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Asymptotic Syzygies of Normal Crossing Varieties

Author
Chun, Daniel

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Asymptotic Syzygies of Normal Crossing Varieties

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Daniel Minha Chun

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Dissertation Committee:
Dr. Ziv Ran, Chairperson
Dr. Carl Mautner
Dr. Jose Gonzalez

The Dissertation of Daniel Minha Chun is approved:

Committee Chairperson

University of California, Riverside

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ABSTRACT OF THE DISSERTATION

Asymptotic Syzygies of Normal Crossing Varieties

by

Daniel Minha Chun

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, March 2018
Dr. Ziv Ran, Chairperson

Asymptotic syzygies of a normal crossing variety follow the same vanishing behavior as one of its smooth components, unless there is a cohomological obstruction arising from how the smooth components intersect each other. In that case, we compute the asymptotic syzygies in terms of the cohomology of the simplicial complex associated to the normal crossing variety.

We combine our results with knowledge of degenerations of certain smooth projective varieties into normal crossing varieties to obtain some results on asymptotic syzygies of those smooth projective varieties.

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1 Introduction

Mark Green's paper [1] introduced a way to interpret syzygies of projective varieties as the cohomology groups of a Koszul type complex. This interpretation allowed for concrete computational results about syzygies that were not possible before. In particular, Lawrence Ein and Robert Lazarsfeld have established interesting results on vanishing and non-vanishing of asymptotic syzygies, as in [5] and [6]. Here, asymptotic refers to the fact that they investigated the syzygies of smooth projective varieties of large enough degree embedding.

Recently, Ziv Ran has extended some of these results to the case of nodal, possibly reducible, curves in [7]. In this paper, we try to generalize his results by analzying the case of normal crossing varieties of arbitrary dimension.

As one would expect intuitively, we find that the asymptotic syzygies of normal crossing varieties depend on the worst behaved smooth component as well as the combinatorics of how the smooth components intersect each other. We use the knowledge of asymptotic syzygies of normal crossing varieties and degenerations in order to answer questions about asymptotic syzygies of smooth varieties in the Applications section.

2 Spectral Sequences

Spectral sequence is a powerful tool to manipulate complicated commutative diagrams to get useful information. One good reference on it is [13] by Ravi Vakil. It is a family $E_r^{p,q}$ of vector spaces, for all integers p, q, r with $r \geq 0$ (for a fixed r, they form the r-th page of the spectral sequence).

If it has a horizontal orientation, for each p, q, r, there is a differential $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ satisfying $d_r^2 = 0$. If it has a vertical orientation, there is a differential $d_r^{p,q} : E_r^{p,q} \to E_r^{p-r+1,q+r}$. We will assume horizontal orientation for the rest of this section, but it will apply to vertical orientation analogously as well.

There are isomorphisms $H^{p,q}(E_r) \to E_{r+1}^{p,q}$ where the homology is given by $H^{p,q}(E_r) = \ker d_r^{p,q}/\mathrm{im} d_r^{p-r,q+r-1}$.

The spectral sequence converges if there is $r(p,q)$ for each p, q such that for all $r \geq r(p,q)$ we have $E_{p,q}^r \cong E_{p,q}^{r(p,q)}$. The bigraded object $E^\infty = \{E_{p,q}^{r(p,q)}\}_{p,q}$, if it exists, is the limit term of the spectral sequence, and we say the spectral sequence abuts to E^∞.

It is called a bounded spectral sequence if all terms except for a finite number of choices p, q vanish. An important fact is that a bounded spectral sequence converges. This is because for large enough r, all of the differentials on the E_r page will either map from zero or to zero, so the pages will stop changing after such a r.

2.1 Filtered Chain Complexes

We will now look at filtered chain complexes, since almost all of the applications of spectral sequences arise from filtered chain complexes.

A filtered chain complex is a chain complex of modules

$$... \stackrel{\partial_{n-1}}{\to} C_n \stackrel{\partial_n}{\to} C_{n+1} \to ...$$

with a filtering $F_\bullet C_n$ on each C_n such that $\partial(F_p C_n) \subset F_p C_{n+1}$. Define $G_p C_n = F_p C_n / F_{p-1} C_n$. Note that ∂ induces chain complexes $F_p C_\bullet$ as well as $G_p C_\bullet$ for each p. The filtration on the complex also induces a filtration on the homology $H_\bullet(C)$, which looks like the following

$$F_pH_n(C) = \mathrm{im}(H_n(F_pC_\bullet) \to H_n(C_\bullet)).$$

Now, define

$$Z^r_{p,q} = \{c \in F_pC_{p+q} | \partial(c) \in F_{p-r}C_{p+q+1}\}/F_{p-1}C_{p+q} \text{ and}$$
$$B^r_{p,q} = \partial(F_{p+r-1}C_{p+q-1}) \cap F_pC_{p+q}/F_{p-1}C_{p+q} \text{ and } E^r_{p,q} = Z^r_{p,q}/B^r_{p,q}.$$

Then the differentials ∂ of $C_\bullet$ induce maps $\partial^r : E^r_{p,q} \to E^r_{p+r,q-r+1}$, so that E^r form a spectral sequence. Furthermore, we can see that by definition, for a fixed choice of p and q, we get that $E^r_{p,q} = G_pH_{p+q}(C)$ for large enough r.

In other words, E^r, our spectral sequence associated to the filtered chain complex, abuts to $E^\infty_{p,q} = G_pH_{p+q}(C)$. Thus, for a fixed n, $H_n(C) = \bigoplus_{p+q=n} E^\infty_{p,q}$. Crucially, we have the following fact

Remark 1. *If we have the same complex with multiple different filtrations, then in general, we will have different limits E^∞, but the direct sum of all the terms on a fixed anti-diagonal of any of these limits will be the same.*

2.2 Double Complexes

Let's now discuss double complexes, because it is in this context that most spectral sequences are used in practice.

A bounded double complex is a collection of vector spaces $C^{p,q}$ $(p, q \in \mathbb{Z})$, which are zero except for a finite number of choices p, q, and horizontal differentials $\partial^{p,q}_h : C^{p,q} \to C^{p,q+1}$ and vertial differentials $\partial^{p,q}_v : C^{p,q} \to C^{p+1,q}$, such that $\partial_h\partial_v + \partial_v\partial_h = 0$.

From the double complex we construct a corresponding single total complex $C_\bullet$ with $C_n = \bigoplus_{p+q=n} C^{p,q}$ with the differential as $\partial = \partial_h + \partial_v$. Then there are two different filtrations. The horizontal filtration on $C_\bullet$ is given by

$$F_p^h C_n = \bigoplus_{n_1+n_2=n, n_1 \leq p} C^{n_1,n_2}$$

And the vertical filtration is given by

$$F_p^v C_n = \bigoplus_{n_1+n_2=n, n_2 \leq p} C^{n_1,n_2}$$

For horizontal filtration, define the zeroth page as follows

$${}'E^0_{p,q} = G_p C_{p+q} \text{ with the differential } {}'E^0_{p,q} \to' E^0_{p,q+1} \text{ induced from } C_\bullet.$$

and for vertical filtration, define the zeroth page analogously as follows

$${}''E^0_{p,q} = G_p C_{p+q} \text{ with the differential } {}''E^0_{p,q} \to'' E^0_{p+1,q} \text{ induced from } C_\bullet.$$

Then by **Remark 1**, we get

Remark 2. *Horizontal and vertical filtrations of the total complex associated to the double complex lead to the same direct sum of all the terms on any fixed anti-diagonal.*

In practice, one of the filtrations usually leads to a simpler result with lots of vanishing, so we get information about the other filtration.

3 Why Koszul Cohomology?

The standard references for the theory of Koszul Cohomology are [1] and [2] by Mark Green. Let X be a projective variety of dimension n defined over $\mathbb{C}$. Let $\mathscr{L}$ be a very ample line bundle on X, and let $\mathscr{B}$ be an arbitrary line bundle on X. $\mathscr{L}$ defines an embedding

$$X \subseteq \mathbb{P}^r = \mathbb{P}H^0(\mathscr{L}),$$

where $r+1 = h^0(\mathscr{O}_X(\mathscr{L}))$. The study of asymptotic syzygies is the study of syzygies of X in $\mathbb{P}^r$ when $\mathscr{L}$ is very positive. Let $S = \mathrm{Sym} H^0(\mathscr{L})$ be the homogeneous coordinate ring of $\mathbb{P}^r$, and let $R = \bigoplus_q H^0(\mathscr{B} \otimes q\mathscr{L}))$, and view R as a finitely generated S-module. R has a minimal graded free resolution $F_\bullet = F_\bullet(X, \mathscr{B}, \mathscr{L})$,

$$0 \to F_r = \bigoplus_q S(-q) \otimes M_{r,q} \to ... \to F_1 = \bigoplus_q S(-q) \otimes M_{1,q} \to F_0 = \bigoplus_q S(-q) \otimes M_{0,q} \to R \to 0$$

where $M_{p,q}$, called syzygies, are finite dimensional vector spaces that keep track of how many copies of $S(-q)$ are in F_p. Intuitively, we can think of them as follows

$M_{0,q}$ = generators of degree q for R as a S-module,

$M_{1,q}$ = primitive relations of weight q among the generators for R,

$M_{2,q}$ = primitive syzygies of weight q among the relations for R,

... and so on.

In other words, if $x_1, x_2, ...$ are generators for R with $\mathrm{deg} x_i = e_i$, then a relation of weight q among the generators is one of the form

$$\sum_i u_i x_i = 0,\ u_i \in S_{q-e_i}.$$

A primitive relation of weight q is one that is not an S-linear combination of relations of lower weight. If $\sum_i u_i^v x_i = 0$ are a basis for the primitive relations of weights e^v respectively, a syzygy of weight q is a relation of the form

$$\sum_v w_v u_i^v = 0 \text{ for all } i \text{ with } w_v \in S_{q-e_v}$$

...and so on.

Since Green's paper [1], much attention has been focused on what we can say about these syzygies. Green's main insight in [1] was to interpret them as the cohomology groups of a Koszul-type complex.

Definition 1. *Let V be a finite dimensional complex vector space, S, the symmetric algebra over V, and $B = \bigoplus_{q\in\mathbb{Z}} B_q$, a graded S-module. Then there is a Koszul Complex*

$$... \to \bigwedge^{p+1} V \otimes B_{q-1} \stackrel{\partial_{p+1,q-1}}{\to} \bigwedge^{p} V \otimes B_q \stackrel{\partial_{p,q}}{\to} \bigwedge^{p-1} V \otimes B_{q+1} \to ... \; ... \quad (a)$$

where the differential is given by $\partial_{p+1,q-1}(v_0 \wedge ... \wedge v_p \otimes m) = \sum_{k=0}^{p}(-1)^k v_0 \wedge ... \wedge \hat{v_k} \wedge ... \wedge v_p \otimes v_k m$.

The cohomology of the above complex is called the Koszul Cohomology, and denoted by $K_{p,q}(B,V)$.

In the special case where $V = H^0(\mathscr{L})$ and $B = R$, it is denoted as $K_{p,q}(X,\mathscr{B},\mathscr{L})$, and if $\mathscr{B} = \mathscr{O}_X$, we omit $\mathscr{B}$ and write $K_{p,q}(X,\mathscr{L})$.

Before establishing a connection between syzygies and Koszul Cohomology, we need a helper lemma which states that the Koszul Complex associated to the projective space is exact.

Lemma 1. *The complex*

$$... \to \bigwedge^{p+1} V \otimes S_{q-1} \stackrel{\partial_{p+1,q-1}}{\to} \bigwedge^{p} V \otimes S_q \stackrel{\partial_{p,q}}{\to} \bigwedge^{p-1} V \otimes S_{q+1} \to ... \; ... \quad (b)$$

with the differentials given by the Koszul differentials, is exact unless $n = 0$, in which case the complex is $0 \to S_0 = \mathbb{C} \to 0$.

Proof. If we construct a chain homotopy between a constant multiple of the identity map and the zero map, then since chain homotopic maps induce the same map on homology, a constant multiple of the identity map and the zero map on homology are the same, meaning the homologies are all zero, which will prove the Lemma.

So, let's construct such a chain homotopy, which we will call h. Set $d+1 = \dim V$.

$$h_p : \bigwedge^{p} V \otimes S_q \to \bigwedge^{p+1} V \otimes S_{q-1} \text{ is defined as}$$
$$h_p(v_0 \wedge ... \wedge v_{p-1} \otimes m) = (-1)^p \sum_{l=0}^{d} v_0 \wedge ... \wedge v_{p-1} \wedge v_l \otimes \frac{\partial m}{\partial v_l}.$$

Then we have

$$(\partial_{p+1} h_p + h_{p-1}\partial_p)(v_0 \wedge ... \wedge v_{p-1} \otimes m) =$$
$$(-1)^p(pv_0 \wedge ... \wedge v_{p-1} \otimes m + v_0 \wedge ... \wedge v_{p-1} \otimes \sum_{l=0}^{d} v_l \frac{\partial m}{\partial v_l}) = (-1)^p(p+q)v_0 \wedge ... \wedge v_{p-1} \otimes m$$

where the second equality is by the Euler identity. We have constructed the appropriate h, so we proved the Lemma.

□

Proposition 1. *We have isomorphisms* $K_{p,q}(X, \mathscr{B}, \mathscr{L}) \cong M_{p,p+q}(X, \mathscr{B}, \mathscr{L})$.

Proof. For a fixed integer l, define a double complex

$$C^{p,q} = \bigwedge^{-p} V \otimes \bigoplus_k (S_k \otimes M_{-q,l+p-k}) \text{ for } q \leq 0,$$
$$= \bigwedge^{-p} V \otimes R_{l+p} \text{ for } q = 1, \text{ and}$$
$$= 0 \text{ otherwise.}$$

As differentials, we take

$$C^{p,q} \xrightarrow{\partial^v} C^{p+1,q} \text{ and } C^{p,q} \xrightarrow{\partial^h} C^{p,q+1}$$

where for $q \geq 0$, ∂^v comes from (b), and for $q = -1$, ∂^v comes from the Koszul differential, and finally, ∂^h comes from the minimal free resolution of R as a S-module.

We can check that $\partial_h \partial_v + \partial_v \partial_h = 0$, so **Remark 2** applies here, and we get two spectral sequences, E' from the horizontal filtration, and E'' from the vertical filtration, with the same abuttment on fixed anti-diagonals and

$$'E_\infty^{p,q} =' E_1^{p,q} = 0 \text{ for all } p, q.$$

and

$$''E_1^{p,q} = K_{-p,l-p}(X, \mathscr{B}, \mathscr{L}) \text{ for } q = 1,$$
$$= M_{-q,l}(X, \mathscr{B}, \mathscr{L}) \text{ for } q \leq 0,\ p = 0, \text{ and}$$
$$= 0, \text{ otherwise.}$$

Now, the differentials on the $''E_1$ page are either zero maps, is $M_{0,l} \to K_{0,l}$, or look like $\partial_{0,-q+1} : M_{-q+1,l} \to M_{-q,l}$ for $q \leq 0$. By the minimality of the minimal free resolution of R, $\partial_{0,-q+1}$ are zero maps.

Thus, the only non-zero map in the $''E_r$ page is $\partial_{0,r} : M_{r,l} \to K_{r,l-r}$, but since the abuttment is to zero, we get the these $\partial_{0,r}$ are isomorphisms.

□

4 Standard Facts About Koszul Cohomology

A useful reference for this topic is [6]. For rest of the paper we will keep the notation of the above **Proposition 1**. We have several useful facts regarding them.

First, we can treat them as coherent cohomology groups of a certain vector bundle on X. Consider a natural evaluation map

$$e_V : V_X =_{\text{def}} V \otimes \mathscr{O}_X \to \mathscr{L}$$

Set $M_V = \ker e_V$ and $\dim V = v$. Then M_V is a vector bundle of rank $v-1$ sitting inside the following short exact sequence of vector bundles

$$0 \to M_V \to V_X \to \mathscr{L} \to 0 \text{ ... (b)}$$

Proposition 2. *Asume that*

$$H^i(\mathscr{B} + m\mathscr{L}) = 0 \text{ for } i > 0 \text{ and } m > 0.$$

Then for $q \geq 2$, we get

$$K_{p,q}(X, \mathscr{B}, V) = H^1(\bigwedge^{p+1} M_V \otimes \mathscr{B} \otimes (q-1)\mathscr{L})$$

If moreover $H^1(\mathscr{B}) = 0$, then the same statement holds also when $q = 1$.

Proof. From (b), we get $V_X \cong M_V \oplus \mathscr{L}$, so taking exterior powers, we get

$$\bigwedge^p V_X \cong \bigoplus_{j=0}^{p} \bigwedge^i M_V \otimes \bigwedge^{p-i} \mathscr{L} \cong \bigwedge^p M_V \oplus \bigwedge^{p-1} M_V \otimes \mathscr{L}$$

where the second isomorphism is because $\mathscr{L}$ is a line bundle so any of its higher exterior products is zero. The isomorphism above gives us a short exact sequence. Twist it by $\mathscr{B} \otimes q\mathscr{L}$ to get

$$0 \to \bigwedge^p M_V \otimes \mathscr{B} \otimes q\mathscr{L} \to \bigwedge^p V_X \otimes \mathscr{B} \otimes q\mathscr{L} \to \bigwedge^{p-1} M_V \otimes \mathscr{B} \otimes (q+1)\mathscr{L} \to 0 \text{ ... } (\text{c})_{p,q}$$

We can splice a bunch of these short exact sequences (for different values of p and q) together then twist to get the following exact sequence

$$\ldots \to \bigwedge^{p+1} V_X \otimes \mathscr{B} \otimes (q-1)\mathscr{L} \overset{\partial_{p+1,q-1}}{\to} \bigwedge^{p} V_X \otimes \mathscr{B} \otimes q\mathscr{L} \overset{\partial_{p,q}}{\to} \bigwedge^{p-1} V_X \otimes \mathscr{B} \otimes (q+1)\mathscr{L} \to \ldots$$

... (d)

In other words, the differential $\partial_{p,q}$ in the complex (d) is the composition $\bigwedge^{p} V_X \otimes \mathscr{B} \otimes q\mathscr{L} \to \bigwedge^{p-1} M_V \otimes \mathscr{B} \otimes (q+1)\mathscr{L} \hookrightarrow \bigwedge^{p-1} V_X \otimes \mathscr{B} \otimes (q+1)\mathscr{L}$, where the first map is from $(c)_{p,q}$ and the second map is simply inclusion.

and if we apply the global sections functor to $\partial_{p,q}$, we get the differential $d_{p,q}$ from **Proposition 1**. Thus, because of the cohomological assumptions in the proposition, from the long exact sequence of cohomology from $(c)_{p,q}$, we get that $\ker d_{p,q} = H^0(\bigwedge^{p} M_V \otimes \mathscr{B} \otimes q\mathscr{L})$, and from the long exact sequence of cohomology from $(c)_{p+1,q-1}$, we get that $H^0(\bigwedge^p M_V \otimes \mathscr{B} \otimes q\mathscr{L})/\mathrm{Im} d_{p+1,q-1} = H^1(\bigwedge^{p+1} M_V \otimes \mathscr{B} \otimes (q-1)\mathscr{L})$. From which we get the desired result.

□

Proposition 3. *Make same assumptions as in **Proposition 2**. Then* $q \geq 2$

$$K_{p,q}(X, \mathscr{B}, V) = H^{q-1}(\bigwedge^{p+q-1} M_V \otimes \mathscr{B} \otimes \mathscr{L})$$

Proof. Because of the cohomological assumptions in **Proposition 2**, from the long exact sequence of cohomology from $(c)_{p+i,q-i}$, we get isomorphisms $H^i(\bigwedge^{p+i} M_V \otimes \mathscr{B} \otimes (q-i)\mathscr{L}) \cong H^{i+1}(\bigwedge^{p+i+1} M_V \otimes \mathscr{B} \otimes (q-i-1)\mathscr{L})$ for $i = 1 \ ... \ q-2$. Combined with $K_{p,q}(X, \mathscr{B}, V) = H^1(\bigwedge^{p+1} M_V \otimes \mathscr{B} \otimes (q-1)\mathscr{L})$ from **Proposition 2**, we get the desired result.

□

Proposition 4. *Assume that X is smooth of dimension n, and the same conditions are held as in **Proposition 2** hold. In addition assume that*

$$H^i(\mathscr{B} \otimes m\mathscr{L}) = 0 \text{ for } 0 < i < n \text{ and all } m \in \mathbb{Z} \ \dots \ (e)$$

and

$$H^0(\mathscr{B} - j\mathscr{L}) = 0 \text{ for } j > 0 \ \dots \ (f)$$

Then for $0 \leq q \leq n+1$ one has isomorphisms

$$K_{p,q}(X, \mathscr{B}, V) \cong K_{v-1-p-n,n+1-q}(X, K_X - \mathscr{B}, V)^{\star}$$

Proof. Let's first deal with the case where $1 \leq q \leq n$. By the same argument given in the proof of **Proposition 3**, using the assumption (e), we get

$$K_{p,q}(X, \mathscr{B}, V) = H^1(\bigwedge^{p+1} M_V \otimes \mathscr{B} \otimes (q-1)\mathscr{L}) = H^{n-1}(\bigwedge^{p+n-1} M_V \otimes \mathscr{B} \otimes (q+1-n)\mathscr{L})$$

which is Serre dual to

$$H^1(\bigwedge^{p+n-1} M_V^{\star} \otimes (K_X - \mathscr{B}) \otimes (n-q-1)\mathscr{L})$$

where $M_V^{\star}$ is the dual of M_V. Note that we have

$$0 \to \bigwedge^{v-1} M_V \to \bigwedge^{v} V_X \to \mathscr{L} \to 0$$

by taking v-th exterior product of the short exact sequence (b). Also, $\mathscr{O}_X \cong \bigwedge^{v} V_X$. That means

$$\bigwedge^{v-1} M_V \cong -\mathscr{L}$$

So we get

$$\bigwedge^{p+n-1} M_V^{\star} \cong \bigwedge^{v-p-n} M_V \otimes \bigwedge^{v-1} M_V \cong \bigwedge^{v-p-n} M_V \otimes \mathscr{L}$$

where the first isomorphism is due to Hodge Duality. Thus, $K_{p,q}(X, \mathscr{B}, V)$ is dual to

$$H^1(\bigwedge^{v-p-n} M_V \otimes (K_X - B) \otimes (n-q)\mathscr{L})$$

and the above is isomorphic to $K_{v-1-p-n,n+1-q}(X, K_X - \mathscr{B}, V)$ by **Proposition 2**, so we're done.

Now for the cases $q = 0,\ n+1$, first note that because of (f), using the same argument as in the proof of **Proposition 2**, we get

$$K_{p,0}(X, \mathscr{B}, V) = H^0(\bigwedge^{p} M_V \otimes \mathscr{B})$$

We can then use the same duality argument as in the case of $1 \leq q \leq n$ to prove the statement for the cases $q = 0,\ n+1$.

□

5 Some Established Results

Green proved in [1]

Proposition 5. *For a smooth curve C of genus g and a line bundle $\mathscr{L}$ of degree d on C,*

$K_{p,q}(C, \mathscr{L}) = 0$ *for* $q \geq 3$ *if* $h^1(\mathscr{L}) = 0$, *and*

$K_{p,2}(C, \mathscr{L}) = 0$ *if* $d \geq 2g + 1 + p$

For higher dimensional varieties, the picture is more complicated, but there are still quite a few established results.

For example, let X be an abelian variety of dimension $n \geq 3$, $\mathscr{L}$ an ample line bundle on X, a an integer with $a \geq 2$, and $\mathscr{B}$ a line bundle on X such that $b\mathscr{L} - \mathscr{B}$ is ample for some integer $b \geq 1$. Set $r_a = h^0(a\mathscr{L}) - 1$, and assume $a \geq b$. M. Aprodu and L. Lombardi prove in [12] that

Proposition 6. *$K_{p,1}(X, \mathscr{B}, a\mathscr{L}) = 0$ for p in the range $r_a - a(n-1) + b(1 - \frac{1}{a}) \leq p \leq r_a - n$*

Our two main computational results in this paper have similar flavor to the above two established results. We prove that

Theorem 1 Let X be a general smooth degree $n+2$ hypersurface in $\mathbb{P}^{n+1}$. Then, $K_{p,q}(X, \mathscr{O}_X(d)) = 0$ if $p \leq (q-1)d - 3$.

and

Theorem 2 Let X be a general smooth degree $4a$ hypersurface in $\mathbb{P}^3$ with $a \geq 2$. Then, $K_{p,1}(X, \mathscr{O}_X(d)) = 0$ if $p \geq h^0(d) - 4d + 4$ where $h^0(d) = h^0(\mathscr{O}_X(d))$

6 Notations

Let $D = D_0 \cup ... \cup D_b$ be a normal crossing variety of dimension n sitting inside an ambient smooth projective variety X, where the D_i are the smooth irreducible components of D. Set $D_{i_0 \ldots i_p} = D_{i_0} \cap ... \cap D_{i_p}$ to be the scheme-theoratic intersection in X (in other words, if $\mathscr{I}_i$ is the ideal sheaf of D_i in X, then, $\mathscr{I}_{i_0} + ... + \mathscr{I}_{i_p}$ is the ideal sheaf of $D_{i_0 ... i_p}$).

Let $\mathscr{B}$ and $\mathscr{P}$ be arbitrary line bundles on D. Let $\mathscr{A}$ be an ample line bundle on D. Set $\mathscr{L}_d = \mathscr{P} \otimes d\mathscr{A}$ where $d >> 0$, and set $V = H^0(\mathscr{L}_d)$.

Set $B^p_q = \bigoplus_{i_0 < ... < i_p} H^0((\mathscr{B} \otimes q\mathscr{L}_d)|_{D_{i_0 ... i_p}})$, and $B^p = \bigoplus_{q \geq 0} B^p_q$. Then B^p is a graded $S(\bigoplus_{i_0 < ... < i_p} H^0(\mathscr{L}_d|_{D_{i_0 ... i_p}}))$-module. Letting $V \to \bigoplus_{i_0 < ... < i_p} H^0(\mathscr{L}_d|_{D_{i_0 ... i_p}})$ be the natural map induced by restriction maps to each component $D_{i_0 ... i_p}$, we see B^p is also a graded $S(V)$-module by the action of $S(V)$ induced by this map.

7 Koszul Cohomology of Normal Crossing Varieties

First, we will need to construct a complex of O_D-modules, which look like the following

$$0 \to \mathscr{O}_D \xrightarrow{\rho} C_0 = \bigoplus_{i_0} \mathscr{O}_{D_{i_0}} \xrightarrow{\partial_0} C_1 = \bigoplus_{i_0 < i_1} \mathscr{O}_{D_{i_0 i_1}} \to ... \to C_{b-2} = \bigoplus_{i_0 < ... < i_{b-2}} \mathscr{O}_{D_{i_0 ... i_{b-2}}} \xrightarrow{\partial_{b-2}} C_{b-1} = \mathscr{O}_{D_{1 ... b}} \to 0 \ ... \ (*)$$

where given an open affine $U \subset X$ and $\alpha = (f_{i_0 ... i_p}) \in C_p(U \cap D)$, then $\partial_p(\alpha)_{i_0 ... i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j (f_{i_0 ... \hat{i_j} ... i_{p+1}})|_{U \cap D_{i_0 ... i_{p+1}}}$.

Furthermore, the map ρ is induced by restriction to each component.

Proposition 7. *The complex (*) is exact.*

Proof. We can work over an open affine $U = \mathrm{Spec} A \subset X$, with $I_{i_0 ... i_p} = I_{i_0} + ... + I_{i_p}$ being the ideal cutting out $D_{i_0 ... i_p} \cap U$ (so $\cap I_i$ cuts out D). Let a be a section over

$U \cap D$. $\rho(a) = 0$ means that $a \in I_i$ for all i, which means $a \in \cap I_i$, so $a = 0 \in A/\cap I_i$, and so ρ is injective.

Now let's prove exactness at C_0. Suppose we're given a closed cycle $\alpha = (f_i) \in C_0(U \cap D)$. Then $f_j - f_i = 0$ on D_{ij} for every $i < j$, so $f_2 - f_1 \in I_{12}$, which means we can write $f_2 - f_1 = a_2 - a_1$ where $a_i \in I_i$. Set

$$f^{(2)} = f_2 - a_2 = f_1 - a_1 \in A$$

which lifts both f_1 and f_2 by construction.

By assumption, we have $f^{(2)} - f_3 = 0$ on D_{13} and D_{23}, which means

$$f^{(2)} - f_3 \in (I_1 \cap I_2) + I_3$$

because $(I_1 \cap I_2) + I_3$ is the ideal cutting out $(D_1 \cup D_2) \cap D_3 = D_{13} \cup D_{23}$. This means we can write $f^{(2)} - f_3 = a_{12} - a_3$ where $a_{12} \in I_{12}$ and $a_3 \in I_3$.

Set

$$f^{(3)} = f^{(2)} - a_{12} = f_3 - a_3 \in A$$

which lifts f_1, f_2, and f_3 by construction.

Continuing similarly, there is some $a_{1\dots b} \in I_{1\dots b}$ such that $f^{(b)} = f^{(b-1)} - a_{1\dots b}$ lifts each f_i. Image of this $f^{(b)}$ is $\alpha = (f_i)$, so the complex is exact at C_0.

For C_p with $p \geq 1$, we induct on the dimension of the ambient X. If $\dim X = 0$, there's nothing to show. For the inductive step, we do another induction on b, which is the number of components of D. For $b = 1$, our complex is $0 \to \mathscr{O}_D \to \mathscr{O}_D \to 0$, which is trivially exact. For the inductive step, set $\alpha = (f_{i_0 \dots i_p}) \in C_p$ to be a closed cycle, and write

$$\alpha = \alpha_{\neq 1} \oplus \alpha_1 = (f_{i_0 \dots i_p})_{i_0 > 1} \oplus (f_{1 i_1 \dots i_p})$$

the differential of the complex associated to $D_2 \cup \dots \cup D_b$ acts the same way as does the differential for D on the $\alpha_{\neq 1}$ component, so the induction hypothesis on b gives us some $\beta_{\neq 1} = (g_{i_0 \dots i_{p-1}})_{i_0 > 1} \in \bigoplus_{1 < i_0 < \dots < i_{p-1}} \mathscr{O}_{D_{i_0 \cdots i_{p-1}}}$ such that $\partial(\beta_{\neq 1})_{i_0 \dots i_p} = (\alpha_{\neq 1})_{i_0 \dots i_p}$ where $i_0 > 1$.

Now, set $D'_i = D_{1i}$ for all $i = 2, \dots\ b$ and $D' = \cup D'_i$. Define $D'_{i_1 \dots i_p} = D'_{i_1} \cap \dots \cap D'_{i_p}$ scheme-theoratically.

Consider the cycle

$$(g_{i_1 \dots i_p}|_{D_{1 i_1 \cdots i_p}} - f_{1 i_1 \dots i_p}) \in \bigoplus_{1 < i_1 < \dots < i_p} \mathscr{O}_{D'_{i_1 \cdots i_p}} \ \dots\ (0)$$

as a $(p-1)$-cycle in the complex associated to D'. It is in fact a closed cycle because for each $1 < i_1 < \dots < i_{p+1}$,

$$((g_{i_2 \dots i_{p+1}} - f_{1 i_2 \dots i_{p+1}}) - (g_{i_1 i_3 \dots i_{p+1}} - f_{1 i_1 i_3 \dots i_{p+1}}) + \dots + (-1)^p (g_{i_1 \dots i_p} - f_{1 i_1 \dots i_p}))|_{D_{1 i_1 \cdots i_{p+1}}} =$$
$$\sum_{j=1}^{p+1} (-1)^{j-1} g_{i_1 \dots \hat{i_j} \dots i_{p+1}}|_{D_{1 i_1 \cdots i_{p+1}}} + \sum_{j=1}^{p+1} (-1)^j f_{1 i_1 \dots \hat{i_j} \dots i_{p+1}}|_{D_{1 i_1 \cdots i_{p+1}}} =$$
$$f_{i_1 \dots i_{p+1}}|_{D_{1 i_1 \cdots i_{p+1}}} + \sum_{j=1}^{p+1} (-1)^j f_{1 i_1 \dots \hat{i_j} \dots i_{p+1}}|_{D_{1 i_1 \cdots i_{p+1}}} = 0$$

where the second to last equality of by the construction of $\beta_{\neq 1}$, and the last equality is because α is a closed p-cycle in the complex associated to D.

Notice D' has $b-1$ components and is embedded in an ambient space D_1 which is one dimension less than that of X. Thus by the inductive assumption on the dimension, the closed cycle from (0) is a boundary, so there exists a $(p-2)$-cycle $\beta_1 = (g_{1 i_1 \dots i_{p-1}}) \in \bigoplus_{1 < i_1 < \dots < i_p} \mathscr{O}_{D'_{i_1 \cdots i_{p-1}}}$ such that for each $1 < i_1 < \dots < i_p$, we have that

$$(g_{1i_2\dots i_p} - g_{1i_1i_3\dots i_p} + \dots + (-1)^{p-1}g_{1i_1\dots i_{p-1}})|_{D_{1i_1\dots i_p}} = g_{i_1\dots i_p}|_{D_{1i_1\dots i_p}} - f_{1i_1\dots i_p}$$

which means we can take $\beta = \beta_{\neq 1} \oplus \beta_1 = (g_{i_0\dots i_{p-1}})_{i_0>1} \oplus (g_{1i_1\dots i_{p-1}})$ so that $\partial(\beta) = \alpha$, so α is a boundary, completing the proof.

□

For $d >> 0$ and for any choice of $i_0 < \dots < i_p$, we have by Serre vanishing,

$$H^i((\mathscr{B} \otimes q\mathscr{L}_d)|_{D_{i_0\dots i_p}}) = 0 = H^i(\mathscr{B} \otimes q\mathscr{L}_d) \text{ for } i > 0,\ q > 0 \ \dots\ (1)$$

We also have

$$H^0((\mathscr{B} \otimes q\mathscr{L}_d)|_{D_{i_0\dots i_p}}) = 0 = H^0(\mathscr{B} \otimes q\mathscr{L}_d) \text{ if } q < 0 \ \dots\ (2)$$

Serre vanishing also gives us $H^1(\mathscr{L}_d(-D_{i_0\dots i_p})) = 0$ where $\mathscr{L}_d(-D_{i_0\dots i_p})$ is the coherent sheaf of sections of $\mathscr{L}_d$ which vanish along $D_{i_0\dots i_p}$, which means

$$\text{Restriction map } \phi_{i_0\dots i_p} : V \to H^0(\mathscr{L}_d|_{D_{i_0\dots i_p}}) \text{ is surjective} \ \dots\ (3)$$

Taking global sections of (*) tensored by $\mathscr{B} \otimes q\mathscr{L}_d$, we get a complex $B_q^\bullet$, with

$$H^0(\mathscr{B} \otimes q\mathscr{L}_d) = H^0(B_q^\bullet) \ \dots\ (4)$$

For any $i > 0$, $q > 0$, we see by (1) that $0 \to \mathscr{B} \otimes q\mathscr{L}_d \to (*)$ is an acyclic resolution of $\mathscr{B} \otimes q\mathscr{L}_d$, thus

$$\text{For any } i > 0,\ q > 0,\ H^i(B_q^\bullet) = H^i(\mathscr{B} \otimes q\mathscr{L}_d) = 0 \ \dots\ (5)$$

Fix some $l \in \mathbb{N}$ (we will specify later on in this report what value we need l to be). Set $C^{p,q} = \bigwedge^{l-q} V \otimes B_q^p$ to be the double complex with vertical differentials coming from $(-1)^p$ times the maps for the complex $B_q^\bullet$ and horizontal differentials coming from the Koszul complex maps. We will consider spectral sequences associated to this double complex.

Remark 3. *Before beginning our analysis, let me note that without the asymptotic assumption (i.e. $d >> 0$), we don't have the Serre vanishing results, and without them, there is not enough simplification in the spectral sequences to say anything meaningful that relates Koszul Cohomology groups of the smooth components to Koszul Cohomology groups of the normal crossing variety. Thus, for the rest of this paper, we will assume the Serre vanishing results.*

We get two spectral sequences, $'E$ starting from horizontal differentials and $''E$ starting from vertical differentials, with same abutment. By (4) and (5),

$$''E_2^{0,q} = K_{l-q,q}(D, \mathscr{B}, \mathscr{L}_d) \text{ for and } ''E_2^{p,0} = \bigwedge^l V \otimes H^p(B_0^\bullet) \text{ for } p > 0$$

with zeroes everywhere else on the $''E_2$ page ... (6)

This means for $q \geq 2$, the only non-zero map on the $''E_q$ page is the map ∂_q : $''E_q^{q-1,0} = \bigwedge^l V \otimes H^{q-1}(B_0^\bullet) \to {}''E_q^{0,q} = K_{l-q,q}(D, \mathscr{B}, \mathscr{L}_d)$. Keeping this notation, we get

$''E_\infty^{0,q} = \mathrm{coker}\partial_q$ for $q \geq 2$ and $''E_\infty^{p,0} = \ker\partial_{p+1}$ for $p \geq 1$ with zeroes everywhere else on the $'E_2$ page ... (7)

We also have

$$'E_1^{p,q} = K_{l-q,q}(B^p, V) \text{ ... (8)}$$

Now, let's start calculating the terms in the $'E_1$ page. $K_{l-q,q}(B^p, V)$ is the cohomology at the middle of

$$... \to \bigwedge^{l-q+1} V \otimes \bigoplus_{i_0<...<i_p} H^0((\mathscr{B}\otimes(q-1)\mathscr{L}_d)|_{D_{i_0\cdots i_p}}) \to \bigwedge^{l-q} V \otimes \bigoplus_{i_0<...<i_p} H^0((\mathscr{B}\otimes q\mathscr{L}_d)|_{D_{i_0\cdots i_p}}) \to \bigwedge^{l-q-1} V \otimes \bigoplus_{i_0<...<i_p} H^0((\mathscr{B}\otimes(q+1)\mathscr{L}_d)|_{D_{i_0\cdots i_p}}) \to ...$$

The above complex is a direct sum over all $i_0 < ... < i_p$ of complexes of the form

$$... \to \bigwedge^{l-q+1} V \otimes H^0((\mathscr{B}\otimes(q-1)\mathscr{L}_d)|_{D_{i_0\cdots i_p}}) \to \bigwedge^{l-q} V \otimes H^0((\mathscr{B}\otimes q\mathscr{L}_d)|_{D_{i_0\cdots i_p}}) \to \bigwedge^{l-q-1} V \otimes H^0((\mathscr{B}\otimes(q+1)\mathscr{L}_d)|_{D_{i_0\cdots i_p}}) \to ...$$

By (3), $\bigwedge^{l-q} V$ has a filtration with quotients $\bigwedge^{j} \ker\phi_{i_0...i_p} \otimes \bigwedge^{l-q-j} H^0(\mathscr{L}_d|_{D_{i_0\cdots i_p}})$, as $j = 0, ..., h^0(\mathscr{L}_d) - h^0(\mathscr{L}_d|_{D_{i_0\cdots i_p}})$, which induces a filtration on the above complex with quotients each of which is a tensor product of a fixed vector space $\bigwedge^{j} \ker\phi_{i_0...i_p}$ with

$$... \to \bigwedge^{l-q+1-j} H^0(\mathscr{L}_d|_{D_{i_0\cdots i_p}}) \otimes H^0((\mathscr{B}\otimes(q-1)\mathscr{L}_d)|_{D_{i_0\cdots i_p}}) \to \bigwedge^{l-q-j} H^0(\mathscr{L}_d|_{D_{i_0\cdots i_p}}) \otimes H^0((\mathscr{B}\otimes q\mathscr{L}_d)|_{D_{i_0\cdots i_p}}) \to \bigwedge^{l-q-1-j} H^0(\mathscr{L}_d|_{D_{i_0\cdots i_p}}) \otimes H^0((\mathscr{B}\otimes(q+1)\mathscr{L}_d)|_{D_{i_0\cdots i_p}}) \to ...$$

Note $K_{l-q-j,q}(D_{i_0...i_p}, \mathscr{B}|_{D_{i_0\cdots i_p}}, \mathscr{L}_d|_{D_{i_0\cdots i_p}})$ is the cohomology at the middle of the above Koszul complex, so combining (8) with the above we get

$$'E_1^{p,q} = \bigoplus_{i_0<i_1<...<i_p} \bigoplus_{j=0}^{l-q} \bigwedge^{j} \ker\phi_{i_0...i_p} \otimes K_{l-q-j,q}(D_{i_0...i_p}, \mathscr{B}|_{D_{i_0\cdots i_p}}, \mathscr{L}_d|_{D_{i_0\cdots i_p}}) ...(9)$$

We thus get the following lemma that relates Koszul Cohomology groups of the smooth components to kernels and cokernels of maps involving Koszul Cohomology groups of the normal crossing variety.

Lemma 2. *Fix any integer $l \in \mathbb{N}$. Then, there are two spectral sequences $'E$ and $''E$ with the same abutment with the following properties:*

$$'E_1^{p,q} = \bigoplus_{i_0<i_1<...<i_p} \bigoplus_{j=0}^{l-q} \bigwedge^j \mathrm{ker}\phi_{i_0...i_p} \otimes K_{l-q-j,q}(D_{i_0...i_p}, \mathscr{B}|_{D_{i_0\cdots i_p}}, \mathscr{L}_d|_{D_{i_0\cdots i_p}})$$

$''E_\infty^{0,q} = \mathrm{coker}\partial_q$ and $''E_\infty^{p,0} = \mathrm{ker}\partial_{p+1}$ with zeroes everywhere else on the $'E_2$ page where the map $\partial_q : {''E}_q^{q-1,0} = \bigwedge^l V \otimes H^{q-1}(B_0^\bullet) \to {'E}_q^{0,q} = K_{l-q,q}(D, \mathscr{B}, \mathscr{L}_d)$ is the only non-zero map on the $''E_q$ page.

8 The Main Results

The main results are as follows.

Corollary 1. *Suppose for each choice of q, we're given a number s_q such that*

$$K_{h^0(\mathscr{L}_d|_{D_{i_0\cdots i_p}})-s,q}(D_{i_0...i_p}, \mathscr{B}|_{D_{i_0\cdots i_p}}, \mathscr{L}_d|_{D_{i_0\cdots i_p}}) = 0 \text{ for all } 0 \le s \le s_q \text{ and for any choice of } i_0 < ... < i_p.$$

Then, for any q and l with $0 \le q \le n+1$ and $l - q \ge h^0(\mathscr{L}_d) - s_q$, we get $K_{l-q,q}(D, \mathscr{B}, \mathscr{L}_d) = 0$ if $q = 0$ or 1 and $K_{l-q,q}(D, \mathscr{B}, \mathscr{L}_d) \cong \bigwedge^l V \otimes H^{q-1}(B_0^\bullet)$ if $2 \le q \le n+1$.

Proof. Fix any l with $l - q \ge h^0(\mathscr{L}_d) - s_q$. We then have

$$K_{l-q-j,q}(D_{i_0...i_p}, \mathscr{B}|_{D_{i_0\cdots i_p}}, \mathscr{L}_d|_{D_{i_0\cdots i_p}}) = 0 \text{ for all } j \le h^0(\mathscr{L}_d) - h^0(\mathscr{L}_d|_{D_{i_0\cdots i_p}}),$$

and for $j > h^0(\mathscr{L}_d) - h^0(\mathscr{L}_d|_{D_{i_0\cdots i_p}})$ we have $\bigwedge^j \mathrm{ker}\phi_{i_0...i_p} = 0$, which means by **Lemma 2**

$$''E_1^{p,q} = 0 =''E_\infty^{p,q} \; ... \; (10)$$

(10) tells us that $K_{l-q,q}(D,\mathscr{B},\mathscr{L}_d)=0$ if $q=0$ or 1. By **Lemma 2**, (10) also tells us that

$$\partial_q : \bigwedge^l V \otimes H^{q-1}(B_0^\bullet) \to K_{l-q,q}(D,\mathscr{B},\mathscr{L}_d).$$

is an isomorphism for $q \geq 2$

□

Corollary 2. *In addition to assumptions in **Corollary 1**, suppose we set* $\mathscr{B}=\mathscr{O}_D$ *and assume that* $H^i(\mathscr{O}_{D_{i_0\ldots i_p}})=0$ *for all* $i>0$ *and* $i_0<\ldots<i_p$.

Then, we have

$$K_{l-q,q}(D,\mathscr{B},\mathscr{L}_d) \cong \bigwedge^l V \otimes H^{q-1}(\Delta(D)) \text{ if } 2 \leq q \leq n+1$$

where $\Delta(D)$ *is the simplicial complex constructed using incidence information of* D *(i.e. each* D_{i_0} *is a 0-face, each* $D_{i_0i_1}$ *is a 1-face, each* $D_{i_0i_1i_2}$ *is a 2-face, etc.).*

Proof. The additional assumptions mean that $B_0^\bullet$ gives us an acyclic resolution of $\mathscr{O}_D$. So, in this case, $H^i(B_0^\bullet)=H^i(\mathscr{O}_D)$. Furthermore, by **Remark 5.5** in [4], $H^i(\mathscr{O}_D)\cong H^i(\Delta(D))$. We're done.

□

In other words, under these assupmtions, the behavior at the tail of a row in the Betti table of $(D,\mathscr{L}_d)$ depends only on the combinatorics of how the pieces $D_{i_0\ldots i_p}$ intersect and on the behavior at the tail of a row in the Betti table of each $(D_{i_0\ldots i_p},\mathscr{L}_d|_{D_{i_0\ldots i_p}})$.

Remark 4. *Recall* $n = dimD$*. Set* $D_i = Proj\ \mathbb{C}[x_0, ..., x_{n+1}]/x_{i-1}$ *and* $D = D_1 \cup ... \cup D_{n+2}$*. Set* $\mathscr{B} = \mathscr{O}_D$*. Then,* $\Delta(D) = S^n$*, the n-sphere. We're in the case of* ***Corollary 2****, so* $H^i(B_0^\bullet) = H^i(\mathscr{O}_D) \cong H^i(S^n, \mathbb{C}) = \mathbb{C}$ *if* $i = 0$ *or* n *and is* 0 *for all other values of* i*. Thus, by* ***Corollary 2****,* $K_{l-q,q}(D, \mathscr{L}_d) = 0$ *where* $l \geq h^0(\mathscr{L}_d)+q-s_q$*.*

In fact, we would be able to use the exact same argument for any normal crossing variety D with $H^i(\Delta(D)) = 0$ for any $1 \leq i \leq n-1$ for any $n = \dim D$.

9 Applications

In this section, we use the upper semicontinuity of dimension of Koszul Cohomology groups in flat families with constant cohomology to deduce vanishing statements for asymtotic syzygies of smooth projective varieties. Specifically, we obtain results on syzygies of smooth hypersurfaces of arbitrary dimension and smooth hypersurfaces of general type in $\mathbb{P}^3$.

Calculation 1: Consider $F \subseteq \mathbb{P}^{n+1} \times \mathbb{P}^1 = \text{Proj } \mathbb{C}[x_0, ..., x_{n+1}] \times \text{Proj } \mathbb{C}[y_0, y_1]$, defined by $y_0 f + y_1 g = 0$, where f is a homogeneous degree $n+2$ polynomial cutting out a smooth hypersurface in $\mathbb{P}^{n+1}$, and $g = x_0 x_1 ... x_{n+1}$.

Then, $F \to \mathbb{P}^1$ is a flat family where general fibers F_t for $t \neq 0$ are smooth Calabi-Yau n-folds, and the special fiber F_0 is a union of $n+2$ copies of $\mathbb{P}^n$ intersecting each other in a spherical configuration.

Let's prove vanishing statements on the special fiber F_0. By **Theorem 2.2** in [2], $K_{p,q}(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(-3), \mathscr{O}_{\mathbb{P}^n}(d)) = 0$ if $(q-1)d - 3 \geq p$. Set $h^0(d) = h^0(\mathscr{O}_{\mathbb{P}^n}(d))$. By duality of Koszul Cohomology groups, this means $K_{h^0(d)-n-1-p,n+1-q}(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) = 0$ if $(q-1)d - 3 \geq p$.

Note that Euler characteristic is locally constant for the fibers of a flat family. In addition, since we're assuming $d >> 0$, by Serre Vanishing, the higher cohomologies of all the fibers vanish. Lastly, $\mathbb{P}^1$ is connected, thus the h^0 term is constant for all fibers of $F \to \mathbb{P}^1$. Set h to be this constant term.

Then, by **Remark 4** and **Corollary 2**, we find for any fixed q with $0 \leq n+1-q \leq n+1$ that $K_{h-n-1-p,n+1-q}(F_0, \mathscr{O}_{F_0}(d)) = 0$ if $(q-1)d-3 \geq p$. By upper semicontinuity, this means $K_{h-n-1-p,n+1-q}(F_t, \mathscr{O}_{F_t}(d)) = 0$ if $(q-1)d-3 \geq p$ for a general fiber F_t.

Sinc, the dualizing sheaf K_{F_t} is $\mathscr{O}_{F_t}$, by duality of Koszul Cohomology, we obtain the following result.

Theorem 1. *Let X be a general smooth degree $n+2$ hypersurface in $\mathbb{P}^{n+1}$. Then, $K_{p,q}(X, \mathscr{O}_X(d)) = 0$ if $p \leq (q-1)d-3$.*

Remark 5. *Before this paper, the best result on vanishing of asymptotic syzygies of smooth Calabi-Yau varieties was **Corollary 1.6** in [8], which states that for a smooth Calabi-Yau n-fold X, $K_{p,q}(X, \mathscr{O}_X(d)) = 0$ for all p and q with $p \leq d-n$ and $q \geq 2$. So **Theorem 1** is an improvement on that result in the particular case of X being a smooth hypersurface.*

Calculation 2: First, note that a general quartic $K3$ hypersurface in $\mathbb{P}^3$ has Picard number 1. Fix a positive integer a. Consider $F \subseteq \mathbb{P}^3 \times \mathbb{P}^1 = \text{Proj}\ \mathbb{C}[x_0, x_1, x_2, x_3] \times \text{Proj}\ \mathbb{C}[y_0, y_1]$, defined by $y_0 f + y_1 g = 0$, where f is a homogeneous degree $4a$ polynomial cutting out a smooth hypersurface in $\mathbb{P}^3$, and $g = g_1 g_2 ... g_a$ where g_i are a general homogeneous polynomials of degree 4 each cutting out a smooth hypersurface in $\mathbb{P}^3$ with Picard number 1.

Then, $F \to \mathbb{P}^1$ is a flat family where general fibers F_t for $t \neq 0$ are smooth surfaces of general type of genus $\binom{4a-1}{3}$, and the special fiber $F_0 = S_1 \cup ... \cup S_a$ where each S_i is a smooth quartic $K3$ hypersurface in $\mathbb{P}^3$.

Let's prove vanishing statements on the special fiber F_0. We will use **Theorem 1.3** in [9], which gives us a complete description of vanishing and non-vanishing of syzygies of K3 surfaces.

Let $\mathscr{L}$ be a line bundle on a $K3$ surface S with $\mathscr{L}^2 = 2g - 2$ where g is the genus of any member of $|\mathscr{L}|$. Note $h^0(\mathscr{L}) = g + 1$. By [11], the Clifford index of any irreducible smooth curve $C \in |\mathscr{L}|$ is constant. Call this constant c. Then, **Theorem 1.3** in [9] tells us that $K_{p,1}(S, \mathscr{L}) = 0$ if and only if $p \geq g - c - 1 = h^0(\mathscr{L}) - c - 2$. Assume that Picard number of S is 1.

By sections 1 and 2 in [10], setting H to be a generator of the Picard group of S, we get $c = H \cdot (C - H) - 2$. In our case, $S = S_i$ and $\mathscr{L} = \mathscr{O}_{S_i}(d)$, thus, $c + 2 = 4d - 4$, which means $K_{p,1}(S_i, \mathscr{O}_{S_i}(d)) = 0$ if and only if $p \geq h^0(\mathscr{O}_{S_i}(d)) - 4d + 4$. Applying **Corollary 2** here, we obtain $K_{p,1}(F_0, \mathscr{O}_{F_0}(d)) = 0$ if and only if $p \geq h - 4d + 4$, where as in **Calculation 1**, h is defined to be the constant h^0 term of all the fibers of the flat family $F \to \mathbb{P}^1$. We can apply **Corollary 2** now to get

Theorem 2. *Let X be a general smooth degree $4a$ hypersurface in $\mathbb{P}^3$ with $a \geq 2$. Then, X is a surface of general type with $K_{p,1}(X, \mathscr{O}_X(d)) = 0$ if $p \geq h^0(d) - 4d + 4$ where $h^0(d) = h^0(\mathscr{O}_X(d))$.*

The above result complements the work of F. J. Gallego and B. P. Purnaprajna on the syzygies of surfaces of general type in [8].

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