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ON A ONE-DIMENSIONAL FINITE STRAIN BEAM THEORY: THE 3-DIMENSIONAL DYNAMIC PROBLEM

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On a One-Dimensional Finite Strain Beam Theory: The Three Dimensional Dynamic Problem

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Abstract

This paper is concerned with a one-dimensional finite strain beam theory which generalizes to the fully three dimensional dynamic situation the formulation originally developed by Reissner for the plane static problem. Our approach proceeds by constraining the general 3-dimensional theory with the introduction a kinematic assumption. The crucial step in the formulation presented is the particular parametrization chosen, which results from the kinematic description of the beam in terms of a *moving orthogonal* frame. This frame does not coincide with the *convected* frame, often used in the formulation of rod theories, unless shear deformation is ignored. The introduction of the moving frame allows a simple geometric interpretation of the strain measure conjugate to the resultant torque as the axial vector of a spatial skewsymmetric tensor associated with the moving frame. In addition, the objective rate which results from the reduced expression for the internal power in terms of the resultant (spatial) force and torque, has a simple physical interpretation involving the *spin* of the moving frame. The formulation presented is particularly useful in a numerical treatment of the 3-dimensional dynamic problem, and forms the basis of the finite element implementation considered in a forthcoming paper.

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1. Basic Kinematics.

In this section we discuss the basic kinematic concepts relevant to present 3-dimensional nonlinear model.

1.1. Moving basis. Kinematic assumption.

Geometrically, the *current configuration* of a rod is described by defining a family of *cross sections* the centroids of which are connected by a curve which we refer to as the *line of centroids*. Notice that a result of shearing of the rod, cross sections *are not normal* to the line of centroids in the current configuration. Accordingly, to specify the current configuration of the beam we formally introduce the following objects

(i) A curve defined in an open interval $\mathrm{I} \subset \mathbb{R}$:

$$S \in \mathrm{I} \rightarrow \boldsymbol{\phi}_o(S) \in \mathbb{R}^3, \tag{1.1}$$

called the *line of centroids*.

(ii) A family of planes defined by the *unit vector field*

$$S \in \mathrm{I} \rightarrow \hat{\mathbf{n}}(S) \in \mathbb{R}^3. \tag{1.2}$$

The planes through $\boldsymbol{\phi}_o(S) \in \mathbb{R}^3$ normal to $\hat{\mathbf{n}}(S)$, $S \in \mathrm{I}$, will be referred to as *cross sections* of the rod.

(iii) A *fiber* within each cross section defined by the *unit vector field*

$$S \in \mathrm{I} \rightarrow \mathbf{t}_1(S) \in \mathbb{R}^3. \tag{1.3}$$

Thus, at each point of the curve $S \rightarrow \boldsymbol{\phi}_o(S)$ we may define an *orthonormal* frame $\{\mathbf{t}_1(S), \mathbf{t}_2(S), \hat{\mathbf{n}}(S)\}$, which we shall refer to as *moving* or *intrinsic frame*, such that

$$|\hat{\mathbf{n}}(S)| = 1, \quad |\mathbf{t}_\Gamma(S)| = 1, \ \hat{\mathbf{n}}(S) \bullet \mathbf{t}_\Gamma(S) = 0, \ (\Gamma=1,2), \quad \mathbf{t}_1(S) \bullet \mathbf{t}_2(S) = 0$$

and

$$\mathbf{t}_3(S) \equiv \hat{\mathbf{n}}(S) = \mathbf{t}_1(S) \times \mathbf{t}_2(S), \qquad S \in \mathrm{I} \subset \mathbb{R}. \tag{1.4}$$

For convenience, the notation $\mathbf{t}_3(S) \equiv \hat{\mathbf{n}}(S)$ will often be employed. Our basic kinematic assumption, then, is that the admissible configurations of the rod, denoted by

$$\boldsymbol{\phi} : \mathrm{I} \times \Omega \rightarrow \mathbb{R}^3, \tag{1.5}$$

where $\Omega \subset \mathbb{R}^2$ is compact, have the following explicit form

$$\mathbf{x} = \boldsymbol{\phi}(\xi_1, \xi_2, S) \equiv \boldsymbol{\phi}_o(S) + \sum_{\Gamma=1}^{2} \xi_\Gamma \mathbf{t}_\Gamma(S). \tag{1.6}$$

The geometric significance of assumption (1.6) is illustrated for the plane case in Figure 1. For simplicity we shall assume herein that the *unstressed* configuration of the rod, which is taken as the *reference configuration*, is such that the line of centroids is a *straight line* so that the moving frame in the reference configuration becomes simply the standard basis in $\mathbb{R}^3$, and is denoted by $\{\hat{\mathbf{E}}_I\}$. This is illustrated in Figure 1. For convenience, we shall often use the notation $\boldsymbol{\xi} = \xi_1 \hat{\mathbf{E}}_1 + \xi_2 \hat{\mathbf{E}}_2$.

Remark 1.1. We emphasize that the unit vector field $S \rightarrow \hat{\mathbf{n}}(S)$ is not tangent to the line of centroids $S \rightarrow \boldsymbol{\phi}_o(S)$ in the current configuration; but *normal* to the cross section passing through $\boldsymbol{\phi}_o(S)$. □

Remark 1.2. The parameter $S \in \mathrm{I}$ represents the *arch* of *length* of the line of centroids in the reference (unstressed) configuration. There is no difficulty in considering "initially curved" geometry. In such an event, the basis $\{\hat{\mathbf{E}}_I\}$ becomes a function of $S \in \mathrm{I}$. □

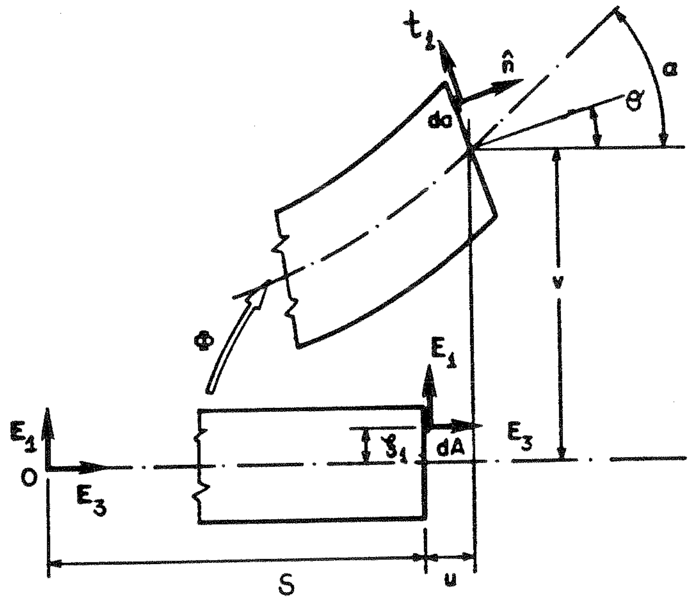

Figure 1. Kinematic assumption. Reference and current configurations (plane problem)

1.2. Derivatives of the moving basis.

Since the moving basis $\{\mathbf{t}_I(S)\}$ is orthonormal for each $S \in \mathrm{I}$, there exists an *orthogonal transformation* $S \rightarrow \mathbf{\Lambda}(S) \in SO(3)$, where $SO(3)$ stands for the orthogonal (Lie) group, such that

$$\mathbf{t}_I(S) = \mathbf{\Lambda}(S)\,\hat{\mathbf{E}}_I, \quad \text{or} \quad \mathbf{t}_I(S) = \Lambda_{iI}\hat{\mathbf{e}}_i, \quad I=(1,2,3), \tag{1.7}$$

where, $\{\hat{\mathbf{e}}_i\}$ denotes the *fixed spatial* frame, not necessarily coincident with $\{\hat{\mathbf{E}}_I\}$, and $\mathbf{\Lambda}(S) = \Lambda_{iI}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{E}}_I$ is a *two-point orthogonal* tensor field. Taking derivative of (1.7) relative to $S \in \mathrm{I}$ we obtain

$$\frac{d}{dS}\mathbf{t}_I(S) = \mathbf{\Omega}(S)\,\mathbf{t}_I(S) \tag{1.8}$$

where:

$$\mathbf{\Omega}(S) \equiv \left[\frac{d}{dS}\mathbf{\Lambda}(S)\right]\mathbf{\Lambda}^T(S) \tag{1.9}$$

is a *skew-symmetric* tensor field; i.e., $\mathbf{\Omega}(S) + \mathbf{\Omega}^T(S) = \mathbf{O}$. Since $\mathbf{\Omega}(S)$ is an *spatial* tensor for each $S \in \mathrm{I}$, its components may be given relative to the *moving frame* $\{\mathbf{t}_I\}$, and expressed in matrix form as

$$[\mathbf{\Omega}(S)] = -\begin{bmatrix} 0 & \kappa_3(S) & -\kappa_2(S) \\ -\kappa_3(S) & 0 & \kappa_1(S) \\ \kappa_2(S) & -\kappa_1(S) & 0 \end{bmatrix}. \tag{1.10}$$

It is convenient to introduce the *axial vector* field $S \rightarrow \boldsymbol{\omega}(S) \in \mathbb{R}^3$ associated with the skew-symmetric tensor $\mathbf{\Omega}(S)$, which is defined by the relation $\mathbf{\Omega}(S)\,\boldsymbol{\omega}(S) \equiv \mathbf{O}$. Thus, relative to the *moving frame* we have the representation

$$\boldsymbol{\omega}(S) = \kappa_1(S)\,\mathbf{t}_1(S) + \kappa_2(S)\,\mathbf{t}_2(S) + \kappa_3(S)\,\hat{\mathbf{n}}(S). \tag{1.11}$$

The derivatives of the moving frame given by (1.8) may then be recast into the alternative expression

$$\frac{d}{dS}\,\mathbf{t}_I(S) = \boldsymbol{\omega}(S)\times\mathbf{t}_I(S)\,, \quad (I=1,2,3)\,. \tag{1.12}$$

This completes the basic kinematic relations needed for subsequent developments. We note the following

Remark 1.3. The moving frame $\{\mathbf{t}_I(S)\}$ should not be confused with the *convected* basis which is often used in the development of rod theories (e.g. Antman [1972], Naghdi [1980]), and is defined as follows. Let $\mathbf{F}(\boldsymbol{\xi},S)$ be the deformation gradient and $\mathbf{F}_o(S)\equiv\mathbf{F}(\boldsymbol{\xi},S)\big|_{\xi=0}$. Then, the convected basis $\{\boldsymbol{\Xi}_I\}$ is defined as

$$\boldsymbol{\Xi}_I(S) = \mathbf{F}_o(S)\,\hat{\mathbf{E}}_I\,, \qquad (I=1,2,3)\,. \tag{1.13}$$

From the basic kinematic assumption (1.6) it easily follows that

$$\mathbf{F}_o(S) = \sum_{\Gamma=1}^{2}\mathbf{t}_\Gamma(S)\otimes\hat{\mathbf{E}}_\Gamma + \frac{d}{dS}\boldsymbol{\phi}_o(S)\otimes\hat{\mathbf{E}}_3 \tag{1.14}$$

The convected base vectors, then, are given by

$$\boldsymbol{\Xi}_1(S) = \mathbf{t}_1(S)\,, \quad \boldsymbol{\Xi}_2(S) = \mathbf{t}_2(S)\,, \quad \boldsymbol{\Xi}_3(S) = \frac{d}{dS}\boldsymbol{\phi}_o(S)\,. \tag{1.15}$$

Thus, the essential difference between the *convected* basis and the *moving* basis $\{\mathbf{t}_I(S)\}$ is that $\boldsymbol{\Xi}_3$ is *tangent* to the line of centroids whereas $\hat{\mathbf{n}}(S)\equiv\mathbf{t}_3(S)$ is *normal* to the cross section. Notice also that the *moving* basis $\{\mathbf{t}_I(S)\}$ is *orthonormal* whereas the *convected* basis is not. If shear deformation is not taken into account the difference between both bases disappears. □

Remark 1.4. It is emphasised that the vector field $S\rightarrow\boldsymbol{\omega}(S)$, although parametrized for convenience by the reference arc length $S\in\mathrm{I}$, takes values *on the current configuration*. Accordingly, its components are given relative to a spatial basis; either $\{\hat{\mathbf{e}}_i\}$ or relative to $\{\mathbf{t}_I\}$ as in (1.11). Alternatively, we may define a *material* vector field by setting

$$S\rightarrow\mathbf{K}(S)\equiv\kappa_I(S)\,\hat{\mathbf{E}}_I \tag{1.16}$$

In view of (1.7), $\boldsymbol{\omega}(S)$ and $\mathbf{K}(S)$ are *spatial* and *material* vector fields related according to

$$\boldsymbol{\omega}(S) = \boldsymbol{\Lambda}(S)\,\mathbf{K}(S)\,. \tag{1.17}$$

The vector $\mathbf{K}(S)$ appears naturally in the *material* form of the reduced expression for the internal power, as shown in Section **4**. □

2. Motion. Linear and Angular Momentum.

In this section we extend the kinematic concepts discussed above to account for dynamic effects. We shall see that the expression for the angular momentum involves the material time derivative of the *vorticity* vector associated with the moving frame.

A *motion* of the rod is a curve of configurations parametrized by time; that is

$$t\rightarrow\boldsymbol{\phi}_t\equiv\boldsymbol{\phi}_o(S,t) + \sum_{\Gamma=1}^{2}\xi_\Gamma\,\mathbf{t}_\Gamma(S,t)\,, \tag{2.1}$$

where $t\in\mathbb{R}^+$ is the *time*. Prior to introducing the linear and angular momentum vector fields associated with the motion (2.1), we need the following result.

Time derivatives of the moving frame. The moving frame $\{\mathbf{t}_I(S,t)\}$ is defined by (1.7) where the orthogonal transformation now depends on time; e.g., $(S,t)\rightarrow\boldsymbol{\Lambda}(S,t)$. Denoting by a superposed "dot" the *material* time derivative, we then have (I=1,2,3)

$$\dot{\mathbf{t}}_I(S,t) = [\dot{\boldsymbol{\Lambda}}(S,t)\,\boldsymbol{\Lambda}^T(S,t)]\,\mathbf{t}_I(S,t)\equiv\mathbf{W}(S,t)\,\mathbf{t}_I(S,t) \tag{2.2}$$

where $\mathbf{W}(S,t) \equiv -\mathbf{W}^T(S,t)$ is a *spatial skew−symmetric* tensor which defines the *spin* of the moving frame. The associated *axial* vector $\mathbf{w}(S,t)$, which satisfies $\mathbf{W}(S,t)\,\mathbf{w}(S,t) = \mathbf{O}$, gives the *vorticity* of the moving frame. In terms of the vorticity vector, equation (2.2) may be written as

$$\dot{\mathbf{t}}_I(S,t) = \mathbf{w}(S,t)\times\mathbf{t}_I(S,t) \tag{2.3}$$

Linear and Angular Momentum. Consider an arbitrary cross section denoted by Ω_t and given by $\Omega_t = \phi_t\big|_{s=Fixed}(\Omega)$, for each $s\in I$. We define the *linear momentum* per unit of *reference* arc *length*, associated with the motion (2.1), by the integral

$$\mathbf{L}_t \equiv \int_\Omega \rho_o(\xi,S)\,\dot{\boldsymbol{\phi}}_t(\xi,S,t)\,d\xi \equiv A_\rho\,\dot{\boldsymbol{\phi}}_o(S,t)\,, \tag{2.4}$$

where $\rho_o(\xi,S)$ is the density in the reference configuration, and we have employed the fact that $S\rightarrow\boldsymbol{\phi}_o(S,t)$ defines the current position of the centroid of the cross section.

Similarly, the *angular momentum* per unit of *reference* arc *length*, associated with the motion (2.1), and relative to the point $\mathbf{x}_o \equiv \boldsymbol{\phi}_o(S,t)$, is defined as

$$\mathbf{H}_t \equiv \int_\Omega \rho_o(\xi,S)\,[\mathbf{x} - \boldsymbol{\phi}_o(S,t)]\times\dot{\boldsymbol{\phi}}(\xi,S,t)\,d\xi\,, \tag{2.5}$$

where $\mathbf{x} = \boldsymbol{\phi}(\xi,S,t)$. To find a reduced expression for $\mathbf{H}_t$ we make use of (2.3) as follows.

Expression for $\mathbf{H}_t$. From the kinematic assumtion (2.1) and (2.3) we have

$$\dot{\boldsymbol{\phi}} - \dot{\boldsymbol{\phi}}_o = \sum_{\Gamma=1}^{2}\xi_\Gamma\,\dot{\mathbf{t}}_\Gamma \equiv \mathbf{w}\times(\boldsymbol{\phi}-\boldsymbol{\phi}_o)\,. \tag{2.6}$$

Substitution of (2.6) into (2.5) together with the fact that $\mathbf{x}_o = \boldsymbol{\phi}_o(S,t)$ defines the centroid of the cross section, yields

$$\begin{aligned}\mathbf{H}_t &= \int_\Omega \rho_o\,(\boldsymbol{\phi}-\boldsymbol{\phi}_o)\times[\mathbf{w}\times(\boldsymbol{\phi}-\boldsymbol{\phi}_o)]\,d\xi\\ &\{\int_\Omega \rho_o\,[|\boldsymbol{\phi}-\boldsymbol{\phi}_o|^2\mathbf{1} - (\boldsymbol{\phi}-\boldsymbol{\phi}_o)\otimes(\boldsymbol{\phi}-\boldsymbol{\phi}_o)]\,d\xi\}\,\mathbf{w} = \mathbf{I}_\rho\,\mathbf{w}\,,\end{aligned} \tag{2.7}$$

where $\mathbf{I}_\rho$ is the *inertia tensor* with the following explicit representation relative to the moving frame

$$\mathbf{I}_\rho = [\sum_{A=1}^{2}\sum_{B=1}^{2}\int_\Omega \rho_o(\xi,S)\,\xi_A\,\xi_B\,d\xi\,]\,(\delta_{AB}\,\mathbf{1} - \mathbf{t}_A\otimes\mathbf{t}_B)\,. \tag{2.8}$$

Notice that the *components* of $\mathbf{I}_\rho$ relative to the moving frame do not depend on time. Taking the material time derivative of (2.7), noting that $\dot{\mathbf{w}} = \dot{w}_I\,\mathbf{t}_I$ and making use of (2.3), we obtain the following expression for $\dot{\mathbf{H}}_t$,

$$\dot{\mathbf{H}}_t = \mathbf{I}_\rho\,\dot{\mathbf{w}} + \mathbf{w}\times\mathbf{H}_t\,. \tag{2.9}$$

The analogy between (2.9) and the expression for the angular momentum in rigid body mechanics is evident.

Remark 2.1. Let us consider the particular case in which the moving frame $\{\mathbf{t}_I(S,t)\}$ is directed along the *principal axes* of inertia of the cross section. Introducing the notation

$$I_1(S) \equiv \int_\Omega [\xi_2]^2\rho_o(\xi,S)\,d\xi\,,\quad I_2(S) \equiv \int_\Omega [\xi_1]^2\rho_o(\xi,S)\,d\xi\,, \tag{2.10}$$

and denoting by $J \equiv I_1 + I_2$ the *polar* moment of inertia of the cross section, expression (2.7) for the inertia tensor then reduces to the familiar form

$$\mathbf{I}_\rho \equiv I_1\,\mathbf{t}_1\otimes\mathbf{t}_1 + I_2\,\mathbf{t}_2\otimes\mathbf{t}_2 + J\,\hat{\mathbf{n}}\otimes\hat{\mathbf{n}}\;\;\Box \tag{2.11}$$

3. Force and Torque. Equations of Motion.

In this section we summarize the equations of motion for the nonlinear beam model. A comprehensive treatment can be found in e.g., Antman [1972], Sect.6. For completeness a simple derivation is included in the Appendix. The component form of these equations in the *material* description take a particularly simply form involving the orthogonal matrix $[\Lambda]$ which is well suited for computational purposes.

Consider a cross section $\Omega_t = \phi_t\big|_{S=Fixed}(\Omega)$ in the current configuration, and let $\mathbf{P}(\xi,S)$ denote the *first Piola-Kirchhoff* stress tensor. We may express the two-point tensor $\mathbf{P}(\xi,S)$ as

$$\mathbf{P}(\xi,S) \equiv \mathbf{T}_1(\xi,S)\otimes\hat{\mathbf{E}}_1 + \mathbf{T}_2(\xi,S)\otimes\hat{\mathbf{E}}_2 + \mathbf{T}_3(\xi,S)\otimes\hat{\mathbf{E}}_3. \tag{3.1}$$

Clearly, $\mathbf{T}_3(\xi,S) = \mathbf{P}(\xi,S)\,\hat{\mathbf{E}}_3$ is the *stress vector* (per unit of reference area) acting on the cross section $\Omega_t \subset \mathbf{R}^2$.

The *resultant contact force per unit of reference length* $\mathbf{f}(S,t)$ over the cross section Ω_t in the current configuration is then given by

$$\mathbf{f}(S,t) \equiv \int_\Omega \mathbf{P}(\xi,S)\,\hat{\mathbf{E}}_3 d\xi = \int_\Omega \mathbf{T}_3(\xi,S)\,d\xi \tag{3.2a}$$

Similarly, the *resultant torque per unit of reference arc length* $\mathbf{m}(S,t)$ over the cross section Ω_t in the current configuration is given by

$$\mathbf{m}(S,t) = \int_\Omega [\mathbf{x} - \phi_o(S,t)]\times\mathbf{T}_3(\xi,S)\,d\xi\,. \tag{3.2b}$$

The linear and angular momentum balance equations then take the form (see Appendix)

$$\frac{\partial}{\partial S}\mathbf{f} + \bar{\mathbf{q}} = \dot{\mathbf{L}}_t \equiv A_\rho\ddot{\phi}_o\,, \tag{3.3a}$$

$$\frac{\partial}{\partial S}\mathbf{m} + \frac{\partial\phi_o}{\partial S}\times\mathbf{f} + \bar{\mathbf{m}} = \dot{\mathbf{H}}_t \equiv \overset{\bullet}{\overline{\mathbf{I}_\rho\mathbf{w}}}\,, \quad S\in\mathrm{I}\,, \tag{3.3b}$$

where $\bar{\mathbf{q}}$ and $\bar{\mathbf{m}}$ are the "applied" force and torque per unit of reference arc length. In applications, the material form of these equations is often more convenient.

3.1. Material description.

The vector fields $\mathbf{f}(S,t)$ and $\mathbf{m}(S,t)$, although parametrized for convenience by the reference arc length $S\in\mathrm{I}$, *take values on the the current configuration;* i.e., their components are given relative to a *spatial* basis, either $\{\hat{\mathbf{e}}_i\}$ or $\{\mathbf{t}_I\}$. Alternatively, we define material vector fields

$$S\rightarrow\mathbf{N}\equiv\mathrm{N}_I\hat{\mathbf{E}}_I, \qquad S\rightarrow\mathbf{M}\equiv\mathrm{M}_I\hat{\mathbf{E}}_I, \quad S\in\mathrm{I}\,, \tag{3.4}$$

by pulling-back[†]the vector fields $\mathbf{f}(S,t)$ and $\mathbf{m}(S,t)$ to the reference configuration $\mathrm{I}\times\Omega\subset\mathbf{R}^3$ with the *orthogonal transformation* $S\rightarrow\Lambda(S,t)$. Accordingly, we have the relations:

$$\mathbf{f} = \Lambda\,\mathbf{N}\,, \qquad \text{and} \qquad \mathbf{m} = \Lambda\,\mathbf{M} \tag{3.5}$$

The geometric meaning of $\mathbf{N}(S,t)$ and $\mathbf{M}(S,t)$ follows from the observation that

$$\mathbf{f} = \mathrm{N}_I\Lambda\,\hat{\mathbf{E}}_I \equiv \mathrm{N}_I\mathbf{t}_I\,, \qquad \text{and} \qquad \mathbf{m} = \mathrm{M}_I\mathbf{t}_I \tag{3.6}$$

Thus, *the components of the force and moment vectors* $\mathbf{f}$ *and* $\mathbf{m}$ *relative to the moving frame* $\{\mathbf{t}_I\}$ *equal those of* $\mathbf{N}$ *and* $\mathbf{M}$ *relative to the reference frame* $\{\hat{\mathbf{E}}_I\}$.

The component form of the equations in the material description are obtained by

[†]For a formal definition of the pull-back operation see, e.g, Abraham, Marsden & Ratiu [1983], *Manifolds, Tensor Analysis and Applications,* Addison-Wesley Co., Ma.

substitution of (3.5) into (3.3a,b).

Remark 3.1. The classical equations of thin rods of Kirchhoff-Love (Love [1944], pp.387-388) may be now recover from (3.3a) and (3.3b) as follows. First, we introduce the *current* arc *length* defined by the map

$$S \rightarrow s(S) \equiv \int_o^S |\partial \boldsymbol{\phi}_o(\mu,t)/\partial\mu|\, d\mu\,, \tag{3.7}$$

which may be regarded as a smooth reparametrization. Next, we note that if *no shearing effect* is considered, we must have

$$\frac{\partial}{\partial S}\boldsymbol{\phi}_o(S,t) = \frac{ds}{dS}\,\hat{\mathbf{n}}(s)\,, \quad |\hat{\mathbf{n}}(s)| = 1 \tag{3.8}$$

which is simply the first Frenet formula. From $(3.6)_2$ we have

$$\frac{\partial \mathbf{m}}{\partial S} = \frac{ds}{dS}\left[\frac{\partial \mathrm{M_I}}{\partial s} - \Omega_{\mathrm{IJ}}\mathrm{M_J}\right]\mathbf{t}_\mathrm{I}. \tag{3.9}$$

Making use of (3.8) and $(3.6)_1$, since $\hat{\mathbf{n}}\times\mathbf{t}_3 = \mathbf{t}_2$ and $\hat{\mathbf{n}}\times\mathbf{t}_2 = -\mathbf{t}_1$, we also have:

$$\frac{\partial \boldsymbol{\phi}_o}{\partial S}\times\mathbf{f} = \frac{ds}{dS}\left[\mathrm{N}_1\mathbf{t}_2 - \mathrm{N}_2\mathbf{t}_1\right] \tag{3.10}$$

Substitution of (3.9) and (3.10) into (3.3b) leads, for the *static case* and with the assumption that $\overline{\mathbf{m}} \equiv \mathbf{o}$, to the Kirchhoff-Love moment equilibrium equations [††] (Love [1944], p.388, Eq.(11)). The force equilibrium equation follows at once from (3.3a) and $(3.6)_1$.□

4. Internal Power and Strain Measures. Constitutive Equations.

Our purpose in this section is to formulate properly invariant reduced constitutive equations in terms of *global* kinetical and kinematical objects. Our first step is to obtain a reduced expression for the *internal power* from the general expression of 3-dimensional theory, by introducing the basic kinematic assumption (2.1). This reduced expression yields the appropriate definition of strain measures conjugate to the resultant force and moment in the *spatial* as well as in the the fully *material* descriptions.

4.1. Internal power. Strain measures.

We first consider the reduced expression for the internal power in terms of the *spatial* force $\mathbf{f}(S,t)$ and torque $\mathbf{m}(S,t)$ defined by (3.2a) and (3.2b), respectively. The basic result is summarized in the following

Proposition 4.1. With the kinematic assumption (2.1) in force, the *internal power* Π may be expressed as

$$\Pi \equiv \int_{\mathrm{I}\times\Omega} \mathbf{P}:\dot{\mathbf{F}}\, d\xi\, dS = \int_{\mathrm{I}} [\mathbf{f}\cdot\overset{\nabla}{\boldsymbol{\gamma}} + \mathbf{m}\cdot\overset{\nabla}{\boldsymbol{\omega}}]\, dS\,, \tag{4.1a}$$

where $\boldsymbol{\omega}(S,t)$ is the *spatial* vector with components given by (1.11), $\boldsymbol{\gamma}(S,t)$ is a *spatial* vector defined by

$$\boldsymbol{\gamma}(S,t) \equiv \frac{\partial\boldsymbol{\phi}_o}{\partial S}(S,t) - \hat{\mathbf{n}}(S,t)\,, \tag{4.1b}$$

and $\overset{\nabla}{(\bullet)}$ stands for the following *objective* rate

[††] Notice that $[\boldsymbol{\Omega}]$ defined by (1.10) would have opposite sign with the convention in Love [1944], Eq. (5), p.384.

$$\overset{\nabla}{(\bullet)} \equiv \frac{\partial}{\partial t}(\bullet) - \mathbf{w}\times(\bullet) \tag{4.1c}$$

Proof. To prove expressions (4.1a,b) we first compute the deformation gradient. From (2.1) and using (1.12) we obtain

$$\mathbf{F} = \sum_{\Gamma=1}^{2} \mathbf{t}_\Gamma \otimes \hat{\mathbf{E}}_\Gamma + [\frac{\partial \boldsymbol{\phi}_o}{\partial S} + \boldsymbol{\Omega}\times(\mathbf{x}-\boldsymbol{\phi}_o)]\otimes\hat{\mathbf{E}}_3 \tag{4.2}$$

Taking the material time derivative and making use of (2.3) we have

$$\dot{\mathbf{F}} = \sum_{\Gamma=1}^{2} (\mathbf{w}\times\mathbf{t}_\Gamma)\otimes\hat{\mathbf{E}}_\Gamma + [\frac{\partial \dot{\boldsymbol{\phi}}_o}{\partial S} + \dot{\boldsymbol{\Omega}}\times(\mathbf{x}-\boldsymbol{\phi}_o)]\otimes\hat{\mathbf{E}}_3 + [\boldsymbol{\Omega}\times\{\mathbf{w}\times(\mathbf{x}-\boldsymbol{\phi}_o)\}]\otimes\hat{\mathbf{E}}_3 \tag{4.3}$$

Since $\mathbf{P} = \sum_{I=1}^{3} \mathbf{T}_I\otimes\hat{\mathbf{E}}_I$, it follows that

$$\mathbf{P}:\dot{\mathbf{F}} = \mathbf{T}_3\bullet\frac{\partial \dot{\boldsymbol{\phi}}_o}{\partial S} + [(\mathbf{x}-\boldsymbol{\phi}_o)\times\mathbf{T}_3]\bullet\dot{\boldsymbol{\Omega}}$$
$$+ \mathbf{T}_3\bullet[\boldsymbol{\Omega}\times\{\mathbf{w}\times(\mathbf{x}-\boldsymbol{\phi}_o)\}] + \sum_{\Gamma=1}^{2}\mathbf{w}\bullet(\mathbf{t}_\Gamma\times\mathbf{T}_\Gamma) \tag{4.4}$$

We now make use of the angular momentum balance condition $\frac{\partial \boldsymbol{\phi}}{\partial \xi_I}\times\mathbf{T}_I = \mathbf{0}$, and (1.12) to express the last term in (4.4) as

$$\sum_{\Gamma=1}^{2}\mathbf{w}\bullet(\mathbf{t}_\Gamma\times\mathbf{T}_\Gamma) = \mathbf{w}\bullet\sum_{\Gamma=1}^{2}\frac{\partial}{\partial\xi_\Gamma}(\mathbf{x}-\boldsymbol{\phi}_o)\times\mathbf{T}_\Gamma$$
$$= -\mathbf{w}\bullet[\frac{\partial\boldsymbol{\phi}}{\partial S}\times\mathbf{T}_3] \equiv -\mathbf{T}_3\bullet[\mathbf{w}\times\frac{\partial\boldsymbol{\phi}_o}{\partial S} + \mathbf{w}\times\{\boldsymbol{\Omega}\times(\mathbf{x}-\boldsymbol{\phi}_o)\}] \tag{4.5}$$

Substitution of (4.5) into (4.4), use of definitions (3.2a) and (3.2b), together with the identity

$$\boldsymbol{\Omega}\times[\mathbf{w}\times(\mathbf{x}-\boldsymbol{\phi}_o)] - \mathbf{w}\times[\boldsymbol{\Omega}\times(\mathbf{x}-\boldsymbol{\phi}_o)]$$
$$= [\mathbf{w}\otimes\boldsymbol{\Omega} - \boldsymbol{\Omega}\otimes\mathbf{w}](\mathbf{x}-\boldsymbol{\phi}_o) \equiv [\boldsymbol{\Omega}\times\mathbf{w}]\times(\mathbf{x}-\boldsymbol{\phi}_o), \tag{4.6}$$

leads to the following reduced expression for the internal power

$$\Pi = \mathbf{f}\bullet[\frac{\partial}{\partial t}(\frac{\partial\boldsymbol{\phi}_o}{\partial S}) - \mathbf{w}\times\frac{\partial\boldsymbol{\phi}_o}{\partial S}] + \mathbf{m}\bullet[\dot{\boldsymbol{\Omega}} - \mathbf{w}\times\boldsymbol{\Omega}], \tag{4.7}$$

which proves the proposition. □

Remark 4.1. The physical significance of the rate $\overset{\nabla}{(\bullet)}$ should be clear. It gives the rate of change of $(\bullet)$ relative to an observer which moves with the spatial frame $\{\mathbf{t}_I\}$, since the effect of the *spin* of the *moving* frame $\{\mathbf{t}_I\}$ given by $\mathbf{w}$ is substracted from the material time derivative. Thus, one often speaks of a *corrotated* rate. This interpretation follows at once from (2.3). □

Alternatively, we may recast the reduced expression (4.1a) for the internal power in terms of *material* fields N and M as follows.

Proposition 4.2. With the kinematic assumption (2.1) in force, the reduced expression for the internal power may be expressed as

$$\Pi \equiv \int_{I\times\Omega}\mathbf{P}:\dot{\mathbf{F}}\,d\xi\,dS = \int_I[\mathbf{N}\bullet\dot{\boldsymbol{\Gamma}} + \mathbf{M}\bullet\dot{\mathbf{K}}]\,dS, \tag{4.8a}$$

where $\mathbf{N}(S,t)$ and $\mathbf{M}(S,t)$, are *material* vector fields defined by (3.5), and $\boldsymbol{\Gamma}(S,t)$, $\mathbf{K}(S,t)$ are *material* vector fields given by

$$\Gamma \equiv \Lambda^T \frac{\partial \phi_o}{\partial S} - \hat{E}_3 \equiv \Lambda^T [\frac{\partial \phi_o}{\partial S} - \hat{n}],$$
$$K \equiv \Lambda^T \Omega \tag{4.8b}$$

Proof. The result follows at once from (2.2)-(2.3) by noting that for any *spatial* vector $\mathbf{h} = h_I \mathbf{t}_I$ we have

$$\overset{\nabla}{\mathbf{h}} \equiv \frac{\partial}{\partial t}\mathbf{h} - \mathbf{w} \times \mathbf{h} = \frac{\partial}{\partial t}\mathbf{h} - \mathbf{W}\mathbf{h} = \Lambda \frac{\partial}{\partial t}[\Lambda^T \mathbf{h}]. \tag{4.9}$$

Since $\mathbf{m}$, $\mathbf{M}$, $\mathbf{f}$ and $\mathbf{N}$ are related according to (3.5), use of proposition 4.1 and (4.8b) suplies the result. □

Remark 4.2. It is interesting to examine the limiting case of the Kirchhoff-Love situation considered in Remark 3.1. As a result of assumption (3.8) the strain measures γ and Γ reduce to

$$\gamma = [\frac{ds}{dS} - 1]\,\hat{n}, \quad \text{and} \quad \Gamma = [\frac{ds}{dS} - 1]\,\hat{E}_3, \tag{4.10}$$

since $\hat{n} \equiv t_3 = \Lambda \hat{E}_3$. Thus, shear deformation of the rod *vanishes* identically. Actually, the situation discussed in Love [1944] (Sects. 255-256, pp.388-393) corresponds to that of a superposed infinitesimal deformation, and may be obtained as a particular case of proposition 4.2 by consistent linearization procedures. □

Remark 4.3. The strain measures (4.8b) could also be obtained by starting with the *material* form of the balance equations and making use of a one dimensional virtual work type of argument as in Reissner [1972]. Our approach, however, proceeds directly from the 3-dimensional theory. □

Next, we formulate *global* constitutive equations.

4.2. Constitutive Equations.

In what follows, attention is restricted to the elastic case and the pure mechanical theory. More general situations including heat conduction may be considered by the methods in e.g., Naghdi [1980] or Antman [1972]. For present purposes we simply note that as a result of proposition 4.1 for *elastic* behavior we may define a stored energy function $\psi(S,\gamma,\Omega)$ such that

$$\mathbf{f} = \frac{\partial \psi(S,\gamma,\Omega)}{\partial \gamma}, \quad \text{and} \quad \mathbf{m} = \frac{\partial \psi(S,\gamma,\Omega)}{\partial \Omega}, \quad S \in I \tag{4.11}$$

Similarly, in the *material* description, as a result of proposition 4.2 we may define a stored energy function $\Psi(S,\Gamma,K)$ such that

$$\mathbf{N} = \frac{\partial \Psi(S,\Gamma,K)}{\partial \Gamma}, \quad \text{and} \quad \mathbf{M} = \frac{\partial \Psi(S,\Gamma,K)}{\partial K}, \quad S \in I \tag{4.12}$$

For computational purposes, particularly for inelasticity, the rate form of constitutive equations (4.11) and (4.12) is often needed. In the material description taking the material time derivative of (4.12) we simply have

$$\begin{Bmatrix} \dot{\mathbf{N}} \\ \dot{\mathbf{M}} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \Psi}{\partial \Gamma \partial \Gamma} & \dfrac{\partial \Psi}{\partial \Gamma \partial K} \\ \dfrac{\partial \Psi}{\partial \Gamma \partial K} & \dfrac{\partial \Psi}{\partial K \partial K} \end{bmatrix} \begin{Bmatrix} \dot{\Gamma} \\ \dot{K} \end{Bmatrix} \equiv \mathbf{C}(S,\Gamma,K) \begin{Bmatrix} \dot{\Gamma} \\ \dot{K} \end{Bmatrix}. \tag{4.13}$$

Making use of (4.9) and the chain rule, equation (4.13) may be expressed in the spatial description as

$$\begin{Bmatrix} \overset{\nabla}{\mathbf{f}} \\ \overset{\nabla}{\mathbf{m}} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \psi}{\partial \gamma \partial \gamma} & \dfrac{\partial \psi}{\partial \gamma \partial \Omega} \\ \dfrac{\partial \psi}{\partial \gamma \partial \Omega} & \dfrac{\partial \psi}{\partial \Omega \partial \Omega} \end{bmatrix} \begin{Bmatrix} \overset{\nabla}{\gamma} \\ \overset{\nabla}{\Omega} \end{Bmatrix} \equiv \mathbf{c}(S,\gamma,\Omega) \begin{Bmatrix} \overset{\nabla}{\gamma} \\ \overset{\nabla}{\Omega} \end{Bmatrix}. \tag{4.14}$$

We refer to $\mathbf{C}$ and $\mathbf{c}$ as the *material* and *spatial elasticity* tensors, respectively. In particular, one often assumes in applications that the material elasticity tensor $\mathbf{C}$ in the rate constitutive equations (4.13) is *diagonal* with *constant* coefficients. This is equivalent to assuming a quadratic (uncoupled) expression for the *material* stored energy function $\Psi(S,\Gamma,\mathrm{K})$. Of this particular type are the constitutive equations of the classical Kirchhoff-Love rod theory. This completes our discussion of constitutive equations.

Remark 4.4. With the assumption that the *material* elasticity tensor $\mathbf{C}$ in (4.13) is diagonal with constant coefficients, one can formulate simple *inelastic* constitutive models which are properly invariant and account for viscoplastic response. Such models are particularly useful in computational applications. See Simo, Hjelmstad & Taylor [1984]. □

5. Concluding Remarks: The Plane Case.

We shall show that for the plane problem the formulation heretofore presented reduces to that proposed by Reissner [1972].

Assume that the motion of the beam takes place in the coordinate plane normal to $\hat{\mathbf{E}}_2 \equiv \hat{\mathbf{e}}_2 = \mathbf{t}_2$, as illustrated in Figure 1. The orthogonal tensor $\Lambda(S)$ then admits, for all $S \in \mathrm{I}$, the matrix representation

$$[\Lambda(S)] = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 1 \end{bmatrix} \tag{5.1}$$

The axial vector $\boldsymbol{\Omega}(S)$ given by (1.11), now coincident with $\mathbf{K}(S)$ defined by (1.16), and the vorticity vector $\mathbf{w}$ have the expressions

$$\boldsymbol{\Omega} \equiv \mathbf{K} = \frac{\partial\theta}{\partial S}\hat{\mathbf{E}}_2, \qquad \mathbf{w} = \frac{\partial\theta}{\partial t}\hat{\mathbf{E}}_2. \tag{5.2}$$

The strain measures Γ defined by (4.8b) take the form

$$\Gamma_3 = (1+u')\cos\theta + v'\sin\theta - 1, \qquad \Gamma_1 = -(1+u')\sin\theta + v'\cos\theta \tag{5.3}$$

where $\dfrac{\partial\boldsymbol{\phi}_o}{\partial S} = (1+u')\hat{\mathbf{e}}_3 + v'\hat{\mathbf{e}}_1$. Introducing the notation

$$\epsilon = \frac{ds}{dS} - 1, \qquad \alpha = \tan^{-1}\frac{v'}{1+u'}, \tag{5.4}$$

where $s(S)$ is the current arc length defined by (3.7) so that

$$\frac{ds}{dS} = \sqrt{(1+u')^2 + (v')^2}, \tag{5.5}$$

the strains Γ_3 and Γ_1 given by (5.3) may be expressed as

$$\Gamma_3 = (1+\epsilon)\cos(\alpha-\theta) - 1, \qquad \Gamma_1 = (1+\epsilon)\sin(\alpha-\theta), \tag{5.6}$$

which coincide with the expressions given in (Reissner [1972,1982]). Notice that $\alpha - \theta$ defines the shear angle in the natural way. Explicit component expressions for the 2-dimensional equilibrium equations in terms of $\mathbf{N}$ and $\mathbf{M}$ follow at once by substitution of (3.5) into (3.3a) and (3.3b).

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Appendix

To develop equations of motion expressed in terms of the resultant force $\mathbf{f}(S,t)$ and the resultant moment $\mathbf{m}(S,t)$, we proceed from the *material* form of the balance of linear and angular momentum principles of the 3-dimensional theory, which may be expressed as

$$\mathrm{DIV}\,\mathbf{P} + \rho_o \mathbf{B} = \rho_o \ddot{\boldsymbol{\phi}}_t, \quad \mathbf{F}\mathbf{P}^T = \mathbf{P}\mathbf{F}^T, \tag{A.1}$$

where $(\xi,S) \rightarrow \mathbf{B}(\xi,S,t)$ is the body force field, and $\mathrm{DIV}\,\mathbf{P} \equiv \partial \mathbf{T}_I/\partial \xi_I$.

Balance of Linear Momentum. From (3.1) and (3.2), making use of $(\mathrm{A.1})_1$, we have:

$$\begin{aligned}\frac{\partial}{\partial S}\mathbf{f}(S,t) &= \int_\Omega \frac{\partial}{\partial S}\mathbf{T}_3 d\xi \\ &= -\int_\Omega \left[\sum_{\Gamma=1}^{2}\frac{\partial \mathbf{T}_\Gamma}{\partial \xi_\Gamma} + \rho_o \mathbf{B}\right] d\xi + \int_\Omega \rho_o \ddot{\boldsymbol{\phi}}_t d\xi\end{aligned} \tag{A.2}$$

Applying the divergence theorem, and defining the applied load as

$$\bar{\mathbf{q}}(S,t) = \sum_{\Gamma=1}^{2}\int_{\partial\Omega}[\mathbf{T}_\Gamma \nu_\Gamma]\, d\Gamma + \int_\Omega \rho_o \mathbf{B}\, d\xi\,, \tag{A.3}$$

where $\boldsymbol{\nu} = \nu_1 \hat{\mathbf{E}}_1 + \nu_2 \hat{\mathbf{E}}_2$ is the vector field normal to the "lateral" contour $\partial\Omega$ of the beam, we obtain the balance equation:

$$\frac{\partial}{\partial S}\mathbf{f}(S,t) + \bar{\mathbf{q}}(S,t) = \dot{\mathbf{L}}_t \equiv A_\rho \ddot{\boldsymbol{\phi}}_o(S,t), \quad S \in \mathrm{I} \ \Box \tag{A.4}$$

Balance of Angular Momentum. From (3.2a) and (3.2b) we have

$$\begin{aligned}\frac{\partial}{\partial S}\mathbf{m}(S,t) &= \int_\Omega \frac{\partial \boldsymbol{\phi}}{\partial S}\times \mathbf{T}_3 d\xi - \frac{\partial \boldsymbol{\phi}_o}{\partial S}\times \mathbf{f} + \int_\Omega [\boldsymbol{\phi}-\boldsymbol{\phi}_o]\times \sum_{\Gamma=1}^{2}\frac{\partial \mathbf{T}_\Gamma}{\partial \xi_\Gamma} d\xi + \int_\Omega \rho_o \ddot{\boldsymbol{\phi}}\, d\xi \\ &= \overset{\bullet}{\overline{\mathbf{I}_\rho \mathbf{w}}} + \int_\Omega \frac{\partial \boldsymbol{\phi}_t}{\partial \xi_I}\times \mathbf{T}_I d\xi - \frac{\partial \boldsymbol{\phi}_o}{\partial S}\times \mathbf{f} - \bar{\mathbf{m}}(S)\,,\end{aligned} \tag{A.5}$$

where use has been made of (2.11), the divergence theorem, and the following notation for the applied moment field:

$$\bar{\mathbf{m}}(S,t) = \sum_{\Gamma=1}^{2}\int_{\partial\Omega}[\mathbf{x}-\boldsymbol{\phi}_o]\times[\mathbf{T}_\Gamma \nu_\Gamma]\, d\Gamma + \int_\Omega \rho_o [\mathbf{x}-\boldsymbol{\phi}_o]\times \mathbf{B}\, d\xi\,. \tag{A.6}$$

From the balance of angular momentum condition $(A.1)_2$ it follows that $\frac{\partial \boldsymbol{\phi}}{\partial \xi_I} \times \mathbf{T}_I \equiv \mathbf{o}$. Thus, (A.5) reduces to

$$\frac{\partial}{\partial S} \mathbf{m}(S,t) + \frac{\partial \boldsymbol{\phi}_o}{\partial S} \times \mathbf{f} + \overline{\mathbf{m}}(S,t) = \mathbf{I}_\rho \dot{\mathbf{w}} + \mathbf{w} \times \mathbf{H}_t, \quad S \in I, \quad \square \tag{A.7}$$