Title
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Journal
Comptes Rendus Mathématique, 353(5)

ISSN
0764-4442

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Publication Date
2015-05-01

DOI
10.1016/j.crma.2015.02.011

Peer reviewed

Dirac families for Loop groups as Matrix factorizations

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September 23, 2014

Abstract

We identify the category of integrable lowest-weight representations of the loop group LG of a compact Lie group G with the linear category of twisted, conjugation-equivariant *curved Fredholm complexes* on the group G: namely, the twisted, equivariant *matrix factorizations* of a super-potential built from the loop rotation action on LG. This lifts the isomorphism of K-groups of [FHT1]–[FHT3] to an equivalence of categories. The construction uses families of Dirac operators.

1. Introduction and background

The group LG of smooth loops in a compact Lie group G has a remarkable class of linear representations whose structure parallels the theory for compact Lie groups [PS]. The defining stipulation is the existence of a circle action on the representation, with finite-dimensional eigenspaces and spectrum bounded below, intertwining with the loop rotation action on LG. We denote the rotation circle by $\mathbb{T}_r$; its infinitesimal generator L_0 represents the *energy* in a conformal field theory.

Noteworthy is the *projective nature* of these representations, described (when G is semi-simple) by a *level* $h \in H^3_G(G;\mathbb{Z})$ in the equivariant cohomology for the adjoint action of G on itself. The representation category $\mathfrak{Rep}^h(LG)$ at a given level h is semi-simple, with finitely many irreducible isomorphism classes. Irreducibles are classified by their *lowest weight* (plus some supplementary data, when G is not simply connected [FHT3, Ch. IV]).

In a series of papers [FHT1]–[FHT3] the authors, jointly with Michael Hopkins, construct $K^0\mathfrak{Rep}^h(LG)$ in terms of a twisted, conjugation-equivariant topological K-theory group. To wit, when G is connected, as we shall assume throughout this paper,[1] we have

$$K^0\,\mathfrak{Rep}^h(LG) \cong K_G^{\tau+\dim G}(G), \tag{1.1}$$

with a twisting $\tau \in H^3_G(G;\mathbb{Z})$ related to h, as explained below.

1.2 Remark. One loop group novelty is a *braided tensor* structure[2] on $\mathfrak{Rep}^h(LG)$. The structure arises from the *fusion product* of representations, relevant to 2-dimensional conformal field theory. The K-group in (1.1) carries a Pontryagin product, and the multiplications match in (1.1).

The map from representations to topological K-classes is implemented by the following *Dirac family*. Calling $\mathscr{A}$ the space of connections on the trvial G-bundle over S^1, the quotient stack $[G\!:\!G]$ under conjugation is equivalent to $[\mathscr{A}\!:\!LG]$ under the gauge action, via the holonomy map $\mathscr{A} \to G$. Denote by $\mathbf{S}^\pm$ the (lowest-weight) modules of spinors for the loop space $L\mathfrak{g}$ of the Lie algebra and

[1] Twisted loop groups show up when G is disconnected [FHT3].
[2] When G is not simply connected, there is a constraint on h.

by $\psi(A) : \mathbf{S}^{\pm} \to \mathbf{S}^{\mp}$ the action of a Clifford generator A, for $d + Adt \in \mathscr{A}$. A representation $\mathbf{H}$ of LG leads to a family of Fredholm operators over $\mathscr{A}$,

$$\not{D}_A : \mathbf{H} \otimes \mathbf{S}^+ \to \mathbf{H} \otimes \mathbf{S}^-, \quad \not{D}_A := \not{D}_0 + \mathrm{i}\psi(A) \tag{1.3}$$

where $\not{D}_0$ is built from a certain Dirac operator [L] on the loop group.[3] The family is projectively LG-equivariant; dividing out by the subgroup $\Omega G \subset LG$ of based loops leads to a projective, G-equivariant Fredholm complex on G, whose K-theory class $\left[(\not{D}_\bullet, \mathbf{H} \otimes \mathbf{S}^{\pm})\right] \in K_G^{\tau+*}(G)$ is the image of $\mathbf{H}$ in the isomorphism (1.1). When $\dim G$ is odd, $\mathbf{S}^+ = \mathbf{S}^-$ and skew-adjointness of $\not{D}_A$ leads to a class in K^1. The twisting τ is the level of $\mathbf{H} \otimes \mathbf{S}$ as an LG-representation, with a (G-dependent) shift from the level h of $\mathbf{H}$.

The degree-shift is best explained in the world of super-categories, with $\mathbb{Z}/2$ gradings on morphisms and objects; odd simple objects have as endomorphisms the rank one Clifford algebra Cliff(1), and contribute, in the semi-simple case, a free generator to K^1 instead of K^0. Consider the τ-projective representations of LG with compatible action of Cliff($L\mathfrak{g}$), thinking of them as modules for the (not so well-defined) crossed product $LG \ltimes \mathrm{Cliff}(L\mathfrak{g})$. They form a semi-simple super-category $\mathfrak{SRep}^\tau$, and the isomorphism (1.1) becomes

$$K^* \,\mathfrak{SRep}^\tau \left(LG \ltimes \mathrm{Cliff}(L\mathfrak{g})\right) \cong K_G^{\tau+*}(G) \tag{1.4}$$

with the advantage of no shift in degree or twisting.[4] This isomorphism is induced by the Dirac families of (1.3): a super-representation $\mathbf{SH}^{\pm}$ of $LG \ltimes \mathrm{Cliff}(L\mathfrak{g})$ can be coupled to the Dirac operators $\not{D}_\bullet$ without a choice of factorization $\mathbf{H} \otimes \mathbf{S}^{\pm}$.

Acknowledgements. This research was partially supported by NSF-funded Focussed Research Group grants DMS-1160461 and DMS-1160328.

2. The main result

There is a curious lack of symmetry in (1.4): the isomorphism is induced by a functor of underlying Abelian categories, from $\mathbb{Z}/2$-graded representations to twisted Fredholm bundles over G, but this functor is far from an equivalence. The category $\mathfrak{SRep}^\tau$ is semi-simple (in the graded sense discussed), but that of twisted Fredholm bundles is not so. We can even produce continua of non-isomorphic objects in any given K-class by perturbing a Fredholm family by compact operators.

Here, we redress this problem via the inclusion of a *super-potential* W, of algebraic geometry and 2-dimensional physics B-model fame. As explained by Orlov[5] [O], this deforms the category of complexes of vector bundles into that of *matrix factorizations*: the 2-*periodic, curved complexes* with curvature W. Our super-potential will have Morse critical points, leading to a semi-simple super-category with one generator for each critical point. The generators are precisely the Dirac families of (1.3) on irreducible LG-representations. This artifice of introducing a super-potential is redeemed by its natural topological origin, in the *loop rotation* action on the stack $[G\!:\!G]$. (Rotation is more evident in the presentation by connections, $[\mathscr{A}\!:\!LG]$.) Namely, for twistings τ transgressed from BG, the action of $\mathbb{T}_r$ refines to a $B\mathbb{Z}$-action on the G-equivariant *gerbe* G^τ over G which defines the K-theory twisting. The logarithm of this lift is $2\pi\mathrm{i}W$.

[3] It is also the square root G_0, in the super-Virasoro algebra, of the infinitesimal circle generator L_0.

[4] For simply connected G, both sides live in degree $\dim\mathfrak{g}$; both parities can be present for general G.

[5] Orlov discusses complex algebraic vector bundles; we found no suitable exposition covering equivariant Fredholm complexes in topology, and a discussion is planned for our follow-up paper.

2.1 Remark. (i) The conceptual description of a super-potential as logarithm of a $B\mathbb{Z}$-action on a category of sheaves is worked out in [P]; the matrix factorization category is the *Tate fixed-point category* for the $B\mathbb{Z}$-action. On varieties, W is a function and $\exp(2\pi\mathrm{i}W)$ defines the $B\mathbb{Z}$-action; on a stack, there can be (as here) a geometric underlying action as well.

(ii) To reconcile our story with [P], we must rescale W_τ so that it takes integer values at all critical points; we will ignore this detail to better connect with the formulas in [FHT2, FHT3].

To spell out our construction, recall that a stack is an instance of a category. A $B\mathbb{Z}$-action thereon is described by its generator, an automorphism of the identity functor. This is a section over the space of objects, valued in automorphisms, which is central for the groupoid multiplication. For $[G\!:\!G]$, the relevant section is the identity map $G \to G$ from objects to morphisms. Intrinsically, $[G\!:\!G]$ is the mapping stack from $B\mathbb{Z}$ to BG, and the $B\mathbb{Z}$-action in question is the self-translation action of $B\mathbb{Z}$. This rigidifies the $\mathbb{T}_r$-action on the homotopy equivalent spaces $LBG \sim BLG \sim \mathscr{A}/LG$.

A class $\hat{\tau} \in H^4(BG;\mathbb{Z})$ transgresses to a $\tau \in H^3_G(G;\mathbb{Z})$, with a natural $\mathbb{T}_r$-equivariant refinement. This can also be rigidified, as follows. The exponential sequence lifts $\hat{\tau}$ uniquely to $H^3(BG;\mathbb{T})$, the group cohomology with smooth circle coefficients. That defines a Lie 2-group $G^{\hat{\tau}}$, a multiplicative $\mathbb{T}$-gerbe over G. (Multiplicativity encodes the original $\hat{\tau}$.) The mapping stack from $B\mathbb{Z}$ to $BG^{\hat{\tau}}$ is the quotient $[G^{\hat{\tau}}\!:\!G^{\hat{\tau}}]$ under conjugation, and carries a natural $B\mathbb{Z}$-action from the self-translations of the latter. Because $B\mathbb{T} \hookrightarrow G^{\hat{\tau}}$ is strictly central, the self-conjugation action of $G^{\hat{\tau}}$ factors through G, and the quotient stack $[G^{\hat{\tau}}\!:\!G]$ is our $B\mathbb{Z}$-equivariant gerbe over $[G\!:\!G]$ with band $\mathbb{T}$. We denote this central circle by $\mathbb{T}_c$, to distinguish it from $\mathbb{T}_r$.

The $B\mathbb{Z}$-action gives an automorphism $\exp(2\pi\mathrm{i}W_\tau)$ of the identity of $[G^{\hat{\tau}} : G]$, lifting the one on $[G : G]$. Concretely, $[G^{\hat{\tau}} : G]$ defines a $\mathbb{T}_c$-central extension of the stabilizer of $[G : G]$, and $\exp(2\pi\mathrm{i}W_\tau)$ is a trivialization of its fiber over the automorphism g at the point $g \in G$; see §3 below. The logarithm W_τ is multi-valued and only locally well-defined; nevertheless, the category of twisted matrix factorizations, $\mathrm{MF}^\tau_G(G;W_\tau)$ is well-defined, and its objects are represented by τ-twisted G-equivariant Fredholm complexes over G curved by $W_\tau + \mathbb{Z}\cdot\mathrm{Id}$.

2.2 Theorem. *The following defines an equivalence of categories from $\mathfrak{SRep}^\tau$ to $\mathrm{MF}^\tau_G(G;W_\tau)$: a graded representation $\mathbf{SH}^\pm$ goes to the twisted and curved Fredholm family $\left(\not{D}_\bullet, \mathbf{SH}^\pm\right)$ whose value at the connection $d + A\,dt \in \mathscr{A}$ is the τ-projective LG-equivariant curved Fredholm complex*

$$\not{D}_A = \not{D}_0 + \mathrm{i}\psi(A) : \mathbf{SH}^+ \rightleftarrows \mathbf{SH}^-$$

2.3 Remark. (i) Matrix factorizations obtained from irreducible representations are supported on single conjugacy classes, the so-called *Verlinde conjugacy classes* in G, for the twisting τ. These are the supports of the co-kernels of the Dirac families (1.3), [FHT3, §12].

(ii) There is a braided tensor structure on $\mathfrak{SRep}^\tau\,(LG \ltimes \mathrm{Cliff}(L\mathfrak{g}))$ (without $\mathbb{T}_r$-action). A corresponding structure on $\mathrm{MF}^\tau_G(G, W_\tau)$ should come from the Pontryagin product. We do not know how to spell out this structure, partly because the $\mathbb{T}_r$-action is already built into the construction of MF^τ, and the Pontryagin product is *not* equivariant thereunder.

(iii) The values of the automorphism $\exp(2\pi\mathrm{i}W_\tau)$ at the Verlinde conjugacy classes determine the *ribbon element* in $\mathfrak{Rep}^h(LG)$; see [FHLT] for the discussion when G is a torus.

Theorem 2.2 has a $\hat{\tau} \to \infty$ scaling limit, which we will use in the proof. In this limit, the representation category of LG becomes that of G. On the topological side, noting that each $\hat{\tau}$ defines an inner product on $\mathfrak{g}$, we zoom into a neighborhood of $1 \in G$ so that the inner product stays fixed. This leads to a G-equivariant matrix factorization category $\mathrm{MF}_G(\mathfrak{g}, W)$ on the Lie algebra. The τ-central extensions of stabilizers near 1 have natural splittings, and the W_τ converge to a super-potential $W \in G \ltimes \mathrm{Sym}(\mathfrak{g}^*)$, which, in a basis ξ_a of $\mathfrak{g}$ with dual basis ξ^a of $\mathfrak{g}^*$, we will calculate in §3 to be

$$W = -\mathrm{i} \cdot \xi_a(\delta_1) \otimes \xi^a + \frac{1}{2} \sum\nolimits_a \|\xi^a\|^2 \tag{2.4}$$

with $\xi_a(\delta_1)$ denoting the respective derivative of the delta-function at $1 \in G$. It is important that (2.4) is central in the crossed product algebra $G \ltimes \mathrm{Sym}(\mathfrak{g}^*)$.

To describe the limiting case, recall from [FHT3, §4] the G-analogue of the Dirac family (1.3). Kostant's *cubic Dirac operator* [K] on G is left-invariant, and the Peter-Weyl decomposition gives an operator $\not{D}_0 : \mathbf{V} \otimes \mathbf{S}^\pm \to \mathbf{V} \otimes \mathbf{S}^\mp$ for any irreducible representation $\mathbf{V}$ of G, coupled to the spinors $\mathbf{S}^\pm$ on $\mathfrak{g}$. As before, it is better to work with graded modules for the super-algebra $G \ltimes \mathrm{Cliff}(\mathfrak{g})$.

2.5 Theorem. *Sending $\mathbf{SV}^\pm$ to $\left(\not{D}_\bullet, \mathbf{SV}^\pm\right)$, the curved complex over $\mathfrak{g}$ given by*

$$\mathfrak{g} \ni \mu \mapsto \not{D}_\mu = \not{D}_0 + \mathrm{i}\psi(\mu) : \mathbf{SV}^+ \leftrightarrows \mathbf{SV}^-$$

provides an equivalence of super-categories from graded $G \ltimes \mathrm{Cliff}(\mathfrak{g})$-modules $\mathbf{SV}^\pm$ to G-equivariant, W-matrix factorizations over $\mathfrak{g}$.

With λ denoting the lowest weight of V and $T(\mu)$ the μ-action on $\mathbf{SV}$, we have [FHT3, Cor. 4.8]

$$\not{D}_\mu^2 = -\|\lambda_V + \rho\|^2 + 2\mathrm{i} \cdot T(\mu) - \|\mu\|^2 \in (-2W) + \mathbb{Z}.$$

3. Outline of the proof

(3.1) Executive summary. The category $\mathrm{MF}_G^\tau(G; W_\tau)$ sheafifies over the conjugacy classes of G. Near any $g \in G$ with centralizer Z, the stack $[G : G]$ is modeled on a neighborhood of 0 in the adjoint quotient $[\mathfrak{z} : Z]$ of the Lie algebra $\mathfrak{z}$, via $\zeta \in \mathfrak{z} \mapsto g \cdot \exp(2\pi\zeta)$. We will compute the local W_τ in the crossed product $Z \ltimes C^\infty(\mathfrak{z})$, recovering (2.4), up to a g-dependent central translation in $\mathfrak{z}$. We then show that MF^τ lives only on *regular* elements g. Assuming for brevity that $\pi_1(G)$ is torsion-free: Z is then the maximal torus $T \subset G$, where the super-potential W_τ turns out to have Morse critical points, located precisely at the Verlinde conjugacy classes. The local category is freely generated by the respective Atiyah-Bott-Schapiro Thom complex.[6] The latter is quasi-isomorphic to our Dirac family for a specific irreducible representation, associated with the Verlinde class [FHT3, §12].

(3.2) The 2-group. We use a *Whitehead crossed module* [W] description for $G^{\hat{\tau}}$. This is an exact sequence of groups

$$\mathbb{T}_c \rightarrowtail K \xrightarrow{\varphi} H \twoheadrightarrow G,$$

equipped with an action of H on K which lifts the self-conjugation of H and factors the self-conjugation of K via φ. Call h an H-lift of g, and C the pre-image of Z in H. Define the central extension $\widetilde{Z}$ by means of a $\mathbb{T}_c$-central extension of C, trivialized over $\varphi(K) \cap C$, as follows. The commutator $c \mapsto hch^{-1}c^{-1}$ gives crossed homomorphism $\chi : C \to \varphi(K)$, with respect to the conjugation action of C on $\varphi(K)$. The action having been lifted to K, χ pulls back the central extension $\mathbb{T}_c \rightarrowtail K \to \varphi(K)$ to C. The h-action on K identifies the fiber of K over any $c \in \varphi(K)$ with that over hch^{-1}, trivializing the pull-back extension over $\varphi(K)$. Finally, $hhh^{-1}h^{-1} = 1$, so the extension is also trivialized over $c = h$, defining our $\exp(2\pi\mathrm{i}W_\tau)$ at $g \in Z$.

[6] The Clifford multiplication acts in both directions, giving a curved complex.

(3.3) Computing the local super-potential. Following [BSCS], take $K = \Omega^\tau G$, the τ-central extension of the group of based smooth maps $[0, 2\pi] \to G$ sending $\{0, 2\pi\}$ to 1, and $H = \mathscr{P}_1 G$, the group of smooth paths $[0, 2\pi] \to G$ starting at $1 \in G$. The requisite H-action on the Lie algebra $\mathrm{i}\mathbb{R} \oplus \Omega\mathfrak{g}$ of K is

$$\gamma.(x \oplus \alpha) = \left(x - \frac{\mathrm{i}}{2\pi} \int_0^{2\pi} \langle \gamma^{-1} d\gamma | \alpha \rangle \;\oplus\; \mathrm{Ad}_\gamma(\alpha) \right) \tag{3.4}$$

extending the Ad-action of $\Omega^\tau G$ [PS, Prop. 4.3.2], and exponentiating to an action on $\Omega^\tau G$. (Acting on other components of ΩG requires topological information from $\hat{\tau}$.)

The equivariant gerbe $[G^{\hat{\tau}} : G]$ is locally trivialized (possibly on a finite cover of Z) uniquely up to discrete choices: the automorphisms of the central extension $\widetilde{Z}$. We spell out W_τ in these terms. Lift g to $\mathscr{P}_1 G$ as $h = \exp(t\mu)$, for a shortest logarithm $2\pi\mu$ of g, and assume for now that Z centralizes μ. Instead of the entire group C, we use in the construction the subgroup $\mathscr{P}_1 Z$ of paths in Z. It centralizes h, and this trivializes our $\mathbb{T}_c$-extension over $\mathscr{P}_1 Z$, with $W_\tau = 0$. However, by (3.4), the extension over $\Omega Z = \varphi(K) \cap \mathscr{P}_1 Z$ is trivialized by the Lie algebra character $\alpha \mapsto -\frac{\mathrm{i}}{2\pi} \int_0^{2\pi} \langle \mu | \alpha \rangle dt$. To trivialize $\widetilde{Z}$, we must therefore extend this to a linear character of $\mathscr{P}_0\, \mathfrak{z}$. The same formula (3.4) does this, supplying the locally constant trivialization of $\widetilde{Z}$. We now get the value $2\pi \mathrm{i} W_\tau(g) = \pi \mathrm{i} \|\mu\|^2 \oplus 2\pi\mu \in \mathrm{i}\mathbb{R} \oplus \mathfrak{g}$.

At the remaining points, W_τ is determined by continuity, but can also be pinned down by the restriction to a maximal torus containing g.

(3.5) Vanishing of singular contributions. When $\mathfrak{z}$ is non-abelian, we show the vanishing of the matrix factorization category localized at g. Take $g = 1, Z = G$, W on $\mathfrak{g}$ as in (2.4), plus possibly a central linear term μ. Koszul duality equates the localized category $\mathrm{MF}^\tau_G(\mathfrak{g}; W)$ with the super-category of $\mathbb{Z}/2$-graded modules over the differential super-algebra $(G \ltimes \mathrm{Cliff}(\mathfrak{g}), [\not{D}_\mu, \_])$; $\not{D}_\mu = \not{D}_0 + \mathrm{i}\psi(\mu)$, with Kostant's cubic Dirac operator of §2. Ignoring the differential, the algebra is semi-simple, with simple modules the $\mathbf{V} \otimes \mathbf{S}^\pm$ of Theorem 2.5, for the irreducible G-representations $\mathbf{V}$. Since $\not{D}_\mu^2 = -\|\lambda_V + \mu + \rho\|^2 < 0$, $[\not{D}_\mu, \not{D}_\mu]$ provides a homotopy between 0 and a central unit in the algebra. This makes the super-category of graded modules quasi-equivalent to 0.

(3.6) Globalization for the torus. We describe the stack $[T^{\hat{\tau}} : T]$ and potential W_τ in the presentation $T = [\mathfrak{t} : \Pi]$ of the torus as a quotient of its Lie algebra by $\Pi \cong \pi_1(T)$. Lifted to $\mathfrak{t}$, the gerbe of stabilizers $\tilde{T}$ is trivial, with band $T \times \mathbb{T}_c$. The descent datum under translation by $p \in \Pi$ is the shearing automorphism of $T \times \mathbb{T}_c$ given by the character $t \mapsto \exp\langle p | \log t \rangle$, $t \in T$. In the same trivialization over $\mathfrak{t}$, the super-potential is

$$2\pi \mathrm{i} W_\tau(\mu) = \pi \mathrm{i} \|\mu\|^2 \oplus 2\pi\mu \in \mathrm{i}\mathbb{R} \oplus \mathfrak{t},$$

the first factor being the Lie algebra of $\mathbb{T}_c$. That is the function "$\frac{1}{2}\|\log\|^2$" on T, invariant under p-translation, save for an additive shift by the integer $\|p\|^2/2$.

With Λ denoting the character lattice of T, the crossed product algebra of the stack $[T^\tau : T]$ can be identified with the functions on $\left(\coprod_{\lambda \in \Lambda} \mathfrak{t}_\lambda\right)/\Pi$, with the action of Π by simultaneous translation on Λ and $\mathfrak{t}$. On the sheet $\lambda \in \Lambda$, $W_\tau = -\langle \lambda | \mu \rangle + \|\mu\|^2/2$ has a single Morse critical point at $\mu = \lambda$.

It follows that the super-category $\mathrm{MF}^\tau_T(T; W_\tau)$ is semi-simple, with one generator of parity $\dim \mathfrak{t}$ at each point in the kernel of the isogeny $T \to T^*$ derived from the quadratic form $\hat{\tau} \in H^4(BT; \mathbb{Z})$. The kernel comprises precisely the Verlinde points in T [FHLT], and this concludes the proof.

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