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Generalized geometry and pluriclosed flow

DISSERTATION

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in Mathematics

by

Joshua Pierce Jordan

Dissertation Committee:
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2023

DEDICATION

To my parents and family; Bruce, Maria, and Carter you have supported me through many highs and lows. This dissertation was only possible because your relentless work ethic and good humor rubbed off.

To Sydney; you're always a breath of fresh air when the road is getting a little difficult. I only stuck this out to impress you.

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ABSTRACT OF THE DISSERTATION

Generalized geometry and pluriclosed flow

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Professor Jeffrey Streets, Chair

This dissertation will show the ways in which generalized geometry elucidates the study of pluriclosed flow. In their 2009 paper [37], Streets and Tian introduce pluriclosed flow – a parabolic flow of pluriclosed metrics – and classify some static solutions. In 2018 [34], Streets expanded this into a geometrization conjecture for compact, complex surfaces. The author is able to use these tools to show an equivalence between pluriclosed flow and a non-linear, coupled Hermitian-Yang-Mills type flow. From there, the author is able to more geometrically prove a result of Streets and Warren – an Evans-Krylov theorem for pluriclosed flow. The author is also able to use this equivalence to prove long-time existence and convergence of the flow on Bismut-flat manifolds and surfaces of non-negative Kodaira dimension.

Chapter 1

Introduction

1.1 Overview

Since the founding of Riemannian geometry in 1854, the question of canonical geometric structures has been of central importance. Questions of "canonical coordinates", "canonical mappings", and "canonical metrics" are ubiquitous in the literature. In the following dissertation, we will focus specifically on canonical metrics.

Riemannian geometry was born from the study of surfaces embedded in $\mathbb{R}^3$. Even at this early point, the question of canonical metrics was raised. Gauss noticed that the embeddings of a surface (and thus metrics induced by embeddings) are highly rigid with respect to Gauss curvature. As his Theorema Egregium puts it (in modern language), the Gauss curvature of the induced metric is diffeomorphism-invariant. This makes the question of distinguishing between two embedded surfaces into a question on metrics with specific curvature properties. In particular, one could ask "which surfaces admit metrics of constant Gauss curvature?"

It was known to Gauss that the only spaces having constant curvature metrics were quotients of the sphere, the plane, and the hyperbolic plane. However, in 1907, Poincaré and Koebe were able to prove an astounding converse.

Theorem 1.1. *Every connected Riemann surface* (Σ^2, g) *admits a complete metric of constant curvature.*

Proving that, in fact, every connected Riemann surface is diffeomorphic to such a quotient. More in the context of the problem as we originally posed it, a constant scalar curvature metrics corresponds to a distinguished embedding within a fixed diffeo-type. There are different methods for proving this theorem, but my favorite involves using a flow equation to construct the metric in question, thereby developing a tool with which to distinguish the diffeo-types of embedded surfaces.

The Poincaré-Koebe theorem and the flow approach to its proof suggest that curvature conditions are a natural way to approach the question of canonical metrics in dimensions larger than two and that one might be able to use curvature flows to construct these metrics. However, the rigidity of constant sectional curvature (the higher-dimensional analogue of Gauss curvature) proves to be too restrictive in higher-dimensional settings. Some spaces – as simple as products of spheres – cannot be covered by a na ive higher-dimensional extension of the Poincaré-Koebe theorem, since they are not even homeomorphic to a space of constant sectional curvature. One is then forced to consider weaker curvature conditions.

If (M^n, g) is a Riemannian manifold and Π_p is a plane in T_pM, then we can denote the sectional curvature of the plane by $K(\Pi_p)$. Notice that a sectional curvature constraint is very strong; it constrains values on all planes at all points. Two tensorial quantities derived from K admitting weaker constraints are the *Ricci curvature* (denoted Ric) and the *scalar* *curvature.*

$$v_p \in \mathcal{S}_p^{n-1}M, \quad \operatorname{Ric}(v_p, v_p) = \sum_{\Pi_p \ni v_p} K(\Pi_p), \quad \operatorname{Scal} = \sum_{\Pi_p} K(\Pi_p) = \operatorname{tr}_g \operatorname{Ric}$$

As it happens, constant scalar curvature is too loose a condition. A result of Kazdan and Warner shows that every smooth manifold of dimension at least three admits negative scalar curvature metrics [24], so it would seem that this condition cannot easily be used to tell the difference between manifolds. This seems to suggest that constant Ricci curvature sits in just the right gap to be more rigid than constant scalar but less rigid than constant sectional, making it a promising candidate for a higher-dimensional Uniformization Theorem. In terms of equations, constant Ricci is also known as the *Einstein equation.*

$$\operatorname{Ric} = \lambda g \tag{1.1}$$

Formulated in terms of flows, we obtain the Ricci flow.

$$\frac{\partial}{\partial t} g = -2 \operatorname{Ric} \tag{1.2}$$

The hope is to use Ricci flow – like the flow in the proof of Poincaré-Koebe – to deform the initial geometry into one of our uniformizers – in this case, Einstein manifolds.

These equations are very difficult to study in general; they are degenerate elliptic / parabolic, quasilinear systems of equations without an obvious convexity. Such equations do not typically have good regularity properties (c.f. [5]). The standard method of dealing with such situations in geometry is to condition the initial data in a way that is respected by the equation im the hopes of introducing some kind of reduction or stronger regularity property. In this case, as the Einstein and Ricci flow equations respect the Kähler condition on Hermitian manifolds, one can restrict the equations to Kähler-Einstein and Kähler-Ricci flow. Kähler manifolds have a wealth of structure that generic smooth manifolds lack, including cohomology classes defined by metrics and Ricci forms, Chern classes, and holomorphic structures.

In his 1978 paper [46], Yau was able to use this structure to show the existence of Kähler-Einstein metrics on manifolds with $c_1 \leq 0$. Cao gave a related proof for the Kähler-Ricci flow; proving long-time existence and exponential convergence to Kähler-Einstein for $c_1 \leq 0$ [4]. However, the resulting theorem, the Calabi-Yau theorem, finds a comfortable home not only in differential geometry and partial differential equations, but also mathematical physics. In superstring theory, one understands spacetime as a product $\mathbb{R}^4 \times M^6$ where M^6 is a 6-dimensional, compact, Hermitian manifold satisfying the Hull-Strominger system [21, 41]. Calabi-Yau 6-manifolds are an especially nice class of these string compactifications, but the Hull-Strominger system also suggests the existence of internal dimensions beyond the Kähler setting – providing natural hypotheses for generalizations of the Calabi-Yau condition.

The first difficulty that one confronts in trying to construct these new canonical geometries and extend the Calabi-Yau theorem to a broader class of manifolds is that there is a lack of cohomological data associated to metrics and Ricci foms on non-Kähler Hermitian manifolds. To recover these notions requires us to institute some kind of integrability condition on the metric which will be preserved by the flow. A theorem of Gauduchon [11] (see also Theorem 2.6) suggests the condition

$$\sqrt{-1}\partial\overline{\partial}\omega = 0. \tag{1.3}$$

Every compact, complex surface admits such a metric and we get cohomological data from the metric in the form of $[\omega] \in H^{1,1}_{Aep}$ and $[\partial\omega] \in H^{2,1}_{\overline{\partial}}$.

A second difficulty is that there is no longer an obvious choice of connection. The Levi-Civita connection is Hermitian if and only if the metric is Kähler, so the Levi-Civita connection will never be a good choice for a non-Kähler metric. This forces us view as natural an often overlooked class of connections, those with torsion! Once we do this, there are infinitely many Hermitian connections on a non-Kähler manifold [13]! One of these connections – independently discovered in the contexts of string theory [41] and of the topology of complex manifolds [3] — is called the *Bismut connection.* It is the unique Hermitian connection D^+ with totally antisymmetric torsion (see Lemma 2.8) and can be defined in terms of the Levi-Civita connection ∇^{LC} via

$$D^+ = \nabla^{LC} - \frac{1}{2} g^{-1} d^c \omega. \tag{1.4}$$

It is this connection and its metric-compatible non-Hermitian counterpart

$$D^- = \nabla^{LC} + \frac{1}{2} g^{-1} d^c \omega \tag{1.5}$$

which will play fundamental roles in defining the Bismut Hermitian-Einstein equation and pluriclosed flow (see §2.2.2), this dissertation's primary focus for candidate non-Kähler extensions of the Kähler-Einstein equation and Kähler-Ricci flow.

A third difficulty is that, whereas the Kähler-Einstein equation and Kähler-Ricci flow can be reduced to a complex Monge-Ampére equation (a fact which was crucial to Yau's 1978 work), the Bismut Hermitian-Einstein equation cannot, in general, be reduced to a scalar equation. There is a reduction in the setting of generalized Kähler manifolds of commuting type [10, Proposition 9.40], but in this situations the reduction is non-concave. Many of the most important results in the elliptic theory are known to fail in such situations (see [29, 5]).

For these reasons, the following questions are of general interest and this dissertation will attempt to present some of the author's results as early steps towards an answer.

Problem 1.1.1. *Under what topological conditions does a complex manifold admit a solution to the Bismut Hermitian-Einstein equation?*

Problem 1.1.2. *Under what topological conditions does a solution to pluriclosed flow exist for all time and converge to a Bismut Hermitian-Einstein metric?*

These questions have been addressed by various authors. Jeffrey Streets has addressed some related apriori estimates on manifolds with globally generated bundles, generalized Kähler manifolds with split tangent bundle, non-positive Chern bisectional curvature, rank one tangent bundle with $c_1 < 0$, and torus fibrations with torus-invariant metric [33, 35, 40, 32, 31].

1.2 Statement of Results

We will prove several results over the course of this dissertation. These results represent some of the most comprehensive existence and convergence results for pluriclosed flow to-date. We begin by using generalized geometric tools to prove an identity dating back to Bismut which will allow us to construct the first counter-examples to a naive generalization of the Calabi-Yau theorem, i.e. vanishing first Chern class is not sufficient to guarantee existence of a Bismut Hermitian-Einstein metric.

Theorem 1.2. *In every dimension there exist infinitely many complex manifolds with vanishing first Chern class which do not admit a Bismut Hermitian-Einstein metric.*

These same tools will then be used to prove a localized higher-regularity theorem for pluriclosed flow using an argument similar to that of Yau [46], giving a fully geometric proof of an extension of a theorem due to Streets [32] (see also [40]).

Theorem 1.3. *Given (M^{2n}, J), fix (ω, β) a solution to pluriclosed flow (4.1) on $[0, \tau), \tau \leq 1$, with associated generalized metric G. Fix a background generalized metric $\widetilde{G}(\widetilde{g}, \widetilde{\beta})$ such that $\Lambda^{-1}\tilde{G} \leq G \leq \Lambda\tilde{G}$. There exists $R > 0$ depending on $\widetilde{g}$ such that for all $0 < r < R$, and*

$k \in \mathbb{N}$, *there exists a constant* $C = C(k, \Lambda, \widetilde{G})$ *such that*

$$\max_{B_{\frac{r}{2}}(p)\times\{t\}} \sum_{i=0}^{k} |\nabla^i \Upsilon|^{\frac{2}{i+1}} \leq C(r^{-4} + \frac{1}{t}).$$

This higher-regularity theorem reduces the question of long-time existence to an estimate on the metric and β-field, which can be made by way of Bochner formulas provided the background curvature tensor is simple. Two cases are especially clear, that of Bismut-flat and non-negative Kodaira dimension.

Theorem 1.4. *Let* (M^{2n}, ω_F, J) *be a compact Bismut-flat manifold. Given* ω_0 *a pluriclosed metric so that* $[\partial\omega_0] = [\partial\omega_F] \in H^{2,1}_{\overline{\partial}}$, *the solution to pluriclosed flow with initial data* ω_0 *exists on* $[0, \infty)$ *and converges to a Bismut-flat metric* ω_∞.

A corollary in the low-dimensional setting is existence and convergence on diagonal Hopf surfaces.

Corollary 1.5. *Given any initial data on a standard Hopf surface, the pluriclosed flow exists for all time and converges to a positive multiple of the Hopf/Boothby metric.*

By a theorem of Gauduchon and Ivanov [14], this resolves the question of Bismut Hermitian-Einstein metrics for compact, complex surfaces. This is also one of the first instances of a non-Kähler metric as an attractor for a for a geometric flow with arbitrary initial data.

The situation for non-negative Kodaira dimension is interesting as these manifolds are precisely of the type used for Theorem 1.2 and admit no Bismut Hermitian-Einstein metric.

Theorem 1.6. *Let* (M^4, J) *be a minimal non-Kähler surface of Kodaira dimension* $\kappa \geq 0$. *Given* ω_0 *a pluriclosed metric on* M, *the solution to pluriclosed flow with initial condition* ω_0 *exists on* $[0, \infty)$.

These will doubtless prove to be useful for studying the kinds of collapsing behavior associated to pluriclosed flow.

Chapter 2

Background & Notation

In this chapter, we introduce some background: Schauder theory, Hermitian geometry, and generalized geometry. In §2.1, we will present the Schauder interior regularity theory, which plays a central role in apriori estimates for the nonlinear elliptic and parabolic PDE in which this dissertation deals. The author will present a proof of X.J. Wang [44] that extends the well-known blow-up proof of L. Simon to Dini continuous functions.

We will then discuss some relevant geometric background in §2.2. As this dissertation is primarily concerned with non-Kähler geometry, it will be important to discuss integrability conditions, connections, and curvature. We will spend some time here to emphasize the disjuncture between the complex and Riemannian geometry that arises in this setting.

Finally, we will introduce the basic material of generalized complex geometry in §2.3. This material is central to both the statement and proof of all of the major results in this paper. Later in this paper (see §3.2), we will see that generalized complex geometry encodes the data of a pluriclosed structure on the complexified tangent bundle very elegantly, tying the discussion here to §2.2.

2.1 PDE Theory

2.1.1 Elliptic Regularity

An important role in the following work is played by the elliptic regularity theory. In particular, the Schauder theory is indispensable. This is a family of results establishing a priori estimates for smooth, uniformly elliptic operators by perturbative means. The theory hinges on proving an estimate relating the Laplacian to the Hessian of a function. We will follow work of Xu Jia Wang for the proof [44]. We will first require a minor lemma known as the Cauchy estimates. The following notation will be useful $B_r(x_0) = \{x \in \mathbb{R}^n \,|\, |x - x_0| < r\}$, $B_r^+(x_0) = B_r(x_0) \cap \{x_n > 0\}$.

Lemma 2.1 ([15, Theorem 3.9]). *Suppose $u \in C^\infty(\overline{B}_1)$ and $\Delta u = 0$ in B_1. Then*

$$|D^k u(0)| \leq C_{n,k} \sup_{B_1} |u|.$$

Proof. Consider the functions $\phi, \psi : \overline{B}_1^+ \to \mathbb{R}$ defined as follows.

$$\begin{aligned}\phi(x', x_n) &= \frac{1}{2}[u(x', x_n) - u(x', -x_n)] \\ \psi(x', x_n) &= \|u\|_{L^\infty}[|x'|^2 + x_n(n - (n-1)x_n)]\end{aligned}$$

The functions have several useful properties.

- They are harmonic in B_1^+.

$$\Delta\phi \equiv 0, \quad \Delta\psi \equiv 0$$

- They have convenient data along the flat boundary.

$$\phi|_{\{x_n=0\}} \equiv 0, \quad \psi|_{\{x_n=0\}} \equiv 0$$

- They are well-behaved along the curved boundary.

$$|\phi|_{\partial B_1^+\setminus\{x_n=0\}}| \leq \|u\|_{L^\infty}, \quad \psi|_{\partial B_1^+\setminus\{x_n=0\}} \geq \|u\|_{L^\infty}$$

But this means that $\psi \pm \phi$ is harmonic in B_1^+ and non-negative on the boundary ∂B_1^+. So by the maximum principle, $\psi \pm \phi \geq 0$ in $\overline{B}_1^+$. Letting $x' = 0$ and $x_n \searrow 0$, we find

$$|D_n u(0)| \leq n\|u\|_{L^\infty}.$$

The same argument works for the other directions and the higher order estimates follow by differentiating the equation. □

We will prove the Schauder estimate using Lemma 2.1 by solving approximating problems on smaller and smaller scales. The continuity hypotheses then allow us to conclude convergence with a certain amount of rate-control.

Theorem 2.2 ([44, Theorem 1]). *Suppose $u \in C^2$ and*

$$a^{ij}(x)u_{ij}(x) = f(x) \tag{2.1}$$

in $B_1(0)$ and f is Dini continuous, then there exists a universal constant $C = C(n, \Lambda)$ such that for any $x, y \in B_{\frac{1}{2}}(0)$

$$\begin{aligned} |D^2u(x) - D^2u(y)| \leq C_n &\left[d \sup_{B_1} |u| + (\int_0^d \frac{\omega_f(r)}{r} dr + d\int_d^1 \frac{\omega_f(r)}{r^2} dr) \right] \\ &+ C_n \sup_{B_1} |\partial_x^2 u| \left[\int_0^d \frac{\omega_a(r)}{r} dr + d\int_d^1 \frac{\omega_a(r)}{r^2} dr \right] \end{aligned}$$

where $d = |x - y|$.

Proof. For the sake of simplicity, we prove the result for $a^{ij} = \delta^{ij}$. Without loss of generality, we can suppose that $f(0) = 0$. We define a decreasing family of balls, $B_k = B_{2^{-k}}(0)$ for $k \in \mathbb{N}$. On each ball, we can define $u_k : B_k \to \mathbb{R}$ solving approximating problems on smaller and smaller balls, i.e.

$$\begin{cases} \Delta u_k(x) = 0, & x \in B_k \\ u_k(x) = u(x), & x \in \partial B_k \end{cases}$$

The difference $v_k = u - u_k$ then solves a Poisson equation on B_k with vanishing boundary data.

$$\begin{cases} \Delta v_k = f \\ v_k|_{\partial B_k} = 0 \end{cases}$$

By the Maximum Principle for Poisson equations, we know

$$\|u - u_k\|_{L^\infty(B_k)} \leq C4^{-k}(\|f\|_{L^\infty(B_k)}) \leq C4^{-k}\,\omega_f(2^{-k}). \tag{2.2}$$

This allows us to estimate $u_k - u_{k+1}$ in B_{k+1}.

$$\begin{aligned} \|u_k - u_{k+1}\|_{L^\infty(B_{k+1})} &\leq \|u_k - u\|_{L^\infty(B_k)} + \|u - u_{k+1}\|_{L^\infty(B_{k+1})} \\ &\leq C4^{-k}\,\omega_f(2^{-k}) \end{aligned}$$

Further, since $u_k - u_{k+1}$ is harmonic in B_{k+1}, we can apply the Cauchy estimates and Equation 2.2 to get further estimates.

$$\begin{aligned} \|Du_k - Du_{k+1}\|_{L^\infty(B_{k+2})} &\leq C2^k\|u_k - u_{k+1}\|_{L^\infty(B_{k+1})} && (2.3) \\ &\leq C2^{-k}\,\omega_f(2^{-k}) && (2.4) \\ \|D^2u_k - D^2u_{k+1}\|_{L^\infty(B_{k+2})} &\leq C4^k\|u_k - u_{k+1}\|_{L^\infty(B_{k+1})} && (2.5) \\ &\leq C\,\omega_f(2^{-k}) && (2.6) \end{aligned}$$

Now, we define the 2nd order approximation of u about 0

$$Q(x) = u(0) + u_i(0)x^i + \frac{1}{2}u_{ij}(0)x^i x^j.$$

This function has several useful properties.

- $u_k - Q$ is harmonic for all k
- By definition of second derivative, $4^k\|u - Q\|_{L^\infty(B_k)} = o(1)$ as $k \to \infty$.
- Letting $v_k = u_k - Q$, we can take a limit.

$$4^k\|v_k\|_{L^\infty(B_k)} \le 4^k\|u - u_k\|_{L^\infty(B_k)} + o(1) \le C\,\omega_f(2^{-k}) + o(1) \to 0$$

By harmonicity, we can apply the Cauchy estimates to $v_k = u_k - Q$ find the following.

$$\begin{aligned}|Du_k(0) - Du(0)| &= |Dv_k(0)| \le C2^k \sup_{B_k}|v_k| \to 0\\ |D^2u_k(0) - D^2u(0)| &= |D^2v_k(0)| \le C4^k \sup_{B_k}|v_k| \to 0\end{aligned}$$

This implies that $Du_k(0) \to Du(0)$ and $D^2u_k(0) \to D^2u(0)$ as $k \to \infty$.

We begin to estimate the modulus of continuity of the second derivatives by fixing $z \in B_{\frac{1}{16}}(0)$ and $k \ge 1$ s.t. $z \in \overline{B}_{k+3} \setminus B_{k+4}$. Then

$$|D^2u(z) - D^2u(0)| \le |D^2u(z) - D^2u_k(z)| + |D^2u_k(z) - D^2u_k(0)| + |D^2u_k(0) - D^2u(0)| =: I_3 + I_1 + I_2.$$

To estimate I_1, consider the functions $h_j = u_j - u_{j-1}$ (harmonic in B_j). We can use the Cauchy Estimates to estimate the second derivatives of the h_j's as follows (where $j \le k$ i.e.

$B_j \supset B_k$)

$$\begin{aligned}
|D^2 h_j(z) - D^2 h_j(0)| &\le \|D^3 h_j\|_{L^\infty(B_{j+3})}|z| \\
&\le C 8^{j+3}|z| \sup_{B_{j+2}} |h_j| \\
&\le C 8^j |z| \sup_{B_j} |u_j - u_{j-1}| \\
&\le C 2^j |z|\, \omega_f(2^{-j})
\end{aligned}$$

We can then exploit these functions for the analysis of I_1.

$$\begin{aligned}
I_1 &\le |D^2 u_k(z) - D^2 u_k(0)| \\
&\le |D^2 u_{k-1}(z) - D^2 u_{k-1}(0)| + |D^2 h_k(z) - D^2 h_k(0)| \\
&\le |D^2 u_0(z) - D^2 u_0(0)| + \sum_{j=1}^{k} |D^2 h_j(z) - D^2 h_j(0)| \\
&\le |D^2 u_0(z) - D^2 u_0(0)| + C|z| \sum_{j=1}^{k} 2^j\, \omega_f(2^{-j}) \\
&\le C|z| (\|u_0\|_{L^\infty(B_1)} + \sum_{j=1}^{k} \frac{\omega_f(2^{-j})}{2^{-2j}} 2^{-j}) \\
&\le C|z| \left(\|u\|_{L^\infty(B_1)} + \int_{|z|}^{1} \frac{\omega_f(r)}{r^2} dr \right)
\end{aligned}$$

For I_2, we refine our estimate a bit.

$$\begin{aligned}
I_2 &= |D^2 u_k(0) - D^2 u(0)| \\
&\le \sum_{j=k}^{\infty} \|D^2 u_j - D^2 u_{j+1}\|_{L^\infty(B_{j+1})} \\
&\le C \sum_{j=k}^{\infty} \omega_f(2^{-j}) \\
&\le C \int_0^{|z|} \frac{\omega(r)}{r} dr
\end{aligned}$$

The same argument holds for I_3 with a translation, so we will not include it.

This then concludes the proof. $\square$

This gives the usual Schauder estimate as a corollary in the following manner.

Corollary 2.3. *Suppose further that u solves Equation 2.1 where a^{ij} is Λ-uniformly elliptic for some $f \in C^\alpha$ then*

$$\|u\|_{C^{2,\alpha}(B_{\frac{1}{4}})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)}\right)$$

Proof. If $f \in C^\alpha(B_1)$ then $\omega_f(r) \leq [f]_\alpha r^\alpha$ for some $\alpha \in (0,1)$ by definition. So we can evaluate the integrals on the right-hand side of the estimate for a function $\omega(r) = Cr^\alpha$.

$$\begin{aligned} C\int_0^d \frac{r^\alpha}{r}dr + Cd\int_d^1 \frac{r^\alpha}{r^2}dr &= C\frac{1}{\alpha}d^\alpha + C\frac{d^\alpha - d}{1-\alpha} \\ &= d^\alpha\left(\frac{C}{\alpha} + C\frac{1-d^{1-\alpha}}{1-\alpha}\right) \\ &\leq d^\alpha \frac{C}{\alpha(1-\alpha)} \end{aligned}$$

Therefore, in this setting, Theorem 2.2 becomes

$$|D^2u(x) - D^2u(y)| \leq C\left[d\|u\|_{L^\infty} + \frac{[f]_\alpha}{\alpha(1-\alpha)}d^\alpha\right].$$

Requiring $x, y \in B_{\frac{1}{4}}(0)$, suitably rescaling the estimate, and rewriting gives

$$[D^2u]_{\alpha,B_{\frac{1}{4}}} \leq C\left(\|u\|_{L^\infty(B_{\frac{1}{2}})} + [f]_{C^\alpha(B_{\frac{1}{2}})}\right). \tag{2.7}$$

However, we have the following interpolation inequality for any $\epsilon > 0$ on closed, bounded domains.

$$\|u\|_{C^2} \le \epsilon[D^2u]_{C^\alpha} + C_\epsilon\|u\|_{L^\infty}$$

Thus implying,

$$\begin{aligned}
\|u\|_{C^{2,\alpha}(\overline{B}_{\frac{1}{4}})} &\le \|u\|_{C^{2,\alpha}(\overline{B}_{\frac{1}{4}})} = \|u\|_{C^2(\overline{B}_{\frac{1}{4}})} + [D^2u]_{C^\alpha(\overline{B}_{\frac{1}{4}})} \\
&\le C(\|u\|_{L^\infty(\overline{B}_{\frac{1}{4}})} + 2[D^2u]_{C^\alpha(\overline{B}_{\frac{1}{4}})}) \\
&\le C(\|u\|_{L^\infty(\overline{B}_1)} + \|f\|_{C^\alpha(B_1)})
\end{aligned}$$

But this is the estimate that was sought. □

2.1.2 Parabolic Regularity

It is often useful to consider parabolic problems to break the symmetry of geometric problems on compact manifolds. We present here important theorems for the sequel and omit their proof because they are either similar to work that has already been presented or too far afield for this dissertation.

Theorem 2.4 ([26, c.f. Theorem 5.6])**.** *Given the Λ-uniformly elliptic Cauchy-Dirichlet problem*

$$\begin{cases} (\frac{\partial}{\partial t} - a^{ij}(x,t)\partial_i\partial_j)u = f(x,t) \in C^\alpha(B_1) \\ u(x,0) = g(x) \in C^\alpha(B_1) \\ u|_{\partial B_1}(x,t) = h(x,t) \in C^\alpha(B_1) \end{cases},$$

there is an $\epsilon = \epsilon(n, \alpha, \Lambda, g, h) > 0$ s.t. the problem admits a unique solution in $u \in C^{2+\alpha,1+\alpha}_{x,t}(B_1 \times [0, \epsilon))$.

The elliptic Schauder estimates have an analogue in the parabolic setting. Let $Q_r = B_r(0) \times (-r^2, 0]$. In particular,

Theorem 2.5 ([44, Theorem 2]). *Let $u \in C^{2,1}_{x,t}$ solving*

$$(\frac{\partial}{\partial t} - a^{ij}(x,t)\partial_i\partial_j)u = f(x,t).$$

If a^{ij} and f are Dini continuous, then for any points $p_1 = (x_1, t_1), p_2 = (x_2, t_2) \in Q_{\frac{1}{2}}$,

$$\begin{aligned} |D_x^2 u(p_1) - D_x^2 u(p_1)| \le C_n &\left[d \sup_{Q_1} |u| + \int_0^d \frac{\omega_f(r)}{r} + d \int_d^1 \frac{\omega_f(r)}{r^2} \right] \\ &+ C_n \sup_{Q_1} |D_x^2 u| \left[\int_0^d \frac{\omega_a(r)}{r} + d \int_d^1 \frac{\omega_1(r)}{r^2} \right], \end{aligned}$$

where $d = |x_1 - x_2| + \sqrt{|t_1 - t_2|}$ and $\omega_\bullet$ is the modulous of continuity of $\bullet$.

Proof. The proof is similar to Theorem 2.2 but substituting parabolic balls for regular balls and getting the estimate on $\partial_t u$ from the equation. □

2.2 Hermitian Geometry

As we are interested in addressing a non-Kähler generalization of the Calabi-Yau theorem, it will be worth highlighting some important places in which the general theory of Hermitian manifolds differs from that of Kähler manifolds. We will first embark on a discussion of metric integrability conditions before proceeding on to connections and curvature.

2.2.1 Integrability Conditions

On a Hermitian manifold (M^{2n}, J, g), we have several natural conditions on the fundamental two-form, $\omega = gJ$. The most studied of these conditions is *Kähler*.

Definition 2.1. A fundamental 2-form (or sometimes the corresponding Riemannian metric) as above is called *Kähler* if

$$d\omega = 0. \tag{2.8}$$

This condition is natural for several reasons. First, this is precisely the condition for the Riemannian-geometric and complex-geometric data to coincide in the sense that the unique Hermitian connection is the Levi-Civita connection. Second, every compact Riemann surface admits such a metric.

While these metrics are incredibly important, they are not completely generic. A standard example of a non-Kähler manifold is provided by Hopf surfaces – complex manifolds diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$ which admit no Kähler metric. These manifolds do however admit a metric having a weaker integrability condition – a *pluriclosed* metric.

Definition 2.2. A fundamental 2-form (or its corresponding metric) as above is called *pluriclosed* if

$$\sqrt{-1}\partial\overline{\partial}\omega = 0. \tag{2.9}$$

In fact, every compact, complex surface admits a metric of this type. This goes back to a theorem of Gauduchon. This theorem asserts the existence of certain kinds of integrable metrics called *Gauduchon metrics*.

Definition 2.3. On a complex manifold (M^{2n}, J), a fundamental two-form (or its associated Hermitian metric) is called *Gauduchon* if

$$\sqrt{-1}\partial\overline{\partial}\omega^{n-1} = 0.$$

Notice that when $n = 2$, Gauduchon is equivalent to pluriclosed, and so the following theorem is sufficient.

Theorem 2.6. *[12, 28] Let (M^{2n}, J, g) be a connected, compact, Hermitian manifold. Then there exists $\tilde{g}$ in the conformal class of g s.t.*

$$\sqrt{-1}\partial\overline{\partial}\tilde{\omega}^{n-1} = 0.$$

This metric is unique up to positive scalar multiples.

Proof. As we want $\tilde{g}$ in the conformal class of g, it suffices to find $u > 0$ and smooth s.t. $\tilde{g} = u^{\frac{1}{n-1}} g$ where $\tilde{g}$ is Gauduchon. We define a Lefschetz-type operator $L_g^{n-1}(u) = u\omega^{n-1}$ and a linear elliptic operator on functions

$$T : \Lambda^0 \to \Lambda^0, \quad u \mapsto \sqrt{-1} *_g \partial\overline{\partial}(L_g^{n-1}u).$$

This reduces the problem to finding a smooth, positive solution of $Tu = 0$ on M.

We will make use of the Fredholm theorem. To do that, we will need to compute T^*.

$$T^*u = \sqrt{-1}L^*\overline{\partial}^*\partial^* * u = \sqrt{-1}L^* * \overline{\partial} * * \partial * * u = \sqrt{-1}L^* * \overline{\partial}\partial u$$

Now we need to compute the formal adjoint of L, denoted L^*, acting on $\mathrm{Im}(*|_{\Lambda^{1,1}})$. For this, let $\eta \in \Lambda^{1,1}$ and $v \in \Lambda^0$. Then

$$\begin{aligned}
\int_M vL^*(*\eta)dv_g &= \int_M \langle *\eta, Lv\rangle dv_g \\
&= \int_M (*\eta) \wedge *(v\omega^{n-1}) \\
&= c\int_M v(*\eta) \wedge \omega \\
&= c\int_M v(*(*\eta \wedge \omega))dv_g \\
&= c\int_M v(\Lambda\eta)dv_g
\end{aligned}$$

This shows that

$$T^* = -\sqrt{-1}\Lambda\partial\overline{\partial} = -\Delta_\omega.$$

Thus we have $\ker T = \operatorname{coker} T^*$ by the Fredholm alternative. But since $T = \Delta_\omega + l.o.t.s$ as well, we have that $\operatorname{ind} T = \operatorname{ind} T^* = 0$, so $\dim \operatorname{coker} T^* = \dim \ker T^* = 1$. Therefore, $\dim \ker T = 1$ and we can choose u, a generator.

By maximum principle, no function in $\mathrm{Im}(T^*)$ has constant sign other than the zero function, so we may choose u s.t. $(u, 1)_{L^2} > 0$ as $\mathrm{Im}(T^*)^\perp = \ker T$. It is possible to claim further that $u \geq 0$. Supposing otherwise, we can choose two functions $\phi, \psi \in C_0^\infty(M)$ s.t. $\operatorname{supp}\phi \subset \operatorname{supp} u_+$ and $\operatorname{supp}\psi \subset u_-$. These functions have supports with disjoint interiors. Then, we define the function

$$\Phi = \left(\int_M u_-\psi dv_g\right)\phi + \left(\int_M u_+\phi dv_g\right)\psi \in C^\infty(M) \cap L^2(M).$$

But this function is orthogonal to u.

$$\int_M u\Phi dv_g = \int_M u_-\psi dv_g \int_M u_+\phi + \int_M u_+\phi dv_g \int_M (-u_-)\phi = 0$$

However, as

$$\ker(T)^{\perp} = (\mathrm{Im}(T^*)^{\perp})^{\perp} = \overline{\mathrm{Im}(\Delta)}^{L^2},$$

we find that $\Phi \in \mathrm{Im}(\Delta)$, which is a contradiction as $\Phi \geq 0$.

Finally, we must show that u never vanishes. However, this follows immediately from the strong maximum principle and $u \geq 0$. As the operator is of the form,

$$\Delta u + \langle b, \nabla u \rangle + uf = 0$$

we will have that if $u(p_0) = 0$ – its minimum – then

$$\Delta u(p_0) \leq 0$$

which is a contradiction if $u \not\equiv 0$. And the theorem is proved. □

2.2.2 Connections & Curvature

Chern Connection on a Hermitian Bundle

On any holomorphic, Hermitian vector bundle $(\mathcal{E}, g) \to (M, J)$ there is a distinguished connection – called the *Chern connection* – that we will denote by ∇ which extends the $\overline{\partial}$-operator on $\mathcal{E}$ and we will typically denote the matrix of connection one-forms by θ.

Lemma 2.7. *There is a unique Hermitian connection on $\mathcal{E}$ with the property that $\pi^{0,1}\nabla = \overline{\partial}_E$.*

Proof. Pick a local trivialization (e_i) by holomorphic sections. As ∇ extends $\overline{\partial}$, we must have that $\theta_i^j \in \Lambda^{1,0}$. Further, by metric compatibility, we can compute as follows.

$$dg(e_i, \overline{e}_j) = \theta_i^l g_{l\bar{j}} + \overline{\theta}_{\bar{j}}^{\bar{l}} g_{i\bar{l}}$$

Decomposing by type, we find the following as an equation in $\Lambda^{1,0}(M)$.

$$\partial g_{i\bar{j}} = \theta_i^l g_{l\bar{j}}$$

Thus, $\theta = \partial g \cdot g^{-1} \in \Lambda^{1,0}(\,\mathrm{End}(\mathcal{E}))$ completely determines the connection one-forms.

Existence follows by starting at these connection one-forms and showing that the define a Hermitian connection. This is obvious from the construction. $\square$

This connection has a curvature tensor which we will denote $\Omega \in \Lambda^{1,1}(\mathrm{End}(\mathcal{E})$ (or Ω^h when the metric is not otherwise implied). We call this the Chern curvature. Notice that such a tensor has two well-defined trace operators.

$$\begin{aligned} \Lambda_h = \mathrm{tr}_{\omega_h} : \Lambda^{1,1}(\mathrm{End}(\mathcal{E})) &\to \Lambda^0(\mathrm{End}(\mathcal{E})) \\ \mathrm{Tr} : \Lambda^{1,1}(\mathrm{End}(\mathcal{E})) &\to \Lambda^{1,1}(M) \end{aligned}$$

This allows us to define two (generally) different curvature quantities $\rho^C = \mathrm{Tr}\,\Omega$ and $S^C = \sqrt{-1}\Lambda_h\Omega$– the first and second Chern-Ricci curvatures.

Bismut and Hull Connections on Complexified Tangent Bundle

We will also often be considering sections of the complexified tangent bundle $T_{\mathbb{C}}M$. The natural connection in this setting was discovered by JM Bismut in investigations into index theory on complex manifolds, during which he needed to introduce an exotic holomorphic structure on the complexified tangent bundle [3].

Lemma 2.8. *(M, g, J) Hermitian. There is a unique Hermitian connection on $T_{\mathbb{C}}M$ with totally anti-symmetric torsion. We will call this the* Bismut connection *and denote it by D^+.*

Proof. Notice that we can define a tensor $H = D^+ - \nabla^{LC} \in \Omega^1(\mathrm{End}(T_{\mathbb{C}}M))$. We begin by compiling a few facts.

1. $T^+ \in \Omega^2(T_{\mathbb{C}}M)$ is related to H in the following way, as a consequence of the symmetry of the Levi-Civita connection.

$$T^+(X, Y) = H(X, Y) - H(Y, X)$$

2. If we let $H(X, Y, Z) = g(H(X, Y), Z)$, then metric-compatibility of D^+ and ∇^{LC} implies

$$H(X, Y, Z) = -H(X, Z, Y)$$

3. As D^+ is Hermitian and ∇^{LC} is metric-compatible (and not Hermitian, in general) we can prove

$$H(X, JY, Z) + H(X, Y, JZ) = -g((\nabla_X^{LC} J), X), \quad T^+(X, JY, Z) = -T^+(X, Y, JZ).$$

Using the above identities and clever grouping, one is able to prove

$$[H(X,JY,Z)+H(X,Y,JZ)] \ +[H(Z,JX,Y)+H(Z,X,JY)]+[H(Y,JZ,X)+H(Y,Z,JX)]$$
$$=T^{+}(Y,Z,JX)-T^{+}(Z,JX,Y)+T^{+}(Y,JX,Z).$$

But $T^{+}\in\Lambda^{3}$ by hypothesis, so the formula reduces to

$$[H(X,JY,Z)+H(X,Y,JZ)]+[H(Z,JX,Y)+H(Z,X,JY)]+ [H(Y,JZ,X)+H(Y,Z,JX)]$$
$$= \ -T^{+}(JX,Y,Z).$$

However, we can apply this identity to JX instead of X, from which we obtain

$$\begin{aligned} T^{+}(X,Y,Z) &= [H(JX,JY,Z)+H(JX,Y,JZ)]+[H(Z,JJX,Y)+H(Z,JX,JY)] \\ &\quad +[H(Y,JZ,JX)+H(Y,Z,JJX)] \\ &= \ -g((\nabla^{LC}_{JX}J)Y,Z)-g((\nabla^{LC}_{Z}J)JX,Y)-g((\nabla^{LC}_{Y}J)Z,JX). \end{aligned}$$

To simplify further, one must recall the following very useful identity.

$$d\omega(X,Y,Z)=g((\nabla^{LC}_{X}J)Y,Z)-g((\nabla^{LC}_{Y}J)X,Z)+g((\nabla^{LC}_{Z}J)X,Y)$$

Combining these gives the torsion uniquely in terms of the metric and complex structure.

$$T^{+}(X,Y,Z)=-d\omega(JX,Y,Z)$$

It only remains to show that $H(X,Y)=-H(Y,X)$ to prove the theorem. By polarization, it suffices to prove that $H(X,X,Z)=0$ for any $X,Z\in T_{\mathbb{C}}M$. We will use the following

Koszul formula for a connection D with torsion T.

$$
\begin{aligned}
2g(D_X Y, Z) = {} & Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\
& - g([X,Z],Y) - g([Y,Z],X) + g([X,Y],Z) \\
& - T(X,Z,Y) + T(Z,Y,X) - T(Y,X,Z)
\end{aligned}
$$

Notice that for ∇^{LC}, $T \equiv 0$ and for D^+, $T^+ \in \Omega^3(M)$. Therefore,

$$
\begin{aligned}
2g(D^+_X X, Z) = {} & Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\
& - g([X,Z],Y) - g([Y,Z],X) + g([X,Y],Z) \\
& = 2g(\nabla^{LC}_X X, Z).
\end{aligned}
$$

Or equivalently, $H(X,X,Z) = 0$ and $H \in \Omega^3$.

Therefore,

$$H(X,Y,Z) = \frac{1}{2}T^+(X,Y,Z) = \frac{1}{2}d\omega(JX,Y,Z).$$

We can thus define the following connection uniquely and the result is proved.

$$g(D^+_X Y, Z) = g(\nabla^{LC}_X Y, Z) - \frac{1}{2}d\omega(JX,Y,Z).$$

$\square$

This suggests the definition of a further connection, the *Hull connection*.

Definition 2.4. On a pluriclosed Hermitian manifold (M,J,g), we can define a metric-compatible, non-Hermitian connection for which

$$g(D^-_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2}d\omega(JX,Y,Z).$$

We prove that D^- is compatible and non-Hermitian below.

Lemma 2.9. *The following defines a metric-compatible connection on $T_{\mathbb{C}}M$ which is Hermitian iff it is Kähler.*

Proof.

$$\begin{aligned}(D_X^- g)(Y,Z) &= Xg(Y,Z) - g(D_X^- Y, Z) - g(Y, D_X^- Z) \\ &= -\frac{1}{2}d\omega(JX,Y,Z) - \frac{1}{2}d\omega(JX,Z,Y) \\ &= 0.\end{aligned}$$

To see that this connection is generally non-Hermitian, notice that

$$g((D_X^- - D_X^+)Y, Z) = d\omega(JX,Y,Z).$$

But we can further compute the action on J.

$$\begin{aligned}g((D_X^- J)Y, Z) &= g([(D_X^- - D_X^+)J]Y, Z) \\ &= g((D_X^- - D_X^+)[JY] - J(D_X^- - D_X^+)Y, Z) \\ &= d\omega(JX,JY,Z) + d\omega(JX,Y,JZ)\end{aligned}$$

This vanishes iff $d\omega = 0$. $\square$

We will denote the respective curvature tensors of $D^\pm$ by $R^\pm$. It is worth noting that since both are metric-compatible $R^\pm \in \Lambda^2 \times \Lambda^2$. Since D^+ is Hermitian, it has further symmetries, i.e., $R^+ \in \Lambda^2 \times \Lambda^{1,1}$. Then using the identity $R^+(X,Y,Z,W) = R^-(Z,W,X,Y)$ (see Lemma 2.10), we find that $R^- \in \Lambda^{1,1} \times \Lambda^2$, i.e. $(D^-)^{0,1}$ defines a Dolbeault operator on $T_{\mathbb{C}}M$. We will prove a slightly more general statement.

Lemma 2.10. *Let $H \in \Lambda^3$ with $dH = 0$ and $D^{\pm} = \nabla^{LC} \pm H$. Then,*

$$R^H(X, Y, Z, W) = R^{-H}(Z, W, X, Y).$$

Proof. We compute the following.

$$\begin{aligned}
R^H(X, Y, Z, W) &= g([D^+_X, D^+_Y]Z - D^+_{[X,Y]}Z, W) \\
&= g([\nabla^{LC}_X + i_X H, \nabla^{LC}_Y + i_Y H]Z - \nabla^{LC}_{[X,Y]}Z - H([X, Y], Z), W) \\
&= g(W, [\nabla^{LC}_X, \nabla^{LC}_Y]Z + [i_X H, \nabla^{LC}_Y]Z + [\nabla^{LC}_X, i_Y H]Z + [i_X H, i_Y H]Z \\
&\quad \nabla^{LC}_{[X,Y]}Z - H([X, Y], Z)) \\
&= R^{LC}(X, Y, Z, W) + g([i_X H, \nabla^{LC}_Y]Z + [\nabla^{LC}_X, i_Y H]Z + [i_X H, i_Y H]Z, W) \\
&\quad - H([X, Y], Z, W) \\
&= R^{LC}(X, Y, Z, W) + (\nabla^{LC}_X H)(Y, Z, W) - (\nabla^{LC}_Y H)(X, Z, W) \\
&\quad + g(H(Y, W), H(X, Z)) - g(H(X, W), H(Y, Z))
\end{aligned}$$

Notice then that by symmetries of R^{LC} and the quadratic terms, we find

$$\begin{aligned}
&R^H(X, Y, Z, W) - R^{-H}(Z, W, X, Y) = \\
&(\nabla^{LC}_X H)(Y, Z, W) - (\nabla^{LC}_Y H)(X, Z, W) - (\nabla^{LC}_Z H)(X, Y, W) + (\nabla^{LC}_W H)(X, Y, Z).
\end{aligned}$$

But it is easy to check that dH is simply the antisymmetric part of $\nabla^{LC} H$ (the result is obvious in normal coordinates). Thus,

$$R^H(X, Y, Z, W) - R^{-H}(Z, W, X, Y) = dH(X, Y, Z, W) = 0.$$

$\square$

It is also worthwhile to note that the Bismut curvature of the tangent bundle can be computed in terms of the Chern, offering us a bridge to a more familiar tensor. To start, we derive a basic relationship between the Chern and Bismut curvatures of an arbitrary Hermitian metric.

Lemma 2.11. *Given (M^{2n}, g, J) a Hermitian manifold, then in any local complex coordinate chart one has*

$$\overline{\partial} T_{ij\overline{k}\overline{l}} = -\Omega_{i\bar{k}j\bar{l}} + (R^+)_{j\bar{l}i\bar{k}} + T_{j\alpha\bar{k}} g^{\overline{\beta}\alpha} \overline{T}_{\overline{l}\overline{\beta}i},$$

where $T = -\sqrt{-1}\partial\omega$ is the torsion of the Chern connection.

Proof. Fix some local holomorphic coordinates, and let Γ denote the associated Chern connection coefficients. It follows from Lemma 2.8 that

$$D_i^+ \partial_j = \Gamma_{ji}^l \partial_l, \qquad D_{\bar{i}}^+ \partial_j = \overline{T}_{\bar{i}\bar{k}j} g^{\bar{k}l} \partial_l. \tag{2.10}$$

Using this, we derive and identity relating the $\Lambda^{1,1} \otimes \Lambda^{1,1}$ part of the Bismut curvature and the Chern curvature:

$$\begin{aligned}
(R^+)_{i\bar{j}k\bar{l}} &= g(D_i^+ D_{\bar{j}}^+ \partial_k - D_{\bar{j}}^+ D_i^+ \partial_k, \overline{\partial}_l) \\
&= \Omega_{k\bar{j}i\bar{l}} + \nabla_i \overline{T}_{\bar{j}\bar{l}k} + T_{ik}^p \overline{T}_{\bar{j}\bar{l}p} - T_{ip\bar{l}} g^{\bar{q}p} \overline{T}_{\bar{j}\bar{q}k} \\
&= \Omega_{k\bar{j}i\bar{l}} + \Omega_{i\bar{l}k\bar{j}} - \Omega_{i\bar{j}k\bar{l}} + T_{ik}^p \overline{T}_{\bar{j}\bar{l}p} - T_{ip\bar{l}} g^{\bar{q}p} \overline{T}_{\bar{j}\bar{q}k}.
\end{aligned}$$

Using this we furthermore obtain

$$
\begin{aligned}
\overline{\partial} T_{ij\overline{kl}} &= \overline{\partial}_k T_{ij\bar{l}} - \overline{\partial}_l T_{ij\bar{k}} \\
&= g_{j\bar{l},i\bar{k}} - g_{i\bar{l},j\bar{k}} - g_{j\bar{k},i\bar{l}} + g_{i\bar{k},j\bar{l}} \\
&= \ -\Omega_{i\bar{k}j\bar{l}} + \Omega_{j\bar{k}i\bar{l}} + \Omega_{i\bar{l}j\bar{k}} - \Omega_{j\bar{l}i\bar{k}} \\
&\quad\ + g^{\bar{q}p} g_{j\bar{q},i} g_{p\bar{l},\bar{k}} - g^{\bar{q}p} g_{i\bar{q},j} g_{p\bar{l},\bar{k}} - g^{\bar{q}p} g_{j\bar{q},i} g_{p\bar{k},\bar{l}} + g^{\bar{q}p} g_{i\bar{q},j} g_{p\bar{k},\bar{l}} \\
&= \ -\Omega_{i\bar{k}j\bar{l}} + \Omega_{j\bar{k}i\bar{l}} + \Omega_{i\bar{l}j\bar{k}} - \Omega_{j\bar{l}i\bar{k}} + \overline{T}^{\bar{q}}_{\overline{kl}} T_{ij\bar{q}} \\
&= \ -\Omega_{i\bar{k}j\bar{l}} + (R^+)_{j\bar{l}i\bar{k}} - T^{\alpha}_{ji} \overline{T}_{\overline{lk}\alpha} + T_{j\alpha\bar{k}} g^{\bar{\beta}\alpha} \overline{T}_{\overline{l\beta}i} + \overline{T}^{\bar{q}}_{\overline{kl}} T_{ij\bar{q}} \\
&= \ -\Omega_{i\bar{k}j\bar{l}} + (R^+)_{j\bar{l}i\bar{k}} + T_{j\alpha\bar{k}} g^{\bar{\beta}\alpha} \overline{T}_{\overline{l\beta}i},
\end{aligned}
$$

as required. □

Proposition 2.1. *Given (M^{2n}, g, J) pluriclosed, one has*

$$
(R^+)_{i\bar{j}k\bar{l}} = (\Omega)_{k\bar{l}i\bar{j}} - T_{ip\bar{l}} g^{\bar{q}p} \overline{T}_{\bar{j}\bar{q}k}.
$$

Proof. This follows from Lemma 2.11, using that $\overline{\partial} T = 0$ since g is pluriclosed. □

Next we compute the $\Lambda^{2,0} \otimes \Lambda^{1,1}$ component of the Bismut curvature.

Proposition 2.2. *Given (M^{2n}, g, J) pluriclosed, one has*

$$
(R^+)_{ijk\bar{l}} = \nabla^C_k T_{ij\bar{l}}.
$$

Proof. We first note that as a consequence of the Bianchi identities for a general Hermitian metric one has

$$
(R^+)_{ijk\bar{l}} = \nabla^C_i T_{kj\bar{l}} - \nabla^C_j T_{ki\bar{l}} + T^{\lambda}_{ij} T_{k\lambda\bar{l}} - T^{\lambda}_{ik} T_{j\lambda\bar{l}} + T^{\lambda}_{jk} T_{i\lambda\bar{l}}.
$$

Since we know that the Chern torsion is given by $T = -\sqrt{-1}\partial\omega$, we have the identity

$$\partial T_{ijk\bar{l}} = 0.$$

This can be rewritten in terms of the Chern connection as

$$\nabla^C_i T_{jk\bar{l}} - \nabla^C_j T_{ik\bar{l}} + \nabla^C_k T_{ij\bar{l}} + T^p_{ij} T_{pk\bar{l}} + T^p_{jk} T_{pi\bar{l}} - T^p_{ik} T_{pj\bar{l}} = 0.$$

Combining this with the first equation of the proof then yields the claim. □

2.3 Generalized Geometry

Generalized geometry is typically used to refer to the study of Courant algebroids. These were introduced by Liu, Weinstein, and Xu in 1997 [27] and their study was stimulated in 2003 when Nigel Hitchin proposed them as a natural setting for a generalization of Calabi-Yau manifolds [19]. Marco Gualtieri's 2003 PhD thesis [17] and Mario Garcia-Fernandez and Jeff Streets' book [10] are standard references. Much of what follows will be patterned on their expositions, but with an eye towards working definitions.

2.3.1 Courant Algebroids

We will begin by discussing Courant algebroids in the smooth category before moving to the holomorphic category, in which we will most often work.

Smooth Courant Algebroids

The catchphrase of Courant algebroids, "treating T and T^* on the same footing", is encoded in the following definition.

Definition 2.5 ([17])**.** A Courant algebroid is a vector bundle $E \to M$ equipped with

1. a non-degenerate, symmetric bilinear form $\langle , \rangle$,
2. a Jacobian (non-skew-symmetric) bracket $[,]$ of sections, and
3. a smooth bundle map $\rho : E \to TM$.

These pieces of data must satisfy the following compatibility conditions where $A, B, C, D \in \Gamma(E)$.

1. $\rho[A, B] = [\rho A, \rho B]$
2. $[A, B] + [B, A] = \mathcal{D}\langle A, B\rangle$
3. $[A, fB] = f[A, B] + \rho(A)fB$
4. $\rho(A)\langle B, C\rangle = \langle [A, B], C\rangle + \langle B, [A, C]\rangle$

Here, $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is defined by by

$$\langle \mathcal{D}f, A\rangle = df \circ \rho(A).$$

This definition can be made more concrete by considering a couple examples.

Example 2.1. Consider the vector bundle $\mathbb{T}M = TM \oplus T^*M$ with $\rho : \mathbb{T}M \to TM$ given by the first factor projection and neutral product $\langle , \rangle$ can be defined

$$\langle X + \xi, Y + \eta\rangle = \frac{1}{2}(\xi(Y) + \eta(X)).$$

To define the bracket, let $H \in \Lambda^3_{\mathrm{cl}}$. Then $[,]_H$ will be given by

$$[X+\xi, Y+\eta]_H = [X,Y]_{\mathrm{Lie}} + \mathcal{L}_X\eta - i_Y d\xi + i_Y i_X H.$$

We denote such a Courant algebroid by (M,H).

It turns out that (M,H) is actually, up to isomorphism, the only sort of Courant algebroid of the following type.

Definition 2.6. A Courant algebroid $E \to M$ is called exact if it fits into an exact sequence of vector bundles.

$$0 \to T^* \xrightarrow{\rho^*} E^* \cong E \xrightarrow{\rho} T \to 0$$

To see this, we present the following theorem and an argument following [10].

Proposition 2.3 ([10, Proposition 2.10])**.** *Given an exact Courant algebroid E with isotropic splitting σ, the map $F : \mathbb{T}M \to E$ defined by*

$$F(X+\xi) = \sigma X + \pi^*\xi$$

is an isomorphism of orthogonal bundles where $\mathbb{T}M$ is equipped with the neutral product $\langle , \rangle$. Further, there is a unique $H \in \Lambda^3 T^*M$ s.t. F is an isomorphism of Courant algebroids between E and the H-twisted Courant algebroid structure (see Example 2.1) on $\mathbb{T}M$.

Proof. We first prove that F is an isomorphism of orthogonal bundles.

$$\langle F(X+\xi), F(X+\xi)\rangle = \langle \sigma X, \sigma X\rangle + \pi^*\xi(\sigma X) + \langle \pi^*\xi, \pi^*\xi\rangle = \xi(X) = \langle X+\xi, X+\xi\rangle$$

Notice that the isotropy of σ is used for $\langle \sigma X, \sigma X\rangle = 0$ and $\ker\pi = \operatorname{Im}\pi^*$ gives $\langle \pi^*\xi, \pi^*\xi\rangle = 0$.

We can also pull-back a lot of the Courant algebroid data fairly easily.

- $\pi_{\mathbb{T}M} = \pi \circ F$ which satisfies the anchor map axioms (as σ is a section of π)
- $\mathcal{D} = \pi^*\mathcal{D}_E = d$ defines a differential operator $C^\infty(M) \to \Gamma(\mathbb{T}M)$
- We get the Dorfman bracket by imposing naturality.

$$F[X+\xi, Y+\eta]_{\mathbb{T}M} = [F(X+\xi), X(Y+\eta)]_E$$

Using this, we can embark on the a more detailed calculation.

$$F[X+\xi, Y+\eta]_{\mathbb{T}M} = [\sigma X, \sigma Y] + [\sigma X, \pi^*\eta] + [\pi^*\xi, \sigma Y] + [\pi^*\xi, \pi^*\eta]$$

Using the axioms, note that

$$\pi[\pi^*\xi, \pi^*\eta] = [0,0] = 0$$

so it suffices to compute for arbitrary $Z \in TM$

$$\langle [\pi^*\xi, \pi^*\eta], \sigma Z\rangle = \pi(\pi^*\xi)\langle \pi^*\eta, \sigma Z\rangle - \langle \pi^*\eta, [\pi^*\xi, \sigma Z]\rangle = -\langle \eta, \pi[\pi^*\xi, \sigma X]\rangle = 0.$$

As the second term is also in the kernel of π, we compute similarly.

$$\begin{aligned}
\langle [\sigma X, \pi^*\eta], \sigma Z\rangle &= \pi(\sigma X)\langle \pi^*\eta, \sigma Z\rangle - \langle \pi^*\eta, [\sigma X, \sigma Z]\rangle \\
&= X(\frac{1}{2}\eta(Z)) - \frac{1}{2}\eta([X,Z]) \\
&= \frac{1}{2}(d\eta(X,Z) + Z(\eta(X))
\end{aligned}$$

Notice also that

$$\langle F(\mathcal{L}_X\eta), \sigma Z\rangle = \langle F(i_X d\eta + d(\eta(X))), \sigma Z\langle = \frac{1}{2}(d\eta(X,Z) + Z(\eta(X))).$$

Therefore, $F(\mathcal{L}_X \eta) = [\sigma X, \pi^* \eta]$.

We also compute the third term.

$$\begin{aligned}
\langle [\pi^*\xi, \sigma Y], \sigma Z\rangle &= \langle (-[\sigma Y, \pi^*\xi] + \mathcal{D}_E\langle \pi^*\xi, \sigma Y\rangle), \sigma Z\rangle \\
&= -\frac{1}{2}(d\xi(Y,Z) + Z(\xi(Y))) + \frac{1}{2}Z(\xi(Y)) \\
&= -\frac{1}{2}(d\xi(Y,Z))
\end{aligned}$$

Therefore, $F(-i_Y d\eta) = [\pi^*\xi, \sigma Y]$.

We can now embark on a computation of the first term. Notice that the tangent part is immediate from the axioms $\pi[\sigma X, \sigma Y] = [X, Y]$. The co-tangent piece can be found by taking the neutral product with σZ like usual, yielding a tensor $H \in \Lambda^3$:

$$H(X, Y, Z) = \langle [\sigma X, \sigma Y], \sigma Z\rangle.$$

We will show C^∞-linearity in the Y variable from the axioms, Z is trivial and X is similar to Y.

$$H(X, fY, Z) = \langle f[\sigma X, \sigma Y] + X(f\sigma Y), \sigma Z\rangle = fH(X,Y,Z) + (Xf)\langle \sigma Y, \sigma Z\rangle = fH(X,Y,Z)$$

To see total anti-symmetry, we compute $H(X, X, Z)$ and $H(X, Z, Z)$.

$$H(X, X, Z) = \langle [\sigma X, \sigma X], \sigma Z\rangle = \langle \frac{1}{2}\mathcal{D}_E\langle \sigma X, \sigma X\rangle, \sigma Z\rangle = 0$$

The last equality follows from the isotropy.

$$
\begin{aligned}
H(X,Z,Z) &= \langle [\sigma X, \sigma Z], \sigma Z\rangle \\
&= -\langle [\sigma Z, \sigma X], \sigma Z\rangle + \langle \mathcal{D}_E \langle \sigma X, \sigma Z\rangle, \sigma Z\rangle \\
&= -Z\langle \sigma X, \sigma Z\rangle + H(Z,Z,X) \\
&= 0.
\end{aligned}
$$

Again, the isotropy plays a crucial role. Putting everything together, we recover the bracket on (M,H) exactly.

Finally, we must check that $dH = 0$. This is a fairly intensive application of the Jacobi identity for the Dorfman bracket, so we will not include it.

As F pulls all of the Courant algebroid data of E back to $\mathbb{T}M$ yielding exactly the data of the H-twisted Courant algebroid on the generalized tangent bundle (M,H), we have proved the theorem. □

Holomorphic Courant Algebroids

A holomorphic Courant algebroid is a Courant algebroid in the holomorphic category. These algebroids can be thought of as complexified smooth Courant algebroids with extra holomorphic data playing the role of a Dolbeault structure. In this way, we get can get moduli of holomorphic structures over a smooth Courant algebroid. To study this, it will be convenient to introduce the notion of *liftings of* $T^{0,1}M$.

Definition 2.7. Let E be a smooth exact Courant algebroid over a complex manifold (M,J). A *lifting* of $T^{0,1}$ to $E \otimes \mathbb{C}$ is an isotropic, involutive subbundle $\ell \subset E \otimes \mathbb{C}$ mapping isomorphically to $T^{0,1}$ under the $\mathbb{C}$-linear extension of the anchor map $\pi\colon E \otimes \mathbb{C} \to T \otimes \mathbb{C}$.

A lifting relates to the complex Courant algebroid $E \otimes \mathbb{C}$ as a Dolbeault operator relates to a smooth complex vector bundle, in the sense that it enable us to construct a Courant algebroid in the holomorphic category out of $E\otimes\mathbb{C}$. More precisely, following [18] we consider the reduction of $E \otimes \mathbb{C}$ by ℓ given by the orthogonal bundle

$$\mathcal{Q}_\ell := \ell^\perp/\ell,$$

where $\ell^\perp$ is the orthogonal complement of ℓ with respect to the symmetric pairing on $E \otimes \mathbb{C}$. Since ℓ is a lifting of $T^{0,1}$ the kernel of $\pi_{|\ell^\perp}$ is $T^*_{1,0}$, and therefore $\mathcal{Q}_\ell$ is an extension of the form

$$T^*_{1,0} \xrightarrow{\pi^*} \mathcal{Q} \xrightarrow{\pi} T^{1,0}. \tag{2.11}$$

The Dolbeault operator on $\mathcal{Q}_\ell$ can be defined as follows: given s a smooth section of $\mathcal{Q}_\ell$, we define

$$\overline{\partial}^\ell_X s = [\tilde{X}, \tilde{s}] \mod \ell$$

where $X \in T^{0,1}$, $\tilde{X}$ is the unique lift of X to ℓ, and $\tilde{s}$ is any lift of s to a section of $\ell^\perp$. The Jacobi identity for the Dorfman bracket on $E\otimes\mathbb{C}$ implies that $\overline{\partial}^\ell \circ \overline{\partial}^\ell = 0$ and that it induces a Dorfman bracket on the holomorphic sections of $\mathcal{Q}_\ell$.

Our next goal is to make the previous construction more explicit by choosing an isotropic splitting of E. For a proof of the next result we refer to [10].

Lemma 2.12 ([10, c.f. Theorem 7.56]). *Let (M, J) be a complex manifold. Given $H_0 \in \Lambda^3$ a closed real three-form, $dH_0 = 0$, consider the exact Courant algebroid (M, H_0) as above (Example 2.1). Then, a lifting*

$$\ell \subset (T \oplus T^*) \otimes \mathbb{C}$$

of $T^{0,1}$ is equivalent to a pair (ω, b), where $\omega \in \Lambda^{1,1}_{\mathbb{R}}$ and $b \in \Lambda^2$, satisfying

$$H_0 = -d^c\omega - db. \tag{2.12}$$

More explicitly, given (ω, b) satisfying (2.12) the lifting is

$$\ell = \ell(\omega, b) := \{e^{b+\sqrt{-1}\omega}(X^{0,1}),\ X^{0,1} \in T^{0,1}\}, \tag{2.13}$$

and, conversely, any lifting is uniquely expressed in this way.

We will now define a prototypical holomorphic Courant algebroid which will also prove to be the generic holomorphic Courant algebroid associated to a lifting $\ell(\omega, b)$.

Definition 2.8. Let (M, J) be a complex manifold. Given $\tau \in \Lambda^{3,0+2,1}$, $d\tau = 0$, we denote by

$$\mathcal{Q}_\tau = T^{1,0} \oplus T^*_{1,0}$$

the exact holomorphic Courant algebroid with Dolbeault operator

$$\overline{\partial}^\tau(X + \xi) = \overline{\partial}X + \overline{\partial}\xi - i_X\tau^{2,1},$$

symmetric bilinear form

$$\langle X + \xi, X + \xi\rangle = \xi(X), \tag{2.14}$$

bracket on holomorphic sections given by

$$[X + \xi, Y + \eta]_\tau = [X, Y] + \partial(\eta(X)) + i_X\partial\eta - i_Y\partial\xi + i_Y i_X\tau^{3,0},$$

and anchor map $\pi(X+\xi) = X$. It is not difficult to check that $\mathcal{Q}_\tau$ so defined satisfies the axioms in Definition 2.5 in the holomorphic category.

Now for the lemma giving the structure of these objects up to isomorphism is stated as follows.

Lemma 2.13 ([10, Lemma 7.55]). *Let (M, J) be a complex manifold endowed with an exact Courant algebroid (M, H_0) as above. Let $\ell(\omega, b)$ be a lifting of $T^{0,1}$ as in Lemma 2.12. Then, using the notation in Definition 2.8, there is a canonical isomorphism*

$$\mathcal{Q}_{\ell(\omega,b)} \cong \mathcal{Q}_{2\sqrt{-1}\partial\omega}.$$

Proof. We have

$$\ell(\omega, b)^\perp = e^{b+\sqrt{-1}\omega}(T^{1,0}) \oplus T^*_{1,0} \oplus \ell$$

and therefore there is a smooth bundle isomorphism

$$\begin{aligned} \mathcal{Q}_{\ell(\omega,b)} &\to T^{1,0} \oplus T^*_{1,0} \\ [e^{b+\sqrt{-1}\omega}Y + \eta] &\mapsto Y + \eta. \end{aligned} \tag{2.15}$$

The agreement of the pairing and the anchor map with the ones on $\mathcal{Q}_{2\sqrt{-1}\partial\omega}$ is straightforward. Let us now express the Dolbeault operator in terms of (2.15). Given $X \in T^{0,1}$ and $Y + \eta \in T^{1,0} \oplus T^*_{1,0}$, we have

$$\begin{aligned} &[e^{b+\sqrt{-1}\omega}X, e^{b+\sqrt{-1}\omega}Y + \eta] \\ &\quad = e^{b+\sqrt{-1}\omega}[X, Y]^{1,0} + L_X\eta + i_Y i_X(H_0 + db + \sqrt{-1}d\omega) && \mod \ell \\ &\quad = e^{b+\sqrt{-1}\omega}\overline{\partial}_X Y + i_X\overline{\partial}\eta - i_X i_Y(2\sqrt{-1}\partial\omega), && \mod \ell \end{aligned}$$

which recovers the Dolbeault operator in Definition 2.8 when $\tau = 2\sqrt{-1}\partial\omega$. Similarly, for $X+\xi, Y+\eta$ holomorphic sections of $T^{1,0} \oplus T^*_{1,0}$, we find

$$
\begin{aligned}
[e^{b+\sqrt{-1}\omega} X + \xi, e^{b+\sqrt{-1}\omega} Y + \eta] &= e^{b+\sqrt{-1}\omega}[X,Y] + L_X\eta - i_Y d\xi + i_Y i_X(2\sqrt{-1}\partial\omega) \\
&= e^{b+\sqrt{-1}\omega}[X,Y] + L_X\eta - i_Y d\xi + i_Y \overline{\partial}\xi \\
&= e^{b+\sqrt{-1}\omega}[X,Y] + \partial(\eta(X)) + i_X\partial\eta - i_X\partial\xi
\end{aligned}
$$

as claimed. □

2.3.2 Generalized Metrics

We introduce next a new ingredient, namely, generalized metrics, which will lead us naturally to the study of Hermitian metrics on exact holomorphic Courant algebroids. Recall that a generalized metric on a smooth exact Courant algebroid E is given by an orthogonal decomposition

$$E = V_+ \oplus V_-$$

such that the restriction of the neutral inner product to V_+ (resp. V_-) is positive definite (resp. negative definite). Recall also that a generalized metric determines uniquely a Riemann metric g on M and an isotropic splitting of E. In particular, it has an associated isomorphism $E \cong (T \oplus T^*, \langle , \rangle, [,]_H, \pi)$ for a uniquely determined closed three-form H (see Proposition 2.3), such that

$$V_\pm = \{X \pm g(X), X \in T\}. \tag{2.16}$$

The basic interaction between generalized metrics and complex geometry is provided by the following definition.

Definition 2.9. Let (M, J) be a complex manifold endowed with a smooth exact Courant algebroid E. We say that a generalized metric $E = V_+ \oplus V_-$ is *compatible with* J if

$$\ell = \{e \in V_+ \otimes \mathbb{C},\ \pi(e) \in T^{0,1}\} \subset E \otimes \mathbb{C}$$

is a lifting of $T^{0,1}$.

Using the splitting of E determined by the generalized metric, it is not difficult to see that Definition 2.9 implies that g is Hermitian and furthermore

$$\ell = e^{\sqrt{-1}\omega} T^{0,1}$$

where $\omega = gJ$ is the associated fundamental two-form. Applying now Lemma 2.12 we obtain the following.

Lemma 2.14. *Let (M, J) be a complex manifold endowed with a smooth exact Courant algebroid E. A generalized metric $E = V_+ \oplus V_-$ is compatible with J if and only if the associated Riemannian metric g is Hermitian and furthermore*

$$H = -d^c\omega. \tag{2.17}$$

In particular g is pluriclosed.

Given a compatible generalized metric, we can find an alternative presentation of the associated holomorphic Courant algebroid $\mathcal{Q}_\ell \cong \mathcal{Q}_{2\sqrt{-1}\partial\omega}$ (see Lemma 2.13) which will naturally endow this bundle with a Hermitian metric. To see this, note that $V_+^\perp = V_-$ implies that

$$\ell^\perp = (V_- \otimes \mathbb{C}) \oplus \ell.$$

Therefore, as a smooth orthogonal bundle $\mathcal{Q}_\ell$ is canonically isomorphic to

$$\mathcal{Q}_\ell := \ell^\perp/\ell \cong V_- \otimes \mathbb{C}.$$

Definition 2.10. Let (M, J) be a complex manifold endowed with a smooth exact Courant algebroid E and a compatible generalized metric $E = V_+ \oplus V_-$. Then, the induced *generalized Hermitian metric* G on $\mathcal{Q}_\ell$ is defined by

$$G([s_1],[s_2]) = -2\,\langle \pi_- s_1, \overline{\pi_- s_2}\rangle$$

for $[s_j] \in \ell^\perp/\ell$ and $\pi_- \colon \ell^\perp \to V_- \otimes \mathbb{C}$ the orthogonal projection.

We are ready to prove the main result of this section, where we calculate the Chern connection of the induced generalized Hermitian metric G in terms of the connection D^- associated to the underlying pluriclosed structure (see Proposition 3.2). This result provides an interpretation of [3, Theorem 2.9] in the language of holomorphic Courant algebroids.

Proposition 2.4. *Let (M, J) be a complex manifold endowed with a smooth exact Courant algebroid E and a compatible generalized metric $E = V_+ \oplus V_-$. Then, the Chern connection of the associated generalized Hermitian metric G on $\mathcal{Q}_\ell$ is given by*

$$i_X \nabla^G s = \pi_-[\sigma_+ X, \pi_- s] \tag{2.18}$$

via the isomorphism $\mathcal{Q}_\ell \cong V_- \otimes \mathbb{C}$. Here, $\sigma_+ X = X + g(X)$ is the inverse of the isomorphism $\pi_{|V_+} \colon V_+ \to T$. More explcitly, via the identification $V_- \cong T$, the Chern connection is given by

$$D^-_X Y = \nabla_X Y + \tfrac{1}{2} g^{-1} d^c\omega(X, Y, \cdot). \tag{2.19}$$

Proof. Observe that the right hand side of (2.18) defines an orthogonal connection on V_-, which can be identified with ∇^- via the isomorphism $\pi_{|V_-}: V_- \to T$ (see e.g. [10, Proposition 3.14]). Therefore, ∇^- extends $\mathbb{C}$-linearly to a G-unitary connection on $V_- \otimes \mathbb{C}$. By the abstract definition of the Dolbeault operator on $\mathcal{Q}_\ell$, we inmediately see that $(\nabla^-)^{0,1}$ coincides with $\overline{\partial}^\ell$. □

In our next result we calculate an explicit formula for the generalized Hermitian metric G in terms of the isomorphism $\mathcal{Q}_\ell \cong \mathcal{Q}_{2\sqrt{-1}\partial\omega}$ in Lemma 2.13.

Lemma 2.15. *Let (M, J) be a complex manifold endowed with a smooth exact Courant algebroid E and a compatible generalized metric $E = V_+ \oplus V_-$. Then, the orthogonal isomorphism $\psi\colon \mathcal{Q}_{2\sqrt{-1}\partial\omega} \to V_- \otimes \mathbb{C}$ induced by Lemma 2.13 is given by*

$$\psi(X + \xi) = e^{\sqrt{-1}\omega} X - \tfrac{1}{2} e^{-\sqrt{-1}\omega} g^{-1}\xi.$$

Consequently,

$$\psi^* G(X + \xi, X + \xi) = 2g(X, \overline{X}) + (2g)^{-1}(\xi, \overline{\xi}).$$

Proof. The first part follows from

$$\psi(X + \xi) = e^{\sqrt{-1}\omega} X + \pi_-\xi = e^{\sqrt{-1}\omega} X + \tfrac{1}{2}(\xi - g^{-1}\xi) = e^{\sqrt{-1}\omega} X - \tfrac{1}{2} e^{-\sqrt{-1}\omega} g^{-1}\xi.$$

The second part is a straightforward calculation and is left to the reader. □

For our applications it will be convenient to fix a background exact holomorphic Courant algebroid and generalized Hermitian metric. This motivates the following definition, which is inspired by [8, 9].

Definition 2.11. Let (M, J) be a complex manifold endowed with an exact holomorphic Courant algebroid $\mathcal{Q}$. A *generalized Hermitian metric* on $\mathcal{Q}$ is given by a triple (E, V_+, φ), where

1. E is an exact Courant algebroid over M,
2. $V_+ \subset E$ is a generalized metric compatible with J,
3. $\varphi\colon \mathcal{Q}_\ell \to \mathcal{Q}$ is an isomorphism of holomorphic Courant algebroids inducing the identity on M.

Observe that a generalized Hermitian metric (E, V_+, φ) on $\mathcal{Q}$ induces a generalized Hermitian metric G' on $\mathcal{Q}_\ell$ as in Definition 2.10. Therefore, via the isomorphism φ we obtain a Hermitian metric

$$G = \varphi_* G'$$

on $\mathcal{Q}$ compatible with the orthogonal structure. By abuse of notation, we will also call G a *generalized Hermitian metric.* We next unravel the previous definition in terms of the model in Definition 2.8.

Lemma 2.16. *Let (M, J) be a complex manifold, and $\tau_0 \in \Lambda^{3,0+2,1}$, $d\tau_0 = 0$. Then, there is a one to one correspondence between the set of generalized Hermitian metrics on $\mathcal{Q}_{2\sqrt{-1}\tau_0}$ and*

$$\left\{\omega + \beta \mid \omega > 0,\ d\beta = \tau_0 - \partial\omega\right\} \subset \Lambda^{1,1}_{\mathbb{R}} \oplus \Lambda^{2,0}.$$

Furthermore, the generalized Hermitian metric $G = \varphi_* G'$ is given by

$$G(X+\xi, X+\xi) = 2g(X, \overline{X}) + (2g)^{-1}(\xi + 2\sqrt{-1}\beta(X), \overline{\xi} - 2\sqrt{-1}\overline{\beta(X)}). \tag{2.20}$$

Proof. The pair (E, V_+) determines $\omega > 0$ and an isomorphism $E \cong (T \oplus T^*, \langle , \rangle , [,]_H)$ for $H = -d^c\omega$. By Lemma 2.13, $\mathcal{Q}_\ell \cong \mathcal{Q}_{2\sqrt{-1}\partial\omega}$ and the isomorphism $\varphi \colon \mathcal{Q}_\ell \to \mathcal{Q}_{\tau_0}$ corresponds to

$$\varphi = e^{-2\sqrt{-1}\beta} \colon T^{1,0} \oplus T^*_{1,0} \to T^{1,0} \oplus T^*_{1,0}$$

for $\beta \in \Lambda^{2,0}$ satisfying $2\sqrt{-1}d\beta = 2\sqrt{-1}\tau_0 - 2\sqrt{-1}\partial\omega$. The last part of the statement is now straightforward from Lemma 2.15. $\square$

Remark 2.5. *For our applications to the pluriclosed flow, we will need to fix an initial pluriclosed Hermitian metric g and a background pluriclosed metric g'. We will not require that the associated holomorphic Courant algebroids $\mathcal{Q}_{2\sqrt{-1}\partial\omega}$ and $\mathcal{Q}_{2\sqrt{-1}\partial\omega'}$ are isomorphic, but the weaker condition of being isomorphic as holomorphic orthogonal bundles. In practice, this boils down to the explicit condition*

$$\overline{\partial}\beta = \partial\omega' - \partial\omega \tag{2.21}$$

for some $\beta \in \Lambda^{2,0}$ or, equivalently,

$$[\partial\omega] = [\partial\omega'] \in H^{2,1}_{\overline{\partial}}(X).$$

One can easily see that a pair (ω, β) as in (2.21) *defines a generalized Hermitian metric in the exact holomorphic Courant algebroid $\mathcal{Q}_{2\sqrt{-1}\partial\omega'+2\sqrt{-1}\partial\beta}$, that is, the twisting of $\mathcal{Q}_{2\sqrt{-1}\partial\omega'}$ by the d-closed $(3,0)$-form $2\sqrt{-1}\partial\beta$ (see Definition 2.8).*

Chapter 3

Obstructions & Counter-Examples

This chapter will deal with obstructions to the existence of Bismut Hermitian-Einstein metrics which can be used to construct some examples of $c_1 = 0$ manifolds which do not admit such metrics. Broadly speaking, generalized geometry (see §2.3) will provide a correspondence between Bismut Hermitian-Einstein metrics and a coupled Hermitian Yang-Mills system on an associated holomorphic Courant algebroid. This will allow us to invoke the necessity of slope stability from the Kobayashi-Hitchin correspondence.

The sections of the chapter will go as follows. In §3.1, the author will discuss the relevant pieces of the Kobayashi-Hitchin correspondence following Lubke-Teleman [28]. In §3.2, the author will re-prove a result of Bismut [3] using the language of holomorphic Courant algebroids (see §2.3.1) which serves to relate these elliptic systems. In §3.3 we derive some corollaries which make possible the construction of counter-examples.

3.1 Kobayashi-Hitchin Correspondence

To begin, we will define a notion of Ricci curvature that arises naturally in the study of Hermitian Yang-Mills connections.

Definition 3.1. Let (M^{2n}, g, J) be a Hermitian manifold and suppose $(\mathcal{E}, h) \to M$ denote a holomorphic vector bundle with Hermitian metric h and associated Chern connection ∇^h. The *second Ricci curvature* is

$$S_g^h := \sqrt{-1}\Lambda_\omega \Omega^h \in \operatorname{End}(E).$$

Furthermore, we will call the Hermitian metric h *g-Hermitian-Einstein* if there exists $\gamma_h \in \mathbb{C}$ s.t.

$$S_g^h = \gamma_h \operatorname{Id}_E .$$

The existence of Hermitian-Einstein metrics, is goverened by slope stability criteria as in the Donaldson-Uhlenbeck-Yau Theorem [6] and its extensions to Hermitian manifolds – the Kobayashi-Hitchin correspondence (see [28]). To state the precise result which we will use, let us recall first some basic definitions. Given a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_M$-modules over M, the determinant $\det \mathcal{F} := (\Lambda^r \mathcal{F})^{**}$, where r denotes the rank of $\mathcal{F}$, is a holomorphic line bundle over M. Given now an Aeppli class $a \in H_A^{n-1,n-1}$, we can define the slope of $\mathcal{F}$ by

$$\mu_a(\mathcal{F}) = \frac{c_1(\det \mathcal{F}) \cdot a}{r}.$$

where $c_1(\det \mathcal{F}) \in H^{1,1}_{BC}(M)$ is the first Chern class of $\det \mathcal{F}$, regarded an element in the Bott-Chern cohomology of M. Here we use the standard duality pairing

$$\begin{aligned} H^{1,1}_{BC} \otimes H^{n-1,n-1}_{A} &\to \mathbb{C} \\ (\alpha, \beta) &\mapsto \int_M \alpha \wedge \beta \end{aligned}$$

.

Definition 3.2. Let (M, J) be a compact complex manifold endowed with an Aeppli class $a \in H^{n-1,n-1}_{A}$. A holomorphic vector bundle $\mathcal{E}$ over M is *a-semistable* if for any subsheaf $\mathcal{F} \subset \mathcal{E}$ one has

$$\mu_a(\mathcal{F}) \leq \mu_a(\mathcal{E}).$$

We say that $\mathcal{E}$ is *a-stable* if the inequality is strict and *a-polystable* if it is a direct sum of a-stable bundles all having the same slope.

The relation between slope stability and the Hermitian-Einstein equation is provided by the following important result (cf [28]):

Theorem 3.1 (Kobayahsi-Hitchin Correspondence)**.** *Let (M, J) be a compact complex manifold. Let $\tilde{\omega}$ be a Gauduchon Hermitian metric on M with Aeppli class $a = [\tilde{\omega}^{n-1}] \in H^{n-1,n-1}_{A}$. A holomorphic vector bundle $\mathcal{E}$ over (M, J) admits a Hermitian metric h solving the Hermitian-Einstein equation*

$$S^h_g = \gamma_h \operatorname{Id}_{\mathcal{E}}$$

for some $\gamma_h \in \mathbb{C}$ if and only if $\mathcal{E}$ is a-polystable.

The full strength of Theorem 3.1 will not be made use of because, as we will see, the Bismut Hermitian-Einstein condition is equivalent only to a certain coupled Hermitian-Einstein equation. Thus, we can only make use of the necessity portion in the sequel. We will prove this following the book of Lübke and Teleman [28].

Theorem 3.2 ([28, Theorem 2.3.2])**.** *Given (M^n, g, J) with g Gauduchon, having associated Aeppli class $a = [\omega_g^{n-1}]$, and $\mathcal{E} \to M$ a holomorphic vector bundle admitting a Hermitian-Einstein metric h, then $\mathcal{E}$ is a-semistable. If $\mathcal{E}$ is not a-stable, then it must be a a-polystable with the induced metric on each subbundle being Hermitian-Einstein and having the same constant.*

To prove this, we will require four intermediate results which we state and prove now.

Lemma 3.3 ([28, Lemma 2.1.8])**.** *If g is Gauduchon with Aeppli class a and h a g-Hermitian-Einstein metric on $\mathcal{E} = (E, \overline{\partial})$, then its Einstein factor γ_h is proportional to the g-slope of $\mathcal{E}$:*

$$\gamma_h = \frac{2\pi}{(n-1)!\operatorname{Vol}_g(M)} \mu_a(\mathcal{E}).$$

Proof. First, recall that by definition $S_g^h = \sqrt{-1}\Lambda_g \Omega^h$. Also, recall that if $\alpha \in \Lambda^{1,1}$ then

$$\alpha \wedge \omega_g^{n-1} = (n-1)!\alpha \wedge \star_g \omega_g = (n-1)!\Lambda_\omega(\alpha)\mathrm{dvol}_g.$$

Then we can compute as follows.

$$\begin{aligned}
\mu_a(\mathcal{E}) &= \frac{1}{\operatorname{rk}\mathcal{E}} \int_M c_1(\det \mathcal{E}) \wedge \omega_g^{n-1} = \frac{\sqrt{-1}}{2\pi \operatorname{rk}(\mathcal{E})} \int_M \operatorname{tr} \Omega^h \wedge \omega_g^{n-1} \\
&= \frac{(n-1)!}{2\pi \operatorname{rk}(\mathcal{E})} \int_M \operatorname{tr}(\sqrt{-1}\Lambda_g \Omega^h) \wedge \omega_g^{n-1} = \frac{\gamma_h (n-1)! \operatorname{Vol}_g(M)}{2\pi}
\end{aligned}$$

□

Lemma 3.4 ([28, Lemma 2.1.4])**.** *If $\mathcal{E}$ (resp. $\mathcal{F}$) are holomorphic vector bundles admitting g-Hermitian-Einstein metrics h' (resp. h'') with Einstein factors γ' (resp. γ''), then*

- *$\mathcal{E}^*$ is g-Hermitian-Einstein with constant $-\gamma'$*
- *$\mathcal{E} \otimes \mathcal{F}$ is g-Hermitian-Einstein with factor $\gamma' + \gamma''$*

- *$\Lambda^p\mathcal{E}$ is g-Hermitian-Einstein with factor $p\gamma'$*

Proof. Notice that by Lemma 3.3 all of these statements reduce to statements about g-slope.

For example, to prove $\gamma(\mathcal{E}^*) = -\gamma'$, it will suffice to prove $\mu_g(\mathcal{E}^*) = -\mu_g(\mathcal{E})$. However, this would be immediate provided $c_1(\mathcal{E}^*) = -c_1(\mathcal{E})$ (modulo $\partial\overline{\partial}$-exact terms). This can be proved relatively simply, though, as $(h')^{-1}$ defines a metric on $\mathcal{E}^*$.

$$\frac{\sqrt{-1}}{2\pi}\rho^C((h')^{-1}) = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\det(h')^{-1} = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\det(h'))^{-1} = -\frac{\sqrt{-1}}{2\pi}\rho^C(h')$$

However, both $[\frac{\sqrt{-1}}{2\pi}\rho^C((h')^{-1})] = c_1(\mathcal{E}^*)$ and $[\frac{\sqrt{-1}}{2\pi}\rho^C(h')] = c_1(\mathcal{E})$ proving the result.

The second statement follows from the fact that $h' \otimes h''$ is a metric on $\mathcal{E} \otimes \mathcal{F}$ and a similar argument on Chern-Ricci forms.

$$\rho^C(h'\otimes h'') = -\partial\overline{\partial}\log\det(h'\otimes h'') = -\partial\overline{\partial}\left(\log(\det h')^{\mathrm{rk}(\mathcal{F})} + \log(\det h'')^{\mathrm{rk}(\mathcal{E})}\right) = \mathrm{rk}(\mathcal{F})\rho^C(h') + \mathrm{rk}(\mathcal{E})\rho^C(h'')$$

This implies that $\mu_a(\mathcal{E} \otimes \mathcal{F}) = \mu_a(\mathcal{E}) + \mu_a(\mathcal{F})$ from which the result follows.

The result on wedge products can be proved similarly. □

Lemma 3.5 ([28, Corollary 2.1.6])**.** *Every line bundle L on a compact, Hermitian manifold (M^{2n}, J, g) admits a g-Hermitian-Einstein metric, which is unique up to a constant positive factor.*

Proof. Notice that given two metrics h', h'' on L, we must have that $h_f = e^{\frac{f}{n}}h'$ for some f. This means that the following identity holds between the second Ricci curvatures.

$$S_g^{h_f} = -\Delta_g(\log\det(e^{\frac{f}{n}}h')) = -\Delta_g f + S_g^{h'}$$

However, since $S_g^{h'}$ is a smooth function in this setting, we can use the Fredholm alternative for the g-Chern Laplacian (as in Theorem 2.6) to find a unique ϕ and $\lambda \in \mathbb{R}$ s.t.

$$S_g^{h'} = \Delta_g \phi + \lambda.$$

This then implies that

$$S_g^{h_\phi} = \lambda \cdot 1.$$

$\square$

Proposition 3.1 ([28, Theorem 2.2.1]). *Let (M, J, g) be Gauduchon and $\mathcal{E} = (E, \overline{\partial})$ be a holomorphic vector bundle admitting a g-Hermitian-Einstein metric h with Einstein factor γ_h. Then:*

- $\gamma_h < 0$ *implies $\mathcal{E}$ has no non-trivial holomorphic sections.*
- $\gamma_h = 0$ *implies that every global holomorphic section of $\mathcal{E}$ is h-Chern-parallel.*

Proof. Recall the Bochner formula for holomorphic sections s.

$$\Delta |s|_h^2 = |\nabla_h s|_{g,h}^2 - \langle S_g^h s, s \rangle_h$$

But by the Hermitian-Einstein condition, this becomes

$$\Delta |s|_h^2 = |\nabla_h s|_{g,h}^2 - \gamma_h |s|_h^2.$$

Then, we can integrate this relation to get

$$\gamma_h \|s\|_{L^2(M)}^2 = \|\nabla_h s\|_{L^2(M)}^2.$$

From this, it is immediately clear that $\gamma_h \geq 0$ whenever a holomorphic section exists. Further, it is also immediately clear that if $\gamma_h = 0$, we get $\nabla_h s \equiv 0$. $\square$

Lemma 3.6 ([28, Proposition 2.3.1])**.** *Over a complex manifold (M, J), suppose we have a short exact sequence of holomorphic vector bundles*

$$0 \to \mathcal{E}' \xrightarrow{i} \mathcal{E} \xrightarrow{j} \mathcal{E}'' \to 0$$

where $\mathcal{E}$ admits g-Hermitian-Einstein metric h. Then,

$$\frac{\Lambda_g c_1(\mathcal{E}', h')}{\operatorname{rk} \mathcal{E}'} \leq \frac{\Lambda_g c_1(\mathcal{E}, h)}{\operatorname{rk} \mathcal{E}} \tag{3.1}$$

with equality iff the short exact sequence splits holomorphically orthogonally and the induced metrics on $\mathcal{E}'$ and $\mathcal{E}''$ are g-Hermitian-Einstein with the same constant as $\mathcal{E}$.

Proof. Recall that for any holomorphic, Hermitian vector bundle, the Chern-Ricci form is a representative of the first Chern class. We will then go ahead and compute the curvature in a local frame in terms of the curvatures of $\mathcal{E}'$ and $\mathcal{E}''$. First, notice that we already know that, as smooth bundles,

$$\mathcal{E} = \mathcal{E}' \oplus (\mathcal{E}')^{\perp}.$$

Further, since the sequence is exact, $j : (\mathcal{E}')^{\perp} \to \mathcal{E}''$ is an isomorphism. So we can fix an adapted local unitary frame (e_i) for $\mathcal{E}$ s.t. $e_1, ..., e_s$ is a unitary frame for $\mathcal{E}'$ and $e_{s+1}, ..., e_{s+r}$ is an adapted local unitary frame for $\mathcal{E}''$. Then if $\theta, \theta', \theta''$ are the connection matrices of the Chern connections on $\mathcal{E}, \mathcal{E}', \mathcal{E}''$ respectively, we get a decomposition in terms of the second fundamental form A

$$\theta = \begin{pmatrix} \theta' & A \\ A^{\dagger} & \theta'' \end{pmatrix}.$$

Cartan's second structural equation then gives

$$\Omega = \begin{pmatrix} \Omega' - A \wedge A^\dagger & * \\ * & \Omega'' - A^\dagger \wedge A \end{pmatrix}. \tag{3.2}$$

But this means that

$$\begin{aligned} \rho^C(h) &= \sum_{i=1}^{r+s} \Omega_{i\bar{i}} \\ \rho^C(h') &= \sum_{i=1}^{s} \Omega_{i\bar{i}} + \mathrm{tr}(A \wedge A^\dagger) \end{aligned}$$

. When $\sqrt{-1}\Lambda\Omega = \gamma\,\mathrm{Id}$, we can use this to compute

$$\begin{aligned} \frac{\Lambda_g c_1(\mathcal{E})}{\mathrm{rk}\,\mathcal{E}} &= \frac{\sqrt{-1}}{2\pi\,\mathrm{rk}\,\mathcal{E}}\Lambda_g \sum_{i=1}^{r+s} \Omega_{i\bar{i}} = \frac{\gamma}{2\pi} \\ \frac{\Lambda_g c_1(\mathcal{E}')}{\mathrm{rk}\,\mathcal{E}'} &= \frac{\gamma}{2\pi} + \frac{\sqrt{-1}}{2\pi}\Lambda_g\,\mathrm{tr}(A \wedge A^\dagger) \end{aligned}$$

But since A must be skew (metric-compatible connection matrices are skew in unitary frames), it must be that $A^\dagger = -A^*$ so that

$$\frac{\gamma}{2\pi} + \frac{\sqrt{-1}}{2\pi}\Lambda_g\,\mathrm{tr}(A \wedge A^\dagger) = \frac{\gamma}{2\pi} - \frac{1}{2\pi}||A|^2 \leq \gamma.$$

This is exactly (3.1).

Finally, (3.1) is saturated iff $A \equiv 0$, but this is the case iff $\mathcal{E}'' \cong (\mathcal{E}')^\perp$ holomorphically and orthogonally. That all the bundles have the same Einstein factor then follows from (3.2) and Lemma 3.3, proving the theorem. □

We now prove the main result of this section.

Proof of Theorem 3.2. Let $\mathcal{F}$ be a coherent subsheaf of $\mathcal{E}$ of rank $0 < s < r$, then we have an inclusion.

$$i : \mathcal{F} \hookrightarrow \mathcal{E}$$

This induces a morphism

$$\det i : \det \mathcal{F} \to \Lambda^s \mathcal{E}.$$

This morphism is injective as a morphism of a sheaves because it is injective as a morphism of subbundles off a closed analytic set of codimension no-smaller than 1 (see [25, Theorem 5.5.8]). If we tensor in $\det \mathcal{F}^*$, we get a holomorphic section

$$\eta : \mathcal{O}_X \hookrightarrow \Lambda^s \mathcal{E} \otimes \det \mathcal{F}^*.$$

Theorem 3.1 then tells us that

$$\gamma(\Lambda^s \mathcal{E} \otimes \det \mathcal{F}^*) \geq 0.$$

By Lemma 3.4, this means that

$$\gamma(\Lambda^s \mathcal{E} \otimes \det \mathcal{F}) = s\gamma(\mathcal{E}) - \gamma(\det \mathcal{F}) \geq 0 \tag{3.3}$$

with equality iff η is parallel. But by Lemma 3.3 and the fact that determinants of sheaves have rank one, we can deduce the inequality of g-slopes $s\mu_a(\mathcal{E}) \geq \mu_a(\det \mathcal{F}) = s\mu_a(\mathcal{F})$. This is precisely the g-semistability of $\mathcal{E}$.

Assuming that $\mathcal{E}$ is not stable, we get that there is a choice of reflexive subsheaf $\mathcal{F}$ (see [28, Proposition 1.4.5.ii]) s.t. $\mu_a(\mathcal{F}) = \mu_a(\mathcal{E})$. But by (3.3), we find that off Σ, a closed analytic set of codimension no smaller than 1,

$$\gamma(\Lambda^s \mathcal{E} \otimes \det \mathcal{F}) = 0$$

and η is g-Chern-parallel. Thus, $\det \mathcal{F}$ is a parallel subbundle of $\Lambda^s \mathcal{E}$ off Σ. Thus, the Hermitian-Einstein metric we chose on $\det \mathcal{F}$ is just the restriction of the induced metric on $\Lambda^s \mathcal{E}$.

We also note that $\det i$ is injective as a map of bundles gives us that i is injective as a map of bundles off Σ. By Lemma 3.6, as we have the following short exact sequence

$$0 \to \mathcal{F}|_{M\setminus\Sigma} \to \mathcal{E}|_{M\setminus\Sigma} \to \mathcal{E}/\mathcal{F}|_{M\setminus\Sigma} \to 0,$$

we have that $\mathcal{E}|_{M\setminus\Sigma}$splits holomorphically and orthogonally and all the involved bundles are Hermitian-Einstein with the same constant.

By a sheaf theoretic argument, the splitting can be extended to a global splitting of sheaves. If any component of the splitting is not g-stable, we simply replace $\mathcal{E}$ with that bundle a re-run the splitting argument. This will terminate after no more that $\operatorname{rk} \mathcal{E} < \infty$ steps leaving a sum of stable bundles.

$\square$

3.2 Bismut's Identity

To make use of these preliminaries, we introduce an isomorphism between $TM_{\mathbb{C}}$ and $\mathcal{Q}_{\sqrt{-1}\partial\omega}$. This isomorphism is very simple. It is just the lowering of the $(0,1)$ piece of a complexified tangent vector into a $(1,0)$-form. However, it is a holomorphic isomorphism! This gives us a way of identifying the usual Chern data on the Courant algebroid with the Chern data of an exotic holomorphic structure on the complexified tangent bundle!

Lemma 3.7. *On a pluriclosed manifold (M, J, g) the map given by*

$$\begin{aligned} \psi_g \colon TM \otimes \mathbb{C} &\to T^{1,0} \oplus T^*_{1,0} \\ X = X^{1,0} + X^{0,1} &\mapsto X^{1,0} - g(X^{0,1}) \end{aligned}$$

induces an isomorphism of Hermitian, holomorphic vector bundles

$$\psi_g : (TM \otimes \mathbb{C}, g, (D^-)^{0,1}) \to (\mathcal{Q}_{\sqrt{-1}\partial\omega}, G)$$

where G is the generalized metric given by

$$G = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}.$$

Proof. We note first that ψ_g is obviously a complex isometry. Then, using the definitions of Bismut and Chern connections one can show that

$$\left\langle D^+_X Y, Z \right\rangle = \langle \nabla_X Y, Z \rangle + \tfrac{1}{2} d\omega(JX, Y, Z) + \tfrac{1}{2} d\omega(JX, JY, JZ),$$

which in turn implies

$$D^-_{X^{0,1}} Y = \nabla_{X^{0,1}} Y - \sqrt{-1} g^{-1} \partial\omega(X^{0,1}, Y^{0,1}, \cdot).$$

Therefore, using that the Chern connection is metric compatible, we obtain

$$\psi_g(D^-_{X^{0,1}} \psi_g^{-1}(Y^{1,0} + \xi^{1,0})) = \nabla_{X^{0,1}}(Y^{1,0} + \xi^{1,0}) + \sqrt{-1}\partial\omega(X^{0,1}, Y^{1,0}, \cdot),$$

which corresponds to the Dolbeault operator in Definition 2.8 for $\tau = \sqrt{-1}\partial\omega$. □

Lemma 3.7 gives as an immediate corollary an identity of Bismut [3] which says that the Chern curvature on the Courant algebroid is completely determined by the Hull connection on the complexified tangent bundle. This is especially interesting because the Hull connection is metric-compatible, but neither torsionfree nor Hermitian.

Proposition 3.2 (Bismut's Identity [3]). *Let (M, g, J) be a pluriclosed Hermitian manifold. Consider the associated holomorphic orthogonal bundle $\mathcal{Q}_{\sqrt{-1}\partial\omega}$, as in Definition 2.8, endowed with the Hermitian metric G'. Then, the Chern connection $\nabla^{G'}$ of G' satisfies*

$$\nabla^{G'} = (\psi_g)_* D^-, \qquad \Omega^{G'} = (\psi_g)_* R^-$$

Proof. Notice first that D^- is the Hull connection and also the Chern connection preserving the exotic holomorphic structure $(D^-)^{0,1}$. The result is then immediate upon invocation of Lemma 3.7 and the fact that Chern connections pull-back to Chern connections. $\square$

However, Proposition 3.2 can be combined with Lemma 2.10 to give the second Chern-Ricci tensor on the Courant algebroid purely in terms of the Bismut-Ricci form on the base manifold.

Proposition 3.3. *Let (M, g_0, J) be a pluriclosed manifold. Consider the associated Hermitian holomorphic vector bundle $\mathcal{Q} = \mathcal{Q}_{\sqrt{-1}\partial\omega_0}$ as in Definition 2.8. Consider a generalized Hermitian metric $G = G(\omega, \beta)$ as in Proposition 2.16. Then the Chern connection ∇^G of the Hermitian metric G, defined as in (2.20) satisfies*

$$S^G_g = \sqrt{-1}(e^{\sqrt{-1}\beta})^* \begin{pmatrix} -g^{-1}\rho_B^{1,1} & g^{-1}\rho_B^{0,2}g^{-1} \\ \rho_B^{2,0} & -\rho_B^{1,1}g^{-1} \end{pmatrix}.$$

Consequently, g is Bismut Hermitian-Einstein if and only if G is Hermitian-Einstein with respect to g, that is, if and only if

$$S^G_g = 0.$$

Proof. This follows from Proposition 3.2, since

$$\begin{aligned} S_g^G &= \sqrt{-1}\Lambda_\omega \Omega_G^C \\ &= \sqrt{-1}(e^{\sqrt{-1}\beta})^* \psi_* \Lambda_\omega R^- \\ &= -\sqrt{-1}(e^{\sqrt{-1}\beta})^* \psi_* g^{-1} \rho_B, \end{aligned}$$

where for the last identity we have used Proposition 2.10 to conclude

$$(\Lambda_\omega R^-)X = \tfrac{1}{2}\sum_{j=1}^{2n} R^-(e_i, Je_i)X = -\tfrac{1}{2}\sum_{j=1}^{2n} g^{-1}R^+(X,\cdot,Je_i,e_i) = -g^{-1}\rho_B(X).$$

Finally, given $X + \xi \in T^{1,0} \oplus T^*_{1,0}$, we calculate

$$\begin{aligned} (\psi_* g^{-1}\rho_B)(X) &= \psi(g^{-1}\rho_B(X)) = g^{-1}\rho_B^{1,1}(X) - \rho_B^{2,0}(X) \\ (\psi_* g^{-1}\rho_B)(\xi) &= -\psi(g^{-1}\rho_B g^{-1}\xi) = \rho_B^{1,1} g^{-1}\xi - g^{-1}\rho_B^{0,2} g^{-1}\xi. \end{aligned}$$

□

3.3 Slope Stability & Consequences

We now apply Theorem 3.2 to the Bismut Hermitian-Einstein equation via Proposition 3.3, deriving the following necessary condition to the existence of pluriclosed Bismut Hermitian-Einstein metrics.

Corollary 3.8. *Let (M, g, J) be a pluriclosed Hermitian manifold. Denote by $a = [\tilde{\omega}^{n-1}] \in H_A^{n-1,n-1}$ the Aeppli class of the unique Gauduchon metric $\tilde{\omega}$ in the conformal class of ω, such that $\int_X \tilde{\omega}^n = \int_X \omega^n$. Consider the associated orthogonal holomorphic vector bundle $\mathcal{Q} = \mathcal{Q}_{\sqrt{-1}\partial\omega}$ as in Definition 2.8. Assume that the metric g is Bismut Hermitian-Einstein.*

Then, for any subsheaf $\mathcal{F} \subset \mathcal{Q}$ one has

$$\mu_a(\mathcal{F}) \leq 0, \tag{3.4}$$

with equality only if $\mathcal{Q}$ splits holomorphically.

Proof. The proof is an easy consequence of Proposition 3.2 and Theorem 3.1, after noting that $\mathcal{Q}$ satisfies $c_1(\mathcal{Q}) = 0 \in H^{1,1}_{BC}(M)$, since $\det \mathcal{Q}$ admits a canonical holomorphic trivialization induced by the holomorphic pairing. □

In general, the stability condition depends in a intricate way on the Bismut Hermitian-Einstein pluriclosed metric. This is due to the fact that the map

$$\omega \mapsto a = [\tilde{\omega}^{n-1}] \in H_A^{n-1,n-1}(M)$$

is typically a complicated function in the space of pluriclosed metrics. In the special case of complex surfaces $n = 2$, this map only depends on the Aeppli class $[\omega] \in H_A^{1,1}$, and is just the identity map.

We can obtain even more concrete implications of Corollary 3.8 for the existence of pluriclosed Bismut Hermitian-Einstein metrics. In particular, our next result provides a clean obstruction to the existence of such metrics on a compact, complex manifold. For this, we exploit the fact that any exact holomorphic Courant algebroid has a canonical isotropic subsheaf, given by the holomorphic cotangent bundle

$$T^*_{1,0} \xrightarrow{\pi^*} \mathcal{Q}.$$

We will say that an Aeppli class $a \in H_A^{n-1,n-1}(M)$ is *positive* if $a = [\tilde{\omega}^{n-1}]$, for some Gauduchon metric $\tilde{\omega}$ on M.

Theorem 3.9. *Let (M, J) be a compact connected complex manifold. Assume that M admits a pluriclosed Hermitian metric g which is Bismut Hermitian-Einstein. Then, there exists a positive Aeppli class $a \in H_A^{n-1,n-1}(M)$ such that, for any complex manifold Z and any holomorphic map $f \colon M \to Z$ such that df is surjective at one point, one has*

$$f^* c_1(Z) \cdot a \geq 0$$

Furthermore, for $Z = (M, J)$ and $f = \mathrm{Id}$, $c_1(M) \cdot a > 0$ unless g is Kähler.

Proof. Let g as in the statement and consider the holomorphic Courant algebroid $\mathcal{Q} := \mathcal{Q}_{\sqrt{-1}\partial\omega}$. Let $a = [\tilde{\omega}^{n-1}]$ be the Aeppli class of the associated (normalized) Gauduchon metric $\tilde{\omega}$. By Corollary 3.8, the holomorphic vector bundle underlying $\mathcal{Q}$ is a-polystable. Let $f \colon M \to Z$ be as in the statement. Then, the differential $df \colon T^{1,0}M \to T^{1,0}Z$ induces a morphism

$$f^* T^*_{1,0} Z \longrightarrow T^*_{1,0} \xrightarrow{\pi^*} \mathcal{Q}.$$

Since df is surjective at one point, by Sard's Theorem there exists a dense open subset $R \subset Z$ of regular values of f. Using that M is connected and that f is holomorphic, it follows that $f^{-1}(R) \subset M$ is also dense (in fact, the complement is an analytic subspace of codimension ≥ 1). Hence, the previous morphism induces a subsheaf

$$f^* T^*_{1,0} Z \hookrightarrow \mathcal{Q}.$$

The inequality of slopes (3.4) now gives

$$0 \geq c_1(f^* T^* Z) \cdot a = -f^* c_1(Z) \cdot a.$$

As for the last part, if $c_1(M) \cdot a = 0$ then by Proposition 3.2 and Corollary 3.8 we have a holomorphic splitting $\mathcal{Q} = T^{1,0} \oplus T^*_{1,0}$ and hence $\partial\omega = 0$. $\square$

We next obtain a more concrete criterion derived from Theorem 3.9.

Corollary 3.10. *Let $f\colon (M,J) \to Z$ be a holomorphic map of compact, connected, complex manifolds. Assume that df is surjective at some point and that Z is Kähler with $c_1(Z) < 0$. Then M does not admit a Bismut Hermitian-Einstein pluriclosed metric.*

Proof. By Aubin-Yau's Theorem [2, 46] there exists a Kähler-Einstein metric ω_Z on Z with negative scalar curvature, that is, such that $\rho_Z = -\omega_Z$. Let $a = [\tilde{\omega}^{n-1}] \in H_A^{n-1,n-1}(M)$ be a positive Aeppli class on M. Then

$$f^* c_1(Z) \cdot a = -\int_X f^*\omega_Z \wedge \tilde{\omega}^{n-1}.$$

By hypothesis there exists $x \in M$ such that f is a submersion, and hence, arguing as in the proof of Theorem 3.9, the preimage of the set of regular values is open and dense. On this locus $f^*\omega_Z \wedge \tilde{\omega}^{n-1} > 0$, and hence $f^*c_1(Z) \cdot a < 0$. $\square$

3.3.1 Counter-examples

As a consequence of Corollary 3.10 we obtain examples of compact pluriclosed manifolds (M^{2n}, J) with $c_1(M) = 0 \in H^2(M,\mathbb{Z})$ which do not admit a Bismut Hermitian-Einstein metric. To the knowledge of the author, this is the first class of such examples in the literature for dimension $n \geq 3$ (the case $n = 2$ is settled by [14]). In order to present our examples we start with some general discussion of principal bundles over complex manifolds. Let Z be a Kähler manifold. Let $T = \mathbb{C}^n/\Lambda$ be an n-dimensional complex torus. Let

$$\delta\colon H^1(T,\mathbb{Z}) \to H^2(Z,\mathbb{Z})$$

be a homomorphism of $\mathbb{Z}$-modules such that $c_1(Z) \in \operatorname{Im}\delta$. We can identify δ with $c \in H^2(Z,\mathbb{Z})\otimes\Lambda$, and hence it determines a topologically non-trivial principal T-bundle $\pi\colon M \to Z$ with characteristic class c. Assuming further that $\operatorname{Im}\delta \subset H^{1,1}(Z)$, M can be endowed with a holomorphic structure. The first Chern class satisfies $c_1(M) = \pi^* c_1(Z) \in H^2(M,\mathbb{Z})$ and hence it vanishes because $c_1(Z) \in \operatorname{Im}\delta$ [20]. Furthermore, M is non-Kähler by Blanchard's Theorem.

Example 3.1. Consider the case that Z is a compact connected Riemann surface with genus ≥ 2, and hence $c_1(Z) < 0$. Let $\pi\colon M \to Z$ be a non-trivial principal T-bundle over M. For dimensional reasons, the condition $c_1(Z) \in \operatorname{Im}\delta \subset H^{1,1}(Z)$ is always satisfied, and hence $c_1(M) = 0 \in H^2(M,\mathbb{Z})$. Choose a principal connection $\theta = (\theta_1, \ldots, \theta_{2n})$ on M and define a T-invariant complex structure on M by $J\theta_{2j-1} = \theta_{2j}$. Choose a Kähler metric ω_Z on Z and consider the Hermitian form

$$\omega = \pi^*\omega_Z + \sum_{j=1}^{n} \theta_{2j-1} \wedge \theta_{2j}.$$

Then, we have

$$dd^c\omega = -d^c\left(\sum_{j=1}^{n} \pi^* F_{\theta_{2j-1}} \wedge \theta_{2j} - \theta_{2j-1} \wedge p^* F_{\theta_{2j}}\right) = 0$$

by dimensional reasons, where F_{θ_j} denotes the curvature of θ_j. Therefore, M is a pluriclosed manifold with vanishing first Chern class. Applying Corollary 3.10, we now conclude that M does not admit a Bismut Hermitian-Einstein metric.

Example 3.2. Let Z be an algebraic complex surface with $c_1(Z) < 0$ and let $T = \mathbb{C}/\Lambda$. By the Aubin-Yau Theorem we have ω_Z a Kähler-Einstein metric on Z with negative scalar curvature and $[\omega_Z] \in H^2(Z,\mathbb{Z})$. Choose α a primitive $(1,1)$-form with $[\alpha] \in H^2(Z,\mathbb{Z})$, and define δ so that its image is spanned by

$$\delta_1 = -[\omega_Z] = c_1(Z), \qquad \delta_2 = [\alpha].$$

On the corresponding T-bundle we can choose a connection $\theta = (\theta_1, \theta_2)$ with curvature

$$\frac{\sqrt{-1}}{2\pi} F_\theta = (-\omega_Z, \alpha),$$

and a T-invariant complex structure such that $J\theta_1 = \theta_2$. For any $u \in C^\infty(Z)$ we define the Hermitian metric $\omega = \pi^* e^u \omega_Z + \theta_1 \wedge \theta_2$. Then, a direct calculation shows that [16]

$$dd^c\omega = \pi^*(\Delta(e^u)\omega_Z^2 - \alpha^2 - \omega_Z^2),$$

and hence the existence of a pluriclosed metric reduces to solve

$$[\omega_Z]^2 = -[\alpha] \cdot [\alpha] \in \mathbb{N}.$$

Taking $\iota \colon Z \hookrightarrow \mathbb{P}^3$ a degree $d \geq 5$ projective hypersurface, we have $c_1(Z) = (4-d)\iota^* H$, for H the hyperplane class, and we obtain the condition

$$(d-4)^2 = -[\alpha] \cdot [\alpha]$$

for a primitive $(1,1)$-class $[\alpha]$. We have $H^2(Z,\mathbb{Z}) \cong \mathbb{Z}^{d(d(d-4)+6)-2}$ and, assuming that d is odd, the intersection pairing is given by the standard symmetric bilinear form with signature $-(d-2)d(d+2)/3$. Using Hirzebruch's formula for the Hodge numbers of projective hypersurfaces to calculate $h^{1,1}(M)$, one can prove that such a class always exists. For example, taking $d = 5$ one has $H^2(Z,\mathbb{Z}) \cong \mathbb{Z}^{53}$, $h^{1,1}(M) = 45$, and signature -35. Therefore, there exists a 36-dimensional subspace of primitive $(1,1)$-classes, and hence there is a primitive $(1,1)$-class $[\alpha]$ with $[\alpha] \cdot [\alpha] = -1$.

Chapter 4

Higher Regularity

A common technique in the regularity theory of fully nonlinear parabolic and elliptic partial differential equations is "bootstrapping". This refers to a technique in which one obtains C^∞-estimates by inductively controlling higher C^k-norms by lower C^k-norms. For example, for smooth, semilinear elliptic or parabolic equations with some structural assumptions, the Schauder estimate implies that control on the C^α-norm suffices to control all C^k norms.

In the case of smooth, fully non-linear equations, the bootstrapping process becomes somewhat more complicated because the coefficients of the linearized equation also depend on the unknown function. This process is usually simplified by the Evans-Krylov theorem for concave, uniformly elliptic / parabolic equations, but we are primarily interested in operators which either have no definite convexity or are not uniformly elliptic / parabolic. We will begin in §4.1 with a discussion of a well-known – non-potential theoretic – apriori estimate in the setting of Kähler-Ricci flow which makes bootstrapping possible. Then we will prove a similar result for pluriclosed flow – without potentials by necessity this time – in §4.2 following prior work of the author and Streets [23].

4.1 Kähler-Ricci Flow

In order to clarify the more general situation of pluriclosed flow, we will begin with some related apriori estimates for Kähler-Ricci flow due to Yau [46] and Cao [4]. To start, we define Käher-Ricci flow; it is simply the restriction of Ricci flow to Kähler initial conditions.

Definition 4.1. On a complex manifold (M, J), a family of metrics ω_t is said to be a solution to *Kähler-Ricci flow* on $[0, T)$ provided ω_0 is Kähler and

$$\frac{\partial}{\partial t}\omega_t = -\rho(\omega_t), \quad \forall t \in (0, T).$$

As a quick aside, this flow preserves the Kähler condition and is reduceable to a parabolic complex Monge-Amperé equation on Kähler manifolds.

$$\frac{\partial}{\partial t}\phi = \log \frac{(\omega_0 + \sqrt{-1}\partial\overline{\partial}\phi)^n}{\omega_0^n}$$

In order to apply make contact with the Schauder theory and prove solvability of this equation by the continuity method, Yau and Cao needed to show that estimates on the metric implied $C^{2,\alpha}$-estimates on the potential. In essence, they needed to show that a C^2-estimate on the potential implied a $C^{2,\alpha}$-estimate. The central insight of the proof of the estimate in the parabolic setting is the application of the rough heat operator of the time-evolving metric to the time-evolving norm of the difference of time-evolving and initial Levi-Civita connections. It turns out that this quantity has a maximum principle whenever the initial geometry is bounded. We state the ultimate estimate below and we present the crucial evolution equation in the following subsection.

Theorem 4.1. *(cf. [4]) Suppose that (M^{2n}, g) is a Kähler manifold with $c_1(M) = 0$ and $h = h(t)$ is a smooth Kähler-Ricci flow on $[0, T)$ with $h(0) = g$. Then, if there is a universal*

$\Lambda > 0$ *s.t.* $\Lambda^{-1} g \leq h \leq \Lambda g$ *on* $[0,T)$*, then* h *is* C^{α} *for* $t \in [0,T)$ *with an estimate.*

$$\|h\|_{C^{\alpha}} \leq C(\Lambda, n, \alpha, g)$$

To prove the estimate, we consider the evolution of $\Upsilon = \nabla - \nabla^0 \in \Lambda^1(\operatorname{End} T^{1,0}M)$ the difference of Levi-Civita connections. We will use the index convention $\Upsilon_{ij}^k = dz^k(\nabla_i \partial_j - \nabla_i^0 \partial_j)$.

Proposition 4.1. *Let* (M,J,g) *be Kähler and* $h = h_t$ *be a Kähler-Ricci flow with* $h_0 = g$*. Then the norm of the difference of Levi-Civita connections* $\Upsilon = \nabla - \nabla^0$ *satisfies the following evolution equation.*

$$(\frac{\partial}{\partial t} - \Delta)|\Upsilon|_h^2 = -|\overline{\nabla}\Upsilon|^2 - |\nabla\Upsilon|^2 + \overline{\Upsilon} * \Upsilon * \nabla\Upsilon + \overline{\Upsilon} * \Upsilon * \operatorname{Rm}^0 + \overline{\Upsilon} * \Upsilon * \operatorname{Ric}^0 + \overline{\Upsilon} * \nabla^0 \operatorname{Ric}^0$$

Proof. We begin computing in Kähler coordinates for h centered at (p,t). In this case, we have the following.

$$\frac{\partial}{\partial t}\Upsilon_{ij}^k = \frac{\partial}{\partial t} dz^k(\nabla_i \partial_j) = \frac{\partial}{\partial t}(h^{\overline{m}k} h_{i\overline{m},j}) = h^{\overline{m}k} \dot{h}_{i\overline{m},j} = h^{\overline{p}k} \nabla_j \dot{h}_{i\overline{p}} = -\nabla_j \operatorname{Ric}_i^k$$

It will also be useful to know how the rough h-Laplacian acts on Υ.

$$\begin{aligned}
(\Delta^h \Upsilon)_{ij}^k &= \partial_p \overline{\partial}_p [dz^k(\nabla_i \partial_j - \nabla_i^0 \partial_j)] \\
&= \partial_p dz^k [\overline{\nabla}_p \nabla_i \partial_j - \overline{\nabla}_p \nabla_i^0 \partial_j] \\
&= \partial_p dz^k [-\operatorname{Rm}_{i\overline{p}j}^l \partial_l - \overline{\nabla}_p^0 \nabla_i^0 \partial_j] \\
&= (\operatorname{Rm}^0)_{i\overline{p}j,p}^l - \operatorname{Rm}_{i\overline{p}j,p}^l \\
&= \nabla_p^0 (\operatorname{Rm}^0)_{i\overline{p}j}^l - \nabla_p \operatorname{Rm}_{i\overline{p}j}^l + \Upsilon * \operatorname{Rm}^0 \\
&= \nabla_i^0 (\operatorname{Rc}^0)_j^k - \nabla_i \operatorname{Rc}_j^k + \Upsilon * \operatorname{Rm}^0
\end{aligned}$$

The conjugated rough h-Laplacian will also be useful to know.

$$\begin{aligned}
\overline{\Delta}\Upsilon &= \Delta\Upsilon + (\overline{\Delta} - \Delta)\Upsilon \\
&= \Delta\Upsilon + (\overline{\nabla}_p\nabla_p - \nabla_p\overline{\nabla}_p)\Upsilon^k_{ij} \\
&= \Delta\Upsilon + \overline{\nabla}_p(\Upsilon^k_{ij,p} - \upsilon^k_{\lambda j}\Gamma^\lambda_{pi} - \Upsilon^k_{i\lambda}\Gamma^\lambda_{pj} + \Upsilon^\lambda_{ij}\Gamma^k_{p\lambda}) - \Upsilon^k_{ij,p\overline{p}} \\
&= \Delta\Upsilon - \Upsilon^k_{\lambda j}\Gamma^\lambda_{pi,\overline{p}} - \Upsilon^k_{i\lambda}\Gamma^\lambda_{pj,\overline{p}} + \Upsilon^\lambda_{ij}\Gamma^k_{p\lambda,\overline{p}} \\
&= \Delta\Upsilon - \Upsilon^k_{\lambda j}\Upsilon^\lambda_{pi,\overline{p}} - \Upsilon^k_{i\lambda}\Upsilon^\lambda_{pj,\overline{p}} + \Upsilon^\lambda_{ij}\Upsilon^k_{p\lambda,\overline{p}} + \Upsilon * \mathrm{Rc}^0 \\
&= \Delta\Upsilon + \Upsilon * \overline{\nabla}\Upsilon + \Upsilon * \mathrm{Ric}^0
\end{aligned}$$

Notice that, in particular, this yields the following rough heat equation.

$$(\frac{\partial}{\partial t} - \Delta_h)\Upsilon = \Upsilon * \mathrm{Rm}^0 - \nabla^0 \mathrm{Rc}^0$$

We can now use this to compute the heat operator acting on $|\Upsilon|^2$.

$$\begin{aligned}
(\frac{\partial}{\partial t} - \Delta)|\Upsilon|^2 = {}& 2\Re(h(\dot{\Upsilon}, \overline{\Upsilon})) + \frac{\partial}{\partial t}(h^{i\overline{p}}h^{j\overline{q}}h_{k\overline{r}})\Upsilon^k_{ij}\overline{\Upsilon}^{\overline{r}}_{\overline{pq}} \\
& - (2\Re h(\Delta\Upsilon, \overline{\Upsilon}) + |\overline{\nabla}\Upsilon|^2 + |\nabla\Upsilon|^2 + \overline{\Upsilon} * \Upsilon * \mathrm{Ric}^0 + \overline{\Upsilon} * \Upsilon * \nabla\Upsilon) \\
= {}& -|\overline{\nabla}\Upsilon|^2 - |\nabla\Upsilon|^2 + 2\Re h(\overline{\Upsilon}, \Upsilon * \mathrm{Rm}^0 - \nabla^0 \mathrm{Ric}^0) + \overline{\Upsilon} * \Upsilon * \mathrm{Ric}^0 + \overline{\Upsilon} * \Upsilon * \overline{\nabla}\Upsilon \\
= {}& -|\overline{\nabla}\Upsilon|^2 - |\nabla\Upsilon|^2 + \overline{\Upsilon} * \Upsilon * \overline{\nabla}\Upsilon + \overline{\Upsilon} * \Upsilon * \mathrm{Rm}^0 + \overline{\Upsilon} * \Upsilon * \mathrm{Ric}^0 + \overline{\Upsilon} * \nabla^0 \mathrm{Ric}^0
\end{aligned}$$

□

This estimate can be combined with the Bochner formula to prove a smoothing inequality of the form $|\Upsilon| = O(t^{-\frac{1}{2}})$ where the constants only depend on background geometry. Notice that $|\Upsilon|$ controls the size of third-derivatives of the form $\phi_{ij\overline{k}}$ and from here we can make contact with the Schauder theory via the scalar reduction.

4.2 Pluriclosed Flow

We begin by defining pluriclosed flow. It will look very similar to Kähler-Ricci flow except it will be necessary to consider pluriclosed metrics, the Bismut connection, and an auxiliary $(2,0)$-form β.

Definition 4.2. On a pluriclosed manifold (M, J, g), a family of pairs $(\omega(t), \beta(t)) \in \Lambda^{1,1}_{\mathbb{R}} \oplus \Lambda^{2,0}$ with $\omega(0) = g$ is said to be a solution of *pluriclosed flow* on $[0, T)$ provided

$$\frac{\partial}{\partial t}\omega = -\rho_B^{1,1}, \qquad \frac{\partial}{\partial t}\beta = -\rho_B^{2,0}. \tag{4.1}$$

For pluriclosed flow, we run into a couple difficulties that are not present in the Kähler-Ricci flow setting. The two that are most pressing are (1) we are forced to consider connections with torsion and (2) we can no longer use Kähler coordinates to simplify computations.

In this section, we will begin by showing in §4.2.2 the analogue of Proposition 4.1. We will then need to discuss a Bochner formula and metric-trace evolution in §4.2.3 before proving the smoothing estimate and bootstrapping in detail in §4.2.4 & 4.2.5.

4.2.1 Generalized Metric Evolution

Proposition 4.2. *Given (M^{2n}, J) and (ω, β) a solution to pluriclosed flow (4.1), the associated generalized metric G satisfies*

$$\frac{\partial}{\partial t}G = -S. \tag{4.2}$$

Proof. We will use the following equations

$$\rho_B^{1,1} = S^g - T^2 \qquad \rho_B^{2,0} = \partial\overline{\partial}^*\omega$$

These can be derived in using the Bianchi identity for the Chern curvature and type decomposing the identity $\rho_B - \rho_C = dd^*\omega$ (cf. [22]).

Furthermore, we can without loss of generality compute at a space-time point where β vanishes.

$$\begin{aligned}
\frac{\partial}{\partial t} G_{Z^i\overline{Z}^j} =& \frac{\partial}{\partial t} g_{i\bar{j}} + \frac{\partial}{\partial t}(\beta_{ik}\overline{\beta}_{\bar{j}\bar{l}} g^{\bar{l}k}) \\
=& -(\rho_B^{1,1})_{i\bar{j}} - \left((\rho_B^{2,0})_{ik}\overline{\beta}_{\bar{j}\bar{l}} + \beta_{ik}(\overline{\rho_B^{2,0}})_{\bar{j}\bar{l}}\right) g^{\bar{l}k} + \beta_{ik}\overline{\beta}_{\bar{j}\bar{l}}(\rho_B^{1,1})_{\mu\bar{\nu}} g^{\bar{l}\mu} g^{\bar{\nu}k} \\
=& -(S^g_{i\bar{j}} - T^2_{i\bar{j}}) \\
=& -S_{Z^i\overline{Z}^j}.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{\partial}{\partial t} G_{Z^i\overline{W}^j} &= \frac{\partial}{\partial t}(\beta_{ip} g^{\bar{j}p}) \\
&= g^{\bar{j}p}\frac{\partial}{\partial t}\beta_{ip} - \beta_{ip} g^{\bar{j}m} g^{\bar{n}p}\frac{\partial}{\partial t} g_{m\bar{n}} \\
&= -g^{\bar{j}p}(\rho_B^{2,0})_{ip} \\
&= -g^{\bar{j}p}(\partial\overline{\partial}^*\omega_{ip}) \\
&= g^{\bar{j}p}\Delta_g\beta_{ip} \\
&= -S_{Z^i\overline{W}^j}.
\end{aligned}$$

Further,

$$\begin{aligned}\frac{\partial}{\partial t} G_{W^i \overline{W}^j} &= \frac{\partial}{\partial t} g^{\bar{j}i} \\ &= g^{\bar{j}m} g^{\overline{n}i} (\rho_B^{1,1})_{m\overline{n}} \\ &= g^{\bar{j}m} g^{\overline{n}i} (S^g_{m\overline{n}} - T^2_{m\overline{n}}) \\ &= -S^G_{W^i\overline{W}^j}.\end{aligned}$$

$\square$

4.2.2 Connection Evolution

In this section, we will typically be working over the smooth vector bundle Q underlying $\mathcal{Q}_{\sqrt{-1}\partial\omega}$. It is easy to see that $Q \cong T^{1,0} \oplus \Lambda^{1,0}$ because $\mathcal{Q}$ is an exact holomorphic Courant algebroid. As such, we mention that upper-case indices range over all directions of Q whereas lower-case indices range only over $T^{1,0}$ directions with conjugation acting as one would expect. We will also find it necessary in some places to distinguish between S^g and S^G the second Ricci curvatures of $T^{1,0}$ and $\mathcal{Q}$ respectively.

Proposition 4.3. *Fix $(M^{2n}, \omega_t, \beta_t, J)$ a solution to pluriclosed flow, with G_t the associated family of generalized metrics on $\mathcal{Q}_{\sqrt{-1}\partial\omega}$. Given $\widetilde{G}$ a Hermitian metric on $\mathcal{Q}_{\sqrt{-1}\partial\omega}$, we have*

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) |\Upsilon(G,\widetilde{G})|^2_{g,G^{-1},G} =& -|\nabla\Upsilon|^2_{g,G^{-1},G} - |\overline{\nabla}\Upsilon + \overline{T}\cdot\Upsilon|^2_{g,G^{-1},G} \\ &+ T * \Upsilon * \Omega^{\widetilde{G}} + \Upsilon^g * \overline{\Upsilon} * \Omega^{\widetilde{G}} + \Upsilon * \nabla^{\widetilde{G}}\Omega^{\widetilde{G}},\end{aligned}$$

where

$$(\overline{T}\cdot\Upsilon)^{B}_{\bar{i}jA} = g^{\bar{k}l}\overline{T}_{\bar{i}\bar{k}j}\Upsilon^B_{lA}.$$

Proof. A general calculation for the variation of the Chern connection associated to a Hermitian metric yields

$$\frac{\partial}{\partial t}\Upsilon_{iA}^{B} = \nabla_i \frac{\partial}{\partial t} G_A^B.$$

Specializing this result using Proposition 4.2 we thus have

$$\frac{\partial}{\partial t}\Upsilon_{iA}^{B} = -\nabla_i S_A^B. \tag{4.3}$$

Using this and the differential Bianchi identity with torsion, we compute

$$\begin{aligned}
\Delta \Upsilon_{iA}^{B} &= g^{\bar{k}l}\nabla_l \nabla_{\bar{k}} \Upsilon_{iA}^{B} \\
&= g^{\bar{k}l}\nabla_l \left(\Omega_{\bar{k}iA}^{B} - \widetilde{\Omega}_{\bar{k}iA}^{B} \right) \\
&= g^{\bar{k}l}\nabla_i \Omega_{\bar{k}lA}^{B} + g^{\bar{k}l} T_{il}^{p} \Omega_{\bar{k}pA}^{B} + g^{\bar{k}l}\nabla_l \widetilde{\Omega}_{i\bar{k}A}^{B} \\
&= \partial_t \Upsilon_{iA}^{B} - g^{\bar{k}l} T_{il}^{p} \Omega_{p\bar{k}A}^{B} + g^{\bar{k}l}\widetilde{\nabla}_l \widetilde{\Omega}_{i\bar{k}A}^{B} + g^{\bar{k}l}\left(\Upsilon_{li}^{q}\widetilde{\Omega}_{q\bar{k}A}^{B} + \Upsilon_{lA}^{D}\widetilde{\Omega}_{i\bar{k}D}^{B} - \Upsilon_{lD}^{B}\widetilde{\Omega}_{i\bar{k}A}^{D} \right).
\end{aligned} \tag{4.4}$$

Next we observe the commutation formula

$$\begin{aligned}
\overline{\Delta \Upsilon}_{\bar{j}\bar{A}}^{\bar{B}} &= g^{\bar{l}k}\nabla_{\bar{l}}\nabla_k \overline{\Upsilon}_{\bar{j}\bar{A}}^{\bar{B}} \\
&= g^{\bar{l}k}\left[\nabla_k \nabla_{\bar{l}} \overline{\Upsilon}_{\bar{j}\bar{A}}^{\bar{B}} - (\Omega^g)_{k\bar{l}\bar{j}}^{\bar{q}} \overline{\Upsilon}_{\bar{q}\bar{A}}^{\bar{B}} - \Omega_{k\bar{l}\bar{A}}^{\bar{C}} \overline{\Upsilon}_{\bar{j}\bar{C}}^{\bar{B}} + \Omega_{k\bar{l}\bar{C}}^{\bar{B}} \overline{\Upsilon}_{\bar{j}\bar{A}}^{\bar{C}} \right] \\
&= \Delta \overline{\Upsilon}_{\bar{j}\bar{A}}^{\bar{B}} - (S^g)_{\bar{j}}^{\bar{q}} \overline{\Upsilon}_{\bar{q}\bar{A}}^{\bar{B}} - S_{\bar{A}}^{\bar{C}} \overline{\Upsilon}_{\bar{j}\bar{C}}^{\bar{B}} + S_{\bar{C}}^{\bar{B}} \overline{\Upsilon}_{\bar{j}\bar{A}}^{\bar{C}}.
\end{aligned} \tag{4.5}$$

Combining (4.3)-(4.5), the evolution equations of Proposition 4.2 and the pluriclosed flow equation we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}|\Upsilon|^2_{g^{-1},G^{-1},G} = {} & \frac{\partial}{\partial t}\left[g^{\bar{j}i}G^{\overline{C}A}G_{B\overline{D}}\Upsilon^B_{iA}\overline{\Upsilon}^{\overline{D}}_{\bar{j}\overline{C}}\right] \\
= {} & -g^{\bar{j}k}\left(-S^g_{k\bar{l}}+T^2_{k\bar{l}}\right)g^{\bar{l}i}G^{\overline{C}A}G_{B\overline{D}}\Upsilon^B_{iA}\overline{\Upsilon}^{\overline{D}}_{\bar{j}\overline{C}} - g^{\bar{j}i}G^{\overline{C}M}\left(-S^G_{M\overline{L}}\right)G^{\overline{L}A}\Upsilon^B_{iA}\overline{\Upsilon}^{\overline{D}}_{\bar{j}\overline{C}} \\
& + g^{\bar{j}i}G^{\overline{C}A}\left(-S^G_{B\overline{D}}\right)\Upsilon^B_{iA}\overline{\Upsilon}^{\overline{D}}_{\bar{j}\overline{C}} + g^{\bar{j}i}G^{\overline{C}A}G_{B\overline{D}}\Big(\Delta\Upsilon^B_{iA} + g^{\bar{k}l}(T^g)^p_{il}\Omega^B_{p\bar{k}A} \\
& -g^{\bar{k}l}\widetilde{\nabla}_l\widetilde{\Omega}^B_{i\bar{k}A} - g^{\bar{k}l}\left[\Upsilon^q_{li}\widetilde{\Omega}^B_{q\bar{k}A}+\Upsilon^M_{lA}\widetilde{\Omega}^B_{i\bar{k}M}-\Upsilon^B_{lM}\widetilde{\Omega}^M_{i\bar{k}A}\right]\Big)\overline{\Upsilon}^{\overline{D}}_{\bar{j}\overline{C}} \\
& + g^{\bar{j}i}G^{\overline{C}A}G_{B\overline{D}}\Upsilon^B_{iA}\Big(\overline{\Delta\Upsilon}^{\overline{D}}_{\bar{j}\overline{C}} - g^{\bar{k}l}(\overline{T}^g)^{\bar{q}}_{\bar{k}\bar{j}}\Omega^{\overline{D}}_{l\bar{q}\overline{C}} + g^{\bar{k}l}\widetilde{\nabla}_{\bar{k}}\widetilde{\Omega}^{\overline{D}}_{l\bar{j}\overline{C}} \\
& -g^{\bar{k}l}\left[\overline{\Upsilon}^{\bar{q}}_{\bar{k}\bar{j}}\widetilde{\Omega}^{\overline{D}}_{\bar{q}l\overline{C}}+\overline{\Upsilon}^{\overline{L}}_{\bar{k}\overline{C}}\widetilde{\Omega}^{\overline{D}}_{\bar{j}l\overline{L}}-\overline{\Upsilon}^{\overline{D}}_{\bar{k}\overline{L}}\widetilde{\Omega}^{\overline{L}}_{\bar{j}l\overline{C}}\right]\Big).
\end{aligned}
$$

We furthermore compute

$$
\begin{aligned}
\Delta|\Upsilon|^2_{g^{-1},G^{-1},G} &= g^{\bar{k}l}g^{\bar{j}i}G^{\overline{C}A}G_{B\overline{D}}\nabla_l\nabla_{\bar{k}}(\Upsilon^B_{iA}\overline{\Upsilon}^{\overline{D}}_{\bar{j}\overline{C}}) \\
&= \langle\Delta\Upsilon,\overline{\Upsilon}\rangle + \langle\Upsilon,\Delta\overline{\Upsilon}\rangle + |\nabla\Upsilon|^2 + |\overline{\nabla}\Upsilon|^2 \\
&= (S^g)^{\bar{q}}_{\bar{j}}\overline{\Upsilon}^{\overline{D}}_{\bar{q}\overline{C}}\Upsilon^B_{iA}g^{\bar{j}i}G^{\overline{C}A}G_{B\overline{D}} + S^{\overline{\Lambda}}_{\overline{C}}\overline{\Upsilon}^{\overline{D}}_{\bar{j}\overline{\Lambda}}\Upsilon^B_{iA}g^{\bar{j}i}G^{\overline{C}A}G_{B\overline{D}} \\
&\quad - S^{\overline{D}}_{\overline{\Lambda}}\overline{\Upsilon}^{\overline{\Lambda}}_{\bar{j}\overline{C}}\Upsilon^B_{iA}g^{\bar{j}i}G^{\overline{C}A}G_{B\overline{D}} + \langle\Delta\Upsilon,\overline{\Upsilon}\rangle + \langle\Upsilon,\overline{\Delta\Upsilon}\rangle + |\nabla\Upsilon|^2 + |\overline{\nabla}\Upsilon|^2.
\end{aligned}
$$

Subtracting the two equations above yields

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)|\Upsilon|^2 = {} & -|\nabla\Upsilon|^2 - |\overline{\nabla}\Upsilon|^2 \\
& + g^{\bar{j}i}g^{\bar{k}l}G^{\overline{C}A}G_{B\overline{D}}\Big(-(T^2)_{i\bar{k}}\Upsilon^B_{lA}\overline{\Upsilon}^{\overline{D}}_{\bar{j}\overline{C}} + T^p_{il}\Omega^B_{p\bar{k}A}\overline{\Upsilon}^{\overline{D}}_{\bar{j}\overline{C}} - \overline{T}^{\bar{q}}_{\bar{k}\bar{j}}\Omega^{\overline{D}}_{l\bar{q}\overline{C}}\Upsilon^B_{iA} \\
& +\Upsilon^B_{iA}\tilde{\nabla}_{\bar{k}}\tilde{\Omega}^{\overline{D}}_{l\bar{j}\overline{C}} - \overline{\Upsilon}^{\overline{D}}_{\bar{j}\overline{C}}\tilde{\nabla}_l\tilde{\Omega}^B_{i\bar{k}A} \\
& -\Upsilon^q_{li}\overline{\Upsilon}^{\overline{D}}_{\bar{j}\overline{C}}\widetilde{\Omega}^B_{q\bar{k}A} - \Upsilon^M_{lA}\overline{\Upsilon}^{\overline{D}}_{\bar{j}\overline{C}}\widetilde{\Omega}^B_{i\bar{k}M} + \Upsilon^B_{lM}\overline{\Upsilon}^{\overline{D}}_{\bar{j}\overline{C}}\widetilde{\Omega}^M_{i\bar{k}A} \\
& -\Upsilon^B_{iA}\overline{\Upsilon}^{\bar{q}}_{\bar{k}\bar{j}}\widetilde{\Omega}^{\overline{D}}_{\bar{q}l\overline{C}} - \Upsilon^B_{iA}\overline{\Upsilon}^{\overline{L}}_{\bar{k}\overline{C}}\widetilde{\Omega}^{\overline{D}}_{\bar{j}l\overline{L}} + \Upsilon^B_{iA}\overline{\Upsilon}^{\overline{D}}_{\bar{k}\overline{L}}\widetilde{\Omega}^{\overline{L}}_{\bar{j}l\overline{C}}\Big).
\end{aligned}
$$

Then, observing that the second through fifth terms above form a perfect square we arrive at

$$\left(\frac{\partial}{\partial t} - \Delta\right)|\Upsilon|^2 = -|\nabla\Upsilon|^2 - |\overline{\nabla}\Upsilon + T\cdot\Upsilon|^2 + T * \Upsilon * \tilde{\Omega} + \Upsilon^g * \overline{\Upsilon} * \tilde{\Omega} + \Upsilon * \tilde{\nabla}\tilde{\Omega},$$

as claimed. $\square$

4.2.3 Bochner Formula and Metric-trace Evolution

In this section we prove various useful formulae derived from the Bochner formula for sections of a holomorphic vector bundle. Let $\mathcal{W}$ be a holomorphic vector bundle over a complex manifold (M, J). We fix a Hermitian metric g on M and define the Chern Laplacian on functions $f \in C^\infty(M)$ by

$$\Delta f := \sqrt{-1}\Lambda_\omega \partial\overline{\partial} f.$$

Given a holomorphic section $w \in H^0(M, \mathcal{W})$, the classical Bochner formula states that

$$\Delta|w|_h^2 = |\nabla^h w|_{g,h}^2 - \left\langle S_g^h w, w\right\rangle_h \tag{4.6}$$

for any choice of Hermitian metric h on $\mathcal{W}$ with Chern connection ∇^h. Note that $|\nabla^h w|_{g,h}^2$ is calculated using the background Hermitian metric g jointly with the given Hermitian metric on the bundle.

We are interested in the application of formula (4.6) to the following general setup: Let $\pi_{\mathcal{E}} \colon \mathcal{E} \to M$ and $\pi_{\mathcal{F}} \colon \mathcal{F} \to N$ denote holomorphic vector bundles over complex manifolds M and N respectively. Suppose $\Phi : \mathcal{E} \to \mathcal{F}$ is a morphism of holomorphic vector bundles covering $\phi : M \to N$, i.e. there is a commutative diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\Phi} & \mathcal{F} \\ {\scriptstyle \pi_{\mathcal{E}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathcal{F}}} \\ M & \xrightarrow{\phi} & N. \end{array}$$

Given such a map, there is a tautological holomorphic section of $\mathcal{E}^* \otimes \phi^*\mathcal{F}$ which, by abuse of notation, we denote also

$$\Phi \in H^0(M, \mathcal{E}^* \otimes \phi^*\mathcal{F}).$$

Furthermore, any pair of Hermitian metrics G and $\widetilde{G}$ on $\mathcal{E}$ and $\mathcal{F}$, respectively, induce a Hermitian metric $\mathcal{G} = G^{-1} \otimes \phi^*\widetilde{G}$ on $\mathcal{E}^* \otimes \phi^*\mathcal{F}$. Observe that, for any choice of frames on $\mathcal{E}$ and $\mathcal{F}$, one has

$$|\Phi|^2_{\mathcal{G}} = (\phi^*\tilde{G}_{\alpha\overline{\gamma}})\Phi^\alpha_j\overline{\Phi^\gamma_k}G^{j\overline{k}}.$$

Furthermore, the induced Chern connection $\nabla^{C,G^{-1},\widetilde{G}}$ acts on Φ, defining a tensor $A(G, \widetilde{G}, \Phi)$ by

$$A(e) = (\nabla^{\mathcal{G}}\Phi)(e) = \phi^*\nabla^{\widetilde{G}}(\Phi e) - \Phi(\nabla^G e). \tag{4.7}$$

for any smooth section e of $\mathcal{E}$. As a direct application of (4.6) we obtain the following:

Lemma 4.2. *Let $\mathcal{E} \to M$ and $\mathcal{F} \to N$ denote holomorphic vector bundles over complex manifolds M and N respectively, and suppose $\Phi : \mathcal{E} \to \mathcal{F}$ is a holomorphic map of vector bundles covering $\phi : M \to N$. Given g a Hermitian metric on M, G a Hermitian metric on $\mathcal{E}$ and $\widetilde{G}$ a metric on $\mathcal{F}$, one has*

$$\Delta|\Phi|^2_{\mathcal{G}} = |A|^2_{g,G^{-1},\tilde{G}} + \left\langle \Phi \circ S^G_g - \phi^* S^{\tilde{G}}_g \circ \Phi, \Phi \right\rangle_{\mathcal{G}}.$$

We next apply next Lemma 4.2 to various situations of our interest. The simplest case is to apply it to the identity map of a fixed holomorphic vector bundle. We note that in this case the tensor $A = \nabla^{\mathcal{G}}\Phi$ is the difference of two Chern connections, and we use some more common notation for this:

Definition 4.3. Given $G, \widetilde{G}$ Hermitian metrics on a holomorphic vector bundle $\mathcal{E}$ over M, let

$$\Upsilon(G, \widetilde{G}) := \nabla^{G} - \nabla^{\widetilde{G}} \in T^*_{1,0} \otimes \mathrm{End}(\mathcal{E}).$$

denote the difference of the associated Chern connections. When taking the norm of Υ, we require a metric on $T^*_{1,0}$ as well as one on $\mathcal{E}$ and $\mathcal{E}^*$. These choices will be denoted explicitly as subscripts, using possibly a given metric and its inverse on both $\mathcal{E}$ and $\mathcal{E}^*$ if it is not explicitly indicated.

Lemma 4.3. *Let $\mathcal{E} \to M$ be a holomorphic vector bundle over a complex manifold M. Given g a Hermitian metric on M and G and $\widetilde{G}$ Hermitian metrics on $\mathcal{E}$, one has that*

$$\Delta(\mathrm{tr}_G \tilde{G}) = |\Upsilon(G, \widetilde{G})|^2_{g, G^{-1}, \widetilde{G}} + \mathrm{tr}_G \left\langle \left(S^G_g - S^{\tilde{G}}_g\right) \cdot, \cdot \right\rangle_{\tilde{G}}.$$

Proof. It follows from Lemma 4.2, setting $M = N$, $\mathcal{E} = \mathcal{F}$, and $\Phi = \mathrm{Id}$. □

We next consider specifically the case of generalized Hermitian metrics on exact holomorphic Courant algebroids, as in Proposition 2.16.

Lemma 4.4. *([7, Lemma 5.4], [45, c.f.]) Let (M, g_0, J) be a pluriclosed manifold. Consider the associated orthogonal holomorphic vector bundle $\mathcal{Q} = \mathcal{Q}_{\sqrt{-1}\partial\omega_0}$ as in Definition 2.8, and a generalized Hermitian metric $G = G(\omega, \beta)$ as in Proposition 2.16. Choose an arbitrary Hermitian metric $\widetilde{G}$ on $\mathcal{Q}$. Then, one has that*

$$\Delta(\mathrm{tr}_G \tilde{G}) = |\Upsilon(G, \widetilde{G})|^2_{g, G^{-1}, \widetilde{G}} + \mathrm{tr}_G \left\langle \left(S^G_g - S^{\widetilde{G}}_g\right) \cdot, \cdot \right\rangle_{\tilde{G}}.$$

Furthermore, provided that we take $\widetilde{G} = G(\omega_0, 0)$, one also has that

$$
\begin{aligned}
\psi_{g_0}^{-1} \circ \Upsilon(G, \widetilde{G}) \circ \psi_g &= (\varphi_{g_0,g,\beta}) \cdot D_g^- - D_{g_0}^- \\
\operatorname{tr}_G \tilde{G} = \operatorname{tr}_{\tilde{G}} G &= \operatorname{tr}_g g_0 + \operatorname{tr}_{g_0} g + |\beta|^2_{g,g_0}
\end{aligned}
$$

where ψ_g, ψ_{g_0} are as in Section 3.2, and $(\varphi_{g_0,g,\beta}) \cdot D_g^-$ denotes the action on D_g^- of the complex gauge transformation $\varphi_{g_0,g,\beta} \in \operatorname{End}(TM \otimes \mathbb{C})$, given by

$$\varphi_{g_0,g,\beta}(X) = X^{1,0} + g_0^{-1}(gX^{0,1} + \sqrt{-1}\beta X^{1,0}).$$

Proof. The first part of the statement is a special case of Lemma 4.3. Assuming now $\widetilde{G} = G(\omega_0, 0)$ and setting $G' = G(\omega, 0)$, one has that

$$\operatorname{tr}_G \tilde{G} = |e^{-\sqrt{-1}\beta}|^2_{G'^{-1},\tilde{G}} = (g_0)_{j\bar{k}} g^{j\bar{k}} + g_{j\bar{k}} g_0^{j\bar{k}} + g_0^{l\overline{m}} g^{j\bar{k}} \beta_{jl} \overline{\beta_{km}} = \operatorname{tr}_g g_0 + \operatorname{tr}_{g_0} g + |\beta|^2_{g,g_0}.$$

Finally, using Proposition 2.16, one has that

$$
\begin{aligned}
\Upsilon(G, \widetilde{G}) &= (e^{\sqrt{-1}\beta})^* \nabla^{G'} - \nabla^{\widetilde{G}} \\
&= (e^{-\sqrt{-1}\beta} \circ \psi_g)_* D_g^- - (\psi_{g_0})_* D_{g_0}^- \\
&= \psi_{g_0} \circ ((\psi_{g_0}^{-1} \circ e^{-\sqrt{-1}\beta} \circ \psi_g) \cdot D_g^- - D_{g_0}^-) \circ \psi_{g_0}^{-1}
\end{aligned}
$$

and also that

$$
\begin{aligned}
\psi_{g_0}^{-1} \circ e^{-\sqrt{-1}\beta} \circ \psi_g(X) &= \psi_{g_0}^{-1} \circ e^{-\sqrt{-1}\beta}(X^{1,0} - gX^{0,1}) \\
&= \psi_{g_0}^{-1}(X^{1,0} - gX^{0,1} - \sqrt{-1}\beta X^{1,0}) \\
&= X^{1,0} + g_0^{-1}(gX^{0,1} + \sqrt{-1}\beta X^{1,0}).
\end{aligned}
$$

$\square$

We can then apply Lemma 4.4 to a solution of pluriclosed flow to get the following evolution of the metric-trace.

Proposition 4.4. *([7, Lemma 5.6]) Fix $(M^{2n}, \omega_t, \beta_t, J)$ a solution to pluriclosed flow, with G_t the associated family of generalized Hermitian metrics on $\mathcal{Q}_{\sqrt{-1}\partial\omega_0}$. Given $\widetilde{G}$ a Hermitian metric on $\mathcal{Q}_{\sqrt{-1}\partial\omega_0}$, we have*

$$\left(\frac{\partial}{\partial t} - \Delta\right) \operatorname{tr}_G \widetilde{G} = -\left|\Upsilon(\widetilde{G}, G)\right|^2_{g, G^{-1}, \widetilde{G}} + \operatorname{tr}_G \left\langle S_g^{\widetilde{G}} \cdot, \cdot \right\rangle_{\widetilde{G}}.$$

4.2.4 Smoothing Estimate

It is fairly easy to spot that – for bounded geometry background data – the unsigned terms in Proposition 4.3 are controlled by a multiple of the negative term in Proposition 4.4, which suggests that a careful combination (along with some assumptions on the background geometry) should be a sub-solution to the heat equation. This turns out to be the case and the allows us to show that the metric gets smoother over time. We begin with an algebraic to ease some later computations.

Lemma 4.5. *Suppose (M^{2n}, J) is a complex manifold, and suppose (ω, β) and $(\widetilde{\omega}, \widetilde{\beta})$ are pluriclosed metrics on M such that the associated generalized Hermitian metrics G and $\widetilde{G}$ are defined on the same holomorphic Courant algebroid $\mathcal{Q}$ and satisfy*

$$\Lambda^{-1}\tilde{G} \leq G \leq \Lambda\tilde{G}, \qquad |\Upsilon(G, \tilde{G})|_{g,G} < \Lambda.$$

Then there exists a constant $A = A(n, \Lambda)$ such that

$$A^{-1}\widetilde{g} \leq g \leq A\widetilde{g}, \qquad |\beta|_{\widetilde{g}} \leq A, \qquad |\Upsilon(g, \widetilde{g})|_{\widetilde{g}} \leq A.$$

Proof. By the assumed uniform equivalence of G and $\tilde{G}$ and their explicit expressions from Proposition 2.16, it follows that for $\xi \in \Lambda^{1,0}$ we have

$$\Lambda^{-1}\tilde{g}^{-1}(\xi,\overline{\xi}) = \Lambda^{-1}\tilde{G}(\xi,\overline{\xi}) \leq G(\xi,\overline{\xi}) = g^{-1}(\xi,\overline{\xi}) \leq \Lambda\tilde{G}(\xi,\overline{\xi}) = \Lambda\tilde{g}^{-1}(\xi,\overline{\xi}).$$

This implies the claimed uniform equivalence of g and $\widetilde{g}$. A similar argument using sections of the tangent bundle then yields the upper bound for $|\beta|_{\widetilde{g}}$.

To estimate the connection, we compute the tangent-cotangent connection coefficients.

$$(\Upsilon^G)_{iW^a}^{Z^b} = g^{\overline{c}b}g^{\overline{q}a}\overline{T}_{\overline{qc}i} - \tilde{g}^{\overline{c}b}\tilde{g}^{\overline{q}a}\overline{\tilde{T}}_{\overline{qc}i}.$$

Taking norms and using the uniform equivalence of g and $\widetilde{g}$ and the estimate for $\Upsilon(G,\widetilde{G})$ we obtain

$$|T|_{\widetilde{g}} \leq C.$$

Now turning to the tangent-tangent components, we find

$$(\Upsilon^g)_{ia}^b = (\Upsilon^G)_{iZ^a}^{Z^b} - \sqrt{-1}g^{\overline{c}b}g^{\overline{q}p}\beta_{ap}\overline{T}_{\overline{qc}i} + \sqrt{-1}\tilde{g}^{\overline{c}b}\tilde{g}^{\overline{q}p}\tilde{\beta}_{ap}\overline{\tilde{T}}_{\overline{qc}i}.$$

We have estimated all terms on the right hand side of this equation, thus the estimate for $\Upsilon(g,\widetilde{g})$ follows. $\square$

Proposition 4.5. *Given (M^{2n},J) compact, fix (ω,β) a solution to pluriclosed flow (4.1), with associated generalized metric G. Fix a background metric generalized metric $\widetilde{G}(\widetilde{g},\widetilde{\beta})$ such that $\Lambda^{-1}\tilde{G} \leq G \leq \Lambda\tilde{G}$, then*

$$\max_{M\times\{t\}} |\Upsilon|^2 \leq C(1+\frac{1}{t}).$$

Proof. Notice that Φ satisfies the following evolution equation.

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\Phi &= |\Upsilon|^2 + t\left(\frac{\partial}{\partial t}-\Delta\right)|\Upsilon|^2 + A\left(\frac{\partial}{\partial t}-\Delta\right)\operatorname{tr}_G \tilde{G} \\ &\leq (1-A)|\Upsilon|^2 + t(T*\Upsilon*\tilde{\Omega}+\Upsilon^g*\overline{\Upsilon}*\tilde{\Omega}+\Upsilon*\tilde{\nabla}\tilde{\Omega}) + A\operatorname{tr}_G\langle S_g^{\tilde{G}}\cdot,\cdot\rangle_{\tilde{G}} \\ &\leq (1-A+C)|\Upsilon|^2 + CA + C.\end{aligned}$$

Thus, by taking A sufficiently large, we find

$$\left(\frac{\partial}{\partial t}-\Delta\right)\Phi \leq C.$$

This then implies

$$\max_{M\times\{t\}}\Phi \leq C(1+t).$$

But then we can simply reorganize to find

$$\max_{M\times\{t\}}|\Upsilon|^2 \leq C\left(1+\frac{1}{t}\right).$$

□

It is worth mentioning that this argument admits localization by way of a smooth cut off function, in which case the estimate becomes

$$\max_{B_{\frac{r}{2}}(p)\times\{t\}}|\Upsilon|^2 \leq C(r^{-4}+\frac{1}{t}).$$

The proof proceeds similarly, but becomes more technical and is for that reason omitted. The interested reader can find the proof of the localized estimate here [23].

4.2.5 Proof of Theorem 1.3

As a generalized metric G coming from a solution to pluriclosed flow satisfies the quasilinear equation

$$\frac{\partial}{\partial t} G_{A\overline{B}} = g^{i\bar{j}} G_{A\overline{B},i\bar{j}} + \partial G * \overline{\partial} G (= -S_{A\overline{B}}), \tag{4.8}$$

it will suffice to get a $C^{1,\alpha}$-estimate on G to make contact with the Schauder theory. We can now use the smoothing estimate, Lemma 4.4, and a blow-up argument to reach $C^{1,\alpha}$ and break through the bootstrapping threshold.

Proof of Theorem 1.3. The case $k = 0$ is established in Proposition 4.5. We now consider a function related to the C^2-norm of G.

$$f := |\Upsilon|^2 + |\nabla \Upsilon|$$

We want to show that $f = O\left(1 + \frac{1}{t}\right)$. Suppose to the contrary that there are $t_i \nearrow 1$ and points $p_i \in M$ so that

$$\sup_{M \times [0,t_i)} \frac{t f(p,t)}{1+t} = \frac{t_i f(p_i, t_i)}{1 + t_i} = \nearrow \infty.$$

Let $\sigma_i = f(p_i, t_i) \nearrow \infty$ and define rescaled metrics G_i on $B_1(0) \times [-1, 0]$ for i sufficiently large.

$$G_i(x,t) = G(p_i + \frac{x}{\sqrt{\sigma_i}}, t_i + \frac{t}{\sigma_i})$$

Define $\tilde{G}_i$ similarly. These metrics are so constructed that

$$\frac{\partial}{\partial t} G_i = -S_i.$$

For this family of flows, we can define f-functions as well. So, define

$$f_i = |\Upsilon(G_i, \tilde{G}_i)|^2 + |\nabla^{G_i}\Upsilon(G_i, \tilde{G}_i)|.$$

By the homogeneity of these quantities, we find that

$$f_i(x,t) = \frac{1}{\sigma_i} f(p_i + \frac{x}{\sqrt{\sigma_i}}, t_i + \frac{t}{\sigma_i}).$$

In particular, $f_i(0,0) = 1$.

However, provided that $t_i\sigma_i \geq 2$ we have that

$$t_i + \frac{t}{\sigma_i} \geq \frac{t_i}{2}, \quad \forall t \in [-1,0].$$

But at such points, we may estimate as follows.

$$\begin{aligned}
\frac{t_i f_i(x,t)}{2(1+t_i)} &\leq \frac{\left(t_i + \frac{t}{\sigma_i}\right) f_i(x,t)}{1 + \left(t_i + \frac{t}{\sigma_i}\right)} \\
&\leq \frac{t_i}{1+t_i} f_i(0,0)
\end{aligned}$$

Thus, $f_i \leq 2$ on $B_1(0) \times [-1,0]$ when i is sufficiently large.

However, this is a $C^{1,\alpha'}$-estimate for G_i uniform in i. But applying the Schauder estimates to Equation 4.8 upgrades our uniform $C^{1,\alpha'}$-estimates to uniform $C^{2,\alpha'}$-estimates on $B_{\frac{1}{2}}(0) \times [-\frac{1}{2}, 0]$. Then, by Arzela-Ascoli we obtain a $C^{2,\alpha''}$-limit G_∞. The strength of this convergence implies $f_i \to f_\infty$ uniformly on $B_{\frac{1}{2}}(0) \times [-\frac{1}{2}, 0]$ where f_∞ is the f-function associated to G_∞. Therefore, in particular

$$f_\infty(0,0) = \lim_{i\to\infty} f_i(0,0) = 1.$$

However, we also have Proposition 4.5 which implies that

$$|\Upsilon(G_i, \tilde{G}_i)|^2(x,t) \leq \frac{1}{\sigma_i}|\Upsilon(G, \tilde{G})|^2(p_i + \frac{x}{\sqrt{\sigma_i}}, t_i + \frac{t}{\sigma_i}) \leq \frac{C}{\sigma_i} \to 0.$$

Thus, where they exist, it must be that G_∞ and $\tilde{G}_\infty$ must have the same connections. However, $\tilde{G}_\infty$ is flat (as it is the blow-up of a bounded geometry metric), so G_∞ must be flat as well. Therefore, $f_\infty(0,0) = 0$, which is a contradiction.

This proves $f = O(1 + \frac{1}{t})$ on $[0, T)$. To see that this applies for derivative estimates of all orders, one need only differentiate Equation 4.8 and apply the Schauder estimates. $\square$

Much like Proposition 4.5, this theorem may be localized as well and one ends up with

$$\max_{B_{\frac{r}{2}}(p)\times\{t\}} \sum_{i=0}^{k} |\nabla^i \Upsilon|^{\frac{2}{i+1}} \leq C(r^{-4} + \frac{1}{t}).$$

Chapter 5

Long-time Existence

For several reasons that have already been discussed, the long-time existence theory of pluriclosed flow is incredibly delicate. Before 2021, the long-time existence and convergence even on the Hopf surface was unknown and all known long-time existence results relied on specific ansatz or potential-theoretic formulations of the flow (c.f. [35, 39]). The results that follow can be found in [7].

5.1 Bismut-Flat Manifolds

Referring back to Propositions 4.4 & 4.3 we see that the evolution equations for the metric-trace and Chern connection of a generalized metric simplify substantially on Bismut-flat backgrounds, due to Bismut's Identity (Proposition 3.2). In fact, these quantities become subsolutions of rough heat equations – improving our previous estimates substantially.

Proof of Theorem 1.4. Let ω_F denote the given Bismut-flat metric, and let $\mathcal{Q}_{\sqrt{-1}\partial\omega_F}$ denote the holomorphic Courant algebroid associated to $[\partial\omega_F]$. Furthermore, let G_F denote the

Hermitian metric on $\mathcal{Q}_{\sqrt{-1}\partial\omega_F}$ associated to ω_F via Proposition 2.16. Now given ω_0 another pluriclosed metric satisfying $[\partial\omega_0] = [\partial\omega_F] \in H^{2,1}_{\overline{\partial}}$, we can choose $\beta \in \Lambda^{2,0}$ such that

$$\overline{\partial}\beta = \partial\omega_F - \partial\omega_0.$$

Let G_0 denote the metric associated to (ω_0, β_0) as in Proposition 2.16. By [38], there exists $\epsilon > 0$ and a solution G_t to pluriclosed flow with initial data G_0 on $[0, \epsilon)$. Since ω_F is Bismut-flat, it follows from Proposition 3.2 that the Chern curvature of G_F vanishes, and thus we obtain from Proposition 4.4 the evolution equations

$$\left(\frac{\partial}{\partial t} - \Delta\right) \operatorname{tr}_G G_F = -\left|\Upsilon(G_F, G)\right|^2_{g,G^{-1},G_F}. \tag{5.1}$$

It follows from the maximum principle that, for any interval $[0, T]$ on which the solution exists,

$$\sup_{M\times[0,T]} \operatorname{tr}_G G_F \leq \sup_{M\times\{0\}} \operatorname{tr}_G G_F.$$

We note that for two generalized Hermitian metrics G, $\widetilde{G}$ on a fixed holomorphic Courant algebroid it follows that $\operatorname{tr}_G \widetilde{G} = \operatorname{tr}_{\widetilde{G}} G$ (see Lemma 4.4). Thus there exists a uniform constant $\Lambda > 0$ so that for any time t one has

$$\Lambda^{-1} G_F \leq G_t \leq \Lambda G_F. \tag{5.2}$$

Furthermore, let $\Upsilon = \Upsilon(G_t, G_F)$ as in Definition 4.3. Again using that the Chern curvature of G_F vanishes, It follows from Proposition 4.3 that

$$\left(\frac{\partial}{\partial t} - \Delta\right) |\Upsilon|^2_{g,G} = -\left|\nabla\Upsilon\right|^2_{g,G} - \left|\overline{\nabla}\Upsilon + \overline{T}\cdot\Upsilon\right|^2_{g,G}. \tag{5.3}$$

It follows from the maximum principle that for any interval $[0, T]$ on which the solution exists,

$$\sup_{M\times[0,T]} |\Upsilon(G, G_F)|^2_{g,G} \leq \sup_{M\times\{0\}} |\Upsilon(G, G_F)|^2_{g,G}.$$

Using Lemma 4.5 we thus obtain uniform equivalence and a C^1 bound for the classical objects (ω_t, β_t). We can now argue as in §4.2.5, there are uniform C^∞ estimates for G_t and g_t for all times. Thus the flow exists for all time, finishing the claim of long-time existence.

To show convergence we first note that by putting together (5.1) and (5.3), and using the uniform equivalence estimate of (5.2), it follows that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(t|\Upsilon(G, G_F)|^2_{g,G} + \operatorname{tr}_{G_F} G\right) \leq 0.$$

It follows from the maximum principle that for any time $t > 0$ one has

$$\sup_{M\times\{t\}} |\Upsilon(G, G_F)|^2_{g,G} \leq \frac{\sup_{M\times\{0\}} \operatorname{tr}_{G_F} G}{t}. \tag{5.4}$$

Using the uniform C^∞ estimates for G_t, every sequence of times $t_j \to \infty$ contains a subsequence which converges to a limiting metric G_∞. By (5.4) it follows that $\Upsilon(G_\infty, G_F) = 0$, and thus G_∞ is Chern-flat. Choosing such a flat limit G_∞, we can repeat the above analysis with G_F replaced by G_∞. In particular, for a large time t such that $|G_t - G_\infty|_{G_\infty} \leq \epsilon$, we have

$$2n \leq \sup_{M\times\{t\}} \operatorname{tr}_{G_\infty} G \leq 2n + C\epsilon$$

for some uniform constant C. These inequalities will be preserved for all times larger than t by (5.1), and then the convergence to G_∞ follows. $\square$

5.1.1 Hopf Surface

As an immediate application of Theorem 1.4, we can now show long-time existence on Hopf surfaces. To get a concrete limit, we will makes use of our knowledge of the cohomology of such surfaces. We will begin by recalling the construction of Hopf surfaces.

Definition 5.1. Given complex numbers α, β satisfying $|\alpha| \leq |\beta| < 1$, we obtain a *Hopf surface* via

$$M_{\alpha\beta} = \mathbb{C}^2 \backslash \{\mathbf{0}\} / \langle (z_1, z_2) \to (\alpha z_1, \beta z_2) \rangle .$$

For any α, β this is a complex manifold diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$. We will call such a surface *standard* if $|\alpha| = |\beta|$. Standard Hopf surfaces admit a natural metric which is pluriclosed with respect to the complex structure induced by the quotient structure, known as the Hopf/Boothby metric.

$$\omega_{HB} = \frac{1}{|z|^2}\omega_{Euc}$$

This metric is the unique (up to scaling) bi-invariant Hermitian metric for $S^3 \times S^1 \cong SU(2) \times U(1)$.

The standard Hopf surfaces equipped with their natural metrics are Bismut-flat. We check this for completeness.

Lemma 5.1. *The Hopf/Boothby metric is Bismut-flat.*

Proof. We can compute the coefficients Γ_{ij}^k of the Chern connection the usual way. The only non-zero coefficients are the following.

$$\Gamma_{11}^1 = \Gamma_{12}^2 = -\frac{\overline{z}^1}{|z|^2}, \qquad \Gamma_{22}^2 = \Gamma_{21}^1 = -\frac{\overline{z}^2}{|z|^2}$$

Then we can compute the non-zero Chern curvatures

$$\Omega_{1\bar{2}1\bar{1}} = \Omega_{1\bar{2}2\bar{2}} = -\frac{\overline{z}^1 z^2}{|z|^6}.$$

The torsion can then be computed using $T_{ij\bar{k}} = (\Gamma^p_{ij} - \Gamma^p_{ji})g_{p\bar{k}}$.

$$T_{12\bar{1}} = \frac{\overline{z}^2}{|z|^4}, \qquad T_{12\bar{2}} = -\frac{\overline{z}^1}{|z|^4}.$$

Applying Lemma 2.11 and the computation of T gives

$$R^+_{1\bar{1}i\bar{j}} = \Omega_{i\bar{j}1\bar{1}} - T_{12\bar{j}} g^{\bar{2}2} \overline{T}_{\bar{1}\bar{2}i}.$$

Thus, $R^+_{1\bar{1}1\bar{1}} = R^+_{1\bar{1}2\bar{2}} = 0$ obviously. Also,

$$R^+_{1\bar{1}1\bar{2}} = \Omega_{1\bar{2}1\bar{1}} - T_{12\bar{2}} g^{\bar{2}2} \overline{T}_{\bar{1}\bar{2}1} = -\frac{\overline{z}^1 z^2}{|z|^6} + \frac{\overline{z}^1 z^2}{|z|^6} = 0.$$

A similar calculation will give $R^+_{2\bar{2}ij} = 0$. In addition,

$$R^+_{1\bar{2}i\bar{j}} = \Omega_{i\bar{j}1\bar{2}} - T_{1\alpha\bar{j}} g^{\alpha\bar{\beta}} \overline{T}_{\bar{2}\bar{\beta}i} = 0.$$

Thus $R^+_{i\bar{j}} = 0$.

Now we consider terms of the form

$$R^+_{ijk\bar{l}} = \nabla_k T_{ij\bar{l}}.$$

Using the computation of T, we get that the only terms not obviously vanishing are

$$R^+_{12k\bar{1}} = \nabla_k T_{12\bar{1}}, \quad R^+_{12k\bar{2}} = \nabla_k T_{12\bar{2}}.$$

Thus, we can compute using the terms of T and the coefficients of ∇

$$\begin{aligned}
R^{+}_{121\bar{1}} &= T_{12\bar{1},1} - T_{12\bar{1}}\Gamma^{1}_{11} - T_{12\bar{1}}\Gamma^{2}_{12} \\
&= -2\frac{\bar{z}^{1}\bar{z}^{2}}{|z|^{6}} - \frac{\bar{z}^{2}}{|z|^{4}}\left(-2\frac{\bar{z}^{1}}{|z|^{2}}\right) \\
&= 0.
\end{aligned}$$

Similar computations hold for the other components of R^{+}_{12}. Thus, $R^{+} \equiv 0$. $\square$

Proof of Corollary 1.5. Recall that $h^{1,1}_{A}(M_{\alpha\beta}) = 1$ (see the proof of [1, Theorem 3.3]) and that the Aeppli class of a pluriclosed Hermitian metric on a compact complex surface is non-zero (see e.g. [30]). Therefore, given any pluriclosed metric ω on $M_{\alpha\beta}$ one has $[\omega] = \lambda[\omega_{\mathrm{Hopf}}]$ for $0 < \lambda \in \mathbb{R}$, where the positivity of λ follows e.g. by integration on a holomorphic curve (cf. below). Using now that $[\partial\omega] \in H^{2,1}_{\bar{\partial}}$ factorizes through the natural map

$$H^{1,1}_{A} \to H^{2,1}_{\bar{\partial}} \colon [\omega] \mapsto [\partial\omega],$$

we have that Theorem 1.4 applies giving convergence of the pluriclosed flow to a Bismut-flat structure for any initial data. Due to the classification by Gauduchon-Ivanov [14] (cf. also [10, Theorem 8.26]), it follows that ω_{Hopf} is the unique Bismut-flat metric in its Aeppli class, thus the limiting metric is a scalar multiple of the Hopf metric. $\square$

5.2 Non-negative Kodaira dimension

In this subsection we prove Theorem 1.6. A key point is to find a background metric with certain curvature properties for every choice of $\mathcal{Q}$. To begin, we show that every class in $H^{1,1}_{BC}$ is represented by an invariant form. We build on an observation of Teleman, which gives an explicit characterization of the failure of the $\partial\bar{\partial}$-lemma on compact complex surfaces.

Lemma 5.2 ([42])**.** *Let (M^4, J) be a compact complex surface, and let*

$$B^{1,1}_{\mathbb{R}} = \{\mu \in \Lambda^{1,1}_{\mathbb{R}} \mid \exists a \in \Lambda^1_{\mathbb{R}} = da\}.$$

Then there exists an exact sequence

$$0 \to \sqrt{-1}\partial\overline{\partial}\Lambda^0_{\mathbb{R}} \to B^{1,1}_{\mathbb{R}} \to \mathbb{R},$$

where the final map is the L^2 inner product with a pluriclosed metric.

Lemma 5.3. *Let (M^4, J) be a compact complex surface which is the total space of a holomorphic principal T^2-bundle. Given ω a pluriclosed metric on M there exists a T^2-invariant metric in $[\omega] \in H^{1,1}_A$.*

Proof. First choose $\hat{\omega}$ a T^2-invariant pluriclosed metric on M, which always exists by averaging an arbitrary pluriclosed metric over the T^2-action. We first use this to show that every class in $H^{1,1}_{BC}$ admits T^2-invariant representatives. Now fix $[\phi] \in H^{1,1}_{BC}$. We can define

$$\hat{\phi} := \int_{g \in T^2} g^* \phi.$$

The form $\hat{\phi}$ is T^2-invariant, and to show that $[\hat{\phi}] = [\phi] \in H^{1,1}_{BC}$, it suffices by a standard chain-homotopy argument to show that for X any vector field tangent to the T^2-action, one has $L_X \phi \in \sqrt{-1}\partial\overline{\partial}\Lambda^0_{\mathbb{R}}$. By Lemma 5.2, it suffices to show that the L^2 inner product with $\hat{\omega}$ vanishes. Using Stokes Theorem and the Cartan formula we compute

$$\langle L_X \phi, \hat{\omega} \rangle_{\hat{\omega}} = \int_M L_X \phi \wedge \hat{\omega} = \int_M L_X (\phi \wedge \hat{\omega}) = \int_M d i_X (\phi \wedge \hat{\omega}) = 0,$$

as required. Now knowing this, we assume that $\hat{\omega}$ is the average of ω over the T^2-action, and show that $[\hat{\omega}] = [\omega] \in H^{1,1}_A$. It suffices to prove that the infinitesimal action preserves Aeppli cohomology classes, and for this we use that integration gives a perfect pairing between $H^{1,1}_A$

and $H^{1,1}_{BC}$. Thus we fix $[\phi] \in H^{1,1}_{BC}$ with a T^2-invariant representative $\hat{\phi}$, and integrate by parts to conclude

$$\int_M L_X\omega \wedge \hat{\phi} = -\int_M \omega \wedge d(i_X\hat{\phi}) = -\int_M \omega \wedge L_X\hat{\phi} = 0,$$

as required. □

Next we record a key lemma computing the Bismut curvature tensor of a T^2-invariant pluriclosed metric. Such invariant metrics are described by the Kaluza-Klein ansatz (cf. [36, Definition 5.1]), and the curvature computation below is implicit in [36, Proposition 5.12].

Lemma 5.4. *Let (M^4, J) be a compact complex surface which is the total space of a holomorphic principal T^2-bundle over a Riemann surface Σ. Let ω denote a T^2-invariant pluriclosed metric on M, expressed as*

$$\omega = \pi^*\omega_\Sigma + \mathrm{tr}_h\, \mu \wedge J\mu,$$

where ω_Σ is a metric on Σ, $\mu + J\mu$ is a Hermitian connection, and h is an inner product on $\mathfrak{t}^2$. Then

$$\Omega^B = \tfrac{1}{2}\left(R_{\omega_\Sigma} - |F_\mu|^2_{g_\Sigma,h}\right)\pi^*\omega_\Sigma \otimes \pi^*\omega_\Sigma + h\left(d\,\mathrm{tr}_{\omega_\Sigma} F_\mu, \cdot\right) \otimes \pi^*\omega_\Sigma.$$

Proof. We begin by computing the Levi-Civita connection coefficients in a co-frame $\{\phi^1, \phi^2 = J\phi^1, \phi^3, \phi^4 = J\phi^2\}$ s.t.

1. $\pi^*\omega_\Sigma = \phi^1 \wedge \phi^2$,
2. $\mathrm{tr}_h(\mu \wedge J\mu) = \phi^3 \wedge \phi^4$,
3. $\pi^1 = \pi^*\psi^1$ and $\phi^2 = \pi^*\psi^2$ with $\psi^2 = J_\Sigma\psi^1$, $\psi^i \in \Lambda^1\Sigma$,

4. $\mu((\phi^1)^\flat) = \mu((\phi^2)^\flat) = 0$, and

5. $\pi_*((\phi^3)^\flat) = \pi_*((\phi^4)^\flat) = 0$.

We adopt the convention that uppercase indices are in $\{1,2,3,4\}$, lowercase Latin indices are in $\{1,2\}$, and lowercase Greek in $\{3,4\}$. Then, by the naturality of the Lie bracket, we have the following identities.

$$\pi_*[e_\alpha, e_A] = \mu[e_i, e_A] = 0$$

We also have

$$F_{ij} = -\mu([e_i, e_j]).$$

Finally, by the fact that $\mathbb{T}^2$ is abelian,

$$[e_\alpha, e_\beta] = 0.$$

Then, we can compute the Levi-Civita connection coefficients η using the Koszul formula and our choice of frame.

$$g(\nabla^{LC}_A e_B, e_C) = -\frac{1}{2}(g([e_A, e_B], e_C) - g([e_A, e_C], e_B) + g([e_B, e_C], e_A))$$

We find that if α is the Levi-Civita connection coefficient of Σ, then

$$\eta = \begin{pmatrix} \pi^*\alpha + \frac{1}{2}F & \frac{1}{2}F \\ -\frac{1}{2}F & 0 \end{pmatrix}.$$

Afterwards, we can compute $d\omega$.

$$d^c\omega_{ABC} = -\omega([e_A, e_B], e_C) + \omega([e_A, e_C], e_B) - \omega([e_B, e_C], e_A)$$

In particular, $d\omega_{ijk} = d\omega_{i\alpha\beta} - d\omega_{\alpha\beta\gamma} = 0$ (first and last for dimensional reasons, second for brackets). The only non-vanishing term is

$$d\omega_{ij\alpha} = -\omega(\mu^{-1}F_{ij}, e_\alpha) = -\operatorname{tr}_h(F_{ij} \wedge J\mu(e_\alpha)) = -F_{ij\alpha}$$

Thus,

$$d^c\omega_{ij\alpha} = -F_{ij\alpha}.$$

But this means that

$$g^{-1}d^c\omega = \begin{pmatrix} -F & F \\ -F & 0 \end{pmatrix}.$$

Putting this together, we find that the connection forms A of the Bismut connection are

$$A = \eta - \frac{1}{2}g^{-1}d^c\omega = \begin{pmatrix} \pi^*\alpha - F & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the structure equations yield

$$\Omega^B = \begin{pmatrix} \operatorname{Rm} + dF - \pi^*\alpha \wedge F - F \wedge \pi^*\alpha & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \operatorname{Rm} + d^{\nabla_\Sigma}F & \\ 0 & 0 \end{pmatrix}.$$

Further, we can break $d^{\nabla_\Sigma}F_i^j$ into pieces. First, consider the horizontal part. Since we consider $F_i^j \in \Lambda^1(V^*)$,

$$d^{\nabla_\Sigma}F_i^j(e_k, e_l) = dF_i^j(e_k, e_l) = -F([e_k, e_l]^V)_i^j = F_{kl}^\alpha F_{\alpha i}^j = F_{ij}^\alpha F_{kl}^\alpha.$$

Then the mixed part, is the following

$$d^{\nabla_\Sigma}F_i^j(e_k, e_\lambda) = F_{ij,k}^\lambda - \alpha_{ki}^p F_{pj}^\lambda + \alpha_{kp}^j F_{ip}^\lambda.$$

Finally, the vertical part is

$$d^{\nabla_\Sigma} F(e_\beta, e_\lambda)_i^j = dF_i^j(e_\beta, e_\lambda) = F^\lambda_{ij,\beta} - F^\beta_{ij,\lambda}.$$

But notice that due to the low-dimensionality and anti-symmetry, the non-vanishing block is completely determined by its $i = 1$, $j = 2$ component. Thus, we know

$$\mathrm{Rm}_1^2 = \frac{1}{2} R_\Sigma \phi^2 \wedge \phi^1.$$

We also find that

$$d^{\nabla_\Sigma} F_1^2 = |F|^2 \phi^1 \wedge \phi^2 + d(\mathrm{tr}_{\omega_\Sigma} F)^\alpha \phi^\alpha.$$

Putting this all together gives

$$(\Omega^B)_1^2 = \frac{1}{2}(R_\Sigma - |F|^2)\phi^2 \wedge \phi^1 + h(d\,\mathrm{tr}_{\omega_\Sigma} F, \cdot)$$

from which the result follows. $\square$

Proof of Theorem 1.6. Fix (M^4, J) a minimal compact complex non-Kähler surface of Kodaira dimension $\kappa \geq 0$. It follows from the Kodaira classification of surfaces (cf. [43] §7) that M must be an elliptic fibration, with only multiple fibers. In particular, it follows that there exists a finite cover of M which admits a holomorphic principal T^2 action, and it suffices to show global existence on such manifolds. In particular we suppose $\pi : M^4 \to \Sigma$ is a holomorphic T^2-bundle over a compact Riemann surface Σ where $\chi(\Sigma) < 0$ if $\kappa = 1$ and $\chi(\Sigma) = 0$ if $\kappa = 0$.

Fix ω_0 a pluriclosed metric on M. By Lemma 5.3, there exists a T^2-invariant metric $\hat{\omega} = \pi^*\omega_\Sigma + \mathrm{tr}_h\, \mu \wedge J\mu \in [\omega_0]$. We can modify the metric on Σ so that R_Σ is constant, and by Hodge theory further modify the principal connection μ to assume that $\operatorname{tr}_{\omega_\Sigma} F_\mu$ is constant. These changes preserve the associated Aeppli cohomology class on M and so we assume without loss of generality that $\hat{\omega}$ satisfies these conditions. The metric $\hat{\omega}$ defines a holomorphic Courant algebroid $\mathcal{Q}_{\sqrt{-1}\partial\hat{\omega}}$ together with a generalized Hermitian metric $\hat{G}$. Now by construction we can choose $\beta \in \Lambda^{2,0}$ such that

$$\overline{\partial}\beta = \partial\hat{\omega} - \partial\omega_0.$$

Let G_0 denote the metric associated to (ω_0, β_0) as in Proposition 2.16. By [38], there exists $\epsilon > 0$ and a solution G_t to pluriclosed flow with initial data G_0 on $[0, \epsilon)$.

We first obtain a partial estimate on the metric using the fibration structure and the Schwarz Lemma. The holomorphic Courant algebroid $\mathcal{Q}$ comes equipped with a natural holomorphic projection map onto $T^{1,0}_M$ which we denote $\pi_{\mathcal{Q}}$. We furthermore obtain from the fibration structure the holomorphic map $d\pi : T^{1,0}_M \to T^{1,0}_\Sigma$. Composing these yields the holomorphic map of vector bundles $\Phi = d\pi \circ \pi_{\mathcal{Q}} : \mathcal{Q} \to T^{1,0}_\Sigma$. It follows from the construction and Proposition 2.16 that

$$|\Phi|^2_{G, g_\Sigma} = \operatorname{tr}_\omega \pi^* \omega_\Sigma.$$

Furthermore, using Lemma 4.2 (where $A = A(G, g_\Sigma, \Phi)$ is defined by (4.7)) we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} \operatorname{tr}_\omega \pi^* \omega_\Sigma &= \frac{\partial}{\partial t} |\Phi|^2_{G, g_\Sigma} \\
&= \left\langle \Phi \circ S^G_g, \Phi \right\rangle_{G^{-1}, g_\Sigma} \\
&= \Delta_g |\Phi|^2_{G^{-1}, g_\Sigma} - |A|^2_{g, G^{-1}, g_\Sigma} + \left\langle S^{g_\Sigma}_g \circ \Phi, \Phi \right\rangle_{G^{-1}, g_\Sigma} \\
&= \Delta_g \operatorname{tr}_\omega \pi^* \omega_\Sigma - |A|^2_{g, G^{-1}, g_\Sigma} + \tfrac{1}{2} R_\Sigma \left(\operatorname{tr}_\omega \pi^* \omega_\Sigma\right)^2 .
\end{aligned}$$

Note that by construction R_Σ is constant, either -2 or 0 depending on whether $\kappa = 1$ or 0. By the maximum principle we conclude for any smooth existence time $T > 0$ the estimate

$$\sup_{M\times\{T\}} \operatorname{tr}_\omega \pi^* \omega_\Sigma \leq \left(C + \tfrac{1}{2}|R_\Sigma| T\right)^{-1}. \tag{5.5}$$

We next establish the uniform equivalence of the metrics G_t along the flow. Combining Proposition 4.4 with Proposition 3.2, the curvature computation of Lemma 5.4, and the estimate (5.5) we obtain for a topological constant λ,

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right) \operatorname{tr}_G \hat{G} =& \; - |\Upsilon(G, \hat{G})|^2_{g,G^{-1},\hat{G}} + \operatorname{tr}_G \left\langle S_g^{\widetilde{G}} \cdot, \cdot \right\rangle_{\widetilde{G}} \\
=& \; - |\Upsilon(G, \hat{G})|^2_{g,G^{-1},\hat{G}} + \operatorname{tr}_G \left\langle \operatorname{tr}_g \left(\lambda \pi^* \omega_\Sigma \otimes \psi_* \pi^* \omega_\Sigma\right) \cdot, \cdot \right\rangle_{\hat{G}} \\
\leq& \; C \left(\operatorname{tr}_g \pi^* \omega_\Sigma\right) \operatorname{tr}_G \hat{G} \\
\leq& \; C \operatorname{tr}_G \hat{G}.
\end{aligned}$$

By the maximum principle we conclude

$$\sup_{M\times\{T\}} \operatorname{tr}_G \hat{G} \leq e^{CT}.$$

This implies that G_t and $\hat{G}$ are uniformly equivalent on any compact time interval, and from Proposition 4.3 we conclude

$$\left(\frac{\partial}{\partial t} - \Delta\right) |\Upsilon(G, \hat{G})|^2_{g,G^{-1},G} \leq C \left(1 + |\Upsilon(G, \hat{G})|^2_{g,G^{-1},G}\right).$$

By the maximum principle we obtain a uniform estimate for $|\Upsilon(G, \hat{G})|^2_{g,G^{-1},G}$ on any finite time interval, and the proof of long-time existence now concludes as in Theorem 1.4. □

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