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Fields and First Order Perturbation Effects in Two-Dimensional Conductor Dominated Magnets

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ABSTRACT

General expressions are given for the field and its expansion coefficients produced by a two dimensional conductor structure surrounded by iron with a circular inside boundary. Saturation effects are described in terms of the tangential field at that boundary. The effects of the following types of perturbations are discussed: displacement, rotation and error excitation of a conductor, change of conductor shape, and modification of the inside contour of the iron. A design criterion is given to minimize the error fields associated with a displacement of the iron shell relative to the conductor structure. Expressions for the force and torque acting on a conductor are derived both for the unperturbed and perturbed magnet. Formulae are presented that allow convenient and fast evaluation of pertinent quantities with a computer when the structure is too complicated for hand computations.

1. Introduction

With the maturing of superconductors as a practical material for the construction of magnets, a considerable amount of work has been done over the last few years on magnets whose fields are dominated not by the iron configuration, as in conventional iron magnets, but by the conductor configuration. A number of theoretical papers have been published, for instance those by Beth,[1] Blewett,[2] Asner,[3] Meuser,[4] that deal with two dimensional conductor-dominated magnets that are surrounded by a circular iron shell. These papers discuss special design considerations for specific types of magnets. The topic of this paper is in a sense the opposite: it will be attempted to give as general a description of conductor-dominated two-dimensional magnets as possible, and to discuss in particular the effects resulting from deviations of the actual magnet from the ideal design.

A good understanding of these perturbation effects is not only important for establishing reasonable manufacturing tolerances, but is also important for the design of a magnet. It will be shown, for instance, that it is advantageous to satisfy a certain design criterion to make the magnet insensitive to a particular very common kind of perturbation. Besides causing unwanted fields or harmonics, perturbations can also produce substantial forces between the conductor structure and the iron shell, and these forces can lead to disastrous results if not anticipated and properly taken into account. In contrast to the discussion of perturbation effects in iron-dominated magnets,[5] perturbation effects are fairly easily expressed to higher than first order of the perturbation parameters; however, this was not done here since the first order perturbation theory will be of sufficient accuracy in the majority of practical magnets.

In developing the general description, an attempt has been made to introduce as few restrictions as possible. Although symmetrical multipole magnets are emphasized, the theory has been kept general enough to also allow its application to other types of magnets. The only essential limitations are the restriction to two-dimensional magnets and the assumption that the boundary between the iron shell and the inside of the magnet is a circular cylinder. Although the latter restriction can in principle be dropped through application of the appropriate conformal transformation, in view of the resulting complications this does not seem be to justifiable at the present time. The exact analysis extends only to the boundary between the iron and the inside of the magnet, and the iron saturation effects are expressed in terms of the azimuthal field component that is produced by saturation effects at that boundary. The information necessary to obtain that field, or to design the outside contour of the shield, is supplied. This approach was preferred over the one including the iron shield and assuming constant permeability of the iron: If the permeability is small enough to be noticeable, the $B(H)$ curve is usually so nonlinear that the latter approach will usually give meaningless results. A drawback of the approach chosen here is that in calculating the effect of perturbations of the conductor structure, the effect of the change of saturation in the iron as a consequence of the perturbation is not taken into account, although this could be done with an iterative procedure. However, these secondary effects should be fairly small unless the iron is driven extremely hard, and seem to be difficult to describe properly no matter what basic approach is chosen. The description of iron-free magnets is obtained by dropping the terms associated with the iron shell, or by letting R (see fig. 1) go to infinity.

Although it is the main purpose of the general description to give a better understanding of various effects, the formulae are simple enough to be easily evaluated by hand for simple structures. To make quantitative evaluation possible when the structures are more complicated, sect. 7 gives explicit formulae that are easily evaluated by a computer. The guiding thought is to convert all surface integrals into contour integrals to make both the logic of the program simpler and to minimize execution time. In most parts of sect. 7 it has been assumed that the current density in the conductor structure is not a continuous function of the space coordinates, but is constant over finite cross sections of the conductors. Although this restriction could be lifted, it seems not worthwhile to do so at the present time.

In order to limit the length of this paper, the author had to leave out many formulae that might be of interest under some circumstances. It is hoped that the presentation is clear enough to allow the reader to derive expressions needed for specific applications, and in some instances it is briefly indicated how to obtain them.

2. Basic formulae, notation, normalization and units

The coordinate system is chosen such that the fields are in the ξ, η - plane of a Cartesian coordinate system, with the center of the iron shell at $\xi = \eta = 0$. MKS units are used throughout. All distances are normalized with the normalization length ρ, which is most conveniently chosen to be the useful aperture radius of the magnet. The actual quantities used to describe the geometry are the dimensionless quantities $x = \xi/\rho$, $y = \eta/\rho$ and $z = x + iy = re^{i\phi}$. This does not preclude dimensional checks on formulae, since at any stage a dimensional check can be performed by assuming that ρ is dimensionless and equals one.

The field components H_x and H_y can be derived in the conductor- and iron-free region from a scalar potential V, and everywhere from a vector potential which needs to have only a component A in the direction perpendicular to the x-y plane. The field components are obtained from the potentials through

$$\rho H_x = \partial A/\partial y = - \partial V/\partial x \tag{1a}$$

$$\rho H_y = - \partial A/\partial x = - \partial V/\partial y \quad . \tag{1b}$$

The eqs. involving V are of course valid only in conductor- and iron-free regions. Introducing the complex quantities $F(z) = A + iV$, and $H = H_x + iH_y$, and indicating the complex conjugate of a quantity by an asterisk, the field components can, because of eqs. (1), be obtained in a conductor- and iron-free region from the complex potential F through

$$\rho H^* = i dF(z)/dz \quad . \tag{2}$$

If the complex potential can be expanded into a Taylor series in z,

$$F(z) = \sum_{n=0}^{\infty} C_n z^n \quad , \tag{3}$$

the coefficients C_n are usually called the multipole coefficients. From eqs. (2) and (3) follows for the Taylor series of H^*:

$$H^* = \sum_{n=1}^{\infty} \frac{inC_n}{\rho} z^{n-1} \quad . \tag{4a}$$

Introducing $i \cdot n \cdot C_n/\rho = c_n$, the series for H^* becomes

$$H^* = \sum_{n=1}^{\infty} c_n z^{n-1} \quad . \tag{4b}$$

In this paper the coefficients c_n will be called the multipole coefficients. The physical significance of c_n and the usefulness of setting ρ equal to the usable aperture radius of the magnet follows from eq. (4b): The absolute value of the contribution of the term $c_n z^{n-1}$ to the total field at $|z| = 1$ equals $|c_n|$. c_n and the expansion coefficients that will be introduced below may sometimes be used to describe only the effects resulting from one part of the total structure. Whenever it is essential for the correctness of a formula that these coefficients describe the whole structure, the coefficient symbol will be underlined. Since this paper contains a large number of equations, the numbers of those equations that are either very important definitions or can be considered as an end result will be underlined. Throughout this paper, n, m, N, M represent integers. The difference between quantities describing the perturbed and unperturbed system will be indicated by Δ.

3. Fields for infinite permeability

3.1. BASIC FORMULAE

To describe the fields generated by the conductor structure, it is convenient to consider first the fields produced by a current filament. If the region of interest is enclosed by an infinite permeability iron shell with normalized inside radius R (fig. 1), the boundary condition at the inside surface is that the tangential field component is zero there. An equivalent condition is the requirement that the scalar potential is constant on that surface. This can obviously only be satisfied if the current return for the filament is also in the region enclosed by the iron. If the current filament with current I in the $\vec{x} \times \vec{y}$ direction is located at z and the current return at the coordinate origin, then it is easy to verify that the complex potential at z_o is given by

$$F(z_o) = -\frac{I}{2\pi}\cdot \ell n \left((z_o - z)(z_o - R^2/z^*)/z_o\right) \qquad (5a)$$

and that this complex potential satisfies the above stated boundary condition. If every current filament in the whole aperature is represented in this manner, the singularity at $z_o = 0$ disappears since the sum of all currents of the whole system must vanish. One can therefore consider the contribution of the current I at z to the total complex potential to be given by

$$F(z_o) = -\frac{I}{2\pi}\cdot \ell n \; (z_o - z)(z_o - R^2/z^*) \quad . \qquad (5b)$$

This potential can also be interpreted as the potential at $|z_o| < R$ produced by I at z if the return current is uniformly distributed over the inside boundary of the iron shell. From eqs. (5b) and (2) follows for H^*:

$$H^*(z_o) = \frac{iI}{2\pi\rho}\left(\frac{1}{z-z_o} + \frac{1}{R^2/z^*-z_o}\right) \quad . \tag{6a}$$

Introducing the normalized area element $d\sigma = d\xi d\eta/\rho^2 = dxdy$, and the current density j, the fields produced by distributed currents become

$$H^* = \frac{i\rho}{2\pi}\int j\left(\frac{1}{z-z_o} + \frac{1}{R^2/z^*-z_o}\right) d\sigma \quad . \tag{6b}$$

When calculating H^* inside a conductor, an infinitesimally small circular disc around z_o has to be excluded from the integration of the first part of the integrand. Although the contribution to the field from that circle is infinitesimally small, eliminating that circle from the integration has the effect that H^* is no longer an analytical function of z_o. This expected result becomes even more apparent in eq. (53).

Expanding H^* into a Taylor series in z_o, with a convergence radius equal to the distance from the origin to its closest conductor, gives for the multipole coefficients c_n from eqs. (4b) and (6b)

$$a_n = \frac{i\rho}{2\pi}\int jz^{-n}d\sigma \tag{7a}$$

$$b_n = \frac{i\rho}{2\pi}\int jz^{*n}d\sigma/R^{2n} \tag{7b}$$

$$c_n = a_n + b_n \quad . \tag{7c}$$

The coefficients c_n, a_n, b_n are intentionally not underlined since eqs. (7) describe obviously the contribution of any part of the structure. If a part of the structure is described by eqs. (7), and if that part is rotated about the coordinate origin by α in the positive sense, the coefficients produced by the rotated part are obtained from eqs. (7) by replacing z by $ze^{i\alpha}$, and one obtains

$$a_n(\alpha) = a_n(0) \cdot e^{-in\alpha} \tag{8a}$$

$$b_n(\alpha) = b_n(0) \cdot e^{-in\alpha} \tag{8b}$$

$$c_n(\alpha) = c_n(0) \cdot e^{-in\alpha} \quad . \tag{8c}$$

If one is dealing with a symmetrical $2N$ pole magnet whose conductor structure is invariant under rotation by π/N, with alternating sign of excitation of the individual sectors, and if the reference sector of angular extent π/N is described by a_n, b_n, c_n, it then follows from eqs. (8) that the whole structure is described by

$$\underline{c}_n = c_n \cdot \sum_{m=0}^{2N-1} e^{-inm\pi/N}(-1)^m = c_n \cdot \sum_{m=0}^{2N-1} e^{-im\pi(1+n/N)} \quad .$$

If every term of this sum equals one, the sum equals $2N$. Otherwise, application of the summation formula for the geometric series shows that the sum vanishes, leading to the following result:

$$\underline{c}_n = 2Nc_n;\ \underline{a}_n = 2Na_n;\ \underline{b}_n = 2Nb_n \quad \text{for} \quad n = N(2m+1)$$

$$\underline{c}_n = \underline{a}_n = \underline{b}_n = 0 \qquad \text{for} \quad n \neq N(2m+1) \quad . \tag{9}$$

Since the odd multiples of N are thus the only harmonics that are possible in a symmetrical 2N-pole magnet, they are called "allowed harmonics". If the reference sector has a symmetry plane and if this sector has such an orientation that it is symmetrical with respect to the x-axis, as is shown in fig. 1 for a dipole magnet, it follows from eqs. (7) that the expansion coefficients for that sector are imaginary. Because of eq. (9) the same is then also true for the expansion coefficients of the whole magnet.

3.2. EFFECT OF BASIC PERTURBATIONS ON THE FIELD

When one is dealing with a symmetrical 2N pole magnet, and the reference section is perturbed in some way, resulting in an effect described by Δc_n, and if all other sections have the same perturbation when rotated into the same position as the reference section, it is clear that the eqs. (9) are valid also if all a, b, c in eq. (9) are replaced by Δa, Δb, Δc. For this reason the emphasis in the following is on describing the effects of perturbations of parts of the system. In general the effects of perturbations are more damaging when they are not identical in all sections than when they are.

If a particular section of the system has for some reason an incorrect excitation, its effect is of course directly described by the coefficients a_n, b_n that describe the contribution of that section to the field.

If a particular section, described by a_n, b_n, is rotated by the small angle α, it follows directly from eqs. (8) that the effect of that rotation is to first order in α given by

$$\Delta a_n = - in\alpha a_n ; \Delta b_n = - in\alpha b_n \quad . \qquad (\underline{10})$$

For the discussion of the effect of a perturbation of the outside contour of a conductor, it is assumed for simplicity that in the conductor block under consideration, the current density j is constant. One then has to distinguish between two cases, namely when the total current is unchanged, and when j is not affected by the contour change. In the latter case, it follows from eqs. (7) that

$$\Delta a_n = \frac{i\rho}{2\pi} j\Delta\left(\int z^{-n} d\sigma \right) ; \Delta b_n = \frac{i\rho}{2\pi} \cdot \frac{j}{R^{2n}} \Delta\left(\int z^{*n} d\sigma \right) \quad . \qquad (\underline{11})$$

When the net current I is fixed, which will be the more frequent occurence, one can replace j in eqs. (7) by $I/\int d\sigma$ and obtains:

$$\Delta a_n = \frac{i\rho}{2\pi} j\Delta\left(\int z^{-n} d\sigma \right) - a_n \cdot \frac{\Delta\sigma}{\sigma} ; \Delta b_n = \frac{i\rho}{2\pi} \cdot \frac{j}{R^{2n}} \Delta\left(\int z^{*n} d\sigma \right) - b_n \frac{\Delta\sigma}{\sigma} \quad . \qquad (\underline{12})$$

If, as it will mostly be, the contour modification consists of the addition of a narrow strip of not necessarily constant thickness t, it is of course sufficient to calculate $\Delta\left(\int z^{*n} d\sigma\right)$ by evaluating $\int z^{*n} t(s) ds$, where ds is the line element. The same procedure is applicable for the calculation of $\Delta\left(\int z^{-n} d\sigma\right)$.

Probably the most important type of perturbation to know is the displacement of a conductor by Δz. Replacing z^{-n} in eq. (7a) by $(z+\Delta z)^{-n}$ and expanding in Δz to first order gives

$$\Delta a_n = - n\Delta z \cdot \frac{i\rho}{2\pi} \cdot \int z^{-(n+1)} d\sigma \quad .$$

Expressing the integral by a_{n+1}, and applying the same procedure to b_n gives

$$\Delta a_n = - n\Delta z a_{n+1} ; \; \Delta b_n = n\Delta z^* b_{n-1}/R^2 \quad . \tag{13}$$

3.3. EFFECTS OF DISPLACEMENT OF THE WHOLE CONDUCTOR STRUCTURE OF A SYMMETRICAL 2N POLE RELATIVE TO THE SHELL

From eqs. (7c), (9) and (13) follows that for displacement of the conductor structure of a 2N pole the only nonvanishing harmonics are described by

$$\Delta \underline{c}_{N(2m+1)-1} = - \Delta z \big(N(2m+1)-1\big) \; \underline{a}_{N(2m+1)} \quad , \tag{14a}$$

$$\Delta \underline{c}_{N(2m+1)+1} = \Delta z^* \big(N(2m+1)+1\big) \; \underline{b}_{N(2m+1)}/R^2 \quad . \tag{14b}$$

If the coordinate origin of a symmetrical 2N pole is displaced by Δz without perturbing the magnet, it follows from eqs. (4b) and (9) that the effect on $\underline{c}_n$ is described by

$$\Delta \underline{c}_{N(2m+1)-1} = \Delta z \big(N(2m+1)-1\big) \; \underline{c}_{N(2m+1)} \quad . \tag{15}$$

If one displaces the coordinate origin together with the conductor structure by Δz, corresponding to moving the shell by $-\Delta z$ relative to the conductors with the coordinate origin remaining at the center of the conductor structure, it follows from addition of eqs. (14a) and (15):

$$\Delta \underline{c}_{N(2m+1)-1} = \Delta z \left(N(2m+1)-1\right) \underline{b}_{N(2m+1)} \tag{16a}$$

$$\Delta \underline{c}_{N(2m+1)+1} = \Delta z^{*} \left(N(2m+1)+1\right) \underline{b}_{N(2m+1)}/R^{2} \quad . \tag{16b}$$

Eqs. (14) and (16) can be of importance for the design of magnets: It is clearly impossible to avoid generation of $\Delta \underline{c}_{N\pm1}$ by a dislocation of the shell relative to the conductor structure. But this kind of perturbation will also cause harmonics directly adjacent to the allowed harmonics unless the conductor structure is so designed that not only the usual design objective $\underline{c}_{N(2m+1)} = 0$ is satisfied for $m > 0$, but that also

$$\underline{a}_{N(2m+1)} = \underline{b}_{N(2m+1)} = 0 \quad \text{for} \quad m > 0 \tag{17}$$

is fulfilled. When eq. (17) is satisfied, the generation of $\underline{c}_{N\pm1}$ by dislocation of the shell can, at least in principle, be turned into an advantage. If the usually quite damaging component $\underline{c}_{N+1}$ is generated by some other asymmetry of the system, it can be compensated by dislocating the shell relative to the conductors. The associated production of $\underline{c}_{N-1}$ can then be eliminated by an appropriate new choice of the coordinate origin. Whether this procedure is practically feasible depends of course on the magnitude of $\underline{c}_{N+1}$ and $\underline{b}_N$, but is certainly worth considering when the magnet is in the design stage.

To satisfy eq. (17) one needs twice as many free parameters compared to satisfying only $\underline{c}_{N(2m+1)} = 0$. However the design process is greatly simplified if one restricts oneself to conductor structures with radius-independent current densities in the range $r_1 \leqslant r \leqslant r_2$, and $j = 0$ outside that range. It follows then from eqs. (7) that $\underline{a}_{N(2m+1)}$ and $\underline{b}_{N(2m+1)}$ depend in the same manner on the azimuthal current distribution, and consequently vanish together when one of them does. Fig. 1 gives a simple example of such a design: if $\alpha_1 = 43.18°$, $\alpha_2 = 52.15°$, $\alpha_3 = 67.27°$, and $|j|$ const. and identical in all current blocks, eq. (17) is satisfied for $2m+1 = 3, 5, 7$. The above mentioned angles have to be divided by N for a 2N-pole.

4. Detailed shell related effects

4.1. SATURATION EFFECTS

Until this point it has been assumed that the iron has infinite permeability, leading to the boundary condition that at the inside iron surface the azimuthal field is zero, requiring that the scalar potential is constant there. Because of the relation between B and H in the iron, there will actually be an azimuthal field component, and associated with it a varying scalar potential. The correct solution for the fields is therefore obtained by adding to the fields that have been described above fields that result from the solution to the field equations that satisfy the boundary conditions established by the iron and have no singularities for $|z| < R$. The solution to this Dirichlet problem in a circular disc is given by Schwarz's integral,[6] and if the scalar potential is used to express the boundary condition one obtains:

$$F(z_o) = \frac{i}{\pi} \int_{-\pi}^{\pi} \frac{z_o}{z-z_o} V(\phi) \cdot d\phi + F(0); \; z = R \cdot e^{i\phi} \quad . \qquad (18)$$

Dropping the unimportant quantity $F(0)$ and expanding in z_o gives

$$F(z_o) = \sum_{n=1}^{\infty} z_o^n \cdot \int_{-\pi}^{\pi} \frac{i}{\pi} z^{-n} V(\phi) d\phi \quad . \qquad (19)$$

From this one obtains with eq. (2) for H^*

$$H^*(z_o) = \sum_{n=1}^{\infty} \underline{d}_n \cdot z_o^{n-1}$$

$$\underline{d}_n = -\frac{n}{\pi\rho}\int_{-\pi}^{\pi} z^{-n} V(\phi)d\phi \quad . \qquad (\underline{20})$$

Integrating by parts and introducing $H_\phi(\phi) = -V'(\phi)/R\rho$ yields

$$\underline{d}_n = -\frac{i}{\pi R^{n-1}} \cdot \int_{-\pi}^{\pi} e^{-in\phi} H_\phi(\phi)d\phi \quad . \qquad (\underline{21})$$

As one expects, the expansion coefficients $\underline{d}_n$ are essentially the Fourier coefficients of the azimuthal field component at the inside surface of the iron shell. For a symmetrical 2N pole, i.e., when $H_\phi(\phi+\pi/N) = -H_\phi(\phi)$ is valid, one obtains of course nonvanishing coefficients only for $n = N(2m+1)$ and eq. (21) reduces to

$$\underline{d}_{N(2m+1)} = -\frac{2Ni}{\pi R^{N(2m+1)-1}} \cdot \int_{-\pi/2N}^{\pi/2N} e^{-iN(2m+1)\phi} H_\phi(\phi)d\phi \quad , \qquad (\underline{22})$$

and if the structure is symmetrical with respect to the x-axis as shown in fig. 1. i.e., if $H_\phi(-\phi) = H_\phi(\phi)$, eq. (22) becomes

$$\underline{d}_{N(2m+1)} = -\frac{4Ni}{\pi R^{N(2m+1)-1}} \cdot \int_{0}^{\pi/2N} \cos\left(N(2m+1)\phi\right) \cdot H_\phi(\phi)d\phi \quad . \qquad (\underline{23})$$

It is clear that in order to avoid generation of undesired harmonics, $H_\phi(\phi)$ should be proportional to $\cos N\phi$, unless the maximum value of H_ϕ is so small and R so large that the undesired harmonics are not harmful.

To know H_ϕ requires the solution to Laplace's equation in nonlinear iron, which is clearly not obtainable analytically. Although one can use one of the many computer programs developed for that purpose, the following procedure should give a reasonable design of the iron shell for operation at a specified field level: From the equations given in sect. 5 for the field in the region adjacent to the iron shell, the flux entering the iron that results from the infinite permeability solution is known. Specifying $H_\phi(\phi)$ allows calculation of the associated flux through the equations given in this section. Although with a specified $H_\phi(R,\phi)$ there will in general be no question that the convergence radius of the power series (eqs. (19) and (20)) is larger than R, one might have to solve problems where there is doubt about convergence. Starting from eq. (18) one can derive expressions that give the vector potential without use of a power series. Since these expressions are not likely to be used frequently, they are given without proof:

$$A(R,\phi_2) - A(R,\phi_1) = \frac{R\rho}{\pi} \cdot \int_{-\pi}^{\pi} H_\phi(R,\phi) \cdot \ell n \left| \frac{\sin\left(\frac{\phi-\phi_2}{2}\right)}{\sin\left(\frac{\phi-\phi_1}{2}\right)} \right| \cdot d\phi \quad . \qquad (\underline{24})$$

If for a 2N-pole $H_\phi(R,\phi+\pi/N) = - H_\phi(R,\phi)$ is valid, one obtains

$$A(R,\phi_2) - A(R,\phi_1) = \frac{R\rho}{\pi} \cdot \int_{-\pi/2N}^{\pi/2N} H_\phi(R,\phi) \cdot \ell n \left| \frac{\tan\left(N\frac{\phi-\phi_2}{2}\right)}{\tan\left(N\frac{\phi-\phi_1}{2}\right)} \right| d\phi \quad . \qquad (\underline{25})$$

The singularities of the integrand are of course integrable and the integration is easily carried out numerically.

With $A(R,\phi)$ thus completely known one is in a position to design the outside contour of the iron shell such that one obtains the specified H_ϕ on the inside. The graphical method given by Meuser,[4]) if modified as stated above, should result in a resonably good design.

4.2. SHIMMING OF THE INSIDE IRON SURFACE

Although small deviations of the inside iron surface from a circle are not very likely to occur, effects of such perturbations are of some interest. One might for instance intentionally introduce additional iron at that surface to modify the fields of a magnet that does not produce quite the desired fields. If one adds locally at R,ϕ an iron sheet of normalized thickness h, its effect can be described to first order in h by changing the scalar potential at the unperturbed iron surface by

$$V(\phi) = -\rho h(\phi) \cdot H_r(\phi) \quad . \tag{26}$$

Using this expression in eqs. (18) and (19), and expressing the resulting field change by $\Delta\underline{c}_n$, $\Delta\underline{c}_n$ resulting from an extended sheet is given by

$$\Delta\underline{c}_n = \frac{n}{\pi R^n} \cdot \int_{-\pi}^{\pi} e^{-in\phi} h(\phi) H_r(\phi) d\phi \quad . \tag{\underline{27}}$$

$H_r(\phi)$ can again be obtained from sect. 5. The simplified expressions for a symmetrical $2N$ pole with symmetrical perturbations are again easily obtained and will not be given here. While eq. (27) would be only a rough approximation if the additional iron has to be applied in a region where H_r is very strong, even this rough approximation would allow a reasonable estimate of the effect of a shim.

5. Field in the region adjacent to the iron

Referring to eq. (6b) it is clear that an expansion of the first part of the parenthesis in z_o is not possible when $|z_o|$ is larger than the distance from the coordinate origin to the closest conductor. If $|z_o|$ is larger than the distance from the origin to the farthest conductor, an expansion in $1/z_o$ is possible, giving together with the contribution from the second part of the integrand in eq. (6b) the Laurent expansion for H^*. Carrying this through and defining b_{-n} for $n \geqslant 0$ through

$$b_{-n} = -\frac{i\rho}{2\pi} \cdot \int j z^n d\sigma, \quad (n \geqslant 0) \quad . \tag{28}$$

$H^*(z_o)$ is given in the above specified region by

$$H^* = \sum_{n=-\infty}^{\infty} b_n \cdot z_o^{n-1} \quad . \tag{29}$$

Since, according to eq. (28), $b_o \equiv 0$, one can also set in this context $b_o = 0$ for any part of the system. Comparison of eq. (28) with eq. (7b) yields

$$b_{-n} = R^{2n} b_n^*, \ (n > 0) \quad . \tag{30}$$

Using eqs. (30) and (29), H^* can be expressed by

$$H^* = z_o^{-1} \sum_{n=1}^{\infty} R^n \left(b_n (z_o/R)^n + b_n^* (R/z_o)^n \right) \quad . \tag{31}$$

From this follows for the field at the iron surface:

$$H_r(R,\phi) = 2\mathrm{Re}\sum_{n=1}^{\infty} R^{n-1} b_n e^{in\phi} \quad . \tag{32}$$

The expression for the vector potential at the iron surface is

$$A(R,\phi) = 2\rho\,\mathrm{Im}\sum_{n=1}^{\infty} R^{n} b_n e^{in\phi}/n \quad . \tag{33}$$

6. Force and torque between conductor structure and iron shell

One can obtain an expression for the force acting on a system by starting from Maxwell's stress tensor, representing the force by a surface integral,[7] and then specializing the result for the two-dimensional case. A more compact derivation of eqs. (34) and (41) is given in Appendix 2. With f_x and f_y describing the x and y component of force per meter length of magnet, and introducing $f = f_x + if_y$, f^* is given by

$$f^* = -\frac{i\mu_o\rho}{2} \oint H^{*2} dz_o \quad . \tag{34}$$

This equation gives the force acting on all parts enclosed by the integration path. It should be noted that for the validity of eq. (34) it is not required that H^* is an analytical function of z_o. To obtain the force acting on the whole conductor structure, the contour has to be somewhere between the conductors and the iron shell. H^* can there be expressed by the Laurent series. To include also saturation effects, it is convenient to introduce the expansion coefficient $\underline{g}_n$, which is defined as follows:

$$\left.\begin{aligned} \underline{g}_n &= \underline{b}_n + \underline{d}_n \qquad n > 0 \\ \underline{g}_{-n} &= R^{2n}\underline{b}_n^* \qquad n > 0 \\ \underline{g}_o &= 0 \quad . \end{aligned}\right\} \tag{35}$$

In the above specified region, H^* is given by

$$H^* = \sum_{n=-\infty}^{\infty} \underline{g}_n z_o^{n-1} \quad , \tag{36}$$

and using this in eq. (34) yields:

$$f^* = -\frac{i\mu_o\rho}{2} \oint \sum_{n,m=-\infty}^{\infty} \underline{g}_n \underline{g}_m z_o^{n+m-2} dz_o \quad .$$

All integrals in the sum disappear unless the exponent of z_o equals -1, giving

$$f^* = 2\pi\mu_o\rho \cdot \sum_{n=1}^{\infty} \underline{g}_{n+1}\underline{g}_{-n} = 2\pi\mu_o\rho \sum_{n=1}^{\infty} \left(\underline{b}_{n+1}+\underline{d}_{n+1}\right)\underline{b}_n^* R^{2n} \quad . \tag{37}$$

For a perturbation of the ideal structure, described by $\Delta\underline{b}$, one obtains to first order for the perturbation of the force

$$\Delta f^* = 2\pi\mu_o\rho . \sum_{n=1}^{\infty} R^{2n} \left(\Delta\underline{b}_n^* \underline{g}_{n+1} + \Delta\underline{b}_{n+1}\underline{b}_n^*\right) \quad . \tag{38}$$

Eq. (37) confirms the a priori known fact that the net force on the conductor structure of an unperturbed 2N-pole magnet is zero. Using eq. (38) to evaluate the effect of the displacement by Δz of the whole conductor structure, one obtains with eq. (13):

$$\Delta f^* = 2\pi\mu_o\rho \cdot \sum_{n=1}^{\infty} R^{2n-2}\left(\Delta z^*(n+1)\left|\underline{b}_n\right|^2 + \Delta z n \underline{b}_{n-1}^* \underline{g}_{n+1}\right) \quad . \tag{39}$$

Since Δz^* is multiplied by a real and positive number, that term represents a force that has the same direction as the displacement of the conductor structure. Applying eq. (39) to a symmetrical 2N-pole, it follows from eq. (9) ($\underline{d}_n$ satisfies the same equations) that the term proportional to Δz vanishes, unless $N = 1$. Although the term proportional to Δz will in general be very small even for $N = 1$, it is at least of academic interest to note that the dipole magnet is the only multipole magnet where a displacement of the conductor structure does not necessarily lead to a force that is parallel to the displacement. Neglecting the term $\sim\Delta z$ for $N = 1$, one obtains for the force resulting from a displacement of the conductors in a 2N-pole magnet:

$$f = 2\pi\mu_o \rho\Delta z \cdot \sum_{m=0}^{\infty} \left(N(2m+1)+1\right) R^{2N(2m+1)-2} \cdot |\underline{b}_{N(2m+1)}|^2 \quad .. \qquad (\underline{40a})$$

To obtain a more practical form of eq. (40a) for a multipole magnet, it is now assumed that only the first term in the sum of eq. (40a) contributes significantly to f. Assuming also that only the term proportional to $\underline{b}_N$ is significant in eq. (32), $\underline{b}_N$ can be expressed through the maximum radial field $H_{r,max}$ at the iron surface for infinite permeability, and f becomes

$$f = \frac{1}{2}\pi\mu_o (N+1)\, H^2_{r,max} \cdot \rho\Delta z \quad .$$

Using more practical units by expressing f in metric tons per meter magnet length, $B_{r,max}$ in Tesla, and the displacement $\rho\Delta z$ in mm, one obtains

$$f = .127 \cdot (N+1) \cdot B^2_{r,max} \cdot \rho \Delta z \quad . \tag{40b}$$

Since the forces can be substantial, and in particular since they are in the <u>same</u> direction as the causing displacement, they have to be taken into account in the design of the support structure. It is also noteworthy that these forces could be used as a diagnostic tool by installing strain gauges at appropriate locations.

The torque, or moment of the force with respect to the axis of the system, has only a component in the $\vec{x} \times \vec{y}$ direction, and its magnitude T per meter magnet length is obtained in a manner similar to the derivation of the force. One obtains from

$$T = \frac{1}{2}\mu_o \rho^2 \mathrm{Re} \oint H^{*2} z_o dz_o \quad : \tag{41}$$

$$T = -2\pi\mu_o \rho^2 \mathrm{Im} \sum_{n=1}^{\infty} R^{2n} \underline{d}_n \underline{b}_n^* \quad . \tag{42}$$

As expected, a torque can appear only as a consequence of saturation of the iron. T is zero for a symmetrical 2N-pole, even if the whole conductor structure is displaced. However perturbations like rotational error of a part of the conductor structure can result in torques and are easily evaluated with eq. (42).

7. Numerical evaluation formulae

It is the purpose of this section to provide expressions that are easily evaluated by a computer for the most important quantities of interest. With the exception of parts of sects. 7.3 and 7.4, it is assumed in this section that j is constant over conductor cross sections of finite size. Despite this fact j is sometimes written after the integral sign to indicate summation of j times the integral over all conductors of the region specified by the subject of the discussion. It is assumed in sections 7.1 and 7.2 that the iron has infinite permeability, since saturation effects are easily taken into account through the expansion coefficients $\underline{d}_n$.

7.1. EVALUATION OF EXPANSION COEFFICIENTS

The expansion coefficients b_n, characterizing a conductor, are given by eq. (7b), and application of eq. (A3a) yields

$$b_n = \frac{j\rho}{4\pi} \cdot \frac{R^{-2n}}{n+1} \cdot \oint z^{*n+1} dz \quad . \tag{43}$$

Applying eq. (A3b) to eq. (7a) gives for $n > 1$:

$$a_n = \frac{j\rho}{4\pi} \cdot \frac{1}{n-1} \cdot \oint z^{1-n} dz^* \quad , \quad n > 1 \quad . \tag{44a}$$

Application of eq. (A3a) to eq. (7a) gives for $n = 1$:

$$a_1 = \frac{j\rho}{4\pi} \cdot \oint \frac{z^*}{z} dz \quad . \tag{44b}$$

As was stated at the end of sect. 3.1, the coefficients are imaginary if the conductor is symmetrical with respect to the x-axis. If that is the case, the integrals above are most easily obtained by integrating only over the upper half of the conductor, dropping the real part and multiplying by 2. A minor reduction in computer time can be obtained for most integrals appearing in sect. 7 by applying the following argument, demonstrated in its application to eq. (44b): writing $z^* = 2x - z = z - 2iy$, it becomes clear that z^* in eq. (44b) can be replaced by $2x$ or $-2iy$.

To evaluate the contour integrals in sect. 7, various techniques can be applied. A simple method consists of specifying the contour by a sufficient number of points and then applying the trapezoid formula or Romberg integration.[8] If substantial parts of the contour are straight lines, integration over these straight lines can often be performed explicitly in the following manner: If starting and end point of the straight line are z_1 and z_2, and $\Delta z = z_2 - z_1$, one can use temporarily the following parameter representation of the straight line (p = real, $0 \leqslant p \leqslant 1$):

$$z = z_1 + \Delta z \cdot p \quad .$$

From this follows

$$z^* = z_1^* + \Delta z^* p = z_1^* + \frac{\Delta z^*}{\Delta z} (z - z_1) \tag{45a}$$

$$dz^*/\Delta z^* = dz/\Delta z \quad . \tag{45b}$$

Applying this to the integral in eq. (43) gives for a straight line:

$$\int_{z_1}^{z_2} z^{*n+1} dz = \frac{\Delta z}{\Delta z^*} \int_{z_1}^{z_2} z^{*n+1} dz^* = \frac{\Delta z}{\Delta z^*} \frac{z_2^{*n+2} - z_1^{*n+2}}{n+2} \quad . \tag{46}$$

One obtains similarly

$$\int_{z_1}^{z_2} z^{1-n} dz^* = \frac{\Delta z^*}{\Delta z} \frac{z_2^{2-n} - z_1^{2-n}}{2-n} \quad , \quad n > 2 \tag{47a}$$

$$\int_{z_1}^{z_2} z^{-1} dz^* = \frac{\Delta z^*}{\Delta z} \ln(z_2/z_1) \tag{47b}$$

$$\int_{z_1}^{z_2} \frac{z^*}{z} dz = \Delta z^* + \left(z_1^* - z_1 \cdot \frac{\Delta z^*}{\Delta z}\right) \cdot \ln(z_2/z_1) \quad . \tag{48}$$

It should be noted that the right side of eq. (48) equals Δz^* when the extension of the straight line goes through $z = 0$.

For a circular arc with its center at the origin, one obtains by application of $z \cdot z^* = r^2$:

$$\int_{z_1}^{z_2} z^{*n+1} dz = -r^2 \left(z_2^{*n} - z_1^{*n}\right)/n \tag{49}$$

$$\int_{z_1}^{z_2} z^{-n+1} dz^* = r^{-2(n-1)} \left(z_2^{*n} - z_1^{*n}\right)/n \qquad (\underline{50a})$$

$$\int_{z_1}^{z_2} \frac{z^*}{z} dz = z_1^* - z_2^* \quad . \qquad (\underline{50b})$$

Expressions for circular arcs with the center not coinciding with the origin can also be derived. However they are somewhat more complicated and will not be given here since they do not seem to appear very frequently.

If the contour of a conductor is bounded by two circular arcs with radius r_1 and r_2 and with their centers at the origin, and by two radial lines at α_1 and α_2, the integrals in eqs. (7a) and (7b) are of course most easily evaluated directly and are given by

$$\int z^{*n} d\sigma = i\left(r_2^{n+2} - r_1^{n+2}\right)\left(e^{-in\alpha_2} - e^{-in\alpha_1}\right)/n(n+2) \qquad (\underline{51})$$

$$\int z^{-n} d\sigma = i\left(r_2^{2-n} - r_1^{2-n}\right)\left(e^{-in\alpha_2} - e^{-in\alpha_1}\right)/n(2-n); \quad n \neq 2 \qquad (\underline{52a})$$

$$\int z^{-2} d\sigma = i \cdot \ell n(r_2/r_1) \cdot \left(e^{-in\alpha_2} - e^{-in\alpha_1}\right)/2 \quad . \qquad (\underline{52b})$$

7.2. EVALUATION OF H^*

Although the expansion coefficients are usually of primary interest, for a variety of reasons it can be of interest to have a direct method for calculation of H^*: H^* can be of interest in the aperture to find the difference between the actual fields and the contributions of the major multipole coefficients. H^* is of interest in the coil regions to find $|H|_{max}$. H^* can also be of interest for the evaluation of f and T with eqs. (34) and (41), unless one wants to use the equations given in sect. 7.3.

Applying eqs. (A3a), (A4), (A5) to the first part of the integrand in eq. (6b), and eq. (A3b) to the second part of the integrand in eq. (6b) gives:

$$H^* = \frac{j\rho}{4\pi}\left(\oint \frac{z^*-z_o^*}{z-z_o}\,dz - \oint \frac{zz^*}{R^2-z_oz^*}\,dz^*\right) \quad . \qquad (\underline{53})$$

As mentioned in Appendix 1, eq. (53) is correct whether or not z_o is inside the contour. The absolute value of the integrand of the first integral is one and has furthermore the convenient property that when z_o is on the contour, the integrand does not change when one goes "through" z_o unless z_o is located at a corner. As in sect. 7.1, integrations over straight lines or circular arcs can be performed explicitly but will not be given here.

The field perturbation ΔH^* resulting from a displacement of a conductor by Δz is obtained from eq. (53) by differentiation and one obtains

$$\Delta H^* = \frac{j\rho}{4\pi}\left(-\Delta z \cdot \oint \frac{z^*}{(z-z_o)^2}\,dz + \Delta z^* \oint \frac{dz}{z-z_o} - R^2\Delta z^* \oint \frac{z}{(R^2-z_oz^*)^2}\,dz^*\right) \quad . \qquad (\underline{54})$$

The second integral contributes obviously only when z_o is inside the conductor and is therefore not of great interest. For the effect of rotation by a small angle α one obtains similarly

$$\Delta H^* = - i\alpha \cdot \frac{j\rho}{4\pi} \left(\oint \frac{zz^*}{(z-z_o)^2} dz - R^2 \oint \frac{zz^*}{(R^2-z_o z^*)^2} dz^* \right) . \quad (\underline{55})$$

Eq. (53) represents the contribution of one conductor to the total field. When one is dealing with a symmetrical 2N-pole magnet, the contribution from the conductor rotated by $m \cdot \pi/N$ with respect to the reference conductor is obtained by replacing in eq. (53) z by $z \cdot e^{im\pi/N}$ and multiplying the whole expression by $(-1)^m$. Doing this and summing up gives in eq. (53) instead of the first integral

$$\oint \sum_{m=0}^{2N-1} \frac{z^* e^{iN\cdot m\pi/N} - z_o^* e^{i(N+1)\cdot m\pi/N}}{z\left(e^{im\pi/N} - z_o/z\right)} dz .$$

Applying eq. (A18) to both parts of this integrand and following the same procedure for evaluation of the second integral in eq. (53) gives

$$H^* = \frac{j\rho}{2\pi} N z_o^{N-1} \left(\oint \frac{z^{N-1}(zz^* - z_o z_o^*)}{z^{2N} - z_o^{2N}} dz - R^{2N} \oint \frac{z^{*N-1} zz^*}{R^{4N} - z_o^{2N} z^{*2N}} dz \right) .$$

$$(\underline{56})$$

7.3. FORCE AND TORQUE ON INDIVIDUAL CONDUCTORS

In order to reduce computer time, the use of transcendental functions has been avoided so far whenever possible. To calculate force and torque on individual conductors, eq. (34) and (41) can be used together with the expressions for H^* given in sect. 7.2. However, when the stored energy has to be computed, extensive use of logarithms seems unavoidable, and if they are available from the energy computation, the following procedure is preferable: In the two-dimensional case, the force f and torque T per meter are given by eqs. (A9) and (A10):

$$f = i\mu_o \rho^2 \int jH(z_o, z_o^*) d\sigma_o \tag{57}$$

$$T = \mu_o \rho^3 \int j \cdot (\vec{r} \cdot \vec{H}) d\sigma_o = \mu_o \rho^3 \mathrm{Re} \int jH(z_o, z_o^*) z_o^* d\sigma_o \quad . \tag{58}$$

The total field is a linear superposition of the following three fields: 1) The vacuum field, i.e., the field produced by the conductor structure without any iron present. This field is derivable from eq. (5b) if the second factor of the argument of the logarithm is set equal to one. 2) The field produced by the image currents, described by the second factor of the argument of the logarithm of eq. (5b). 3) The fields caused by the saturation effects, described by the expansion coefficients $\underline{d}_n$ (eq. (21)). Although the forces and torques resulting from the last two sources can be obtained in a manner similar to the one used to obtain the effects stemming from the vacuum fields, the following procedure is simpler and quite adequate.

Using eq. (36) to describe the total fields caused by image currents and saturation, one obtains

$$f = i\mu_o \rho^2 \int \sum_{n=1}^{\infty} \underline{g}_n^* j z_o^{*n-1} d\sigma_o \quad . \tag{59}$$

Expressing the integrals through the coefficients b_n (eq. (7b)), describing the part of the conductor structure under consideration for the calculation of f and T, one obtains

$$f = 2\pi\mu_o \rho \sum_{n=1}^{\infty} \underline{g}_n^* b_{n-1} R^{2n-2} \quad . \tag{60}$$

Using the same procedure to calculate T, one obtains

$$T = 2\pi\mu_o \rho^2 \mathrm{Im} \sum_{n=1}^{\infty} \underline{g}_n^* b_n R^{2n} \quad . \tag{61}$$

It should be noted that contrary to what was said in sect. 5, b_o is not necessarily zero in this context.

Although general explicit formulae for b_n are given in sect. 7.1. only for conductors with constant current density over finite areas of conductors, eqs. (60) and (61) are valid even for nonuniform current distributions.

To obtain the contribution of the vacuum field to force and torque, it is convenient to transform eqs. (57) and (58) first into contour integrals. To do so, H is expressed by spatial derivatives of A (eq. (1)). Considering

A as a function of z_o and z_o^* and using eqs. (A2), one obtains

$$H = (-2i/\rho) \cdot \partial A/\partial z_o^* \quad . \tag{62}$$

Using this in eq. (57) and applying eq. (A3a) yields

$$f = - i\mu_o \rho j \oint A(z_o, z_o^*) dz_o \quad . \tag{63}$$

Proceeding similarly to obtain T from eq. (58), and utilizing $z_o^* \partial A/\partial z_o^* = \partial(z_o^* A)/\partial z_o^* - A$ and the fact that A is real yields

$$T = - \mu_o \rho^2 j \mathrm{Re} \oint A(z_o, z_o^*) z_o^* dz_o \quad . \tag{64}$$

It should be noted that in eq. (64), any integral over a circular arc with its center at $z = 0$ does not contribute to T.

To evaluate eqs. (63) and (64), A has to be known on the contour of the conductor under consideration. Since one would also consider a part of a conductor block to obtain the internal stresses, part of the contour will in this general case be inside a conductor. From eq. (5b) follows for the contribution of a current filament at z to the vector potential A at z_o:

$$A = - \frac{I}{4\pi} \left(\ell n(z-z_o) + \ell n(z^*-z_o^*) \right) \quad . \tag{65}$$

The logarithms are declared real for real positive arguments and are made single-valued with a branch cut so that the imaginary part of each logarithm is between $-\pi$ and π.

From this follows for the contribution from all conductors

$$A = -\frac{\rho^2}{4\pi}\int j\left(\ln(z-z_o) + \ln(z^*-z_o^*)\right)d\sigma \quad . \tag{66}$$

Converting this into a contour integral gives with eq. (A8)

$$A = \frac{i\rho^2}{8\pi}\oint j(z^*-z_o^*)\left(\ln(z-z_o) + \ln(z^*-z_o^*)-1\right)dz \quad . \tag{67}$$

Since $\oint z_o^* dz = 0$, $\oint z^* dz = 2i\sigma$, and the total current equals zero, the term -1 in the parenthesis of eq. (67) does not contribute to A so that A can be written as follows:

$$A(z_o,z_o^*) = \frac{i\rho^2}{8\pi}\oint j(z^*-z_o^*)\ln|z-z_o|^2 \cdot dz \quad . \tag{68}$$

Eq. (68), together with eqs. (63) and (64) allows thus the evaluation of f and T. The fact that A is known to be real allows a simple check of at least some parts of the program to evaluate A. To obtain the order of magnitude of computer time needed to evaluate f and T, the following numbers seem reasonable for a quadrupole: if the conductor contours in each sector are specified by 100 points, calculation of A requires computation of 400 logarithms. If the contour of the conductor under consideration is also specified by 100 points, evaluation of f and T requires computation of 4.10^4 logarithms. 35 μsec execution time per logarithm on the CDC 6600 under the Chippewa operating system thus leads to a total time of a few seconds.

7.4. TOTAL STORED ENERGY

Since the vector potential represents the flux in a two dimensional magnet, the power $\dot{E}$ per meter magnet length delivered from the power supply to the magnet is given by

$$\dot{E} = \mu_o \rho^2 \int j\dot{A}(z_o, z_o^*) d\sigma_o \quad .$$

Starting at $t = 0$ with $E = 0$, $j = 0$, and integrating by parts over time yields

$$E = \mu_o \rho^2 \cdot \int \left(Aj - \int_0^j A(j) dj \right) d\sigma_o \quad . \tag{69}$$

It has been assumed here that the current density has the same time dependence everywhere, and the time dependence of A is expressed through its dependence on j. The contributions to A from the vacuum field and the image currents are linear in j and therefore contribute $j\,A/2$ to the integrand in eq. (69). The contribution to A from saturation of the iron requires integration over the past as indicated in eq. (69). This means that for the contribution of saturation to the energy, $\int A_{sat} j d\sigma_o$ has to be known for all past excitation levels. Of course not all contributions to E resulting from saturation are recoverable, so that the term "stored energy" for E is really a misnomer.

To get the contributions to $\int A j d\sigma_o$, and ultimately E, that are caused by saturation effects and the image currents, the procedure is similar to the evaluation of their contribution to f and T:

$$H^{*} = iF'/\rho = \sum_{n=1}^{\infty} \underline{g}_n z_o^{n-1}$$

$$F = A + iV = - i\rho \sum_{n=1}^{\infty} \underline{g}_n z_o^{n}/n$$

$$\int Aj d\sigma_o = \mathrm{Re} \sum_{n=1}^{\infty} - \int i\rho j z_o^n d\sigma_o \underline{g}_n/n$$

$$\int Aj d\sigma_o = 2\pi \cdot \mathrm{Re} \sum_{n=1}^{\infty} R^{2n} \underline{b}_n^{*} (\underline{b}_n + \underline{d}_n)/n \quad .$$

Using this in eq. (69) then gives for the contribution of the image currents and saturation effects to E:

$$E = \mu_o \pi \rho^2 \sum_{n=1}^{\infty} R^{2n} \left(|\underline{b}_n|^2 + 2\mathrm{Re}\left(\underline{b}_n^{*} \underline{d}_n(j) - \frac{\underline{b}_n^{*}}{j} \int_0^j \underline{d}_n(j) dj \right) \right) \Big/ n \quad . \qquad (\underline{71})$$

It is interesting to note that the energy given by eq. (71) is smaller than the vacuum field energy given below if $\underline{d}_n = 0$, and will be still smaller if saturation effects are present. This is most easily seen as follows: if the infinite permeability shell is replaced by a superconducting shell, one obtains the complex potential from a current filament by <u>dividing</u> the first factor of the argument of the logarithm in eq. (5b) by the second factor instead of multiplying by it. The energy resulting from the image currents is then again given by eq. (71), except the right side is multiplied by -1. Since the total energy must be positive, the energy given by eq (71) for $\underline{d}_n = 0$ must be smaller than the vacuum field energy. This statement is of course correct only if the conductor structure is surrounded by a circular shell.

The procedure to obtain the contribution to E that results from the first factor of the argument of the logarithm on the right side of eq. (5b) follows the same pattern as the calculation of the contribution of that term to f and T.

For the contribution of one current filament at z to E one obtains

$$E = \frac{1}{2}\mu_o\rho^2 \int j \cdot \frac{-I}{4\pi}\left(\ln(z_o - z) + \ln(z_o^* - z^*)\right) d\sigma_o \quad .$$

Applying eqs. (A3a) and (A8), and taking into account that $\int j d\sigma_o = 0$ one obtains

$$E = \frac{1}{2}\mu_o\rho^2 \oint jG_1(z_o, z_o^*) dz_o \quad , \tag{72}$$

with

$$G_1(z_o, z_o^*) = \frac{iI}{8\pi} \cdot (z_o^* - z^*)\left(\ln(z_o - z) + \ln(z_o^* - z^*)\right) \quad .$$

The effect of all currents is described by

$$G_1(z_o,z_o^*) = \frac{i\rho^2}{8\pi} \cdot \int j(z_o^*-z^*)\left(\ell n(z_o-z) + \ell n(z_o^*-z^*)\right)d\sigma \quad . \tag{73}$$

Applying eq. (A3b) to this expression gives in the same manner as the derivation of eq. (A8):

$$G_1(z_o,z_o^*) = \frac{\rho^2}{16\pi} \cdot \oint j(z_o^*-z^*)(z_o-z)\left(\ell n(z_o-z) + \ell n(z_o^*-z^*)-1\right)dz^* \quad .$$

It is again easy to see that the term -1 in the parenthesis of this equation does not contribute to E, so that E can be calculated from

$$E = \frac{\mu_o\rho^4}{32\,\pi} \cdot \oint j \cdot G(z_o,z_o^*)dz_o \tag{74}$$

$$G(z_o,z_o^*) = \oint j|z-z_o|^2 \cdot \ell n|z-z_o|^2 \cdot dz^* \quad . \tag{75}$$

For a symmetrical 2N-pole, the integral in eq. (74) has to be evaluated for only one sector since each sector contributes equally to E; for evaluation of G, the integration has to be carried out over all conductors of the system.

Appendix 1

Although eqs. (A3) can be found in the literature,[9] they are briefly derived because they are applied here in a not quite trivial manner, which also needs some explanation.

From Stoke's theorem ($\int \text{curl}\, \vec{v} \cdot d\vec{\sigma} = \oint \vec{v} \cdot d\vec{s}$), applied to a vector in the x - y plane, follows

$$\int \frac{\partial F(x,y)}{\partial x}\, d\sigma = \oint F dy \tag{A1a}$$

$$\int \frac{\partial F(x,y)}{\partial y}\, d\sigma = - \oint F dx \quad . \tag{A1b}$$

Expressing x and y by z and z^* and considering F now as a function of z and z^*, the operators $\partial/\partial x$, $\partial/\partial y$ become

$$\partial/\partial x = \partial/\partial z + \partial/\partial z^* \tag{A2a}$$

$$\partial/\partial y = i(\partial/\partial z - \partial/\partial z^*) \quad . \tag{A2b}$$

Using this in eqs. (A1), multiplying eq. (A1a) by i, and first subtracting and then adding eq. (A1b) gives

$$\int \frac{\partial F}{\partial z^*}\, d\sigma = \frac{1}{2i} \oint F dz \tag{A3a}$$

$$\int \frac{\partial F}{\partial z}\, d\sigma = - \frac{1}{2i} \oint F dz^* \quad . \tag{A3b}$$

It has been assumed in this derivation that F and its first derivatives are in the integration area single-valued and have no singularities there. Considering from now on only eq. (A3a) with the understanding that the equivalent considerations apply to eq. (A3b), the case is now discussed where $\partial F/\partial z^*$ has a singularity of the type $1/(z-z_o)$ at z_o inside the integration area. Considering an infinitesimal circular disc around z_o, and carrying out the integration of $\partial F/\partial z^*$ over that disc, it is clear that that integral is infinitesimally small. After removing that disc from the integration area, eq. (A3a) can be applied. However, the boundary of the integration area consists then of two parts, namely the outer contour and the circle around z_o, as indicated in fig. 2a. When calculating F from $\partial F/\partial z^*$ one can clearly add as "integration constant" any function of z that is analytic in the integration area and can use this to make the contour integral over the circle around z_o vanish. This is most easily seen with the following example: Assuming

$$\partial F(z,z^*)/\partial z^* = F_1(z) \cdot F_2'(z^*) \quad , \tag{A4}$$

and $F_1(z)$ to be proportional to $1/(z-z_o)$, $F(z,z^*)$ can be chosen to be

$$F(z,z^*) = F_1(z)\left(F_2(z^*) - F_2(z_o^*)\right) \quad . \tag{A5}$$

With this choice of F, the contour integral over the infinitesimal circle around z_o vanishes so that the contour integration has to be carried out only over the outer boundary of the integration area.

If $\partial F/\partial z^*$ is given by eq. (A4) and $F_1(z)$ has single poles at several locations z_n, the following choice of $F(z,z^*)$ eliminates the need for contour integration around each individual pole:

$$P_n(z) = \left(\prod_m (z-z_m)\right)/(z-z_n); \; Q_n(z) = P_n(z)/P_n(z_n) \quad , \tag{A6a}$$

$$F(z,z^*) = F_1(z) \left(F_2(z^*) - \sum_n F_2(z_n^*) \cdot Q_n(z)\right) \quad . \tag{A6b}$$

It should be noted that although all poles that are in the integration area have to be treated as indicated, it does of course make no difference when a pole that lies outside is treated in eqs. (A6) as if it were inside.

To transform

$$J = \int \left(\ell n(z-z_o) + \ell n(z^*-z_o^*)\right) d\sigma \tag{A7}$$

into a contour integral, it is indicated to make each logarithm with a branch cut single-valued, with the convention that the logarithm of a positive number is real. Because of the singularity at z_o, which is assumed to be inside the integration area, a circular disc is again removed around z_o without changing the value of J. The complete contour for the integration is indicated in fig. 2b. The strip to the left of z_o has zero thickness and therefore does not change J either. Using eq. (A3a) and choosing

$$F = (z^*-z_o^*)\left(\ell n(z-z_o) + \ell n(z^*-z_o^*)-1\right)$$

gives

$$\int \left(\ell n(z-z_o) + \ell n(z^*-z_o^*)\right) d\sigma = \frac{1}{2i} \oint (z^*-z_o^*)\left(\ell n(z-z_o) + \ell n(z^*-z_o^*)-1\right) dz. \quad (A8)$$

Since the integral around the small circle vanishes, and the integrals along the branch cut cancel each other, the contour for evaluation of the integral on the right side of eq. (A8) can be simply the outer boundary of the integration region. It is easy to see that this would not have been the case if F would have been chosen as

$$F = (z^*-z_o^*)\left(\ell n(z^*-z_o^*)-1\right) + z^* \ell n(z-z_o) \quad .$$

It is of course not necessary for the validity of eq. (A8) that z_o is inside the integration area. It is noteworthy that choosing F such that one has to integrate only over the outer boundary of the integration region is more than a convenience: it makes it for instance possible to perform in the normal manner operations like differentiation under the integral sign, which is not possible when one has to integrate along branch cuts or when singularities are excluded in the manner described above. Finally it should be noted that the freedom to choose either eq. (A3a) or (A3b) for the transformation of the integral, as well as some freedom in selecting the "integration constant", can lead to very dissimilar looking contour integrals, and one should select that form of the contour integral that is most easily evaluated.

Appendix 2

From the general expression for the force density $\mu_o \vec{j} \times \vec{H}$ and torque density $\mu_o \vec{r} \times (\vec{j} \times \vec{H})$ acting on a conductor, it follows immediately in the two-dimensional case for the force f and torque T per meter magnet length

$$f = i\mu_o\rho^2 \int jHd\sigma \tag{A9}$$

$$T = \mu_o\rho^3 \mathrm{Re} \int jH^* z d\sigma \quad . \tag{A10}$$

To convert these integrals into contour integrals, it is indicated to consider H as a function of z and z^*. With eqs. (A2), div $\vec{H} = 0$ and curl $\vec{H} = \vec{j}$ reduce to

$$\partial H/\partial z + \partial H^*/\partial z^* = 0 \tag{A11}$$

$$\rho j = 2i\partial H^*/\partial z^* = -\, 2i\partial H/\partial z \quad . \tag{A12}$$

Eliminating j in eq. (A9) and (A10) with eq. (A12), and then applying eqs. (A3) gives for f and T:

$$f = \mu_o\rho \cdot \int 2H \frac{\partial H}{\partial z} d\sigma = -\frac{\mu_o\rho}{2i} \oint H^2 dz^* \tag{A13}$$

$$T = \mu_o\rho^2 \mathrm{Re} \int i \cdot 2H^* \frac{dH^*}{dz^*} \cdot z d\sigma = \frac{1}{2} \mu_o\rho^2 \mathrm{Re} \oint H^{*2} z dz \quad . \tag{A14}$$

It should be noted that eqs. (A13) and (A14) are valid even when iron is enclosed by the contour or when part of the contour goes through iron, since one can consider the permeability to be caused by Ampèrean currents. For this reason, H has been used exclusively. Over those parts of the contour that go through iron, one has to consider the iron removed over a strip of infinitesimal thickness so that the contour goes in principle through vacuum. The field components parallel and perpendicular to the contour then follow from the boundary conditions for B and H.

Appendix 3

Consider

$$S = \frac{1}{2\pi i} \cdot \oint \frac{z^{n-1}}{(z-a)(z^M-1)} dz \quad , \tag{A15}$$

with a representing some complex number, M an integer, n an integer satisfying

$$1 \leqslant n \leqslant M \quad , \tag{A16}$$

and the contour chosen to be a circle that encloses both $z = a$ and $z = 1$. By either introducing $1/z$ as new integration variable, or by letting the radius of the integration circle go to infinity, it is easily seen that $S = 0$. Since the integrand has singularities at $z_a = a$, $z_m = e^{im2\pi/M}$, $m = 0, 1, \ldots, M - 1$, application of the residue theorem gives

$$\frac{a^{n-1}}{a^M-1} + \sum_{m=0}^{M-1} \frac{z_m^{n-1}}{(z_m-a)Mz_m^{M-1}} = 0 \quad . \tag{A17}$$

Since one could also have chosen $z_m = e^{-im2\pi/M}$, from eq. (A17) follows that

$$\sum_{m=0}^{M-1} \frac{e^{\pm in\cdot m\cdot 2\pi/M}}{e^{\pm im2\pi/M} - a} = M \frac{a^{n-1}}{1 - a^M} \quad . \tag{A18}$$

It should be noted that eq. (A16) does not really represent a restriction of the value of n since on the left side of eq. (A18) any multiple of M can be added to or subtracted from n without changing the value of the sum.

It is also noteworthy that by differentiating eq. (A18) with respect to a, one can obtain an expression for a sum of the same type as the left side of eq. (A18), but with the denominator raised to some integer power.

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Figure captions

Fig. 1. Conductor and iron configuration for dipole magnet.

Fig. 2. Integration contours.

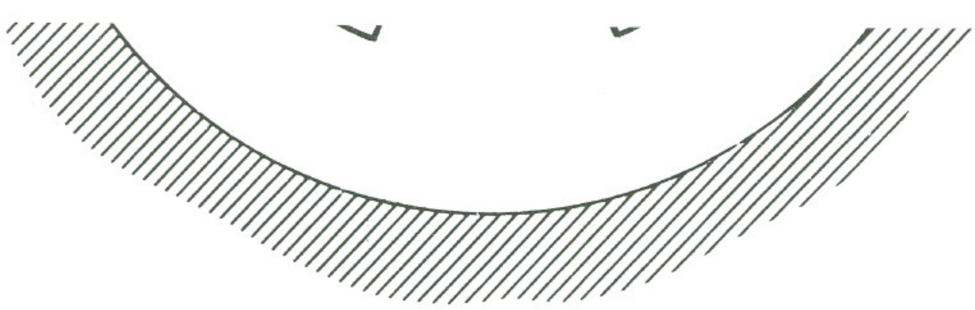

Fig. 1