int d\Omega\; F_{\pm}(s,t')\, F_{\pm}^{*}(s,t'') \; ; \qquad \text{(II-12)}$$

F_+ and F_- are not coupled by unitarity.

Since F_+ and F_- are each unitary, we could carry out the derivations of the preceding sections in terms of them. That is, if we could find the potentials $V_{\pm}(s,t)$, we could hope to put them into Eq. (II-10) (dropping the $2s^{-\frac{1}{2}}$, since (II-12) shows that $F_{\pm}$ satisfy a nonrelativistically-normalized unitarity) and solve the Schrödinger equation for amplitudes $F_{\pm}$ which a) were correctly unitary, and b) had their t discontinuity correct, for small t. The potentials

will, as is customary, be constructed from the single-particle poles in the crossed channels, which in the spinless case will mean that $V(s,t)$ will be a delta function in t. However, the price we pay for eliminating spin from the πN unitarity equations is that the crossing relations become complicated. In the appendix, the following points are established: the exchange of a particle contributes to $F_\pm(s,t)$ not only a pole, but also a cut extending from the pole position to $t = +\infty$. Thus $V_\pm$ corresponding to a single-particle exchange force can be written

$$V_\pm(s,t) = g_\pm(s)\,\delta(t - t_p) + h_\pm(s)\,\varphi_\pm(s,t)\,, \qquad \text{(II-13)}$$

where $g_\pm$ and $h_\pm$ are kinematic factors, t_p is the position of the pole in $F_\pm$, and $\varphi_\pm$ is zero if $t < t_p$. Also at large t, $|\varphi_+(s,t)| < \text{const.} \times t^{-3/2}$ and $\varphi_-(s,t) > \text{const.} > 0$.

From (II-10), this means that the potential corresponding to V_- behaves like r^{-3} at the origin. Thus we can not use a (nonsingular) Schrödinger equation to satisfy requirements (I-3) applied to F_-. Actually, this could already have been seen from the fact that F_- has no s wave, and is essentially the same trouble that Balázs found.[12] For this reason the equivalent-potential method does not enable us to calculate scattering in those πN states with $J = \ell - \frac{1}{2}$. On the other hand, the potential corresponding to V_+ behaves like r^{-1} at the origin, and so we are able to produce an amplitude satisfying (I-3) by this method. In fact, it will turn out that for the range of

energies considered, the effect on the amplitude of φ_+ is very small and might as well have been neglected; this is consistent with the hope that for moderate energies we need only consider the long-range parts of forces.

It might be thought that, since we will have an expression for $f_{\ell+}$, we should make a continuation in the energy, and by virtue of the MacDowell symmetry arrive at an expression for $f_{(\ell+1)^-}$. However, because of the considerations of the preceding section, this continuation can not be done.

The equivalent-potential method may be expected to apply to any amplitude which satisfies a unitarity equation such as (II-12), and which has all of its partial waves. Another example of such an amplitude is the spin-singlet NN amplitude. In this paper I am especially interested in calculating amplitudes for which a simple approximation to the potential can be written with few if any parameters. This is not the case with the NN problem, since the couplings of the vector mesons are not very well known, and especially since the $\pi\pi$ s-wave exchange seems to be very important,[13] and so I have not included a calculation of NN scattering.

E. Calculational Methods and Conventions

For the one-channel problem, the partial wave Schrödinger equation is

$$\left(\frac{d^2}{dr^2} + q^2 + \frac{\ell(\ell+1)}{r^2}\right) \psi(r) = V(r,s)\, \psi(r) . \qquad \text{(II-14)}$$

A sufficiently simple way to obtain the scattering amplitude is to solve (II-14) numerically (one Schrödinger equation takes about 30 μ sec on the CDC 6600 computer) and get the phase shift δ_ℓ from the asymptotic form of ψ. If R is so large that $V(r,s)$ is negligible for $r > R$, then

$$\tan \delta_\ell(s) = \frac{\psi(R)\,[j_\ell(qR) + qR\, j_\ell'(qR)] - \psi'(R)\, Rj_\ell(qR)}{\psi(R)\,[n_\ell(qR) + qR\, n_\ell'(qR)] - \psi'(R)\, Rn_\ell(qR)}\ , \qquad \text{(II-15)}$$

where j_ℓ and n_ℓ are spherical Bessel functions and primes denote differentiation with respect to r. It is convenient to use Bessel functions rather than sines and cosines to avoid having to integrate (II-14) up to an R so large that the centrifugal term is also negligible.

For the two-channel calculations, the partial wave Schrödinger equations are[14]

$$\left(\frac{d^2}{dr^2} + q_i^2 + \frac{\ell(l+1)}{r^2}\right) \psi_i(r) = \sum_j V_{ij}(r,s)\, \psi_j(r)\ ; \qquad \text{(II-16)}$$

below the threshold of the kth channel, $q_k = +\,i\,|q_k^2|^{\frac{1}{2}}$. We will need two independent regular solutions of (II-16), which I will label with a second subscript; thus $\psi_{ij}(r)$ is the jth solution for the ith channel. If we were lucky enough to pick our two solutions so that the incoming waves were diagonal, that is, if for R large enough,

$$(\psi_{ij})_{\text{lucky}} = R\left(\delta_{ij}\, h_\ell^2(q_iR) + S'_{ij}\, h_\ell^1(q_iR)\right)\ , \qquad \text{(II-17)}$$

then we could read off the S matrix, which is given by

$$S_{ij} = q_i \, q_i^{-1} \, S'_{ij} \; ; \qquad \text{(II-18)}$$

the h_ℓ in (II-17) are spherical Hankel functions: $h_\ell^{\,1} = j_\ell + i\, n_\ell$, $h_\ell^{\,2} = j_\ell - i\, n_\ell$. In general, the asymptotic form of ψ is given by

$$\psi_{ij} = R\left(A_{ij}\, h_\ell^{\,2}(q_i R) + B_{ij}\, h_\ell^{\,1}(q_i R)\right). \qquad \text{(II-19)}$$

Comparing (II-17) and (II-19), we see

$$(\psi)_{\text{lucky}} = \psi\, A^{-1} \text{ , and so } \quad S' = BA^{-1} \, . \qquad \text{(II-20)}$$

The matrices A and B can be obtained from the asymptotic form of ψ_{ij} , by differentiating and inverting (II-19). Finally, in the two-channel case the eigenphase shifts are obtained from

$$2i\, \delta_\ell^{\pm}(s) = \log S^{\pm} \; ; \; S^{\pm} = \frac{1}{2}\left[S_{11} + S_{22} + \left((S_{11} - S_{22})^2 + 4\, S_{12}^{\,2}\right)^{\frac{1}{2}}\right]. \qquad \text{(II-21)}$$

The "resonant energies" that I will list are the energies at which δ passes through $\pi/2$. The output reduced widths are defined by

$$\Gamma = \left[\frac{s^{\frac{1}{2}}}{8\, q^{2\ell+1}} \left(\frac{d\delta}{ds}\right)^{-1}\right]_{s=s_{\text{resonance}}} , \qquad \text{(II-22)}$$

and have the dimensions of $(\text{GeV})^{2-2\ell}$. In the narrow-width approximation, $(d\delta/ds)^{-1}$ is the imaginary coordinate of the pole in the s plane.

Since the above discussion is not limited to integral values of ℓ, Regge trajectories $\alpha(s) = \alpha_R(s) + i\,\alpha_I(s)$ can be obtained through the approximation that $\alpha_R(s)$ is that value of ℓ for which $\delta_\ell(s) = \pi/2$, and $\alpha_I(s) = (d\alpha_R/ds)\,(d\delta/ds)^{-1}$. As one of several checks on the numerical accuracy of these computations, some of the one-channel equations were also analyzed with a computer program written by Burke and Tate,[15] which directly finds the pole in the S matrix for complex ℓ; the agreement was excellent.

F. Summary of Equivalent-Potential Method

The calculations to be described in the next two sections proceed as follows: first we pretend that we know the generalized potentials; we shall in fact take them to be the simplest single-particle exchange terms. For the case of spinless external particles $V(s,t)$ will thus be a delta function: for the πN case $V_+(s,t)$ will be more complicated. We will then construct the Schrödinger equation potential $V(r,s)$ from Eq. (II-10), or in the πN case from its analogue,

$$V_+(r,s) = -\frac{1}{\pi}\int_{t_0}^{\infty} dt\; V_+(s,t)\; e^{-r\sqrt{t}}/r\;. \qquad \text{(II-10')}$$

The Schrödinger equation can then be solved numerically to give the scattering amplitude of which $V(s,t)$ is the generalized potential.

Our amplitude will satisfy unitarity and have the correct t discontinuity for small t. In addition, the structure in t will be that dictated by the Mandelstam unitarity iteration, and so the partial waves of our amplitude will have correct threshold behavior. There are no free parameters in the calculation that are not contained in the potential itself; in particular there is no cutoff, even though the exchanged particles will have non-zero spin. This means that when the generalized potential is known unambiguously, the scattering is predicted unambiguously.

III. EXAMPLES: $\pi\pi$ AND πK SCATTERING

To be valuable in helping us to judge the validity of the equivalent-potential method, an example should have the following three properties: First, the equivalent-potential method must be applicable to it; this eliminates most but not all examples of particles with non-zero spin. Second, we should know how to construct with a minimum of free parameters a generalized potential which has some chance of being a good approximation in the resonance region. Third, there should be relatively unambiguous experimental results with which to compare the calculations. The examples I have chosen to discuss are $\pi\pi$, πK, and πN scattering. In this section I present two calculations of the $\pi\pi$ amplitude, the first assuming elastic unitarity and the second including one other channel, and then a calculation of πK scattering. In the following section I present a calculation of πN scattering in the states with $J = \ell + \frac{1}{2}$.

A. Single Channel $\pi\pi$ Calculation

In the $\pi\pi$ calculation I will let the generalized potential be given by the ρ meson exchange term. For a resonance pole in the t channel, the amplitude will have the form

$$A(t,s) = (2\ell + 1)\, P_\ell(1 + s/2q_t^{\,2})\, A_\ell(t)\,, \qquad \text{(III-1)}$$

where $A_\ell(t)$ is the t channel partial wave in which the resonance occurs. Near the resonance,

$$A_\ell(t) \approx \frac{\gamma/\rho}{t - t_R - i\gamma} . \qquad \text{(III-2)}$$

The phase-space factor ρ equals $2q_t t^{-\frac{1}{2}}$, by the normalization which has been established in (I-1). From Eq. (II-22), $\Gamma_{in} = t^{\frac{1}{2}} \gamma/(8q_t^{2\ell+1})$, and so

$$A_\ell(t) = \frac{4\, q_t^{2\ell}\, \Gamma_{in}}{t - t_R - i\gamma} . \qquad \text{(III-3)}$$

In the narrow-width approximation, by combining (III-3) and (III-1) we see

$$\text{Im } A(t,s) = 4\pi\, \Gamma_{in}(2\ell + 1)\, P_\ell(1 + s/2q_R^2)\, q_R^{2\ell}\, \delta(t - t_R) , \qquad \text{(III-4)}$$

q_R being the value of q_t when $t = t_R$.

To cross into the s channel, we have only to multiply by an isotopic spin crossing matrix element β, so the contribution to the generalized potential of the pole in the t channel is

$$V(s,t) = 4\pi\, \beta\, \Gamma_{in}(2\ell + 1)\, P_\ell(1 + s/2q_R^2)\, q_R^{2\ell}\, \delta(t - t_R) . \qquad \text{(III-5)}$$

For ρ exchange, $\ell = 1$ and $t_R = m_\rho^2$. There is an identical contribution from exchange in the u channel; the two contributions cancel for those states in which scattering is forbidden by Bose statistics and add in the allowed states, so if we remember to compute only allowed states, we can write

$$V(s,t) = 12 \pi \beta \Gamma_{in}(s + 2 q_R^2) \delta(t - m_\rho^2) . \qquad \text{(III-6)}$$

The parameters appearing in (III-6) are all reasonably well known. If we take the physical ρ to have a mass of 750 MeV and a width of 120 MeV, then $\Gamma_{in} \approx 0.20$. There are thus no free parameters, and we can get an unambiguous prediction of the $\pi\pi$ amplitude, which can be compared with experiment; in particular, we can see how well the ρ is reproduced in the output.

Not surprisingly, this prediction fails miserably. The ρ does not appear in the output, and the $\ell = 1$ phase shift is only 10° at the mass at which the ρ should be. This is consistent with the feeling that one gets from N/D calculations[16,17,18] that the force due to the ρ is just not strong enough to produce the ρ. Having failed to reproduce the ρ with the physical input, suppose we now treat the input width Γ_{in} as a parameter, and vary it until the $\ell = 1$ phase shift does equal $\pi/2$ at the ρ mass. This results in $\Gamma_{in} = 0.46$ (corresponding to a width of 270 MeV); the output width turns out to be $\Gamma_{out} = 0.50$, and so this calculation is very nearly self consistent.

In fact, it is a common approach in $\pi\pi$ calculations to determine the ρ parameters by requiring approximate self-consistency.[18,19] In the present case this would not work, as there are many self consistent solutions; even with Γ_{in} fixed at 0.46, the input and output are

consistent to within 10% as the input mass varies from 600 MeV to 1 GeV. To require exact self consistency in an approximate calculation does not seem very meaningful; among other reasons, this would depend on the way in which the output width was defined. Therefore all that I can contribute to discussions of self-consistency is to report that it is possible to achieve self-consistency using the physical ρ mass.

We have, within the limitations of the equivalent-potential approximation, confirmed that the force due to the ρ is not strong enough, that other forces (including perhaps the effects of closed coupled channels) must be included. Below I will try to deal with the problem of closed coupled channels more or less directly; for the time being, let me hope that all of these other effects can in some sense be approximated by increasing the input ρ width, so that the potential will indeed be given by Eq. (III-6), with the physical ρ mass but with $\Gamma_{in} = 0.46$. Having thus adjusted the potential to give the output ρ mass correctly, I can now see how closely the other parameters of the $\pi\pi$ amplitude agree with experiment.

With this potential, for $I = 0$ there are two Regge trajectories above $\ell = 0$ at threshold, both of which pass through $\ell = 2$ and thus produce resonances. For $I = 1$ there is one trajectory, which passes through the ρ; for $I = 2$ the force is repulsive. The real parts of the trajectories continue to rise as the energy is increased, and in fact do eventually go through higher physical values of ℓ, but well

above $s = 300\ m_\pi^2$. However, the approximations we have made can be justified, if at all, only at low energy, and so it would be entirely unjustified to attach any significance to these recurrences.

The two $I = 0$, 2^+ resonances occur at 1070 and 1900 MeV; I will identify them with the $f^o(1250)$ and the $f^{o'}(1500)$. These and other parameters are summarized in Table I.

The calculation of s-wave scattering by the equivalent potential method is less reliable than the calculation of higher partial waves, for the usual reason that the s wave depends more strongly on the shorter-range parts of the potential. In this case where we know that our potential is wrong, where we are representing unknown forces by an increase in the input width, it would perhaps be asking too much to expect an accurate calculation of the s-wave phase shift. On the other hand, as at the present time there is no firm evidence on this phase shift, any result we might calculate seems safe from immediate experimental refutation. For completeness, the s-wave scattering length is included in Table I; the negative sign corresponds to a decreasing phase shift. This is the sign that would be expected from the existence of trajectories above $\ell = 0$ at threshold, even though the forces are attractive.

As explained above, the calculated amplitude can not be continued to $s = 0$. However, for some distance above threshold the trajectories are very nearly linear; these linear trajectories have been extrapolated to $s = 0$, and the intercepts obtained in this way

entered into Table I. Since crossing symmetry is not imposed on this calculation, there is no constraint forcing Regge intercepts to lie below 1. At one time it was supposed that the f^o lay on the P trajectory and the $f^{o'}$ on the P';[20] this is the basis for the "experimental" intercepts appearing in Table I. If the f^o should lie on the P' trajectory, the agreement between experiment and calculation in this respect would of course be worse.

B. Coupled $\pi\pi$ - $K\bar{K}$ Scattering

It is often suggested[16], for example that closed coupled channels are important constituents of the ρ, and therefore must be included in any dynamical calculation of it. Three of the more likely channels are $\pi\omega$, $N\bar{N}$, and $K\bar{K}$. Since I am prevented from treating the first two of these because of spin complications (the ρ couples to the spin-triplet states of the $N\bar{N}$ system), I will attempt to calculate the coupled $\pi\pi$ - $K\bar{K}$ system, ignoring the other channels. This approximation will mean that I can not hope to get the $K\bar{K}$ scattering at all correct, but perhaps the inclusion of even one additional channel will improve the results for the $\pi\pi$ elastic amplitide.

In analogy to the single channel calculation, I will take the generalized potential to be given by vector meson exchange. The appropriate force diagrams are presented in Fig. 1. The $\rho K\bar{K}$ coupling is not directly accessible to experiment, but is determined if we assume that the reduced VPP couplings are related by SU3.[21]

Using these relations, and assuming for simplicity that the φ is the member of the octet (although any other ω - φ mixing angle could have been assumed), the 2 x 2 matrix potential can be expressed in terms of the reduced $\rho\pi\pi$ coupling Γ_{in} :

$$V_{\pi\pi,\pi\pi}(s,t) = 12\pi\beta\,\Gamma_{in}(s + 2q_R^2)\,\delta(t - m_\rho^2) \qquad \text{(III-7a)}$$

$$V_{\pi\pi,K\bar{K}}(s,t) = V_{K\bar{K},\pi\pi}(s,t) = (9\pi/\sqrt{2})\,\beta\,\Gamma_{in}(s + 2\,q_R^2)\,\delta(t - m_{K^*}^2) \qquad \text{(III-7b)}$$

$$V_{K\bar{K},K\bar{K}}(s,t) = 9\,\pi\,\beta\,\Gamma_{in}(s + 2\,q_R^2)\,\delta(t - m_\varphi^2)$$

$$+ 3\,\pi\,\beta\,\Gamma_{in}(s + 2\,q_R^2)\,\delta(t - m_\rho^2)\,. \qquad \text{(III-7c)}$$

Here q_R is the value of q_t at the position of the exchanged resonance: in (III-7a), $q_R^2 = (m_\rho^2 - 4m_\pi^2)/4$; in (III-7b),

$$q_R^2 = \left(m_{K^*}^2 - (m_K - m_\pi)^2\right)\left(m_{K^*}^2 - (m_K + m_\pi)^2\right) / (4m_{K^*}^2)\,, \text{ etc.}$$

With the physical value $\Gamma_{in} = 0.20$, the $\pi\pi$ $\ell = 1$ phase shift is again 10° at the mass at which the ρ should be, just as it was in the single-channel calculation. A resonance in this amplitude does appear, at about 1600 MeV. Adjusting Γ_{in} to force this resonance down to 750 MeV, we find $\Gamma_{in} = 0.38$, which is a somewhat smaller value than was necessary in the single channel calculation. The output width is improved even more than the input, and is brought down to

$\Gamma_{out} = 0.32$; the calculation is still nearly self consistent.

With $\Gamma_{in} = 0.38$, the rest of the amplitude was calculated, and the results entered in Table I. The widths of the f^o and the $f^{o'}$ are both substantially reduced from what they were in the one-channel calculation; unfortunately, the $f^{o'}$ as well as the f^o is coupled primarily to the $\pi\pi$ channel, in defiance of experimental results.

One resonance appears in this calculation which did not appear at all in the single-channel calculation: a $I = \ell = 0$ resonance at 685 MeV, with a full width of 20 MeV. It is amusing to note that, because of the asymmetry in the decay of the ρ^o, a resonance with these quantum numbers has been conjectured to lie underneath the ρ.[22] However, it is possible to follow the trajectory on which our new resonance lies, and see that above the $K\bar{K}$ threshold it couples primarily to the $K\bar{K}$ channel; so at $\ell = 0$ it is primarily a bound state of $K\bar{K}$, which explains its small width. But this means that its existence depends largely on the dynamics in the $K\bar{K}$ channel, and that is just the part of the problem which we knew we had no chance of getting correctly. Thus the appearance of this resonance so near to the ρ mass must be regarded as fortuitous. For the same reasons, the failure of the $f^{o'}$ to couple strongly to $K\bar{K}$ is not serious.[22a]

C. πK Scattering

For the calculation of πK scattering, let us again take the forces to be given by vector meson exchange, the ρ in the t channel and the K^* in the u channel. We again need the $\rho K\bar{K}$ coupling,

and we can again relate all of the reduced couplings by SU3.[21] This time let us say that the one independent input parameter is the K^* reduced width Γ'_{in}, since this is what can be compared with the output.

For $I = 3/2$ the total force is repulsive, so we need only consider the $I = \frac{1}{2}$ states. Because of the unequal π and K masses, the potential looks slightly different than (III-6);[23] the potential appropriate to angular momentum ℓ is given by

$$V(s,t) = V_\rho(s,t) + (-1)^\ell V_{K^*}(s,t) \qquad \text{(III-8a)}$$

$$V_\rho(s,t) = 16\pi \Gamma'_{in}(s + \alpha)\,\delta(t - m_\rho^2) \qquad \text{(III-8b)}$$

$$V_{K^*}(s,t) = 4\pi \Gamma'_{in}\left(s + 2q_R^2 - \beta(m_{K^*}^2)\right)\delta\left(t - m_{K^*}^2 + \beta(s)\right) \qquad \text{(III-8c)}$$

In (III-8b), $\alpha \equiv m_{K^*}^2/4 - (m_\pi^2 + m_k^2)/2$, and in (III-8c), $\beta(x) \equiv (m_K^2 - m_\pi^2)^2/x$. In a preliminary report of this calculation,[6] the approximation $\beta = 0$ was made; the numerical differences in the results were completely negligible.

Γ'_{in} can be determined by requiring the output mass of the K^* to be 890 MeV; this requirement gives $\Gamma'_{in} = 0.57$ (the observed width of 50 MeV[24] implies $\Gamma = 0.22$). The output width $\Gamma_{out} = 0.55$ is very nearly the same as the input. There is one other resonance, at $\ell = 2$ at 1265 MeV; I identify this with the $K^{**}(1405)$. Parameters for these two resonances are presented in Table II.

IV. EXAMPLE: π N SCATTERING

For the π N example, I will construct the generalized potential from nucleon and $N^*(1238)$ exchange. The couplings of the ρ are somewhat uncertain, and in any case the ρ contribution is expected to be small.[25] Since in the rest of the potential there are no free parameters, I simply neglect the ρ force.

The contributions of N and N^* exchange in the u channel to the invariant amplitudes A and B have been given by Ball and Wong.[26] For the amplitudes of isotopic spin (1/2, 3/2) these contributions are

$$A(s,t,u) = (4/3,\ 1/3)\ g_{N^*}{}^2(a_1 - a_2 s)/(\Delta^2 - u) \qquad \text{(IV-1a)}$$

$$B(s,t,u) = -(4/3,\ 1/3)\ g_{N^*}{}^2(b_1 - b_2 s)/(\Delta^2 - u)$$

$$+ (1,\ -2)\ g_N{}^2/(M^2 - u) \qquad \text{(IV-1b)}$$

where in units in which $m_\pi = c = \hbar = 1$, the N^* mass Δ is 8.9, $a_1 = 842$, $a_2 = 23.4$, $b_1 = -157$, $b_2 = 1.5$, and the values of the couplings are $g_N{}^{*2}/8\pi = 0.06$, $g_N{}^2/4\pi = 14.4$. Projecting the partial waves of definite exchange parity from (IV-1), we have

$$A_\ell{}^{\pm}(s) = \pm (4/3,\ 1/3)\ g_{N^*}{}^2[(a_1 - a_2 s)/q^2]\ Q_\ell(1 + t_{N^*}/2q^2) \qquad \text{(IV-2a)}$$

$$B_\ell^{\pm}(s) = \mp (4/3,\ 1/3)\ g_{N^*}^2[(b_1 - b_2 s)/q^2]\ Q_\ell(1 + t_{N^*}/2q^2)$$

$$\pm (1,\ -2)\ g_N^2[1/q^2]\ Q_\ell(1 + t_N/2q^2)\ , \qquad \text{(IV-2b)}$$

where, because of the unequal-mass kinematics, t_{N^*} and t_N, the positions of the poles in the amplitudes of definite exchange parity, depend on s;

$$t_N = M^2 - (M^2 - m_\pi^2)^2/s$$

$$t_{N^*} = \Delta^2 - (M^2 - m_\pi^2)^2/s\ ; \qquad \text{(IV-3)}$$

compare the quantity $\beta(s)$ in (III-8c).

Equations (IV-2) and the crossing relations derived in the appendix enable us to compute the contributions to $V_+(s,t)$ of N and N^* exchange, each of which will have the form (II-12). With a superscript $\pm$ to indicate exchange parity, so that V_+^+ is physical for $J - \frac{1}{2}$ even, and V_+^- for $J - \frac{1}{2}$ odd, we see from Eqs. (IV-2) and (A-13).

$$V_+^{\pm}(s,t) = \pm \Big\{(1,-2)\ g_N^2[(C_{12} - C_{22}/\beta_R)\ 2\pi\ \delta(t - t_N)$$

$$- (C_{22}/q^2)\ \mathrm{Im}\ \hat{S}(z, 1 + t_N/2q^2)\ \theta(t - t_N)]$$

Equation (IV-4) continued

$$+ (4/3,\ 1/3)\ g_{N^*}^2 \left[\left((a_1 - a_2 s)\ (C_{11} - C_{21}/\beta_R)\right.\right.$$

$$+ (b_1 - b_2 s)\ (-C_{12} + C_{22}/\beta_R)\Big)\ 2\pi\ \delta(t - t_{N^*})$$

$$+ (1/q^2)\left(-(a_1 - a_2 s)\ C_{21} + (b_1 - b_2 s)\ C_{22}\right)\ \mathrm{Im}\ \hat{S}(z,\ 1 + t_{N^*}/2q^2)$$

$$\left.\left. \times\ \theta(t - t_{N^*})\right]\right\} \qquad \text{(IV-4)}$$

where $z = 1 + t/2q^2$, $\beta_R = z + (z^2 - 1)^{\frac{1}{2}}$, the C's are given by Eq. (A-3), and $\hat{S}$ is given by Eq. (A-12).

Since only states with $J = \ell + \frac{1}{2}$ can be calculated, we can not look for the nucleon pole; the only low energy resonance we expect to find is the $N^*(1238)$ in the P_{33} state. With the physical values of the input couplings this resonance appears at 1100 MeV, just above threshold. Its full width is 1.5 MeV; this corresponds to a reduced width Γ of 0.8, to be compared with the experimental value of 1.7 (120 MeV). These results have been entered into Table III.

It thus appears that there is no need to depart from the physical values of the input parameters to obtain results which look reasonably similar to experiment. The non-resonant phase shifts also agree in a general way with the results of phase-shift analysis, even though there were no parameters that were adjusted to make them agree. In Figs. 2 and 3 the non-resonant phase shifts with $\ell \leqslant 2$ are compared with the 0 to 700 MeV phase shift analysis by Roper.[27]

At higher energies the results were not consistent with experiment. Between 1500 and 2500 MeV, the only resonances to appear were a second P_{33} at 1600, and an S_{11} at 2140, and this is clearly wrong. In particular, the Regge recurrences of the $N^*(1238)$ never appear; the trajectory rises only to $\ell = 2.1$ at 1920 MeV. However, the slope of the trajectory at the resonant energy is 0.9 $(\text{GeV})^{-2}$, which is the same slope as that obtained from a straight-line fit to the N^* and its observed recurrences[28] (and so if we extrapolate back to $s = 0$, our intercept would be about the same as in that fit). This result is not surprising if we believe that, while at low energies the Δ trajectory is primarily coupled to the π N channel, at higher energies channels with higher thresholds (and probably higher external spin) are important.

The reported results were all obtained from the potential given by (IV-4). If the cut contribution had been neglected ($\hat{S}$ set equal to zero), none of the qualitative features would have been changed. In order to have another estimate of the degree of agreement between the calculated results and experiment, the value of the input π N $\bar{N}$ coupling necessary to give the N^* in the right place was determined. (The force due to the exchanged N^* is very small in this state.) The result was $g_N^2/4\pi = 11.0$, to be compared with the known value of 14.4.

V. DISCUSSION

The most important numerical results from the calculations described in the preceding two sections appear in Tables I-III and Figs. 2-3. In the π N calculation there were no free parameters, and so these results are pure "predictions;" in each of the other calculations there was one free parameter, as the input couplings were allowed to vary from their physical values. The agreement between these results and experiment compares favorably with that of other dynamical calculations having comparably few free parameters.

There were several areas in which the calculated results were not reasonable at all. One such area was at high energy; in the π N calculation the results were not good above 1600 MeV total energy. However, at high energies we should not expect the method used to be a good approximation, for at least three reasons: at high energy (i) the simple form of the generalized potential is not justified, (ii) the assumption of elastic unitarity is wrong, and (iii) the differences between this method and the full Mandelstam unitarity iteration become more acute. Indeed, it would be rather puzzling if we were able to get good results by this method in a region where we knew[27] that the amplitide was quite inelastic. And so while our failures above 1600 MeV certainly do indicate a limitation on what we may hope to do, they do not necessarily cast doubt on the better results obtained at lower energies.

A more serious failure was indicated by our inability to get any sensible results at all in the $\pi\pi$ and π K calculations with

the physical values of the input couplings. This failure seems to confirm the idea that a simple force, single channel model of the ρ is not very realistic. The fact that the output widths were substantially reduced (see Table I) when one additional channel was added indicates that at least a large part of the discrepancy between this model and reality arises in the neglect of the higher-threshold channels. That the inclusion of coupled channels will increase the force (equivalently, decrease the input width) is well known; that it will significantly decrease the output widths has been suspected, but in N/D calculations it has not always turned out that way.[17] Unfortunately, at the present time the equivalent-potential method is not able to handle the other channels which might be important constituents of the ρ. In any case, if we accept the result that closed channels are very important in reducing the ρ width, it follows that no strictly single-channel calculation should be expected to produce the correct ρ width.

In order to understand the relationship between the calculations reported here and analogous N/D calculations, let us suppose for the moment that the assumptions common to both methods, such as elastic unitarity and the particular choice of the generalized potential, were exactly correct. Under this supposition, the approximation involved in N/D is to neglect contributions to the left-hand cut of all but the lowest term of the unitarity iteration. As was mentioned in Section II, solving the Schrödinger equation means including all of the terms of that iteration. Only the lowest term is included exactly

(with relativistic kinematics) --to include them all would require an infinite iteration just to construct the potential--but it seems reasonable to hope that this is better than neglecting them altogether. If this be correct, then a given (attractive) generalized potential should produce larger scattering in equivalent-potential calculations than in N/D, which neglects the attraction produced by iterations of the potential. For a repulsive potential, the iterations alternate in sign, and hence tend to cancel; taking only the lowest-order contribution to the left-hand cut means using too much repulsion. We would expect, then, that in the equivalent-potential method attractive forces would appear stronger, and repulsive forces weaker, than in N/D calculations.

Unfortunately, this comparison is made difficult by the fact that with an adjustable cutoff, the force due to exchange of particles of spin $\geqslant 1$ can be made as strong as one likes. We then have to push the argument further: if N/D neglects important attractive contributions to the left-hand cut, then in order to obtain a resonance or a bound state at the correct mass, it is necessary to make the cutoff higher than if the additional attraction were included. This means that the D function would change more slowly with energy; residues would thus be greater, and trajectories flatter. We conclude that if the equivalent-potential method is a reliable approximation to the unitarity iteration, it should produce smaller residues and steeper trajectories than does the N/D method.

Indeed, this is the case. The residue of the N^* reported here is only one half of the physical value, while N/D calculations predict it to be too large.[26,29] In the single channel $\pi\pi$ calculation, the residue, although larger than the experimental value, came out smaller than in most N/D calculations; also the trajectories rose sharply enough to make resonances at $\ell = 2$. It is interesting to compare this feature of the $\pi\pi$ results with the calculation done by Bali[30] and with the N/D calculation of Collins and Teplitz.[18] The input coupling used by Collins and Teplitz was similar to the value used in this paper; however, their output ρ trajectory did not quite make it to $\ell = 1$, and no trajectory rose above $\ell = 1.5$. Bali, on the other hand, by directly examining the relativistic unitarity iteration for the simple ρ model, found trajectories that did rise sharply, and could produce an $\ell = 2$ resonance.

There is another way to compare the effective strengths of forces in the equivalent-potential and in the N/D methods. Consider the case of two forces, of opposite signs but comparable magnitudes. If the forces are of the same range, they will cancel within $V(s,t)$. If they are of different ranges then the iteration will make the attractive one stronger, and the repulsive one weaker, although changing the cutoff might not affect the relative strengths of the two forces.

An example of such a case can be found in the πN S_{11} state, where N^* exchange is repulsive and N exchange is attractive. Abers and Zemach[25] estimate the magnitude of the N^* force to be 1.1

times that of the N force; the point is that they are comparable. Also the ranges are quite different: because of the unequal π and N masses, the ratio of the ranges of the two forces is not $M/\Delta \approx 0.75$, but rather is t_N/t_{N^*} which at threshold is about 0.44. The facts that the energy dependence of the two forces is different, and that the dispersion relation in W includes contributions from the singularities of the P_{11} amplitude, which although far away are strong, make difficult the application of the above reasoning to the N/D calculation of the S_{11}. Nevertheless, N/D calculations do produce a negative scattering length[26,29] (which in this case indicates a net repulsive force), although it is known that the scattering length is positive.[27,31] Coulter and Shaw[29] obtained a negative scattering length even when they took account of inelasticity.

The potential that I used differed from that used in Refs. 26 and 29 in that I did not include the force due to ρ exchange. However, since this force is attractive in the S_{11} state, including it would not have decreased the attraction, and hence would not have altered the fact that the equivalent-potential calculation predicts that the scattering length is positive.

One might suspect that the failure of N/D calculations for the S_{11} state indicates a failure of the assumptions, in particular that unknown short-range forces are very important, at least for the s wave. The results presented here suggest the opposite: that when iterations of the potential are taken into account, simple N and N^* exchange is adequate to obtain a reasonable fit to low-energy π N scattering.

ACKNOWLEDGMENTS

I would like to thank Professor Geoffrey F. Chew both for suggesting this investigation and also for providing guidance throughout my entire Ph.D. program. I am indebted for helpful discussions to many members of the Berkeley Theoretical Group, especially to Professor Stanley Mandelstam and Drs. Peter Collins, Michael DerSarkissian, and Naren Bali. This work was performed under the auspices of the United States Atomic Energy Commission.

APPENDIX

We need to know the contribution to $V_{\pm}$ of a particle pole in the crossed reaction. The invariant amplitudes A and B satisfy simple crossing relations, so this contribution to A and B is a pole in t or u. The partial waves will then be given by expressions of the form

$$A_\ell(s) = K_1(s)\, Q_\ell(z_0) ,$$
$$B_\ell(s) = K_2(s)\, Q_\ell(z_0) . \qquad \text{(A-1)}$$

Comparison with Eqs. (IV-2) shows, for example, for nucleon exchange, $K_1(s) = 0$, $K_2(s) = \pm (1,-2)\, g_N^2/q^2$, $z_0 = 1 + t_N(s)/2q^2$. In this appendix we determine $V_{\pm}$ when A_ℓ and B_ℓ are given by (A-1).

From Ref. 11,

$$f_{\ell\pm}(s) = C_{11}\, A_\ell(s) + C_{12}\, B_\ell(s) + C_{21}\, A_{\ell\pm1}(s) + C_{22}\, B_{\ell\pm1}(s) , \qquad \text{(A-2)}$$

the matrix C being given by

$$C = \frac{1}{32\pi W^2} \begin{pmatrix} (W+M)^2 - \mu^2, & [(W+M)^2 - \mu^2]\,[W - M] \\ -(W-M)^2 + \mu^2, & [(W-M)^2 - \mu^2]\,[W + M] \end{pmatrix}, \qquad \text{(A-3)}$$

with $W^2 = s$ and μ = pion mass. Substituting (A-1) and (A-2) into the definition

$$F_{\pm}(s,t) = \sum (2\ell + 1)\, P_\ell(1 + t/2q^2)\, f_{\ell\pm}(s)\,, \qquad \text{(A-4)}$$

we get

$$F_{\pm}(s,t) = (K_1C_{11} + K_2C_{12}) \sum (2\ell + 1)\, P_\ell(1 + t/2q^2)\, Q_\ell(z_0) + (K_1C_{21} + K_2C_{22}) \times \sum (2\ell + 1)\, P_\ell(1 + t/2q^2)\, Q_{\ell\pm1}(z_0)\,. \qquad \text{(A-5)}$$

The first sum is easy:

$$\sum_{\ell=0}^{\infty} (2\ell + 1)\, P_\ell(z)\, Q_\ell(z_0) = 1/(z_0 - z)\,. \qquad \text{(A-6)}$$

The second sum is not so easy. Define

$$S_{\pm}(z,\, z_0) = \sum (2\ell + 1)\, P_\ell(z)\, Q_{\ell\pm1}(z_0)\,. \qquad \text{(A-7)}$$

For s fixed in the physical region, z_0 will be fixed and greater than 1. We shall need to evaluate the discontinuities and the asymptotic behavior of $S_{\pm}$ in the z plane for fixed z_0. The sums in (A-7) converge only in an ellipse passing through z_0, so it is necessary to do the sums where they converge, and then continue in z [this continuation is implied in writing (A-6)].

Let us first sum S_+ for $1 < z < z_0$. We can use Laplace's integral representations for P_ℓ and Q_ℓ:

$$P_\ell(z) = (1/\pi) \int_0^\pi d\theta\, [\alpha(z,\theta)]^\ell\,, \quad \alpha \equiv z + (z^2 - 1)^{1/2} \cos\theta\,,$$
$$Q_\ell(z_0) = \int_1^\infty dx\, (x^2 - 1)^{-1/2}\, [\beta(z_0,x)]^{-\ell-1},\ \beta \equiv z_0 + (z_0^2 - 1)^{1/2} x\,. \tag{A-8}$$

Although P_ℓ is an entire function, let us choose to stay on the sheet of $(z^2 - 1)^{1/2}$ in which $(z^2 - 1)^{1/2} \to z$ at large $|z|$. From (A-6) and (A-7),

$$\hat{S}(z,z_0) \equiv S_+(z,z_0) - \frac{1}{\beta_R}\,\frac{1}{z_0 - z} = \sum (2\ell + 1)\, P_\ell(z)\, Q_{\ell+1}(z_0) - \frac{1}{\beta_R} \sum (2\ell + 1)\, P_\ell(z)\, Q_\ell(z_0)\,. \tag{A-9}$$

The choice $\beta_R = z + (z^2 - 1)^{1/2}$ will mean that $\hat{S}$ has no pole. Now substitute (A-8) into (A-9):

$$\hat{S}(z,z_0) = \frac{1}{\pi} \sum (2\ell + 1) \int_1^\infty \frac{dx}{(x^2 - 1)^{1/2}} \int_0^\pi d\theta \left(\frac{1}{\beta} - \frac{1}{\beta_R}\right) \alpha^\ell\, \beta^{-\ell-1}\,. \tag{A-10}$$

The sum can be done inside the integrals,

$$\sum (2\ell + 1)\, \alpha^\ell\, \beta^{-\ell-1} = \frac{1}{\beta - \alpha} + \frac{2\alpha}{(\beta - \alpha)^2}\,, \tag{A-11}$$

since $|\alpha/\beta| < 1$ throughout the region of double integration. We can now do the integral over θ to get

$$\hat{S}(z,z_0) = \frac{-1}{\beta_R} \int_1^\infty \frac{dx}{(x^2-1)^{1/2}} \frac{\beta^2 - 1}{\beta(\beta^2 - 2\beta z + 1)^{1/2}} \frac{1}{\beta - z + (z^2-1)^{1/2}} . \tag{A-12}$$

It is straightforward to show that this integral exists for all z except for $z = z_0$ and $z\epsilon[-1,+1]$, that $|\hat{S}| \to \text{const} \times |z|^{-3/2}$ at large $|z|$, and that $\hat{S}$ has a cut from -1 to $+1$ (which does not appear in S_+) and another from z_0 to $+\infty$. Combining (A-5), (A-6), and (A-9), we have

$$F_+(s,t) = [(K_1C_{11} + K_2C_{12}) + (K_1C_{12} + K_2C_{22})/\beta_R]/(z_0 - z) + (K_1C_{21} + K_2C_{22})\,\hat{S}(z,z_0) , \tag{A-13}$$

with $z = 1 + t/2q^2$ and $\beta_R = z + (z^2-1)^{1/2}$. $\hat{S}$ can be evaluated numerically from (A-12). V_+ is the imaginary part of (A-13), which is 0 if $z < z_0$.

We can sum $S_-(z,z_0)$ in a similar way, and obtain

$$S_-(z,z_0) = \frac{\beta_R}{z_0 - z} + 4 \int_1^\infty \frac{dx}{(x^2-1)^{1/2}} \frac{1}{(\beta^2 - 2\beta z + 1)^{1/2}} \times \left[\frac{\beta z - 1}{\beta - z + (z^2-1)^{1/2}} - \frac{(z^2-1)^{1/2}(\beta - z - (z^2-1)^{1/2})}{\beta - z - (z^2-1)^{1/2} + (\beta^2 - 2\beta z + 1)^{1/2}} \right], \tag{A-14}$$

but the imaginary part of this integral is $\geq \pi/4$ at large z.

FOOTNOTES AND REFERENCES

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5. When I refer to an amplitude as "correct" I mean "identical to that amplitude implied by Eqs. (I-1) and (I-2), together with the requirements of polynomial boundedness and analyticity in ℓ, for a given choice of $V(s,t)$;" I do not necessarily mean "as indicated by experiment." Whether these two notions have anything in common is a subject of much speculation, including this paper.

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Table I. Parameters of the $\pi\pi$ Amplitude

	One-Channel Calculation	Two-Channel Calculation	Experiment
Reduced ρ width: input	0.46	0.38	0.20(120MeV)
output	0.50	0.32	0.20
Intercept of ρ trajectory	0.7	0.7	0.54[a]
f^o mass (MeV)	1070	1140	1250[b]
Reduced f^o width	0.50	0.35	0.25(100MeV)[b]
Intercept of f^o trajectory	1.3	1.2	1[c]
$f^{o'}$ mass (MeV)	1900	1870	1500[d]
Reduced $f^{o'}$ width	0.55	0.39	?
Intercept of $f^{o'}$ trajectory	0.7	0.5	0.69[c]
Other resonances below 2500 MeV	none	$0^+(685)$	none established
Scattering length (m_π^{-1})	-0.8	-2.0	?

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c. That is, if the f^o lies on the P trajectory, and the $f^{o'}$ on the P'. The P' intercept is from J. J. G. Scanio, New Determination of the P' Regge Trajectory Intercept, Phys. Rev. (to be published).

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Table II. Parameters of the π K Amplitude

	Calculation	Experiment
Reduced K^* width: input	0.57	0.22 (50 MeV)[a]
output	0.55	0.22
Intercept of K^* trajectory	0.4	?
K^{**} mass (MeV)	1265	1405[a]
Reduced K^{**} width	0.16	0.12 (95 MeV)[a]
Intercept of K^{**} trajectory	0.75	?
Other resonances below 2500 MeV	none	none established

a. See Ref. 24.

Table III. Parameters of the π N Amplitude

	Calculation	Experiment
N^* mass (MeV)	1100	1238
Reduced N^* width	0.8	1.7 (120MeV)
Other resonances below 2500 MeV	$P_{33}(1600)$, $S_{11}(2140)$	many
Non-resonant phase shifts	See Figs. 2 and 3	
$I = \frac{1}{2}$ scattering length (m_π^{-1})	0.29	0.17[a]

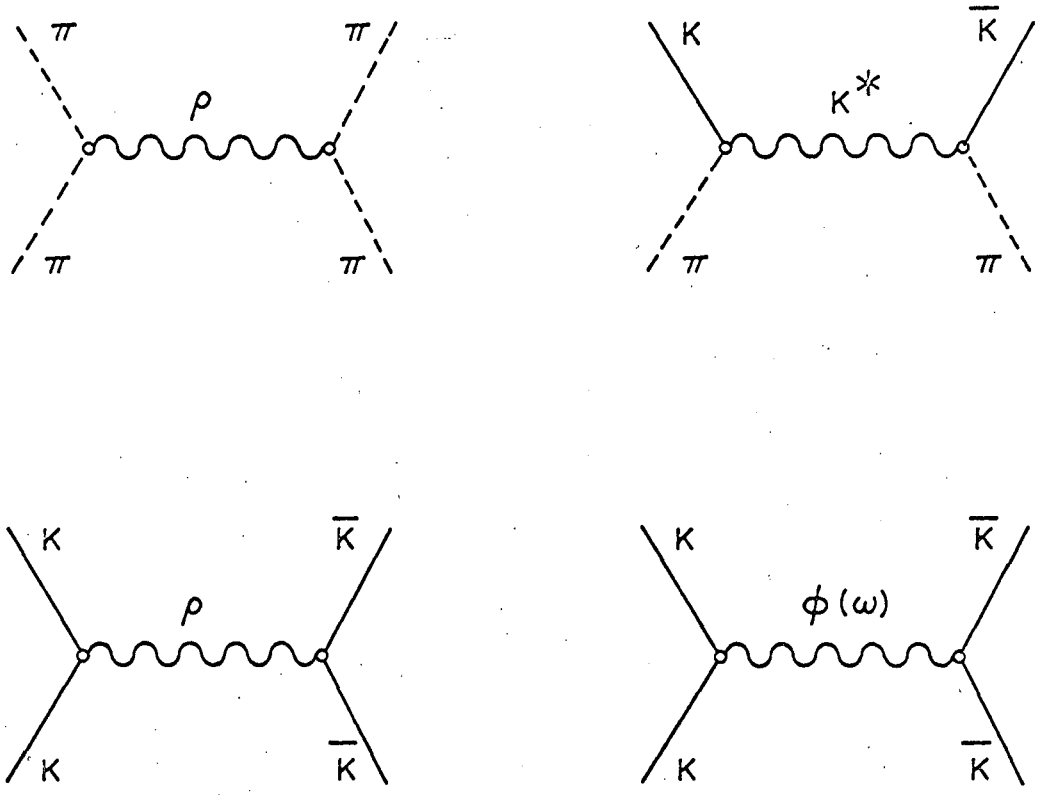
a. See Ref. 31.

FIGURE LEGENDS

Fig. 1. The generalized potential for the coupled $\pi\pi$ - K K calculation.

Fig. 2. π N $I = \frac{1}{2}$ phase shifts for $\ell \leq 2$ and $J = \ell + \frac{1}{2}$. Solid lines are the phase shifts calculated in this paper; dashed lines the results of Roper, Wright, and Feld (Ref. 27). (a) S_{11} phase shift; (b) P_{13} phase shift; (c) D_{15} phase shift.

Fig. 3. Same as Fig. 2 for nonresonant $I = 3/2$ phase shifts. (a) S_{31} phase shift; (b) D_{35} phase shift.



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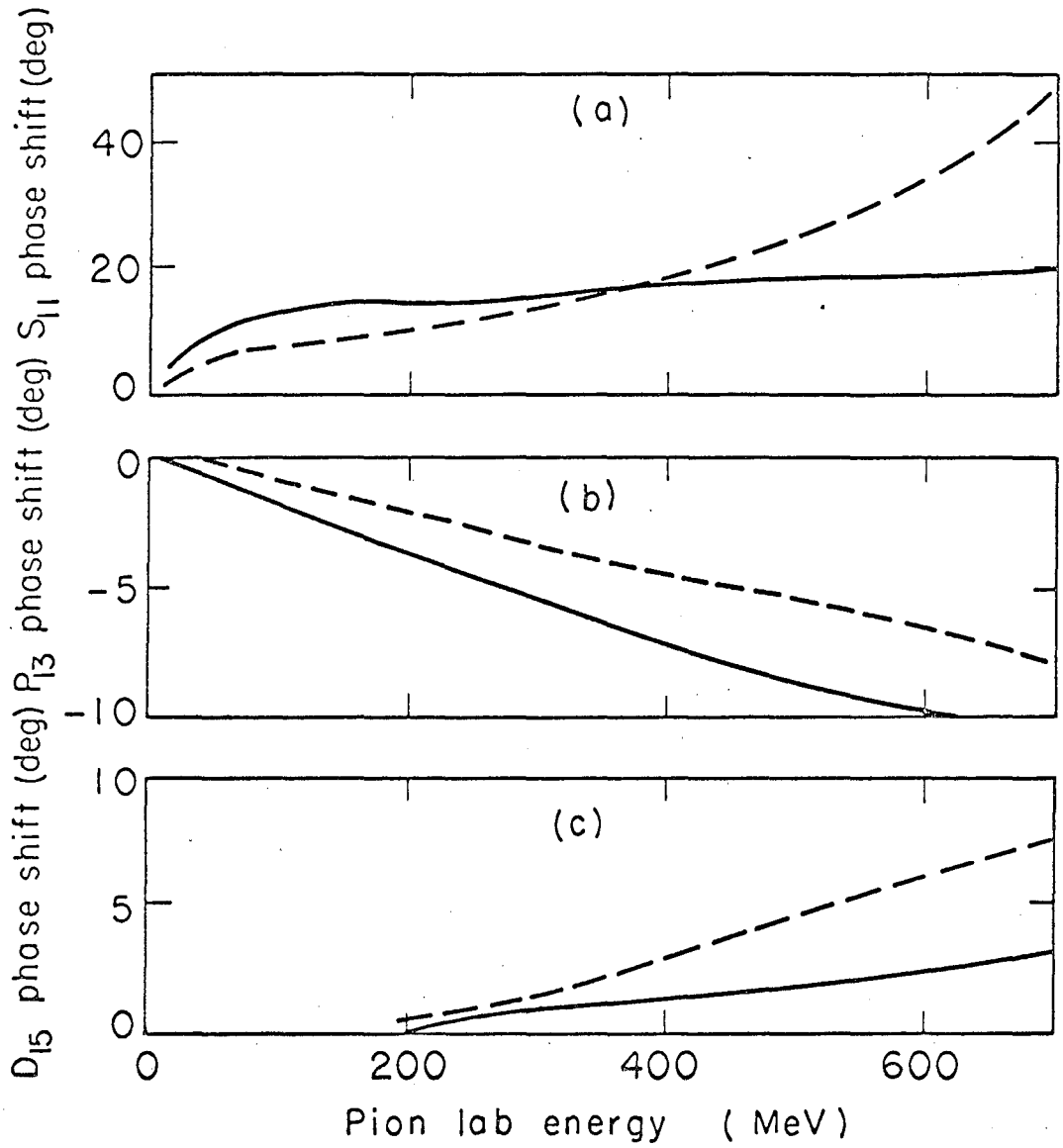
(a)
(b)
(c)
40
20
0
-5
-10
10
5
0
200
400
600
Pion lab energy (MeV)
S_{11} phase shift (deg)
P_{13} phase shift (deg)
D_{15} phase shift (deg)

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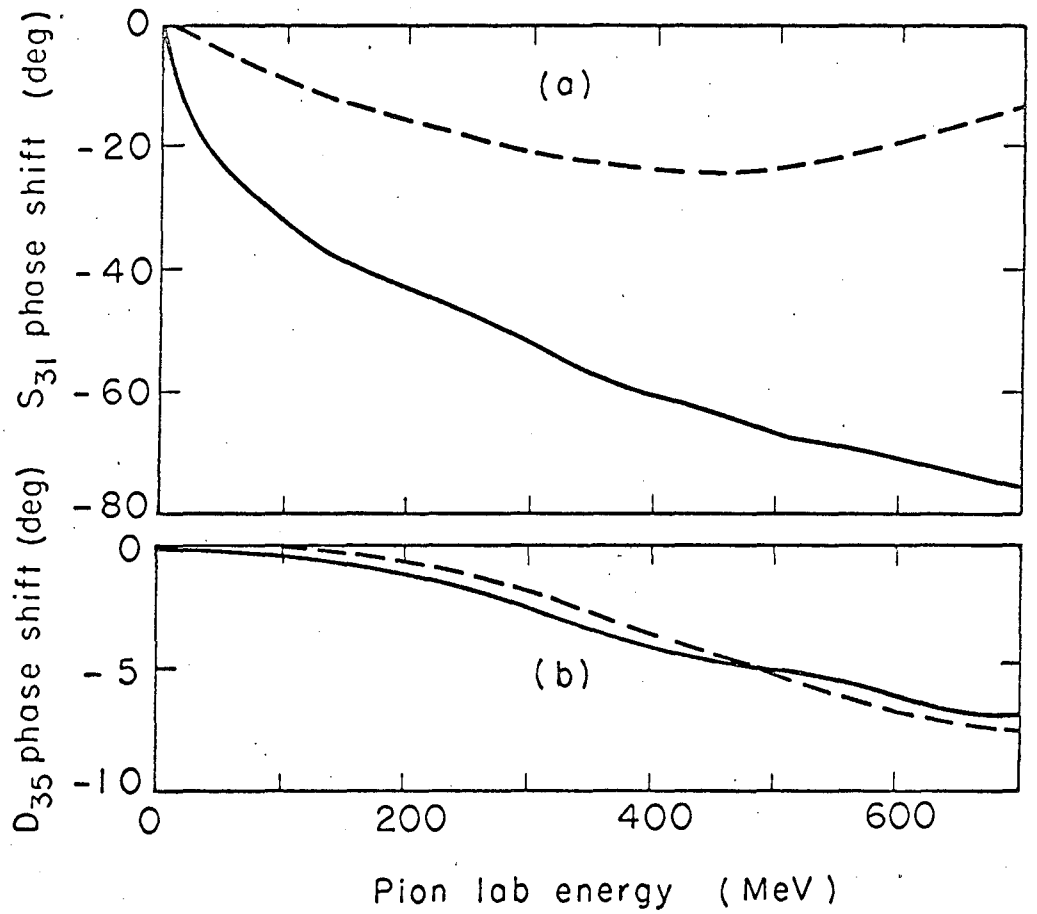
0
-20
-40
-60
-80
(a)
S_{31} phase shift (deg)
0
-5
-10
(b)
D_{35} phase shift (deg)
0
200
400
600
Pion lab energy (MeV)

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