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Author
Grinblatt, Mark

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Mark Grinblatt
Anderson Graduate School of Management
University of California, Los Angeles

AN ANALYTIC SOLUTION FOR INTEREST RATE SWAP SPREADS

by

Mark Grinblatt*

*UCLA Anderson Graduate School of Management
Los Angeles, CA 90024

Comments Welcome

Abstract

AN ANALYTIC SOLUTION FOR INTEREST RATE SWAP SPREADS

This paper argues that liquidity differences between government securities and short term Eurodollar borrowings account for interest rate swap spreads. It then models liquidity as a linear function of two mean-reverting state variables and values it. The interest rate swap spread for a swap of particular maturity is the annuitized equivalent of this value. It has a closed form solution: a simple integral. Special cases examined include the Vasicek (1977) and Cox-Ingersoll-Ross (1985) one-factor term structure models. Numerical values for the parameters in both special cases illustrate that many realistic "swap spread term structures" can be replicated. Model parameters are estimated using weekly data from January 1988 through February 1992 on the "term structure of swap spreads." Some simple tests of the model are performed using this data.

Interest rate swaps, which are contracts to periodically exchange fixed for floating payments, are one of the most important financial instruments in the world. Aside from the sheer size of the swap market (over six trillion dollars in notional amount outstanding at the end of 1993),[1] and their importance as hedging instruments, swaps offer data that is of great use to financial modellers. For example, many sophisticated banking houses use the "all-in-cost," which is the yield on the fixed side of the swap, to generate risk-free rates for their derivatives models.

Despite the wide use of swaps by finance practitioners for these various purposes, little quantitative modelling of them exists. Sundaresan (1991) and Longstaff and Schwartz (1995) model interest rate swap spreads as a risk premium.[2] In Sundaresan's model, the risk is due to the risk of default in the Eurodollar market. The floating rate of the swap is generally derived from the London Interbank Offered Rate (LIBOR)[3], which, in Sundaresan's model, represents the yield on a risky financial instrument. The fixed payment is based on the yield of the most recently issued Treasury issue of the same maturity as the swap. If the two counterparties to the swap merely exchanged the Treasury yield for the LIBOR rate, the fixed payer would have essentially borrowed at the risk-free Treasury rate and invested at the risky LIBOR rate. If the two counterparties to the swap have symmetric (or no) default risk, this would be unfair to the payer of the floating rate. Thus, to make the swap fair, the swap's fixed rate payment has to

[1]Source, ISDA. Notional amount is roughly equivalent to open interest.

[2]The swap spread is the difference between the yield of a recently issued government bond of identical maturity as the swap and the yield associated with the fixed rate of the swap.

[3]This is also known as the Eurodollar borrowing rate.

exceed the Treasury yield. Hence, the swap spread in Sundaresan's model is always positive.

However, the empirical evidence does not seem to be consistent with corporate default risk

being a major factor in determining swap spreads.[4] In Longstaff and Schwartz (1995), there is

no asymmetric risk premium between the rates paid on the floating side or the fixed side.

Instead, swap spreads arise because of the possibility of counterparty default on the contract

itself.[5] However, they show that for realistic parameters, the swap spreads that arise from

counterparty default risk are small, on the order of one to two basis points.[6,7]

If default is not driving swap spreads, then what is? There is evidence that liquidity can

have large price effects in the fixed income market.[8] In this paper, we model swap spreads as

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[4]Litzenberger (1992), in his presidential address to the American Finance
Association, argues that industry practice, which prices swaps as though they were not
terribly credit sensitive, makes sense. Minton (1993) finds that swap spreads are
unrelated to an aggregate default risk factor. Evans and Bales (1991) finds that swap
spreads are not as cyclical as A-rated corporate spreads. Chen and Selender (1994) find
that AA-AAA corporate spreads have some marginal explanatory power for swap
spreads, but the explanatory power is extremely weak and only valid for long term swap
spreads. Other credit spreads have no explanatory power in their paper.

[5]The correlation between a "default factor" and an interest rate factor can make the
term structure of swap spreads upward sloping or downward sloping, depending on the
sign of the correlation.

[6]Cooper and Mello (1991) find larger swap spreads generated by counterparty default
risk, but only with parameters that generate swap defaults more frequently than they
actually occur.

[7]Credit risk differences between counterparties will alter the bid-ask spread of the
swap, but should have no effect on a bank's reported mid-market swap spreads, which
we study here.

[8]Boudoukh and Whitelaw (1990), for example, comment that "on-the-run" U.S.
Treasury notes and bonds (which are the most recent issues of each Treasury maturity),
have yields that are lower than those of comparable "off-the-run" Treasury securities by
about 10 basis points. Warga (1992) argues that the duration-adjusted difference

compensation for a liquidity-based convenenience yield associated with Treasury notes. This convenience yield is lost to an investor wishing to receive fixed rate payments, who--in lieu of purchasing a Treasury note--enters into a swap to receive fixed payments.

The convenience yield is assumed to depend on two stochastic factors, one of which is related to short term interest rates. The assumed stochastic process for this convenience yield allows us to derive a closed form solution for the term structure of interest rate swap spreads.[9] Most stylized facts about swap spreads and about the curvature of the term structure of swap spreads can be captured with reasonable parameter values. For example, U-shaped and humped swap spread curves, with deviations on the order of 5 to 10 basis points, can be generated by the model, as well as the more common upward sloping and downward sloping curves.

The paper is organized as follows. Section 1 analyzes explanations for differences in the yields of two default-free or near default-free securities of the same maturity. It concludes that a convenience yield, tied to liquidity, is the only plausible explanation for this difference. Section 2 models swap spreads as the annuity payment that is equivalent to the present value of

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between on and off-the-run Treasuries is on the order of 50 basis points and that liquidity, but not taxes, are a potential explanation. In addition, Amihud and Mendelson (1991) show that the yields of U.S. Treasury bills are lower than those of otherwise identical government notes in their final coupon period by 70 to 110 basis points. Kamara (1994) argues that this difference can be driven by liquidity differences, but not taxes, assuming that demand for Treasury notes is perfectly elastic. Daves and Erhardt (1992) find that U.S. Treasury interest only and principal only strips that are otherwise identical differ in their prices. The tax treatment of the two types of strips is identical.

[9]Brennan (1991) values convenience yields for commodities using the standard contingent claims methodology. Despite some similarities, the stochastic processes for the convenience yields valued by Brennan are not the same as those valued here.

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the liquidity-based convenience yield. Section 3 estimates the parameters of the model and assesses the model's performance. Section 4 concludes the paper.

1. Differences Between Yields on Government Securities and other Risk-free Securities

U.S. Dollar interest rate swap spreads are tied to spreads between short term borrowing rates in the Eurodollar market and the borrowing rates of the U.S. Treasury. In this section, we argue that the former rate is a corporate "risk-free"[10] rate and the latter is a government risk-free rate and explore a variety of reasons for why the two risk-free rates might differ. The nature of these explanations enables us to model variables that can determine interest rate swap spreads.

A six month LIBOR loan is a loan to an AA or AAA-rated borrower.[11] Such a loan is essentially risk-free. And yet, the "TED spread" -- the difference between the yield on the LIBOR loan and the yield on a Treasury bill of identical maturity--is substantial.[12] It is almost always more than 25 basis points and is often more than 100 basis points. Such large differences cannot be explained by the infinitesimal differences in default risk between the two

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[10]We use the term "risk-free" interchangeably with the term "default-free." Obviously, the fixed income securities discussed here contain interest rate risk.

[11]Occasionally, LIBOR is the rate charged to an A borrower.

[12]We refer to LIBOR as if it is a single rate, when in fact, different banks can in theory quote different LIBOR lending rates. In practice, the variation in quoted rates is small with the vast majority of banks having identical LIBOR rates. The rates at which simultaneous transactions take place exhibit even less variation. Hence, it is appropriate to treat LIBOR as if it were a single rate. Also, many researchers and practitioners use the difference between the nearest maturing Eurodollar and T-bill futures prices as the unique TED spread.

investments. Similarly large differences between government and essentially risk-free corporate investments at longer maturities also exist. While Amihud and Mendelson (1991) have shown that there are a variety of short term government risk-free rates, and thus, the TED spread need not be the only spread to focus on here, it is these kinds of yield spreads that must intimately be tied to interest rate swap spreads.

There are several non-risk-based explanations for these differences. Among them:

1) If the marginal investor in Treasury securities faces high state income taxes, the non-taxability of Treasury interest at the state and municipal level should result in lower yields to the Treasury securities.

2) Treasury securities are usable as collateral for margin investments (e.g. futures accounts), while corporate securities are not. Thus, if cash in a margin account earns a lower return than Treasuries do, there is some advantage to Treasuries as collateral.

3) The marginal investor in Treasury securities may have some kind of regulatory advantage in Treasuries, such as a different capital or reserve requirement for holding Treasuries and corporates (as, for example, implemented by the Basle accord), or a binding investment constraint (like no more than 30% of fixed income investments may be in the corporate sector).

4) The Treasury and corporate markets may be segmented. For example, international demand for Treasuries relative to corporates may overprice the Treasuries relative to the corporates. While some segment of the investing public may recognize this, they cannot arbitrage the difference away because of market frictions like transaction costs.

5) There is a relative liquidity advantage to holding Treasury securities.

The tax equilibrium argument is implausible because the state tax advantage does not apply to broker-dealers, tax exempt investors like pension funds, or international investors. It is unlikely that the marginal corporate-Treasury investor is a wealthy individual in New York or California. In addition, the extant empirical evidence on the Treasury market argues that differences in the tax treatment of some Treasury securities are an unlikely sources of pricing

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anomalies.[13]

The margin explanation fails because the difference between the interest earned on Treasuries and cash in margin accounts is large only for small investors. In addition, many of these margin accounts do not permit longer term Treasury securities as collateral. Yet, the yield difference between Treasuries and risk-free corporates (computed from the interest rate swap curve) does not generally decline with maturity. Finally, the fraction of Treasury securities used as collateral in this manner is miniscule compared to the amount outstanding. The yield difference between corporates and all Treasuries is unlikely to be generated by a benefit taken for only a few Treasury securities.

The regulatory contraint argument cannot explain why many unconstrained investment funds and mutual funds still invest in Treasuries. Thus, either a large segment of the market is irrational or (more likely), the constrained investors are not the marginal ones.

In addition to these inconsistencies, the first three explanations cannot account for the much smaller spread between corporates and "off-the-run" Treasury securities. For example, off-the-run Treasuries are treated the same as on-the-run Treasuries for the purpose of computing regulatory capital. Moreover, with a larger yield, off-the-runs have larger state tax advantages and advantages as collateral for margin accounts than the "on-the-run" (most recently issued) treasuries.

As with the third explanation, market segmentation requires a degree of irrationality that is not acceptable in most financial modelling. For example, international investors may favor

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[13]See, for example, Warga (1992) and Kamara (1994).

U.S. Treasury securities over U.S. corporate fixed income securities (for a variety of reasons that we will not explore here). In this case, all domestic investors who are free to invest in both the corporate and Treasury markets should only be investing in corporates. Yet, we observe many of the savviest domestic investors holding both corporates and Treasuries.

This leaves a Treasury liquidity advantage as the remaining plausible explanation for the corporate-Treasury risk-free yield difference. This is not so much a matter of lower bid-ask spreads and immediacy for purchases and sales of Treasuries (although this could alter the yield on Treasuries as well). Rather, it is a liquidity advantage driven by the needs of savvy investors. Because of the large volume of daily transactions taking place in the Treasury market, Treasury securities are the preferred vehicle for hedging interest rate risk associated with interest rate sensitive positions. A corporate bond buyer who acquires a bond because of a belief that the bond price reflects an overestimate of its default risk will often short a Treasury note of similar maturity to reduce the interest rate risk of the position. Those willing to lend out the Treasury note to such an investor in the Treasury securities repurchase (or "repo") market typically receive a loan at an abnormally low interest rate.

In this sense, there is a daily cash flow to holding Treasury notes if one is big enough and sophisticated enough to be a repo market participant. The cash flow is a premium paid to avoid a "short squeeze," a phenomenon that has been noted and modelled by Duffie (1994).[14] In this sense, the term liquidity advantage is something of a misnomer, in that when the likelihood of a short squeeze is high, (and thus, the Treasury note is in some sense less liquid),

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[14]On rare occasions, this interest rate may even be negative.

the convenience yield from lending out the Treasury security is greatest. However, we term it a liquidity advantage because securities that are ex-ante illiquid (e.g. most corporate bonds) would never be shorted in the first place as interest rate hedging vehicles. Because the likelihood of short squeezes with such illiquid securities is so high, they are inframarginal, and thus never carry the liquidity benefit of the Treasuries.

Below, we model an exogenously given stochastic process[15] that generates this liquidity-based "convenience yield" of Treasuries to derive an analytic formula for interest rate swap spreads in LIBOR-based swaps.

2. The Model

Consider two riskless financial investments that last for T years:

Investment L: \$1 invested in a zero coupon bond that has maturity T/N and an interest rate of LIBOR, which is rolled over every T/N years,[16] combined with a zero net present value interest rate swap with a notional amount of \$1 that pays a floating amount equal to (T/N maturity) LIBOR and receives a fixed amount every T/N years.

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[15]A formal model of the general equilibrium behind this process, while obviously desirable, is beyond the scope of this paper. A sketch of the features of such a model would include: (i) asymmetric information about the default risk (or fair market credit spread) of corporate fixed income securities, (ii) an inability to short corporate securities, perhaps because of the likelihood of frequent "short squeezes," (iii) an insufficient supply of highly liquid Treasury notes and bonds in the repo market to meet the interest rate hedging needs of investors buying undervalued corporate securities, (iv) a general repo rate that is below LIBOR in equilibrium, which, along with special repo rates due to occasional liquidity crises in some Treasury issues, quantifies the convenience yield of Treasuries, and (v) equilibrium between the Treasury-repo market and the interest rate swap market.

[16]It is not essential that this investment be rolled over. Since no additional out-of-pocket cash is required to roll the investment over, the rolled over investment has the same present value as a short term LIBOR investment that terminates at T/N.

Investment G: $1 invested in the "on the run" note--a fixed rate Treasury note issued at date 0 that pays coupons every T/N years and matures at year T.[17] For simplicity, we assume that the government note is trading at par at date 0.[18]

Investments L and G are, in effect, investments of equal amount in par fixed rate bonds[19] with identical principal but different coupons. For investment L, the floating rate interest payments cancel with floating payments from the swap, and all principal from the LIBOR investments is rolled over until one is left with $1 principal paid at year T. Hence, one is left with the fixed-rate payments from the swap and the principal from a sequence of rolled over short-term LIBOR investments.

Investment L would be equivalent to a LIBOR bond except for the fact that a LIBOR bond of maturity T has risk. This is because a AA or AAA investor at date 0 may not be a AA or AAA investor two or three years from now, but the bond contract will still be in place. By contrast, a sequence of short term LIBOR investments has the AA or AAA credit rating of the borrower "refreshed" periodically. In addition, while the swap is not risk-free per se because there is counterparty default risk, its pricing will be virtually equivalent to a default-free swap

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[17]In reality, this date of issue will usually be slightly prior to date 0 and the note will be the most recently issued treasury security with a maturity between T-1/2 years and T+1/2 years at its issue date. For simplicity, we assume away this complication.

[18]In modelling swap spreads, we assume that there is an on-the-run note for the exact maturity date of every swap. In practice, there are a limited number of on-the-run notes and swap spreads of all maturities are quoted relative to interpolated yields from these on-the-run notes.

[19]The term "bond" here generically refers to the stream of cash flows generated from the purchase of a fixed income security.

if the counterparties have symmetric credit risk.[20] We noted earlier that Longstaff and

Schwartz (1995), modelling this counterparty default risk, show that for most reasonable

scenarios, there is a negligible deviation in swap pricing with default risk from the pricing of a

default-free swap. Thus, the yield on investment L, which is also the all-in-cost of the swap,

will be smaller than the yield of a LIBOR bond of comparable maturity.[21]

Our strategy is to derive the interest rate swap spread by comparing investments L and G

and analyzing how they are affected by the liquidity-based convenience yield. The model used

assumes that the zero coupon rates implied by L investments of various maturities should be

used to compute present values of riskless cash flows. Hence, zero coupon rates derived from

Treasury notes of different maturities are inappropriate for obtaining the present values of risk-

free cash flows that have no liquidity-based convenience yield.

2.1 The General Case

The derivation of the term structure of interest rate swap spreads employs the following

notation:

$r(t)$ =   stochastic short term interest rate factor at date t.

$q(t)$ =   vector of other state variables that determine the term structure of interest rates, which, without loss of generality, we take to have differentials that are orthogonal to the other state variables

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[20]There may be a shortening of duration as a consequence of symmetric counterparty default risk, but since it is a small effect, we will treat default as if it does not exist. In addition, the cross-collateralization and mark-to-market dissolution of many swaps (see Minton (1993)) eliminates even this small effect.

[21]Sun, Sundaresan, and Wang (1993) have verified this empirically.

y(t) = instantaneous convenience yield from holding Investment G (government note)
= $\beta r(t) + x(t)$, where $\beta > 0$.

L(r,q,t) = date t fair market value of investment L (a portfolio comprised of the interest rate swap and the position in short term LIBOR that is periodically rolled over).

G(r,q,x,t) = date t fair market value of investment G (the government note).

P(r_0,q_0,t) = date 0 fair market value of a riskless zero coupon bond (with no convenience yield) paying \$1 at date t when the initial state variables have values $r(0) = r_0$, $q(0) = q_0$.

The convenience yield, y(t), is stochastic and is assumed to be independent of the maturity of Investment G. We assume that one can translate this yield into an instantaneous net cash flow by lending out investment G for an instant. We model the convenience yield as a linear function of r and another state variable, x. This is consistent with the stylized observation that swap spreads are positively related to the level of interest rates. It can simply be due to the fact that expected inflation changes the price numeraire for everything, including liquidity benefits. In addition, as real interest rates rise, uninformed noise investors may naively move from the equity markets to the bond markets, creating larger opportunities for informed "arbitrageurs." These arbitrageurs, who hedge interest rate risk by shorting Treasury securities, may find that as their arbitrage activities increase, their added hedging needs may increase the value of the liquidity possessed by Treasury securities.

There are two reasons that we allow x(t) to be stochastic. First, it is possible that the liquidity advantage of investment G over investment L, measured in liquidity units, may change as the Treasury securities become relatively more or less liquid. These changes may depend on factors like the amount of Treasury notes outstanding, a change in the projected

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fiscal deficit, or the time since issue of the Treasury security.[22] However, it is also possible that this liquidity advantage is relatively constant and that the convenience yield changes because the preferences for liquidity change stochastically. In this case, a given liquidity unit translates into a different amount of convenience yield units, since the latter is measured in monetary units.

The latter interpretation of $x(t)$ may explain why certain risk proxies (like the yield spread between lower rated and higher rated corporate bonds) are often correlated with interest rate swap spreads. It is not that swaps, per se, have become riskier or less risky, but that the liquidity preference of the marginal investor has changed, depending on the estimates placed on a "credit crunch," or a run on their bank.

To derive the fair market interest rate swap spread, we need only compare the cash flows of investments L and G. The cash flows of investment L are the cash flows of a fixed rate bond. By definition, the coupons of this bond exceed those of the Treasury note in Investment G by the fixed swap spread (see Figure 1). Thus, a short position in investment L and a long position in investment G results in a constant net cash outflow every T/N years. This constant net cash outflow is the swap spread, and it pays for the stream y, the liquidity advantage gained by holding investment G. However, if two counterparties entered into a contract to merely exchange the cash flows of the two investments with each other, the payer of investment L's cash flows would pay the swap spread every T/N years and not receive the liquidity advantage.

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[22]For a description of how and why liquidity changes as a Treasury security ages, see Sarig and Warga (1989).

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It follows that the present value of the swap spread cash flows is equivalent to the present value of the liquidity-generated convenience yield of investment G. We employ the risk-neutral martingale valuation methodology to compute the present value of this convenience yield. At each point in time, we consider every pair consisting of a path for r and a convenience yield outcome $y = \beta r + x$. We discount y by the sequence of risk-free rates along the path for r, weighting each discounted y in a y path by the martingale probability measure for that pair generated by the stochastic processes for r and x.[23] The weighted sum is the present value of the liquidity benefit at that point in time. Integrating over all points in time from date 0 to the maturity date gives the present value of the liquidity benefit. We state and prove this formally as follows:

Proposition 1: Assume
(1) State variables x, r, and the vector q follow diffusion processes

$$dr = \kappa(r,q,t)dt \qquad + \sigma_r(x,r,q,t)dz \qquad\qquad (1)$$
$$dx = \theta(x,r,q,t)dt \qquad + \sigma_x(x,r,q,t)dw \qquad\qquad (2)$$
$$dq_i = a_i(r,q,t)dt \qquad + \sigma_i(r,q,t)du_i, \qquad\qquad (3)$$

where du_i is independent of everything and $E(dzdw) = \rho(x,r,q,t)$.

(2) The interest rate swap spread is the cash flow from a portfolio that is long \$1 in investment L (described above) and short \$1 in investment G (described above)

(3) The instantaneous liquidity benefit of investment G, y dt, satisfies
   $y = \beta r + x$,

(4) trading is continuous

(5) markets are frictionless

(5) counterparty default risk has no effect on swap pricing

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[23]In the literature on derivatives, this is a standard methodology used to solve these kinds of problems.

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(6) there is no arbitrage

Then, the fair market annualized swap spread is the annualized annuity equivalent of the present value of the liquidity benefit,

Swap Spread(T) = (N/T) PV(liquidity benefit)/Σ_i P($r_0,q_0,iT/N$), where

$$PV(\text{liquidity benefit}) = \int_0^T E(e^{-\int_0^t \tilde{r}(\tau)d\tau} (\beta\tilde{r}(t) + \tilde{x}(t)) \,|r_0, x_0)dt$$

$$= \beta(1 - P(r_0,q_0,T)) + \int_0^T [P(r_0,q_0,t)E(\tilde{x}(t)|x_0) + cov(e^{-\int_0^t \tilde{r}(\tau)d\tau} \tilde{x}(t)|r_0, x_0)]dt$$

and where expectations, E(), and covariances are taken with respect to the "risk neutral" martingale probability measures generated by modifying the stochastic processes for r and x.[24]

Proof: Applying Ito's Lemma to derive the stochastic differential equations for investments L and G, with respective values L(r,q,t) and G(r,q,x,t) we obtain

$$dL = L_t dt + L_r dr + \sum L_{qi} dq_i + \tfrac{1}{2}(L_{rr}\sigma_r^2 + \Sigma L_{qi,qi}\sigma_i^2)dt \tag{4}$$

$$dG = G_t dt + G_r dr + G_x dx + \tfrac{1}{2}(G_{rr}\sigma_r^2 + 2G_{rx}\rho\sigma_x\sigma_r + G_{xx}\sigma_x^2 + \Sigma G_{qi,qi}\sigma_i^2)dt, \tag{5}$$

It is well known that Equation 4, in combination with no arbitrage conditions for as many type L investments as state variables determining L, gives rise to the differential equation

$$L_t + L_r[\kappa(r,q,t) - \lambda(r,q,t)] + \Sigma L_{qi}[a_i(r,q,t) - \eta_i(r,q,t)] + \tfrac{1}{2}(L_{rr}\sigma_r^2 + \Sigma L_{qi,qi}\sigma_i^2) = Lr. \tag{6}$$

where $\lambda(r,q,t)$ and $\eta_i(r,q,t)$ are free parameters.

We now turn to Investment G. Consider a sufficiently large number of investments of this type, each with different maturities, along with a type L investment of any maturity. Since all of these investments depend on at most r, x, and q, the instantaneous returns of a properly weighted replicating portfolio of the G type investments is perfectly correlated with the L type investment. Equations (4) and (5) can be used to show that arbitrage is prevented iff

$$E(dG) = G(r - y + \lambda(r,q,t)G_r/G + \mu(r,q,x,t)G_x/G + \Sigma\eta_i(r,q,t)G_{qi}/G)]$$

where $\mu(r,q,x,t)$ is a set of free parameters, or equivalently,

$$G_t + G_r[\kappa(r,q,t) - \lambda(r,q,t)] + G_x[\theta(r,q,x.t) - \mu(r,q,x,t))] + \Sigma G_{qi}[a_i(r,q,t) - \eta_i(r,q,t))]$$
$$+ \tfrac{1}{2}(G_{rr}\sigma_r^2 + 2G_{rx}\rho\sigma_x\sigma_r + G_{xx}\sigma_x^2 + \Sigma G_{qi,qi}\sigma_i^2) = G(r-y) \tag{7}$$

Equations (6) and (7) suggest that investments G and L are priced as if they were traded in a risk neutral financial market where the stochastic processes that generate r, q, and x are

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[24]The term that multiplies β in the numerator represents the present value of the interest cash flows of a floating rate bond that pays instantaneous LIBOR. With a one dollar principal payment at the end, such a bond (can be shown with some trivial inductive logic) to have a value of one dollar. Without the the principal payment, its value is one dollar less the present value of the principal payment. The annuitized value of this term can be shown to be equal to the all-in-cost of the swap.

modified to be

$$dr = [\kappa(r,q,t) - \lambda(r,q,t)]dt + \sigma_r(x,r,q,t)dz \tag{8}$$

$$dx = [\theta(x,r,q,t) - \mu(r,q,x,t)]dt + \sigma_x(x,r,q,t)dw \tag{9}$$

$$dq_i = [a_i(x,r,q,t) - \eta_i(r,q,t)]dt + \sigma_i(x,r,q,t)du_i, \tag{10}$$

In such a risk neutral market, investment L appreciates at the instantaneous rate rdt. Investment G appreciates at the rate (r-y)dt, which is less than the riskless rate, reflecting the additional cash benefit obtainable by lending out investment G in the repo market.[25]

Any stream of cash flows that depend on x, r, q, and time has a value that can be replicated by a portfolio of investments L and G. Following a standard result from the derivatives literature, we can value such cash flows by the expectation of their risk-free discounted values, where expectations are computed with respect to the risk-neutral processes (8), (9), and (10). The interest rate swap spread is such a cash flow stream, and thus, trivially, has the present value expressed in the proposition.

<div align="center">Q.E.D.</div>

The valuation formula becomes a closed form integral if we make r and x independent, and if the risk neutral expectation of x can be computed. One interesting class of such cases occurs when the modified risk neutral process for x is

$$dx = \theta^*(X^* - x)dt + \sigma_x(x,r,q,t)dw, \tag{11}$$

where θ^* and X^* are constants. Here, we can substitute

$$E(x(t)) = e^{-\theta^* t}x_0 + (1 - e^{-\theta^* t})X^* \tag{12}$$

into the Proposition 1's swap spread formula and set the covariance in the formula to zero.

Alternatively, if the stochastic processes permit analytic computation of both the risk neutral expectation and covariance, we also obtain an analytic solution.

In the next two subsections, we discuss two specific cases that fall into these two classes.

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[25]We need not worry about coupons here. At ex-coupon dates, both investments L and G have prices that drop by the amount of the coupon, which is known with certainty at that time. In essence, the expected returns of investment L and G at the ex-coupon date, are still rdt and (r-y)dt at each date if we account for these cash flows.

<div align="center">15</div>

2.2. The Cox-Ingersoll-Ross (CIR) Special Case

In this subsection, we explore a Cox-Ingersoll-Ross (1985) version of the model that generates an analytic solution. Here, r and x are assumed to follow a jointly independent square root process, i.e.,

$$dr = \kappa(r^* - r) + \sigma_r\sqrt{r}\,dz$$
$$dx = \theta(x^* - x) + \sigma_x\sqrt{x}\,dw,$$

where $E(dzdw) = 0dt$. One can deduce from the technology and utility assumptions of this model that the interest rate risk premium parameter, $\lambda(r,t) = \lambda r$ and the risk premium for volatility in x, $\mu(x,r,t) = \mu x$. In addition, zero coupon bond prices are generated by a single factor, r.

For the CIR special case, the interest rate swap spread for a swap of maturity T with N equally spaced payments is given by

$$Swap\ Spread\ (T) = \frac{\beta(1 - P(r_0,T)) + \int_0^T P(r_0, t)\left[X^* + e^{-\theta^* t}(x_0 - X^*)\right]dt}{T/N\sum_{i=1}^N P(r_0, iT/N)}, \tag{13}$$

where

$$P(r_0,t) = A(t)\exp(-B(t)r_0), \text{ the date 0 value of \$1 paid at t,}$$
$$\text{with } A(t) \text{ and } B(t) \text{ given by the CIR term structure model}$$
$$X^* = \theta x^*/(\theta+\mu)$$
$$\theta^* = \theta + \mu$$

The risk neutral expectation operator used to derive this formula thus has probabilities generated by the modified stochastic processes

$$dr = (\kappa+\lambda)(\kappa r^*/(\kappa+\lambda) - r)dt + \sigma_r\sqrt{r}\,dz \tag{14}$$
$$dx = (\theta+\mu)(\theta x^*/(\theta+\mu) - x)dt + \sigma_x\sqrt{x}\,dw. \tag{15}$$

2.3. The Vasicek Special Case

In this section, we explore a Vasicek (1977) version of the model. This case assumes that r(t) and x(t) follow a bivariate Ornstein-Uhlenbeck (AR1) process:

$$dr = \kappa(r^* - r)dt + \sigma_r dz \qquad (16)$$
$$dx = \theta(x^* - x)dt + \sigma_x dw, \text{ where} \qquad (17)$$

$$dw = \rho\, dz + \sqrt{1 - \rho^2}\, du,$$

and where z and u follow independent standard Wiener processes. It also assumes that zero coupon bond prices are generated by a single factor, r, and that the risk premia for return volatility generated by z and w, are constant, i.e., $\lambda(r,t) = \lambda$ and $\mu(x,r,t) = \mu$. Note that with this process, we can relax the independence assumption of the CIR version of the model, and allow dz and dw to be correlated.

The interest rate swap spread for a swap of maturity T with N equally spaced payments is then given by

$$\textit{Swap Spread } (T) = \frac{\beta(1 - P(r_0, T)) + \int_0^T P(r_0, t)\left[X^* + e^{-\theta t}(x_0 - X^*) - \dfrac{\rho\,\sigma_r\sigma_x}{\kappa}A(t)\right]dt}{T/N \sum_{i=1}^{N} P(r_0, iT/N)},$$

$$(18)$$

where

$$A(t) = \frac{1 - e^{-\theta t}}{\theta} - \frac{1 - e^{-(\theta + \kappa)t}}{\theta + \kappa},$$

$P(r_0,t) \quad = \quad$ A(t)exp(-B(t)r_0), the date 0 value of \$1 paid at t,
with A(t) and B(t) given by the Vasicek term structure model
$X^* \quad = \quad x^* - \mu/\theta.$

The risk neutral expectation operator used to derive this formula thus has probabilities generated by the modified stochastic processes

$$dr = \kappa(r^* - \lambda/\kappa - r)dt + \sigma_r dz \text{ and} \tag{19}$$

$$dx = \theta(x^* - \mu/\theta - x)dt + \sigma_x dw. \tag{20}$$

The Appendix shows that the risk neutral covariance

$$cov\left(e^{-\int_0^t r(\tau)d\tau}, x(t)\right) = -P(r_0,t)\frac{\rho\,\sigma_r\sigma_x}{\kappa}\left[\frac{1-e^{-\theta t}}{\theta} - \frac{1-e^{-(\theta+\kappa)t}}{\theta+\kappa}\right]$$

The expression in the large brackets in Equation (18) is the certainty equivalent of the x

portion of the liquidity-based convenience yield of the government note. If $\rho = 0$, this

certainty equivalent is the expected value of x, where the expectation is computed with the

risk-neutral probability measure generated by the stochastic processes described in equations

(19) and (20). However, if ρ is non-zero, realizations of particular liquidity benefits are

associated with different probability measures for the interest rate path. The certainty

equivalent must be adjusted to account for this.

2.4. Numerical Values for Swap Spreads

In this section, we explore a number of parametrizations of the model that illustrate a

variety of swap spread curves that can be generated by the model. These will enable us to

develop intuition about the effects of various model parameters. The integration necessary for

the results reported in this section was approximated by a monthly summation.[26] The swap

spread curves from ten parametrizations of the Vasicek version of the model and the associated

term structure of interest rates are given in Table 1. Each parametrization corresponds to a

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[26]The approximation typically, but not always, generates a slight downward bias in
the swap spread model value. The distortion is no more than two basis points and
usually less than one basis point. The effect on the slope of the term structure is an
order of magnitude smaller.

column in the table. Spreads for years 1-5, 7, and 10 are reported. Graphs of this table are

provided in Figure 2.

Table 1: Ten Parametrizations of the Interest Rate Swap Model, Resulting Swap Spreads (in basis points (bp)), and Vasicek Term Structure (below swap spreads in %)										
Params	1	2	3	4	5	6	7	8	9	10
R^* (%)	6	6	6	6	6	6	10	6	6	4
X^* (bp)	70	70	0	0	80	40	-25	-25	100	-150
r_0 (%)	6	6	6	6	6	6	6	10	14	12
x_0 (bp)	70	70	0	0	40	80	-25	-25	30	-400
κ	.2	.2	.2	.2	.2	.2	.2	.2	.4	.2
θ	.2	.2	.2	.2	.2	.2	.2	.2	.12	.4
σ_r (%)	2	2	2	2	2	2	2	2	2	2
σ_x (%)	1	1	1	1	1	1	1	1	1	1
ρ	0	.8	0	.5	0	0	0	0	0	0
β	0	0	.1	.1	0	0	.1	.1	.05	.4
swapspreads and term structure:										
1 yr spread: yld %:	71 5.99	71 5.99	61 5.99	61 5.99	45 5.99	77 5.99	39 6.37	73 9.62	100 12.59	102 11.25
2 yr spread: yld (%):	71 5.98	70 5.98	61 5.98	60 5.98	48 5.98	74 5.98	43 6.68	69 9.28	98 11.49	108 10.57
3 yr spread: yld (%):	71 5.96	69 5.96	61 5.96	60 5.96	51 5.96	71 5.96	45 6.95	67 8.97	97 10.63	110 9.98
4 yr spread: yld (%):	71 5.94	68 5.94	60 5.94	59 5.94	53 5.94	68 5.94	47 7.19	64 8.69	97 9.95	109 9.45
5 yr spread: yld (%):	71 5.92	68 5.92	60 5.92	58 5.92	55 5.92	66 5.92	49 7.39	62 8.44	97 9.41	107 8.97

Table 1: Ten Parametrizations of the Interest Rate Swap Model, Resulting Swap Spreads (in basis points (bp)), and Vasicek Term Structure (below swap spreads in %)										
7 yr spread: yld (%)	71 5.87	66 5.87	60 5.87	57 5.87	58 5.87	63 5.87	52 7.72	58 8.02	98 8.62	100 8.18
10yrspread: yld (%)	71 5.81	64 5.81	59 5.81	55 5.81	62 5.81	59 5.81	55 8.08	54 7.54	100 7.88	88 7.27
Description:	flat	ρ effect	β effect	$\beta\rho$ effect	x up slope	x dn slope	r up slope	r dn slope	u shape	hump

Parametrization 1 illustrates a flat swap spread term structure. The swap spread generated is a constant seventy-one basis points. This swap spread curve is generated by parameters where x_0 and r_0, the initial values of the state variables, are at their risk-neutral long run equilibrium levels, X^* (not necessarily x^*), and R^*. The associated term structure of interest rates in this case is slightly downward sloping because bond prices are convex functions of interest rates. In this case, altering κ and θ have no effect on the swap spread curve. The larger is ρ, the more downward sloping the swap spread term structure. A negative ρ generates an upward sloping term structure. With $\rho = 0$, σ_r and σ_x have no effect on the term structure of swap spreads.

Parametrization 2 illustrates a slightly downward sloping swap spread term structure, driven by a large positive correlation between changes in the liquidity-based convenience yield and changes in the interest rate factor. Essentially, the paths that lead to large liquidity-based convenience yields in the future are discounted at higher (lower) rates if ρ is positive (negative) and those paths that have low convenience yields have low discount rates. Here, in contrast to Parametrization 1, σ_r and σ_x have an effect. The larger either is, the larger the covariance between changes in the two state variables and the more downward sloping the swap spread term structure. However, the effect is

rather slight. Even at the unreasonably large correlation of .8, the spread declines by 7 basis points as maturity goes from one year to ten years.

Parametrization 3 illustrates the effect of a positive β, which is the coefficient on interest rates that, in part, determines the level of the convenience yield. The more positive is β, the larger is the swap spread. However, if, because of bond convexity, the term structure of interest rates is downward sloping, as is the case here, the swap spread term structure will also be downward sloping. Here the resulting slope is slight, with only two basis points separating the spread in a ten year swap from a one year swap. However, a larger β, a smaller κ, and a larger σ_r can magnify the downward slope. Altering θ or σ_x has no effect in this scenario, (assuming $X^* = x^* - \mu/\theta$ is held constant).

Parametrization 4 illustrates what happens if we combine a positive β with a positive ρ. There is, once again, a downward slope to the swap spread term structure, but the one year-ten year spread difference is now six basis points rather than two basis points. An increase in θ or κ makes the negative slope less steep, while an increase in either of the two standard deviations exacerbates the steepness of the slope.

Parametrizations 5 and 6 illustrate what happens if X^* deviates from x_0. If, as in Parametrization 5, $X^* > x_0$, the swap spread term structure is upward sloping. If the opposite is true, as in Parametrization 6, it is downward sloping. An increase in the liquidity mean reversion parameter θ increases the steepness of the slope, whether it is positive or negative. However, the interest rate mean reversion parameter, κ, and volatility σ_r, have no effect (assuming $\rho = 0$). An increase in ρ makes the positive slope

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in Parametrization 5 less steep and the negative slope in Parametrization 6 more steep. A change in R^* essentially has no effect on the swap spreads. An increase in r_0 has a slight effect, both raising short maturity swap spreads and lowering long term swap spreads a bit.

Parametrizations 7 and 8 illustrate what happens if R^* deviates from r_0. These parametrizations have a different term structure of interest rates than the first six. For a deviation between R^* and r_0 to have an effect on the slope of the term structure of swap spreads, β must be non-zero. Here, it is assumed to be positive. Parametrization 7 assumes that $R^* > r_0$, which, for positive beta, generates an upward sloping swap spread term structure.

Parametrization 8 assumes $R^* < r_0$ and generates a downward sloping swap spread term structure. An increase in ρ or σ_r, because of the convexity effect and the positive β, makes the upward sloping term structure less upward sloping and the downward sloping term structure more downward sloping. σ_x and ρ, by contrast, have no effect (unless ρ is non-zero). An increase in κ exacerbates the steepness of the upward and downward sloping curves. Consistent with Parametrizations 5 and 6, an increase in X^* makes a positively sloped swap spread curve steeper and a negatively sloped one less steep. The opposite is true for an increase in x_0.

Parametrizations 9 and 10 illustrate odd-shaped swap spread term structures. These examples demonstrate the flexibility of the model to generate patterns of swap spreads that may be rare, but which have been observed on occasion. Parametrization 9 has a U-shaped swap spread term structure. Parametrization 10 illustrates a hump in the swap

spread term structure, with the peak spread occurring for three year swaps. Both are generated by downward sloping term structures of interest rates, a positive beta, and an x_0 that is below X*. The difference between the two shapes is driven largely by the ratio of the two mean reversion parameters. In the U-shaped case, interest rates revert much faster than the liquidity-based convenience yield. In the hump case, the reverse is true.

Table 2, using seven parametrizations, explores the effect of the CIR square root processes on the interest rate swap spread. The parametrization numbers at the top of the table correspond to those in Table 1. For comparison purposes, we use parameter values that are somewhat comparable (in a risk-neutral pricing environment) to those used in Table 1. This is not possible for parametrizations 3 and 4, which have a non-zero ρ, nor for parametrization 10, which has extreme negative values of x_0 and X*, which are not permitted in a square root process. For the two parametrizations, 7 and 8, that have small, identical, negative values of x_0 and X*, we use their absolute values, .0025. We employ the same values for parametrization 2, which had zero values for x_0 and X*. The remaining four parametrizations, 1, 5, 6, and 9, essentially use the same parameters as found in Table 1, with one caveat: In all seven parametrizations, the two volatility parameters are not directly comparable between the Ornstein-Uhlenbeck model and the square root model. To make the comparison as close as possible, we convert the Ornstein-Uhlenbeck σ_r and σ_x into square root volatilities by dividing each respectively by $\sqrt{R}$* and $\sqrt{X}$*, where $R^* = \kappa r^*/(\kappa + \lambda)$ and $X^* = \theta x^*/(\theta + \mu)$.

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Params	1	2	5	6	7	8	9
Table 2: Seven Parametrizations of the Interest Rate Swap Model, Resulting Swap Spreads (in basis points (bp)), and Cox-Ingersoll-Ross Term Structure (below swap spreads in %)							
R^* (%)	6	6	6	6	10	6	6
X^* (bp)	70	25	80	40	25	25	100
r_0 (%)	6	6	6	6	6	10	14
x_0 (bp)	70	25	40	80	25	25	30
κ	.2	.2	.2	.2	.2	.2	.4
θ	.2	.2	.2	.2	.2	.2	.12
σ_r (%) (CIR)	8.165	8.165	8.165	8.165	8.165	8.165	8.165
σ_x (%) (CIR)	11.952	11.952	11.952	11.952	11.952	11.952	11.952
ρ	0	0	0	0	0	0	0
β	0	.1	0	0	.1	.1	.05
swapspreads and term structure:							
1 yr spread: yld %:	71 5.99	86 5.99	45 5.99	77 5.99	90 6.37	124 9.62	100 12.58
2 yr spread: yld (%):	71 5.98	86 5.98	48 5.98	74 5.98	93 6.69	121 9.26	98 11.48
3 yr spread: yld (%):	71 5.96	86 5.96	51 5.96	71 5.96	96 6.97	117 8.95	97 10.61
4 yr spread: yld (%):	71 5.94	86 5.94	53 5.94	68 5.94	98 7.21	115 8.66	97 9.92
5 yr spread: yld (%):	71 5.92	85 5.92	55 5.92	66 5.92	100 7.41	112 8.40	97 9.38
7 yr spread: yld (%)	71 5.87	85 5.87	58 5.87	63 5.87	103 7.76	108 7.97	98 8.58
10yr spread: yld (%)	71 5.82	85 5.82	62 5.82	59 5.82	106 8.13	104 7.48	100 7.85

Table 2: Seven Parametrizations of the Interest Rate Swap Model, Resulting Swap Spreads (in basis points (bp)), and Cox-Ingersoll-Ross Term Structure (below swap spreads in %)							
Description:	flat	β effect	x up slope	x dn slope	r up slope	r dn slope	u shape

The results for the four parametrizations that are directly comparable--1, 5, 6, and 9--are remarkably similar to those found in Table 1. Those for which we needed to raise the values of x_0 and X^* to positive numbers, specifically 2, 7, and 8, have term structures that have been shifted up, but which have essentially the same slope as those in Table 1. This suggests that the model is not terribly sensitive to the particular functional form of the stochastic process.

3. Estimation of Parameters of the Model

Weekly data from 1/1/88 to 2/28/92 on U.S. dollar fixed-LIBOR floating interest swap spreads for maturities of one through five years (inclusive) was obtained from Salomon Brothers, Inc.[27] The data represents Salomon's swap traders' Friday 3:00 PM mid-market estimates of the swap spreads for these maturities. Along with this data, Salomon provided 3:00 PM mid-market quotes for one-month and six-month T-bills and one-month and six-month LIBOR.

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[27]Salomon also provided data on 7 and 10 year swap spreads. However, in contrast to the highly liquid 1-5 year maturities, trades in the 7 and 10 year maturities rarely took place over our sample period. Hence, we do not consider these trader indications to be as representative of market prices as the spreads quoted for the other five maturities. A similar problem plagues data from 11/1/85 through 12/31/87, also provided by Salomon, at which time the swap market was still in its infancy (less than $800 billion in notional even by the end of '87) and there was much less competition. In addition, because this period encompasses a stock market crash (October 87) and a bond market crash (April 87) of unprecedented proportions, parameter estimation is likely to be unduly affected by this period and we excluded it from our study.

In this section, we use this data to estimate the parameters of the general model assuming x and r are independent. Recall from the discussion at the end of section 2.1 that when x and r are independent and Equation 11 describes the risk neutral stochastic process for x, the swap spread for a swap of maturity T is

$$
Swap\ Spread\ (T) = \frac{\beta(1 - P(r_0,T)) + \int_0^T P(r_0,\ t)\left[X^* + e^{-\theta^* t}(x_0 - X^*)\right]dt}{T/N\sum_{i=1}^{N} P(r_0,\ iT/N)},
\tag{21}
$$

This encompasses both the CIR and Vasicek versions of the model, although the zero prices $P(r_0, t)$ and the risk-neutral parameters, X^* and θ^* for a given X and θ, would vary between the models. Rather than second guess the functional form of the term structure, we use the actual term structure to compute the zero coupon prices $P()$. Specifically, using data on 6 month LIBOR, as well as the 1,2,3,4, and 5 year all-in-cost, we use a cubic spline to interpolate the all-in-cost for maturities 1.5, 2.5, 3.5, and 4.5 years and then compute ten semi-annual zero coupon bond prices from the all-in-cost rates. We then apply a second cubic spline to the semi-annual zero coupon prices to obtain interpolated zero coupon prices for each interval of time necessary for the numerical integration.[28]

Equation 21 is a fairly simple regression that is nonlinear in only one parameter, θ^* and, given θ^*, for a time series of length τ, is linear in the coefficients $x_0(1),...x_0(\tau),\beta$, and X^*. There is no constant in the regression. To estimate parameters, we pool cross-section and time series and minimize the total sum of squares. This gives us 1080 observations and 219

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[28]This numerical integration was performed with the routine provided with Gauss.

parameters to estimate. To obtain parameters that minimize the total sum of squares, we first orthogonalize x and the term structure of interest rates (as represented by the LIBOR all-in-cost curve), by regressing the pooled cross-section and time series of swap spreads on the all-in-cost of the swap and a constant. The slope coefficient in this univariate regression is β (see footnote 24). We then use the intercept plus the residuals from this regression as dependent variables in a nonlinear multiple regression.[29] We first guess a value for θ^* and then obtain OLS estimates of the remaining parameters, $x_0(1),...x_0(\tau)$, and X^*.[30] Iteration on θ^* is guided by the Newton-Raphson algorithm. Table 3 summarizes the parameter estimates.

Table 3: Parameter Estimates for the General Model (with standard errors in parentheses) based on Weekly Swap Spreads from 1/1/88-2/28/92.	
β	.075 (.003)
X^* (basis points)	-.7 (.8)
mean of x_0 (basis points)	-7.1 (1.1)
θ^*	.628
θ from slope of $x_0(t)$ on $x_0(t-1)$	3.967 (1.294)
ρ(predicted,actual) 1 year	.925
ρ(predicted,actual) 2 year	.961
ρ(predicted,actual) 3 year	.938
ρ(predicted,actual) 4 year	.818
ρ(predicted,actual) 5 year	.696

[29]The intercept in the second regression is suppressed.

[30]Since these coefficient estimates are linear functions of dependent variables that are constructed to be orthogonal to interest rates, we would expect them to be reasonably orthogonal to the r's that generate the interest rates.

Table 3. Parameters are obtained by minimizing the total sum of squared errors in Equation 21, using data from a pooled cross-section and Friday 3PM time series. For a given θ^*, OLS estimates of the remaining parameters are obtained in a 2 step regression--the first step a univariate regression of swap spreads on the all-in-cost and the second step a regression of residuals plus the constant from the first regression on the remaining variables in the swap spread equation. Iteration on θ^* is guided by the Newton-Raphson algorithm. Standard errors for X^* are obtained assuming that θ^* is estimated without error and using the OLS formulae for standard errors on the remaining coefficients. Standard errors of the average x_0 are computed as the standard error of the mean of the reproduced time series.

Figure 3 graphs the actual and predicted values for the five swap spreads through time.

The model is fit only with swap spread data and can therefore only determine the "risk-neutral" parameters. However, with x following an AR1 process, we expect the average x_0 to differ from X^* by μ/θ^*, the ratio of the risk premium parameter to the mean reversion parameter. The implicit estimate of the risk premium parameter that arises out of this comparison is -4.2 (basis points).

We use three methods to assess whether the model is reasonable. First, a glance at the five graphs in Figure 3 illustrates that the predicted swap spreads track the actual swap spreads quite reasonably. A notable exception is March 89 for 1, 4, and 5 year swap spreads. The correlations between actual and predicted swap spreads range from .70 to .96 depending on the maturity of the swap.

A good model will also generate parameters that are consistent with the assumed stochastic processes. No contraints are placed on the estimation of x_0's, (as would be the case, for example, with Kalman filter estimation) other than that they fit the swap spread data. However, if x follows an AR1 process, $\theta = \theta^*$. A comparison of the mean reversion

parameter obtained from regressing the x_0's on their lag1 values with the mean reversion parameter that provides the best fit for the swap spreads indicates that they are within two standard errors of one another.[31]

A final check of the model is provided by looking at observable data that might proxy for the state variables and comparing parameter estimates from such observable data with those obtained by fitting the swap spread model. Let us assume that r and y are observable and are respectively proxied for by one-month LIBOR (in CD equivalent form, which is the way it is quoted) and by the one month TED spread, which is computed here as the difference between one month LIBOR and the yield on a one-month Treasury bill (both in CD equivalent form). An X-Y plot of the TED spread against LIBOR (not provided here) indicates that the relation between the two variables is fairly linear.

The choice of the T-bill rate as the benchmark for the risk-free government rate is somewhat arbitrary. Over this sample period, T-bill T-note interest rate swaps have mid-market swap spreads of between ten and fifteen basis points. Hence, the TED spread probably overestimates the liquidity advantage of the on-the-run government note by ten to fifteen basis points.

We use this data to estimate the parameters of Equation 17 with a regression that has a non-linear constraint, using weekly data. Let

LIB_t = one month LIBOR at week t (CD equivalent)
$BILL_t$ = one month T-bill rate at week t (CD equivalent)
TED_t = LIB_t - $BILL_t$,

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[31]This is an overly conservative statement since the comparison assumes that $\theta*$ is estimated without error.

$$\Delta t \quad = 4/52$$

and for an arbitrary variable denoted VAR, denote the four week change[32] in the variable as

$$\Delta VAR_t = VAR_t - VAR_{t-4}.$$

Using the regression,

$$\Delta TED_t = \alpha_0 + \alpha_1 \Delta LIB_t + \alpha_2 LIB_{t-4}\Delta t + \alpha_3 TED_{t-4}\Delta t + \delta_t, \tag{22}$$

constrained so that $\alpha_2 = -\alpha_1\alpha_3$, we estimate

$$\beta \quad = \alpha_1,$$
$$\theta \quad = -\alpha_3,$$
$$x^* \quad = -\alpha_0/(\alpha_3\Delta t),$$

The parameter estimates obtained are $x^* = -78$ (basis points), $\beta = .27$, $\theta = 7.89$. These are very different from the estimates obtained by fitting the swap spreads. The risk premium parameter, μ, that minimizes the total sum of squared prediction errors for swap spreads using these parameter estimates is 1082 basis points. It generates an X^* of -169 basis points. This compares with a $\mu = -4.2$ and $X^* = -.7$ basis points from the fitted model described in Table 3. It is difficult to ascertain the reasons for the discrepancy between the results where liquidity is assumed to be directly measured and those where liquidity is inferred by goodness of fit. One possibility is that measurement error in the change in LIBOR appears on both sides of Equation 22 and tends to overstate the absolute magnitude of α_1 and α_3. The former bias, in turn, may force α_0 to be more negative than it really is. Another possibility is that the TED spread is not really a good measure of the liquidity based convenience yield. As such, we are

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[32]We use four weeks rather than one week because the LIBOR quotes are rounded to the nearest 1/16th and thus tend to have understated covariances with any random variable. The resulting overlapping data does not bias the coefficient estimates.

merely measuring the parameters of a different stochastic process. Since Equation 17 and Equation 11 are very different, it may also be that an AR1 process does not describe the true process for x, even though the risk neutral stochastic process for x is adequately captured by Equation 11. Finally, we have no method for measuring the true standard error of the difference in the estimates between the two methodologies. It may simply be that the standard error of the discrepancies in pairs of parameter estimates using the two estimation methodologies are huge.

4. Conclusion

This paper is a first attempt to develop a model where liquidity considerations alone generate interest rate swap spreads. It may also be the first paper to value liquidity with the valuation techniques developed for derivatives. For certain parameter restrictions, closed form solutions can be found for a number of interest rate processes, including multifactor ones.

An alternative way to think about these issues is to note that we have implicitly added a factor to traditional models of the Treasury yield curve. This paper takes the yield curve generated by the all-in-cost of the swap (also known as the LIBOR term structure). The Treasury yield curve is then derived from the LIBOR term structure by annuitizing the present value of a stochastic liquidity factor (the swap spread) and subtracting it from the LIBOR term structure. Hence, a 1-factor model of the yield curve can be thought of, not as a 1-factor model of the Treasury yield curve but as 1-factor model of the all-in-cost curve (the LIBOR yield curve). The Treasury yield curve is then generated by considering the effect of the additional liquidity factor.

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The model can generate term structures for interest rate swap spreads that have a variety of shapes, including U-shaped and humped curves. Elementary inspections of the swap spreads generated by the estimated model indicate that it is adequate at explaining the existing empirical data. It is not, however, consistent, with results based on measurements of the stochastic process generating the TED spread.

One important issue that the model does not account for is the effect of special repo. The model makes the assumption that the liquidity advantage of Treasury notes of all maturities is the same. Realistically, differences in the liquidity-based convenience yield of the Treasuries, due to some issues being "more special" than others at certain times needs to be accounted for. However, a model where there are different liquidity-based state variables for different maturities is not testable without further restrictions. One alternative is to keep the model in its current form and treat the effect of special repo as serially correlated measurement error in tests. This presents identification problems of its own, in that it may be impossible to determine if the serial correlation is driven by special repo or by some other misspecification of the model.

APPENDIX

Proof that

$$cov\left(e^{-\int_0^t r(\tau)d\tau}, x(t)\right) = -P(r_0,t)\frac{\rho\,\sigma_r\sigma_x}{\kappa}\left[\frac{1 - e^{-\theta t}}{\theta} - \frac{1 - e^{-(\theta+\kappa)t}}{\theta+\kappa}\right]$$

Define for all points in time τ

$$\phi = e^{\kappa\tau}r.$$
$$\psi = e^{\theta\tau}x.$$

By Ito's Lemma

$$d\phi = \kappa e^{\kappa\tau}R^* dt + e^{\kappa\tau}\sigma_r dz \text{ and}$$
$$d\psi = \theta e^{\theta\tau}X^* dt + e^{\theta\tau}\sigma_x dw,$$

By Stein's Lemma,

$$cov\left(e^{-\int_0^t r(\tau)d\tau}, x(t)\right) = -E\left(e^{-\int_0^t r(\tau)d\tau}\right)cov\left(\int_0^t r(\tau)d\tau, x(t)\right)$$

$$= -P(r_0,t)cov\left(\int_0^t e^{-\kappa\tau}\phi(\tau)d\tau, e^{-\theta t}\psi(t)\right)$$

$$= -P(r_0,t)E[(\int_0^t e^{-\kappa\tau}\int_0^\tau e^{\kappa s}\sigma_r dz(s)d\tau)(e^{-\theta t}\int_0^t e^{\theta\tau}\rho\,\sigma_x dz(\tau))] ,$$

where s is a variable indexing time. Reversing the order of integration for the double

(partially stochastic) integral above,

$$cov\left(e^{-\int_0^t r(\tau)d\tau}, x(t)\right) = -P(t)E[(\int_0^t e^{\kappa\tau}(\int_\tau^t e^{-\kappa\tau}\sigma_r d\tau)dz(\tau))(e^{-\theta t}\int_0^t e^{\theta\tau}\rho\,\sigma_x dz(\tau))]$$

$$= -P(t)e^{-\theta t}E[(\int_0^t e^{\kappa\tau}\frac{-\sigma_r}{\kappa}(e^{-\kappa t}-e^{-\kappa\tau})dz(\tau))(\int_0^t e^{\theta\tau}\rho\,\sigma_x dz(\tau))]$$

$$= -P(t)e^{-\theta t}\frac{-\rho\,\sigma_r\sigma_x}{\kappa}\int_0^t e^{(\kappa+\theta)\tau}(e^{-\kappa t}-e^{-\kappa\tau})d\tau ,$$

where the latter equality follows from the independence of the non-contemporaneous dz's and

the Brownian motion assumption that $E(dz^2) = d\tau$. Completing the integration,

$$cov\left(e^{-\int_0^t r(\tau)d\tau}, x(t)\right) = -P(t)e^{-\theta t}\frac{-\rho\,\sigma_r\sigma_x}{\kappa}\left[\frac{e^{(\kappa+\theta)t}-1}{\kappa+\theta}e^{-\kappa t} - \frac{e^{\theta t}-1}{\theta}\right]$$

$$= -P(t)\frac{\rho\,\sigma_r\sigma_x}{\kappa}\left[\frac{1-e^{-\theta t}}{\theta} - \frac{1-e^{-(\theta+\kappa)t}}{\theta+\kappa}\right].$$

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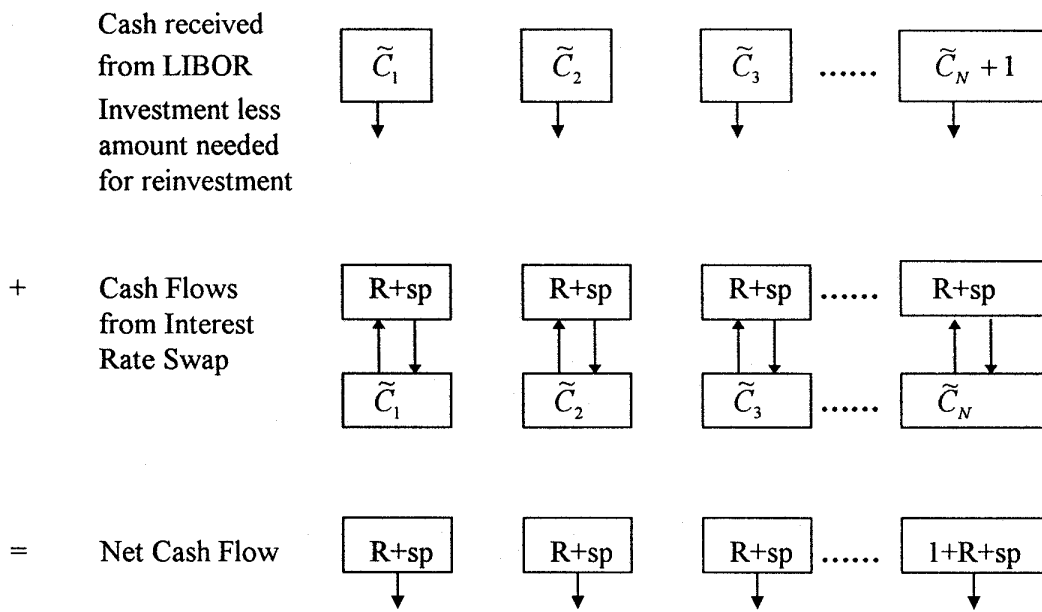
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FIGURE 1

Period:	1	2	3	.... N
Year:	$\dfrac{T}{N}$	$\dfrac{2T}{N}$	$\dfrac{3T}{N}$	...... T

Investment L

Cash received
from LIBOR
Investment less
amount needed
for reinvestment

$$\boxed{\widetilde{C}_1} \quad \boxed{\widetilde{C}_2} \quad \boxed{\widetilde{C}_3} \quad ...... \quad \boxed{\widetilde{C}_N + 1}$$

+ Cash Flows
from Interest
Rate Swap

$$\boxed{R+sp} \quad \boxed{R+sp} \quad \boxed{R+sp} \quad ...... \quad \boxed{R+sp}$$
$$\boxed{\widetilde{C}_1} \quad \boxed{\widetilde{C}_2} \quad \boxed{\widetilde{C}_3} \quad ...... \quad \boxed{\widetilde{C}_N}$$

= Net Cash Flow

$$\boxed{R+sp} \quad \boxed{R+sp} \quad \boxed{R+sp} \quad ...... \quad \boxed{1+R+sp}$$

Investment G

Net Cash Flows

$$\boxed{R} \quad \boxed{R} \quad \boxed{R} \quad ...... \quad \boxed{1+R}$$
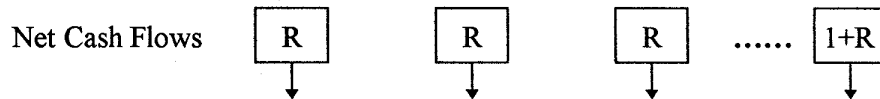
$\widetilde{C}_i$ = LIBOR rate at beginning of period
R   = Treasury yield for note of maturity T
sp = Swap spread for swap of maturity T
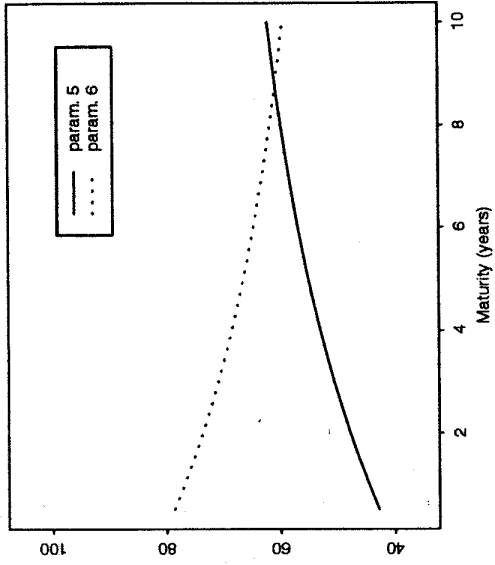N   = Number of coupons

FIGURE 2
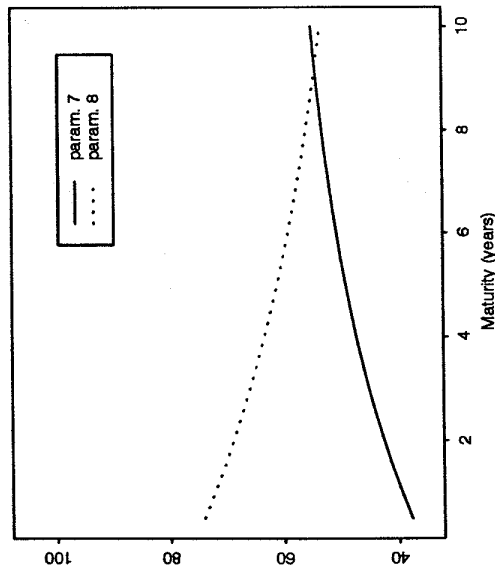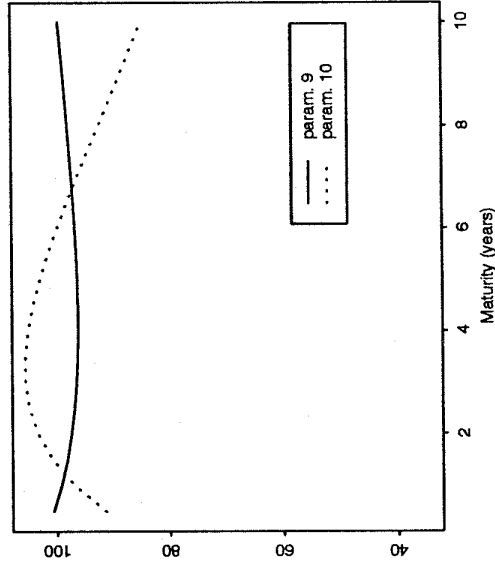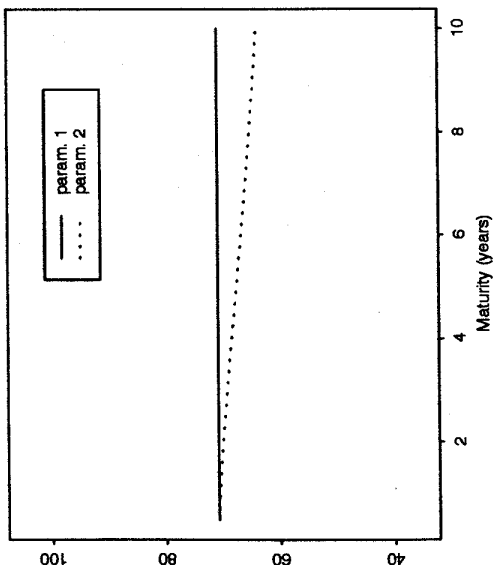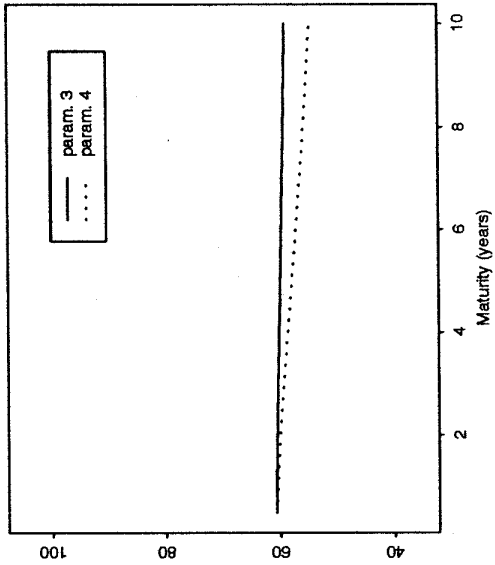
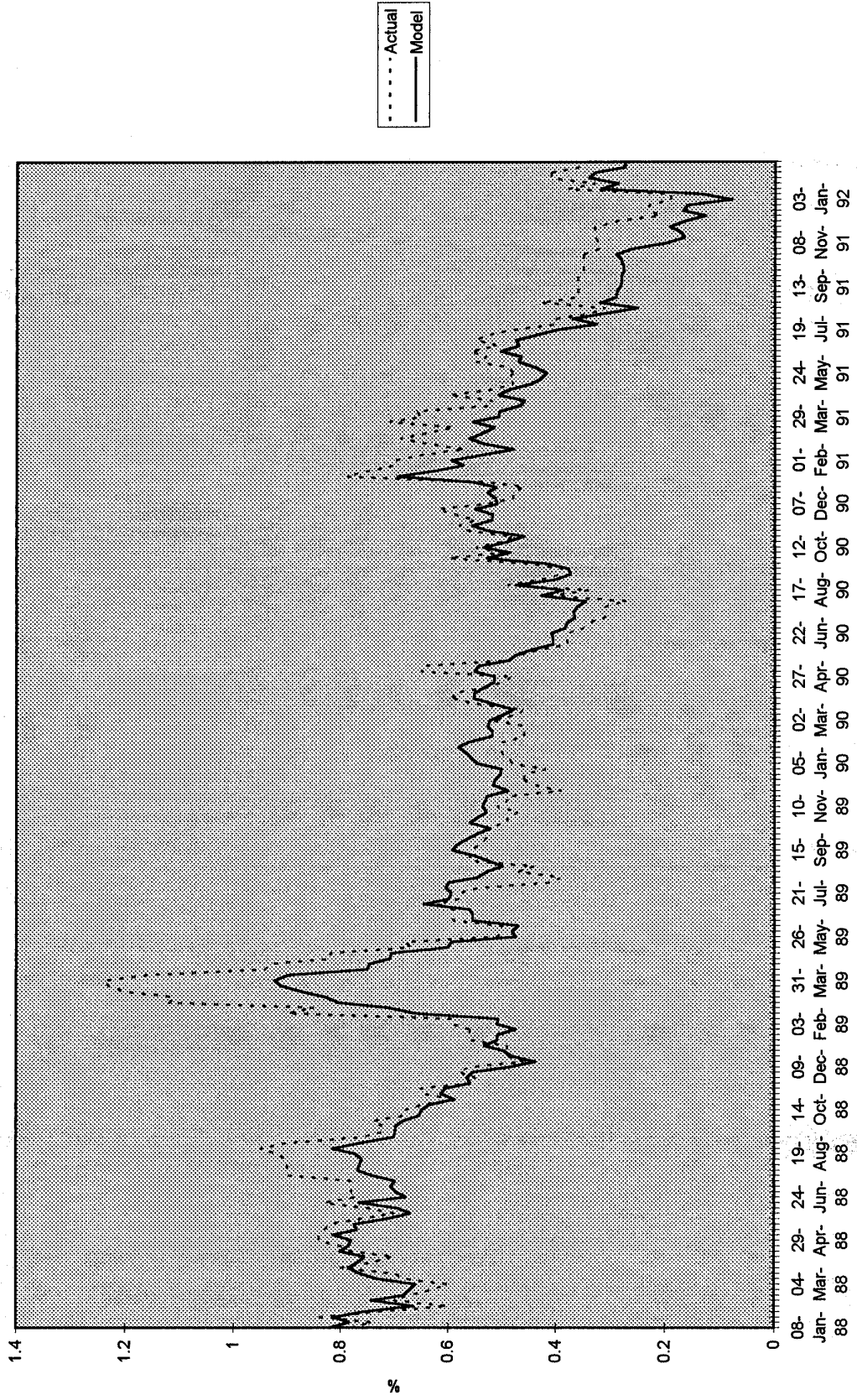Figure 3 - Panel A: 1 Year Swap-Spread

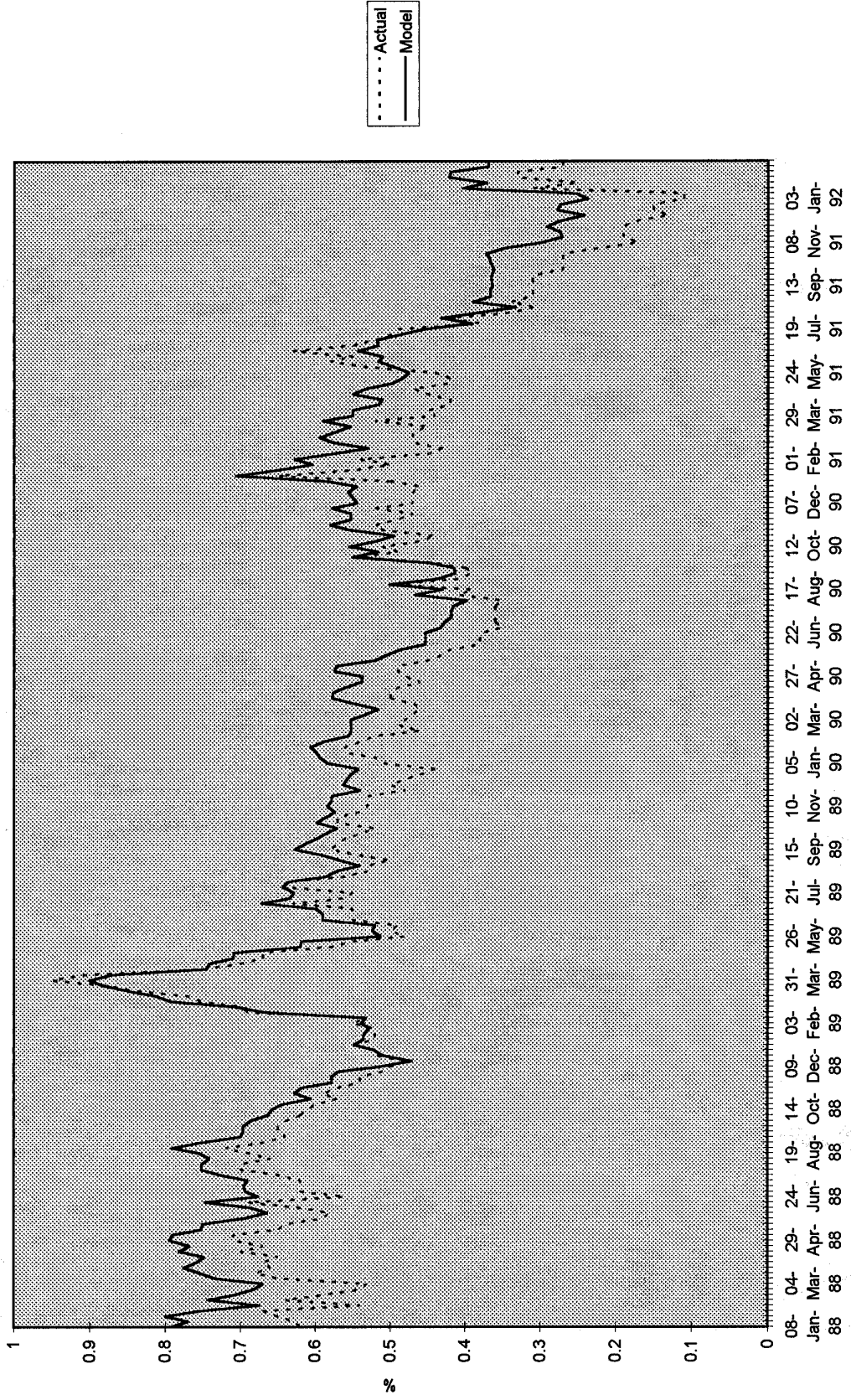**Figure 3 - Panel B: 2 Year Swap-Spread**
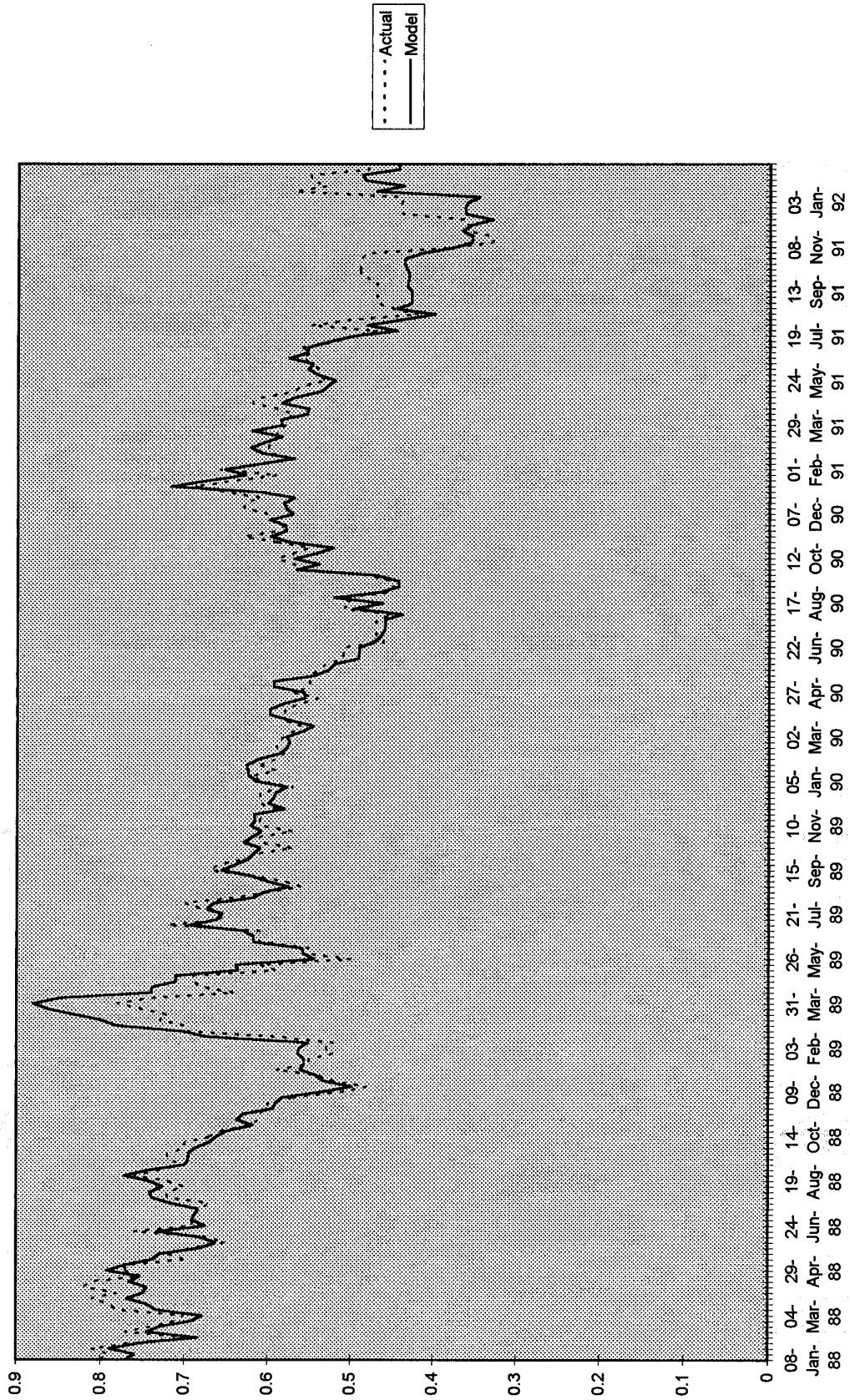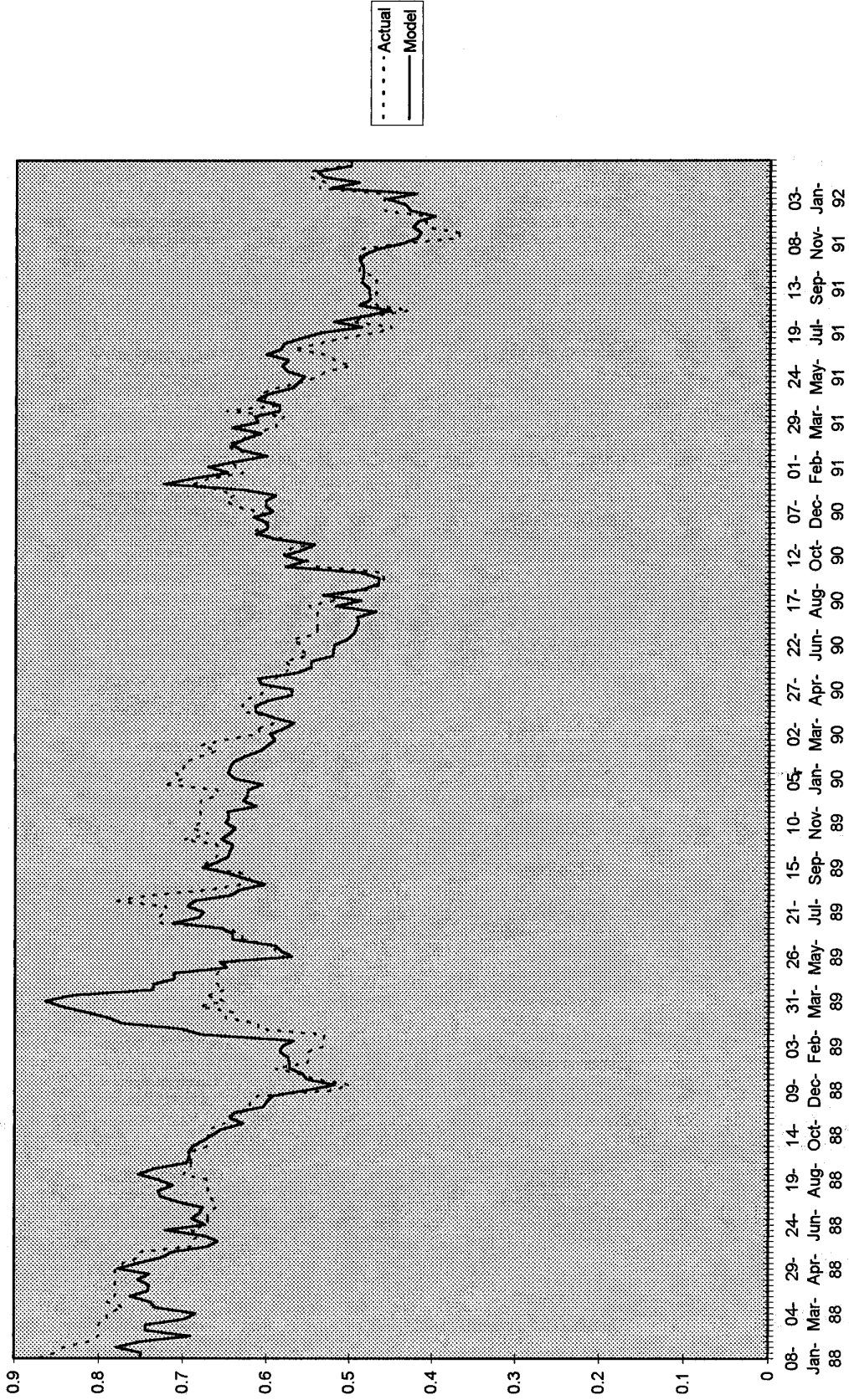
Figure 3 - Panel C: 3 Year Swap-Spread

Figure 3 - Panel D: 4 Year Swap-Spread

Figure 3 - Panel E: 5 Year Swap-Spread