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Asset Prices and Efficiency in a Krebs Economy

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Abstract

I study the asset pricing implications and the efficiency of a tractable dynamic stochastic general equilibrium model with heterogeneous agents and incomplete markets along the lines of Krebs (2003a). Contrary to previous applications of these types of models, I find that generically the distribution of idiosyncratic shocks affects the risk premia of aggregate shocks and that the equilibrium is constrained inefficient in the sense that a planner can Pareto improve the equilibrium outcome by assigning different portfolio choices to agents. The inefficiency is caused by a 'portfolio externality': the average portfolio of the economy affects the portfolio return of each agent. The constrained efficient outcome can be achieved through linear taxes and subsidies that I characterize in closed-form.

Keywords: AK models, constrained efficiency, externality, idiosyncratic risk, incomplete markets, optimal taxation.

JEL codes: D52, D58, E21, E22, G11, H21, H23.

1 Introduction

Since the theoretical work of Bewley (1986) and the quantitative work of Huggett (1993), Aiyagari (1994), and Krusell and Smith (1998), heterogeneous-agent general equilibrium models with incomplete markets have been widely applied in economics. Two particular applications are asset pricing and welfare economics. The former literature is largely motivated by the inability of the representative-agent, consumption-based asset pricing model to explain various asset pricing puzzles, most notably the equity premium puzzle and the risk-free rate puzzle. The latter literature, for instance Dávila et al. (2012), is motivated by the theoretical result that the general equilibrium with incomplete markets (GEI) is generically constrained inefficient.[1]

Since heterogeneous-agent models are typically analytically intractable, few theoretical results are known about the asset pricing and welfare implications of incomplete market heterogeneous-agent models, apart from a few exceptions. By

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[1]A property holds "generically" if it holds for all parameter values except those in a set with measure zero.

judiciously constructing individual income processes, Constantinides and Duffie (1996) show that any arbitrage-free asset prices, dividends, and aggregate consumption can be explained. Krueger and Lustig (2010), on the other hand, show that idiosyncratic labor income risk has no effect on the equity premium if idiosyncratic shocks are independent of aggregate shocks and aggregate consumption growth is independent over time. With regard to efficiency, Krebs (2003a,b, 2006) develops a tractable dynamic general equilibrium model with heterogeneous agents who are subject to uninsurable idiosyncratic human capital risk[2] but finds that the equilibrium is nevertheless constrained efficient. This paper extends these two literatures by generalizing the model of Krebs (2006).

The contribution of this paper is twofold. First, I show that the irrelevance result of Krueger and Lustig does not generally hold unless there is a single source of aggregate risk. When there are multiple sources of aggregate risk, as in Constantinides and Duffie (1996), risk premia on assets are typically affected by idiosyncratic shocks. This is because when there are multiple sources of aggregate risk, the aggregate component of individual consumption growth (which prices assets through the Euler equation) depends on the portfolio choice, which in turn is affected by idiosyncratic shocks.

Second, I study the efficiency of the Krebs economy. Both in the baseline model of Krebs (2006) (with one technology, two inputs, and idiosyncratic human capital depreciation shocks) and the extension of Toda (2014) (with multiple AK technologies), the equilibrium is constrained efficient. However, in the more general case with one or more technologies/inputs and general idiosyncratic technological shocks, I find that the equilibrium is generically constrained inefficient. The inefficiency is cause by a pecuniary externality (Greenwald and Stiglitz, 1986): when there are multiple technologies and inputs, the return on an individual portfolio depends on other agents' portfolio (weighted average portfolio). Nevertheless, the constrained efficient outcome can be achieved through linear taxes and subsidies that can be characterized in closed-form. This result is in sharp contrast to those in the literature. For instance, the mathematical economics literature[3] typically studies only two period models, and while it is possible to show the existence of a Pareto improving intervention (Citanna et al., 1998), it is rarely possible to characterize the optimal intervention. Dávila et al. (2012) numerically solve a stochastic growth model with idiosyncratic labor income risk and find that the optimal tax rates depend on the wealth level and idiosyncratic states of agents. Being a quantitative work, however, they neither prove that the equilibrium is actually constrained inefficient nor that an optimal intervention exists.

I also provide a numerical example calibrated to the U.S. economy. The effect of idiosyncratic shocks on asset prices is substantial (the equity premium ranges from 0.63% to 3.74% by changing the idiosyncratic labor income volatility from 0% (representative-agent model) to 30%), and the welfare loss is moderate (1% in consumption equivalent). In this example (regardless of the parameter

[2]Similar models (AK models with idiosyncratic investment risk) have been used by Saito (1998), Angeletos (2007), and Toda (2014), among many others.

[3]The notion of constrained efficiency was first defined by Diamond (1967). Geanakoplos and Polemarchakis (1986) first proved the generic constrained inefficiency of equilibrium in a two period exchange economy. Geanakoplos et al. (1990) and Carvajal and Polemarchakis (2011) treat the case with production and idiosyncratic risk, respectively.

values) there is always an over-investment in capital, and the optimal tax rate on the capital stock is 0.5% in the baseline specification.

Related literature

This paper is closest to Krebs (2003a,b, 2006), Krueger and Lustig (2010), Gottardi et al. (2011), and Toda (2014). Krebs (2006) is probably the first tractable dynamic general equilibrium with a continuum of heterogeneous agents and incomplete markets in a general (non-i.i.d.) Markov setting, which is particularly useful for various applications such as the current paper. His earlier works (Krebs, 2003a,b) study the growth and welfare effects of human capital risk in specialized models. Krueger and Lustig (2010) study a consumption-based asset pricing model with heterogeneous agents. They find that under certain assumptions, idiosyncratic shocks have no impact on the equity premium. My paper shows that their result holds only in a knife-edge case where there is only one source of aggregate shock. Gottardi et al. (2011) study the Ramsey problem of finding the optimal public debt and linear taxes in a model similar to the current paper (but with no aggregate risk and i.i.d. idiosyncratic risk). Although I do not consider public debt, my paper is complementary since I study a more general model and characterize the optimal tax rates (which happen to be linear) in closed-form. Toda (2014) studies the theoretical properties such as equilibrium existence and uniqueness with general preferences and shocks (which the current paper applies) and characterizes the stationary wealth distribution and the power law exponents.

2 A simple example

In this section I present a simple specialized model in order to build the intuition for the main results. The exposition is deliberately informal since formal definitions, theorems, and proofs in a more general model will be given in subsequent sections.

2.1 Settings

There is an "all purpose" good that can be either consumed or saved as physical and human capital.[4] Time is infinite and is denoted by $t = 0, 1, \dots$. A perfectly competitive firm produces the good using the production function $F(K, H) = AK^\alpha H^{1-\alpha}$, where K, H are the efficiency units of physical and human capital, $A > 0$, and $0 < \alpha < 1$. Suppressing the time subscript, the firm's problem at each period is

$$\max_{K,H} AK^\alpha H^{1-\alpha} - rK - \omega H,$$

where $r > 0$ is the rental rate of physical capital and $\omega > 0$ is the wage.

The economy is populated by a continuum of agents with mass 1 indexed by $i \in I = [0, 1]$ with identical Epstein-Zin constant relative risk aversion, constant

[4] If the reader is uncomfortable with the assumption that physical and human capital can be converted 1:1, one can interpret human capital as private equity.

elasticity of intertemporal substitution (CRRA/CEIS) recursive preferences

$$U_t = \left(c_t^{1-1/\varepsilon} + \beta \, \mathrm{E}_t[U_{t+1}^{1-\gamma}]^{\frac{1-1/\varepsilon}{1-\gamma}}\right)^{\frac{1}{1-1/\varepsilon}},$$

where $\beta > 0$ is the discount factor, $\gamma > 0$ is the relative risk aversion (RRA) coefficient, and $\varepsilon > 0$ is the elasticity of intertemporal substitution (EIS).[5]

Each agent derives income by renting physical and human capital to the firm, which depreciate at rate $\delta, \eta \leq 1$ after production, respectively.[6] Then agents make consumption and investment decisions by allocating the good as physical and human capital. Between time t and $t+1$, the efficiency unit of physical capital grows at gross growth rate Z_{t+1}, which is an i.i.d. random variable. Similarly, the human capital of agent i grows by $y_{i,t+1}Y_{t+1}$, where $y_{i,t+1}$ is the purely idiosyncratic component[7] (so $\mathrm{E}_{t+1}[y_{i,t+1}] = 1$) and Y_{t+1} is the aggregate component. Assume that $y_{i,t+1}$ is i.i.d. over time and across agents and Y_{t+1} is i.i.d. over time. Z, y, Y can be broadly interpreted as capital-augmenting technological shocks or obsolescence. Let k_{it}, h_{it} be agent i's stock of physical and human capital at the beginning of time t, c_{it} consumption, and x_{it}^k, x_{it}^h the physical and human capital (dis)investment at time t. Then the budget constraint is

$$c_{it} + x_t^k + x_t^h = r_t k_{it} + \omega_t h_{it} \tag{2.1}$$

and the accumulation equations for physical and human capital are

$$k_{i,t+1} = Z_{t+1}[(1-\delta)k_{it} + x_{it}^k], \tag{2.2a}$$

$$h_{i,t+1} = y_{i,t+1}Y_{t+1}[(1-\eta)h_{it} + x_{it}^h]. \tag{2.2b}$$

It is convenient to define agent i's total wealth at time t *after production* by

$$w_{it} = (1 + r_t - \delta)k_{it} + (1 + \omega_t - \eta)h_{it} \tag{2.3}$$

and the fraction of wealth held as physical capital *after consumption* by

$$\theta_{it} = [(1-\delta)k_{it} + x_{it}^k]/(w_{it} - c_{it}).$$

Adding $(1-\delta)k_{it} + (1-\eta)h_{it}$ to both sides of (2.1), the fraction of wealth held as human capital turns out to be

$$[(1-\eta)h_{it} + x_{it}^h]/(w_{it} - c_{it}) = 1 - \theta_{it}.$$

Combining (2.2), (2.3), and using the definition of θ_{it}, agent i's dynamic budget constraint becomes

$$w_{i,t+1} = R_{i,t+1}(\theta_{it})(w_{it} - c_{it}),$$

where the gross return on wealth is

$$R_{i,t+1}(\theta) = (1 + r_{t+1} - \delta)Z_{t+1}\theta + (1 + \omega_{t+1} - \eta)y_{i,t+1}Y_{t+1}(1-\theta). \tag{2.4}$$

[5] As usual, $\gamma = 1$ corresponds to log utility. Readers uncomfortable with recursive preferences may set $\varepsilon = 1/\gamma$ to obtain the additive CRRA preference, in which case $U_t' := \frac{1}{1-\gamma}U_t^{1-\gamma}$ is the usual expected lifetime utility. Note that the value function using U_t is proportional to the wealth w, whereas the value function using U_t' is proportional to $w^{1-\gamma}$.

[6] There can be "appreciation" of human capital by "learning by doing", in which case $\eta < 0$.

[7] In Krebs (2006), the idiosyncratic risk is present only in the depreciation rate η and not in the technological shock y. This difference is important in obtaining the inefficiency result.

2.2 Equilibrium

A sequential equilibrium is defined by a plan of consumption and portfolio choice $\{(c_{it}, \theta_{it})_{i\in I}\}_{t=0}^{\infty}$ and rental rates $\{(r_t, \omega_t)\}_{t=0}^{\infty}$ such that (i) agents maximize utility subject to budget constraints, (ii) firms maximize profit, and (iii) rental markets for physical and human capital clear. By the i.i.d. assumption and the homotheticity of the utility function and the budget constraints, it should be clear that in equilibrium every agent will choose the same constant consumption rate $\tilde{c} = c_{it}/w_{it} \in [0, 1]$ and portfolio $\bar{\theta} \in [0, 1]$.

The equilibrium can be obtained as follows. Summing (2.2) across agents in equilibrium and using the definition of $\bar{\theta}$, we obtain

$$\begin{aligned} K_{t+1} &= Z_{t+1}\bar{\theta}(W_t - C_t), \\ H_{t+1} &= Y_{t+1}(1-\bar{\theta})(W_t - C_t), \end{aligned}$$

where W_t, C_t are aggregate wealth and consumption. By the first-order condition for profit maximization of the firm, we obtain

$$\begin{aligned} r_{t+1} &= A\alpha(H_{t+1}/K_{t+1})^{1-\alpha} = A\alpha\left(\frac{Y_{t+1}}{Z_{t+1}}\frac{1-\bar{\theta}}{\bar{\theta}}\right)^{1-\alpha}, \\ \omega_{t+1} &= A(1-\alpha)(H_{t+1}/K_{t+1})^{-\alpha} = A(1-\alpha)\left(\frac{Y_{t+1}}{Z_{t+1}}\frac{1-\bar{\theta}}{\bar{\theta}}\right)^{-\alpha}. \end{aligned}$$

Combining with (2.4), the gross return on individual portfolio θ given the equilibrium portfolio $\bar{\theta}$ is

$$\begin{aligned} R_{i,t+1}(\theta, \bar{\theta}) = &\left(1 + A\alpha\left(\frac{Y_{t+1}}{Z_{t+1}}\frac{1-\bar{\theta}}{\bar{\theta}}\right)^{1-\alpha} - \delta\right) Z_{t+1}\theta \\ &+ \left(1 + A(1-\alpha)\left(\frac{Y_{t+1}}{Z_{t+1}}\frac{1-\bar{\theta}}{\bar{\theta}}\right)^{-\alpha} - \eta\right) y_{i,t+1}Y_{t+1}(1-\theta). \end{aligned} \tag{2.5}$$

Again by homotheticity and the assumption of i.i.d. shocks, it should be clear that the value function of each agent is of the form $V(w) = bw$, where $b > 0$ is a constant. Therefore the Bellman equation is

$$bw = \max_{c,\theta}\left(c^{1-1/\varepsilon} + \beta\left(b(w-c)\,\mathrm{E}[R(\theta,\bar{\theta})^{1-\gamma}]^{\frac{1}{1-\gamma}}\right)^{1-1/\varepsilon}\right)^{\frac{1}{1-1/\varepsilon}}, \tag{2.6}$$

where I have suppressed the individual and time subscripts. Thus the equilibrium portfolio satisfies

$$\bar{\theta} = \arg\max_{\theta} \frac{1}{1-\gamma}\,\mathrm{E}[R(\theta,\bar{\theta})^{1-\gamma}]. \tag{2.7}$$

Letting $\rho = \mathrm{E}[R(\bar{\theta},\bar{\theta})^{1-\gamma}]^{\frac{1}{1-\gamma}}$, carrying out the maximization over c in (2.6), and comparing coefficients, we get (after some algebra)

$$b = \left(1 - \beta^{\varepsilon}\rho^{\varepsilon-1}\right)^{\frac{1}{1-\varepsilon}}, \tag{2.8a}$$

$$c = (1 - \beta^{\varepsilon}\rho^{\varepsilon-1})w. \tag{2.8b}$$

2.3 Asset prices and efficiency

Having solved for the equilibrium in (almost) closed-form, I state the main results for this particular example.

2.3.1 Asset prices

Suppose that agents can trade finitely many assets in zero net supply indexed by $k \in K = \{1, \dots, K\}$. Let R^k be the one period gross return in equilibrium. Since agents are symmetric and assets are in zero net supply, in equilibrium agents choose not to trade (have zero holdings) of these assets. Suppose that a typical agent invests a fraction of wealth α in asset k and $1-\alpha$ in the equilibrium wealth portfolio. By the same argument used to derive (2.7), we obtain

$$0 = \arg\max_{\alpha} \frac{1}{1-\gamma} \mathrm{E}[((1-\alpha)R(\bar{\theta}, \bar{\theta}) + \alpha R^k)^{1-\gamma}].$$

Taking the first-order condition with respect to α, we obtain

$$0 = \mathrm{E}[R(\bar{\theta}, \bar{\theta})^{-\gamma}(R^k - R(\bar{\theta}, \bar{\theta}))] \iff \mathrm{E}[MR^k] = 1,$$

where the stochastic discount factor (SDF) is

$$M = \frac{R(\bar{\theta}, \bar{\theta})^{-\gamma}}{\mathrm{E}[R(\bar{\theta}, \bar{\theta})^{1-\gamma}]}.$$

By a standard argument in asset pricing, the excess return of an asset is

$$\mathrm{E}[R^k] - R_f = -\frac{\mathrm{Cov}[M, R^k]}{\mathrm{E}[M]} = -\frac{\mathrm{Cov}[R(\bar{\theta}, \bar{\theta})^{-\gamma}, R^k]}{\mathrm{E}[R(\bar{\theta}, \bar{\theta})^{-\gamma}]}, \tag{2.9}$$

where the risk-free rate is $R_f = 1/\mathrm{E}[M] = \mathrm{E}[R(\bar{\theta}, \bar{\theta})^{1-\gamma}]/\mathrm{E}[R(\bar{\theta}, \bar{\theta})^{-\gamma}]$.

Thus we obtain the first result: idiosyncratic risk generally matters for asset pricing. To see this, note that the portfolio return $R(\theta, \bar{\theta})$ in (2.5) depends both on aggregate shocks (Y, Z) and the idiosyncratic shock y. Since y appears only in the second term of (2.5), as we increase the riskiness of y, all else equal agents will typically shift capital from human to physical capital by increasing θ. But then the composition of aggregate shocks (Y, Z) in $R(\bar{\theta}, \bar{\theta})$ will change, and therefore unless Y and Z are perfectly correlated, the idiosyncratic shock y will affect the risk premium of assets through (2.9) and its effect on $\bar{\theta}$ (unless, of course, the asset return R^k is independent of (Y, Z)). Thus even in a simple i.i.d. setting such as this model, idiosyncratic risk generally matters for asset pricing whenever there are two or more sources of aggregate risk (in this case, Y and Z). This result does not contradict Krueger and Lustig (2010), since their model has only one aggregate shock. The point is that their irrelevance result is not robust.

2.3.2 Efficiency

Next I turn to efficiency. Since the value function of any agent in equilibrium is of the form $V(w) = bw$ with b determined by (2.8a), it is clear that the welfare of any agent is a monotonic function of $\rho = \mathrm{E}[R(\bar{\theta}, \bar{\theta})^{1-\gamma}]^{\frac{1}{1-\gamma}}$. Here the portfolio return is defined by (2.5) and the equilibrium portfolio $\bar{\theta}$ satisfies (2.7). However, since the objective function of the individual optimal portfolio problem (2.7) depends on the equilibrium portfolio, the equilibrium portfolio $\bar{\theta}$ will typically not satisfy

$$\bar{\theta} \in \arg\max_{\theta} \frac{1}{1-\gamma} \mathrm{E}[R(\theta, \theta)^{1-\gamma}]. \tag{2.10}$$

In essence, there is a 'portfolio externality', or a pecuniary externality caused by the portfolio choices of other agents. Thus if a planner assigns the portfolio θ^* that solves (2.10) to every agent, with corresponding ρ denoted by ρ^*, then we will have $\rho < \rho^*$ and therefore the planner can improve welfare. In other words, the equilibrium is constrained inefficient in the sense of Diamond (1967). It turns out that the equilibrium in the original model of Krebs (2006) is nevertheless constrained efficient. However, this is a knife-edge case in which the only idiosyncratic human capital shock is in depreciation or appreciation η and not in technological change or obsolescence y.

If the equilibrium portfolio $\bar{\theta}$ is constrained inefficient, a natural question is whether $\bar{\theta}$ invests too much or too little in physical capital, and whether the constrained efficient portfolio can be implemented in a decentralized way through taxes and subsidies. For this example, we can prove that there is always an over-investment in physical capital and that the optimal tax rate on physical capital is positive.

Proposition 2.1. *Let $\bar{\theta}$ be the equilibrium portfolio share of physical capital that satisfies* (2.7) *and θ^* be the constrained efficient portfolio that solves* (2.10). *Then $\bar{\theta} \geq \theta^*$. Furthermore, the optimal tax rate on physical capital is positive.*

Proof. See Appendix. □

2.4 Numerical example

In this subsection I explore the quantitative impact of idiosyncratic risk on asset prices and efficiency.

The parameter values are as follows. The physical capital share in the production function is $\alpha = 0.36$ and the depreciation rate is $\delta = 0.08$, which are standard in the literature. Since human capital is likely to depreciate more slowly, I set $\eta = 0.04$. The relative risk aversion coefficient is $\gamma = 4$, which is considered by many researchers the upper bound for a "reasonable" degree of risk aversion. Vissing-Jørgensen (2002) finds that the elasticity of intertemporal substitution (EIS) is 0.8–1.0 for bond holders,[8] so I set $\varepsilon = 0.9$.[9]

I assume that Z, Y, y are two-point, symmetrically distributed random variables, so for example Z takes the value $\mu_Z \pm \sigma_Z$ with probability 1/2. The aggregate shocks Z, Y may well be correlated. Once the correlation coefficient ρ is set, the probability of each state is automatically determined. For instance, the "high Z, high Y" state has probability $p_{HH} = \frac{1+\rho}{4}$. Since by (2.4)

[8] Vissing-Jørgensen (2002) finds that EIS is 0.3–0.4 for stock holders. However, since she estimates EIS using a model with additive utility, for which relative risk aversion is tied to EIS by $\varepsilon = 1/\gamma$, her EIS estimate for stock holders is likely to reflect risk aversion.

[9] Readers uncomfortable with recursive preferences may set $\varepsilon = 1/\gamma$ since in an i.i.d. setting the recursive utility model is observationally equivalent to the additive utility model (Kocherlakota, 1990). Doing so will only affect the value of the discount factor β.

the labor income growth is proportional to $y_{i,t+1}$, following Storesletten et al. (2007) I set $\sigma_y = 0.17$. For simplicity I set the means to 1 (no exogenous growth): $\mu_Z = \mu_Y = \mu_y = 1$. The volatilities of physical and human capital are $\sigma_Z = 0.15$ and $\sigma_Y = 0.05$. I pick a high volatility in physical capital so as to obtain volatile stock market returns.[10]

The parameters yet to be determined are the discount factor β, the coefficient of the production function A, and the correlation coefficient ρ. I set these parameters to match the U.S. gross risk-free rate (1.013), aggregate consumption growth (1.018), and volatility of aggregate consumption (0.033), which are again taken from Storesletten et al. (2007). The implied parameter values are $\beta = 0.988$, $A = 0.2311$, and $\rho = -0.8789$. It may be counterintuitive that the correlation between physical and human capital shocks is highly negative, but Lustig and Van Nieuwerburgh (2008) obtain a similar value ($Corr(DR^y_\infty, DR^a_\infty) = -0.63$ in their Table 4).

With this parametrization, the equilibrium portfolio (physical capital share) is $\bar{\theta} = 0.3867$, the expected stock market return (the term $(1 + r_{t+1} - \delta)Z_{t+1}$ in (2.4)) is 3.13%, and the stock market volatility is 14.1%. The stock market return is a little low but has the same order of magnitude as in the data. The expected return on human capital (the term $(1+\omega_{t+1}-\eta)y_{i,t+1}Y_{t+1}$ in (2.4)) is 8.47%, which is in line with the returns to schooling. The constrained efficient portfolio is $\theta^* = 0.3748$, so there is an over-investment in physical capital ($\bar{\theta} > \theta^*$), as predicted by Proposition 2.1. The ratio of the coefficient of the value function b to the efficient level b^* is 0.9907, so there is a 1% welfare loss in terms of permanent consumption equivalent.

To verify the claim that idiosyncratic shocks matter for asset pricing and efficiency, I conduct a comparative statics exercise by changing the idiosyncratic volatility of human capital σ_y from 0 to 0.3. Note that setting σ_y equal to zero corresponds to a representative-agent model, since in this case there are no idiosyncratic shocks.

Figure 1 shows the effect of idiosyncratic shocks on asset prices. In the representative-agent model ($\sigma_y = 0$), the equity premium is low (0.63%) and the risk-free rate is high (over 6%), consistent with the equity premium puzzle of Mehra and Prescott (1985) and the risk-free rate puzzle of Weil (1989). However, we can see that increasing the idiosyncratic risk lowers the risk-free rate and raises the equity premium, which help explain these asset pricing puzzles.

Figure 2 shows the effect of idiosyncratic shocks on the portfolio choice and welfare. The figure plots the ratios $\bar{\theta}/\theta^*$ and b/b^*, the equilibrium level relative to the constrained efficient level. As idiosyncratic risk increases, agents invest more in physical capital, which is free from idiosyncratic risk. Then the higher wage amplifies the idiosyncratic human capital risk, which lowers welfare. The magnitude of the welfare loss at the baseline specification is moderate (1% in consumption equivalent), but the welfare loss deepens sharply as we increase the idiosyncratic risk further, up to more than 10%.

Figure 3 shows the effect of idiosyncratic shocks on optimal taxes. As the idiosyncratic risk increases, the equilibrium becomes more inefficient, and therefore we need larger taxes to achieve the constrained efficient outcome. The optimal tax rate on physical capital is 0.5% in the baseline specification.

[10] Note that the physical and human capital are measured in efficiency units, not in book value, so it is natural to assume that they are volatile. See the discussion in Black (1995).

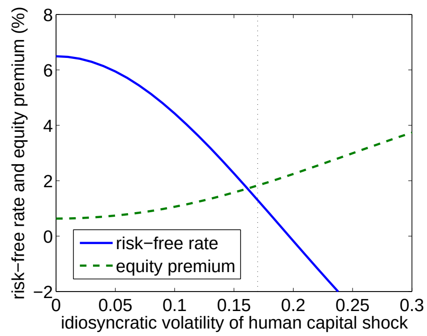

Figure 1: Effect of idiosyncratic shocks on asset prices. Blue solid: risk-free rate, green dashed: equity premium, vertical dotted: baseline specification.

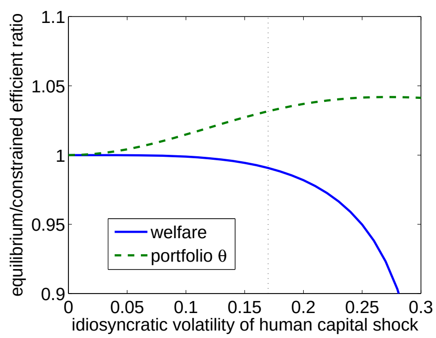

Figure 2: Effect of idiosyncratic shocks on welfare and portfolio. Blue solid: relative welfare b/b^*, green dashed: relative portfolio share of capital θ/θ^*, vertical dotted: baseline specification.

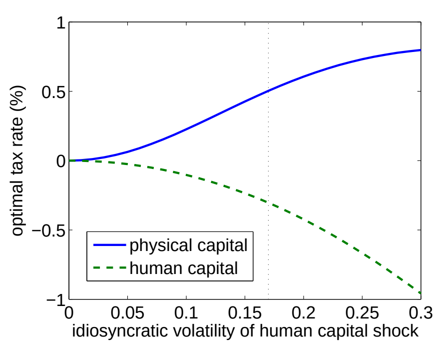

Figure 3: Effect of idiosyncratic shocks on optimal taxes. Blue solid: physical capital, green dashed: human capital, vertical dotted: baseline specification.

3 AK model

In this section I introduce the formal model to present the first main result, that generally idiosyncratic shocks affect asset prices. Since the presence of firms is not necessary to obtain this result, for simplicity I work with an AK model with multiple technologies that is identical to Toda (2014), reproduced below in order to make the paper self-contained.[11]

3.1 Settings

Time is infinite and is denoted by $t = 0, 1, \dots$. All random variables are defined on a probability space $(\Omega, \mathcal{F}, P)$. In the economy there are a continuum of agents with mass 1 indexed by $i \in I = [0, 1]$.

Preferences Agents have (identical) recursive preferences defined over (finite) consumption plans from time t onwards $\{c_{t+s}\}_{s=0}^{T-1}$ (where $t = 0, 1, \dots$ and $T = 1, 2, \dots$), constructed as follows. The one period utility at time t is $U_t^1 = u(c_t)$, where $u : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing. Given the T period recursive utility at time t, denoted by U_t^T, the $T + 1$ period recursive utility is defined by

$$U_t^{T+1} = f(c_t, \mu_t(U_{t+1}^T)),$$

where $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ is the aggregator, c_t is consumption, and $\mu_t(U_{t+1}^T)$ is the certainty equivalent of the distribution of time $t + 1$ utility conditional on time t information. Throughout the paper I maintain the following assumptions regarding the aggregator and the certainty equivalent.

Assumption 1. *The terminal utility is consumption itself: $u(c) = c$. The aggregator $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ is upper semi-continuous, weakly increasing in both arguments, strictly quasi-concave, and homogeneous of degree 1,* i.e., *$f(\lambda c, \lambda v) = \lambda f(c, v)$ for any $\lambda > 0$.*

Assumption 2 (CRRA certainty equivalent)**.** *The certainty equivalent μ_t exhibits constant relative risk aversion (CRRA),* i.e.*,*

$$\mu_t(U) = \begin{cases} \mathrm{E}_t[U^{1-\gamma}]^{\frac{1}{1-\gamma}}, & (\gamma \neq 1) \\ \exp\left(\mathrm{E}_t[\log U]\right), & (\gamma = 1) \end{cases}$$

where $\gamma > 0$ is the coefficient of relative risk aversion.

The recursive preferences satisfying Assumptions 1 and 2 nest the Epstein-Zin CRRA/CEIS (constant elasticity of intertemporal substitution) preference by setting $f(c, v) = (c^{1-1/\varepsilon} + \beta v^{1-1/\varepsilon})^{\frac{1}{1-1/\varepsilon}}$, where $\varepsilon > 0$ is the elasticity of intertemporal substitution and $\beta > 0$ is the discount factor. We get the standard additive CRRA preference when $\varepsilon = 1/\gamma$.

[11] The focus of Toda (2014) is establishing theoretical properties such as equilibrium existence and uniqueness, as well as characterizing the stationary wealth distribution and power law exponents. On the other hand, the current paper is mainly interested in asset pricing, efficiency, and the optimal tax policy.

Technology and asset Agent i is endowed with initial wealth (capital) $w_{i0} > 0$ in period 0 but nothing thereafter. However, each agent has access to stochastic, constant-returns-to-scale technologies (investment projects) indexed by $j \in J = \{1, \dots, J\}$, which are subject to aggregate and idiosyncratic risks. When agent i invests one unit of the good in technology j at the end of period t, he will receive $A^j_{i,t+1} > 0$ units of the good at the beginning of time $t+1$ (AK model). Let $\mathbf{A}_{i,t+1} = (A^1_{i,t+1}, \dots, A^J_{i,t+1})$ be the vector of productivities of agent i. We can interpret technologies with idiosyncratic risks as human capital investment, farming in private land, private equity, etc.

There are also publicly traded assets in zero net supply (such as financial derivatives, Arrow securities, risk-free assets of various maturities, etc.) indexed by $k \in K$. The set of asset K need not be finite, but I assume that at any point in time the number of assets traded is finite. One share of asset $k \in K$ pays out dividend D^k_{t+1} at the beginning of time $t+1$ without default, independent of the identity of the asset holder. Therefore the dividend D^k_{t+1} is a measurable function of aggregate shocks alone. Let $\mathbf{D}_{t+1} = (D^k_{t+1})_{k \in K}$ be the collection of dividends.

The asset price P^k_t is to be determined in equilibrium and induces the asset return by $R^k_{t+1} = (P^k_{t+1} + D^k_{t+1})/P^k_t$, which is common across agents. Markets are incomplete in the sense that there is no insurance for the idiosyncratic component of investment returns, which can arise for a number of reasons but I take it as exogenous.

Information and distributional assumptions Agent i's information is represented by the filtration (increasing sequence of σ-algebras) $\{\mathcal{F}_{it}\}_{t=0}^{\infty}$. The public information is denoted by $\mathcal{F}_t = \bigcap_i \mathcal{F}_{it}$. Of course, productivity $\mathbf{A}_{it}$ is $\mathcal{F}_{it}$-measurable and dividend $\mathbf{D}_t$ is $\mathcal{F}_t$-measurable. I assume that agents are symmetric in the following sense.

Assumption 3. *Productivities $\{\mathbf{A}_{it}\}_{i \in I}$ are i.i.d. conditional on public information $\mathcal{F}_t$.*

I refer to the (common) conditional mean $A^j_t = \mathrm{E}\left[A^j_{it} \,\middle|\, \mathcal{F}_t\right]$ as the aggregate component of productivity of technology j. Letting $a^j_{it} = A^j_{it}/A^j_t$ be the purely idiosyncratic component, the productivity decomposes into the aggregate and idiosyncratic components as $A^j_{it} = a^j_{it} A^j_t$. Let $\mathbf{A}_t = (A^1_t, \dots, A^J_t)$ and $\mathbf{a}_{it} = (a^1_{it}, \dots, a^J_{it})$ be the vectors of aggregate and idiosyncratic shocks.

The second assumption concerns public and private information.

Assumption 4. *The distribution of productivity and dividends $(\mathbf{A}_{i,t+1}, \mathbf{D}_{t+1})$ conditional on private information $\mathcal{F}_{it}$ is the same as the distribution conditional on public information $\mathcal{F}_t$.*

Assumption 4 implies that the current and past idiosyncratic shocks do not predict future idiosyncratic shocks, which might appear unrealistic. However, note that $a^j_{i,t+1}$'s are *rates* of return and hence shocks are permanent in terms of the *level* of capital.

Finally, I need a Markov assumption.

Assumption 5. *The aggregate state of the economy at time t is denoted by $s_t \in S$, where $\{s_t\}$ follows an exogenous stationary Markov process. The portfolio* *constraint Π_t and the distribution of productivity $\mathbf{A}_{i,t+1}$ conditional on time t public information depend only on s_t.*

The support S of the Markov process s_t need not be finite.

Budget and portfolio constraints I denote the portfolio share (relative position) in investments and asset holdings by a vector $(\theta, \phi) \in \mathbb{R}^J_+ \times \mathbb{R}^K$, where $\sum_{j=1}^J \theta^j + \sum_{k\in K} \phi^k = 1$. $\phi^k > 0$ ($\phi^k < 0$) means a long (short) position in asset k. An agent's portfolio share is constrained to be in the set $\Pi_t \subset \mathbb{R}^J_+ \times \mathbb{R}^K$ at time t, which can be interpreted as a constraint on leverage or other institutional constraints (limits on shortsales, restrictions on access to certain capital markets, etc.). The assumption that only finitely many assets are traded at any point in time is mathematically represented by $(\theta, \phi) \in \Pi_t$ implies $\phi^k = 0$ for all but finitely many $k \in K$. Letting $\pi_t = (\theta_t, \phi_t)$ and

$$R_{i,t+1}(\pi_t) = \sum_{j=1}^J A^j_{i,t+1}\theta^j_t + \sum_{k\in K} R^k_{t+1}\phi^k_t$$

be the gross return on portfolio of investments and assets, individual i faces the budget constraint

$$w_{i,t+1} = R_{i,t+1}(\pi_t)(w_{it} - c_{it}). \tag{3.1}$$

If shortsales are allowed, it may be the case that $R_{i,t+1}(\pi_t) \le 0$ in some states, leaving the agent with negative wealth. I rule out this possibility by letting an agent with negative wealth bankrupt and get utility $-\infty$, so agents choose only portfolios that satisfy $R_{i,t+1}(\pi_t) > 0$ almost surely. By redefining the portfolio constraint if necessary, I assume that $R_{i,t+1}(\pi) > 0$ almost surely for any $\pi \in \Pi_t$.

3.2 Equilibrium

In this subsection I briefly discuss the definition and properties of equilibrium. For a more complete account, see Toda (2014). As usual the general equilibrium is defined by individual optimization and market clearing.

Definition 3.1. $\left\{(c_{it}, w_{it}, \theta_{it}, \phi_{it})_{i\in I}, (P^k_t)_{k\in K}\right\}_{t=0}^\infty$ is a sequential general equilibrium with incomplete markets and heterogeneous agents if

1. given the asset returns $R^k_{t+1} = (P^k_{t+1} + D^k_{t+1})/P^k_t$ individual consumption c_{it} and portfolio $\pi_{it} = (\theta_{it}, \phi_{it})$ are optimal subject to the budget constraint (3.1) and the portfolio constraint $\pi_{it} \in \Pi_t$,
2. the markets for assets in zero net supply clear, *i.e.*, $\int_I \phi^k_{it}(w_{it} - c_{it})\mathrm{d}i = 0$ for all $k \in K$, and
3. individual wealth w_{it} evolves according to the budget constraint (3.1).

Let $V(w, s)$ be the value function of an agent with wealth w in state s.[12] The Bellman equation is

$$V(w,s) = \max_{\substack{0\le c\le w\\ \pi\in\Pi_s}} f\left(c, \mathrm{E}\left[V(R(\pi)(w-c), s')^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}}\right), \tag{3.2}$$

[12] That the value function depends only on wealth w and aggregate state s follows from Assumptions 3–5.

where I have suppressed the individual and time subscript on portfolio return and s' denotes the aggregate state of the next period. Let $L^+(S)$ be the space of positive functions defined on S. Since preferences are homothetic (Assumptions 1 and 2) and shocks are multiplicative, it follows that the value function is linear: $V(w,s) = b(s)w$ for some $b \in L^+(S)$. Substituting this into the Bellman equation (3.2), letting $c = \tilde{c}w$, and using the homotheticity of the aggregator f, it follows that b is a fixed point of the mapping $B : L^+(S) \to L^+(S)$ defined by

$$(Bx)(s) = \max_{0 \le \tilde{c} \le 1} f\left(\tilde{c}, (1-\tilde{c}) \max_{\pi \in \Pi_s} \mathrm{E}\left[(x(s')R(\pi))^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}}\right), \tag{3.3}$$

where $x \in L^+(S)$. Of course, in order for B to be well-defined, the maximum with respect to $\tilde{c}$ and π in (3.3) must be attained, but this follows from the upper semi-continuity of f and imposing mild restrictions on the portfolio constraint Π_s and the portfolio return $R(\pi)$.

The following theorem provides a sufficient condition for the existence of an equilibrium and an algorithm for computing the equilibrium.

Theorem 3.2. *Suppose that (i) Assumptions 1–5 hold, (ii) for all $s \in S$ the set $\{\theta \,|\, (\theta, 0) \in \Pi_s\}$ is nonempty, compact, convex, and (iii) for all $s \in S$ we have $\mathrm{E}\left[\sup_{(\theta,0)\in\Pi_s} R(\theta,0)^{1-\gamma} \,\middle|\, s\right] < \infty$. Let $\rho_s := \sup_{(\theta,0)\in\Pi_s} \mathrm{E}\left[R(\theta,0)^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}}$ and suppose further that there exists $0 < \epsilon < 1$ such that either*

$$(\forall s) \qquad f(\epsilon, \rho_s) < 1 \le \sup_{0 \le c \le 1} f(c, \rho_s(1-c)), \quad \text{or}$$

$$(\forall s) \qquad \sup_{0 \le c \le 1} f(c, \rho_s(1-c)) \le 1 < \sup_{0 \le c \le 1} f(c/\epsilon, \rho_s(1-c))$$

holds. Define $\left\{b^T\right\}_{T=1}^{\infty} \subset L^+(S)$ by $b^1 = 1$ and $b^T = Bb^{T-1}$ for $T \ge 2$. Then $\left\{b^T\right\}_{T=1}^{\infty}$ is well-defined, monotonically converges pointwise to some $b \in L^+(S)$, and the value function in state s is $V(w,s) = b(s)w$. Letting

$$\theta_s \in \operatorname*{arg\,max}_{(\theta,0)\in\Pi_s} \frac{1}{1-\gamma} \mathrm{E}\left[(b(s')R(\theta,0))^{1-\gamma} \,\middle|\, s\right], \tag{3.4a}$$

$$\tilde{c}_s = \operatorname*{arg\,max}_{0 \le \tilde{c} \le 1} f\left(\tilde{c}, (1-\tilde{c})\, \mathrm{E}\left[(b(s')R(\theta_s,0))^{1-\gamma} \,\middle|\, s\right]^{\frac{1}{1-\gamma}}\right), \tag{3.4b}$$

$(c_{it}, \theta_{it}, \phi_{it}) = (\tilde{c}_{s_t} w_{it}, \theta_{s_t}, 0)$ is the equilibrium consumption-portfolio.

Proof. See Toda (2014). ∎

Toda (2014) shows further that (i) if there is an equilibrium (which is not necessarily Markovian), there is an equilibrium with the same consumption allocation and a common portfolio choice across agents (hence no trade in assets in zero net supply), and (ii) all equilibria with no trade in assets have the same consumption allocation. Combining these two facts, the equilibrium in Theorem 3.2 is essentially the unique equilibrium of the economy in the sense that there may be indeterminacy in the portfolio choice (due to redundant assets or technologies) or asset prices (due to binding portfolio constraints), but the consumption allocation is unique.

4 Asset pricing

In this section I derive asset pricing implications of the general equilibrium model of Section 3.

4.1 Asset pricing implications

Fix an aggregate state s and let θ_s be the equilibrium portfolio. Let $\pi = (\theta, \phi) \in \mathbb{R}^J \times \mathbb{R}^K$ with $\sum \theta^j + \sum \phi^k = 1$ be any portfolio. I say that $\pi = (\theta, \phi)$ is *infinitesimally tradable in equilibrium* in state s if for sufficiently small $|\alpha|$ we have

$$(1-\alpha)(\theta_s, 0) + \alpha(\theta, \phi) = ((1-\alpha)\theta_s + \alpha\theta, \alpha\phi) \in \Pi_s,$$

that is, if an agent can take a small long or short position in the portfolio π and invest the remaining wealth in the equilibrium portfolio $(\theta_s, 0)$. In particular, I say that asset $k \in K$ is infinitesimally tradable (in equilibrium) if the portfolio consisting of entirely asset k is infinitesimally tradable.

The following theorem gives a formula for pricing an asset.

Proposition 4.1. *Let everything be as in Section 3 and suppose that asset k is infinitesimally tradable. Then the asset price P_t^k satisfies the recursive formula*

$$P_t^k = \frac{\mathrm{E}\left[b(s_{t+1})^{1-\gamma} R(\theta_{s_t}, 0)^{-\gamma} (P_{t+1}^k + D_{t+1}^k) \,\middle|\, s_t\right]}{\mathrm{E}\left[b(s_{t+1})^{1-\gamma} R(\theta_{s_t}, 0)^{1-\gamma} \,|\, s_t\right]}, \tag{4.1}$$

where $b(s)$ is as in Theorem 3.2. In particular, the one period gross risk-free rate in state s is given by

$$R_{f,s} = \frac{\mathrm{E}\left[b(s')^{1-\gamma} R(\theta_s, 0)^{1-\gamma} \,\middle|\, s\right]}{\mathrm{E}\left[b(s')^{1-\gamma} R(\theta_s, 0)^{-\gamma} \,|\, s\right]}. \tag{4.2}$$

Furthermore, the risk premium of an infinitesimally tradable portfolio π satisfies the covariance pricing formula

$$\mathrm{E}\left[R(\pi) \,|\, s\right] - R_{f,s} = -\frac{\mathrm{Cov}\left[b(s')^{1-\gamma} R(\theta_s, 0)^{-\gamma}, R(\pi) \,\middle|\, s\right]}{\mathrm{E}\left[b(s')^{1-\gamma} R(\theta_s, 0)^{-\gamma} \,|\, s\right]}. \tag{4.3}$$

Proof. See Appendix. □

Proposition 4.1 can be viewed as a generalization of Rubinstein (1976), who obtains similar results under the assumptions of a representative-agent and serially independent returns.

4.2 (Ir)relevance of market incompleteness

At least since Mankiw (1986), the relevance of market incompleteness and/or agent heterogeneity for asset pricing has been recognized, but their quantitative importance has been found to be small when idiosyncratic shocks are transitory (Telmer, 1993). Constantinides and Duffie (1996) show that idiosyncratic shocks have a significant impact on asset pricing when they are permanent.

In a recent paper, Krueger and Lustig (2010) showed that, under some assumptions, the absence of insurance markets for idiosyncratic labor income risk has no effect on the premium for aggregate risk. The assumptions are (i) a continuum of agents, (ii) identical CRRA utility, (iii) idiosyncratic labor income risk that is independent of aggregate risk, (iv) a constant capital share of income, and (v) solvency constraints or borrowing constraints on total financial wealth that are proportional to aggregate income. They also assume that aggregate consumption growth is independent over time and the labor income shock can be decomposed multiplicatively into the idiosyncratic and aggregate components. These assumptions are parallel to mine: in my model I assume (i) a continuum of agents, (ii) identical homothetic CRRA recursive utility, (iii) idiosyncratic investment risk that is conditionally independent of aggregate risk, (iv) the portfolio share is common across all agents (which is a consequence, not an assumption), and (v) portfolio constraints that are independent of the wealth level. In this section I show that the key assumption leading to the irrelevance result of Krueger and Lustig is that there is only one source of aggregate shock.

Krueger and Lustig (2010) assume that aggregate consumption growth is independent over time. Let us assume the following similar but weaker 'conditional independence assumption (CIA)' in our model of Section 3: the next period's state s_{t+1} and productivities $\mathbf{A}_{i,t+1}$ are independent conditional on the current state s_t. In that case since $b(s_{t+1})$ and $\mathbf{A}_{i,t+1}$ (hence $R_{i,t+1}(\theta,0) = \sum_{j=1}^{J} A_{i,t+1}^{j}\theta^{j}$) are conditionally independent, the equilibrium portfolio condition (3.4a) becomes

$$\theta_s \in \mathop{\arg\max}_{(\theta,0)\in\Pi_s} \frac{1}{1-\gamma} \operatorname{E}\left[R(\theta,0)^{1-\gamma} \,\middle|\, s\right]. \tag{4.4}$$

Since Proposition 4.1 is entirely derived by (3.4a), under CIA (4.4) implies that Proposition 4.1 holds without the term $b(s')$. In particular, the covariance pricing formula (4.3) becomes

$$\operatorname{E}[R(\pi)\,|\,s] - R_{f,s} = -\frac{\operatorname{Cov}[R(\theta_s,0)^{-\gamma}, R(\pi)\,|\,s]}{\operatorname{E}[R(\theta_s,0)^{-\gamma}\,|\,s]}. \tag{4.5}$$

In general, (4.5) still depends on the idiosyncratic shocks. The following proposition gives a necessary condition for the irrelevance of idiosyncratic risk.

Proposition 4.2. *Suppose that the risk premium of an infinitesimally tradable portfolio π does not depend on idiosyncratic shocks. If there is no portfolio constraint on technology (for any $\theta \geq 0$ with $\sum_j \theta^j = 1$ we have $(\theta,0) \in \Pi_s$), then*

$$-\frac{\operatorname{Cov}\left[(A^j)^{-\gamma}, R(\pi) \,\middle|\, s\right]}{\operatorname{E}[(A^j)^{-\gamma}\,|\,s]} = \operatorname{E}[R(\pi)\,|\,s] - R_{f,s} \tag{4.6}$$

for all j, where A^j is the aggregate component of technology j.

Proof. See Appendix. □

Since the aggregate component of productivities $\mathbf{A} = (A^1, \dots, A^J)$ is exogenous in the model, (4.6) generically fails when there are multiple technologies. Thus idiosyncratic risks generically affect risk premia. This 'generic relevance of idiosyncratic risk' is intuitive. When there are multiple sources of aggregate risk, the aggregate component of individual consumption growth (which prices assets through the Euler equation) depends on the portfolio choice, which in turn is affected by idiosyncratic shocks. However, if there is a single source of aggregate risk, the irrelevance result of Krueger and Lustig survives.

Proposition 4.3. *Suppose that there exists a random variable A such that the distribution of A^j/A conditional on the aggregate state s is constant for each j. Then the risk premium does not depend on idiosyncratic shocks.*

Proof. By assumption, $A_i^j = a_i^j m_s^j A$ for some constant m_s^j. Then (4.5) becomes

$$\begin{aligned}
\mathrm{E}\left[R(\pi) \mid s\right] - R_{f,s} &= -\frac{\mathrm{E}\left[\left(\sum_j a_i^j m_s^j \theta_s^j\right)^{-\gamma} \,\middle|\, s\right] \mathrm{Cov}\left[A^{-\gamma}, R(\pi) \mid s\right]}{\mathrm{E}\left[\left(\sum_j a_i^j m_s^j \theta_s^j\right)^{-\gamma} \,\middle|\, s\right] \mathrm{Cov}\left[A^{-\gamma}, R(\pi) \mid s\right]} \\
&= -\frac{\mathrm{Cov}\left[A^{-\gamma}, R(\pi) \mid s\right]}{\mathrm{E}\left[A^{-\gamma} \mid s\right]},
\end{aligned}$$

which depends only on aggregate shocks A and s. $\square$

Thus we can interpret the irrelevance result of Krueger and Lustig as deduced from the presence of only one aggregate shock, namely the growth rate of aggregate consumption.

5 Model with firms

It is well-known that the equilibrium is generically constrained inefficient when markets are incomplete (Geanakoplos and Polemarchakis, 1986; Citanna et al., 1998). A striking property of the incomplete market general equilibrium models of Krebs (2006) and Section 3 (Toda, 2014) is that the equilibrium is nevertheless constrained efficient. However, both of these models are rather special: in Krebs (2006) the only idiosyncratic shock is in human capital depreciation, and in Section 3 production is linear and employs only one input (AK model). Thus it is not clear whether constrained efficiency in these models is a robust property. To explore this issue further, in this section I generalize the Krebs model to the case with multiple firms and idiosyncratic technological shocks.

5.1 Settings

In the economy there is an "all purpose" good which can either be consumed or invested as physical or human capital. The supply side of the economy consists of J firms indexed by $j = 1, 2, \dots, J$. Firm j has a constant-returns-to-scale neoclassical production function $F_t^j(K, H)$ at time t, where K, H denote the input of the efficiency unit of physical and human capital, respectively. The production function F_t^j may not only be time-dependent but state-dependent, in which case F_t^j is a random function. I adopt all standard assumptions for F_t^j, namely, that F_t^j is twice continuously differentiable, increasing and strictly concave in both arguments, $\frac{\partial}{\partial K} F_t^j(0, H) = \frac{\partial}{\partial H} F_t^j(K, 0) = \infty$, and $\frac{\partial}{\partial K} F_t^j(\infty, H) = \frac{\partial}{\partial H} F_t^j(K, \infty) = 0$. At each period, each firm rents physical and human capital from consumers. Thus a firm's decision problem is

$$\max_{K, H \geq 0} [F_t^j(K, H) - r_t^j K - r_t^0 H],$$

a static problem, where r_t^j denotes the rental rate of physical capital firm j faces at period t and r_t^0 is the rental rate of human capital (wage per efficiency unit of labor). Since F_t^j is constant-returns-to-scale, firms make zero profit and hence we need not worry about the ownership of firms.

If a consumer rents physical capital k_t^j to firm j at period t and invests x_t^j, then the physical capital at the beginning of the next period will be

$$k_{t+1}^j = Z_{t+1}^j[(1-\delta_t^j)k_t^j + x_t^j],$$

where δ_t^j is the depreciation rate of physical capital used by firm j, and Z_{t+1}^j denotes the shock to the efficiency unit of physical capital to firm j that occurs between periods t and $t+1$, both of which are random variables. Z_{t+1}^j may represent capital obsolescence or capital-augmenting technological change.

If consumer $i \in I = [0,1]$ has human capital k_t^0 at period t and invests x_t^0, then the human capital at the beginning of the next period will be

$$k_{t+1}^0 = Z_{i,t+1}^0[(1-\delta_{it}^0)k_t^0 + x_t^0],$$

where δ_{it}^0 is the depreciation rate and $Z_{i,t+1}^0$ denotes the shock to the efficiency unit of human capital that occurs between periods t and $t+1$, both assumed to be i.i.d. across individuals conditional on the history of aggregate shocks (Assumption 3). Here I allow disinvestment of human capital ($x_t^0 < 0$), which can be interpreted as cutting work hours or switching to a less (mentally or physically) demanding job. I assume that the joint distribution of shocks $\left\{(Z_{t+1}^j, \delta_{t+1}^j)\right\}_{j=0}^J$ (where I have suppressed the subscript i for human capital) conditional on private information $\mathcal{F}_{it}$ is the same as the distribution conditional on public information $\mathcal{F}_t = \bigcap_{i\in I} \mathcal{F}_{it}$ (Assumption 4). Because the physical capital of each firm evolves stochastically, the rental rate r_t^j may differ across firms. However, since human capital is not firm-specific but individual-specific, the wage (per efficiency unit of human capital) r_t^0 must be common across all firms.

Finally, there are an arbitrary number of assets in zero net supply. Since by the nature of the model there will be no trade in assets in zero net supply, in what follows I shall ignore these assets.

Each consumer maximizes his recursive utility as in Section 3 subject to the constraints

$$c_t + \sum_{j=0}^J x_t^j = \sum_{j=0}^J r_t^j k_t^j, \tag{5.1a}$$

$$k_{t+1}^j = Z_{t+1}^j[(1-\delta_t^j)k_t^j + x_t^j],\ j = 0, 1, \dots, J. \tag{5.1b}$$

(5.1a) is the budget constraint: the left-hand side is the sum of consumption and investment, which must be equal to the right-hand side, the income from all sources. (5.1b) is the equation of motion for human capital ($j = 0$) or physical capital ($j \geq 1$) invested in each firm. Note that my formulation in (5.1) is more general than that in Krebs (2006): Equation (2) on p. 510 of his paper only allows for depreciation after production, but through Z_{t+1}^j I allow for factor obsolescence or factor-augmenting technological change. Thus Krebs's model is nested within the current framework by setting $Z_{t+1}^j \equiv 1$ and $J = 1$ (since there is a single technology).

The definition of equilibrium is similar to that in Section 3: given the initial distribution of physical and human capital $((k_{i0}^j)_{j=0}^J)_{i\in I}$, a sequential equilibrium is defined by a sequence of quantities

$$\left\{(c_{it}, (k^j_{i,t+1})^J_{j=0})_{i\in I}, (K^j_{t+1}, H^j_{t+1})^J_{j=1}\right\}^\infty_{t=0}$$

and rental rates and wages $\left\{(r^j_t)^J_{j=0}\right\}^\infty_{t=1}$ such that (i) consumers and firms optimize, and (ii) markets clear.

5.2 Equilibrium

Let Δ^J be the unit simplex in $\mathbb{R}^{J+1}$, that is, the set consisting of $\theta \in \mathbb{R}^{J+1}_+$ with $\sum_{j=0}^J \theta^j = 1$. The following lemma shows that the model has a similar structure to that in Section 3.

Lemma 5.1. *Let $\left\{(r^j_t)^J_{j=0}\right\}^\infty_{t=1}$ be rental rates. Define $w_{i0} = \sum_{j=0}^J k^j_{i0}$ and*

$$w_{it} = (1 + r^0_t - \delta^0_{it})k^0_{it} + \sum_{j=1}^J (1 + r^j_t - \delta^j_t)k^j_{it} \tag{5.2}$$

for all $t \geq 1$. Then the individual problem reduces to maximizing the recursive utility subject to the budget constraint

$$w_{i,t+1} = R_{i,t+1}(\theta_{it})(w_{it} - c_{it}), \tag{5.3}$$

where $\theta_{it} \in \Delta^J$ and

$$R_{i,t+1}(\theta) = (1 + r^0_{t+1} - \delta^0_{i,t+1})Z^0_{i,t+1}\theta^0 + \sum_{j=1}^J (1 + r^j_{t+1} - \delta^j_{t+1})Z^j_{t+1}\theta^j. \tag{5.4}$$

Proof. For notational simplicity drop the individual subscript i. Let $k'^j_t = (1 - \delta^j_t)k^j_t + x^j_t$ be the amount of physical capital allocated to firm j (if $j \geq 1$) or the amount of human capital (if $j = 0$) *after* production and investment. Since the investment x^j_t is unrestricted, so is k'^j_t. Adding total capital after depreciation $\sum_{j=0}^J (1 - \delta^j_t)k^j_t$ to the budget constraint (5.1a), we obtain

$$c_t + \sum_j k'^j_t = \sum_j (1 + r^j_t - \delta^j_t)k^j_t = w_t, \tag{5.5}$$

where w_t is the wealth of the consumer including the production in period t defined by (5.2).

Define the "portfolio share" at period t, $\theta_t \in \Delta^J$, by $k'^j_t = \theta^j_t(w_t - c_t)$ for $j = 0, 1, \dots, J$. Using (5.1b), the physical or human capital at the beginning of period $t + 1$ becomes

$$k^j_{t+1} = Z^j_{t+1}\theta^j_t(w_t - c_t). \tag{5.6}$$

Letting $\theta_t = (\theta^0_t, \dots, \theta^J_t) \in \Delta^J$, by (5.5) and (5.6) the consumer's wealth in period $t + 1$ is

$$w_{t+1} = \sum_j (1 + r^j_{t+1} - \delta^j_{t+1})k^j_{t+1} = R_{t+1}(\theta_t)(w_t - c_t),$$

which is precisely (5.3) and (5.4). ∎

The budget constraint (5.3) has the same form as the budget constraint (3.1). By Assumptions 3 and 4, the optimal consumption-portfolio problem is common across all agents. It follows that if there is an equilibrium, there exists an equivalent equilibrium in which agents hold a common portfolio (value weighted average portfolio) and the consumption allocation is the same as in the original equilibrium. Therefore without loss of generality we may restrict attention to symmetric equilibria.

Next I characterize the equilibrium rental rates $\left\{(r_t^j)_{j=0}^J\right\}_{t=1}^\infty$. Let $\bar{\theta}_t$ be the (symmetric) equilibrium portfolio. Let K_t^j, H_t^j be the amount of physical and human capital employed by firm j at period t and $H_t = \sum_j H_t^j$ the total human capital in the economy. Adding (5.6) across all individuals in the economy, for $j \geq 1$ we obtain

$$K_{t+1}^j = Z_{t+1}^j \bar{\theta}_t^j (1 - \tilde{c}_t) W_t, \tag{5.7}$$

where $\tilde{c}_t$ is the common consumption rate and W_t is the aggregate wealth in period t. For human capital ($j = 0$), since the human capital shock $Z_{i,t+1}^0$ is conditionally i.i.d. across individuals, by the law of large numbers we obtain

$$H_{t+1} = K_{t+1}^0 = \mathrm{E}_{t+1}[Z_{i,t+1}^0] \bar{\theta}_t^0 (1 - \tilde{c}_t) W_t. \tag{5.8}$$

Letting $\phi_t^j = H_t^j / H_t$ be the fraction of human capital employed by firm j, by firm optimization, (5.7), (5.8), and homogeneity of the production functions, the rental rates satisfy

$$r_t^j = \frac{\partial}{\partial K} F_t^j(K_t^j, H_t^j) = \frac{\partial}{\partial K} F_t^j(Z_t^j \bar{\theta}_{t-1}^j, \mathrm{E}_t[Z_{it}^0] \bar{\theta}_{t-1}^0 \phi_t^j), \tag{5.9a}$$

$$r_t^0 = \frac{\partial}{\partial H} F_t^j(K_t^j, H_t^j) = \frac{\partial}{\partial H} F_t^j(Z_t^j \bar{\theta}_{t-1}^j, \mathrm{E}_t[Z_{it}^0] \bar{\theta}_{t-1}^0 \phi_t^j). \tag{5.9b}$$

By the standard assumptions on the production function, (5.9b) can be solved for ϕ_t^j as a function of r_t^0, $\bar{\theta}_{t-1}^j$, and $\bar{\theta}_{t-1}^0$. Using $\sum_j \phi_t^j = 1$, we can solve for r_t^0 as a function of $\bar{\theta}_{t-1}$. Hence ϕ_t^j is also a function of $\bar{\theta}_{t-1}$, and so is r_t^j by (5.9a). Write this dependency as

$$r_t^j = r_t^j(\bar{\theta}_{t-1}), \ j = 0, 1, \dots, J. \tag{5.10}$$

Although r_t^j is a random variable, it is random only through $\bar{\theta}_{t-1}$, $(F_t^j, Z_t^j)_{j=1}^J$, and $\mathrm{E}_t[Z_{it}^0]$. Substituting (5.10) into the budget constraint (5.3), we obtain the following corollary.

Corollary 5.2. *Individual wealth evolves according to the budget constraint*

$$w_{i,t+1} = R_{i,t+1}(\theta_{it}, \bar{\theta}_t)(w_{it} - c_{it}),$$

where the portfolio return is given by

$$\begin{aligned} &R_{i,t+1}(\theta, \bar{\theta}) \\ &\quad = (1 + r_{t+1}^0(\bar{\theta}) - \delta_{i,t+1}^0) Z_{i,t+1}^0 \theta^0 + \sum_{j=1}^J (1 + r_{t+1}^j(\bar{\theta}) - \delta_{t+1}^j) Z_{t+1}^j \theta^j. \end{aligned} \tag{5.11}$$

Hence the portfolio return $R_{i,t+1}$ (conditional on time t information) is random only through the technology $(F^j_{t+1}, Z^j_{t+1}, \delta^j_{t+1})_{j=1}^J$, aggregate shock in human capital $\mathrm{E}_{t+1}[Z^0_{i,t+1}]$, and idiosyncratic shocks in human capital $(Z^0_{i,t+1}, \delta^0_{i,t+1})$.

We can derive necessary conditions for equilibrium as follows.

Theorem 5.3. *Suppose that a symmetric equilibrium exists. Then the value function is linear in wealth, $V_t(w) = b_t w$, and the equilibrium portfolio θ_t, the consumption rate $\tilde{c}_t$, and the coefficient b_t satisfy*

$$\theta_t \in \operatorname*{arg\,max}_{\theta \in \Delta^J} \frac{1}{1-\gamma} \mathrm{E}_t[b_{t+1}^{1-\gamma} R_{t+1}(\theta, \theta_t)^{1-\gamma}], \tag{5.12a}$$

$$\tilde{c}_t = \operatorname*{arg\,max}_{0 \le c \le 1} f\left(\tilde{c}, (1-\tilde{c}) \mathrm{E}_t[b_{t+1}^{1-\gamma} R_{t+1}(\theta_t, \theta_t)^{1-\gamma}]^{\frac{1}{1-\gamma}}\right), \tag{5.12b}$$

$$b_t = f\left(\tilde{c}_t, (1-\tilde{c}_t) \mathrm{E}_t[b_{t+1}^{1-\gamma} R_{t+1}(\theta_t, \theta_t)^{1-\gamma}]^{\frac{1}{1-\gamma}}\right). \tag{5.12c}$$

The equilibrium with a continuum of agents generally differs from the one with a single agent.

Proof. The proof that the equilibrium satisfies (5.12) is similar to Section 3 and Toda (2014) and therefore I omit it.

The single agent equilibrium corresponds to replacing $\mathrm{E}_t[Z^0_{it}]$ in (5.9) by just Z^0_{it} because the expression $\mathrm{E}_t[Z^0_{it}]$ was derived from the law of large numbers, which does not apply for the single agent problem. Hence unless $Z^0_{it} \equiv Z^0_t$ is common across all consumers, *i.e.*, unless there is no idiosyncratic component in the shock corresponding to human capital obsolescence or human capital-augmenting technological change (in which case $\mathrm{E}_t[Z^0_{it}] = Z^0_t$), the equilibrium with a continuum of agents generally differs from the one with a single agent. $\square$

Unlike the case with linear technologies in Section 3, since in Theorem 5.3 the multiple agent equilibrium differs from the single agent equilibrium, there is some risk sharing in equilibrium. Although agents face exactly the same idiosyncratic shocks in the market (continuum of agents) economy and the autarky (single agent) economy, the agents in the market economy can insure against the shock in wages (income flow from human capital) by pooling their human capital (the term with $\mathrm{E}_t[Z^0_{it}]$ in (5.9)), but cannot insure against the shock to the individual human capital stock itself. In the autarky economy, on the other hand, the agent cannot insure against either. This conclusion differs from that of Krebs (2006), where the single agent and multiple agent problems are exactly the same and there is no risk sharing because $Z^0_{it} \equiv Z^0_t$ is common across all agents by construction.

6 Efficiency

Because the equilibrium with AK technologies in Section 3 is equivalent to a single agent (planning) problem, the equilibrium is constrained efficient. This is not necessarily the case when there are firms. In this section, I show that the equilibrium with firms is generically constrained inefficient if there are idiosyncratic human capital obsolescence shocks, but that nevertheless the constrained efficient allocation can be achieved through linear taxes and subsidies.

Recall that the original model of Krebs (2006) is constrained efficient. The following proposition shows that the same is true in a more general model with multiple firms, provided that the only idiosyncratic human capital shock is in depreciation.

Proposition 6.1. *Suppose that the human capital shock Z^0_{it} is common across all agents,* i.e.*, the only idiosyncratic shock is in human capital depreciation δ^0_{it}. Then the equilibrium is constrained efficient.*

Proof. See Appendix. □

The crucial assumption in proving the special case that the equilibrium is constrained efficient is that there is no idiosyncratic component in the shock for human capital obsolescence or human capital-augmenting technological change, that is, $\mathrm{E}_t[Z^0_{it}] = Z^0_{it} = Z^0_t$ for all agents. Under this assumption we can replace $\mathrm{E}_t[Z^0_{it}]$ appearing in the definition of market rental rates (5.9) by just Z^0_t, and using the constant-returns-to-scale property of the production functions, we can show that the multiple agent market economy is identical to an autarky economy. Because there are no gains from trade, the equilibrium is constrained efficient.

Constrained efficiency, however, holds only in the knife-edge case that the only idiosyncratic shock is in human capital depreciation. Once we allow for idiosyncratic technological shocks, the equilibrium becomes generically constrained inefficient. The proof employs standard transversality techniques in mathematical economics, which are easiest to apply in finite dimensional spaces. Therefore I assume henceforth that the Markov process describing all shocks has a finite support:

Assumption 6. *The Markov process describing all shocks has a finite support.*

Remember from Section 5 that the human capital shock is denoted by $(Z^0_{it}, \delta^0_{it})$, where $Z^0_{it} > 0$ is the obsolescence (technological) shock that occurs between periods and $\delta^0_{it} \le 1$ is the depreciation shock after production. As in Section 3, each shock can be decomposed into the aggregate and purely idiosyncratic components. Let

$$Z^0_{it} = z^0_{it} Z^0_t$$

be such a decomposition, where z^0_{it} is the purely idiosyncratic component so $\mathrm{E}_t[z^0_{it}] = 1$. Since by Proposition 6.1 the equilibrium is constrained efficient if $z^0_{it} \equiv 1$, I make the following assumption.

Assumption 7. *There exists an aggregate state in which the support of the conditional distribution of z^0_{it} contains more than one point.*

Combining Assumptions 6, 7, $z^0_{it} > 0$, and the fact that idiosyncratic shocks are constrained to be mean 1 ($\mathrm{E}_t[z^0_{it}] = 1$), the support of the idiosyncratic obsolescence shock for all aggregate states can be identified as an element u of an open subset U of some Euclidean space.[13] It is u that I shall perturb to show the generic constrained inefficiency.

[13] In fact, U is the product of the interior of simplexes in some Euclidean spaces, where the product is taken over aggregate states in which the conditional distribution of idiosyncratic shock is non-degenerate.

Theorem 6.2 (Generic constrained inefficiency)**.** *Let everything be as above. Then there exists an open subset U^* of U with full measure (*i.e., *$U \backslash U^*$ is closed in U and has Lebesgue measure 0; in particular, $U \backslash U^*$ is nowhere dense) such that for every $u \in U^*$, the equilibrium associated with u is constrained inefficient in the sense that the planner can Pareto improve by intervening in agents' portfolio choices.*

Proof. See Appendix. □

The intuition for the generic constraint inefficiency result is straightforward. Unlike the case with only AK technologies, in the case with firms that have nonlinear production functions, the return on an individual's portfolio depends on the portfolio choice of other agents through the effect on rental rates. In essence, there is a 'portfolio externality', which makes the economy inefficient. This externality also exists in a representative-agent model as well as the model of Krebs (2006). However, in these models, combining the constant-returns-to-scale property of the production function and the absence of idiosyncratic multiplicative shocks, the individual optimization problem in the market economy turns out to be equivalent to a planning problem in autarky (Proposition 6.1). This is why the equilibrium was constrained efficient in earlier studies, but it is a knife-edge case.

As soon as we prove that the equilibrium is generically constrained inefficient, a natural question that arises is whether the planner can achieve the constrained efficient allocation through some intervention. Citanna et al. (1998) show how to Pareto improve the equilibrium allocation in an abstract general equilibrium model with incomplete markets. Due to the special structure of my model, the planner can actually achieve the constrained efficient allocation through linear taxes and subsidies on physical and human capital.

Theorem 6.3. *Let everything be as above. Let θ_t^* be the constrained efficient portfolio at time t and b_t^* be the corresponding coefficient of the value function satisfying the analog of* (5.12c)*. Define the random variables $A_{t+1}^0, \dots, A_{t+1}^J$ by $R_{t+1}(\theta, \theta_t^*) = \sum_{j=0}^J A_{t+1}^j \theta^j$, where the portfolio return is defined by* (5.11) *and I drop subscript i. Then the constrained efficient outcome can be decentralized by linear taxes and subsidies. The optimal tax rate on capital j (human capital if $j = 0$) is*

$$\tau_t^j = 1 - \frac{\mathrm{E}_t[(b_{t+1}^*)^{1-\gamma} R_{t+1}(\theta_t^*, \theta_t^*)^{1-\gamma}]}{\mathrm{E}_t[(b_{t+1}^*)^{1-\gamma} R_{t+1}(\theta_t^*, \theta_t^*)^{-\gamma} A_{t+1}^j]}.$$

Proof. See Appendix. □

7 Concluding remarks

AK-type models are analytically tractable and free from the 'curse of dimensionality' even with heterogeneous agents and incomplete markets when preferences are homothetic and all shocks are multiplicative. This is because agents choose the same portfolio regardless of the wealth level, and hence the wealth distribution is not a relevant state variable for describing the equilibrium. The key to make the analysis tractable is to recast the individual decision problem as an optimal consumption-portfolio problem. Taking advantage of these features, I studied the asset pricing implications and efficiency of a dynamic stochastic general equilibrium model with heterogeneous agents and incomplete markets with multiple technologies/inputs and general preferences/shocks. Unlike commonly used models with a single source of aggregate risk, the model predicts that generically idiosyncratic shocks are quantitatively important in pricing assets and the equilibrium is generically constrained inefficient.

The significant contribution of Krebs (2006) is that it is probably the first tractable general equilibrium model with a continuum of heterogeneous agents, incomplete markets, and production that allows for arbitrary stochastic processes (at least in terms of aggregate shocks). Toda (2014) further generalized the model to allow for arbitrary number of technologies and assets with arbitrary portfolio constraints, and derived an efficient algorithm for computing the equilibrium. This type of models has recently been applied in the quantitative general equilibrium/finance context, where the trade-off between the computational complexity and the desire to develop a quantitative model with a complex financial structure with incomplete markets necessitates resorting to analytically tractable models. For instance, Toda (2013) considers the role of securitization for sharing idiosyncratic risks; Walsh (2014) analyzes capital flows and default of emerging economies in international capital markets.

A Proofs

Proof of Proposition 2.1. The proof is fairly complicated and we need several steps.

Step 1. If $f, g : \mathbb{R} \to \mathbb{R}$ are increasing (decreasing) functions and X is a random variable, then $\mathrm{E}[f(X)g(X)] \geq \mathrm{E}[f(X)]\,\mathrm{E}[g(X)]$.[14]

To see this, let X' be an i.i.d. copy of X. Since f, g are monotonic, we have $(f(X) - f(X'))(g(X) - g(X')) \geq 0$. Taking expectations of both sides, noting that X, X' are i.i.d., and rearranging terms, we obtain the desired inequality.

Step 2. Let f be an increasing concave function and g be a concave function. Then $f \circ g$ is concave.

Take any x_1, x_2 and $\alpha \in [0, 1]$. Since g is concave, we have

$$g((1-\alpha)x_1 + \alpha x_2) \geq (1-\alpha)g(x_1) + \alpha g(x_2).$$

Since f is increasing and concave, applying f to both sides we get

$$\begin{aligned} f(g((1-\alpha)x_1 + \alpha x_2)) &\geq f((1-\alpha)g(x_1) + \alpha g(x_2)) \\ &\geq (1-\alpha)f(g(x_1)) + \alpha f(g(x_2)). \end{aligned}$$

Step 3. The function $R_{i,t+1}(\theta, \theta)$ defined by (2.5) *(with $\bar{\theta} = \theta$) is continuous and strictly concave in θ.*

Substituting $\bar{\theta} = \theta$ in (2.5), we can see that $R_{i,t+1}(\theta, \theta)$ is the sum of a linear function of θ and a positive multiple of $\theta^\alpha (1-\theta)^{1-\alpha}$. Therefore continuity is

[14] This inequality is known as the Chebyshev inequality.

trivial and it suffices to prove that $f(\theta) = \theta^\alpha (1-\theta)^{1-\alpha}$ is strictly concave. Taking logs and differentiating twice, we obtain

$$\frac{f'}{f} = \frac{\alpha}{\theta} - \frac{1-\alpha}{1-\theta} \quad \text{and} \quad \frac{f''f - (f')^2}{f^2} = -\frac{\alpha}{\theta^2} - \frac{1-\alpha}{(1-\theta)^2},$$

so

$$\frac{f''}{f} = \left(\frac{\alpha}{\theta} - \frac{1-\alpha}{1-\theta}\right)^2 - \frac{\alpha}{\theta^2} - \frac{1-\alpha}{(1-\theta)^2} = -\frac{\alpha(1-\alpha)}{\theta^2(1-\theta)^2} < 0.$$

Step 4. The objective function in (2.10) *is continuous and strictly concave and therefore the constrained efficient portfolio θ^* uniquely exists.*

From now on let us drop the $i, t+1$ subscripts from $R_{i,t+1}$. The objective function is obtained by applying the (strictly increasing and strictly concave) function $x \mapsto \frac{x^{1-\gamma}}{1-\gamma}$ to $R(\theta,\theta)$ and taking expectations. By Step 3, $R(\theta,\theta)$ is continuous and strictly concave. By Step 2, the function $\frac{1}{1-\gamma} R(\theta,\theta)^{1-\gamma}$ is continuous and strictly concave, and so is its expectation, which is the objective function in (2.10).

Step 5. For any θ, we have

$$\mathrm{E}\left[R(\theta,\theta)^{-\gamma} \frac{\partial}{\partial\theta} R(\theta,\theta)\right] \geq \mathrm{E}\left[R(\theta,\theta)^{-\gamma} \frac{\mathrm{d}}{\mathrm{d}\theta} R(\theta,\theta)\right],$$

where the left-hand side contains the partial derivative $\frac{\partial}{\partial\theta} R(\theta,\bar{\theta})$ evaluated at $\bar{\theta} = \theta$ and the right-hand side contains the derivative of $\theta \mapsto R(\theta,\theta)$.

Let $D_1 R$ denote the partial derivative of R with respect to the first argument, and similarly $D_2 R$. Then by the chain rule we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta} R(\theta,\theta) = D_1 R(\theta,\theta) + D_2 R(\theta,\theta),$$

so it suffices to prove $\mathrm{E}[-R(\theta,\theta)^{-\gamma} D_2 R(\theta,\theta)] \geq 0$. By carrying out the differentiation using the definition (2.5), after some algebra we get

$$D_2 R(\theta,\theta) = A\alpha(1-\alpha) Z^\alpha Y^{1-\alpha} \theta^{\alpha-1} (1-\theta)^{-\alpha} (y-1), \tag{A.1}$$

which is clearly an increasing function of y. Furthermore, $R(\theta,\theta)$ in (2.5) is increasing in y, and so is $-R(\theta,\theta)^{-\gamma}$. Hence by Step 1, conditioning on Y, Z we get

$$\mathrm{E}\left[-R(\theta,\theta)^{-\gamma} D_2 R(\theta,\theta) \,\middle|\, Y, Z\right] \geq \mathrm{E}\left[-R(\theta,\theta)^{-\gamma} \,\middle|\, Y, Z\right] \mathrm{E}\left[D_2 R(\theta,\theta) \,|\, Y, Z\right] = 0,$$

where the last equality follows from (A.1) and $\mathrm{E}[y \,|\, Y, Z] = 1$. We get the desired inequality by taking the unconditional expectations.

Step 6. There is always over-investment in physical capital, i.e., $\bar{\theta} \geq \theta^*$.

Since $\bar{\theta}$ solves (2.7), by the first-order condition and Step 5, we get

$$0 = \mathrm{E}\left[R(\theta,\theta)^{-\gamma} \frac{\partial}{\partial\theta} R(\theta,\theta)\right]\Bigg|_{\theta=\bar{\theta}} \geq \mathrm{E}\left[R(\theta,\theta)^{-\gamma} \frac{\mathrm{d}}{\mathrm{d}\theta} R(\theta,\theta)\right]\Bigg|_{\theta=\bar{\theta}} = v'(\bar{\theta}),$$

where $v(\theta) = \frac{1}{1-\gamma} \mathrm{E}[R(\theta,\theta)^{1-\gamma}]$. Since θ^* maximizes v, we have $v'(\theta^*) = 0$. Since v is strictly concave by Step 4, v' is decreasing, so $\bar{\theta} \geq \theta^*$.

Step 7. The optimal tax rate on physical capital is positive.

We can decentralize the constrained efficient outcome as follows. Let θ, ϕ be the after- and pre-tax portfolios. Let $\tau_1, \tau_2 < 1$ be the tax rates on physical and human capital. Then $(1-\tau_1)\phi = \theta$ and $(1-\tau_2)(1-\phi) = 1-\theta$. The budget balance requires $\tau_1\phi + \tau_2(1-\phi) = 0$. Define the random variables $A^1, A^2 > 0$ by $R(\theta, \theta^*) = A^1\theta + A^2(1-\theta)$, where the portfolio return $R(\theta, \theta^*)$ is given by (2.5). The individual optimization problem is then

$$\max_{\phi} \frac{1}{1-\gamma} \mathrm{E}[(A^1(1-\tau_1)\phi + A^2(1-\tau_2)(1-\phi))^{1-\gamma}].$$

The first-order condition is

$$\mathrm{E}[R(\theta^*, \theta^*)^{-\gamma}(A^1(1-\tau_1) - A^2(1-\tau_2))] = 0. \tag{A.2}$$

Using the definition of ϕ and budget balance, we get

$$1-\tau_2 = 1 + \frac{\phi}{1-\phi}\tau_1 = 1 + \frac{\frac{\theta^*}{1-\tau_1}}{1-\frac{\theta^*}{1-\tau_1}}\tau_1 = \frac{(1-\tau_1)(1-\theta^*)}{1-\tau_1-\theta^*}.$$

Substituting into (A.2), we obtain

$$\begin{aligned} &\mathrm{E}[R(\theta^*, \theta^*)^{-\gamma}(A^1(1-\tau_1-\theta^*) - A^2(1-\theta^*))] = 0 \\ \iff &\tau_1 = (1-\theta^*)(1 - \mathrm{E}[R^{-\gamma}A^2]/\mathrm{E}[R^{-\gamma}A^1]), \end{aligned}$$

where $R \equiv R(\theta^*, \theta^*)$. By Step 5, it follows that

$$\begin{aligned} \mathrm{E}[R(\theta^*, \theta^*)^{-\gamma}(A^1 - A^2)] &= \mathrm{E}\left[R(\theta, \theta)^{-\gamma}\frac{\partial}{\partial\theta}R(\theta, \theta)\right]\bigg|_{\theta=\theta^*} \\ &\ge \mathrm{E}\left[R(\theta, \theta)^{-\gamma}\frac{\mathrm{d}}{\mathrm{d}\theta}R(\theta, \theta)\right]\bigg|_{\theta=\theta^*} = v'(\theta^*) = 0, \end{aligned}$$

so $\tau_1 \ge 0$. □

Proof of Proposition 4.1. Let $R^k_{t+1} = (P^k_{t+1} + D^k_{t+1})/P^k_t$ be the return of asset k and drop the time subscript. Consider the return $(1-\alpha)R(\theta_s, 0) + \alpha R^k$, which can be attained by investing the fraction of wealth $1-\alpha$ in the optimal portfolio and α in asset k. Since by assumption the portfolio constraint is not binding, the consumer can choose a small positive or negative α, and of course $\alpha = 0$ is optimal. Hence for small enough $\epsilon > 0$ by (3.4a) we obtain

$$0 = \arg\max_{\alpha\in[-\epsilon,\epsilon]} \frac{1}{1-\gamma} \mathrm{E}\left[b(s')^{1-\gamma}[(1-\alpha)R(\theta_s, 0) + \alpha R^k]^{1-\gamma} \,\middle|\, s\right].$$

The first-order condition with respect to α at $\alpha = 0$ is

$$\mathrm{E}\left[b(s')^{1-\gamma}R(\theta_s, 0)^{-\gamma}(R^k - R(\theta_s, 0)) \,\middle|\, s\right] = 0. \tag{A.3}$$

Using $R^k_{t+1} = (P^k_{t+1} + D^k_{t+1})/P^k_t$ and rearranging terms, we obtain (4.1).

By setting $D_t = 1$ and zero thereafter (hence $P_{t+1} = 0$) in (4.1), we obtain the price of the one period risk-free bond as the reciprocal of (4.2).

By the definition of infinitesimal tradability, (A.3) holds with $R(\pi)$ in place of R^k. Rearranging terms and dropping time subscripts, we obtain

$$1 = \frac{\mathrm{E}\left[b(s')^{1-\gamma} R(\theta_s, 0)^{-\gamma} R(\pi) \,\middle|\, s\right]}{\mathrm{E}\left[b(s')^{1-\gamma} R(\theta_s, 0)^{1-\gamma} \mid s\right]}.$$

Using $\mathrm{E}[XY \mid s] = \mathrm{Cov}[X, Y \mid s] + \mathrm{E}[X \mid s] \mathrm{E}[Y \mid s]$ for $X = b(s')^{1-\gamma} R(\theta_s, 0)^{-\gamma}$ and $Y = R(\pi)$, we obtain

$$1 = \frac{\mathrm{Cov}\left[b(s')^{1-\gamma} R(\theta_s, 0)^{-\gamma}, R(\pi) \,\middle|\, s\right] + \mathrm{E}\left[b(s')^{1-\gamma} R(\theta_s, 0)^{-\gamma} \,\middle|\, s\right] \mathrm{E}\left[R(\pi) \mid s\right]}{\mathrm{E}\left[b(s')^{1-\gamma} R(\theta_s, 0)^{1-\gamma} \mid s\right]}.$$

Using (4.2) and rearranging terms, we obtain (4.3). □

Proof of Proposition 4.2. It suffices to prove the case $j = 1$. Let $\Delta^{J-1} = \left\{\theta \in \mathbb{R}^J_+ \,\middle|\, \sum_j \theta^j = 1\right\}$ be the unit simplex. Consider the idiosyncratic shocks $a^1 = 1$ (no idiosyncratic shock in technology 1) and

$$a^2 = \cdots = a^J = z =: \begin{cases} \frac{1-p}{p}, & \text{(with probability } p\text{)} \\ \frac{p}{1-p}, & \text{(with probability } 1-p\text{)} \end{cases}$$

where $0 < p < 1$, independent of any other random variable. Take any $\theta \in \Delta^{J-1}$ with $\theta^1 < 1$ and let $Z = \sum_{j=2}^J A^j \theta^j$ be the aggregate component of the portfolio return excluding technology 1. Then $R(\theta, 0) = \theta^1 A^1 + zZ$ since there is no idiosyncratic risk in technology 1. By the definition of the idiosyncratic shock z, we obtain

$$\begin{aligned}
&\frac{1}{1-\gamma} \mathrm{E}\left[R(\theta, 0)^{1-\gamma} \,\middle|\, s\right] \\
&= \frac{1}{1-\gamma} \mathrm{E}\left[p\left(\theta^1 A^1 + \frac{1-p}{p} Z\right)^{1-\gamma} + (1-p)\left(\theta^1 A^1 + \frac{p}{1-p} Z\right)^{1-\gamma} \,\middle|\, s\right] \\
&= \frac{1}{1-\gamma} \mathrm{E}\left[p^\gamma \left(p\theta^1 A^1 + (1-p)Z\right)^{1-\gamma} + (1-p)\left(\theta^1 A^1 + \frac{p}{1-p} Z\right)^{1-\gamma} \,\middle|\, s\right] \\
&\to \frac{1}{1-\gamma} \mathrm{E}\left[(\theta^1 A^1)^{1-\gamma} \,\middle|\, s\right] < \frac{1}{1-\gamma} \mathrm{E}\left[(A^1)^{1-\gamma} \,\middle|\, s\right]
\end{aligned}$$

as $p \to 0$ since $\gamma > 0$ and $\theta^1 < 1$. Thus, any portfolio with $\theta^1 < 1$ is dominated by investing entirely in technology 1 as the idiosyncratic shock gets larger. Letting $\theta_{s,p}$ be the optimal portfolio corresponding to p, it follows that $\lim_{p \to 0} \theta_{s,p} = (1, 0, \dots, 0)$. Substituting $\theta_s = \theta_{s,p}$ into (4.4) and letting $p \to 0$, we obtain (4.6). □

Proof of Proposition 6.1. Instead of the optimization problem in the market economy, consider the autarky problem and drop the individual subscript i. Then the budget constraint (5.1a) is replaced by the resource constraint

$$c_t + \sum_{j=0}^J x_t^j = \sum_{j=1}^J F_t^j(k_t^j, \phi_t^j k_t^0),$$

where ϕ_t^j denotes the fraction of human capital allocated to production with technology j. Then (5.5) is accordingly replaced by

$$c_t + \sum_j k_t'^j = w_t := \sum_{j=1}^J [F_t^j(k_t^j, \phi_t^j k_t^0) + (1-\delta_t^j)k_t^j + (1-\delta_t^0)\phi_t^j k_t^0],$$

and the budget constraint (5.3) becomes

$$\begin{aligned} w_{t+1} = \sum_{j=1}^J [&F_{t+1}^j(Z_{t+1}^j \theta_t^j, Z_{t+1}^0 \theta_t^0 \phi_{t+1}^j) \\ &+ Z_{t+1}^j(1-\delta_{t+1}^j) + Z_{t+1}^0(1-\delta_{t+1}^0)\theta_t^0 \phi_{t+1}^j](w_t - c_t). \end{aligned} \tag{A.4}$$

Since the allocation of human capital ϕ_{t+1}^j can be chosen after observing time $t+1$ shocks, the agent will choose it so as to maximize the right-hand side of (A.4). Hence (5.3) holds with

$$\begin{aligned} R_{t+1}(\theta_t) = \max_{\phi_{t+1}\in\Delta^{J-1}} \sum_{j=1}^J [&F_{t+1}^j(Z_{t+1}^j \theta_t^j, Z_{t+1}^0 \theta_t^0 \phi_{t+1}^j) \\ &+ Z_{t+1}^j(1-\delta_{t+1}^j)\theta_t^j + Z_{t+1}^0(1-\delta_{t+1}^0)\theta_t^0 \phi_{t+1}^j]. \end{aligned} \tag{A.5}$$

The budget constraint (A.4) (maximized with respect to ϕ_{t+1}) has precisely the same form as the budget constraint (3.1). The only difference is that $R_{t+1}(\theta)$ is not necessarily linear in θ in (A.4), but the linearity plays no role for the proof of Theorem 3.2. Therefore we can construct a single agent equilibrium with optimal portfolio θ_t.

Define the rental rate of physical and human capital at period $t+1$ by

$$r_{t+1}^j = \frac{\partial}{\partial K} F_{t+1}^j(Z_{t+1}^j \theta_t^j, Z_{t+1}^0 \theta_t^0 \phi_{t+1}^j), \tag{A.6a}$$

$$r_{t+1}^0 = \frac{\partial}{\partial H} F_{t+1}^j(Z_{t+1}^j \theta_t^j, Z_{t+1}^0 \theta_t^0 \phi_{t+1}^j), \tag{A.6b}$$

which are parallel to (5.9) and $\phi_{t+1} \in \Delta^{J-1}$ is the maximizer of (A.5). The right-hand side of (A.6b) does not depend on j by considering the first-order condition of the maximization (A.5). Let $R_{t+1}(\theta)$ be the return on portfolio in the autarky economy defined by (A.5) and $R_{t+1}(\theta, \theta_t)$ be the return on portfolio in the market economy defined by

$$R_{t+1}(\theta, \theta_t) = \sum_{j=0}^J (1 + r_{t+1}^j - \delta_{t+1}^j) Z_{t+1}^j \theta^j, \tag{A.7}$$

where r_{t+1}^j is given by (A.6). Since production functions exhibit constant returns to scale, by (A.5)–(A.7) we obtain

$$\begin{aligned} R_{t+1}(\theta_t, \theta_t) &= \sum_{j=1}^J [(1 + r_{t+1}^j - \delta_{t+1}^j) Z_{t+1}^j \theta_t^j + (1 + r_{t+1}^0 - \delta_{t+1}^0) Z_{t+1}^0 \theta_t^0 \phi_{t+1}^j] \\ &= \sum_{j=1}^J [F_{t+1}^j(Z_{t+1}^j \theta_t^j, Z_{t+1}^0 \theta_t^0 \phi_{t+1}^j) + Z_{t+1}^j(1-\delta_{t+1}^j) + Z_{t+1}^0(1-\delta_{t+1}^0)\theta_t^0 \phi_{t+1}^j] \\ &= R_{t+1}(\theta_t). \end{aligned}$$

Furthermore, we obtain $D_\theta R_{t+1}(\theta_t, \theta_t) = D_\theta R_{t+1}(\theta_t)$ by a straightforward calculation. Therefore θ_t (the optimal portfolio in autarky) satisfies (5.12), that is, θ_t is the equilibrium portfolio in the market economy. Since the equilibrium portfolio is also optimal in autarky, the equilibrium is constrained efficient. ☐

Proof of Theorem 6.2. First let us show that the equilibrium is constrained efficient if and only if

$$\theta_t = \operatorname*{arg\,max}_{\theta \in \Delta^J} \mathrm{E}_t[b_{t+1}^{1-\gamma} R_{t+1}(\theta, \theta)^{1-\gamma}]^{\frac{1}{1-\gamma}} \tag{A.8}$$

for all date-events, where b_{t+1} is given in Theorem 5.3 and $R(\theta, \bar{\theta})$ is given by (5.11).

If θ_t does not satisfy (A.8) for some date-event, then

$$\mathrm{E}_t[b_{t+1}^{1-\gamma} R_{t+1}(\theta^*, \theta^*)^{1-\gamma}]^{\frac{1}{1-\gamma}} > \mathrm{E}_t[b_{t+1}^{1-\gamma} R_{t+1}(\theta_t, \theta_t)^{1-\gamma}]^{\frac{1}{1-\gamma}} \tag{A.9}$$

for some $\theta^* \in \Delta^J$. Suppose that at this date-event all agents switch to the portfolio θ^* instead of θ_t simultaneously, but stick to the equilibrium portfolio thereafter. Since the value function is given by $V_t(w) = b_t w$ and $R_{t+1}(\theta, \theta)$ is the return on the common portfolio θ, if a typical agent has wealth w *after* consumption, by (A.9) the future utility term in the recursive utility, $\mathrm{E}_t[V_{t+1}(R_{t+1}(\theta, \theta)w)^{1-\gamma}]^{\frac{1}{1-\gamma}}$, becomes

$$\mathrm{E}_t[b_{t+1}^{1-\gamma} R_{t+1}(\theta^*, \theta^*)^{1-\gamma}]^{\frac{1}{1-\gamma}} w > \mathrm{E}_t[b_{t+1}^{1-\gamma} R_{t+1}(\theta_t, \theta_t)^{1-\gamma}]^{\frac{1}{1-\gamma}} w,$$

so everybody is better off by simultaneously switching to the portfolio θ^*. Therefore the equilibrium is constrained inefficient.

Conversely, if the equilibrium is constrained inefficient, the planner can Pareto improve by intervening in the portfolio choice. Since the objective function is the same for every agent and quasi-concave, by taking the average portfolio intervention weighted by wealth, the planner can Pareto improve by choosing an alternative symmetric portfolio. Therefore (A.8) fails for some date-events.

Next, let us show that the equilibrium is generically constrained inefficient. Pick an aggregate state s in which the conditional distribution of idiosyncratic shock is non-degenerate. Parametrize the idiosyncratic shock by $u \in U_s$, where U_s is the interior of some simplex in a Euclidean space. By the equilibrium condition (5.12a), we have

$$\mathrm{E}\left[b_{s'}^{1-\gamma} R(\theta_s, \theta_s)^{-\gamma} D_\theta R(\theta_s, \theta_s) \,\middle|\, s\right] - \lambda \mathbf{1} = 0, \tag{A.10}$$

where D_θ denotes the vector of partial derivatives with respect to θ (the first argument of $R(\theta, \bar{\theta})$), θ_s is the equilibrium portfolio, and λ is the Lagrange multiplier for the portfolio constraint $\sum_{j=0}^J \theta^j = 1$. If the equilibrium is constrained efficient, by (A.8) and the chain rule we have

$$\mathrm{E}\left[b_{s'}^{1-\gamma} R(\theta_s, \theta_s)^{-\gamma} \left(D_\theta R(\theta_s, \theta_s) + D_{\bar{\theta}} R(\theta_s, \theta_s)\right) \,\middle|\, s\right] - \mu \mathbf{1} = 0, \tag{A.11}$$

where μ is the Lagrange multiplier. Hence by (A.10) and (A.11) we obtain

$$\mathrm{E}\left[b_{s'}^{1-\gamma} R(\theta_s, \theta_s)^{-\gamma} D_{\bar{\theta}} R(\theta_s, \theta_s) \,\middle|\, s\right] - (\mu - \lambda) \mathbf{1} = 0. \tag{A.12}$$

By the Inada condition human capital is clearly positive, so $\theta_s^0 > 0$. Since at least one production technology must operate, there is $j \geq 1$ with $\theta_s^j > 0$. Taking the difference of (A.12) for these two cases, we obtain

$$\mathrm{E}\left[b_{s'}^{1-\gamma} R(\theta_s, \theta_s)^{-\gamma} (D_{\bar{\theta}^j} R(\theta_s, \theta_s) - D_{\bar{\theta}^0} R(\theta_s, \theta_s)) \,\middle|\, s\right] = 0. \tag{A.13}$$

Now consider the system of equations consisting of (A.10), $\sum_{j=0}^J \theta^j - 1 = 0$, and (A.13). Let the left-hand side of these equations be $G(\theta, \lambda; u)$, where $u \in U_s$ is the parametrization of idiosyncratic human capital obsolescence shock. Since

$$G : \mathbb{R}_{++}^{J+1} \times \mathbb{R}_{++} \times U_s \to \mathbb{R}^{J+1} \times \mathbb{R} \times \mathbb{R},$$

Given b_{t+1} and $u \in U_s$, there are more equations $(J+3)$ than unknowns $(J+2)$.

The rest of the proof is a standard transversality argument as in Citanna et al. (1998) and therefore I only provide the sketch. First we show that 0 is a regular value for G. This can be done by showing that the $(J+3) \times (J+2+\dim U_s)$ Jacobian matrix DG has full row rank (this is where I use $\dim U_s \geq 1$). I omit this step since the calculation is straightforward but tedious. By the transversality theorem (Villanacci et al., 2002, p. 151, Theorem 26), then, there exists a full measure set U_s^* such that for every $u \in U_s^*$, 0 is a regular value of the function G_u defined by $(\theta, \lambda) \mapsto G(\theta, \lambda, u)$. But since the Jacobian matrix DG_u is $(J+3) \times (J+2)$ and therefore there are more rows than columns, DG_u can never have full row rank. By the definition of a regular value, it follows that $G_u^{-1}(\{0\}) = \emptyset$, so $G(\theta, \lambda, u) = 0$ has no solutions. Since (A.13) is necessary for constrained efficiency, the equilibrium is generically constrained inefficient.

To show that U_s^* is open, I show that $(U_s^*)^c$ is closed. Let $u^n \in (U_s^*)^c$ and $u^n \to u$. By the definition of U_s^*, (A.8) holds for the portfolio θ_t^n corresponding to u^n. Since Δ^J is compact, $\{\theta_t^n\}$ has a convergent subsequence, and by the maximum theorem its limit also satisfies (A.8), so $u \in (U_s^*)^c$. Letting $U = \prod_s U_s$ and $U^* = \prod_s U_s^*$, we obtain the conclusion. ☐

Proof of Theorem 6.3. Drop time subscripts since there is no risk of confusion. Let $\tau^j < 1$ be the tax rate on capital j and $\phi = (\phi^0, \dots, \phi^J)$ be the pre-tax portfolio. In order to achieve the constrained efficient portfolio, we need $\theta^{*j} = (1-\tau^j)\phi^j$. Let $m^j = 1 - \tau^j$ be the fraction of capital that remains after tax. Then the optimal portfolio problem of an agent is

$$\max_{\phi \in \Delta^J} \frac{1}{1-\gamma} \mathrm{E}\left[(b^*)^{1-\gamma} \left(\textstyle\sum_j A^j m^j \phi^j\right)^{1-\gamma}\right].$$

The first-order condition for optimality is

$$\begin{aligned}
\lambda &= \mathrm{E}\left[(b^*)^{1-\gamma} \left(\textstyle\sum_j A^j m^j \phi^j\right)^{-\gamma} A^j m^j\right] = \mathrm{E}[(b^*)^{1-\gamma} R(\theta^*, \theta^*)^{-\gamma} A^j m^j], \\
\iff \frac{\lambda}{m^j} &= \mathrm{E}[(b^*)^{1-\gamma} R(\theta^*, \theta^*)^{-\gamma} A^j],
\end{aligned}$$

where λ is the Lagrange multiplier for the constraint $\sum_j \phi^j = 1$ and I used $\theta^{*j} = m^j \phi^j$ and the definition of A^j. Multiplying both sides by θ^{*j}, again using $\theta^{*j} = m^j \phi^j$ and the definition of A^j, and noting that $\sum_j \phi^j = 1$, we obtain

$$\lambda = \sum_{j=0}^J \mathrm{E}[(b^*)^{1-\gamma} R(\theta^*, \theta^*)^{-\gamma} A^j \theta^{*j}] = \mathrm{E}[(b^*)^{1-\gamma} R(\theta^*, \theta^*)^{1-\gamma}].$$

Therefore

$$\tau^j = 1 - m^j = 1 - \frac{\mathrm{E}[(b^*)^{1-\gamma} R(\theta^*, \theta^*)^{1-\gamma}]}{\mathrm{E}[(b^*)^{1-\gamma} R(\theta^*, \theta^*)^{-\gamma} A^j]}. \quad \square$$

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