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Publication Date

2006-05-01

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RESEARCH REPORT
UCB-ITS-RR-2006-3

May 2006
ISSN 0192 4095

Matroid Intersection and its application to a Multiple Depot, Multiple TSP

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Abstract

This paper extends the Held-Karp's lower bound available for a single Travelling Salesman Problem to the following symmetric Multiple Depot, Multiple Travelling Salesman Problem (MDMTSP): Given k salesman that start at different depots, k terminals and n destinations, the problem is to choose paths for each of the salesmen so that (1) each vehicle starts at its respective depot, visits atleast one destination and reaches any one of the terminals not visited by other vehicles, (2) each destination is visited by exactly one vehicle and (3) the cost of the paths is a minimum among all possible paths for the salesmen. The criteria for the cost of paths considered is the total cost of the edges travelled by the entire collection. This MDMTSP is formulated as a minimum cost constrained forest problem subject to side constraints. By perturbing the costs of each edge from C_{ij} to $\overline{C}_{ij} := C_{ij} + \pi_i + \pi_j$, one obtains an infinite family of lower bounds, denoted by $w(\pi)$, for the MDMTSP. Each lower bound, $w(\pi)$, involves calculating a minimum cost constrained forest. The minimum cost constrained forest problem is posed as a weighted two matroid intersection problem and hence can be solved in polynomial time. Since the polyhedra corresponding to the matroid intersection problem has the integrality property, it follows that the optimal cost corresponding to the LP relaxation of the MDMTSP is actually equal to $\max_\pi w(\pi)$.

I. Introduction

Let $V = \{1, 2, 3...n\}$ be the set of vertices that represent the destinations to be visited. There are k ($k \leq n$) salesmen initially located at distinct depots represented by vertices $S = \{s_1, s_2...s_k\}$. Each salesmen is required to visit at least 1 vertex in $V = \{1, 2, 3...n\}$ and reach a terminal. There are k possible terminals denoted by the set of vertices, $T = \{t_1, t_2...t_k\}$. A feasible set of paths for the salesmen consists of k vertex disjoint paths that start at S and reach T such that all vertices are visited exactly once. An example of a set of feasible paths for 3 salesman problem is shown in Fig. 1. There exists no edges between any two vertices in S, T or between S and T. Any other edge joining vertices i and j, if present, has a cost C_{ij} associated with it. Costs are symmetric, i.e., $C_{ij} = C_{ji}$. The cost of a path is the sum of the costs of the edges present in it. The objective of the **MDMTSP** is to find a feasible set of paths such that the sum of the costs of the corresponding paths is minimum. To formulate this as an integer programming problem, two additional root vertices r and r' are added as shown in Fig. 2. Additional edges are added such that they connect root vertex r (r') to each of the vertices in S (T). Also, these additional edges are assigned zero cost. That is, for all $i \in S$, $C_{ri} = 0$ and for all $i \in T$, $C_{r'i} = 0$. The **MDMTSP** is formulated as follows:

Problem I.1: The objective is to find an incidence matrix $\mathbf{x}$ such that the following cost given by

$$C(\mathbf{x}) = \sum_{i\in S, j\in V} C_{ij}x_{ij} + \sum_{i\in V, j\in V, i<j} C_{ij}x_{ij} + \sum_{i\in T, j\in V} C_{ij}x_{ij} \tag{1}$$

is minimized subject to the following constraints.

i. $\sum_{j\in V} x_{ij} = 1$ for all $i \in S$,

ii. $\sum_{j\in S} x_{ij} + \sum_{j\in V, i<j} x_{ij} + \sum_{j\in V, j<i} x_{ji} + \sum_{j\in T} x_{ij} = 2$ for all $i \in V$,

iii. $\sum_{j\in V} x_{ij} = 1$ for all $i \in T$,

iv. $x_{ri} = 1$ for all $i \in S$,

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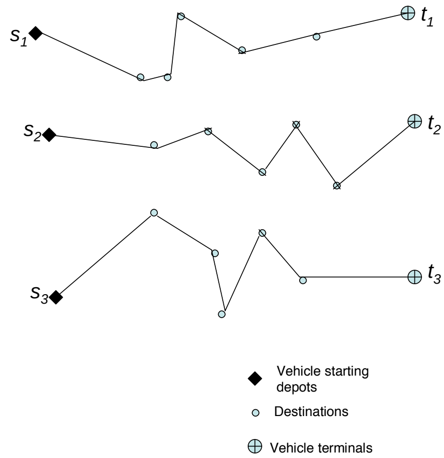

Fig. 1. An example of MDMTSP for 3 salesmen.

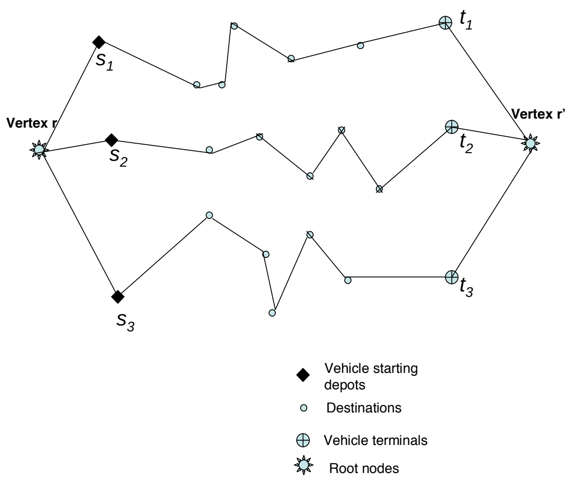

Fig. 2. The MDMTSP with root vertices for 3 salesmen.

v. $\sum_{\{i,j\}\in U_1} x_{ij} \leq |U_1| - 1$ for all $U_1 \in \{\{r\} \bigcup U : U \subseteq S \bigcup V \bigcup T\}$,

vi. $x_{r'i} = 1$ for all $i \in T$,

vii. $\sum_{\{i,j\}\in U_2} x_{ij} \leq |U_2| - 1$ for all $U_2 \in \{\{r'\} \bigcup U : U \subseteq S \bigcup V \bigcup T\}$,

viii. $\sum_{\{i,j\}\in U} x_{ij} \leq |U| - 1$ for all $U \subseteq S \bigcup V \bigcup T$,

ix. $x_{ij} \in \{0, 1\}$ for all $i, j \in \{r, r'\} \bigcup S \bigcup V \bigcup T$ and $i < j$ if $i, j \in V$.

x_{ij} is a decision variable that is equal to 1 if the edge between vertex i to j is chosen and x_{ij} equal to 0 otherwise. Constraints i,ii,iii enforce the degree constraints on each vertex. Constraint iv and v removes any possibility of a path joining vertices corresponding to the depots. Similarly, constraints vi and vii removes any possibility of a path joining vertices corresponding to the terminals. Constraint viii doesn't allow any cycle in the graph induced by the vertices $S \bigcup V \bigcup T$. The MDMTSP is closely related to the Multi-Depot TSP that has been addressed in GuoXing (1995). GuoXing is motivated by a Chinese truck company service where there are three depots and a set of trucks available at each depot. Each truck has to accomplish atleast one task and return to any of the three depots. The constraint here is that each depot should retain the same number of trucks after the service. The main difference between MDMTSP and the problem addressed in GuoXing (1995) is that MDMTSP allows for the possible set of terminals to be distinct from the set of depots. GuoXing provides a transformation to a standard single TSP wherein most applicable literature for the single TSP can be put to good use. An integer programming formulation of a generalized version of the problem discussed by GuoXing is presented in Kara and Bektas (2005). In the formulation of Kara and Bektas, there is an upper and lower bound on the number of vertices visited by each salesman. As pointed out in Kara and Bektas, an earlier version of this problem also appears in Kulkarni and Bhave (1985). In this paper, we formulate MDMTSP as a minimum cost constrained forest problem subject to side constraints. To facilitate further analysis, an additional constraint is added to MDMTSP without changing its set of feasible solutions. This is stated in the following lemma.

Lemma I.1: The following additional constraint can be added to problem I.1 without changing its set of feasible solutions:

$$\sum_{i\in S, j\in V} x_{ij} + \sum_{i\in V, j\in V, i<j} x_{ij} + \sum_{i\in T, j\in V} x_{ij} = n + k \tag{2}$$

Proof: Refer to the appendix. ■

Now, **MDMTSP** can be reformulated with the additional constraint as follows,

Problem I.2: The objective is to find the incidence matrix $\mathbf{x}$ such that the following cost given by

$$C(\mathbf{x}) = \sum_{i\in S, j\in V} C_{ij}x_{ij} + \sum_{i\in V, j\in V, i<j} C_{ij}x_{ij} + \sum_{i\in T, j\in V} C_{ij}x_{ij} \tag{3}$$

is minimized subject to the following constraints.

i. $\sum_{j\in V} x_{ij} = 1$ for all $i \in S$,

ii. $\sum_{j\in S} x_{ij} + \sum_{j\in V, i<j} x_{ij} + \sum_{j\in V, j<i} x_{ji} + \sum_{j\in T} x_{ij} = 2$ for all $i \in V$,

iii. $\sum_{j\in V} x_{ij} = 1$ for all $i \in T$,

iv. $\sum_{i\in S, j\in V} x_{ij} + \sum_{i\in V, j\in V, i<j} x_{ij} + \sum_{i\in T, j\in V} x_{ij} = n + k$,

v. $x_{ri} = 1$ for all $i \in S$,

vi. $\sum_{\{i,j\}\in U_1} x_{ij} \leq |U_1| - 1$ for all $U_1 \in \{\{r\} \bigcup U : U \subseteq S \bigcup V \bigcup T\}$,

vii. $x_{r'i} = 1$ for all $i \in T$,

viii. $\sum_{\{i,j\}\in U_2} x_{ij} \leq |U_2| - 1$ for all $U_2 \in \{\{r'\} \bigcup U : U \subseteq S \bigcup V \bigcup T\}$,

ix. $\sum_{\{i,j\}\in U} x_{ij} \leq |U| - 1$ for all $U \subseteq S \bigcup V \bigcup T$,

x. $x_{ij} \in \{0, 1\}$ for all $i, j \in \{r, r'\} \bigcup S \bigcup V \bigcup T$ and $i < j$ if $i, j \in V$.

Let the constraints i,ii,iii be denoted by $A_1\mathbf{x} = B_1$. The constraints defined by iv,v,vi,vii,viii,ix can be written as $A_2\mathbf{x} \leq B_2$. Since the cycle elimination constraints ensure that each x_{ij} can never exceed 1, the above minimization problem can be restated as

$$C_{opt} = \min_{\mathbf{x}}\{C(\mathbf{x}) : A_1\mathbf{x} = B_1, A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0, \mathbf{x} \text{ is an integer}\}. \tag{4}$$

The LP relaxation of this problem is:

$$C_{lp} = \min_{\mathbf{x}}\{C(\mathbf{x}) : A_1\mathbf{x} = B_1, A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0\}. \tag{5}$$

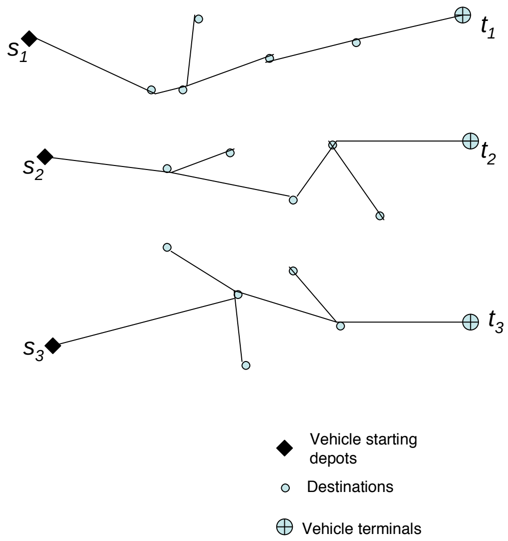

Fig. 3. An example of a constrained forest with 3 trees.

Held and Karp (1970) formulated the single TSP as a minimum cost 1-tree problem with side constraints. Just as how the 1-tree played an important role in the single TSP, a forest with k disjoint trees satisfying the following constraint plays an important role in MDMTSP: Each tree in the forest spans exactly one vertex from S, exactly one vertex from T and a subset of vertices from V. An illustration of such a forest with 3 trees is shown in Fig. 3. Using the notation in equation 4, the constrained forest problem, denoted by **CF**, can be formulated as:

$$C_f = \min_{\mathbf{x}}\{C(\mathbf{x}) : A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0, \mathbf{x} \text{ is an integer}\}. \tag{6}$$

One can now see that the MDMTSP is actually CF with additional degree constraints present in $A_1\mathbf{x} = B_1$. Similar to the results for the single TSP given in Held and Karp (1970), this paper presents the following results for MDMTSP. A review of the generalizations of the Held-Karp's work to other variants of TSP's can be seen in Westerlund et al. (2006).

- By perturbing the costs C_{ij} to $\overline{C}_{ij} := C_{ij} + \pi_i + \pi_j$, where π_i is a weight assigned to vertex i, in MDMTSP and CF one can obtain an infinite family of lower bounds for MDMTSP, namely $w(\pi)$.
- For a given weight vector π, each lower bound $w(\pi)$ can be calculated using any weighted two matroid intersection algorithm.
- The optimal cost corresponding to the LP relaxation of the MDMTSP denoted by C_{lp} is equal to $\max_\pi w(\pi)$.

II. Lower Bounds

Let $\mathbf{P}$ denote the set of all feasible solutions for the MDMTSP as defined in equation 4. That is, $\mathbf{P} := \{A_1\mathbf{x} = B_1, A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0, \mathbf{x} \text{ is an integer}\}$. Similarly, let $\mathbf{F}$ denote the set of all feasible solutions for the constrained forest problem. That is, $\mathbf{F} := \{A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0, \mathbf{x} \text{ is an integer}\}$. Now, let us perturb the costs C_{ij} to $\overline{C}_{ij} := C_{ij} + \pi_i + \pi_j$ where π_i is a weight assigned to vertex i. The objective function of MDMTSP in problem I.2 gets modified to $\min_{\mathbf{x}\in\mathbf{P}} \overline{C}(\mathbf{x})$, where,

$$\overline{C}(\mathbf{x}) = C(\mathbf{x}) + \sum_{i\in S}\pi_i + \sum_{i\in V} 2\pi_i + \sum_{i\in T}\pi_i. \tag{7}$$

Similarly, the objective function for the constrained forest problem gets modified to $\min_{\mathbf{x}\in\mathbf{F}} \widetilde{C}(\mathbf{x})$, where,

$$\widetilde{C}(\mathbf{x}) \quad = \quad C(\mathbf{x}) + \sum_{i\in S}\pi_i\sum_{j\in V}x_{ij} + \sum_{i\in V}\pi_i(\sum_{j\in S}x_{ij} + \sum_{j\in V, i<j}x_{ij} + \sum_{j\in V, j<i}x_{ji} + \sum_{j\in T}x_{ij}) + \sum_{i\in T}\pi_i\sum_{j\in V}x_{ij}$$

$$(8)$$

Note that any feasible solution for the MDMTSP is also a feasible solution for the constrained forest problem. Hence we must have, $\min_{\mathbf{X}\in\mathbf{F}} \widetilde{C}(\mathbf{x}) \leq \min_{\mathbf{X}\in\mathbf{P}} \overline{C}(\mathbf{x})$. Substituting for $\overline{C}(\mathbf{x})$ and $\widetilde{C}(\mathbf{x})$ using equations (7), (8) and rearranging the terms we get the following lower bound for MDMTSP.

$$\min_{\mathbf{X}\in\mathbf{F}} \widetilde{C}(\mathbf{x}) - \sum_{i\in S} \pi_i - \sum_{i\in V} 2\pi_i - \sum_{i\in T} \pi_i \leq \min_{\mathbf{X}\in\mathbf{P}} C(\mathbf{x}) \tag{9}$$

Since the above equation is true for any π, we get the following lemma:
Lemma II.1:

$$\max_{\pi} w(\pi) \leq C_{opt}, \tag{10}$$

where $w(\pi)$ is defined as follows:

$$\begin{aligned} w(\pi) &= \min_{\mathbf{X}\in\mathbf{F}}[C(\mathbf{x}) + \sum_{i\in S}\pi_i(\sum_{j\in V} x_{ij} - 1) + \sum_{i\in V}\pi_i(\sum_{j\in S} x_{ij} + \sum_{j\in V, i<j} x_{ij} + \sum_{j\in V, j<i} x_{ji} + \sum_{j\in T} x_{ij} - 2) \\ &\quad + \sum_{i\in T}\pi_i(\sum_{j\in V} x_{ij} - 1)], \\ &= \min_{\mathbf{X}\in\mathbf{F}}[C(\mathbf{x}) + \pi^T(A_1\mathbf{x} - B_1)]. \end{aligned} \tag{11}$$

The left hand side of equation 10 provides a lower bound to the MDMTSP. Note that for any fixed π, the inner minimization problem in equation 11 is that of calculating an optimal constrained forest.

III. The main result

Theorem III.1: Let C_{opt}, C_{lp} and $w(\pi)$ be given by equations 4, 5 and 11 respectively. Then,

$$C_{opt} \geq C_{lp} = \max_{\pi} w(\pi). \tag{12}$$

Before we prove this, a useful property of the set of feasible solutions denoted by $\{A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0,\ \mathbf{x} \text{ is an integer}\}$ is stated in the following theorem.

Theorem III.2: The optimal solutions of the problem, $\min_{\mathbf{X}}\{C(\mathbf{x}) : A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0\}$, are integers. This implies that $\min_{\mathbf{X}}\{C(\mathbf{x}) : A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0\} = \min_{\mathbf{X}}\{C(\mathbf{x}) : A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0,\ \mathbf{x} \text{ is an integer}\}$. Also the constraint forest problem (CF) can be solved using any weighted two matroid intersection algorithm.

Proof: Let $\overline{V} := S \bigcup V \bigcup T$. Let E be the set of all edges present in MDMTSP. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two collections of subsets of E as follows:

$E \supseteq A_1 \in \mathcal{I}_1$ if and only if graph $(\overline{V}, A_1)$ is free of cycles and free of paths connecting any pair of vertices in S.

$E \supseteq A_2 \in \mathcal{I}_2$ if and only if graph $(\overline{V}, A_2)$ is free of cycles and free of paths connecting any pair of vertices in T.

It is known in the literature (Cerdeira (1994), Cordone and Maffioli (2004)) that $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ are matroids. Note that if there exists no independent set in either M_1 or M_2 with $n+k$ elements, then there exists no feasible solution in $\{A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0, \mathbf{x} \text{ is an integer}\}$. Hence, without loss of generality, one can assume that both matroids M_1 and M_2 have atleast one independent set with $n+k$ elements. Any $\mathbf{x}$ is feasible in $\{A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0, \mathbf{x} \text{ is an integer}\}$ if and only if the set of edges corresponding to $\mathbf{x}$ is a common independent set of matroids M_1 and M_2 with $n+k$ elements. Hence CF can be posed as a maximum weighted two matroid intersection problem. Lawler (2000) has shown that the set of feasible solutions corresponding to the intersection of two matroid polyhedra has its extremal points as integers. Hence, the integer constraint on the decision variables can be removed in the constrained forest problem. This essentially implies the first result in the theorem that the optimal solutions of the problem, $\min_{\mathbf{x}}\{C(\mathbf{x}) : A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0\}$, are integers. Also the optimal solution for CF can be found using any one of the weighted matroid intersection algorithms available in Brezovec et al. (1986), Frank (1981) or Lawler (2000). Hence proved. ■

A. *Proof of theorem III.1*

Proof: Let $C(\mathbf{x}) = c^T\mathbf{x}$.

$$\begin{aligned} C_{lp} &= \min_{\mathbf{x}}\{C(\mathbf{x}) : A_1\mathbf{x} = B_1, A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0\} \\ &= \min_{\mathbf{x}}\{c^T\mathbf{x} : A_1\mathbf{x} = B_1, A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0\} \end{aligned}$$

Dualizing the problem, we have,

$$\begin{aligned} C_{lp} &= \max_{\pi,v}\{-B_1\pi - B_2 v : A_1^T\pi + A_2^T v \geq -c, v \geq 0\}, \\ &= \max_{\pi}\max_{v}\{-B_1\pi - B_2 v : A_1^T\pi + A_2^T v \geq -c, v \geq 0\}. \end{aligned}$$

Again dualizing the inner maximization problem, we have,

$$\begin{aligned} C_{lp} &= \max_{\pi}\min_{\mathbf{x}}\{c^T\mathbf{x} + \pi^T(A_1\mathbf{x} - B_1) : A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0\}, \\ &= \max_{\pi}(-\pi^T B_1 + \min_{\mathbf{x}}\{(c^T + \pi^T A_1)\mathbf{x} : A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0\}), \\ &= \max_{\pi}(-\pi^T B_1 + \min_{\mathbf{x}}\{\widehat{C}(\mathbf{x}) : A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0\}). \end{aligned}$$

where $\widehat{C}(\mathbf{x}) = (c^T + \pi^T A_1)\mathbf{x}$. Now, using theorem III.2 and the definition of $w(\pi)$ given in equation (11),

$$\begin{aligned} C_{lp} &= \max_{\pi}(-\pi^T B_1 + \min_{\mathbf{x}}\{\widehat{C}(\mathbf{x}) : A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0, \mathbf{x} \text{ is an integer}\}), \\ &= \max_{\pi}\min_{\mathbf{x}}\{c^T\mathbf{x} + \pi^T(A_1\mathbf{x} - B_1) : A_2\mathbf{x} \leq B_2, \mathbf{x} \geq 0, \mathbf{x} \text{ is an integer}\}, \\ &= \max_{\pi} w(\pi). \end{aligned}$$

Hence proved. ■

IV. Conclusions

The well known Held-Karp's lower bound available for a single Travelling Salesman Problem is extended to a Multiple Depot, Multiple Travelling Salesman Problem (MDMTSP). By perturbing the costs of each edge from C_{ij} to $\overline{C}_{ij} := C_{ij} + \pi_i + \pi_j$, one obtains a infinite family of lower bounds, namely $w(\pi)$, for the MDMTSP. Each lower bound $w(\pi)$ involves calculating a minimum cost constrained forest that can solved using any weighted, two matroid intersection algorithm. Similar to the single vehicle case, it is also shown that the optimal cost corresponding to the LP relaxation of the MDMTSP is actually equal to $\max_\pi w(\pi)$.

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V. Appendix

Lemma V.1: The following additional constraint can be added to problem I.1 without changing its set of feasible solutions:

$$\sum_{i\in S, j\in V} x_{ij} + \sum_{i\in V, j\in V, i<j} x_{ij} + \sum_{i\in T, j\in V} x_{ij} = n + k \tag{13}$$

Proof: Summing the constraint i in problem I.1 for all vertices in S, we get $\sum_{i\in S, j\in V} x_{ij} = k$. Similarly, summing constraint iii, we get $\sum_{i\in T, j\in V} x_{ij} = k$. Summing constraint ii for all vertices in V and using the fact that $\sum_{i\in V, j\in V, i<j} x_{ij} = \sum_{i\in V, j\in V, j<i} x_{ji}$, we get,

$$\begin{aligned} \sum_{i\in S, j\in V} x_{ij} + 2 \sum_{i\in V, j\in V, i<j} x_{ij} + \sum_{i\in T, j\in V} x_{ij} &= 2n \\ \Rightarrow \sum_{i\in V, j\in V, i<j} x_{ij} &= n - k. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{i\in S, j\in V} x_{ij} + \sum_{i\in V, j\in V, i<j} x_{ij} + \sum_{i\in T, j\in V} x_{ij} &= k + (n-k) + k \\ &= n + k. \end{aligned}$$

■