Title

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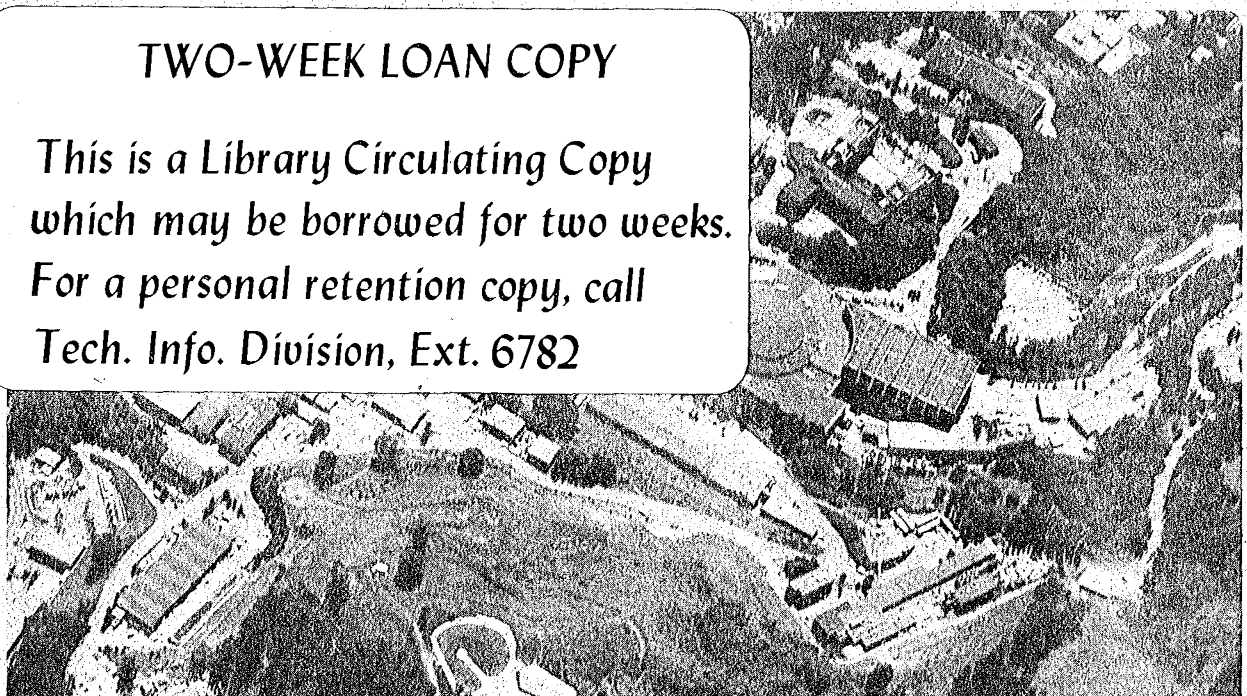

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An Analysis of the Effect
of Operator Splitting and of the Sampling Procedure
on the Accuracy of Glimm's Method

By

Phillip Colella

Abstract

We investigate Glimm's method, a method for constructing approximate solutions to systems of hyperbolic conservation laws in one space variable by sampling explicit wave solutions. It is extended to several space variables by operator splitting. We consider two fundamental problems:

1) We propose a highly accurate form of the sampling procedure, in one space variable, based on the van der Corput sampling sequence. We derive error bounds for Glimm's method, with van der Corput sampling, as applied to the inviscid Burgers' equation: for sufficiently small times, the error in shock locations, speeds, and strengths, is no greater than $O(h^{\frac{1}{2}}|\log h|)$, and the error in the continuous part of the solution, away from shocks, is $O(h|\log h|)$. Here h is the spatial increment of the grid, with the estimates holding in the limit of $h \rightarrow 0$. We test the improved

sampling procedure numerically in the case of inviscid compressible flow in one space dimension and find that it gives high resolution results both in the smooth parts of the solution, as well as at discontinuities.

2) We investigate the operator splitting procedure by means of which the multidimensional method is constructed. An O(1) error stemming from the use of this procedure near shocks oblique to the spatial grid is analyzed numerically in the case of the equations for inviscid compressible flow in two space dimensions, and a method for eliminating this error, by the use of suitable artificial viscosity, is proposed and tested.

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Introduction

The problem which motivates this study is the numerical calculation of time-dependent, discontinuous solutions to compressible fluid flow problems in one or more space variables. There are three criteria which such approximate solutions must simultaneously satisfy.

1) The approximate solution must be reasonably accurate in regions where the flow is smooth. Continuous waves should move at the correct speed, have the correct shape, steepen or spread at the correct rate.

2) Discontinuities which are transported along characteristics should be modelled in the approximate solution by sharp jumps which are transported at the correct speed. Examples of such discontinuities are: contact discontinuities (across which the density and temperature have jump discontinuities while the pressure and velocity remain continuous); the interface between two different materials, or between two different thermodynamic phases of the same material; lines or surfaces across which the solution is continuous, but some derivative of the solution is not.

3) Nonlinear discontinuities should be computed stably and accurately. Such discontinuities occur, for example, when there is mass transfer across the discontinuity, as in the case of shock fronts in an ideal gas.

The main method for computing such solutions has been to solve a set of finite difference equations approximating the differential equations of motion. However, it is difficult to construct difference methods which satisfy all three of the above criteria. For example, it is well-known that a high-order difference method, which would perform well in smooth regions, may generate oscillations in the presence of discontinuities in the solution or its first derivative, or, as discussed in Harten, et. al. [20], even introduce some "unphysical" discontinuity (i.e., one violating the entropy condition; see § 1.1). In spite of these difficulties, many problems have been solved by the use of difference methods over the last thirty years. A good deal of effort has been spent refining them, and they have often been used in conjunction with some specialized technique to adopt them to a specific problem or class of problems in applied physics. For a cross-section of the application of difference methods to a variety of compressible flow problems, see volumes 3 and 4 in the Methods of Computational Physics series [2], as well as the Proceedings of the International Conferences on Numerical Methods in Fluid Dynamics [5], [22], [44], [53] for more recent work.

We will be examining here an alternative approach to computing discontinuous fluid flows, known as Glimm's method. The method was first used by Glimm [15] as part of a constructive existence proof for solutions to systems of nonlinear hyperbolic conservation laws.

It was developed by Chorin [6], [7], into an effective numerical method in the case of gas dynamics. In the first reference Chorin also introduced a multi-dimensional version of the scheme; in the second, he applied the method to reacting gas flow in one space variable. Since that time, the method has been used to compute compressible flow in cylindrical or spherical geometry (Sod [47], [49], [50]), and in applications to some problems in petroleum engineering (Concus and Proskurowski [8], Albright, Concus and Proskurowski [1]).

Although one computes solutions on a grid with Glimm's method, it is not a difference method. Rather than computing a weighted sum to arrive at the value of the solution at a grid point, one samples values from an explicit wave solution. Thus, the method has built into it an approximate form of wave transport and interaction, without the smoothing of such information inherent in averaging. The introduction of such a sampling technique as a numerical method is quite recent, compared to the length of time difference methods have been in use, and has not been subject to the extensive scrutiny and application from which the latter has benefitted. One of the purposes of this study is to indicate some of the features of Glimm's method which might make developing it worth the effort, as well as a few of the directions the development might go.

We consider in this study two fundamental problems.

1) We introduce a more accurate form of the one-dimensional sampling procedure than that used in [6], which uses the van der Corput sampling sequence (see § 2.1), and analyze some simple examples, comparing van der Corput and random sampling. We perform a rigorous error analysis of the approximate solutions obtained using Glimm's method with van der Corput sampling for the inviscid Burgers' equation, and obtain the following result: if the initial data is piecewise C^2 then, for sufficiently small times, the error in the shock location for the approximate solution is bounded by a constant times $h^{1/2}|\log h|$, uniformly in compact time intervals, and the sup norm error in the approximate solution away from discontinuities is bounded by a constant times $h|\log h|$. Here h denotes the spatial increment of the grid, with the estimates holding for $h \to 0$. (For a more precise statement of the results, see Theorem 2.4.) Unlike Glimm's theorem, which is simultaneously a proof of existence and convergence, Theorem 2.4 has as one of its hypotheses the existence of a sufficiently regular solution. However, the information obtained in the latter is more useful from a computational point of view. Finally, we study numerically the dependence of the solution on the sampling sequence in the case of gas dynamics.

2) We investigate the operator splitting procedure by means of which Chorin constructs a multi-dimensional scheme from the one-dimensional method. A source of error stemming from this procedure, not noticed in [6], is analyzed here numerically in the case of gas dynamics. A method for eliminating it is proposed and tested.

For both the one-dimensional and two-dimensional cases we obtain, in the end, results which, for the test problems considered here, are competitive with or superior to those obtained by difference methods, in meeting the three criteria above.

This thesis is divided into four Chapters. In Chapter 1, we give brief introduction to the theory of Hyperbolic Conservation Laws in one space variable. Chapter 2 is devoted to Glimm's method in one space variable: in § 2.1 we define Glimm's method, and the various sampling strategies, and analyze some simple examples; § 2.2 contains the statement and proof of the error bounds for Glimm's method as applied to the inviscid Burgers' equation; and § 2.3 contains some numerical experiments performed using Glimm's method with various sampling strategies for gas dynamics in one space dimension. Chapter 3 contains the discussion of the operator splitting technique. Chapter 4 is devoted to a general discussion of the results, some comparisons of Glimm's method to difference methods, our conclusions, and some suggestions for future work.

Chapter 1 Hyperbolic Systems of Conservation Laws

§ 1.1 Definitions

We wish to consider the initial value problem for hyperbolic conservation laws in one space variable:

$$\begin{aligned} &\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}(F(U)) = 0 \\ &U(x,t) = U : \mathbb{R}^2_+ = \mathbb{R} \times [0,\infty) \longrightarrow \mathbb{R}^N \qquad (1.1.1) \\ &U(x,0) = \varphi(x) \end{aligned}$$

where the flux function $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a C^2 map satisfying the condition that the Jacobian matrix $D_U F = A(U)$ has N real distinct right eigenvalues $\lambda_1(U) < \lambda_2(U) \ldots < \lambda_N(U)$, known as the characteristic velocities. The function $\varphi : \mathbb{R} \to \mathbb{R}^N$ is the given initial data.

Example 1.1 The inviscid Burgers' equation was first studied by Hopf [23] as a model equation for discontinuous fluid flows.

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \qquad u : \mathbb{R}^2_+ \longrightarrow \mathbb{R} \qquad (1.1.2)$$

The single characteristic velocity for 1.1.2 is $\lambda(u) = u$.

Example 1.2 Euler's equations for the one-dimensional motion of an ideal compressible gas are the best-known physical example of a system of hyperbolic conservation laws.

$$\frac{\partial \rho}{\partial t} + \frac{\partial m}{\partial x} = 0 \qquad\qquad U = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix}, \; U : \mathbb{R}^2_+ \to \mathbb{R}^3$$

$$\frac{\partial m}{\partial t} + \frac{\partial}{\partial x}\left(\frac{m^2}{\rho} + p\right) = 0 \tag{1.1.3}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left(\frac{m}{\rho}(E + p)\right) = 0$$

Here ρ is the density, m is the momentum per unit volume, and E is the total energy per unit volume. We can express in terms of these variables the more familiar quantities u the velocity, and ϵ the internal energy per unit mass of the gas: $u = \frac{m}{\rho}$ and $\epsilon = \frac{E}{\rho} - \frac{u^2}{2}$. The pressure p which appears in the equations is a function of ρ, ϵ: $p = (\gamma - 1)\rho\epsilon$ where the constant $\gamma > 1$ is the ratio of specific heats. The relations for ρ, u, p in terms of the conserved quantities ρ, m, E can be inverted: $m = \rho u$, $E = \frac{p}{\gamma - 1} + \frac{\rho u^2}{2}$, so that the state of the gas at a point is uniquely determined by the values of ρ, u, p at that point. Another quantity of interest is the thermodynamic entropy, defined (up to an additive and a multiplicative constant) as $S = \log(p\rho^{-\gamma})$.

The three characteristic velocities for the system are

$\lambda_1(U) = u - c$, $\lambda_2(U) = u$, $\lambda_3(U) = u + c$

where $c = \sqrt{\frac{\gamma p}{\rho}}$ is the sound speed.

It is well-known (see, for example, Courant and Friedrichs [9]) that continuous solutions to the problem (1.1.1) may not exist for all time $t>0$ even if the initial data is C^∞ We admit piecewise smooth φ as initial data, and look for piecewise smooth weak solutions, i.e., ones which satisfy the equations (1.1) in the sense of distribution:

$$\int_{\mathbb{R}^2_+} \frac{\partial\psi}{\partial t} U + \frac{\partial\psi}{\partial x} F(U)\, dx\,dt + \int_{-\infty}^{\infty} \varphi(x)\,\psi(x,0)\,dx = 0 \qquad (1.1.1a)$$

for all $\psi \in C^1(\mathbb{R}^2_+)$.

If we admit this wider class of solutions, there may be more than one solution to (1.1.1) - (1.1.1a) for a given φ A set of additional constraints on the solution, the so-called entropy conditions, were proposed, in varying degrees of generality, by Hopf [23], Lax [26], and Oleinik [42], in the case of a single equation, and by Lax [26] (later extended by Liu [30], [31]) in the case of systems, in order to make (1.1.1) - (1.1.1a) well-posed. This was shown to be the case for a single equation in [23], [26], [42]; for systems of two equations, uniqueness of solutions to (1.1.1) - (1.1.1a) has been proven in the general case by Liu [32], see also [13], [17], [24], [43], [46].

In examples 1.1 and 1.2, the entropy condition reduces to the following conditions, given by Lax [26]. Assume that the set of points in $\mathbb{R}^2_+$ where $U(x,t)$ is discontinuous consists of a collection of piecewise smooth curves $(\ell(t),t)$. We say that such a curve is a shock associated with the characteristic velocity λ_k (or, more simply, a k-shock) if

$$\lambda_{k+1}(U_L(\ell(t),t)) > \lambda_k(U_L(\ell(t),t)) \geq s(t) \qquad (1.1.4)$$
$$\geq \lambda_k(U_R(\ell(t),t)) > \lambda_{k-1}(U_R(\ell(t),t))$$

wherever $s(t) = \frac{d\ell}{dt}$ and the limits

$$\lim_{x \uparrow \ell(t)} U(x,t) = U_L(\ell(t),t) \;,\; \lim_{x \downarrow \ell(t)} U(x,t) = U_R(\ell(t),t)$$

are defined. Lax's condition says that all discontinuties in the solution must be k-shocks, for some $k = 1, \ldots, N$. For gas dynamics, it can be shown (see [9]) that the shock conditions are equivalent to the restriction that the thermodynamic entropy of any particle of fluid is non-decreasing as a function of time.

Example 1.3 Consider the inviscid Burgers' equation (1.1.2) with initial data consisting of a piecewise constant function having a single jump discontinuity at the origin:

$$\varphi(x) \quad \begin{array}{ll} = u_L & x < 0 \\ = u_R & x > 0 \end{array}$$

For $u_L < u_R$, there are at least two weak solutions to (1.1.2), one continuous, one discontinuous (in fact, there are an infinite number).

$$u_c(x,t) \quad \begin{array}{ll} = u_L & x/t < u_L \\ = x/t & u_R \geq x/t \geq u_L \\ = u_R & x/t > u_R \end{array}$$

$$u_D(x,t) \quad \begin{array}{ll} = u_L & x/t < \frac{u_L + u_R}{2} \\ = u_R & x/t > \frac{u_L + u_R}{2} \end{array}$$

However, $u_D(x,t)$ does not satisfy the entropy condition, since $u_L < \frac{u_L+u_R}{2}$, $u_R > \frac{u_L+u_R}{2}$. The only solution which does satisfy the entropy is $u_C(x,t)$ which satisfies the entropy condition trivially, since it has no discontinuities for $t>0$. For $u_L > u_R$, there is no solution continuous for $t>0$; the only discontinuous solution satisfying the entropy condition is

$$u_S(x,t) \begin{array}{ll} = u_L & x/t \leq \frac{u_L+u_R}{2} \\ = u_R & x/t \geq \frac{u_L+u_R}{2} \end{array}$$

§1.2 The Riemann Problem

The simplest piecewise smooth initial data for (1.1.1) - (1.1.1a) is a single jump discontinuity at the origin.

$$\varphi_0(x) \begin{array}{ll} = U_L & x<0 \\ = U_R & x>0 \end{array} \qquad U_L, U_R \in \mathbb{R}^N$$

This problem, known as Riemann's problem, was first considered by Riemann [45], (but solved incorrectly) in the case of gas dynamics. Every piecewise smooth weak solution satisfying the entropy conditions has the following properties:

Self-similarity : $U(x,t) = h(x/t)$, $t>0$ for some piecewise continuous $h : \mathbb{R} \to \mathbb{R}^N$

Finite propagation speed There exist $C_{L,R} = C_{L,R}(U_L, U_R) > 0$ such that

$$U(x,t) \begin{array}{ll} = U_L & x/t < -C_L \\ = U_R & x/t > C_R \end{array}$$

Additivity For any $\xi \in \mathbb{R}$, let U_M be the vector $U(\xi, t\xi) \in \mathbb{R}^N$ Then the function

$$U_1(x,t) \begin{array}{ll} = U(x,t) & \frac{x}{t} \le \xi \\ = U_M & \frac{x}{t} \ge \xi \end{array}$$

is a solution to the Riemann problem with left and right states U_L, U_M . Similarly, the function

$$U_2(x,t) \begin{array}{ll} = U_M & \frac{x}{t} \le \xi \\ = U(x,t) & \frac{x}{t} \ge \xi \end{array}$$

is a solution to the Riemann problem with left and right states U_M, U_R . Geometrically, this says that the solutions fit together to form U (figure 1.1).

Given any choice of the flux function F in (1.1.1) the Riemann problem can be solved if $\|U_L - U_R\|$ is sufficiently small. For some special choices of F the Riemann problem can be solved without restriction on U_L, U_R.

Example 1.4 In Example 1.3, we solved the Riemann problem for in inviscid Burgers' equation: If u_L, u_R are the left and right states, then

for $u_L \le u_R$

$$u(x,t) \begin{cases} = u_L & \frac{x}{t} < u_L \\ = \frac{x}{t} & u_R \geq \frac{x}{t} \geq u_L \\ = u_R & \frac{x}{t} > u_R \end{cases}$$

for $u_L > u_R$

$$u(x,t) \begin{cases} = u_L & \frac{x}{t} > s \\ = u_R & \frac{x}{t} < s \end{cases}$$

where $s = (u_L + u_R)/2$ (figure 1.2).

Example 1.5 The Riemann problem for the system (1.1.3) is discussed extensively in Chorin [6], Courant and Friedrichs [9], Godunov [18], and Sod [48]; the first and last references also give detailed instructions for constructing the solution numerically. We will describe only qualitatively the structure of the solution.

We denote by

$$U_{L,R} = \begin{pmatrix} \rho_{L,R} \\ m_{L,R} \\ E_{L,R} \end{pmatrix} = \begin{pmatrix} \rho_{L,R} \\ \rho_{L,R} u_{L,R} \\ \frac{p_{L,R}}{\gamma - 1} + \frac{\rho_{L,R}}{2} u_{L,R}^2 \end{pmatrix}$$

The special case of the Riemann problem for gas dynamics in which $u_L = u_R = 0$ is often referred to as the shock tube problem. It is named after the experimental situation which it models; that of instantaneously removing at $t = 0$ the partition in a long tube with the gas on the two sides of the partition in different thermodynamical states.

The solution to the Riemann problem for (1.1.3) (figure 1.3) is made up of four regions I, II, III, IV where $U(r,t)$ is constant. These four regions are connected by three waves, each associated with one of the characteristic speeds. These are: a backwards facing hydrodynamic wave (associated with $u - c = \lambda_1(U)$) between $l_{1,b}$ and $l_{2,b}$; a contact discontinuity (associated with $u = \lambda_2(U)$), occuring across the line l_5 ; and a forward facing hydrodynamic wave (associated with $u + c = \lambda_3(U)$) between $l_{1,f}$ and $l_{2,f}$ The pressure and velocity are continuous across the line l_5 so they are equal to some fixed values p^*, u^* in II and III Only the density ρ changes across $l_5(t) = u^* t$, from ρ_L^* to ρ_R^* .

As was discussed in [9], the hydrodynamic waves are uniquely determined by knowing the state of the gas on one side of the wave, and only the pressure on the other. For the backwards facing wave, for example, there are two possibilities. If $p^* > p_L$ then $u^* < u_L$, $\rho_L^* > \rho_L$, $\ell_{1,b} = \ell_{2,b}$ and the wave is a shock associated, in the sense of (1.1.4), with the characteristic velocity $u - c$. If $p^* < p_L$, then we have a backwards facing centered rarefaction wave (see [9]): $\ell_{1,b} \neq \ell_{2,b}$; $p(x,t)$ and $\rho(x,t)$ are continuous strictly monotone decreasing functions of x/t, and $u(x,t)$ a continuous strictly monotone increasing function of x/t, for (x,t) between $\ell_{1,b}$ and $\ell_{2,b}$. The description of the forward facing wave is the same, replacing U_L by U_R, u by $-u$ and $u+c$ by $u-c$.

In figure 1.4 we show the solution at a fixed time to the shock tube problem

$$p_L = 1.0 \qquad p_R = .1$$

$$\rho_L = 1.0 \qquad \rho_R = 0.125$$

$$u_L = 0 \qquad u_R = 0 \qquad \gamma = 1.4$$

The waves which occur are a backward facing rarefaction wave (A), a forward facing shock (B), and a contact discontinuity (C).

Chapter 2 Glimm's Method for One Space Variable

§2.1 Definitions

We want to construct approximate solutions to (1.1.1) - (1.1.1a) which take values on a grid in (x,t) space. Let $h = \Delta x/2$ be a spatial increment, $k = \Delta t$ a time increment. We assume that, at time nk, the approximate solution is constant on intervals of length $2h$:

$$U^{(h)}(x,nk) = U_j^n \in \mathbb{R}^N$$

$$(j-1)h \leq x < (j+1)h \qquad j+n \text{ even}.$$

We wish to compute an approximate solution at time having the same property.

$$U^{(h)}(x,(n+1)k) = U_{j-1}^{n+1}$$

$$(j-2)h \leq x < jh \qquad j+n \text{ even}.$$

(Note the shift in the grid by h at each time step.)

The procedure is as follows:

1) Compute $U^e_{n,j-1}(x,t)$ the exact solution to the Riemann problem with left and right states U^n_{j-2}, U^n_j centered at $((j-1)h, nk)$ (figure 2.1). Assume k/h is sufficiently small; then the waves generated by the adjacent Riemann problems don't intersect and we have an exact solution to (1.1.1) - (1.1.1a) for $nk \leq t \leq (n+1)k$ with initial data $\varphi(x) = U^{(h)}(x,nk)$.

A condition which guarantees this is

$$\frac{k}{h} = \lambda < \Big(\sup_{\substack{i=1,\dots,N \\ x \in \mathbb{R} \\ t \geq nk}} \lambda_i(U^{(h)}(x,t)) \Big)^{-1} \tag{2.1.1}$$

When doing computations, one usually uses the more easily verified

$$\frac{k}{h} = \lambda < \alpha \Big(\sup_{\substack{j,\, j+n \text{ even} \\ i=1,\dots,N}} |\lambda_i(U_j^n)| \Big)^{-1} \tag{2.1.2}$$

where α is a constant, $0 < \alpha < 1$.

2) Pick $a^{n+1} \in [-1,1)$ and take

$$U_{j-1}^{n+1} = U_{j-1,n}^{e}(((j-1)+a^{n+1})h, (n+1)k).$$

(figure 2.2). Notice that the procedure gives an approximate solution for all $(x,t) \in \mathbb{R}_+^2$ which is an exact solution in the strip $nk < t < (n+1)k$. If $U_{j-1,n}^{e}$ is discontinuous at $(((j-1)+a^{n+1})h, (n+1)k)$, then we adopt the convention of setting U_{j-1}^{n+1} equal to the right limit of $U_{j-1,n}^{e}$.

This procedure was first used by Glimm to prove the existence of global (in time) solutions to (1.1.1) - (1.1.1a), assuming the initial data was sufficiently close (in total variation and sup norms) to a constant. He showed that, under these assumptions, for any choice of sampling sequence $\vec{a} = (a^1, a^2, \dots)$ that the set of approximate solutions is precompact in the space of all functions which are in L^1_{loc} in the x-variable, uniformly on compact t-intervals, with respect to the topology given by these conditions. Then he showed that, for almost all $\vec{a}$ (in a suitable measure-theoretic sense), some subsequence, converged in the above topology to a weak solution to (1.1.1) - (1.1.1a). For this proof to hold, the flux function F in (1.1.1) was subject to some technical restrictions (genuine nonlinearity: see Lax [26]).

Since then, several authors ([3], [11], [12], [16], [34], [35], [36], [38], [39], [40]) have extended the range of F for which the estimates leading to precompactness hold, as well as weakening the restrictions on the initial data. For the case of gas dynamics, see Liu [34], [35].

In order to study the dependence of the solutions constructed using Glimm's method on the sampling sequence $\vec{a}$ we introduce the following notation to measure the regularity properties of $\vec{a}$. Let H be a subset of the positive integers, I^g, $g \varepsilon H$ a family of subsets of $[-1,1)$ We denote by $N\{g \varepsilon H, a^g \varepsilon I^g\}$ the number of g contained in H such that a^g is contained in I^g. If I^g is independent of g, we write $N\{g \varepsilon H, a^g \varepsilon I\}$.

N has the following properties: (2.1.3)

i) If $I^g \subset \tilde{I}^g$ for all $g \varepsilon H$, then

$$N\{g \varepsilon H, a^g \varepsilon I^g\} \leq N\{g \varepsilon H, a^g \varepsilon \tilde{I}^g\}.$$

ii) If H_1, H_2 are disjoint, I^g defined for $g \varepsilon H_1 \cup H_2$, then

$$N\{g \varepsilon H_1 \cup H_2, a^g \varepsilon I^g\} =$$

$$N\{g \varepsilon H_1, a^g \varepsilon I^g\} + N\{g \varepsilon H_2, a^g \varepsilon I^g\}$$

iii) If I_1, I_2 are disjoint and independent of q, then
$$N\{q \varepsilon H;\ a^q \varepsilon I_1 \cup I_2\} = N\{q \varepsilon H,\ a^q \varepsilon I_1\} + N\{q \varepsilon H, a^q \varepsilon I_2\}$$

<u>Definition</u> Let $I \subset [-1,1)$ be an interval, $|I|$ = length of I normalized so that $|[-1,1)| = 1$. We define the residual
$$\delta(\vec{a}: n_1, n_2, I) = \frac{1}{n_2 - n_1} N\{q \varepsilon \{n_1+1, \ldots, n_2\},\ a^q \varepsilon I\} - |I|,$$

and we say that $\vec{a}$ is equidistributed if
$$\lim_{n_2 - n_1 \to \infty} \delta(\vec{a}: n_1, n_2, I) = 0$$
for all intervals I, I independent of n_1, n_2.

The following is an immediate consequence of (2.1.3).

<u>Lemma 2.1</u> Let $\vec{a}$ be equidistributed, and let $I^q, q = n_1+1, \ldots, n_2$ be subsets of $[-1,1)$ such that
$$[y+\rho, z-\rho) \subset I^q \subset [y-\rho, z+\rho)$$
for all q, for some y, z, ρ, $z > y + 2\rho$. Then
$$\frac{1}{n_2-n_1} N\{q \varepsilon \{n_1+1, \ldots, n_2\},\ a^q \varepsilon I^q\} - \frac{z-y}{2}\}$$
$$\leq \delta(\vec{a}: n_1, n_2, [y+\rho, z-\rho)) + \delta(\vec{a}: n_1, n_2, [y-\rho, z+\rho)) + \rho$$

A fundamental result, due to Liu [35], is particularly relevant from the point of view of this study. Liu proves that, whenever Glimm's estimates hold, and if $\vec{a}$ is equidistributed, then all the accumulation points of $\{U^{(h)}, h>0\}$, the set of approximate solutions obtained using Glimm's method, are weak solutions to (1.1.1) - (1.1.1a). Thus equidistribution is a sufficient condition on the sampling sequence $\vec{a}$; as we will see below, it is a necessary condition as well.

We will be most concerned with the following two sampling procedures.

Random Sampling Let $(\tilde{A}, d\mu) = \prod_{i=1}^{\infty} ([-1,1), dm)$ be the Cartesian product of an infinite number of copies of the interval $[-1,1)$ each bearing Lebesgue measure dm, normalized so that $m([-1,1)) = 1$, $d\mu$ being the infinite produce measure (Dunford and Schwartz [14]). If $A = \{\vec{a} \in \tilde{A} : \vec{a}$ is equidistributed$\}$ then by the ergodic theorem (Breiman [4]) $\mu(A) = 1$.
A random sampling sequence is some $\vec{a}$ chosen at random, the probability that $\vec{a}$ will be in some subset $B \subset A$ being $\mu(B)$.
A sequence $\hat{a}$ for which the first r elements have statistical properties close to those of $\vec{a}$ can be constructed on a computer by means of a pseudorandom number generator (Hammersley and Handscomb [19], Lehmer [29]). A (k_1, k_2) stratified random sampling is a particular type of random sampling, defined as follows. Let k_1, k_2 be integers, $k_1 > k_2$, k_1, k_2

relatively prime; we construct $\vec{a}'$ from $\vec{a}$ by the rule $a^{i\prime} = 2\rho^i\left(\frac{a^i}{2}+\frac{1}{2}\right)/k_1 - 1$, where ρ^i is defined recursively by $\rho^i \equiv \rho^{i-1} + k_2$, MOD k_1 : ρ_1 is an arbitrary integer $0 \leq \rho_1 < k_1$. For both random and stratified random sampling, it is a consequence of the central limit theorem that $\delta(\vec{a}:0,n,I) = O(n^{-\frac{1}{2}})$ for almost all $\vec{a}$. As is discussed in [19] and in Chorin [7], the stratification of a random sampling sequence reduces $\delta(\vec{a}:0,n,I)$ by a factor $k_1^{-\frac{1}{2}}$:

$$\delta(\vec{a}':0,n,I) = k^{-\frac{1}{2}}\delta(\vec{a}:0,n,I).$$

In the next example we construct sampling sequences $\vec{a}$, $a^i \varepsilon [0,1)$. These are easily turned into sequences taking values in $[-1,1)$ by a simple scaling $a^i \mapsto 2a^i - 1$ which leaves their distribution properties intact.

Quasirandom Sampling The simplest form of this sampling procedure is due to van der Corput (see [19]). Let $\sum_{k=0}^{m} i_k 2^k = n$, $i_k = 0,1$ be the binary expansion of $n = 1,2,\ldots$. Then $a^n = \sum_{k=0}^{m} i_k 2^{-(k+1)}$.

In the following, we will refer to this as van der Corput sampling, to emphasize its nonrandom nature.

The easiest way to see how the sequence is constructed is to write down the first few elements in it:

$$\begin{array}{llll}
1 = 1_2 & a^1 = .5 & = .1_2 \\
2 = 10_2 & a^2 = .25 & = .01_2 \\
3 = 11_2 & a^3 = .75 & = .11_2 \\
4 = 100_2 & a^4 = .125 & = .001_2 \\
5 = 101_2 & a^5 = .625 & = .101_2 \\
6 = 11\ \ _2 & a^6 = .375 & = .0\ 1_2 \\
7 = 111_2 & a^7 = .875 & = .111_2 \\
8 = 1000_2 & a^8 = .0625 & = .0001_2
\end{array}$$

So $a^i \lesseqgtr .5$ if i is $\begin{smallmatrix}\text{even}\\ \text{odd}\end{smallmatrix}$; $\frac{k}{4} \le a^i < \frac{k+1}{4}$ if $i \equiv f(k) \bmod 4$, $k = 0, 1, 2, 3$, where $f(0) = 0$, $f(1) = 2$, $f(2) = 1$, $f(3) = 3$. In general, if one divides the unit interval into the subintervals $[r2^{-s}, (r+1)2^{-s})$, $r = 0, \ldots 2^s - 1$, then for each r there is exactly one g for which $g_0 \le g < g_0 + 2^s$, such that $a^g \in [r2^{-s}, (r+1)2^{-s})$.

We will have need of a variant of this procedure for use in multi-dimensional problems. Let $k_1, k_2 > 0$ be integers, $k_1 > k_2$, k_1, k_2 relatively prime. The (k_1, k_2) van der Corput sampling sequence $\vec{a}$ is given by $a^n = \sum_{\ell=0}^{m} g_\ell (k_1)^{-(\ell+1)}$, where $g_\ell \equiv k_2 i_\ell \bmod k_1$, and $\sum_{\ell=0}^{m} i_\ell k_1^\ell = n$

is the base k_1 expansion of n. Thus the binary van der Corput sampling sequence given above is the special case $k_1 = 2$, $k_2 = 1$. The van der Corput sampling sequences are all equidistributed; the detailed distribution properties of the binary sequence are given in the following lemma.

<u>Definition</u> Let $\sum_{k=0}^{m} i_k 2^k = r$ be the binary expansion of a positive integer r Then $\gamma(r) \equiv \sum_{k=0}^{r} i_k$.

<u>Lemma 2.2</u> Let $\vec{a}$ be the scaled binary van der Corput sequence on $[-1,1)$, I an interval

1) $|a^j - a^{j'}| \geq 2^{-(m-1)}$ for $, j, j' \leq 2^m - 1$ (2.1.4)

2) $N\{g \in \{n_1+1,\ldots,n_2\}, a^g \in I\} - (n_2-n_1)|I|\} \leq 2\,\gamma(n_2-n_1)$ (2.1.5)

3) If $|I| \leq 2^{-\ell}$ and $n_2 - n_1 \leq 2^m$,

then $N\{g \in \{n_1+1,\ldots,n_2\}, a^g \in I\} \leq 2^{m-\ell} + 2$ (2.1.6)

To prove this lemma, we will prove the analogous facts about the original sequence $\vec{a}$ on $[0,1)$. It is trivial to show that the results obtained imply the results given in the lemma for the scaled sequence.

It follows immediately from the fact that the first 2^m-1 sample points are all of the form $\frac{g}{2^m}$, g an integer that

$$|a^g - a^{g'}| \geq \frac{1}{2^m} \quad , g \neq g' \quad , 1 \leq g, g' < 2^m ,$$

from which follows (2.1.4).

To prove (2.1.5), let I be an interval, $I \subset [0,1)$ $|I|_{[0,1)}$ = length of I normalized so that $|[0,1)| = 1$ Let $\sum_{k=0}^{m_0} i_k 2^k = n_2 - n_1$ be the binary expansion of $n_2 - n_1$. We write

$$\left| N\{q \in \{n_1+1,\dots,n_2\}, a^q \in I\} - |I|_{[0,1)}(n_2-n_1) \right|$$
$$\leq \Big| \left(N\{q \in \{n_1+i_0\}, a^q \in I\} - |I|_{[0,1]}\right) i_0$$
$$+ \Big(\sum_{k=0}^{m_0-1} N\{q \in \{n_1 + \sum_{j=0}^{k} i_j 2^j + 1, \dots, \sum_{j=0}^{k+1} i_j 2^j + n_2\}, a^q \in I\}$$
$$- 2^{k+1} |I|_{[0,1)} \Big) i_{k+1} \Big|$$

In every sequence of 2^s consecutive elements of $\vec{a}$ there is exactly one in each interval of the form

$[r2^{-s},(r+1)2^{-s})$ $r=0,\dots,2^s-1$

thus, each summand in the above expression is bounded by 2, and

$$N\{q \in \{n_1+1,\dots,n_2\}, a^q \in I\} - (n_2-n_1)|I|_{[0,1]}\}$$
$$\leq 2 \sum_{k=0}^{m_0} i_k = 2\gamma'(n_2-n_1), \text{ from}$$

which follows (2.1.5).

In the following, we will use (2.1.5) and (2.1.6) in conjunction with (2.1.3), i.e., if I^q, $q \in \{n_1+1,\dots,n_2\}$ satisfy

$$[y+\rho, z-\rho) \subset I^q \subset [y-\rho, z+\rho),$$

then

$$\left| N\{q \in \{n_1+1,\dots,n_2\}, a^q \in I^q\} - \frac{z-y}{2} \right|$$
$$\leq \rho + 2\gamma'(n_2-n_1) \qquad (2.1.7)$$

If $I^q \subset I$, $q \in \{n_1+1,\ldots,n_2\}$, $|I| \leq 2^{-\ell}$, $n_2 - n_1 \leq 2^m$,

then

$$N\{q \in \{n_1+1,\ldots,n_2\}, a^q \in I^q\} \leq 2^{m-\ell+2} \tag{2.1.8}$$

Lax [27] was the first to propose the use of a non-random equidistributed sampling sequence in Glimm's method, the well-equipartitioned sequence of Richtmeyer and Ostrowski. We shall not discuss this sampling sequence here. In numerical experiments, and for simple analytical models, one obtains results using the Richtmeyer-Ostrowski sequence similar to those obtained using van der Corput sampling. However, the techniques used to prove Theorem 2.4 make use of some special properties of the van der Corput sequence which do not hold for the Richtmeyer - Ostrowski sequence.

We now analyze some simple examples.

Example 2.1 Consider the initial value problem for

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \qquad u : \mathbb{R}^2_+ \to \mathbb{R} \tag{2.1.9}$$

$$u(x,0) = \varphi(x)$$

φ piecewise C^1, c a real constant.

The exact solution to this problem is $\varphi(x-ct) = u(x,t)$

In particular, the solution to (2.1.9) in the case

$$\varphi(x) \begin{matrix} = u_L & x<0 \\ = u_R & x>0 \end{matrix} \qquad u_L, u_R \in \mathbb{R}$$

(i.e. the Riemann problem) is

$$u(x,t) \begin{array}{ll} = u_L & \frac{x}{t} < c \\ = u_R & \frac{x}{t} > c \end{array}$$

The discontinuity propagates with velocity c.

Let u_j^n $n \geq 0$, $j+n$ even be the solution to (2.1.9) obtained using Glimm's method. The Courant condition (2.1.1) reduces to $k/h = \lambda \leq \frac{1}{c}$

One easily finds that

$$u_{j+1}^{n+1} \begin{array}{ll} = u_j^n & \text{if} \quad a^{n+1} \varepsilon [-1, \lambda c) \\ = u_{j+2}^n & \text{if} \quad a^{n+1} \varepsilon [\lambda c, 1) \end{array}$$

so that the whole solution is shifted to the left by h if $a^{n+1} < \lambda c$, or to the right by h if $a^{n+1} \geq \lambda c$

If we define

$$\ell(n) = N\{k \varepsilon \{1,\ldots,n\}, a^k \varepsilon [-1, \lambda c)\} - N\{k \varepsilon \{1,\ldots,n\}, a^k \varepsilon [\lambda c, 1)\}$$

then $u_j^n = u_{j-\ell(n)}^0 = \varphi((j - \ell(n))h)$

Thus Glimm's method models the travelling wave nature of the exact solution a discrete wave whose location is determined by the sampling (figure 2.3)

If we look at the limit, for (x,t) fixed, $h \to 0$, $0 < \frac{k}{h} = \lambda \le \frac{1}{c}$

$$n = \left[\frac{t}{k}\right] \qquad j = \left[\frac{x}{2h}\right], \ n \text{ even}$$

$$j = \left[\frac{x}{2h}\right] + 1, \ n \text{ odd}$$

then the error

$$\begin{aligned} u^{(h)}(jh, nk) - u(x,t) &= u_j^n - u(x,t) \\ &= \varphi(jh - \ell(n)h) - \varphi(x - ct) \end{aligned}$$

will be completely determined by $(x - ct) - (jh - \ell(n)h)$ whether the solution is continuous or not.

If $\vec{a}$ is equidistributed, then

$$\begin{aligned} |\ell(n)h - \lambda cnh| \le nh(&\delta(\vec{a}; 0, n, [-1, \lambda c)) \\ &+ \delta(\vec{a}; 0, n, [\lambda c, 1))) \end{aligned}$$

and

$$\begin{aligned} |(x - ct) - (jh - \ell(n)h)| \le \frac{t}{\lambda}(&\delta(\vec{a}; 0, n, [-1, \lambda c)) \\ &+ \delta(\vec{a}; 0, n, [\lambda c, 1))) + (4 + \lambda)h \\ &\to 0 \quad \text{as } h \to 0 \end{aligned}$$

If $\vec{a}$ is a random sequence,

$$|(x - ct) - (jh - \ell(n)h)| \le \frac{C(\vec{a})}{\sqrt{n}} = O(h^{\frac{1}{2}})$$

If $\vec{a}$ is van der Corput,

$$|(x-ct)-(jh-\ell(n)h)| \le C\frac{\gamma(n)}{n} = O(h/|\log h|)$$

Other than the error involved in discretizing the solution at $t=0$, the only error introduced by the sampling is an error in the location of the discretized wave, which is proportional to the residual of the sampling procedure.

Example 2.2 The initial value problem

(2.1.10)

$$\frac{\partial u}{\partial t} + p(x)\frac{\partial u}{\partial x} = 0 \qquad u: \mathbb{R}_+^2 \to \mathbb{R}$$

$$u(x,0) = \varphi(x) \quad \begin{array}{ll} = u_L & x<0 \\ = u_R & x>0 \end{array} \qquad u_L, u_R \in \mathbb{R}$$

$$p(x) \in C^2(\mathbb{R}), \quad \sup_x |p(x)|, |p'(x)|, |p''(x)| \le C_0 < \infty$$

is not a conservation law; nevertheless it can be solved using sampling and error estimates derived.

The exact (distribution) solution to (2.1.10) is given by (Courant and Hilbert [10])

$$u(x,t) \quad \begin{array}{ll} = u_L & x<\ell(t) \\ = u_R & x>\ell(t) \end{array}$$

where $\ell(t)$ satisfies the ordinary differential equation

$$\frac{d\ell}{dt} = p(\ell(t)) \qquad \ell(0) = 0$$

We solve (2.1.10) using Glimm's method, sampling an approximation to the Riemann problem.

For short times,

$$\ell(t) = \ell(t_0) + (t - t_0)\, p(\ell(t_0)) + O((t-t_0)^2)$$

using this approximation to the Riemann problem in Glimm's method, we obtain (figure 2.4),

$$u_j^n \begin{array}{lll} = u_{j-1}^{n-1} & \text{IF} & a^n \varepsilon [-1, \lambda p(jh)) \\ u_{j+1}^{n-1} & \text{IF} & a^n \varepsilon [\lambda p(jh), 1) \end{array}$$

The solution at time step n to (2.1.10), using Glimm's method with the above approximate Riemann solution is

$$u_j^n \begin{array}{lll} = u_L & \text{IF} & j < \ell^{(h)}(n) \\ = u_R & \text{IF} & j > \ell^{(h)}(n) \end{array}$$

where

$$\ell^{(h)}(n) \begin{array}{lll} = \ell^{(h)}(n-1) + 1 & \text{IF} & a^n \varepsilon [-1, \lambda p(\ell^{(h)}(n-1)h)) \\ = \ell^{(h)}(n-1) - 1 & \text{IF} & a^n \varepsilon [-1, \lambda p(\ell^{(h)}(n-1)h)) \end{array}$$

$$\ell^{(h)}(0) = 1$$

The approximate solution converges to the exact solution if $h\ell^{(h)}(n)$, the location of the approximate discontinuity, converges to $\ell(t)$, the exact location of the discontinuity; that is, if in the limit $h \to 0$ with $\frac{k}{h} = \lambda < \frac{1}{C_0}$, $n = \left[\frac{t}{k}\right]$, that $\ell^{(h)}(n)h - \ell(t) \to 0$, uniformly on compact t-intervals.

Let $\alpha \varepsilon (0,1)$ and assume that there exist $C_2 > 0$ such that $\sup_{\substack{j \le C_1 m^\alpha \\ m \ge m_0}} \delta(\vec{a}, jm, (j+1)m, I_{j,m}) \le C_2 m^{-\alpha}$ where the constant C_1 and the sequence of subsets $I_{j,m}$ are to be specified below. Then there exists $C_3 > 0$, independent of h for h sufficiently small, such that

$$|h\ell^{(h)}(n) - \ell(t)| \le C_3 e^{C_0 t} h^{\frac{\alpha}{1+\alpha}}, \qquad (2.1.11)$$

uniformly in compact t-intervals.

The idea here is quite simple. We bound the differences

$$|\ell((n_0+m)k) - \ell(n_0 k) - mkp(\ell(n_0 k))| \le \varepsilon_T(m)$$

$$|\ell^{(h)}(n_0+m)h - \ell^{(h)}(n_0)h - mkp(\ell^{(h)}(n_0)h)| \le \varepsilon_S(m)$$

the first using Taylor's theorem, the second using the assumed bounds on the sampling sequence, choosing m so that $\varepsilon_T(m) \sim \varepsilon_S(m)$. Then the total error in the location of the discontinuity after n time steps is bounded by a constant times $e^{nk} \frac{n}{m} \varepsilon_S(m)$.

Let $m = \left[h^{-\frac{1}{1+\alpha}}\right]$, $n = \left[\frac{t}{k}\right] + 1$, $C_1 = \left[\frac{m^{1+\alpha}}{n}\right] + 1$

Then

$$|\ell^{(h)}(s + jm) - \ell^{(h)}(jm)| \le m$$

$$|p(\ell^{(h)}(s+jm) - p(\ell^{(h)}(jm)h)| \le C_0 mh \le C_4 h^{\frac{\alpha}{1+\alpha}}, \quad 0 < s \le m$$

Take the intervals $I_{j,m}$ to be $I_{j,m} = [-1, \lambda p(\ell^{(h)}(jm)h))$.
By our assumption concerning the sampling sequence, and Lemma 2.1,

$$|\ell^{(h)}((j+1)m)h - \ell^{(h)}(jm)h - \lambda mh\, p(\ell^{(h)}(jm)h)|$$
$$\le C_4 h^{\frac{\alpha}{1+\alpha}} mh + 2m^{1-\alpha}h \le C_5 h^{\frac{2\alpha}{1+\alpha}}$$

By Taylor's theorem, if $0 \le s \le m$

$$|\ell((s+jm)k) - \ell(jmk) - sk\,p(\ell(jmk))| \le \frac{C_0}{2} m^2 k^2 \le C_6 h^{\frac{2\alpha}{1+\alpha}}$$

Thus

$$|\ell^{(h)}((j+1)m)h - \ell((j+1)mk)|$$
$$\le |\ell(jmk) - \ell^{(h)}(jm)h + mk(p(\ell(jmk)) - p(\ell^{(h)}(jm)h))|$$
$$+ (C_5 + C_6) h^{\frac{2\alpha}{1+\alpha}}$$
$$\le (1 + C_0 mk)|\ell(jmk) - \ell^{(h)}(jm)h| + C_7 h^{\frac{2\alpha}{1+\alpha}}$$

If we take $j_0 = [\frac{n}{m}]$, $s_0 = n - j_0 m$,
then

$$|\ell(t) - \ell^{(h)}(n)h| \le |\ell(t) - \ell(nk)|$$
$$+ |\ell(nk) - \ell^{(h)}(n)h - (\ell(j_0mk) - \ell^{(h)}(j_0m)h)|$$
$$+ |\ell(j_0mk) - \ell^{(h)}(j_0m)h|$$

$$
\begin{aligned}
&\le C_0 k + \frac{C_0}{2} m^2 k^2 + mh + \sum_{j=1}^{j_0} (1 + C_0 mk)^j C_7 h^{\frac{2\alpha}{1+\alpha}} \\
&\quad + (1 + C_0 mk)^{j_0} \left(\ell(0) - \ell^{(h)}(0) h \right) \\
&\le C_8 h^{\frac{\alpha}{1+\alpha}} + j_0 e^{C_0 j_0 mk} C_7 h^{\frac{2\alpha}{1+\alpha}} \le C_3 e^{C_0 t} h^{\frac{\alpha}{1+\alpha}}.
\end{aligned}
$$

If $\vec{a}$ is taken to be the van der Corput sampling sequence, then the above hypothesis concerning the sampling sequence holds for all α, $0 < \alpha < 1$ giving a bound in (2.1.11) proportional to $h^{\frac{1}{2}-\varepsilon}$, for ε arbitrarily small. By using special properties of the van der Corput sequence, one can actually obtain a bound proportional to $h^{\frac{1}{2}}$.

Example 2.3 We want to look qualitatively at how Glimm's method models the initial value problem for the inviscid Burgers' equation (1.1.2) for the following special cases.

1) Rarefaction Wave: Let

$$u(x,0) = \varphi(x) \quad \begin{aligned} &= 1 && x \geq 1 \\ &= x && -1 < x < 1 \\ &= -1 && x \leq -1 \end{aligned}$$

The exact solution to this problem is

$$u(x,t) \quad \begin{aligned} &= 1 && x \geq (t+1) \\ &= x/(1+t) && -(t+1) < x < (t+1) \\ &= -1 && x \leq -(t+1) \end{aligned}$$

so that the wave is spreading (figure 2.5).

The initial data for Glimm's method given by $u_j^0 = \varphi(jh)$ j even, satisfies $u_j^0 \leq u_{j+2}^0$, $|u_j^0| \leq 1$ for all so that for the first time step, we may take $k/h = \lambda \leq 1$ Then there are three possibilities. If $a' \varepsilon [-1, -\lambda)$ then the sampling point lies in the left state of all the Riemann problems, which are all rarefaction waves, so the solution is shifted to the right by h: $u_{j+1}^1 = u_j^0$ for all j.

If $a' \varepsilon [\lambda, 1)$ then, by similar reasoning, the solution is shifted to the left by h and $u_{j-1}^1 = u_j^0$. If $a' \varepsilon [-\lambda, \lambda)$, then there is exactly one j_0

such that $\lambda u^0_{j_0} \le a^1 < \lambda u^0_{j_0+2}$. Then $a^1 \ge \lambda u^0_j$ for $j \ge j_0$, $a^1 \ge \lambda u^0_j$ for $j \le j_0$) and we have

$$u^1_{j-1} = u^0_j \qquad j \le j_0$$

$$u^1_{j+1} = u^0_j \qquad j \ge j_0+2$$

$$u^1_{j_0+1} = \frac{a^1}{\lambda} .$$

(figure 2.6). So the sampling procedure models the spreading of the wave by moving part of the approximate solution to the left, part of the approximate solution to the right, and inserting a new value $\frac{a^1}{\lambda}$ in the gap, in such a way that u^1_j still satisfies $u^1_j \le u^1_{j+2}$, $|u^1_j| \le 1$ and therefore, at time step n, $u^n_j \le u^n_{j+2}$, $|u^n_j| \le 1$.

In the exact solution, $\frac{\partial u}{\partial x} = \frac{1}{1+t}$ at all points where $u(x,t) \ne \pm 1$ so, at each time t the amount of spreading is spatially uniform. In order to obtain the same behavior in the approximate solution, one wants to use a sampling sequence such that the $\frac{a^k}{\lambda}$ are distributed as evenly as possible throughout the interval, i.e., that the residual $\delta(\vec{a}, 0, n, I)$ is as small as possible for as many intervals I as possible, uniformly in I. Otherwise, one obtains results as in figure 2.7 where the solution to the problem is computed using random sampling, pieces of the wave spread apart from each other at an uneven rate, leaving flat spots in between. Van der Corput sampling has more desirable distribution properties, as seen in the results in figure 2.8.

2) Compression Wave: Let

$$u(x,0) = \varphi(x) \begin{cases} = -1 & x \geq 1 \\ = -x & -1 < x < 1 \\ = 1 & x \leq -1 \end{cases}$$

Then for $t < 1$

$$u(x,t) \begin{cases} = -1 & x \geq 1-t \\ = x/(t-1) & t-1 < x < 1-t \\ = 1 & x \leq t-1 \end{cases}$$

for $t \geq 1$

$$u(x,t) \begin{cases} = -1 & x > 0 \\ = 1 & x < 0 \end{cases}$$

So $u(x,t)$ consists of a continuous wave, which gets steeper until $t=1$ when it becomes a shock (figure 2.9).

The approximate initial data used in Glimm's method $u_j^0 = \varphi(jh)$, j even, satisfies $u_j^0 \geq u_{j+2}^0$ $|u_j^0| \leq 1$ with $u_j^0 > u_{j+2}^0$ for $j_L^0 \leq j \leq j_R^0$, $u_j^0 = u_{j+2}^0$ otherwise. Those Riemann problems for which $u_j^0 \neq u_{j+2}^0$ have as their solutions shock discontinuities propagating at $s_{j+1}^0 = (u_j^0 + u_{j+2}^0)/2$ so that we may take $k/h = \lambda \leq 1$

As was the case with the rarefaction wave, if

$$a' \in [\lambda s_+, 1) \quad (a' \in [-1, \lambda s_-))$$

the solution is shifted to the left (right) by h. Here

$$s_+ = \frac{1}{2}\left(1 + \max_j \{u_j^0 : u_j^0 \neq 1\}\right)$$

$$s_- = \frac{1}{2}\left(-1 + \min_j \{u_j^0 : u_j^0 \neq -1\}\right)$$

If $a' \in [\lambda s_-, \lambda s_+)$ then there is a j_0 such that

$$u_{j_0-2}^0 > u_{j_0}^0 > u_{j_0+2}^0, \quad a' < s_{j-1}^0 \text{ for } j \leq j_0,$$

$$a' > s_{j+1}^0 \text{ for } j \geq j_0$$

(figure 2.10), so that

$$u_{j+1}^1 = u_j^0 \qquad j < j_0$$

$$u_{j-1}^1 = u_j^0 \qquad j > j_0$$

and the value $u_{j_0}^0$ does not appear in the solution at time step 1. As before, $|u_j^1| \leq 1$, $u_{j-1}^1 \geq u_{j+1}^1$ with $u_{j-1}^1 > u_{j+1}^1$, $j_L^1 \leq j \leq j_R^1$,

$$u_{j-1}^1 = u_{j+1}^1 \qquad \text{otherwise},$$

but with $j_R^1 - j_L^1 = (j_R^0 - j_L^0) - 2$.

The steepening of the wave seen in the exact solution is modeled in Glimm's method by successively removing intermediate values taken on by the wave.

The modeling of a compression wave is rather sensitive to the choice of sampling. For $t < 1$ the exact solution is continuous; so that one wants the approximate solution to also be smooth, i.e., if $nk = 1 - \varepsilon$ one wants

$$u^n_{j-2} - u^n_j = O\left(\frac{h}{\varepsilon}\right)$$

$$(u^n_{j-2} + u^n_j)/2 = u^n_{j-1} + O\left(\frac{h}{\varepsilon}\right) = u^n_{j+2} + O\left(\frac{h}{\varepsilon}\right)$$

If one creates larger jumps between states by fluctuation in the sampling procedure, the effect will be amplified; artificially large jumps lead to incorrect shock speeds

$$(u^n_{j-2} + u^n_j)/2 - u^n_{j\pm 2} >> h \quad \text{for} \quad nk < 1$$

and eventually cause the compression wave to steepen into a shock prematurely. Figure 2.11 shows an example of this having occurred for the above problem. We used random sampling in this run, and the wave has steepened into several strong shocks. Van der Corput sampling was introduced specifically to control this problem. In figure 2.12 we show the problem at the same time as that shown in figure 2.11; the wave is much smoother. In the next section, we prove that, if one uses van der Corput sampling with Glimm's method, then for u^n_{j-2}, u^n_j in a compression wave $|u^n_j - u^n_{j-2}| \leq Ch$ for all n, $nk < T_0$ for T_0 sufficiently small, but independent of h.

§ 2.2 The Inviscid Burgers' Equation

In this section, we will analyze and derive error estimates for Glimm's method, as applied to the initial value problem for the inviscid Burgers' equation.

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \qquad u(x,t) = u : \mathbb{R} \times [0, T_0] \longrightarrow \mathbb{R} \tag{2.2.1}$$

$$u(x,0) = \varphi(x)$$

We will make some simplifying assumptions on the initial data. We require φ to be C^2 except at a finite number of points; at those points where φ fails to be C^2, $\varphi, \varphi', \varphi''$ may have jump discontinuities, so that $\|\varphi\|_\infty, \|\varphi'\|_\infty, \|\varphi''\|_\infty < \infty$ where $\|f\|_\infty = \sup_{x:\, f \text{ is continuous at } x} |f(x)|$. We will also make a rather serious restriction on T_0. Let $C_{INF} = -\inf_x \varphi'(x)$; then we will require $T_0 < T_{CRIT}$ where T_{CRIT} is the largest time satisfying the condition $1 - T_{CRIT} C_{INF} \geq 0$. This restricts us to times sufficiently small such that no compression wave has a chance to steepen into a shock: the only discontinuities in the problem are those present in the initial data. This is a restriction imposed by the limitations of our error analysis: the problem (2.2.1) is known to be well posed, and Glimm's method known to converge for it, without restrictions on T_0.

Hopf [23] first studied discontinuous solutions to (2.2.1): his results were extended and generalized by Lax [26] and Oleinik [42]. In the following, we will state without proof some basic facts about the solutions to 2.2.1; for further details, see Hopf [23]. The lectures by Lax [28] give a good general introduction to the subject of discontinuous solutions of a single quasilinear hyperbolic conservation law.

In the following, we will denote by C_i, K_i constants which may depend on T_0, the initial data φ and the parameter $\lambda = k/h$ in Glimm's method, but are independent of the spatial increment h and the grid locations (jh, nk) $nk \le T_0$. The C_i's will denote constants whose values remain fixed throughout this section; the K_i's will be fixed during the proof of any given Lemma, but may be reused in each Lemma.

Weak solutions to 2.2.1 which are piecewise continuous and satisfy the entropy condition exist, are unique and have a finite number of shocks, i.e., the set of all points in $\mathbb{R} \times (0, T_0]$ where u fails to be continuous can be represented as the union of finite number of continuous connected curves

$$(\ell(t), t) \in \mathbb{R} \times (0, T_0] \qquad \ell : [0, T_0] \longrightarrow \mathbb{R}$$

at all points of which u has a jump discontinuity. We call such curves shocks. The solutions are also uniformly bounded:

$$\sup_{(x,t) \in \mathbb{R} \times [0,T_0]} |u(x,t)| \le \|\varphi\|_\infty . \tag{2.2.2}$$

Let (x_0, t_0) be a point in $\mathbb{R} \times [0, T_0]$ where is continuous. We define a characteristic through (x_0, t_0) to be the line segment in $\mathbb{R} \times [0, T_0]$

$$(c(u(x_0,t_0), x_0, t_0; t), t)$$

$$c(u(x_0,t_0), x_0, t_0; t) = x_0 - u(x_0,t_0)(t_0 - t)$$

such that $0 \le t \le t_0$, or $t > t_0$ and for all t', $t_0 \le t' \le t$, u is continuous at $(c(u(x_0,t_0),x_0,t_0;t'),t')$.

Characteristics have the following properties.

1) There is exactly one through each point where u is continuous.

2) $u(x_0,t_0) = u(c(u(x_0,t_0),x_0,t_0;t),t)$

for all $t > 0$ where the characteristic is defined. If u is continuous at $(c(u(x_0,t_0),x_0,t_0;0),0)$, then u satisfies the functional equation

$$u(x_0,t_0) = \varphi(c(u(x_0,t_0),x_0,t_0;0)) = \varphi(x_0 - u(x_0,t_0)t_0) .$$

3) If φ is not continuous at $c(u(x_0,t_0),x_0,t_0;0) = x_1$, then $\lim_{x \uparrow x_1} \varphi(x) \equiv \varphi_L(x_1) < \varphi_R(x_1) \equiv \lim_{x \downarrow x_1} \varphi(x)$.

We call x_1 a rarefaction center. Conversely, if x_1 is a rarefaction center, then for every v, $\varphi_L(x_1) \le v \le \varphi_R(x_1)$ there is a characteristic passing through $(x_1,0)$ which we denote by $(c(v,x_1,0;t),t)$ where $c(v,x_1,0;t) = x_1 + tv$ defined so long as u is continuous at $(c(v,x_1,0;t'),t')$ for all $t' \le t$ and satisfying $u(c(v,x_1,0;t),t) = v$ for all $t > 0$. We can invert the relation $x = x_1 + u(x,t)t$ to find $u(x,t)$ explicitly as a function of (x,t):

$$u(x,t) = (x - x_1)/t$$

Let $\ell(t)$ be a shock. At all but a finite number of times $t_i, i=1,\ldots,N$, the limits

$$\lim_{\substack{(x,t)\to(\ell(t_0),t_0)\\ x<\ell(t)}} u(x,t) \equiv u_L(\ell(t),t)$$

$$\lim_{\substack{(x,t)\to(\ell(t_0),t_0)\\ x>\ell(t)}} u(x,t) \equiv u_R(\ell(t),t)$$

exist, and

$$\left.\frac{d\ell}{dt}\right|_{t_0} = s(\ell(t_0),t_0) = \tfrac{1}{2}\left(u_L(\ell(t_0),t_0) + u_R(\ell(t_0),t_0)\right)$$

with $s(\ell(t_0),t_0)$ satisfying the entropy condition

$$u_L(\ell(t_0),t_0) > s(\ell(t_0),t_0) > u_R(\ell(t_0),t_0) \qquad (2.2.3)$$

A useful parameter is the shock strength

$$str(\ell(t_0),t_0) = \tfrac{1}{2}(u_L(\ell(t_0),t_0) - u_R(\ell(t_0),t_0))$$

At times t_i, $i=1,\ldots,N$, ℓ is overtaken by one or more shocks, i.e., there exists $\ell_1(t),\ldots,\ell_M(t)$, $\ell(t)=\ell_k(t)$ for some k, for which $\ell_1(t)<\ldots<\ell_M(t)$ for $t<t_i$ and $\lim_{t\uparrow t_i} \ell_j(t) = x_i$ independent of j. In that case, $\ell(t) = \ell_j(t)$ for all $j=1,\ldots,M$, $t>t_i$. No shock ever disappears, although the number of non-coincident shocks decreases by at least one at each time t_i.

The following limits also exist

$$\lim_{t \uparrow t_i} u_{L,R}(\ell_j(t),t) \equiv u^-_{L,R}(\ell_j(t),t)$$

$$\lim_{t \downarrow t_i} u_{L,R}(\ell_j(t),t) = u^+_{L,R}(\ell_j(t),t) = u^+_{L,R}(\ell_k(t),t)$$

for all j,k $\quad 1 \le j,k \le M$ with the entropy conditions (2.2.3) continuing to hold in the limit:

$$u^\pm_L(\ell_j(t_i),t_i) > s^\pm(\ell_j(t_i),t_i) > u^\pm_R(\ell_j(t_i),t_i)$$

$$s^\pm(\ell_j(t_i),t_i) = 1/2\,\left(u^\pm_L(\ell_j(t_i),t_i) + u^\pm_R(\ell_j(t_i),t_i)\right)$$

(2.2.4)

We also have

$$u^+_L(\ell_1(t_i),t_i) = u^-_L(\ell_1(t_i),t_i) = u_L(\ell_1(t_i),t_i)$$

$$u^+_R(\ell_M(t_i),t_i) = u^-_R(\ell_M(t_i),t_i) = u_R(\ell_M(t_i),t_i)$$

(2.2.5)

Given a shock $\ell(t)$, $u_{L,R}(\ell(t),t)$ is a continuous function of t except at times when it is overtaken by another shock, when $u_{L,R}(\ell(t),t)$ may have a jump discontinuity.

By (2.2.2), $\sup_t |s(\ell(t),t)| \le \|\varphi\|_\infty$

so $\ell(t)$ is a piecewise C^1, uniformly Lipschitz function of t:

$|\ell(t)-\ell(t')| \le \|\varphi\|_\infty |t-t'|$ for all $t,t' \le T_0$

The strength of a shock $str(\ell(t),t)$ is a positive continuous function of t except at the times when the shock is overtaken by another shock. At those times it has a jump discontinuity, which, by (2.2.4) and (2.2.5), increases the strength of the shock. Therefore, there exists $\Delta > 0$ such that, for all shocks $(\ell(t),t)$ $0 \le t \le T_0$

$$str(\ell(t),t) \ge \Delta \tag{2.2.6}$$

Let $\ell(t_0)$ be a shock. Define, for $0 \le t \le t_0$

$$c(u_{L,R}^{(+,-)}(\ell(t_0),t_0),\ell(t_0),t_0;t) = \ell(t_0) + (t-t_0)\, u_{L,R}^{(+,-)}(\ell(t_0),t_0)$$

We call the line segment

$$(c(u_{L,R}^{(+,-)}(\ell(t_0),t_0),\ell(t_0),t_0;t),t) \quad 0 \le t \le t_0$$

a backwards characteristic from $(\ell(t_0),t_0)$

For $t \ne t_0$ it coincides with some characteristic of the continuous solution, and satisfies the relation

$$u_{L,R}^{(+,-)}(\ell(t_0),t_0) = u(c(u_{L,R}^{(+,-)}(\ell(t_0),t_0),\ell(t_0),t_0;t),t)$$

If C_0 is the minimum distance between discontinuities in the initial data i.e.

$$C_0 = \min\{|x_i - x_j| : \varphi \text{ is not continuous at } x_i, x_j\}$$

and if $c(u_{L,R}^{(+,-)}(\ell(t_0), t_0), \ell(t_0), t_0; 0)$

is a rarefaction center, then

$$C_0 \leq t_0 \sup_{0 \leq t' \leq t_0} \left| \frac{d}{dt}(\ell(t) - c(u_{L,R}^{(+,-)}(\ell(t_0), t_0), \ell(t_0), t_0; t))\Big|_{t=t'} \right|$$

$$\leq t_0 2 \|\varphi\|_\infty$$

More generally, if $u(x_0, t_0)$ is a point in a centered rarefaction wave satisfying $|x_0 - \ell(t_0)| \leq \eta$

then

$$C_0 - \eta \leq 2 t_0 \|\varphi\|_\infty \tag{2.2.7}$$

We can represent graphically, in the (x,t) plane a typical solution to (2.2.1) (figure 2.13). The bold lines represent shocks, the light lines characteristics. There is a rarefaction center at (A). The two shocks (B) overtake one another at point (C) to form the shock (D).

We can now derive some smoothness results, given the structure of the solution as described above. If u is continuous at (x,t) and φ continuous at $c(u(x,t), x, t; 0)$, then u has the same smoothness properties at (x,t) as φ does at $c(u(x,t), x, t; 0)$. For example, assume φ is C^1 at $c(u(x,t), x, t; 0)$

Then

$$\frac{\partial u}{\partial x} = \lim_{\delta \to 0} \frac{u(x+\delta,t) - u(x,t)}{\delta} \tag{2.2.8}$$

$$= \lim_{\Delta c \to 0} \frac{\Delta\varphi}{\Delta c} \frac{1}{1 + t\frac{\Delta\varphi}{\Delta c}} = \varphi'/(1+t\varphi')\Big|_{c(u(x,t),x,t;0)}$$

$$\Delta c = c(u(x+\delta,t),x,t;0) - c(u(x,t),x,t;0)$$

$$\Delta\varphi = \varphi(c(u(x+\delta,t),x,t;0)) - \varphi(c(u(x,t),x,t;0))$$

A similar calculation yields

$$\frac{\partial u}{\partial t} = \varphi\varphi'/(1+t\varphi')\Big|_{c(u(x,t),x,t;0)} ;$$

and in the case where φ' has a jump discontinuity at $c(u(x,t),x,t;0)$,

$$\lim_{\delta \downarrow 0} \frac{u(x+\delta,t) - u(x,t)}{\delta} = \lim_{\Delta c \downarrow 0} \frac{\Delta\varphi}{\Delta c} \frac{1}{1+t\frac{\Delta\varphi}{\Delta c}}$$

$$= \lim_{x_0 \downarrow c(u(x,t),x,t;0)} \varphi'/(1+t\varphi')\Big|_{x_0}$$

The same results as obtained by formally differentiating the relation $u(x,t) = \varphi(x - tu(x,t))$ and solving for $\frac{\partial u}{\partial x}$:

$$\frac{\partial u}{\partial x} = \varphi' - t\frac{\partial u}{\partial x}\varphi' \Rightarrow \frac{\partial u}{\partial x} = \varphi'/(1+t\varphi')$$

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} = \varphi\varphi'/(1+t\varphi')$$

Hereafter, when deriving such formulas, we will do so by such a formal manipulation, omitting the calculus proof.

More generally, if φ is a continuous, piecewise C^k function in the interval $[c_1, c_2]$ having only a finite number of jump discontinuities in its derivatives, and u is continuous in the region

$\{(x,t) : c_1 + t\varphi(c_1) \le x \le c_2 + t\varphi(c_2), t \le t_0\}$

then u is also a piecewise C^k function of (x,t) in that region. If φ is C^k at $c(u(x,t), x, t; 0)$ then

$$\frac{\partial^{r+s} u}{\partial x^r \partial t^s} = Q_{r,s}(t, \varphi, \varphi', \ldots, \varphi^{(r+s)}) \Big/ (1 + t\varphi')^{2(r+s)-1} \Big|_{c(u(x,t),x,t;0)} \tag{2.2.9}$$

for all $r+s \le k$ where $Q_{r,s}$ is a polynomial in $r+s+2$ variables. If $\varphi^{(r+s)}$ has a jump discontinuity at $c(u(x,t), x, t; 0)$, then $\frac{\partial^{r+s} u}{\partial x^r \partial t^s}$ has a jump discontinuity at (x,t) with

$$\lim_{\delta \downarrow 0} \frac{\partial^{r+s} u}{\partial x^r \partial t^s}\Big|_{(x\pm\delta,t)} \tag{2.2.10}$$

$$= \lim_{\delta \downarrow 0} Q_{r,s}(t, \varphi, \ldots, \varphi^{(r+s)}) \Big/ (1 + t\varphi')^{2(r+s)-1} \Big|_{c(u(x,t),x,t;0)\pm\delta}$$

If the interval $[c_1, c_2]$ contains a rarefaction center at c_0, $c_1 \le c_0 \le c_2$ and u is continuous in the region

$\{(x,t) : (x,t) \ne (c_0, 0), c_1 + \varphi_L(c_1)t \le x \le c_2 + \varphi_R(c_2)t, t \le t_0\}$

then, for $\varphi_L(c_0) < v < \varphi_R(c_0)$, u is a C^∞ function at $(c(v,c_0,0;t),t)$, since $u(x',t') = \frac{x'-c_0}{t'}$ for all (x',t') near $(c(v,c_0,0;t),t)$. If $v = \varphi_L(c_0), \varphi_R(c_0)$, then $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$ have jump discontinuities at $(c(v,c_0,0;t),t)$ since, for example, the formula (2.2.9) for

$$\lim_{\delta \downarrow 0} \frac{\partial u}{\partial x}\bigg|_{(c(\varphi_R(c_0),c_0,0;t)+\delta,t)}$$

cannot be expressed as

$$\lim_{\delta \downarrow 0} \frac{\partial u}{\partial x}\bigg|_{(c(\varphi_R(c_0),c_0,0;t)-\delta,t)} = \frac{1}{t}$$

for any finite value of

$$\lim_{\delta \downarrow 0} \varphi'(c_0+\delta)$$

At a shock, we denote by $\left(\frac{\partial^{r+s}u}{\partial x^r \partial t^s}\right)_L$, $\left(\frac{\partial^{r+s}}{\partial x^r \partial t^s}\right)_R$ the limits (if they are defined)

$$\lim_{\substack{(x,t)\to(\ell(t_0),t_0)\\ x<\ell(t)}} \frac{\partial^{r+s}u}{\partial x^r \partial t^s}\bigg|_{(x,t)} \equiv \left(\frac{\partial^{r+s}u}{\partial x^r \partial t^s}\right)_L (\ell(t_0),t_0)$$

$$\lim_{\substack{(x,t)\to(\ell(t_0),t_0)\\ x>\ell(t)}} \frac{\partial^{r+s}u}{\partial x^r \partial t^s}\bigg|_{(x,t)} \equiv \left(\frac{\partial^{r+s}u}{\partial x^r \partial t^s}\right)_R (\ell(t_0),t_0)$$

$\left(\frac{\partial^{r+s} u}{\partial x^r \partial t^s}\right)_{L,R}(l(t_0), t_0)$ exists if $u_{L,R}(l(t_0), t_0)$ exists, and if either φ is C^{r+s} at $c(u_{L,R}(l(t_0), t_0), l(t_0), t_0; 0)$ or $c(u_{L,R}(l(t_0), t_0), l(t_0), t_0; 0)$ is a rarefaction center x_0 with

$$u_{L,R}(l(t_0), t_0) \neq \varphi_L(x_0)$$
$$u_{L,R}(l(t_0), t_0) \neq \varphi_R(x_0)$$

In the former case

$$\left(\frac{\partial^{r+s} u}{\partial x^r \partial t^s}\right)_{L,R}(l(t_0), t_0) = Q_{r,s}(t, \varphi, \ldots, \varphi^{(r+s)}) \Big/ (1 + t\varphi')^{2(r+s)-1} \Big|_{c(u_{L,R}(l(t_0), t_0), l(t_0), t_0; 0)} \tag{2.2.11}$$

where $Q_{r,s}$ is as in (2.2.8) and in the latter case

$$\left(\frac{\partial^{r+s} u}{\partial x^r \partial t^s}\right)_{L,R}(l(t_0), t_0) = \frac{\partial^{r+s}}{\partial x^r \partial t^s}\left(\frac{x - x_0}{t}\right)\Big|_{(l(t_0), t_0)}$$

If $\varphi, \ldots, \varphi^{(r+s)}$ are uniformly bounded, then, by (2.2.11), and our restriction on T_0

$$\left(\frac{\partial^{r+s} u}{\partial x^r \partial t^s}\right)_{L,R}(l(t_0), t_0)$$

is bounded independent of $t_0 \leq T_0$.

If φ is piecewise C^{r+s} with only a finite number of jump discontinuities in then there are at most a finite number of times t_0 when

$$\left(\frac{\partial^{r+s} u}{\partial x^r \partial t^s}\right)_{L,R}(l(t_0), t_0)$$

can fail to exist: when $l(t_0)$ is overtaken by another shock;

when φ is continuous at $c(u_{L,R}(\ell(t_0),t_0),\ell(t_0),t_0;0)$ but $\varphi^{(r+s)}$ isn't; and when $c(u_{L,R}(\ell(t_0),t_0),\ell(t_0),t_0;0)$ is a rarefaction center x_0 with $u_{L,R}(\ell(t_0),t_0) = \varphi_L(x_0)$ or $u_{L,R}(\ell(t_0),t_0)=\varphi_R(x_0)$. The total number of times this can occur is finite, since a characteristic intersects a shock at most once. At such times the limits

$$\left(\frac{\partial^{r+s}u}{\partial x^r \partial t^s}\right)^{\pm}_{L,R} = \lim_{\delta\downarrow 0} \left(\frac{\partial^{r+s}u}{\partial x^r\partial t^s}\right)_{L,R}(\ell(t\pm\delta),t\pm\delta)$$

exist, and can be computed in a similar fashion as the limits (2.2.10) were.

For the initial data we are considering, the above discussion implies that the solution $u(x,t)$ is piecewise with

$$\sup_{(x,t)\in D}\left|\frac{\partial^{r+s}u}{\partial x^r\partial t^s}\bigg|_{(x,t)}\right| \le C = C(D) < \infty,\quad 0\le r+s\le 2$$

for any compact subset $D\subset \mathbb{R}\times[0,T_0]$ such that D does not contain a rarefaction center. The set where u fails to be C^2 can be represented as the union of finite number of continuous curves $(d(t),t)$, $0\le t\le T_0$ which are either characteristics or shocks.

Finally, we need some knowledge of the smoothness properties of $\ell(t)$, $u_{L,R}(\ell(t),t)$ as functions of t. We already know that ℓ is a piecewise C^1 function of t; $u_{L,R}(\ell(t),t)$ piecewise continuous, with only a finite number of jump discontinuities. If $\ell(t)$ is C^1 and

$u_{L,R}(\ell(t),t)$, $\left(\frac{\partial u}{\partial x}\right)_{L,R}(\ell(t),t)$

continuous near t_0 , then

(2.2.12)

$$\frac{d}{dt}\left(u_{L,R}(\ell(t),t)\right)\Big|_{t_0} = \left(\frac{\partial u}{\partial x}\right)_{L,R}\frac{d\ell}{dt} + \left(\frac{\partial u}{\partial t}\right)_{L,R}$$

$$= \left(\frac{\partial u}{\partial x}\right)_{L,R}(s - u_{L,R})\Big|_{(\ell(t_0),t_0)}$$

So

$$\frac{d^2\ell}{dt^2}\Big|_{t_0} = \frac{ds}{dt} = \frac{d}{dt}\left(\frac{u_L+u_R}{2}\right)$$

$$= \left(\left(\frac{\partial u}{\partial x}\right)_R - \left(\frac{\partial u}{\partial x}\right)_L\right)\left(\frac{u_L-u_R}{2}\right)\Big|_{(\ell(t_0),t_0)}$$

Similarly, if $\left(\frac{\partial^2 u}{\partial x^2}\right)_{L,R}, \left(\frac{\partial u}{\partial x}\right)_{L,R}, u_{L,R}$

are continuous functions of t near t_0

$$\frac{d^2}{dt^2}(u_{L,R})\Big|_{(\ell(t_0),t_0)}$$

$$= P_{L,R}\left(\left(\frac{\partial^2 u}{\partial x^2}\right)_L, \left(\frac{\partial^2 u}{\partial x^2}\right)_R, \left(\frac{\partial u}{\partial x}\right)_L, \left(\frac{\partial u}{\partial x}\right)_R, u_L, u_R\right)\Big|_{(\ell(t_0),t_0)}$$

where $P_{L,R}$ is a polynomial.

Lemma 2.3 We can choose constants $C_1 \geq 1$, $0 < C_2 \leq C_1$ independent of $t_1, t_2 \leq T_0$ such that the following holds:

1) $|\ell(t_1) - \ell(t_2)| \leq C_1 |t_1 - t_2|$

If $u_{L,R}(\ell(t), t)$ is a continuous function of $t \in [t_1, t_2]$ then

$$|u_{L,R}(\ell(t), t) - u_{L,R}(\ell(t'), t')| \leq C_1 |t - t'| \tag{2.2.13}$$

for all $t, t' \in [t_1, t_2]$

If $u_L, u_R, \left(\frac{\partial u}{\partial x}\right)_L, \left(\frac{\partial u}{\partial x}\right)_R$ are continuous functions of $t \in [t_1, t_2]$ then

$$|\ell(t') - \ell(t) - s(\ell(t), t)(t' - t)| \leq C_1 (t - t')^2 \tag{2.2.14}$$

$$\Big| u_{L,R}(\ell(t'), t') - u_{L,R}(\ell(t), t) - \left(\frac{\partial u}{\partial x}\right)_{L,R} (\ell(t), t) \, (s(\ell(t), t) - u_{L,R}(\ell(t), t))(t - t') \Big| \leq C_1 (t - t')^2$$

2) Let u be continuous at (x_0, t_0) with

$$\min_{\ell} |x_0 - \ell(t_0)| \leq \max\left\{ \frac{C_0}{2},\ \Delta \Big/ \left(2 \max\left\{ \|\varphi'\| / (1 - T_0 C_{INF}),\ \frac{4\|\varphi\|_\infty}{C_0} \right\}\right) \right\}$$

then there exists a shock ℓ and time $t > t_0$ such that

$$c(u_{L,R}^{(\cdot,\cdot)}(\ell(t),t), \ell(t), t; t_0) = x_0$$

with

$$|t - t_0| \leq C_2 \min_{\ell} |x_0 - \ell(t_0)| \qquad (2.2.15)$$

The bounds in part 1 of the Lemma follow from Taylor's Theorem, (2.2.12) and (2.2.10).

To prove part 2, assume for example, that there is a shock ℓ, $\ell(t_0) \geq x_0$ such that

$$\ell(t_0) - x_0 = \eta_0 = \min_{\tilde{\ell}} |\tilde{\ell}(t_0) - x_0|$$

and that, for $0 < (t - t_0) < C_2 \eta_0$, $c(u(x_0, t_0), x_0, t_0; t)$ is defined and does not intersect a shock. Then it suffices to show that C_2 is bounded above independent of η_0. We also assume that $s(\ell(t), t)$ is a C^1 function of t for $0 \leq t - t_0 \leq C_2 \eta$. The extension to the case when this fails at a finite number of times is straightforward.

If $x_0 < x < \ell(t_0)$ and $u(x, t_0)$ is in a rarefaction fan centered at $(x_1, 0)$ then, by (2.2.7), $t_0 \geq 4 \dfrac{\|\varphi\|}{C_0}$

So for all (x, t), $c(u(x_0, t_0), x_0, t_0; t) \leq x \leq \ell(t)$, $t \geq t_0$ we have

$$\left|\frac{\partial u}{\partial x}\Big|_{(x,t)}\right| \le \mathrm{MAX}\left\{\frac{\|\varphi\|_\infty}{4C_0}, \frac{\|\varphi'\|_\infty}{1-T_0 C_{INF}}\right\}$$

$$\frac{d}{dt}\left(\ell(t) - c(u(x_0,t_0),x_0,t_0;t)\right)$$
$$= -\mathrm{str}(\ell(t),t) + u_L(\ell(t),t) - u(c(u(x_0,t_0),x_0,t_0;t),t)$$
$$\le -\Delta + (\ell(t) - c(u(x_0,t_0),x_0,t_0;t))\,\mathrm{MAX}\left\{\frac{\|\varphi\|_\infty}{4C_0}, \frac{\|\varphi'\|_\infty}{1-T_0 C_{INF}}\right\}$$

where Δ is the minimum shock strength, defined in (2.2.6).

If we set $\eta(t) = \ell(t) - c(u(x_0,t_0),x_0,t_0;t)$

then $\eta(t)$ satisfies a differential inequality of the form

$$\frac{d\eta}{dt} \le -\Delta + \eta\,\mathrm{MAX}\left\{\frac{\|\varphi\|_\infty}{4C_0}, \frac{\|\varphi'\|}{1-T_0C_{INF}}\right\}$$

with $\frac{d\eta}{dt}\Big|_{t_0} \le -\Delta/2$.

This inequality is easily integrated to show that

$$\eta(t) \le \eta(t_0) - \tfrac{2}{\Delta}(t-t_0) \Rightarrow$$

$$\eta(t) \le 0 \quad \text{IF} \quad \eta(t_0)\frac{\Delta}{2} > (t-t_0)$$

so that $C_2 \le \frac{2}{\Delta}$,

To apply Glimm's method to the initial value problem (2.2.1), we take $u_j^0 = \varphi(jh)$ for even, and assume $k = \lambda h$, where $\lambda \le 1/\|\varphi\|_\infty$. The solution to Riemann's problem for the inviscid Burgers' equation is given in Example (1.4): since

$$|u(\tfrac{x}{t})| \le \max\{|u_L|, |u_R|\}$$

it follows easily from the definition of Glimm's method that the approximate solution $u_j^n = u^{(h)}(jh, nk)$, $j+n$ even, at time step n satisfies the bound

$$\sup_j |u_j^n| \le \sup_j |u_j^0| \le \|\varphi\|_\infty$$

The condition (2.1) reduces to $k/h = \lambda \le 1/\sup|u_j^n|$ (see Example (1.4)), which is clearly satisfied. In particular, this insures that waves from adjacent Riemann problems do not overlap.

We are going to want to trace various wave structures in the approximate solution; the first step in doing so is to identify such structures in the initial data. By our assumptions on φ, the approximate initial data may be partitioned into a finite number of intervals of four different types. We define C_3 and integers $k_1, \ldots, k_d$ with $-\infty = k_1 < k_2 < \ldots < k_d = +\infty$, and C_3, d independent of h for h sufficiently small, by the following conditions:

(2.2.16)

1) For each interval $[k_j, k_{j+1}]$ $j = 1, \ldots, d-1$ one of four possible conditions is satisfied:

i)

$$u^0_k > u^0_{k+2}\ , \quad k = k_j, k_j+2, \ldots, k_{j+1}-2$$
$$(k_{j+1} - k_j)h \geq C_3$$
$$u^0_k - u^0_{k+2} \leq 2h\|\varphi'\|_\infty$$

We say that approximate initial data consists of a compression wave between k_j and k_{j+1}.

ii)

$$u^0_k \leq u^0_{k+2} \quad k = k_j, k_j+2, \ldots, k_{j+1}-2$$
$$(k_{j+1} - k_j)h \geq C_3$$
$$u^0_{k+2} - u^0_k \leq 2h\|\varphi'\|_\infty$$

We say that the approximate initial data consists of a rarefaction wave between k_j and k_{j+1}.

iii) There is a rarefaction center at k_j+1:

$$k_{j+1} = k_j + 2 \qquad u^0_{k_{j+1}} - u^0_{k_j} \geq C_3$$

iv) There is a shock discontinuity at k_j+1:

$$k_{j+1} = k_j + 2 \qquad u^0_{k_j} - u^0_{k_{j+1}} \geq C_3$$

2) If the interval $[k_j, k_{j+1}]$ satisfies i) or ii) there are no larger intervals $[k_j', k_{j+1}'] \supsetneq [k_j, k_{j+1}]$ such that those conditions are satisfied.

There is a one-to-one correspondence between shock discontinuities and rarefaction centers in the initial data for the approximate solution, and the corresponding discontinuities in the exact initial data. Let $x_1 < \ldots < x_s$ be the points where φ has a jump discontinuity. We can associate with an integer p_i, $i = 1, \ldots, s$ such that

$$|\varphi_L(x_i) - u^0_{p_i - 1}|, \; |\varphi_R(x_i) - u^0_{p_i + 1}| \leq 2h \|\varphi'\|_\infty \qquad (2.2.17)$$

$$|p_{i+1}h - p_i h| \geq C_0 - 4h$$

$$|p_i h - x_i| \leq 4h$$

We can define and track in the approximate solution the approximate analogues of the shocks and characteristics of the exact solution.

Denote by (j, n) the point $(jh, nk) \in \mathbb{R} \times [0, T_0]$, $j + n$ even. Then we can define $b(j, n; q)$ the approximate backwards characteristic from (j, n) at time step q, $0 \leq q \leq n$, as follows:

Define $s^q_k = (u^q_{k+1} + u^q_{k-1})/2$ if $u^q_{k-1} > u^q_{k+1}$ and

$$v^q_k \begin{array}{ll} = s^q_k & \text{IF } u^q_{k-1} > u^q_{k+1} \\ = u^q_{k+1} & \text{IF } u^q_{k-1} \leq u^q_{k+1} \end{array}$$

Then we define

$$b(j, n; n-1) = j \pm 1 \qquad \text{IF} \quad a^{n-1} \gtrless \lambda v^{n-1}_j$$

and

$$b(j, n; q) = b(b(j, n; q+1), q+1; q)$$

It follows easily from the definitions that

$$|b(j,n;q) - j| \le n-q \tag{2.2.18}$$

and that $u_j^n \ge u_{b(j,n;q)}^q$.

It is also easy to show that approximate backwards characteristics cannot cross, i.e., if $j \le j'$, then

$$b(j,n;q) \le b(j',n,q) \tag{2.2.19}$$

for all $q < n$. From the definitions, we have

$$b(j',n;q) - b(j,n;q) - (b(j',n;q+1) - b(j,n;q+1)) = 0, \pm 2$$

since $j'-j$ is even, the result follows from the second part of the definition of $b(j,n;q)$.

Similarly, we define approximate forward characteristics $f(j,n;q)$ from (j,n) for $q \ge n$:

1) If $u_{j-2}^n > u_j^n$, then

$$f(j,n;n+1) = j \pm 1 \text{ if } a^{n+1} \lesseqgtr \lambda s_{j-1}^n$$

If $u_{j-2}^n \le u_j^n$, then

$$f(j,n;n+1) = j \pm 1 \text{ if } a^{n+1} \lesseqgtr \lambda u_j^n$$

2) $f(j,n;q) = f(f(j,n;q-1),q-1;q)$

As was the case with approximate backwards characteristics, approximate forward characteristics cannot cross, i.e.

$$j' \le j \Rightarrow f(j', n; q) \le f(j, n; q)$$

although they can coincide. We also have

$$f(b(j,n;n-1), n-1; n) \le j$$
$$b(f(j,n;n+1), n+1; n) \ge j$$

from which follows, by induction

$$f(b(j,n;q), q; n) \le j \qquad q < n \tag{2.2.20}$$
$$b(f(j,n;q), q; n) \ge j \qquad q > n$$

We can also trace approximate shock paths forward in time. Let $\ell^{(k)}(n)$ be such that $u^n_{\ell^{(k)}(n)-1} > u^n_{\ell^{(k)}(n)+1}$; we say that at time step n there is a shock located at $\ell^{(k)}(n)$ Then we define

$$\ell^{(k)}(n+1) = \ell^{(k)}(n) \pm 1 \quad \text{if} \quad a^{n+1} \lesseqgtr \lambda s^n_{\ell^{(k)}(n)} \; .$$

We need to make sure that there is a shock located at i.e., that $u^{n+1}_{\ell^{(k)}(n+1)-1} > u^{n+1}_{\ell^{(k)}(n+1)+1}$ There are only the following possibilities for $u^{n+1}_{\ell^{(k)}(n+1)-1}$, $u^{n+1}_{\ell^{(k)}(n+1)+1}$:

If $u^n_{\ell^{(n)}(n)-3} > u^n_{\ell^{(n)}(n)-1}$, then

$$u^{n+1}_{\ell^{(n)}(n+1)-1} = \begin{cases} u^n_{\ell^{(n)}(n)-3} & \text{if } \lambda s^n_{\ell^{(n)}(n)} \le a^n < \lambda s^n_{\ell^{(n)}(n)-2} \\ u^n_{\ell^{(n)}(n)-1} & \text{otherwise} \end{cases}$$

If $u^n_{\ell^{(n)}(n)-3} \le u^n_{\ell^{(n)}(n)-1}$, then

$$u^{n+1}_{\ell^{(n)}(n+1)-1} = \begin{cases} \dfrac{a^{n+1}}{\lambda} & \text{if } \lambda \operatorname{MAX}\{ s^n_{\ell^{(n)}(n)}, u^n_{\ell^{(n)}(n)-3} \} \le a^n < \lambda u^n_{\ell^{(n)}(n)-1} \\ u^n_{\ell^{(n)}(n)-1} & \text{otherwise} \end{cases}$$

If $u^n_{\ell^{(n)}(n)+1} > u^n_{\ell^{(n)}(n)+3}$, then

$$u^{n+1}_{\ell^{(n)}(n+1)+1} = \begin{cases} u^{n+1}_{\ell^{(n)}(n)+3} & \text{if } \lambda s^n_{\ell^{(n)}(n)+2} \le a^n < s^n_{\ell^{(n)}(n)} \\ u^n_{\ell^{(n)}(n)+1} & \text{otherwise} \end{cases}$$

If $u^n_{\ell^{(n)}(n)+1} \le u^n_{\ell^{(n)}(n)+3}$, then

$$u^{n+1}_{\ell^{(n)}(n+1)+1} = \begin{cases} \dfrac{a^{n+1}}{\lambda} & \text{if } \lambda u^n_{\ell^{(n)}(n)+1} \le a^n < \lambda \operatorname{MIN}\{ s^n_{\ell^{(n)}(n)}, u^n_{\ell^{(n)}(n)+3} \} \\ u^n_{\ell^{(n)}(n)+1} & \text{otherwise} \end{cases}$$

From this, it is easy to see that it is impossible for
$u^{n+1}_{\ell^{(h)}(n+1)-1} \leq u^{n+1}_{\ell^{(h)}(n+1)+1}$.
Thus, given an $\ell^{(h)}(0)$ such that $u^{0}_{\ell^{(h)}(0)-1} > u^{0}_{\ell^{(h)}(0)+1}$
we can define, by the above procedure, the location of the discontinuity $\ell^{(h)}(n)$ at time step n. However, we will reserve the term approximate shock for those discontinuities for which $\ell^{(h)}(0)$ is a shock discontinuity in the initial data, in the sense of (2.2.16).

By similar arguments as those used for backwards characteristics it is easy to show that

$$|\ell^{(h)}(n) - \ell^{(h)}(n+q)| \leq q \tag{2.2.21}$$

Similarly, if k^0+1 is a rarefaction center, then

$$b(\ell^{(h)}(q) \pm 3, q; 0) = k^0$$

only if

$$q \geq q_{MIN} = \left[\frac{C_3}{2h}\right] - 4 \quad ; \tag{2.2.22}$$

and if $\ell^{(h)}_1, \ell^{(h)}_2$ are two approximate shocks for which
$\ell^{(h)}_1(q_0) \geq \ell^{(h)}_1(q_0)$ then $\ell^{(h)}_1(q) \geq \ell^{(h)}_2(q)$
for $q \geq q_0$.

So approximate shocks may neither disappear nor cross, although they may coincide after some time.

We need to define several quantities associated with an approximate shock $\ell^{(h)}(q)$

$$s(\ell^{(h)}(q)) = s^q_{\ell^{(h)}(q)} = (u^q_{\ell^{(h)}(q)+1} + u^q_{\ell^{(h)}(q)-1})/2$$

$$str(\ell^{(h)}(q)) = (u^q_{\ell^{(h)}(q)-1} - u^q_{\ell^{(h)}(q)+1})/2$$

$$u_L(\ell^{(h)}(q)) = u^q_{\ell^{(h)}(q)-3}$$

$$u_R(\ell^{(h)}(q)) = u^q_{\ell^{(h)}(q)+3}$$

The latter two being defined only if
$\min_{\bar{\ell}^{(h)} \neq \ell^{(h)}} |\ell^{(h)}(q) - \bar{\ell}^{(h)}(q)| \geq 4$.

By the correspondence given in (2.2.17) there is a one-to-one correspondence between approximate shocks $\ell^{(h)}$ and shocks in the exact solution $\ell(t)$ given by the condition

$$|h\ell^{(h)}(0) - \ell(0)| \leq 4h \tag{2.2.23}$$

Finally, as was the case for characteristics in the exact solution, we restrict the definition of approximate forward characteristics to the situation where they never intersect a shock. Specifically $f(j, n; n+1)$ will be defined only if $\min_{\ell^{(h)}} |\ell^{(h)}(n) - j| \geq 5$

Given a shock ℓ, we will denote by $\Psi(\ell(t))$ the vector $(\ell(t), u_L(\ell(t),t), u_R(\ell(t),t))$ whenever u_L, u_R are defined. Similarly, given $\ell^{(h)}(q)$ a shock in the approximate solution, we will denote by $\Psi^{(h)}(\ell^{(h)}(q))$ the vector $(h\ell^{(h)}(q), u_L(\ell^{(h)}(q)), u_R(\ell^{(h)}(q)))$ whenever those quantities are defined.

The following Theorem is the main result in this section.

Theorem 2.4 Let $T_0 = \lambda\alpha T_{CRIT}$ where $\alpha < \frac{1}{2}$ and

$$T_{CRIT} = \sup\{t : 1 + t \inf_{x:\ \varphi' \text{ is continuous at } x} \varphi'(x) > 0\} .$$

Let $u(x,t)$ be the exact solution to (2.2.1) for $(x,t) \in \mathbb{R}\times[0,T_0]$, and let u^n_j be the approximate solution obtained using Glimm's method, using the binary van der Corput sampling sequence $\vec{a} = (a^1, a^2, \ldots)$.

1) Let D be a compact subset of $\mathbb{R}\times(0,T_0]$ such that u is continuous at (x,t) for all $(x,t) \in D$. Then there exists $A > 0$ such that, for all h sufficiently small, $|u^n_j - u(x,t)| \leq Ah|\log h|$ where (j,n) are chosen such that
$nk \leq t < (n+1)k$, $jh \leq x < (j+2)h$
with the bound holding uniformly for all $(x,t) \in D$.

2) Let $\ell(t)$ be a shock, and $[t_1,t_2]$ an interval such that $\ell(t)$ is not overtaken by another shock at any time $t \in [t_1,t_2]$. Then there exists $B > 0$ such that, for all h sufficiently small,
$$|\psi(\ell(t)) - \psi^{(h)}(\ell^{(h)}(q))| \leq B h^{\frac{1}{2}} |\log h|$$
where $\ell^{(h)}$ is the approximate shock associated with ℓ by (2.2.23), $q = [\frac{t}{k}]$, with the bound holding uniformly in $[t_1,t_2]$.

We will prove this Theorem in a series of Lemmas. Lemma 2.5 says that Glimm's method is stable in Lipschitz norm (see (2.2.24)), and that, as a consequence, approximate characteristics possess the main properties of the characteristics in the exact solution, modulo small error terms. The approximate solution differs from being constant along characteristics by $O(h)$, and the approximate characteristics themselves differ from straight lines with slope equal to the solution along them by $O(h|\log h|)$.

Lemma 2.5 Assume that the strength of all the approximate shocks $\ell^{(h)}(g)$, $g \le g_0$, $g_0 k \le T_0$, is greater than or equal to $\delta/2$ independent of h, g where δ is as in (2.2.6). Then there exists a C_4 independent of h such that the following holds:

1) If $j = \ell^{(h)}(g)$ for any approximate shock path $\ell^{(h)}$ and $b(j-1, g; 0)$ is not the left state of a rarefaction center, then

$$|u^g_{j+1} - u^g_{j-1}| \le C_4 h \tag{2.2.24}$$

If $b(j-1, g; 0)$ is the left state of a rarefaction center, then

$$|u^g_{j+1} - u^g_{j-1}| \le 2^{-[\log_2 g]}$$

2) If $\max_{\ell^{(h)}} |\ell^{(h)}(g) - j| \ge 3$ (resp. 5), then

$$\max_{\ell^{(h)}} |b(j, g; g') - \ell^{(h)}(g')| \ge 3 \text{ (resp. } 5) \tag{2.2.25}$$

for all $g' \le g$.

3) Let (j,q) be such that $|j-\ell^{(k)}(q)| \ge 3$ for all shocks $\ell^{(k)}$.

If $b(j,q;0)$ is not the left state of a rarefaction center, then

$$|u_j^q - u^{q'}_{b(j,q;q')}| \le C_4 h$$

If $b(j,q;0)$ is the left state of a rarefaction center, then (2.2.26)

$$|u_j^q - u^{q'}_{b(j,q;q')}| \le 2^{-[\log_2 q']}$$

In either case

$$\begin{aligned} |b(j,q;q') - \lambda(q-q')u_j^q| &\le 2\gamma(q-q') + C_4 h(q-q') \\ &\le C_4 |\log h| \end{aligned} \tag{2.2.27}$$

4) If $f(j,q;q')$ is defined, then there exists J_0 independent of such that

$$J_0 \ge |f(j,q;q'') - b(f(j,q;q''),q'';q)| \tag{2.2.28}$$

for all q'', $q \le q'' \le q'$.

5) Let k^0+1 be a rarefaction center, $q < q_{MIN}$. Then for every v, $u^0_{k^0} \le v \le u^0_{k^0+2}$ there exists j such that $|u_j^q - v| \le 2^{-[\log_2 q]} + C_4 h$,

$$b(j,q;0) = k_0$$

There also exist j_L, j_R such that

$$0 < u^q_{j_L} - u^0_{k^0}, \; u^0_{k^0+2} - u^q_{j_R} < C_4 h \tag{2.2.29}$$

$$b(j_L, q; 0) = k^0$$

$$b(j_R, q; 0) = k^0, k^0 + 2$$

In order to prove this Lemma, we have to analyze the propagation in time of the wave structures in the initial data (2.2.16).

Let k^q, r^q be integers, $r^q \geq 0$, $k^q + q$ even. We call $u^q_{k^q+2j}$ $j = 0, \ldots r^q + 1$ a compression wave (at time step q) if, for all $j = 0, \ldots, r^q$

$$u^q_{k^q+2j} > u^q_{k^q+2(j+1)}$$

and if $k^q + 2j + 1 \neq \ell^{(h)}(q)$ for all approximate shocks $\ell^{(h)}$.

Given a compression wave $u^q_{k^q+2j}$ $j = 0, \ldots, r^q + 1$ at time step q one would like to define

$$u^{q+1}_{k^{q+1}+2j} \quad j = 0, \ldots, r^{q+1} + 1$$

a compression wave at time step $q + 1$ which is the wave

$$u^q_{k^q+2j} \quad j = 0, \ldots, r^q + 1$$

advanced by one time step. We shall do so, first under the following assumption about the Riemann problems at the left and right endpoints

$$k^q = k^q_L \; , \; k^q_R = k^q + 2(r^q + 1) \; .$$

i) The Riemann problem just beyond the endpoint results in a rarefaction fan: (2.2.30)

$$u^q_{k^q_L-2} \le u^q_{k^q_L} \quad , \quad u^q_{k^q_R} \le u^q_{k^q_R+2}$$

ii) Condition i) doesn't hold, and the endpoint state of the wave does not change during the time step:

$$u^q_{k^q_L-2} > u^q_{k^q_L} \quad , \quad a^{q+1} \notin [\lambda s^q_{k^q_L+1} , \lambda s^q_{k^q_L-1})$$

$$u^q_{k^q_R} > u^q_{k^q_R+2} \quad , \quad a^{q+1} \notin [\lambda s^q_{k^q_R+1} , \lambda s^q_{k^q_R-1})$$

We then define the successor at time step $q+1$ to the compression wave in terms of a partitioning of the interval $[-1,1)$. Let

$$[-1,1) = \beta^q_L \cup \beta^q_R \cup \alpha^q_{k^q} \cup \alpha^q_{k^q+2} \cdots \cup \alpha^q_{k^q+2(r^q+1)}$$

where

$$\alpha^q_{k^q+2j} = [\lambda s^q_{k^q+2j+1} , \lambda s^q_{k^q+2j-1}) \quad , \; j = 1, \ldots , r^q \;\; ;$$

$$\alpha^q_{k^q} = [\lambda \max\{s^q_{k^q+1} , u^q_{k^q-2}\} , \lambda u^q_{k^q})$$

if (2.2.30, i) holds at the left endpoint;

$$\alpha^q_{k^q} = \phi$$

if (2.2.30,ii) holds at the left endpoint;

$$\alpha^{g}_{k^g+2(r^g+1)} = \left[\lambda u^g_{k^g+2(r^g+1)}, \lambda \operatorname{MIN}\{s^g_{k^g+2r^g+1}, u^g_{k^g+2(r^g+2)}\}\right)$$

if (2.2.30, i) holds at the right endpoint;

$$\alpha^{g}_{k^g+2(r^g+1)} = \phi$$

if (2.2.30, ii) holds at the right endpoint.

Notice that $\alpha^g_{k^g+2j}$ are arranged in decreasing order, as increases, i.e., if $x_i \in \alpha^g_{k^g+2j_i}$, $i = 1, 2$,

then $j_1 > j_2$ implies $x_1 < x_2$.

We define β^g_L, β^g_R by

$$\beta^g_L = \left\{a \in [-1,1) : a \text{ LIES TO THE LEFT OF } \bigcup_{j=0}^{r^g+1} \alpha^g_{k^g+2j}\right\}$$

$$\beta^g_R = \left\{a \in [-1,1) : a \text{ LIES TO THE RIGHT OF } \bigcup_{j=0}^{r^g+1} \alpha^g_{k^g+2j}\right\}$$

Then, reasoning as in Example 2.3, we can define the new compression wave $u^{g+1}_{k^{g+1}+2j}$, $j = 0, \ldots, r^{g+1}+1$

at time step $g+1$ as follows:

1) If $a^{g+1} \in \beta^g_L$ $(a^{g+1} \in \beta^g_R)$

set $k^{g+1} = k^g + 1$ $(k^g - 1)$, $r^{g+1} = r^g$

The wave remains unchanged, except for a translation:

$$u^{g+1}_{k^{g+1}+2j} = u^g_{k^g+2j} \qquad j = 0, \ldots, r^{g+1}+1$$

2) If $a^{q+1} \geq \alpha^q_{k^q}$ $\left(a^{q+1} \geq \alpha^q_{k^q+2(r^q+1)}\right)$,

so (2.2.30, i) holds at k^q $\left(k^q+2(r^q+1)\right)$ we define

$$k^{q+1} = (k^q-1) \quad \left(k^{q+1} = (k^q+1)\right) \qquad r^{q+1} = r^q \; ;$$

then

$$u^{q+1}_{k^{q+1}+2j} = u^q_{k^q+2j} \quad , j = 1, \ldots, r^{q+1} \; ,$$

with

$$u^{q+1}_{k^{q+1}} = \frac{a^{q+1}}{\lambda} \; , \quad u^{q+1}_{k^{q+1}+2(r^{q+1}+1)} = u^q_{k^q+2(r^q+1)}$$

$$\left(u^{q+1}_{k^{q+1}} = u^q_{k^q} \; , \quad u^{q+1}_{k^{q+1}+2(r^{q+1}+1)} = \frac{a^{q+1}}{\lambda}\right)$$

3) If $a^{q+1} \geq \alpha^q_{k^q+2j_0}$ $\quad j_0 = 1, \ldots, r^q$

we define $k^{q+1} = k^q+1$, $r^{q+1} = r^q-1$;

then

$$\begin{aligned} u^{q+1}_{k^{q+1}+2j} &= u^q_{k^q+2(j-1)} \qquad 0 \leq j \leq j_0 \\ &= u^q_{k^q+2j} \qquad\qquad j_0 < j \leq r^{q+1}+1 \end{aligned}$$

In all three cases we obtain again a compression wave at time step $q+1$ between k^{q+1} and $k^{q+1}+2(r^{q+1}+1)$

In particular, the inequalities

$$\begin{aligned} & u^q_{k^q} \geq u^{q+1}_{k^{q+1}} > u^q_{k^q+2} \, , u^{q+1}_{k^{q+1}+2} \\ & u^q_{k^q+2(r^q+1)} \leq u^{q+1}_{k^{q+1}+2(r^{q+1}+1)} < u^q_{k^q+2r^q} \, , u^{q+1}_{k^{q+1}+2r^{q+1}} \end{aligned} \tag{2.2.31}$$

hold.

Given $u^{q_1}_{k^{q_1}+2j}$ $j=0,\dots r^{q_1}+1$, a compression wave at time step q_1 we can define by induction the successive compression waves $u^{q}_{k^q+2j}$, $j=0,\dots,r^q+1$ $q_1 \le q \le q_2$ provided (2.2.30) is satisfied for $q_1 \le q \le q_2$, and we call $u^q_{k^q+2j}$, $j=0,\dots,r^q+1$, $q_1 \le q \le q_2$ a compression wave.

Let $u^q_{k^q+2j}$ $j=0,\dots,r^q+1$, $0 \le q \le P$ be a compression wave for which there exists q_L, q_R such that

$$\begin{aligned} &(2.2.30,\text{i}) \text{ holds at } k^q_{L,R} \text{ for } q \le q_{L,R} \\ &(2.2.30,\text{ii}) \text{ holds at } k^q_{L,R} \text{ for } q > q_{L,R} \end{aligned} \tag{2.2.32}$$

Then it follows from the definitions and (2.2.28) that, for all q', q , $q' < q \le P$

$$k^{q'}+4 \le b(k^q+2j, q; q') \le k^{q'}+2(r^{q'}-1)$$

$$u^{q'}_{b(k^q+2j,q;q')} = u^q_{k^q+2j}$$

$$\alpha^q_{k^q+2j} \supset \alpha^{q'}_{b(k^q+2j,q;q')}$$

for all $j=2,\dots,r^q-1$; if $a^{q+1} \not\subset \alpha^q_{k^q+2j}$) (2.2.33)
for some $j=2,\dots,r^q-1$ then $f(k^q+2j, q; q+1)$ is defined, and

$$b(f(k^q+2j, q; q+1), q+1; q) = k^q+2j \; ;$$

$$\bigcup_{j=2}^{r^g-1} \alpha^g_{k^g+2j} \subset \bigcup_{j=b(k^g+4,g;g')}^{b(k^g+2(r^g-1),g;g')} \alpha^{g'}_{k^{g'}+2j}$$

$$\alpha^g_{k^g} \cup \alpha^g_{k^g+2} \subset \alpha^{g'}_{k^{g'}} \cup \alpha^{g'}_{k^{g'}+2}$$

$$\alpha^g_{k^g+2r^g} \cup \alpha^g_{k^g+2(r^g+1)} \subset \alpha^{g'}_{k^{g'}+2r^{g'}} \cup \alpha^{g'}_{k^{g'}+2(r^{g'}+1)} .$$

Let $m_0 = \log_2\left[\frac{T_0}{k}\right]+1$, and let $u^g_{k^g+2j}$, $j=0,\dots,r^g+1$, $0 \le g \le p < 2^{m_0}$ be a compression wave for which (2.2.32) holds. Then we want to show that, for $j=2,\dots,r^g-1$

$$a^{g+1} \varepsilon\, \alpha^g_{k^g+2j} \quad \text{if and only if} \quad a^{g+1} \varepsilon\, \alpha^0_{b(k^g+2j,g;0)} \tag{2.2.34}$$

where $\vec{a}$ is the binary van der Corput sampling sequence. That $a^{g+1} \varepsilon\, \alpha^0_{b(k^g+2j,g;0)}$ implies $a^{g+1} \varepsilon\, \alpha^g_{k^g+2j}$ follows immediately from (2.2.33). We prove that $a^{g+1} \varepsilon\, \alpha^g_{k^g+2j}$ implies $a^{g+1} \varepsilon\, \alpha^0_{b(k^g+2j,g;0)}$ by induction. Trivially, the implication is true for $g=0$; assume that it is true for $g' < g$ but $a^{g+1} \varepsilon\, \alpha^g_{k^g+2j}$, $a^{g+1} \not\varepsilon\, \alpha^0_{b(k^g+2j,g;0)}$: we show that this leads to a contradiction. By (2.2.33) there exists an s,

$$b(k^g+4,g;0) \le k^0+2s \le b(k^g+2(r^g-1),g;0),$$
$$k^0+2s \ne b(k^g+2j,g;0)\ , \quad a^{g+1} \varepsilon\, \alpha^0_{k^0+2s} .$$

Since $|\alpha^{o}_{k^o+2s}| \leq C_{INF}\,2h < 2^{-(m_o-1)}$, $a^{\ell} \varepsilon \alpha^{o}_{k^o+2s}$

for at most one ℓ $1 \leq \ell \leq 2^{m_o}-1$, by (2.1.4). So $a^{\ell} \not\varepsilon \alpha^{o}_{k^o+2s}$, $\ell \leq g$ and it follows from the induction hypothesis, and (2.2.33) that there exists $j' \neq j$, $2 \leq j' \leq r^{g}-1$, such that $b(k^{g}+2j', g; 0) = k^{o}+2s$. Thus $\alpha^{o}_{k^o+2s} \subset \alpha^{g}_{k^g+2j'}$, $j' \neq j$, which provides the required contradiction.

Using the above assertion we can derive a bound, uniform in h, g on

$$b(k^{g}+2j, g; 0) - b(k^{g}+2(j-1), g; 0), \quad j = 2, \ldots, r^{g}.$$

Assume that, for every $j = j_1, \ldots, j_1+j_o$ there exists a^{s}, $1 \leq s \leq g \leq 2^{m_o}-1$ such that $a^{s} \varepsilon \alpha^{o}_{k^o+2j}$, where $4 \leq j_1, j_1+j_o \leq r^{o}-1$. By (2.2.34) and (2.1.4)

$$\frac{j_o}{2^{m_o-1}} \leq \sum_{j=j_1}^{j_1+j_o} |\alpha^{o}_{k+2j}| \leq 2h\, C_{INF}(j_o+1)$$

which implies

$$j_o < 2h\,C_{INF} \Big/ \left(2^{-(m_o-1)} - 2h\,C_{INF}\right) \leq C_{INF} \Big/ \left(\frac{2\lambda}{T_o} - C_{INF}\right)$$

It follows from (2.2.33) and (2.2.34) that for every s,

$$b(k^{g}+2(j-1), g; 0) < s < b(k^{g}+2j, g; 0),$$

there exists $g' \leq g$ such that $a^{g'} \varepsilon \alpha^{g'-1}_{f(s,0;g'-1)}$ with $b(f(s,0;g'-1), g'-1; 0) = s$,

$$k^{g'-1}+4 \leq f(s,0;g'-1) \leq k^{g'-1}+2(r^{g'}-1).$$

Then it follows immediately that

$$|b(k^q+2j,q;0) - b(k^q+2(j-1),q;0)| \leq 2C_{INF}/(\tfrac{2\lambda}{T_0} - C_{INF}) + 2 \leq J_1$$

From (2.2.34) also follow bounds on

$$b(k^q+2,q;0) - k^0$$

$$k^0+2(r^0+1) - b(k^q+2r^q,q;0) \qquad r^q > 0 .$$

Assume that $a^{q_i} \varepsilon \alpha^{q_i-1}_{k^{q_i-1}+2}$, but $a^q \not\varepsilon \alpha^{q-1}_{k^{q-1}+2}$ for $q_i \leq q' \leq q < q_{i+1}$. Then

$$b(k^q+2,q;0) - k^0 = b(k^{q_i-1}+2,q_i;0) - k_0 + b(k^q+2,q;0) - b(k^{q_i-1},q_i-1;0)$$

But $b(k^{q_i}+2,q_i;q_i-1) = k^{q_i-1}+4$; thus,
$b(k^{q_i}+2,q_i;0) - b(k^{q_i-1}+2,q_i-1;0) \leq J_1$, and

by induction,

$$b(k^q+2,q;0) - k^0 \leq k^0+2 - k^0 + (J_1+1)N\{\ell\,\varepsilon\,\{1,\ldots,q-1\}, a^\ell \varepsilon \alpha^\ell_{k^\ell+2}\}$$

$$\leq 2 + (J_1+1)N\{\ell\,\varepsilon\,\{1,\ldots,q-1\}, a^\ell \varepsilon \alpha^0_{k^0+2} \cup \alpha^0_{k^0}\}$$

$$\leq J_2$$

where J_2 is independent of k, $q \leq 2^{m_0}-1$ by (2.1.8) and (2.2.16).

Similar arguments show that

$$k^0+2(r^0+1) - b(k^g+2r^g, g; 0) \le J_2 \quad \text{IF } r^g > 0$$
$$b(k^g+2, g; 0) - b(k^g, g; 0) \le 2J_2 \quad \text{IF } r^g = 0$$

The following bounds follow easily from the above:

$$0 < u^g_{k^g+2(j-1)} - u^g_{k^g+2j} = u^0_{b(k^g+2(j-1),g;0)} - u^0_{b(k^g+2j,g;0)} \le J_1 C_{1VF} h \quad j=2,\ldots,r^g \tag{2.2.35}$$

$$0 < u^g_{k^g} - u^g_{k^g+2} \le u^0_{k^0} - u^0_{b(k^g+2,g;0)} \le 2J_2 C_{1VF} h$$

$$0 < u^g_{k^g+2r^g} - u^g_{k^g+2(r^g+1)} \le u^0_{b(k^g+2r^g,g;0)} - u^0_{k^0+2(r^0+1)} \le 2J_1 C_{1VF} h$$

$$0 < b(k^g+2j, g; g') - b(k^g+2(j-1), g; g') \le b(k^g+2j, g; 0) - b(k^g+2(j-1), g; 0) \le J_1 \quad j=2,\ldots,r^g.$$

We can extend the results in the above discussion to the case of a compression wave overtaken from one or both sides by an approximate shock. If $u^g_{k^g+2j}$, $j=0,\ldots r^g+1$ is a compression wave at time step g, then we can define r^{g+1}, k^{g+1} for the compression wave $u^{g+1}_{k^{g+1}+2j}$ $j=0,\ldots,r^{g+1}+1$ in the following cases:

1) If $l^{(k)}(g) = k^g - 1$, (2.2.30) holds at k^g+2r^g+1 and $a^{g+1} \in [\lambda s^g_{l^{(k)}(g)}, \lambda s^g_{k^g+1})$, set $k^{g+1} = k^g+1$, $r^{g+1} = r^g - 1$;

2) If $\ell^{(L)}(q) = k^q + 2r^q + 3$, (2.2.30) holds at k^q and $a^{q+1} \in [\lambda s^q_{k^q+2r^q+1}, \lambda s^q_{\ell^{(L)}(q)})$, set $k^{q+1} = k^q + 1$, $r^{q+1} = r^q - 1$.

Otherwise, we define k^{q+1}, r^{q+1} as before, since, if

$$a^{q+1} \notin [\lambda s^q_{\ell^{(L)}(q)}, s^q_{k^q+1}) \quad (a^{q+1} \notin [\lambda s^q_{k^q+2r^q+1}, \lambda s^q_{\ell^{(L)}(q)})$$

then (2.2.30, ii) holds at k^q $(k^q + 2(r^q+1))$. This definition also covers the case when $\ell_1^{(L)}(q) = k^q - 1$, $\ell_2^{(L)}(q) = k^q + 2r^q + 3$ since

$$[\lambda s^q_{\ell_1^{(L)}(q)}, \lambda s^q_{k^q+1}) \cap [\lambda s^q_{k^q+2r^q+1}, \lambda s^q_{\ell_2^{(L)}(q)}) = \phi .$$

Since it is possible that $r^{q+1} < 0$ (corresponding to the shock completely overtaking the compression wave), $u^{q+1}_{k^q+2j}$, $j = 0, \ldots, r^{q+1} + 1$ is well-defined as a compression wave at time step $q+1$ only if $r^{q+1} \geq 0$ otherwise, the compression wave is not defined for $q' > q$.

If $u^{q+1}_{k^{q+1}+2j}$, $j = 0, \ldots, r^{q+1}$ is defined, then $\ell^{(L)}(q) = k^q - 1$ implies $\ell^{(L)}(q+1) = k^{q+1} - 1$ and $\ell^{(L)}(q) = k^q + 2r^q + 3$ implies $\ell^{(L)}(q+1) = k^{q+1} + 2r^{q+1} + 3$.

Let $u^q_{k^q+2j}$, $j = 0, \ldots, r^q + 1$, $0 < q \leq p$ be a compression wave, and assume there exists q_L, q_R such that, for $q < q_L$, $u^q_{k^q-2} \leq u^q_{k^q}$ $(q < q_R,\ u^q_{k^q+2(r^q+2)} \geq u^q_{k^q+2(r^q+1)})$

and that for $q \geq q_L$, $k^q - 1 = \ell^{(L)}(q)$ $(q \geq q_R,\ k^q + 2r^q + 3 = \ell^{(L)}(q))$

Then it follows from the definitions there exists a compression wave $u^q_{\tilde{k}^q+2j}$, $j=0,\ldots,\tilde{r}^q+1$, $0\le q\le p$ with $\tilde{k}^p=k^p$, $\tilde{r}^p=r^p$, which satisfies (2.2.32) so that (2.2.35) holds without modification for the compression wave $u^p_{\tilde{k}^p+2j}$, $j=0,\ldots\tilde{r}^p+1$.

Let k^q, r^q be integers, $r^q\ge 0$, k^q+q even. We say that $u^q_{k^q+2j}$ $j=0,\ldots,r^q+1$ is a rarefaction wave at time step q if $u^q_{k^q}\le u^q_{k^q+2}\le\ldots\le u^q_{k^q+2(r^q+1)}$

As was the case for compression waves in the approximate solution, we want to define k^{q+1}, r^{q+1} such that $u^{q+1}_{k^{q+1}+2j}$, $j=0,\ldots,r^{q+1}+1$ is the wave $u^q_{k^q+2j}$, $j=0,\ldots,r^q+1$ advanced by one time step. We do so in two steps.

1) If

$$a^{q+1}\in[\lambda u^q_{k^q+2j_0},\lambda u^q_{k^q+2(j_0+1)})$$

for some $j_0=0,\ldots,r^q$, then define

$$\tilde{k}^{q+1}=k^q-1,\quad \tilde{r}^{q+1}=r^q+1$$

$$\tilde{u}^{q+1}_{\tilde{k}^{q+1}+2j}=u^q_{k^q+2j}\qquad j<j_0$$

$$\tilde{u}^{q+1}_{\tilde{k}^{q+1}+2j}=u^q_{k^q+2(j-1)}\qquad j>j_0$$

$$\tilde{u}^{q+1}_{\tilde{k}^{q+1}+2j_0}=\frac{a^{q+1}}{\lambda}$$

If

$$a^{\ell+1} \notin [\lambda u^{\ell}_{k^\ell}, \lambda u^{\ell}_{k^\ell+2(r^\ell+1)})$$

Set

$$\tilde{k}^{\ell+1} = k^{\ell}+1 \quad (\tilde{k}^{\ell+1} = k^{\ell}-1), \; \tilde{r}^{\ell+1} = r^{\ell}, \; \tilde{u}^{\ell+1}_{\tilde{k}^{\ell+1}+2j} = u^{\ell}_{k^\ell+2j}, \; j=0,\ldots,r^{\ell}+1$$

for $a^{\ell+1}$ to the left (right) of $[\lambda u^{\ell}_{k^\ell}, \lambda u^{\ell}_{k^\ell+2(r^\ell+1)})$.

2) (a) If

$$u^{\ell}_{k^\ell-2} \le u^{\ell}_{k^\ell}, \quad u^{\ell}_{k^\ell+2(r^\ell+1)} \le u^{\ell}_{k^\ell+2(r^\ell+2)}$$

then we set $k^{\ell+1} = \tilde{k}^{\ell+1}$, $r^{\ell+1} = \tilde{r}^{\ell+1}$

(b) Assume that there is a shock or compression wave just to the left or right, or on both sides of the rarefaction wave, i.e.,

$$u^{\ell}_{k^\ell-2} > u^{\ell}_{k^\ell}, \quad u^{\ell}_{k^\ell+2(r^\ell+1)} > u^{\ell}_{k^\ell+2(r^\ell+2)}$$

or both. Then we have the following four cases:

i) If

$$a^{\ell+1} \in [\lambda u^{\ell}_{k^\ell+2}, \lambda s^{\ell}_{k^\ell-1}), \; a^{\ell+1} \in [\lambda s^{\ell}_{k^\ell+2r^\ell+3}, \lambda u^{\ell}_{k^\ell+2r^\ell})$$

(so that $u^{\ell}_{k^\ell+2} < s^{\ell}_{k^\ell-1}$ and $u^{\ell}_{k^\ell+2r^\ell} > s^{\ell}_{k^\ell+2r^\ell+3}$

define $k^{\ell+1} = \tilde{k}^{\ell+1}+2$, $\tilde{r}^{\ell+1} - 2 = r^{\ell+1}$.

ii) If i) doesn't hold and

$$a^{\ell+1} \in [\lambda u^{\ell}_{k^\ell+2}, \lambda s^{\ell}_{k^\ell-1})$$

(so that $u^{\ell}_{k^\ell+2} < s^{\ell}_{k^\ell-1}$),

define $k^{\ell+1} = \tilde{k}^{\ell}+2$, $r^{\ell+1} = \tilde{r}^{\ell+1} - 1$.

iii) If i) doesn't hold and

$a^{g+1} \varepsilon [\lambda s^{g}_{k^g+2r^g+3}, \lambda u^{g}_{k^g+2r^g})$

(so that $u^{g}_{k^g+2r^g} > s^{g}_{k^g+2r^g+3}$),

define $k^{g+1} = \tilde{k}^{g+1}$, $r^{g+1} = \tilde{r}^{g+1} - 1$

iv) i) - iii) do not hold, then k^{g+1}, r^{g+1} is defined as in (a).

As was the case with compression waves, it is possible that $r^{g+1} < 0$. So $u^{g+1}_{k^{g+1}+2j}$, $j = 0, \ldots, r^{g+1}+1$ is defined as a rarefaction wave only if $r^{g+1} \geq 0$ otherwise, it is not defined.

It is easy to check that, with the above definitions,

$u^{g+1}_{k^{g+1}+2j}$, $j = 0, \ldots, r^{g+1}+1$

is a rarefaction wave at time step $g+1$ if $r^{g+1} \geq 0$, and

$u^{g+1}_{k^{g+1}+2j} = \tilde{u}^{g+1}_{k^{g+1}+2j}$, $j = 0, \ldots, r^{g+1}+1$

Thus, given a rarefaction wave at time step we can define by the above procedure the successive rarefaction waves $u^{g}_{k^g+2j}$, $j = 0, \ldots, r^g+1$, $g_1 \leq g \leq g_2$ provided $r^{g+1} \geq 0$ $g_1 \leq g \leq g_2 - 1$ We call

$u^{g}_{k^g+2j}$ $j = 0, \ldots, r^g+1$, $g_1 \leq g \leq g_2$

a rarefaction wave.

Let $u^{g}_{k^g+2j}$, $j = 0, \ldots, r^g+1$ be a rarefaction wave, as defined above for $g_1 \leq g \leq g_2$ Then the following facts follow immediately from the definitions for g' and for general g', $g_1 \leq g < g' \leq g_2$ by induction.

1) For every $j'=0,\ldots,r^{q'}+1$ there exists a j, $0\le j\le r^{q}+1$ such that $b(k^{q'}+2j',q';q)=k^{q}+2j$.

We also have

$$b(k^{q'}+2j',q';q)\ge \mathrm{MAX}\{k^{q}+2j : j=0,\ldots,r^{q}+1,\ u^{q}_{k^{q}+2j}<u^{q'}_{k^{q'}+2j'}\} \quad (2.2.36)$$

2) If $r^{q'}+1\ge j_1>j_2\ge 0$, then

$$b(k^{q'}+2j_1,q';q)-b(k^{q'}+2j_2,q';q)\le j_1-j_2 \quad (2.2.37)$$

3) $u^{q'}_{k^{q'}}\ge u^{q}_{k^{q}}$, $u^{q'}_{k^{q'}+2(r^{q'}+1)}\le u^{q}_{k^{q}+2(r^{q}+1)}$.

4) Consider the set of real numbers

$$S_{q,q'}=\{u^{q}_{k^{q}+2j} : j=0,\ldots,r^{q}+1\}\cup\{\frac{a^{k}}{\lambda} : k=q+1,\ldots,q'\}$$

Then

$$\{s\in S_{q,q'} : u^{q'}_{k^{q'}}<s<u^{q'}_{k^{q'}+2(r^{q'}+1)}\} \quad (2.2.38)$$

$$=\{u^{q'}_{k^{q'}+2j} : u^{q'}_{k^{q'}}<u^{q'}_{k^{q'}+2j}<u^{q'}_{k^{q'}+2(r^{q'}+1)}),\ j=1,\ldots,r^{q'}\}$$

5) Let $u^{q}_{\bar{k}^{q}+2j}$ $j=0,\ldots,\bar{r}^{q}+1$ $q_1\le q\le q_2$ be a rarefaction wave or a compression wave, such that $k^{q_1}+2(r^{q_1}+1)=\bar{k}^{q_1}$, or $\bar{k}^{q_1}+2(\bar{r}^{q_1}+1)=k^{q_1}$

Then we have, respectively,

$$k^{q}+2(r^{q}+1)=\bar{k}^{q},\quad \bar{k}^{q}+2(r^{q}+1)=k^{q} \quad (2.2.39)$$

for all q, $q_1\le q\le q_2$. If $\ell^{(h)}(q_1)=k^{q_1}-1$, or $\ell^{(h)}(q_1)=k^{q_1}+2r^{q_1}+3$, then $\ell^{(h)}(q)=k^{q}-1$, $\ell^{(h)}(q)=k^{q}+2r^{q}+3$, respectively, for all $q_1\le q\le q_2$.

Consider now the special case of a rarefaction wave

$$u^q_{k^q+2j}, \quad j=0,\dots,r^q+1, \quad 0\le q\le q_1$$

which, at time step $q=0$ has $r^0=0$ and k^0+1 a rarefaction center in the sense of (2.2.16). We call such a wave an (approximate) centered rarefaction fan. It has all the properties of rarefaction waves; in particular, (2.2.36) holds, from which it follows that $b(k^q+2j, q; 0) = k^0$ for all $j=0,\dots,r^q$.

We turn now to the proof of Lemma 2.5. Consider the partition (2.2.16) of the initial data into compression waves, rarefaction waves, rarefaction centers, and shocks. We can follow these structures forward in time using the above discussion, if we assume that compression waves satisfy (2.2.32). They do so at time step $q=0$. We shall assume that they do so for all $qk\le T_0$, deferring the demonstration of this claim to the end of the proof of the Lemma.

By (2.2.16), (2.2.39), every
satisfies one of the following four conditions:

1) $j=k^n+2s$, $s=1,\dots,r^n$ for some compression wave $u^q_{k^q+2j}$, $j=0,\dots r^q+1$, $q=0,\dots,n$) with (2.2.16) holding for $u^0_{k^0+2j}$, $j=0,\dots,r^0+1$.

2) $j=k^n+2s$, $s=0,\dots,r^n+1$ for some rarefaction wave $u^q_{k^q+2j}$ $j=0,\dots,r^q+1$, $q=0,\dots,n$ with (2.2.16) holding for $u^0_{k^0+2j}$, $j=0,\dots,r^0+1$.

3) $j = k^n + 2s \quad s=0,\dots,r^n+1$ for some centered rarefaction wave $u^q_{k^q+2j}$, $j=0,\dots,r^q$, $q=0,\dots,n$.

4) $j = \ell^{(k)}(n) \pm 1$ for some shock $\ell^{(k)}(n)$.

Similarly, for any pair of states u^n_{j-1}, u^n_{j+1} we have $k^n \le j-1, j+1 \le k^n + 2(r^n+1)$, where $u^q_{k^q+2s}, s=0,\dots r^q+1$, $q=0,\dots,n$ is some compression wave, noncentered rarefaction wave, or centered rarefaction wave satisfying (2.2.16) at time step 0, or $j = \ell^{(k)}(n)$ for some approximate shock $\ell^{(k)}$.

The inequalities (2.2.24) follow immediately from (2.2.35) when u^n_{j-1}, u^n_{j+1} are in a compression wave, from (2.2.37) when u^n_{j-1}, u^n_{j+1} are in a rarefaction wave.

To prove (2.2.25) it suffices to show that, for example if $j - \ell^{(k)}(q) = 5$ with $j - \tilde{\ell}^{(k)}(q) = 3, 5, \pm 1$ for any $\tilde{\ell}^{(k)}(q) \neq \ell^{(k)}(q)$, then $b(j,q;q-1) - \ell^{(k)}(q-1) \neq 3$.
In order that $b(j,q;q-1) - \ell^{(k)}(q-1) = 3$,
we must have $b(j,q;q-1) = j-1$, $\ell^{(k)}(q-1) = \ell^{(k)}(q)+1$
so that $a^q \in [-1, \lambda v^{q-1}_{j-1}) \cap [\lambda s^{q-1}_{\ell^{(k)}(q-1)}, 1)$
Since there is no shock $\tilde{\ell}^{(k)}(q-1) \neq \ell^{(k)}(q-1)$ such that

$$\ell^{(k)}(q-1) \le \tilde{\ell}^{(k)}(q-1) \le b(j,q;q-1),$$

by (2.2.24), and our assumption concerning the strength of $\ell^{(k)}$

$$v^{q-1}_{j-1} \le u^{q-1}_{j-1} \le u^{q-1}_{\ell^{(k)}(q-1)+1} + 2C_4 h \le s^{q-1}_{\ell^{(k)}(q-1)} - \delta/2 + 2C_4 h ;$$

thus it is impossible for $[-1, \lambda v^{q-1}_{j-1}) \cap [\lambda s^{q-1}_{\ell^{(k)}(q-1)}, 1) \neq \phi$, and the assertion is proven. The remaining cases $j - \ell^{(k)}(q) = -5, \pm 3$ are dealt with similarly.

If u^q_j is not in a centered rarefaction wave, then either $u^q_j = u^q_{b(j,q;q')}$ in the case when $u^q_j = u^q_{k^q+2s}$, $s = 1, \ldots, r^q$ for some compression wave $u^q_{k^q+2s}$ or

$u^0_{b(j,q;0)} \leq u^{q'}_{b(j,q;q')} \leq u^0_{b(j,q;0)+2}$

by (2.2.36), which implies $|u^q - u^{q'}_{b(j,q;q')}| \leq \|\varphi'\|_\infty 2h$

so that (2.2.26) holds.

In both cases, we have

$$b(j,q;q') = j + N\{k \varepsilon \{q+1, \ldots, q'\},\ a^k \varepsilon [\lambda v^{k-1}_{b(j,q;k)}, 1)\} - N\{k \varepsilon \{q+1, \ldots, q'\},\ a^k \varepsilon [-1, \lambda v^{k-1}_{b(j,q;k)})\}$$

By (2.2.25), $b(j,q;k) \neq \ell^{(k)}(k-1)$

for any approximate shock $\ell^{(k)}$ so,

$$\begin{aligned} |v^{k-1}_{b(j,q;k)} - u^q_j| &\leq |v^{k-1}_{b(j,q;k)} - u^{k-1}_{b(j,q;k-1)}| \\ &\quad + |u^{k-1}_{b(j,q;k-1)} - u^q_j| \\ &\leq C_4 h + \|\varphi'\|_\infty 2h \end{aligned}$$

so that (2.2.27) follows from (2.1.7)

If $b(j,q;0)+1$ is a rarefaction center, then (2.2.26) follows from (2.2.36) and (2.2.38). To prove (2.2. 27) in the case of $j = k^n + 2s$ for some centered rarefaction wave $u^p_{k^p+2s}$ $s = 0, \ldots, r^p + 1$, $p = 0, \ldots, n$)

we partition $\{q+1,\ldots,n\}$ into three sets:

$$S_1 = \{p : b(j,n;p)-1 = k^{-1}+2(r^{p-1}+1),\ q+1 \le p \le n\}$$

$$S_2 = \{p : b(j,n;p)+1 = k^{p-1},\ q+1 \le p \le n\}$$

$$S_3 = \{q+1,\ldots,n\} - (S_1 \cup S_2)$$

If $p \varepsilon S_1$, then $b(j,n;p-1) = b(j,n;p)-1$

$$\ell^{(k)}(p-1) \ne b(j,n;p)$$

by (2.2.25); thus $u^{p-1}_{b(j,n;p)+1}$, $u^{p-1}_{b(j,n;p)-1}$

are points in a compression wave or noncentered rarefaction wave,

and

$$0 \le v^{p-1}_{b(j,n;p)} - u^{p-1}_{b(j,n;p)-1} \le C_4 h$$

by (2.2.24), and

$$u^0_{b(j,n;0)+1} + C_4 h \le u^{p-1}_{b(j,n;p)-1} \le u^n_j \le u^0_{b(j,n;0)+1}$$

by (2.2.35) and (2.2.38). Thus $|v^{p-1}_{b(j,n;p)} - u^n_j| \le 2C_4 h$

and

$$|N\{p\varepsilon S_1 ; a^p \varepsilon [-1, \lambda v^{p-1}_{b(j,n;p)})\} - N\{p\varepsilon S_1 ; a^p \varepsilon [-1, \lambda u^n_j)\}|,$$

$$|N\{p\varepsilon S_1 ; a^p \varepsilon [\lambda v^{p-1}_{b(j,n;p)}, 1)\} - N\{p\varepsilon S_1 ; a^p \varepsilon [\lambda u^n_j, 1)\}|$$

$$\le 4C_4 h(n-q) \le K_1$$

by (2.1.8).

If $p \in S_2$ we find, similarly to the previous case, that

$b(j,n;p-1) = b(j,n;p)+1$;

$u^{p-1}_{b(j,n;p)+1}$, $u^{p-1}_{b(j,n;p)-1}$

are either in a compression or noncentered rarefaction wave, and

$$|u^0_{k^0} - v^{p-1}_{b(j,n;p)}|, \ |u^0_{k^0} - u^{p-1}_{b(j,n;p-1)}| \le C_4 h .$$

Let $|u^n_j - u^0_{k^0}| = \eta$ by (2.2.34), either $\eta = 0$, or $\eta \ge 2^{-[\log_2 \frac{I_0}{k}]-1}$; $|u^n_j - u^{p-1}_{k^{p-1}}|$, $|u^n_j - v^{p-1}_{b(j,n;p)}| \le \eta + C_4 h$

However, by (2.2.38), $(p-1)(\eta + C_4 h) \le 4$,

and the number of elements in S_2 is bounded, independent of g by $\left[\frac{4}{\eta + C_4 h}\right] + 1 = g_{MAX}$. By (2.1.8),

$$\begin{aligned} |N\{p \in S_2 : a^p \in [-1, \lambda v^{p-1}_{b(j,n;p)})\} - N\{p \in S_2 : a^p \in [-1, \lambda u^0_{k^0})\}| \\ \le N\{p \in S_2 : a^p \in [\lambda u^0_{k^0} - C_4 h, \lambda u^0_{k^0} + C_4 h)\} \\ \le 4n C_4 h \le K_2 \end{aligned}$$

$$\begin{aligned} |N\{p \in S_2 : a^p \in [-1, \lambda u^0_{k^0})\} - N\{p \in S_2 : a^p \in [-1, \lambda u^n_j)\}| \\ = N\{p \in S_2 : a^p \in [\lambda u^0_{k^0}, \lambda u^n_j)\} \\ \le N\{p \in \{1, \dots, g_{MAX}\} : a^p \in [\lambda u^0_{k^0}, u^n_j)\} \le 2 g_{MAX} \eta \le K_3 \end{aligned}$$

so that

$$\begin{aligned} |N\{p \in S_2 : a^p \in [-1, \lambda v^{p-1}_{b(j,n;p)})\} - N\{p \in S_2 : a^p \in [-1, \lambda u^n_j)\} \\ \le K_2 + K_3 = K_4 \end{aligned}$$

Similarly, we obtain

$$|N\{p \in S_2 : a^p \in [\lambda v^{p-1}_{b(j,n;p)}, 1)\} - N\{p \in S_2 : a^p \in [\lambda u^n_j, 1)\}| \le K_4$$

If $q \in S_3$, then $k^{p-1} \le b(j,n;p) \pm 1 \le k^{p-1} + 2(r^{p-1}+1)$ and it follows from (2.2.36) and (2.2.18) $a^p \gtrless v^{p-1}_{b(j,n;p)}$ if and only if $a^p \gtrless u^n_j$. Since $a^p = u^n_j$ for at most one p,

$$|N\{p \in S_3 : a^p \in [\lambda v^{p-1}_{b(j,n;p)}, 1)\} - N\{p \in S_3 : a^p \in [\lambda u^n_j, 1)\}|,$$

$$|N\{p \in S_3 : a^p \in [-1, \lambda v^{p-1}_{b(j,n;p)})\} - N\{p \in S_3 : a^p \in [-1, \lambda u^n_j)\}|$$
$$\le 1$$

Combining the above results we obtain, using (2.1.2a)

$$|N\{p \in \{q+1,\dots,n\} : a^p \in [\lambda v^{p-1}_{b(j,n;p)}, 1)\}$$
$$- N\{p \in \{q+1,\dots,n\} : a^p \in [\lambda u^n_j, 1)\}|,$$

$$|N\{p \in \{q+1,\dots,n\} : a^p \in [-1, \lambda v^{p-1}_{b(j,n;p)})\}$$
$$- N\{p \in \{q+1,\dots,n\} : a^p \in [-1, \lambda u^n_j)\}|$$
$$\le K_1 + K_4 + 1$$

and the result follows from (2.1.7).

To prove (2.2.28), we first claim that

$$b(f(j,q;q')-2,q';q) \le j \le b(f(j,q;q'),q';q)$$

for if $j < b(f(j,q;q')-2,q';q) \equiv j_L$

then $f(j_L,q;q') \le f(j,q;q')$

with $j_L > j$ which contradicts (2.2.20). Thus (2.2.28) follows from (2.2.37) when

$$k^{q'}+2 \le f(j,q;q') \le k^{q'}+2(r^{q'}+1), \text{ for } u^{q'}_{k^{q'}+2j'},\ j'=0,\dots,r^{q'}+1$$

a rarefaction wave, and from (2.2.35) and (2.2.37) if

$$k^{q'} \le f(j,q;q') \le k^{q'}+2(r^{q'}+1), \text{ for } u^{q'}_{k^{q'}+2j'}\ \ j'=0,\dots,r^{q'}+1$$

a compression wave.

If $u^{q}_{k^q+2s}$ $s=0,\dots,r^q+1$ is a centered rarefaction wave, then it follows from (2.2.22) that for $q < q_{MIN}$,

$$|k^q-\ell^{(h)}(q)|,\ |k^q+2(r^q+1)-\ell^{(h)}(q)| > 3$$

for any approximate shock $\ell^{(h)}$ so $u^q_{k^q}$, $u^q_{k^q+2(r^q+1)}$ must be either part of a compression wave or a rarefaction wave. If the former holds, then $|u^q_{k^q}-u^0_{k^0}| \le 2J_1 h$, by (2.2.35): if the latter holds, then $u^q_{k^q}=u^0_{k^0}$. In particular

$$u^q_{k^q+2(r^q+1)} - u^q_{k^q} \ge (u^0_{k^0+2}-u^0_{k^0}) - K_6 h,$$

and an induction argument implies that the wave is defined for all $q < q_{MIN}$ and, by (2.2.38), that (2.2.29) holds.

Finally, we need to prove the claim that a compression wave initially satisfying (2.2.16) satisfies the assumption

(2.2.32). To this end, we prove the following fact. Assume that, for $q \le q_0$ the claim is true, and let

$u^q_{k^q_i+2j}$ $j=0,\dots,r^q_i+1$ $0 \le q \le q_1 \le q_0$ $i=1,2$

be two compression waves, with $k^0_1 + 2(r^0_1+1) < k^0_2$, $u^0_j \le u^0_{j+2}$ for all j, $k^0_1+2(r^0+1) < j < k^0_2$

Then, for all $q_1 \ge q \ge 0$ at least one of the following two inequalities hold:

1) $(k^q_2 - (k^q_1 + 2(r^q_1+1)))h \ge K_7$

2) $u^q_{k^q_2} - u^q_{k^q_1+2(r^q_1+1)} \ge K_7$

At $q=0$, u^0_j, $j=k^0_1+2(r^0_1+1),\dots,k^0_2$ consists of one or more rarefaction waves separated by rarefaction centers, so one of these alternatives hold. If (1) holds for $q=0$, then it holds for $q_1 \ge q > 0$ by (2.2.35) and (2.1.7), if (2) holds at $q=0$, then it holds for $q_1 \ge q > 0$ by (2.2.35). In either case

$u^q_j \le u^q_{j+2}$, $j=k^q_1+2(r^q_1+1),\dots,k^q_2$.

To prove the claim, we notice that if (1) or (2) hold then there are one of three possibilities for the approximate solution adjacent to a compression wave $u^{q_1}_{k^{q_1}_c+2j}$ $j=0,\dots,r^{q_1}_c+1$

For example, there exists $\bar{k}^{q_1} \ge k^{q_1}_c + 2(r^{q_1}_c+1)$,

with $u^{q_1}_j \le u^{q_1}_{j+2}$ for $\bar{k}^{q_1} > j \ge k^{q_1}_c + 2(r^{q_1}_c+1)$

such that either $u^{q_1}_{\bar{k}^{q_1}} - u^{q_1}_{k^{q_1}_c+2(r^{q_1}_c+1)} \ge K_7$,

$(\bar{k}^{q_1} - (k^{q_1}_c+2(r^{q_1}_c+1)))h \ge K_7$, or $\bar{k}^{q_1}+1 = \ell^{(k)}(q_1)$

Similar conditions hold to the left of $u^{g_1}_{k^{g_1}_c}$. It follows easily that the assumption (2.2.32) holds for a compression wave at time step g_1+1, $u^{g_1+1}_{k^{g_1+1}_c+2j}$, $j=0,\dots,r^{g_1+1}_c+1$) if it exists. Since (1) or (2) hold at time step $g=0$ the claim is seen to be true by induction.

It follows from Lemma 2.5 that the various quantities associated with the approximate shocks are well-behaved as functions of the time step. If

$$\min_{\ell^{(h)}(g)\neq \tilde{\ell}^{(h)}(g)} |\ell^{(h)}(g) - \tilde{\ell}^{(h)}(g)| \geq 4,$$

then

$$|(u_L(\ell^{(h)}(g)) + u_R(\ell^{(h)}(g)))/2 - s(\ell^{(h)}(g))| \leq C_4'h .$$

If $\min_{\tilde{\ell}(g)\neq \ell^{(h)}(g)} |\tilde{\ell}^{(h)}(g) - \ell^{(h)}(g)| \geq 6$, then $\qquad$ (2.2.40)

$$|u_L(\ell^{(h)}(g)) - u_L(\ell^{(h)}(g+1))| \leq C_4'h$$

$$|u_R(\ell^{(h)}(g)) - u_R(\ell^{(h)}(g+1))| \leq C_4'h .$$

Here $C_4' = \max\{C_4, K_{CRF}\}$, $\infty > K_{CRF} = \max\limits_{h<\frac{1}{2}} \dfrac{2^{-[\log_2 g_{MIN}]}}{h}$ with g_{MIN} as in (2.2.22).

The following Lemma is the estimate of the error in the position, speed, and strength of a shock on either side of which the solution is smooth. The argument is an elaboration of that used in Example 2.2.

Let $A \subset \mathbb{R}$, $A = \{x : \varphi$ is C^2 at x or x is a rarefaction center $\}$; then $\mathbb{R} - A$ consists of a finite number of points. We can partition A into a finite number of disjoint sets $S_1, \dots, S_N$, each of which either consists of a single point, which is a rarefaction center, or is an open interval. In the latter case, S_j is uniquely determined by the requirement that φ is C^2 on S_j, and that there is no larger interval on which φ is C^2 containing S_j

If u is continuous at (x,t) we say that $u(x,t)$ satisfies condition r if $c(u(x,t),x,t;0) \in S_r$ and, if $S_r = \{x_0\}$ that $u(x,t) \neq \varphi_L(x_0), \varphi_R(x_0)$. We say that $u_{L,R}(\ell(t),t)$ satisfies condition $r_{L,R}$ if $c(u_{L,R}(\ell(t),t),\ell(t),t;0) \in S_{r_{L,R}}$ and, if $S_{r_{L,R}} = \{x_0\}$, that $u_{L,R}(\ell(t),t) \neq \varphi_L(x_0)$, $u_{L,R}(\ell(t),t) = \varphi_R(x_0)$.

A shock $\ell(t)$ satisfies (r_L, r_R) if $u_L(\ell(t),t)$, $u_R(\ell(t),t)$ exist, and satisfy, respectively, conditions r_L, r_R.

Given a shock $\ell(t)$, $0 \le t \le T_0$, there are, according to our discussion of the exact solution, at most a finite number of times $\bar{t}$ such that $\ell(\bar{t})$ doesn't satisfy (r_L, r_R) for any possible r_L, r_R Since characteristics can't cross one another, if $\ell(t_1)$, $\ell(t_2)$ satisfy conditions (r_L, r_R) then $\ell(t)$, $t_1 \le t \le t_2$ satisfy conditions (r_L, r_R) So that for each (r_L, r_R) the set of all t, $0 < t < T_0$ such that $\ell(t)$ satisfy conditions (r_L, r_R) is an open interval.

We define, with slight modification, what it means for u_j^n, $\min_{\ell^{(k)}(n)} |j - \ell^{(k)}(n)| \ge 3$ to satisfy condition r. If S_r is an interval, then u_j^n satisfies condition r if

$$b(j,n;0) \pm C_4 h|\log h| \in S_r .$$

If S_r is a rarefaction center $\mathcal{r}_0$, then u^n_j satisfies condition r if $b(j,n;0) = k^0 - 1$, where k^0 is an approximate rarefaction center associated with $\mathcal{r}_0$ by (2.2.20). We say that an approximate shock $\ell^{(k)}(n)$ satisfies conditions (r_L, r_R) if $\min_{\tilde{\ell}^{(k)}(n) \neq \ell^{(k)}(n)} |\tilde{\ell}^{(k)}(n) - \ell^{(k)}(n)| \geq 4$, and $u_L(\ell^{(k)}(n))$, $u_R(\ell^{(k)}(n))$ satisfy, respectively, conditions r_L, r_R.

Lemma 2.6 Let $\ell, \ell^{(k)}$ be a shock in the exact solution and the associated approximate shock. Assume that, for all q, $q_1 \leq q \leq q_2$, $\ell^{(k)}(q), \ell(qk)$ satisfy conditions (r_L, r_R), that the hypotheses under which Lemma 2.5 was proven hold, and that $|\psi^{(k)}(\ell^{(k)}(q_1)) - \psi(\ell(q_1 k))| \leq \delta \leq h^{\frac{1}{3}}$.

Then there exists $C_5 \geq 1$ independent of h, q_1, q_2, for h sufficiently small, such that

$$|\psi^{(k)}(\ell^{(k)}(q)) - \psi(\ell(qk))| \leq C_5 \max\{\delta, h^{\frac{1}{2}}|\log h|\} \quad (2.2.41)$$

We define $m = [\frac{1}{2}\log_2 \frac{T_0}{k}]$ and without loss of generality assume $\delta \geq h^{\frac{1}{2}}|\log h|$. We want to find an expression for $\psi^{(k)}(\ell^{(k)}((j+1)2^m)) - \psi^{(k)}(\ell^{(k)}(j2^m))$ analogous to that found for the exact solution in Lemma 2.3. Recall that one of the assumptions under which Lemma 2.5 was proven was that, for $j2^m < q \leq (j+1)2^m$, the strength of $\ell^{(k)}(q)$ is greater than or equal to $\delta/2$

By (2.2.40),

$$|\ell^{(h)}(q)-\ell^{(h)}(j2^m)|\le 2^m, \tag{2.2.42}$$

$$|\psi^{(h)}(\ell^{(h)}(q))-\psi^{(h)}(\ell^{(h)}(j2^m))|\le C_4' h 2^m \quad j2^m\le q<(j+1)2^m$$

$$\begin{aligned}\ell^{(h)}((j+1)2^m) &= \ell^{(h)}(j2^m)\\ &+N\{q\in\{j2^m+1,\dots,(j+1)2^m\},\ a^q\in[-1,\lambda s^{(h)}(\ell^{(h)}(q-1))]\}\\ &-N\{q\in\{j2^m+1,\dots,(j+1)2^m\},\ a^q\in[\lambda s(\ell^{(h)}(q-1)),1)\}\end{aligned}$$

By (2.1.7) and (2.2.42),

$$|\ell^{(h)}((j+1)2^m)-\ell^{(h)}(j2^m)-\lambda 2^m s(\ell^{(h)}(j2^m))|\le 2^{2m+1}hC_4+2$$

$$\le K_1 \tag{2.2.43}$$

Next we look for an appropriate expression for $u_L(\ell^{(h)}((j+1)2^m))-u_L(\ell^{(h)}(j2^m))$.

There are two cases, corresponding to whether S_{r_L} is an interval or a rarefaction center.

First, we consider the case where S_{r_L} is an interval. Introducing the notation $b(k_1,k_2)=b(\ell^{(h)}(k_1 2^m)-3,k_1 2^m;k_2 2^m)$,

for $k_1>k_2\ge 0$; $\ell_k=\ell^{(h)}(k2^m)$, $u_{L,k}=u_L(\ell^{(h)}(k2^m))$

$s_k=s(\ell^{(h)}(k2^m))$ for $k\ge 0$,

We write

$$\begin{aligned}u_{L,j+1}-u_{L,j} &= h(b(j+1,j)-(\ell_j-3))\\ &\times\frac{b(j+1,0)-b(j,0)}{b(j+1,j)-(\ell_j-3)}\times\frac{u_{L,j+1}-u_{L,j}}{(b(j+1,0)-b(j,0))h}\end{aligned} \tag{2.2.44}$$

assuming the denominators of the last two factors are nonzero, and estimating each term in the product separately. By (2.2.27)

$$|b(j+1,j)-(\ell_{j+1}-3)-\lambda 2^m u_{L,j+1}| \le 2+C_4'h\,2^{2m} \le K_2$$

$$\begin{aligned}|b(j+1,j)-(\ell_j-3)-\lambda 2^m(s_j-u_{L,j})| \\ \le 2^m h C_4' + K_1+K_2 \le K_3\end{aligned} \tag{2.2.45}$$

Since the strength of $\ell^{(\lambda)}(j2^m)$ was assumed to be greater than $\delta/2$,

$$\begin{aligned}|b(j+1,j)-(\ell_j-3)| &\ge \delta/2\cdot\lambda 2^m - K_3-\lambda 2^m C_4'h \\ &\ge K_4 2^m\end{aligned} \tag{2.2.46}$$

Since $b(b(j+1,j),j2^m;0) = b(j+1,0)$

we have, by (2.2.23), (2.2.24)

$$\begin{aligned}|b(j+1,j)-(\ell_j-3)-\lambda_j 2^m(u^0_{b(j+1,0)}-u^0_{b(j,0)}) \\ -(b(j+1,0)-b(j,0))| \\ \le C_4'h+2\gamma(j)+\lambda_j 2^m h C_4 \le K_5\gamma(j)\end{aligned} \tag{2.2.47}$$

We also have that

$$|b(j+1,0)-b(j,0)| \ge K_6 2^m,$$

since, if

$$u^0_{b(j+1,0)}-u^0_{b(j,0)} > \frac{K_4}{2T_0}2^m h,\quad b(j,0)-b(j+1,0) \ge \frac{K_4 2^m}{2T_0\|\varphi'\|_\infty}$$

by (2.2.21); otherwise, by (2.2.24)

$$b(j,0)-b(j+1,0) \ge K_4 2^m - K_5\gamma(j)-\lambda_j 2^m\frac{K_4}{2T_0}2^m h \ge K_6 2^m$$

Denoting by $\tau = \lambda 2^m h = 2^m k$, $t_j = j\tau$,

$$\left| \frac{b(j+1,0)-b(j,0)}{b(j-1,j)-(\ell_j-3)} - 1\Big/\left(1+t_j\varphi'(\ell_j h - t_j u_{L,j})\right)\right| \tag{2.2.48}$$

$$\leq \left| \frac{b(j+1,0)-b(j,0)}{b(j+1,j)-(\ell_j-3)} - 1\Big/\left(1+\lambda j 2^m h\left(\frac{u^0_{b(j+1,0)}-u^0_{b(j,0)}}{h(b(j+1,0)-b(j,0))}\right)\right)\right|$$

$$+\left| 1\Big/\left(1+t_j\left(\frac{u^0_{b(j+1,0)}-u^0_{b(j,0)}}{h(b(j+1,0)-b(j,0))}\right)\right) - 1\Big/\left(1+t_j\varphi'(b(j,0)h)\right)\right|$$

$$+\left| 1\Big/\left(1+t_j\varphi'(b(j,0)h)\right) - 1\Big/\left(1+t_j\varphi'(\ell_j h - t_j u_{L,j})\right)\right|$$

$$\leq \frac{K_5\gamma(j)}{K_4 2^m}\frac{1}{1-T_0C_{INF}} + \frac{\|\varphi''\|_\infty K_6 2^m h}{2(1-T_0C_{INF})} + \frac{(2\gamma(j)+2C_4 k j 2^m\|\varphi''\|_\infty)h}{(1-T_0C_{INF})^2}$$

$$\leq \frac{K_8\gamma(j)}{2^m}$$

$$\left|\frac{u_{L,j+1}-u_{L,j}}{(b(j+1,0)-b(j,0))h} - \varphi'(\ell_j h - t_j u_{L,j})\right| \tag{2.2.49}$$

$$\leq \left|\frac{u_{L,j+1}-u_{L,j}}{(b(j+1,0)-b(j,0))h} - \frac{u^0_{b(j+1,0)}-u^0_{b(j,0)}}{(b(j+1,0)-b(j,0))h}\right|$$

$$+\left|\frac{u^0_{b(j+1,0)}-u^0_{b(j,0)}}{(b(j+1,0)-b(j,0))h} - \varphi'(b(j,0)h)\right|$$

$$+\left|\varphi'(b(j,0)h) - \varphi'(\ell_j h - t_j u_{L,j})\right|$$

$$\leq \frac{2C_4 L}{K_6 2^m h} + \frac{\|\varphi''\|_\infty}{2}(1+C_4 T_0)2^m h$$

$$+\|\varphi''\|_\infty\left(2M(j)h + 2C_4\frac{T_0}{\lambda}\right)$$

$$\leq \frac{K_9}{2^m}$$

In both (2.2.48) and (2.2.49) we have used the fact that $u_L(l^{(h)}((j+1)2^m))$, $u_L(l^{(h)}(j2^m))$ satisfy condition r_L in that $b(j,0)h$, $b(j+1,0)h$, $l_j h - t_j u_{L,j} \in S_{r_L}$. Combining (2.2.43), (2.2.48) and (2.2.49) in (2.2.44), we obtain

(2.2.50)

$$\begin{aligned}
&\left| u_{L,j+1} - u_{L,j} - \tau(s_j - u_{L,j})\,\varphi'(l_j h - t_j u_{L,j}) \Big/ \left(1 + t_j \varphi'(l_j h - t_j u_{L,j})\right)\right| \\
&\quad \le K_3 h \left(\|\varphi'\|_\infty + \frac{K_9}{2^m} \right) \left(\frac{1}{1 - T_0 C_{INF}} + \frac{K_8 \gamma(j)}{2^m} \right) \\
&\quad + \frac{K_9}{2^m} \left(\tau \|\varphi\|_\infty + K_3 h \right) \left(\frac{1}{1 - T_0 C_{INF}} + \frac{K_8 \gamma(j)}{2^m} \right) \\
&\quad + \frac{K_8 \gamma(j)}{2^m} \left(\tau \|\varphi\|_\infty + K_3 h \right) \left(\|\varphi'\|_\infty + \frac{K_9}{2^m} \right) \\
&\quad \le K_{10} h |\log h|
\end{aligned}$$

If S_{r_L} is a rarefaction center with associated approximate center $k^0 + 1$, $j 2^m \ge q_{MIN}$. By (2.2.39) and (2.2.40)

$$\begin{aligned}
&|u_{L,j+1} - u^{j2^m}_{b(j+1,j)}| \le C_4' h \\
&|k^0 + \lambda j 2^m u^{j2^m}_{b(j+1,j)} - b(j+1,j)|, \\
&|\lambda j 2^m u_{L,j} + k^0 - (l_j - 3)| \le C_4 |\log h|
\end{aligned}$$

$$\begin{aligned}
&|(u_{L,j+1} - u_{L,j})\lambda j 2^m - b(j+1,j) - (l_j - 3)| \\
&\quad \le 2 C_4 |\log h| + j 2^m C_4' h \le K_{11} |\log h|
\end{aligned}$$

Thus we have, by (2.2.43)

$$
\begin{aligned}
&\left| u_{L,j+1} - u_{L,j} - \frac{(s_j - u_{L,j})}{\lambda_j 2^m} \lambda 2^m \right| \\
&\le \left| u_{L,j+1} - u_{L,j} - \frac{(b(j+1,j) - (\ell_j - 3))}{\lambda_j 2^m} \right| \\
&\quad + \left| \frac{\ell_{j+1} - 3 - u_{L,j+1} \lambda 2^m - b(j+1,j)}{\lambda_j 2^m} \right| \\
&\quad + \left| \frac{\lambda 2^m s_j - (\ell_{j+1} - \ell_j)}{\lambda_j 2^m} \right| \\
&\quad + \left| \frac{(u_{L,j+1} - u_{L,j}) \lambda 2^m}{\lambda_j 2^m} \right| \\
&\le (K_{12} |\log h| + C_4 |\log h| + K_1 + \lambda C_4' h 2^{2m}) \frac{1}{q_{MIN}} \\
&\le K_{13} h |\log h|
\end{aligned}
$$

$$
\left| u_{L,j+1} - u_{L,j} - (s_j - u_{L,j}) \tau / t_j \right| \le K_{13} h |\log h| \qquad (2.2.51)
$$

Similarly, we obtain

$$\left| u_{R,j+1} - u_{R,j} - \tau (s_j - u_{R,j}) \varphi'(l_j h - t_j u_{R,j}) \Big/ (1 + t_j \varphi'(l_j h - t_j u_{R,j})) \right| \leq K_{11} h |\log h| \qquad (2.2.52)$$

if S_{r_R} is an interval,

$$\left| u_{R,j+1} - u_{R,j} - \tau (s_j - u_{R,j}) / t_j \right| \leq K_{13} h |\log h| \qquad (2.2.53)$$

if S_{r_R} is a rarefaction center.

Define $V(l, u_L, u_R, t) = V : \mathbb{R}^4 \to \mathbb{R}^3$

by

$$V(l, u_L, u_R, t) = \left(\frac{u_L + u_R}{2}, \frac{u_R - u_L}{2} \xi_L(l, t, u_L), \frac{u_L - u_R}{2} \xi_R(l, t, u_R) \right)$$

where $\xi_{L,R}(l, t, x) = \varphi'(l - tx) / (1 + t\varphi'(l - tx))$

if S_{r_L, r_R} is an interval,

$\xi_{L,R}(l, t, x) = 1/t$

if S_{r_L, r_R} is a rarefaction center.

Using V we can express the estimates (2.2.50) -(2.2.53) as

$$\begin{aligned} &|\psi^{(h)}(l^{(h)}((j+1)2^m) - \psi^{(h)}(l^{(h)}(j2^m)) \\ &\qquad - \tau V(\psi^{(h)}(l^{(h)}(j2^m)), t_j)| \leq K_{14} h |\log h| \end{aligned}$$

By Lemma 2.3 we have

$$|\psi(l(t_{j+1})) - \psi(l(t_j)) - \tau V(\psi(l(t_j)), t_j)| \leq C_1 \tau^2 \leq K_{15} h$$

Here $|v| = \max_{i=1,2,3} |v_i|$, $(v_1, v_2, v_3) = v \in R^3$. Since $\xi_{L,R} = 1/t$ only if $g_{m10} \leq j2^m$, and since $\ell(t)$, $\ell^{(h)}(g)$ are assumed to satisfy (r_L, r_R) we have

$$|V(\psi(\ell(t_j)), t_j) - V(\psi^{(h)}(\ell^{(h)}(j2^m)), t_j)| \leq K_{16} |\psi(\ell(t_j)), t_j) - \psi^{(h)}(\ell^{(h)}(j2^m)), t_j)|. \tag{2.2.54}$$

We turn now to the proof of the Lemma. By (2.2.42)

$$\begin{aligned} &|\psi(\ell(gk)) - \psi^{(h)}(\ell^{(h)}(g))| \\ &\quad \leq |\psi(\ell(gk)) - \psi(\ell(g'k))| + |\psi^{(h)}(\ell^{(h)}(g')) - \psi^{(h)}(\ell^{(h)}(g))| \\ &\qquad + |\psi^{(h)}(\ell^{(h)}(g')) - \psi(\ell(g'k))| \\ &\quad \leq K_{17} h^{\frac{1}{2}} + |\psi^{(h)}(\ell^{(h)}(g')) - \psi(\ell(g'k))| \end{aligned} \tag{2.2.55}$$

if $0 < g - g' < 2^{m+1}$, so it suffices to assume $g_2 - g_1 > 2^{m+1}$.

Let j_1, j_2 satisfy $g_1 < j_1 2^m < j_2 2^m \leq g_2$, $j_1 2^m - g_1$, $g_2 - j_2 2^m < 2^m$.

By assumption, the strength of $\ell^{(h)}(g)$ is greater than $\Delta/2$, so that (2.2.46) holds. Thus, by (2.2.51) - (2.2.54)

$$\begin{aligned} &|\psi^{(h)}(\ell^{(h)}(j+1)2^m) - \psi(\ell(t_{j+1}))| \\ &\quad \leq (1 + \tau K_{16}) |\psi^{(h)}(\ell^{(h)}(j2^m)) - \psi(\ell(t_j))| + K_{18} h |\log h| \end{aligned}$$

By induction,

$$|\psi^{(h)}(\ell^{(h)}(j2^m)) - \psi(\ell(t_j))| \leq K_{19} \sum_{s=0}^{(j_2 - j_1)-1} (1 + \tau K_{16})^s h |\log h|$$

$$+ (1+K_{18}\tau)^{j-j_1} |\psi(\ell(t_{j_1})) - \psi^{(h)}(\ell^{(h)}(j_1 2^m))|$$

$$\leq e^{K_{19}(j_2-j_1)\tau}\left(h|\log h|(j_2-j_1)K_{19} + K_{18}h^{\frac{1}{2}}\right)$$

$$\leq K_{20}e^{K_{19}T_0}\delta$$

Thus (2.2.41) follows from (2.2.55).

In the following Lemma, we show that, for points $(x_0, q_0 k)$ sufficiently far from any shock, there exist j_0 such that the approximate backwards characteristic from $(j_0 h, q_0 k)$ is a distance no more than $O(h|\log h|)$ from the backwards characteristic from $(x_0, q_0 k)$ in the exact solution; and that

$$u_{j_0}^{q_0} = u(x_0, q_0 k) + O(1)\, 2^{-[\log_2 q_0]}$$

if $(x_0, q_0 k)$ is in a centered rarefaction fan,

$$u_{j_0}^{q_0} = u(x_0, q_0 k) + O(h)$$

otherwise.

Lemma 2.7 Assume that the hypotheses under which Lemma 2.5 was proven hold; that, for all q, $q \le q_0$, $q_0 k \le T_0$ and for all shocks ℓ that

$$|\ell(qk) - \ell^{(h)}(q)h| \le \delta \le h^{\frac{1}{3}}$$

where $\ell^{(h)}$ is the approximate shock associated with ℓ by (2.2.23); and that, for $x_0 \in \mathbb{R}$, $\eta > 0$

$$\min_{\ell} |x_0 - \ell(q_0 k)| \ge \eta + \max\left\{1 + T_0\|\varphi'\|_\infty, \frac{\|\varphi\|_\infty}{4C_0}\right\}(\delta + C_4' h|\log h| + C_6 h)$$

where C_6 is a positive constant to be determined. Then, for h sufficiently small there exist C_7, j_0 such that

$$|b(j_0, q_0; 0)h - c(u(x_0, q_0 k), x_0, q_0 k; 0)| \le (J_0 + 3)h$$

where J_0 is as in part 4 of Lemma 2.5;

$$\inf_{\ell^{(L)}} |j_0 - \ell^{(L)}(q_0)|h \geq \eta ;$$

$$|u(x_0, q_0 k) - u_{j_0}^{q_0}|,\ |j_0 h - x_0| \leq \max\{2^{-[\log_2 q_0]}, C_7 h|\log h|\}$$

if $u(x_0, q_0 k)$ is in a centered rarefaction fan,

$$|u(x_0, q_0 k) - u_{j_0}^{q_0}|,\ |j_0 h - x_0| \leq C_7 h|\log h|$$

otherwise.

First we wish to show that, for h sufficiently small,

$$\min_{\ell(t'),\, t' \leq t_0} |c(u(x_0,t_0),x_0,t_0;t') - \ell(t')| > \delta + C_4' h|\log h| + C_6 h . \qquad (2.2.56)$$

Since shocks cannot cross one another, it suffices to show that, for $\ell_R(t_0) > x_0 > \ell_L(t_0)$

$$c(u(x_0,t_0),x_0,t_0;t') - \ell_L(t'),\ \ell_R(t') - c(u(x_0,t_0),x_0,t_0;t') > \delta + C_4' h|\log h| + C_6 h$$

where ℓ_L, ℓ_R are defined by the condition, for all $t' < t_0$ there are no shocks $\tilde{\ell}$ such that $\ell_R(t') > \tilde{\ell}(t') > \ell_L(t')$. We prove the bound for ℓ_L ; the proof of the other inequality is identical.

There are two cases. First, assume there are no rarefaction centers in the interval

$$[\, c(u_R(\ell_L(t_0),t_0),\ell_L(t_0),t_0;0),\ c(u(x_0,t_0),x_0,t_0;0)\,]$$

then it suffices to show that

$$c(u(x_0,t_0),x_0,t_0;t') - c(u_R(\ell_L(t_0),t_0),\ell(t_0),t_0;t') > \delta + C_4' h|\log h| + C_6 h .$$

since $\ell_L(t') < c(u_R(\ell(t_0), t_0), \ell(t_0), t_0; t')$

for all $t' < t_0$. We have

$$
\begin{aligned}
x_0 - \ell_L(t_0) &= c(u(x_0,t_0), x_0, t_0; t') \\
&\quad - c(u_R(\ell_L(t_0), t_0), \ell(t_0), t_0; t') \\
&\quad + (t_0 - t')(u(x_0,t_0) - u_L(\ell(t_0), t_0)) \\
&\le (c(u(x_0,t_0), x_0, t_0; t') - c(u_R(\ell_L(t_0), t_0), \ell_L(t_0), t_0; t')) \\
&\qquad \times \left(1 + \left(\sup_{x \in I_{t'}} \frac{\partial u}{\partial x}\bigg|_{(x,t')}\right)(t_0 - t')\right) \\
&\le (c(u(x_0,t_0), x_0, t_0; t') - c(u_R(\ell_L(t_0), t_0), \ell_L(t_0), t_0; t')) \\
&\qquad \times (1 + T_0 \|\varphi'\|_\infty)
\end{aligned}
$$

Here

$$I_{t'} = [c(u_R(\ell_L(t_0), t_0), \ell_L(t_0), t_0; t'), c(u(x_0,t_0), x_0, t_0; t')]; \quad (2.2.56)$$

follows immediately from the hypotheses of the Lemma.

If

$$[c(u_R(\ell_L(t_0), t_0), \ell(t_0), t_0; 0), c(u(x_0,t_0), x_0, t_0; 0)]$$

contains a rarefaction center $\bar{x}$, then, for $t' \ge \frac{c_0}{4\|\varphi\|_\infty}$

we have, as before

$$
\begin{aligned}
x_0 - \ell_L(t_0) &\le (c(u(x_0,t_0), x_0, t_0; t') - c(u_R(\ell_L(t_0), t_0), \ell_L(t_0), t_0; t')) \\
&\qquad \times \left(1 + \left(\sup_{x \in I_{t'}} \frac{\partial u}{\partial x}\bigg|_{(x,t')}\right) T_0\right)
\end{aligned}
$$

with the second factor bounded by $1 + T_0 \max\left\{\|\varphi'\|_\infty, \frac{4\|\varphi\|_\infty}{c_0}\right\}$

and the argument proceeds as before.

If $t' \le \frac{c_0}{4\|\varphi\|_\infty}$, then there is an x'

$$\ell_L(t') < x' < c(u(x_0,t_0),x_0,t_0;t')$$

for which $c(u(x',t'),x',t';0) = \bar{x}$

so that

$$c(u(x_0,t_0),x_0,t_0;t') - \ell_L(t') > x' - \ell_L(t') \ge \frac{c_0}{2}$$

by (2.2. 7).

To prove the Lemma, we consider two cases, depending on whether or not

$$[c(u(x_0,t_0),x_0,t_0;0)-(J_0+3)h,\ c(u(x_0,t_0),x_0,t_0;0)+(J_0+3)h]$$

contains a rarefaction center; here J_0 is as in (2.2.28).

If

$$[c(u(x_0,t_0),x_0,t_0\ 0)-(J_0+3)h,\ c(u(x_0,t_0),x_0,t_0;0)+(J_0+3)h]$$

does not contain a rarefaction center, then Lemma 2.7 follows immediately from Lemma 2.5 if we can show

$f(\bar{j},0;\varrho_0) \equiv f_0$ is defined, where

$$\bar{j} = 2\left[\frac{c(u(x_0,t_0),x_0,t_0;0)}{2h}\right]$$

If $f(\bar{j},0;\varrho)$ is defined, then

$$|hf(\bar{j},0;\varrho) - c(u(x_0,t_0),x_0,t_0;0) - \varrho k u(x_0,t_0)|$$

$$\leq |h\, b(f(\bar{j},0,q),q;0) - c(u(x_0,t_0),x_0,t_0;0)| + |u(x_0,t_0) - u_j^c|\, qk$$
$$\leq C_4 h|\log h| + (J_0+2)h + T_0(C_4 + 2\|\varphi'\|_\infty)h$$

by Lemma 2. 5

$$|f(\bar{j},0;q) - \ell^{(h)}(q)|h$$
$$\geq |c(u(x_0,t_0),x_0,t_0;qk) - \ell(qk)| - C_4 h|\log h| - ((J_0+2) + 2T_0\|\varphi'\|_\infty)h - \delta$$
$$> 5h$$

by (2.2.56), with the appropriate choice of C_6 : thus $f(j,0;q+1)$ if defined, and the result follows by induction on q.

If

$$[c(u(x_0,t_0),x_0,t_0;0) - (J_0+3)h,\ c(u(x_0,t_0)x_0,t_0;0) + (J_0+3)h]$$

contains a rarefaction center $\bar{x}$ with associated approximate rarefaction center $k^0 + \frac{1}{2}$, the result follows immediately from (2.2.26), (2.2.29) if $q \leq q_{MIN}$.
If $q > q_{MIN}$ then by (2.2.29) there exists $\bar{j}$ such that
$|u_{\bar{j}}^{q_{MIN}} - u(x_0,t_0)| \leq (C_4' + (J_0+3)\|\varphi'\|_\infty + C_4)h$
If $f(\bar{j},q_{MIN};q_0)$ is defined, again, the Lemma follows from parts 3 and 4 of Lemma 2.5 by taking $j_0 \equiv f(\bar{j},q_{MIN};q_0)$.

The argument showing that $f(\bar{J}, \xi_{MIN}; \xi_o)$ is defined is identical to the one showing the approximate forward characteristic existed when there was no rarefaction center in

$$[c(u(x_o,t_o), x_o, t_o; 0) - (J_o+3)h, c(u(x_o,t_o), x_o, t_o; 0) + (J_o+3)h];$$

details are omitted.

According to the discussion preceeding Lemma 2.6, at all but a finite number of times, shocks in solutions to (2.2.1) satisfy conditions (r_L, r_R) for some choice of r_L, r_R In the following Lemma, we prove a corresponding fact about shocks in the approximate solutions, bounding the length of time that approximate shocks do not satisfy conditions (r_L, r_R) for any choice of r_L, r_R which implies a bound, by Lemma 2.3 and (2.2.40), on the amount the error can increase at those times.

Lemma 2.8 1) Assume that the hypotheses under which Lemma 2.5 was proven hold, and that there exists times $t_o < t_1 < t_2$ and a constant $\delta \le h^{\frac{L}{3}}$ such that the following holds:

For every shock $\ell(t)$ there exists (r_L, r_R), (r_L', r_R') ξ_o, $0 < \xi_o k - t_o < \delta$ such that $\ell(t)$ satisfies condition (r_L, r_R) for $t_o < t < t_1$, (r_L', r_R') for $t_1 < t < t_2$; that the associated approximate shock $\ell^{(k)}$

satisfies condition (r_L, r_R) at time step ϱ_0; and that, if $\ell^{(h)}(\varrho')$ satisfies condition (r_L, r_R) for $\varrho_0 \le \varrho' \le \varrho$, $\varrho k < t_1$, then $|\psi^{(h)}(\ell^{(h)}(\varrho')) - \psi(\ell(\varrho k))| \le \delta$, and $|\ell^{(h)}(\varrho'') - \ell(\varrho'' k) \le \delta$ for all $\varrho'' \le \varrho$. Then there exists $C_8 \ge 1$ independent of h, ϱ for h sufficiently small, such that the following holds:

There exists

$$0 < t_1 - \varrho_{IN} k,\ \varrho_{OUT} k - t_1 < C_8 \delta \tag{2.2.57}$$

such that $\ell^{(h)}(\varrho)$ satisfies condition (r_L, r_R) for $\varrho_0 \le \varrho \le \varrho_{IN}$, and that $\ell^{(h)}(\varrho_{OUT})$ satisfies condition (r_L', r_R') with

$$|\psi^{(h)}(\ell^{(h)}(\varrho_{OUT})) - \psi(\ell(\varrho_{OUT} k))| \le C_8 \delta \tag{2.2.58}$$

and

$$|\ell^{(h)}(\varrho) - \ell(\varrho k)| \le C_8 \delta \tag{2.2.59}$$

for all $\varrho \le \varrho_{OUT}$.

If ℓ is not overtaken by one or more shocks at time then

$$|\psi^{(h)}(\ell^{(h)}(\varrho)) - \psi(\ell(\varrho k))| \le C_8 \delta \quad \text{for } \varrho_{IN} \le \varrho \le \varrho_{OUT}.$$

2) If $\ell(t)$ satisfies conditions (r_L, r_R) for $0 < t < t_1$ then there exist $\mathring{\varrho}_{OUT}$, $\mathring{\varrho}_{OUT} k < C_8 h |\log h|$, such that $\ell^{(h)}(\mathring{\varrho}_{OUT})$ satisfies (r_L, r_R), and

$$|\psi(\ell(\varrho k)) - \psi^{(h)}(\ell^{(h)}(\varrho))| \le C_8 h |\log h|$$

for $\varrho \le \mathring{\varrho}_{OUT}$.

Let $q_{1N}+1$ be the first time step such that $\ell^{(h)}(q_{1N}+1)$ does not satisfy conditions (r_L, r_R) for some shock $\ell^{(h)}$ with $q_{1N} \geq q_0$, $q_{1N}k < t_1$.

Then one of the following three cases hold:

1) There exists $\tilde{\ell}^{(h)}$ such that

$$|\tilde{\ell}^{(h)}(q_{1N}+1) - \ell^{(h)}(q_{1N}+1)| = 4$$

2) (1) doesn't hold, S_{r_L} (S_{r_R}) is an interval, and

$$|\alpha_{L,r_L} - h\,b(\ell^{(h)}(q_{1N}+1)-3, q_{1N}+1;\, 0)| \leq C_4 h/\log h|$$

$$(|\alpha_{R,r_R} - h\,b(\ell^{(h)}(q_{1N}+1)+3, q_{1N}+1;\, 0)| \leq C_4 h/\log h|)$$

where $\alpha_{L,R_L}(\alpha_{R,r_R})$ is the left (right) endpoint of $S_{r_L}(S_{r_R})$.

3) (1) doesn't hold, S_{r_L} (S_{r_R}) is a rarefaction center $\alpha^0_{r_L}$ $(\alpha^0_{r_R})$ with associated approximate center $k^0_{r_L}+1$ $(k^0_{r_R}+1)$, and

$$b(\ell^{(h)}(q_{1N}+1)-3, q_{1N}+1;0) \neq k^0_{r_L}$$

$$(b(\ell^{(h)}(q_{1N}+1)-3, q_{1N}+1;0) = k^0_{r_R})$$

To prove (2.2.4), we show that each of these three cases imply a bound on $t_1 - q_{1N}k$ of the form (2.2.57).

Case 1) Assume, for example, that $\ell^{(h)}(q_{1N}) = \tilde{\ell}^{(h)}(q_{1N}) - 6$.

Then $\quad \ell(q_1 wk) < \tilde{\ell}(q_1 wk)$,

$$|\ell(q_1 wk) - \tilde{\ell}(q_1 wk)| \le |h\ell^{(h)}(q_1 w) - \ell(q_1 wk)|$$
$$+ |h\tilde{\ell}^{(h)}(q_1 w) - \tilde{\ell}(q_1 wk)|$$
$$+ |h\ell^{(h)}(q_1 w) - h\tilde{\ell}^{(h)}(q_1 w)|$$
$$\le \delta + \delta + 6h = 2\delta + 6h$$

Thus $\quad \ell(t_1) = \tilde{\ell}(t_1)$, and

$$\ell(q_1 wk) < c(u_L^{\circ}(\tilde{\ell}(t_1), t_1), \tilde{\ell}(t_1), t_1; q_1 wk) < \tilde{\ell}(q_1 wk)$$

and (2.2.57) holds by (2.2.15).

Case 2) If condition Γ_L fails to hold for $u_L(\ell^{(h)}(q_1 w+1))$,

then

$$|c(u_L(\ell(q_1 wk), q_1 wk), \ell(q_1 wk), q_1 wk; 0) - r_{L,\Gamma_L}|$$
$$\le |c(u_L(\ell(q_1 wk), q_1 wk), \ell(q_1 wk), q_1 wk; 0) - hb(\ell^{(h)}(q_1 w) - 3, q_1 w; 0)|$$
$$+ |hb(\ell^{(h)}(q_1 w) - 3, q_1 w; 0) - r_{L,\Gamma_L}|$$
$$\le \delta + 2h + \delta q_1 wk + 2C_4 h|\log h| + 2C_4 h q_1 wk$$
$$\le K_1(h|\log h| + \delta)$$

Since

$$t_{L,r_L} \le c(u_L^-(l(t_1),t_1),l(t_1),t_1;0) \le c(u_L(l(\varrho_{IN}k),\varrho_{IN}k),l(\varrho_{IN}k),\varrho_{IN}k;0)$$

we have

$$|l(\varrho k) - c(u_L^-(l(t_1),t_1),l(t_1),t_1;\varrho_{IN}k)| \le K_1(\delta + h/|\log h|)/(1-T_0C_{INF})$$

which implies (2.2.57) by (2.2.15).

Case 3) If condition Γ_L fails to hold for $u_L^{(h)}(l^{(h)}(\varrho_{IN}+1))$

then

$$\begin{aligned} |u_L(l(\varrho_{IN}k),\varrho_{IN}k) - \varphi_L(x^0_{r_L})| &\le |u\ (l^{(h)}(\varrho_{IN})) - \varphi_L(x^0_{r_L})| \\ &\quad + |u_L(l^{(h)}(\varrho_{IN})) - u_L(l(\varrho_{IN}k),\varrho_{IN}k)| \\ &\le 4C_4'h + \delta \end{aligned}$$

Since

$$\begin{aligned} 0 \le l(\varrho_{IN}k) &- c(u_L^-(l(t_1),t_1),l(t_1),t_1;\varrho_{IN}k) \\ &\le \frac{\delta + 4C_4'h}{\varrho_{MIN}k} \end{aligned}$$

and (2.2.57) follows from (2.2.15).

The proofs of (2.2.57) when condition Γ_R fails to hold for $u_R(l^{(h)}(\varrho))$ are similar, and are omitted.

According to our discussion of the exact solution, and by (2.2.41), either $l = l_k$, for some $k = 1,\ldots,M$ with $l_1,\ldots,l_M$ overtaking one another at time t_1; or $|l^{(h)}(\varrho_{IN})h - \tilde{l}^{(h)}(\varrho_{IN})h| \ge |l(\varrho_{IN}k) - \tilde{l}(\varrho_{IN}k)| - 2\delta \ge K_D$ for all shocks $\tilde{l}$ such that $l(t_0) \ne \tilde{l}(t_0)$.

Next, we show that, if $\ell = \ell_k$ for some $k = 1, \ldots, M$ such that $\ell_1(t) < \ldots < \ell_M(t)$ overtake one another at time t_1, then $\ell_1^{(h)}(q) = \ldots = \ell_M^{(h)}(q)$ for all $q \ge q_{SHOCK}$, where $t_1 + K_3\delta \ge q_{SHOCK}$.

$$
\begin{aligned}
|\ell_1^{(h)}(q_{1N})h - \ell_M^{(h)}(q_{1N})h| &\le |h\ell_1^{(h)}(q_{1N}) - \ell_1(q_{1N}k)| \\
&\quad + |h\ell_M^{(h)}(q_{1N}) - \ell_M(q_{1N}k)| \\
&\quad + |\ell_1(q_{1N}k) - \ell_M(q_{1N}k)| \\
&\le K_4\delta
\end{aligned}
$$

$$u_L(\ell_i^{(h)}(q_{1N})) \ge u_R^{(h)}(\ell_i(q_{1N})) - \Delta + \delta \quad i = 1, \ldots, M$$

$$u_j^{q_{1N}} \ge u_L(\ell_i^{(h)}(q_{1N})) - C_4' K_4 \delta, \text{ for}$$
$$\ell_i^{(h)}(q_{1N}) < j < \ell_{i+1}^{(h)}(q_{1N}), \quad i = 1, \ldots, M-1$$

by (2.2.24).

If $K_3\delta + t_1 \ge qk \ge q_{1N}k$, then, for h sufficiently small, $u_R(\ell_1^{(h)}(q)) \ge u_L(\ell_M^{(h)}(q)) + (M-2)(\Delta - C_4' K_4 \delta) \ge 0$

$$(u_L(\ell_i^{(h)}(q)) - u_R(\ell_i^{(h)}(q)))/2 \ge \Delta - (K_3 + C_8) C_4 \delta$$

for $i = 1, M$ by (2.2.40).

Thus

$$s(\ell_1^{(k)}(q)) - s(\ell_M^{(k)}(q)) \geq 2(s - (K_3 + C_8)C_4'\delta)$$

and the result follows from (2.1.7), for an appropriate choice of K_3.

Define $q_1 = \max\{q_{SHOCK}, [\frac{t_1}{k}]+1\}$ if ℓ is overtaken by one or more shocks at time t_1; $q_1 = [\frac{t_1}{k}]+1$ otherwise. Since $(q_1 - q_{1N})k \geq (K_3 + C_8)\delta$ we have, for $q_{1N} \leq q \leq q_1$

$$\begin{aligned} |\ell^{(k)}(q)h - \ell(qk)| &\leq |\ell(qk) - \ell(q_{1N}k)| + |\ell(q_{1N}k) - \ell^{(k)}(q_{1N})h| \\ &\quad + h|\ell^{(k)}(q_{1N}) - \ell^{(k)}(q)| \\ &\leq K_6\delta \end{aligned}$$

Let

$$\begin{aligned} x_1 = \ell(q_1k) - \big((K_6\delta + C_4'h|\log h| + C_c h) \\ \times \max\{1 + T_0\|\varphi'\|_\infty, \|\varphi\|/4C_0\}\big) - \eta \end{aligned}$$

where $\eta = (J_0+2)h + (C_4 h|\log h| + (J_0+3)h)(1 + \|\varphi'\|_\infty T_6)$ if $S_{r_L'}$ is an interval; $\eta = (J_0+2)h$ if $S_{r_L'}$ is a rarefaction center.

Then $|\ell(q_1k) - x_1| \leq K_7\delta$,

and it follows from (2.2.29) that x_1 satisfies condition Γ_L'.

In the case where $S_{r_L'}$ is an interval, the argument used to show (2.2.56) can be applied to show that

$$x_{R,r_L'} - c(u(x_1,q,k),x_1,q,k;0)$$
$$\geq c(u_L(\ell(q,h),q,k),\ell(q,k),q,k;0) - c(u(x_1,q,k),x_1,kq_1,0)$$
$$\geq C_4 h|\log h| + (J_0+3)h$$

So that the interval

$$[c(u(x_1,q,k),x_1,q,k;0) - C_4 h|\log h| - (J_0+3)h,$$
$$c(u(x_1,q,k),x_1,q,k;0) + C_4 h|\log h| + (J_0+3)h]$$

is contained in $S_{r_L'}$. In either the case of $S_{r_L'}$ an interval or rarefaction center (2.2.27), (2.2.28) and Lemma 2.7 imply that there exists j_1 such that $\ell^{(h)}(q_1) - j_1 \geq 3$, and $u^q_{f(j_1,q_1;q)}$ satisfies condition r_L' for all $q \geq q_1$ for which $f(j_1,q_1;q)$ is defined. We also have $|j_1 h - \ell(q,k)| \leq K_8 \delta$.

We shall prove (2.2.58) and (2.2.59) with

$$q_{OUT} = \left[\frac{k_8\delta}{k} \max\{2/\lambda, 1\}\right] + 1 + q_1$$

if we denote by

$$\bar{\ell}_{L,R}(t) = \ell_{1,M}(t)$$
$$\bar{u}_{L,R}(t) = u_{L,R}(\bar{\ell}_{L,R}(t),t)$$
$$\bar{\ell}^{(h)}_{L,R}(q) = \ell^{(h)}_{1,M}(q)$$
$$\bar{u}^{(h)}_{L,R}(q) = u_{L,R}(\bar{\ell}^{(h)}(q))$$

if ℓ is overtaken by $\ell_1,\dots,\ell_M$ at time t_1;

$$\bar{\ell}_{L,R}(t) = \ell(t)$$
$$\bar{u}_{L,R}(t) = u_{L,R}(\ell(t),t)$$
$$\bar{\ell}^{(k)}_{L,R}(q) = \ell^{(k)}(q)$$
$$\bar{u}^{(k)}_{L,R}(q) = u_{L,R}(\ell^{(k)}(q))$$

otherwise.

Then $\bar{u}_{L,R}(t)$, $\bar{\ell}_{L,R}(t)$ are Lipschitz functions of t for $q_{IN}k \le t \le q_{OUT}k$; similarly

$|\bar{\ell}^{(k)}_{L,R}(q+1) - \ell^{(k)}_{L,R}(q)| \le 1$, $|\bar{u}^{(k)}_{L,R}(q+1) - \bar{u}^{(k)}_{L,R}(q)| \le C_4' h$

for all $q_{IN} \le q \le q_{OUT}$ by (2.2.40) and, for $t > t_1$ $(q > q_1)$

$$\bar{\ell}_{L,R}(t) = \ell(t) \quad , \bar{u}_{L,R}(t) = u_{L,R}(\ell(t),t)$$
$$\left(\bar{\ell}^{(k)}_{L,R}(q) = \ell^{(k)}(q) \ , \quad \bar{u}^{(k)}_{L,R}(q) = u_{L,R}(\ell^{(k)}(q)) \right)$$

Thus $|\psi^{(k)}(\ell^{(k)}(q)) - \psi(\ell(qk))| \le K_{10}\delta$

for $q_1 \le q \le q_{OUT}$.

To show that $u_L(\ell^{(k)}(q_{OUT}))$ satisfies condition Γ_L' it suffices to show that, for some q, $q_1 \le q \le q_{OUT}$, $u(\ell^{(k)}(q))$ satisfies condition Γ_L', where q may vary from shock to shock. If this is true, then an argument similar to the one used to from (2.2.57) for q_{IN} shows that $u_L(\ell^{(k)}(q_{OUT}))$ satisfies Γ_L'.

Proof is by contradiction. If $u_L(\ell^{(h)}(\rho))$ does not satisfy r_L' for all ρ, $\rho_1 \le \rho \le \rho_{OUT}$) then $f(j_1, \rho_1; \rho)$ is defined for all ρ, $\rho_1 \le \rho \le \rho_{OUT}$) and

$$5 \le \ell^{(h)}(\rho) - f(j_1, \rho_1; \rho) \le \frac{2K_9\delta}{\lambda} + K_8\delta + 1 \le K_{11}\delta$$

$$\ell^{(h)}(\rho_{OUT}) - f(j_1, \rho_1; \rho_{OUT})$$

$$\le \ell^{(h)}(\rho_1) - b(f(j_1, \rho_1; \rho_{OUT}), \rho_{OUT}; \rho_1)$$

$$- \lambda(\rho_{OUT} - \rho_1)(u^{\rho_{OUT}}_{f(j_1, \rho_1; \rho_{OUT})} - s(\ell^{(h)}(\rho_{OUT}))$$

$$+ (\rho_{OUT} - \rho_1)K_{10}\delta + \gamma(\rho_{OUT} - \rho_1) + C_4$$

$$\le K_8\delta - \lambda(\rho_{OUT} - \rho_1)(4 - K_{11}C_4'\delta - K_{10}\delta)$$

$$+ K_{12}\delta$$

$$< 0$$

for h sufficiently small, thus obtaining the required contradiction.

The proof of part 2 of the Lemma proceeds along similar lines to that of part 1: the details are omitted.

Proof of Theorem 2.4: First, it suffices to prove the Theorem under the assumption that the strength of the approximate shocks are always greater than $\delta/2$, so that Lemmas 2.5 - 2.8 hold. Then the bounds proven below, combined with an induction on the time step q similar to that used at the end of the proof of Lemma 2.5, show that the assumption holds for all $qk \leq T_0$.

Let $0 = t_0 < t_1 < \ldots < t_N = T_0$ be times such that for every shock ℓ, there exists $r_L(i,\ell), r_R(i,\ell)$ such that $\ell(t)$ satisfies conditions $(r_L(i,\ell); r_R(i,\ell))$ for $t_i < t < t_{i+1}$ and take h sufficiently small so that

$$(C_4' C_8)^N h^{\frac{1}{2}} |\log h| \leq h^{\frac{1}{3}}$$

and that Lemmas 2.6 - 2.8 hold. Then, by Lemma 2.8, there exists q^0_{OUT}, $q^0_{OUT} k \leq C_8 h |\log h|$, such that every shock ℓ and associated approximate shock $\ell^{(h)}$ satisfies $(r_L(1,\ell), r_R(1,\ell))$, and

$$|\psi(\ell(qk)) - \psi^{(h)}(\ell^{(h)}(qk))| \leq C_8 h |\log h|$$

for $q \leq q^0_{OUT}$. By Lemmas 2.6 and 2.8, for every i there exist q^i_{IN}, q^{i-1}_{OUT}, with

$$0 < t_i - q^i_{IN} k,\ q^{i-1}_{OUT} k - t_{i-1} \leq (C_4' C_8)^i h^{\frac{1}{2}} |\log h|$$

such that, for $q^{i-1}_{OUT} \leq q \leq q^i_{IN}$, $\ell(qk), \ell^{(h)}(q)$ satisfy conditions $(r_L(i,\ell), r_R(i,\ell))$, and

$$|\psi^{(h)}(\ell^{(h)}(q)) - \psi(\ell(qk))| < (C_4' C_8)^i h^{\frac{1}{2}} |\log h| \leq h^{\frac{1}{3}}$$

If $q_{IN}^{i} \le q \le q_{OUT}^{i}$ then, by Lemma 2.8,

$$|\ell^{(h)}(q) - \ell(qk)| \le C_4'^{\,i} C_8^{\,i+1} h^{\frac{1}{2}} |\log h| \le h^{\frac{1}{3}},$$

if ℓ is overtaken by another shock at t_i otherwise,

$$|\psi^{(h)}(\ell^{(h)}(q)) - \psi(\ell(qk))| \le C_4^{\,i} C_8^{\,i+1} h^{\frac{1}{2}} |\log h|;$$

thus part 2 of the Theorem follows immediately.

To prove part (i) of the Theorem, let

$$0 < t_{INF} = \mathrm{INF}\{t : (x,t) \in D\}$$
$$0 < d_{INF} = \mathrm{INF}\{|\ell(t) - x : (x,t) \in D\}$$

Choose h sufficiently small so that

$$d_{INF} > 2C_4' h|\log h| + \mathrm{MAX}\left\{\frac{C_0}{4\|\varphi\|_\infty}, (1 + T_0\|\varphi'\|_\infty)\right\} \times \left(h^{\frac{1}{3}} + C_4' h|\log h| + C_6 h\right)$$

then, by Lemma 2.7 for every $(x,t) \in D$ there exists j_0 such that

$$|u(x,qk) - u_j^q| \le \mathrm{MAX}\left\{C_4' h,\ 2^{-[\log_2 \frac{t_{INF}}{k}]}\right\} \le K_1 h$$
$$|j_0 - j| \le C_4$$

where q, j are as in the statement of the Theorem.

Since $\min_{\ell^{(h)}} |\ell^{(h)}(q_0) - j_0| \ge 2C_4' |\log h|$

there is no approximate shock $\ell^{(h)}(q_0)$ between j_0 and j

thus, by (2.2.21)

$$|u(x,t) - u_j^q| \le |u(x,t) - u(jh,qk)| + |u(jh,qk) - u_{j_0}^q| + |u_{j_0}^q - u_j^q|$$
$$\le h\left(2\,\mathrm{MAX}\left\{\frac{1}{t_{INF}}, \|\varphi'\|/(1 - T_0 C_{INF})\right\}\right.$$
$$\left. 2\,\mathrm{MAX}\left\{\frac{1}{t_{INF}^2}, \|\varphi\|\|\varphi'\|/(1 - T_0 C_{INF})\right\}\right)$$

$$+ K_1 h + C_4^{1/2} h |\log h|$$
$$\leq A h |\log h|$$

The techniques used to prove Theorem 2.4 do not generalize to the case of random sampling. In particular, it is not clear what the appropriate analogue to Lemma 2.5 is in the case of random sampling. However, in the case where the solution consists entirely of rarefaction waves, we can apply the central limit theorem more or less directly.

Theorem 2.9 Let $\varphi(x)$ be a monotone increasing function such that for some $L > 0$, $u_-, u_+ \varepsilon \mathbb{R}$ $\varphi(x) \equiv u_-$ for $x < -L$, $u_+ = \varphi(x)$ for $x > L$. If $u^{(h)}(x, t; \vec{a})$ is the approximate solution to (2.2.1) obtained using Glimm's method with random sampling, with initial data φ, then

$$\lim_{h \to 0} \mu \left\{ \vec{a} : \int_{-(L + \frac{t}{\lambda} u_-)}^{(L + \frac{t}{\lambda} u_+)} \left(u(x,t) - u^{(h)}(x, nk; \vec{a}) \right) dx < \frac{r}{\sqrt{h}} \right\}$$
$$= \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{r} e^{-r'^2/2\sigma} \, dr'$$

for some $\sigma < \infty$, where $\frac{k}{h} = \lambda < \|\varphi\|_\infty^{-1}$, $n = \left[\frac{t}{k}\right]$ and the measure μ is the one defined in the discussion of random sampling.

Proof: Let $S_q(\vec{a}) = \int_{-(L+\frac{t}{\lambda}u_-)}^{(L+\frac{t}{\lambda}u_+)} u^{(h)}(x, qk; \vec{a})\, dx$

Reasoning as in Example 2.3, for $qk < t$

$$S_q(\vec{a}) - S_{q-1}(\vec{a}) = X_q(\vec{a}) = X_q(a^q)$$

where

$$X_q(a^q) = (u_- - u_+)h \qquad \text{if } a^q \in [-1, \lambda u_-)$$

$$= \left(\frac{2a^q}{\lambda} - (u_+ + u_-)\right)h \quad \text{if } a^q \in [\lambda u_-, \lambda u_+)$$

$$= (u_+ - u_-)h \qquad \text{if } a^q \in [\lambda u_+, 1)$$

We have also used here the fact that $u^{(h)}(-(L+\frac{t}{\lambda}u_-), qk; \vec{a}) = u_-$

$u^{(h)}(L+\frac{t}{\lambda}u_+, qk; \vec{a}) = u_+$ for all $qk < t$.

The random variables X_q are independent, identically distributed

$$E(X_q) \equiv \int X_q(\vec{a})\, d\mu = \frac{1}{2}\int X_q(a^q)\, da^q = -k\frac{(u_+^2 - u_-^2)}{2}$$

If we define $Y_q = X_q/h$, then Y_q are independent, identically distributed, with $E(Y_q)$, $E((Y_q - E(Y_q))^2) = \sigma^2$ finite and independent of h;

$$E(S_q) = \int u^{(h)}(x, 0)\, dx - qk\frac{(u_+^2 - u_-^2)}{2}$$

Since u is a weak solution to (2.2.1),

$$\int_{-(L+\frac{t}{\lambda}u_-)}^{(L+\frac{t}{\lambda}u_+)} u(x,t)\,dx = \int u(x,0)\,dx$$

$$+ \frac{1}{2}\int_0^t u\left(-\left(L+\frac{t}{\lambda}u_-\right),t'\right)^2 dt'$$

$$- \frac{1}{2}\int_0^t u\left(L+\frac{t}{\lambda}u_+,t'\right)^2 dt'$$

$$= \int u(x,0)\,dx - t\left(\frac{u_+^2 - u_-^2}{2}\right)$$

Since $\left| \int u(x,t)\,dx - E(S_n) \right| \le Kh$,

for some K independent of h the result follows from the central limit theorem.

§ 2.3 Gas Dynamics in One Dimension

In the case of systems of equations, we have no results corresponding to Theorem 2.4 for any general initial data. For the Riemann problem, analogues of Theorems 2. 4 and 2.9 can be proven rather easily using the additivity property In this case we have $O(h|\log h|)$ accuracy both in the continuous part of the solution and for the location of discontinuities using van der Corput sampling, and the central limit theorem holding for random sampling. This is a very special situation, however, since there is no wave interaction. In an effort to understand the errors introduced by the interaction of the sampling and the coupling between the modes, we looked at the following test problem for the gas-dynamical system (1.1.3). The initial data consists of a shock and a rarefaction wave of the same family (forward facing) next to one another (figure 2.14). The shock overtakes the rarefaction, the cancellation between them weakening both (figure 2.15, A). The nonlinear coupling between the modes produces waves of the other two families in back of the shock and moving to the left, away from the shock. These are, a weak backwards facing compression wave (figure 2.15, B), with a weak gradient in the pressure and velocity, and a strong density/entropy wave (figure 2.15, C), advected passively by the velocity field $u(x,t)$.

In figures 2.16 - 2.18 we display the calculation of this problem by Glimm's method using, respectively, the random (7,3) stratified random, and binary van der Corput sampling procedures. All calculations were done on the spatial interval $[0,1]$ with boundary conditions at 0 and 1 obtained by assuming the solutions satisfy $\frac{\partial U}{\partial x}\Big|_{x=0,1} = 0$. The various solutions being compared were computed with $\Delta x = .01$ and are represented graphically by circles for the computed values at mesh points, interpolated by a dotted line. Also plotted on each of the graphs with a solid line, is a solution obtained using Glimm's method, with van der Corput sampling and $\Delta x = .0025$. Having compared the latter solution with a similar one done for $\Delta x = .005$ we found that the two results differed by less than .5%, so that the method has converged for $\Delta x = .0025$. For the purposes of comparing the various $\Delta x = .01$ solutions, we treat the $\Delta x = .0025$ solutions as exact, against which the $\Delta x = .01$ solutions can be compared.

The sampling governs the rate at which the shock and rarefaction interact. If s_{ℓ}^{n} is the speed of the shock, located at mesh point ℓ at time step n, and $\lambda_{+}^{n} = u_{\ell+1}^{n} + c_{\ell+1}^{n}$, then the shock will cancel with a piece of the rarefaction wave, and produce more wave of the other two families, at time step n + 1 if and only if $a^{n+1} \in [\frac{k}{h}\lambda_{+}^{n}, \frac{k}{h} s_{\ell}^{n})$. Thus the loss of gradient information observed in the randomly sampled solution (figure 2.16) is a result of random fluctuations in the rate of interaction between the shock and rarefaction which is producing the wave. The use of stratified random sampling (figure 2.17) produces smoother profiles, but the shape of the entropy wave is incorrect; in particular, there is a sizable deviation in the density profile, a failure to conserve mass. The profile obtained using van der Corput sampling (figure 2.18) is in much closer agreement with the $\Delta r = .0025$ result, the rate of wave production being modeled much better than in the other two cases. In fact, if one uses van der Corput sampling, one can use a much coarser mesh and still get good results for this problem. In figure 2.19, we present the results obtained on this problem with binary van der Corput sampling, and $\Delta r = 1/30$. The absolute locations of the waves, and their locations relative to each other, are to within Δr; more important, the size and shape of the waves, which are more sensitive to the cumulative error introduced by the sampling, are in very close agreement with the $\Delta r = .0025$ result. In all the calculations, the shock discontinuity is sharp, as guaranteed by Glimm's method.

Chapter 3 Operator Splitting

In [6], Chorin proposed a method for computing multi-dimensional unsteady compressible flow using Glimm's method by means of operator splitting. We can write the equations of motion for an ideal gas in two space dimensions as

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}(F(U)) + \frac{\partial}{\partial y}(G(U)) = 0$$
$$U(x,y,t) = U : \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}^4$$
$$U(x,y,0) = \varphi(x,y) \qquad \varphi : \mathbb{R}^2 \to \mathbb{R}^4$$

$$U = \begin{pmatrix} \rho \\ m \\ n \\ E \end{pmatrix}, \quad F(U) = \begin{pmatrix} m \\ \frac{m^2}{\rho} + p \\ \frac{mn}{\rho} \\ \frac{m}{\rho}(E+p) \end{pmatrix}, \quad G(U) = \begin{pmatrix} n \\ \frac{mn}{\rho} \\ \frac{n^2}{\rho} + p \\ \frac{n}{\rho}(E+p) \end{pmatrix}$$

Here ρ is the density, m is the x-component of momentum, n is the y-component of momentum, and E is the total energy. We can express the velocity $\vec{v}$ and internal energy ϵ in terms of the above variables: $v_x = \frac{m}{\rho}$ is the x-component of the velocity, $v_y = \frac{n}{\rho}$ is the y-component of the velocity, and $\epsilon = \frac{E}{\rho} - \frac{1}{2}(v_x^2 + v_y^2)$. The pressure p is a function of ρ and ϵ : $p = (\gamma - 1)\rho\epsilon$ where γ, the ratio of specific heats, is a constant assumed to be greater than 1. As was the case for one space variable, the value of U at a given point is uniquely determined by the values of ρ, P and $\vec{v}$ at that point:

$$m = \rho v_x \ , \ n = \rho v_y \ , \ E = \frac{P}{\gamma - 1} + \frac{\rho}{2}(v_x^2 + v_y^2)$$

We wish to construct approximate solutions

$$U^{h_x, h_y}(x, y, nk) = U^n_{ij} \in \mathbb{R}^4$$

$$(i-1)h_x \le x < (i+1)h_x$$
$$(j-1)h_y \le y < (j+1)h_y$$

where $\Delta x = 2h_x$, $\Delta y = 2h_y$ are spatial increments, $\Delta t = k$ is a time increment, and i, j, n are integers, with $j+n$, $i+n$ even, $n \ge 0$.

Assume we know $U^n_{i,j}$ and want to find $U^{n+1}_{i+1,j+1}$, the procedure is as follows:

1) For each j perform one time step of Glimm's method for the equations $\frac{\partial V}{\partial t} + \frac{\partial}{\partial x}(F(V)) = 0$, taking as initial data $V^0_\ell = U^n_{\ell,j}$. Set the result $V^1_{\ell+1} = U^{n+\frac{1}{2}}_{\ell+1,j}$ (we denote this procedure by $(L^x_k U^n)_{i+1,j} = U^{n+\frac{1}{2}}_{i+1,j}$), which is now piecewise constant on

$$i h_x < x < (i+2) h_x$$
$$(j-1)h_y < y < (j+1) h_y \ .)$$

2) For each fixed i perform one time step of Glimm's method for the equations $\frac{\partial W}{\partial t} + \frac{\partial}{\partial x}(G(W)) = 0$, taking as initial data $W^0_\ell = U^{n+\frac{1}{2}}_{i+1,\ell}$, time step k.

Set the result $W^{1}_{\ell+1} = U^{n+1}_{i+1,j+1}$ (we denote this procedure by $\left(L^{y}_{k} U^{n+\frac{1}{2}}\right)_{i+1,j+1} = U^{n+1}_{i+1,j+1}$)

The solution thus derived at time $(n+1)k$ is interpreted as being the piecewise constant function

$$U^{h_x,h_y}(x,y,(n+1)k) = U^{n+1}_{i+1,j+1} \quad i+n,\ j+n \text{ even}$$

$$ih_x < x < (i+2)h_x$$

$$jh_y < y < (j+2)h_y \ .$$

A necessary condition on the time step k is that it must satisfy (2.1.3) for each of the one-dimensional calculations. In practice, a somewhat different condition was used:

$$k \sup_{i,j} \left(|v^{n}_{i,j}| + c^{n}_{i,j}\right) \leq \min\{h_x, h_y\}$$

where $c = \sqrt{\frac{\gamma p}{\rho}}$ is the sound speed, and $|\vec{v}| = (v_x^2 + v_y^2)^{\frac{1}{2}}$.

The above procedure is formally the same as is done to construct multi-dimensional difference methods from one-dimensional ones (Sod [51], Strang [52]). However, the mechanism by which Glimm's method propagates the solution to the equations in one dimension is rather different than that of difference methods, as it requires many time steps for the cumulative effect of the sampling to give the correct wave speeds; therefore the actual justification of the splitting procedure, currently unknown, is likely to be quite different than the usual truncation error analysis for difference methods.

The Riemann problems in question are easily solved, given the solution for one-dimensional gas dynamics. For example, to solve Riemann's problem for

$$\frac{\partial V}{\partial t} + \frac{\partial}{\partial x}(F(V)) = 0$$

$$V = \left(\bar{\rho}, \bar{\rho}\bar{v}_x, \bar{\rho}\bar{v}_y, \frac{\bar{p}}{\gamma-1} + (\bar{v}_x^2 + \bar{v}_y^2)\frac{\bar{\rho}}{2}\right)$$

take the solution $\rho(x,t)$, $p(x,t)$, $u(x,t)$ in example 1.5 with

$$\rho_{L,R} = \bar{\rho}_{L,R}, \quad p_{L,R} = \bar{p}_{L,R}, \quad u_L = \bar{v}_{x,L}, \quad u_R = \bar{v}_{x,R}$$

$$\bar{\rho}(x,t) = \rho(x,t), \quad \bar{p}(x,t) = p(x,t), \quad \bar{v}_x(x,t) = u(x,t);$$

$$\bar{v}_y(x,t) = \bar{v}_{y,L}$$

if (x,t) is to the left of the contact discontinuity ℓ_s

$$\bar{v}_y(x,t) = \bar{v}_{y,R}$$

if (x,t) is to the right of the contact discontinuity ℓ_s. Thus in the x-sweep, we have ordinary 1-D gas dynamics, with the discontinuity in v_y passively advected. To solve the Riemann problem for $\frac{\partial W}{\partial t} + \frac{\partial}{\partial y}(G(W)) = 0$, interchange the roles of v_x and v_y.

To test the validity of this procedure, we looked at the simplest two-dimensional test problem possible (figure 3.1). We took our computational domain to be the unit square with the computational mesh aligned with the x- and y-axes, and took the initial conditions to be

$$\varphi(x,y) \begin{array}{ll} = U_R & x > y \\ = U_L & x < y \end{array}$$

$$U_L = \begin{pmatrix} \rho_L \\ \rho_L v_{x,L} \\ \rho_L v_{y,L} \\ P_L/(\gamma-1) + \frac{\rho_L}{2}(v_{x,L}^2 + v_{y,L}^2) \end{pmatrix} \qquad U_R = \begin{pmatrix} \rho_R \\ \rho_R v_{x,R} \\ \rho_R v_{y,R} \\ P_R/(\gamma-1) + \frac{\rho_R}{2}(v_{x,R}^2 + v_{y,R}^2) \end{pmatrix}$$

This is the Riemann problem, for which we have an analytic solution. Computationally, it is a two-dimensional problem, since the initial discontinuity is at a 45° angle to the mesh directions.

We denote by v_N the component of the velocity normal to $x = y$, v_T the component parallel to $x = y$.

$$v_N = \frac{v_x - v_y}{\sqrt{2}} \qquad v_T = \frac{v_x + v_y}{\sqrt{2}}$$

$$v_{N,R} = \frac{v_{x,R} - v_{y,R}}{\sqrt{2}} \qquad v_{T,R} = \frac{v_{x,R} + v_{y,R}}{\sqrt{2}}$$

$$v_{N,L} = \frac{v_{x,L} - v_{y,L}}{\sqrt{2}} \qquad v_{T,L} = \frac{v_{x,L} + v_{y,L}}{\sqrt{2}}$$

Throughout these test calculations we will set $v_{T,L} = v_{T,R} = 0$ i.e., we will be looking at problems for which there is no slip line in the exact solution. Unless otherwise indicated, the calculations shown were done on a 50 x 50 grid: $\Delta x = \Delta y = .02$. The results of the calculations are displayed by plotting the profiles of various quantities along the line $y = 1 - x$, and comparing them with the exact solution. In these plots, the computed values at the mesh points are graphed as circles, interpolated by a dotted line: the exact solution is plotted as a solid line. When boundary conditions are required, we assume the solution is constant on lines parallel to the initial jump. This was quite effective in preserving the symmetry of the solution, and enabled us to run for long times without noise from the boundary affecting the results.

The one-dimensional calculations using Glimm's method in the x and y directions require sampling sequences $\vec{a}_x, \vec{a}_y$ which we took to be two independent van der Corput sampling sequences: $\vec{a}_x$ was the (3,2) van der Corput sequence, and $\vec{a}_y$ was the (5,3) van der Corput sequence. This insured optimal distribution in the square $[-1,1) \times [-1,1)$.

In figure (3.2), we show the results for the following problem

$$\begin{array}{lll} \rho_L = .353 & \rho_R = .1 & \\ p_L = 14.0 & p_R = .5 & \\ v_{N,L} = -1.78 & v_{N,R} = -11.6 & \gamma = 1.667 \end{array} \tag{3.1}$$

The exact solution is a strong, right-facing shock. It is almost stagnant (after 175 time steps, the exact shock point has moved only two zones). By this time, the oscillations (±80% of the exact post-shock value in the pressure) have begun to make themselves known by a three zone error in the shock location, the shock moving a distance more than two times greater than it should have. We see substantial values (±60% of $|\bar{v}_{N,L}-\bar{v}_{N,R}|$) for $v_T(x,y,t)$ the tangential component of the velocity appearing. Finally, the density profile shows a substantial deviation from conservation of mass.

The fundamental reason why large errors occur in this problem is that, although each half-step L^x_k, L^y_k models the resulting one-dimensional gas dynamics well, the problem it is modeling is $O(1)$ incorrect from the point of view of the two-dimensional flow. For example, consider the problem one solves (one for each value of j) in the first x-pass in the test problem 3.1. They are each the same Riemann problem for a one-dimensional gas flow, with the jump taking place along the diagonal. The left and right states

$$\bar{V}_{L,R} = \begin{pmatrix} \bar{\rho}_{L,R} \\ \bar{\rho}_{L,R}\bar{u}_{L,R} \\ \bar{\rho}_{L,R}(\bar{v}_y)_{L,R} \\ \frac{\bar{p}_{L,R}}{\gamma-1} + (\bar{u}^2_{L,R} + (\bar{v}_y)^2_{L,R})\frac{\bar{\rho}_{L,R}}{2} \end{pmatrix}$$

for the one-dimensional problem are

$$\bar{p}_{L,R} = p_{L,R} \qquad \bar{\rho}_{L,R} = \rho_{L,R}$$

$$\bar{u}_L = \frac{v_{N,L}}{\sqrt{2}} \qquad \bar{u}_R = \frac{v_{N,R}}{\sqrt{2}} \qquad \bar{v}_{y,L} = \frac{v_{N,L}}{\sqrt{2}} \qquad v_{y,R} = \frac{v_{N,R}}{\sqrt{2}}$$

The jump in the velocity $\bar{u}_L - \bar{u}_R$ is less than $v_{N,L} - v_{N,R}$ so a weaker forward-facing shock than that of the original two-dimensional problem is produced, as well as a backwards-facing rarefaction wave. If we sample anywhere in the fan other than the left or right states, we get a $(v_y)^{n+\frac{1}{2}}_{i,j} > \bar{u}_L, \bar{u}_R$ The new values $(v_y)^{n+\frac{1}{2}}_{i,j}$, $\rho^{n+\frac{1}{2}}_{i,j}$, $p^{n+\frac{1}{2}}_{i,j}$ depend only on the sampling value a'_x and the ratio k/h_x but not on k and h_x separately. So the difference between these and the exact answer is an $O(1)$ quantity relative to the mesh spacing. In particular, there is an $O(1)$ contribution to the tangential component of the velocity. Since there has been an $O(1)$ change in the thermodynamic variables p and ρ, there is no reason for the y-pass to produce a tangential velocity to cancel the one produced by the x-pass, and in fact it does not. Similar phenomena occur for a shock tube, (figure 3.3) or even a Riemann problem whose solution consists of two (continuous) rarefaction waves (figure 3.4); in the latter case, there is an $O(1)$ error introduced due to the incorrect modeling of the discontinuity at t = 0 in the first few time steps. This is a start-up error and does not get amplified at later times, in contrast to the error produced at shocks, which is produced and propagated as long as the shock exists.

The above failures in the splitting procedure in situations when there are discontinuities in $\rho, \vec{v}$ can be viewed as a consequence of an invalid interchange of limiting procedures. Analytically, shock solutions are obtained as limits of viscous solutions as some set of diffusion coefficients go to zero. One might try to obtain the shocked solutions by using an operator splitting method to solve the viscous equations; the splitting procedure is then known to converge as $\Delta t \to 0$. Then, in the inviscid limit, the viscous solutions converge to the physically correct shocked solutions. In a difference method, the two limiting procedures take place simultaneously, with the coefficients multiplying the numerical diffusion approaching zero with Δt. The use of operator splitting with Glimm's method corresponds to letting the diffusion coefficients vanish for nonzero Δt. This interchange of limits is valid for continuous solutions, or near contact discontinuities, but near discontinuities in ρ or $\vec{v}$ the two limiting procedures are singular with respect to each other, and cannot be interchanged freely.

In an effort to solve this problem, we introduce some artificial viscosity into Glimm's method, to be used in the presence of large pressure or density gradients, but designed so as not to diffuse contact discontinities. In terms of the above discussion, this will introduce into Glimm's method the limiting procedure used in difference methods, but only near discontinuities in ρ or v.

The general form of the viscosity used here is due to Lapidus [25], who developed it for use with the Lax-Wendroff difference method; see also Sod [48]. If $U^{n+\frac{1}{2}}$ is the array of conserved quantities after half-step in the x-direction, we define

$$\begin{aligned}(L^x_{vis}U^{n+\frac{1}{2}})_{i,j} &= U^{n+\frac{1}{2}}_{i,j} \\ &+ \lambda_x \vec{C}_0 \cdot \Delta'_x \{ \chi_{\delta_0}((v_x)^{n+\frac{1}{2}}_{i+2,j} - (v_x)^{n+\frac{1}{2}}_{i,j}) \, |\Delta'_x (v_x)^{n+\frac{1}{2}}_{i+2,j}| \, \Delta' U^{n+\frac{1}{2}}_{i+1,j} \} \\ &+ \lambda_x \vec{C}_1 \cdot \Delta'_x \{ \chi_{\delta_1}((v_x)^{n+\frac{1}{2}}_{i+1,j} - (v_x)^{n+\frac{1}{2}}_{i+1,j}) \, \Delta' U^{n+\frac{1}{2}}_{i+2,j} \}\end{aligned}$$

$$\Delta'_x f_{i,j} = f_{i,j} - f_{i-2,j}$$

$$\chi_\delta(x) \quad \begin{matrix} = 0 & x > -\delta \\ = 1 & x \le -\delta \end{matrix}$$

$$\lambda_x = k/h_x$$

We define L^y_{vis} similarly, interchanging the roles of i and j, replacing v_x by v_y, h_x by h_y, and Δ'_x by Δ'_y the latter defined by $\Delta'_y f_{i,j} = f_{i,j} - f_{i,j-2}$. The vectors $\vec{C}_0, \vec{C}_1$ and the numbers δ_0, δ_1 are parameters to be set at the beginning of the calculation. If U^n is the approximate solution array, U^{n+1} at time step n + 1, using Glimm's method and artificial viscosity, is

$$U^{n+1} = L^{\eta}_{vis} L^{\eta}_{k} L^{\nu}_{vis} L^{\nu}_{k} U^{n}$$

Note that it may be necessary to make further restrictions on the time step k than (2.1.2) for the scheme to be stable; see [25]. When applying artificial viscosity with Glimm's method, one wants to choose the parameters $\vec{C}_0, \vec{C}_1, \delta_0, \delta_1$ such that the artificial viscosity will not spread the sharp contact discontinuities generated by Glimm's method.

There are some difficulties with the use of the above artificial viscosity with Glimm's method in the presence of extremely strong shocks. Since it acts on the conserved quantities momentum and total energy, it is possible, in one time step, to diffuse a large amount of momentum across a shock. If there is little or no diffusion of mass, this leads to an artificially large preshock velocity, which, when used to compute the pressure from the total energy, may yield a low or even negative pressure. One way to alleviate this problem is to use a smaller time step; another is to allow the artificial viscosity to act on the density, but only near strong discontinuities in ρ or $\vec{v}$.

In figure 3.7 we display the result of using Glimm's method with the above artificial viscosity for the shock tube problem.

$$\rho_L = 1.0 \qquad \rho_R = .125 \tag{3.2}$$

$$p_L = 1.0 \qquad p_R = .1$$

$$v_{N,L} = v_{N,R} = 0 \qquad \gamma = 1.4$$

The artificial viscosity parameters in this calculation were

$$\vec{C}_0 = (0, .5, .5, .5) \quad \vec{C}_1 = (0, .25, .25, .25) \quad \delta_0 = \delta_1 = .02$$

Since $C_0^1 = C_0^0 = 0$, the numerical viscosity was not applied to the density, which insured that the contact discontinuity remained sharp. Comparing this solution with the one obtained for this problem using Glimm's method without artificial viscosity (figure 3.3), we see that the post-shock oscillations seen in the latter solution are, in the artificial viscosity solution, virtually eliminated from the pressure and density, and strongly attenuated in the velocity and internal energy. The shock is spread over three mesh points, but the contact discontinuity remains sharp (for a comparison to difference methods, see Chapter 4).

Chapter 4 Discussion and Conclusions

In one space variable, Glimm's method has directly built into it an approximate form of the propagation of information along characteristics, without the smoothing of such information, as occurs in most difference methods, and without any complicated bookkeeping; the sampling procedure determining the weakest wave or wave interaction to be resolved. If a pair of characteristics have speeds $c_1, c_2, c_1 > c_2$, the waves carried by each of them move toward each other at time step g if $a^g \varepsilon [\lambda c_2, \lambda c_1)$. To model smooth flow correctly using this scheme, one needs $\vec{a}$ to have good distribution properties with respect to all intervals to the above form, even if the length of the interval is approaching zero, as $n \to \infty$: $\delta(\vec{a}, n_0, n_0+n, I_n) \xrightarrow[n\to\infty]{} 0$ where $I_n = [d, d+\frac{f(n)}{n})$ for as many $f(n)$ as possible, independent of d, n_0. This is the motivation for using van der Corput sampling. A comparison of Theorem 2.9 and part 1 of Theorem 2.4 indicates the gain in accuracy in going from random to van der Corput sampling in the absence of any interaction. The numerical examples in figures 2.16 and 2.17 show how the randomness can cause a loss of information in a wave interaction situation.

The assumptions of Theorem 2.4 explicitly exclude the case of a compression wave steepening into a shock. This is likely to be a failure in technique, rather than in the result itself, as seen in the following example. Let $u(x,t)$ be the solution to the inviscid Burgers' equation, with initial data

$$u(x,0) \begin{cases} = 1 & x \leq -1 \\ = -x & -1 < x < 1 \\ = -1 & x \geq 1 \end{cases}$$

Then

$t < 1$

$$u(x,t) \begin{cases} = 1 & x < -(1-t) \\ = -x/(1-t) & -(1-t) \leq x \leq (1-t) \\ = -1 & x > (1-t) \end{cases}$$

$t > 1$

$$u(x,t) \begin{cases} = 1 & x < 0 \\ = -1 & x > 0 \end{cases}$$

It is easy to show that there exists T_0 independent of h for all h sufficiently small, such that Glimm's method with van der Corput sampling applied to this problem gives a solution $u^{(h)}(x,t)$ consisting of a single jump discontinuity between 1 and -1 for all $t \geq T_0 > 1$ But an estimate in Liu [35], specialized to this case and van der Corput sampling, says that the integral of $u^{(h)}(x,t)$ across the shock differs from the exact answer by no more than $O(h^{1/2}|\log h|)$ so the location of the discontinuity is correct to $O(h^{1/2}|\log h|)$

In fact, there are many similarities, both technical and conceptual, between the proof of our Theorem 2.2 and those in [35], as both are based on the idea of tracing approximate characteristics. But we are willing to assume the existence of a sufficiently regular solution and prove that the approximate solution converges to it. This yields more detailed results for a single equation; also it is likely that any extension of these results to systems would not require the restriction on the initial data needed in [35].

There do not appear to be analogous results to Theorem 2.3 in the case of difference schemes. The study of the accuracy of difference schemes in the presence of discontinuities for linear equations (Majda and Osher [36], Mock and Lax [37]) gives some indication of the situation. It is shown in [36] for example that, in the presence of a discontinuity, the true order of accuracy of the approximate solution obtained from a difference scheme can be substantially less than the formal order of accuracy defined in terms of the truncation error of the scheme.

Sod [48] compared the performance of a number of the more widely used difference schemes along with Glimm's method, on a one-dimensional shock tube problem for gas dynamics. The results obtained there using Glimm's method were not the best possible, due to the use od stratified random sampling. On the other hand, comparing difference schemes to Glimm's method on this problem is not entirely fair either, since the latter has the exact solution built into it. In any case, we present in figure 4.1 the calculation done with Glimm's method but using van der Corput sampling. The result obtained is clearly superior to any of those in [48]. It would be interesting to compare the schemes in [48] on the test problem in § 2.3.

The original proposal in [6] for using Glimm's method with operator splitting for multidimensional gas dynamics was seen to give incorrect results for flows in which there occur large jumps in the pressure or velocity along surfaces oblique to the mesh directions. The inclusion of artificial viscosity appears to be successful in eliminating these errors, without degrading the rest of the solution.

For the purpose of comparison with the results in Sod [48] obtained by the various difference methods, we computed the shock tube problem (3.2) using Glimm's method with artificial viscosity, but on a 100 x 100 grid ($\Delta x = .01$, $\Delta y = .01$) (figures 4.2, 4.3). In principle, the problem solved here is more difficult than the one solved in [48], since in the latter it is solved as a one-dimensional problem. But the exact answer is the same for both, and the results are worth comparing.

The calculation of the rarefaction, and the width of the shock transition in the results obtained with Glimm's method compare favorably to the best results by the difference methods.

There are still some small amplitude post-shock oscillations that the artificial viscosity fails to damp. However, the treatment of the contact discontinuity is clearly superior to that given by any of the difference methods. The latter, with one exception, spread the contact discontinuity over 4 - 10 mesh points, with the number of mesh points increasing as a function of time. At the earlier time displayed (figure 4.2), we calculated one intermediate value in the contact discontinuity; but this small degree of spreading is transient, since at later times (figure 4.3) the contact discontinuity is sharp. The only difference methods in [48] to obtain nearly this resolution are those to which Harten's artificial compression method [21] was applied. But our results do not exhibit the oscillations on either side of the contact discontinuity that appeared when artificial compression was applied.

There are several directions in which further work is indicated. For one-dimensional flows, Glimm's method with van der Corput sampling is quite effective in modeling the interaction of discontinuities with the smooth parts of the flow, without introducing unacceptable errors in the latter. The fact that the solutions to the Riemann problem we use in the numerical scheme satisfy exactly the conservation laws is probably not essential to the accuracy of the method, since much of that information is lost in the sampling procedure. What is essential is that the solution which is sampled has built into it the physically correct waves and wave speeds to some reasonable order of accuracy. Thus it is feasible to try to model with Glimm's method the dynamics of media other than an ideal gas in Cartesian coordinates: for example, gas dynamics with source terms or unusual equations of state, or elastic-plastic flow.

The central advantage of Glimm's method for multi-dimensional flow is its treatment of contact discontinuities. They are computed automatically as sharp discontinuities, and do not spread as time progresses. This is especially crucial in shock interaction problems where the contact discontinuities are not present in the initial flow field, but come into existence at some later time on account of shock reflections and interactions. In this case, it is impossible to use a material interface-following technique, such as in [41], to prevent the discontinuity from spreading. In order for Glimm's method to be effective in such situations, the artificial viscosity for controlling the errors near shocks must be introduced in such a way so as not to degrade unacceptably the rest of the solution, particularly the contact discontinuities. In a specific test problem, we were able to accomplish this, but more extensive experiments are required to determine the optimum form and strength for the viscosity for some broad class of problems.

Finally, there are some analytical results which would be interesting to have. One would be elimination of the restriction on time in Theorem 2.3, by some combination of our techniques and those in [35]. Another problem is to prove the appropriate analogue to Theorem 2.3, in the case of systems. This can probably be done by an adaptation of Glimm's perturbation theory for the Riemann problem in [15] to the characteristic equations, currently being looked at by the author in the case of the isentropic flow equations. Finally, one would like to see some analytic justification of the use of splitting and Glimm's method in the case of continuous flow, even for a simple rarefaction wave oblique to the mesh.

We have attempted to assess the effectiveness of Glimm's method as a method for computing time dependent discontinuities compressible fluid flows. In one space variable, we obtain results which easily satisfy the three criteria given at the beginning of the Introduction. In two or more space variables, Glimm's method, with the inclusion of a suitable artificial viscosity, has the potential for surpassing the performance of difference methods because of its treatment of contact discontinuities. However, this modification of the method requires further investigation to determine the limits of its applicability.

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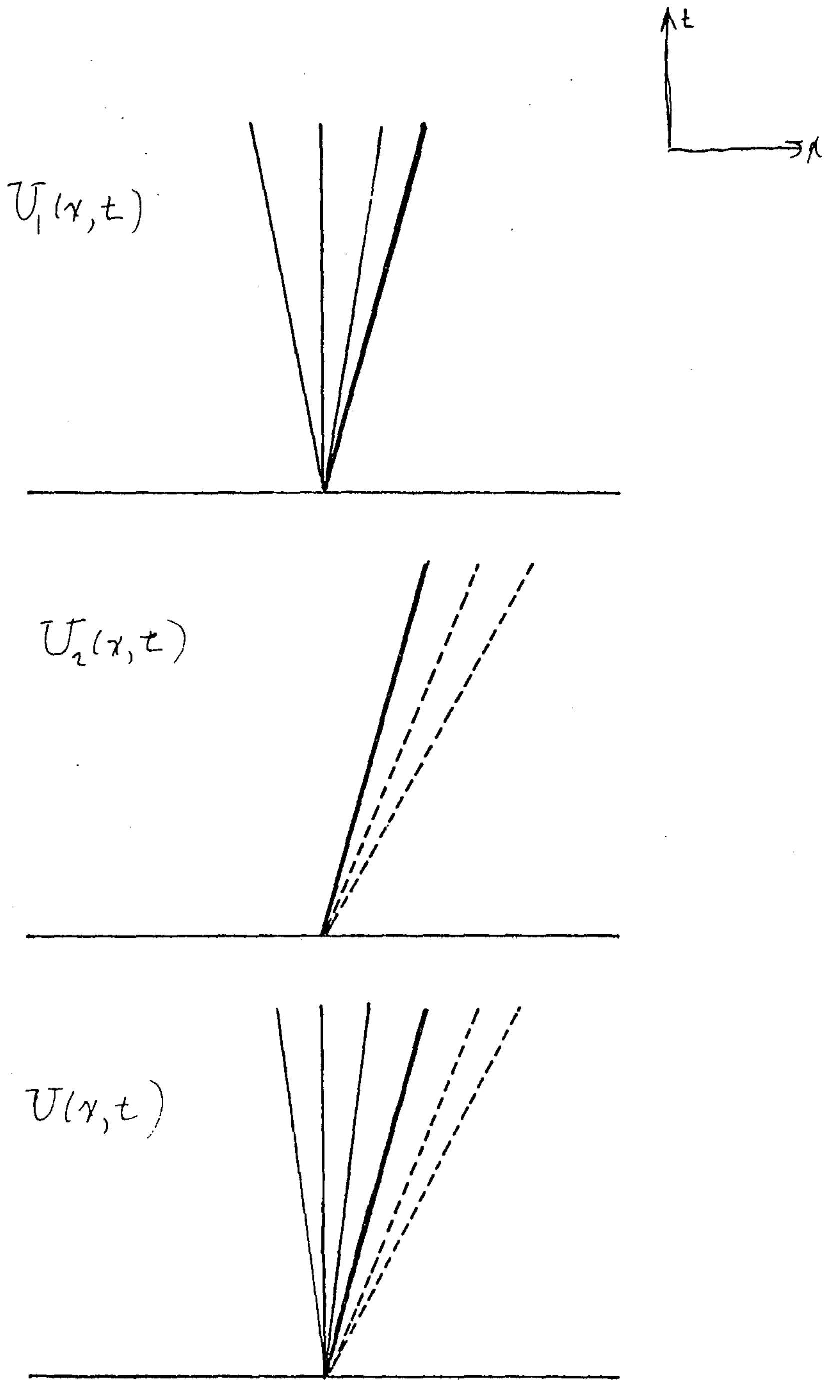


Figure 1.1

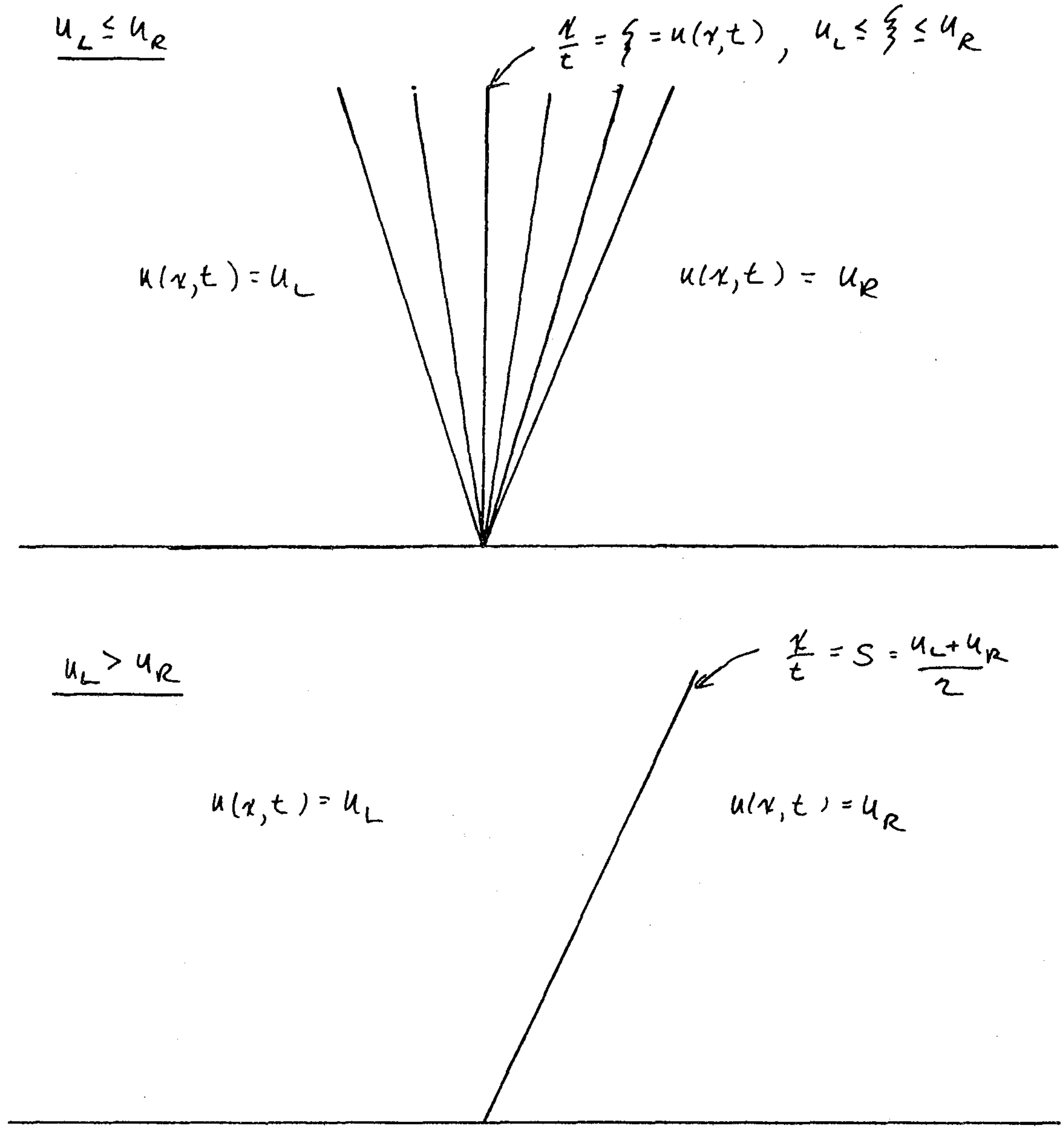


Figure 1.2 Riemann Problem for the Inviscid Burgers' Equation

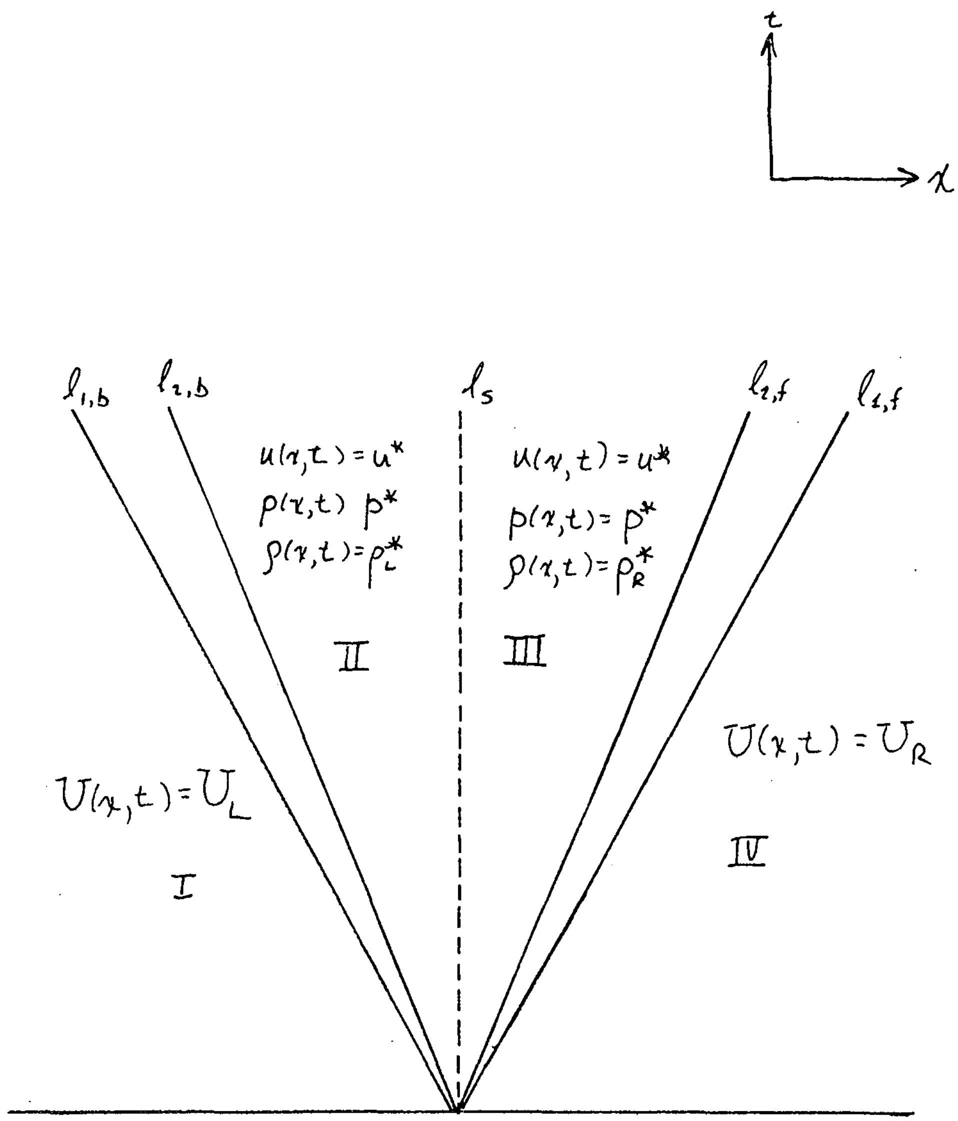


Figure 1.3 Riemann Problem for Gas Dynamics

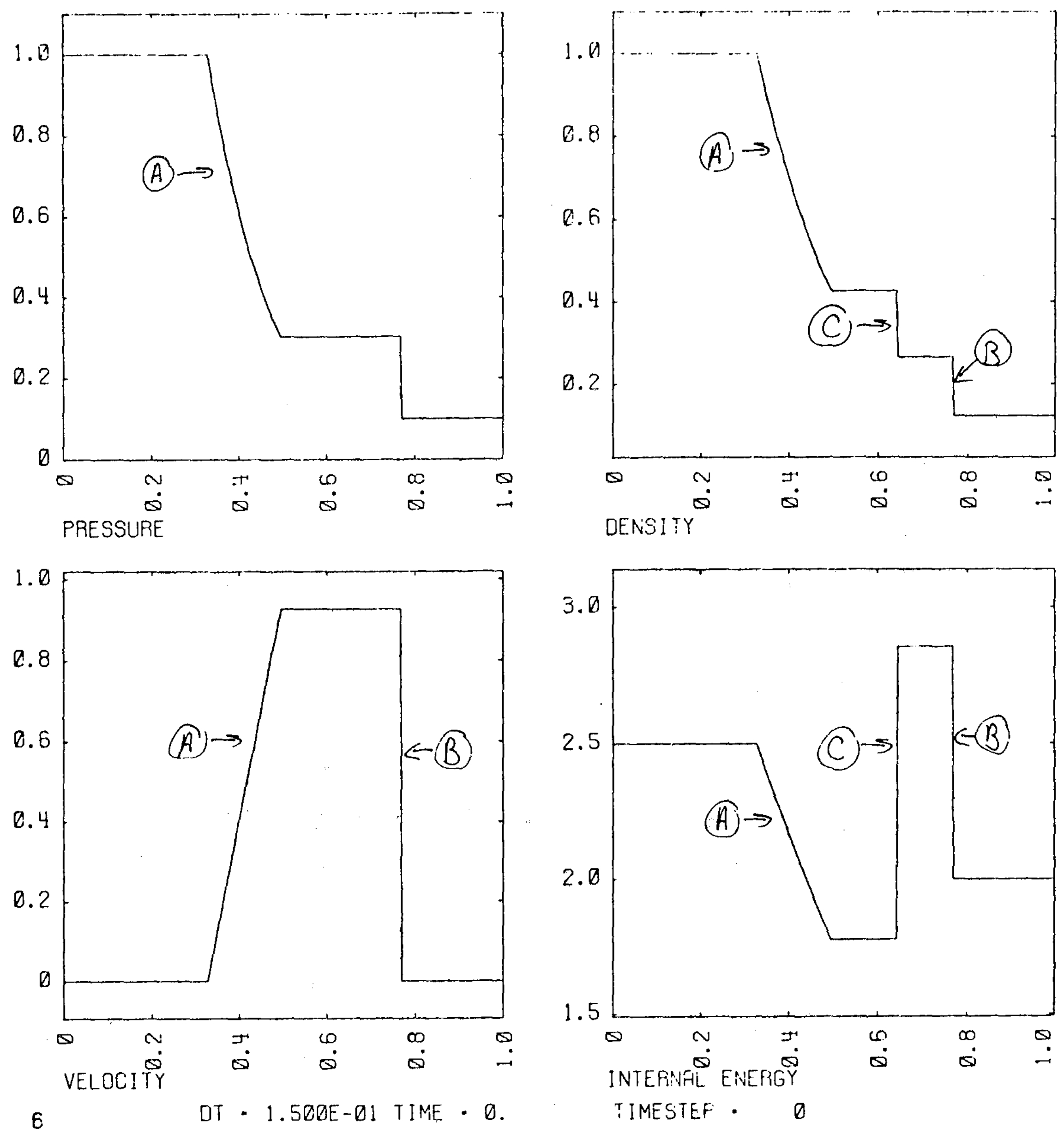
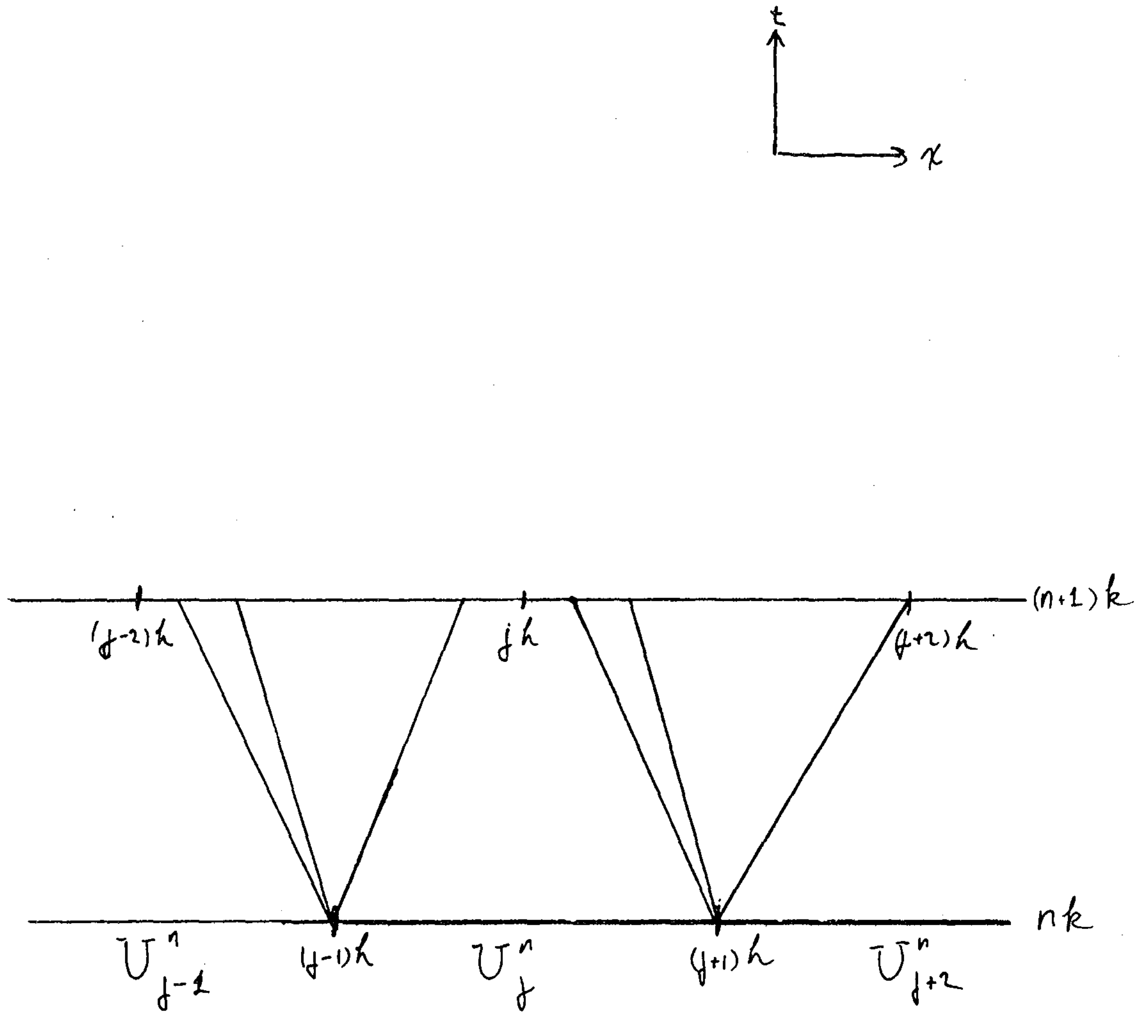

1.0
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1.0
PRESSURE
A
DENSITY
A
C
B
VELOCITY
A
B
3.0
2.5
2.0
1.5
INTERNAL ENERGY
A
C
B
6
DT • 1.500E-01 TIME • 0.
TIMESTEP • 0

Figure 1.4



$$U^{(h)}(x, nk) = U^n_j \quad , (j-1)h < x < (j+1)h$$
$$j+n \text{ even}$$

Figure 2.1 Exact Solution to Initial Value Problem

for $nk \le t \le (n+1)k$

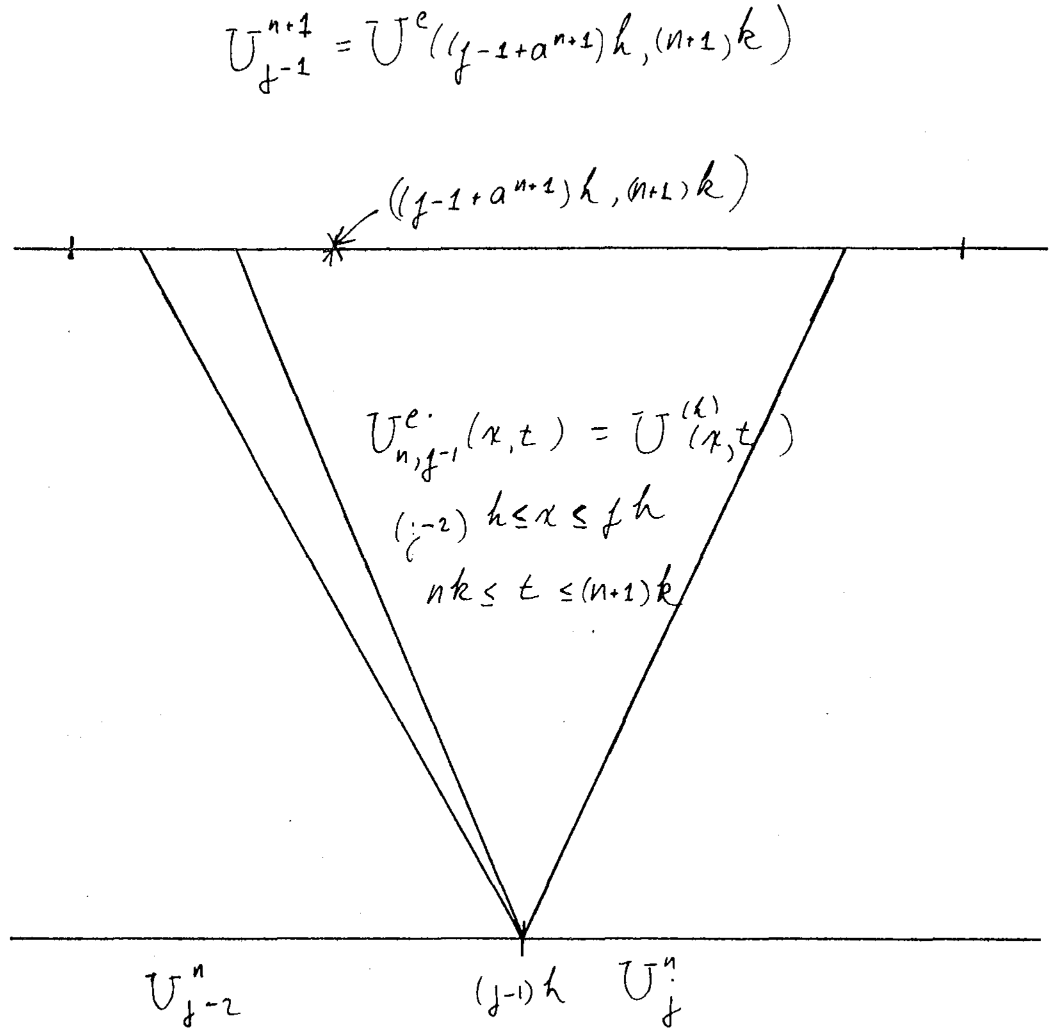


Figure 2.2 Sampling the Riemann Problem

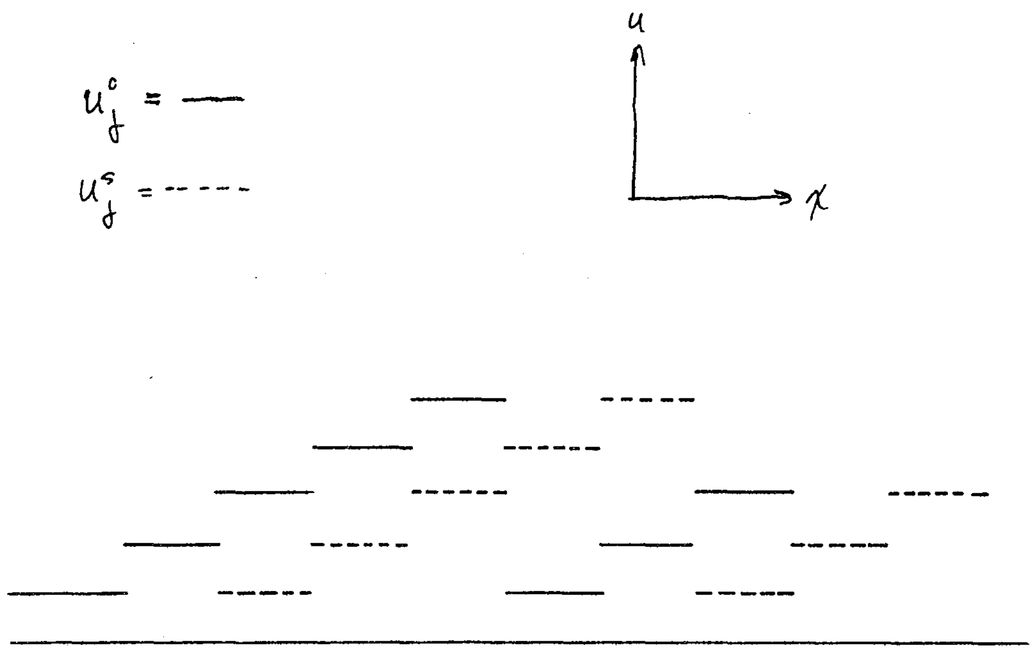


Figure 2.3 Discrete Travelling Wave Solutions Obtained Using Glimm's Method for (2.1.9)

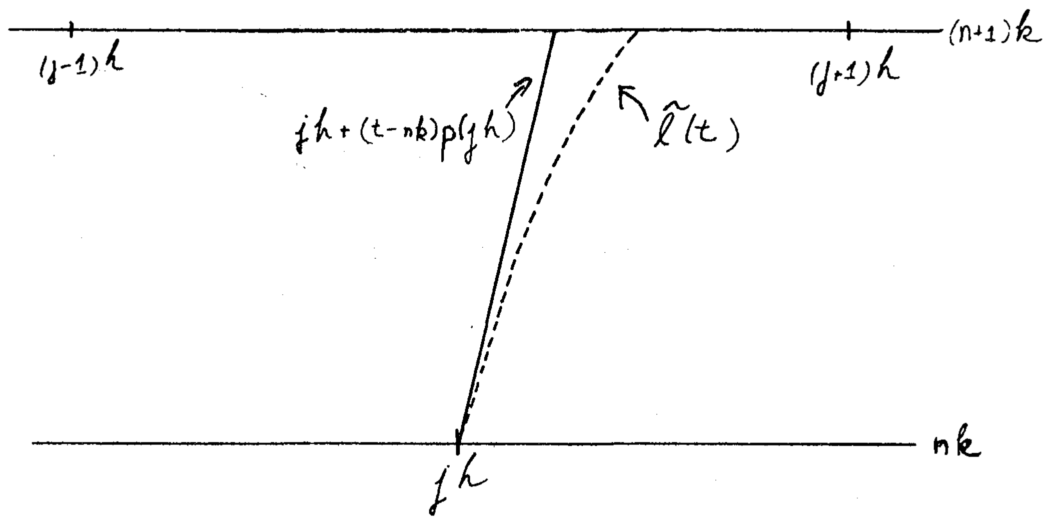


Figure 2.4 Exact and Approximate Solutions to the Riemann Problem for (2.1.10)

$$\frac{d\tilde{\ell}}{dt} = p(\tilde{\ell}(t)) \quad \tilde{\ell}(nk) = jh$$

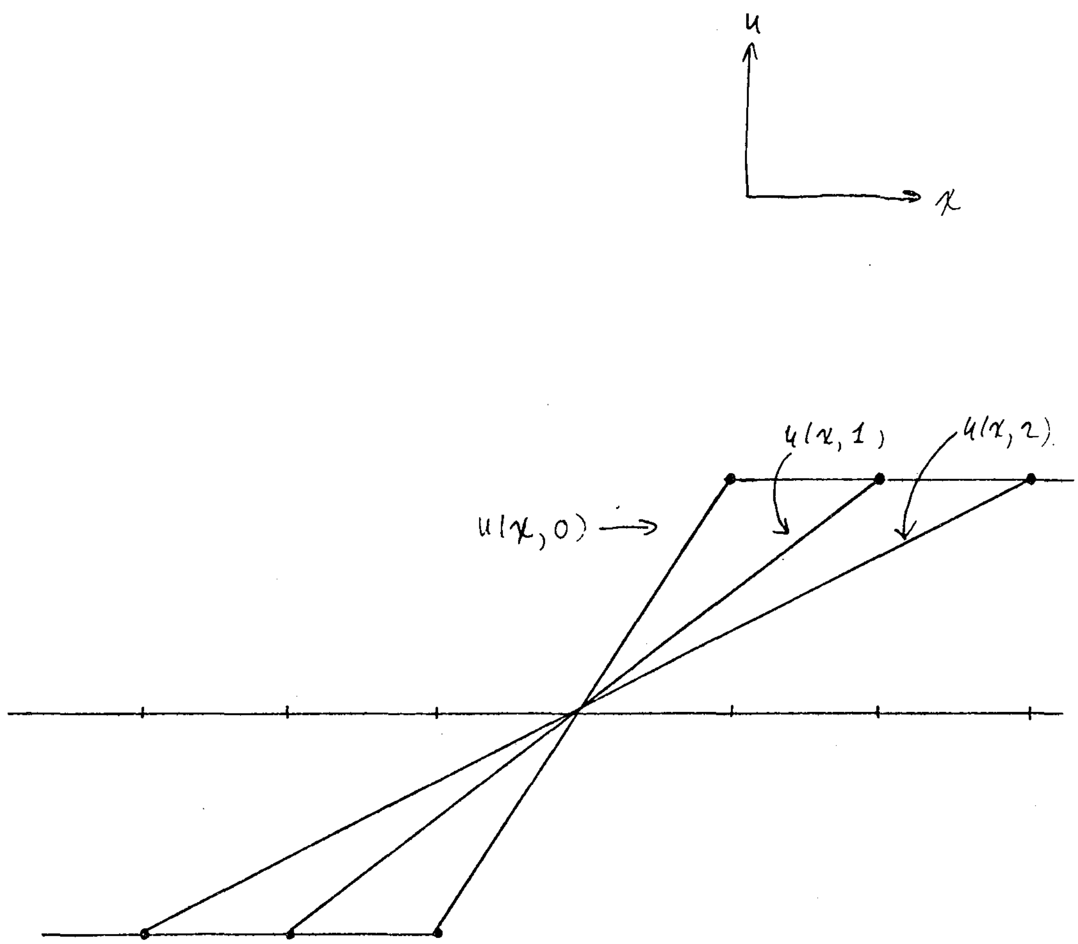


Figure 2.5 Rarefaction Wave Solution to the Inviscid Burgers' Equation

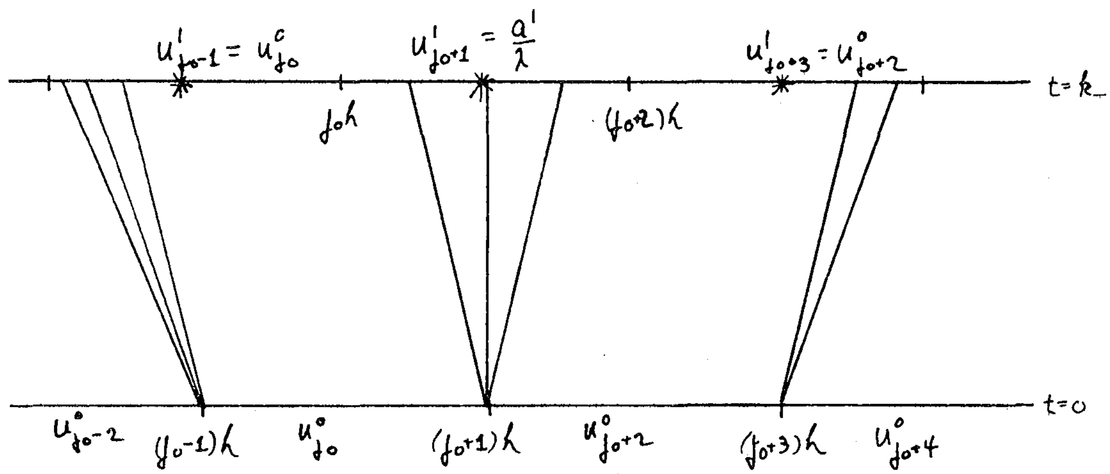


Figure 2.6 Sampling an Approximate Rarefaction Wave

(* Denotes Sampled Point)

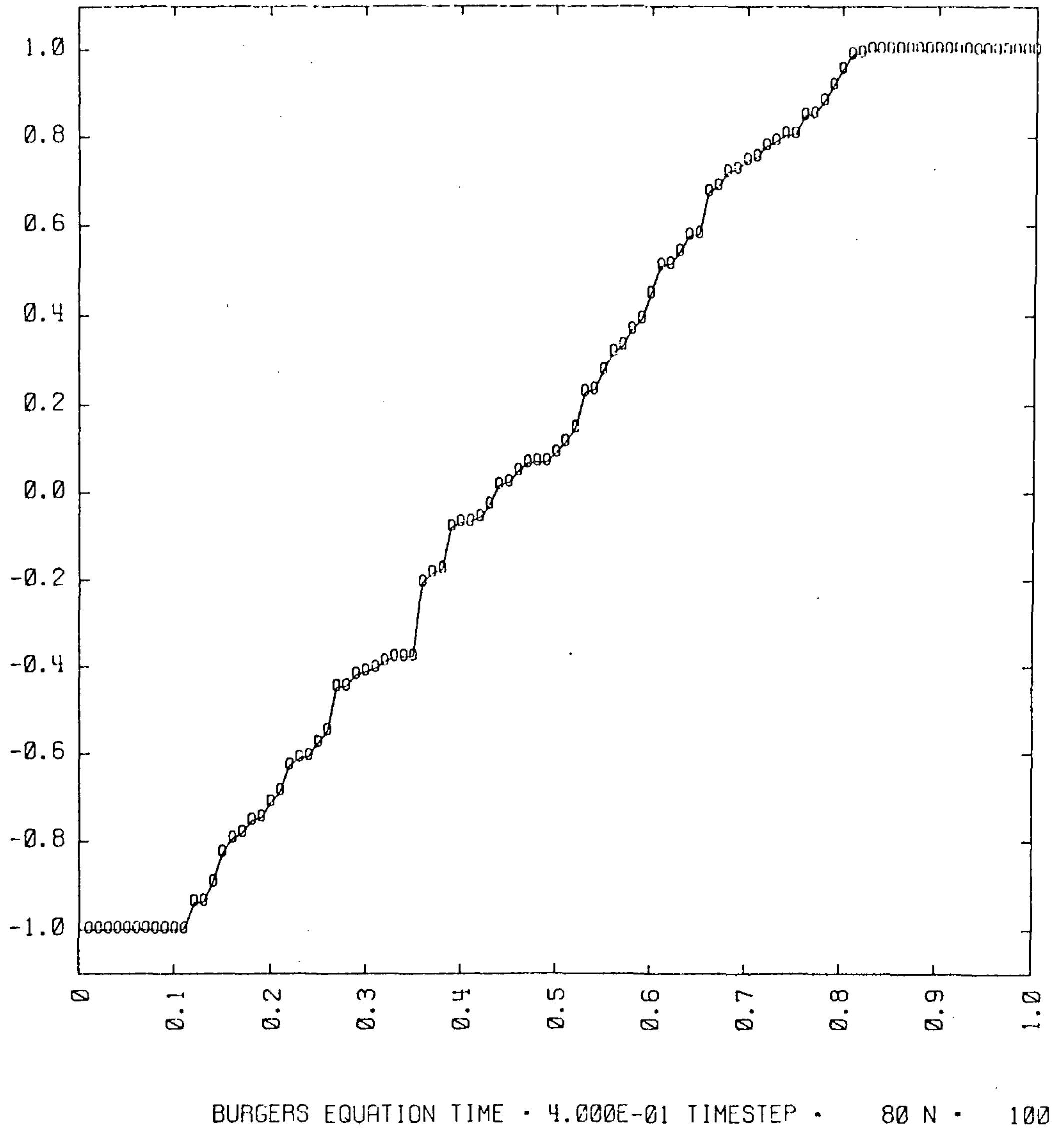


Figure 2.7 Inviscid Burgers' Equation, Rarefaction Wave Random Sampling

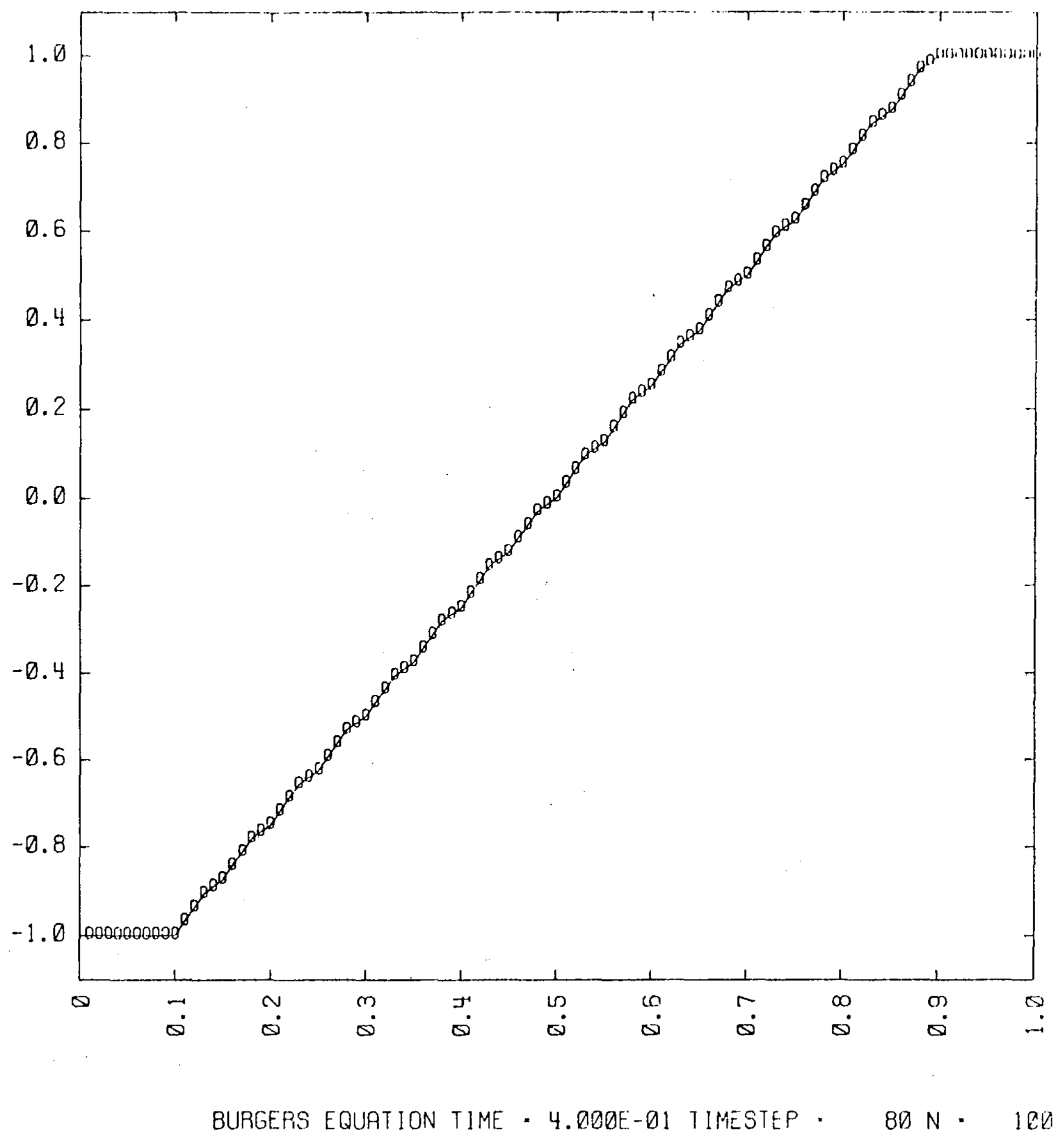


Figure 2.8 Inviscid Burgers' Equation, Rarefaction Wave van der Corput Sampling

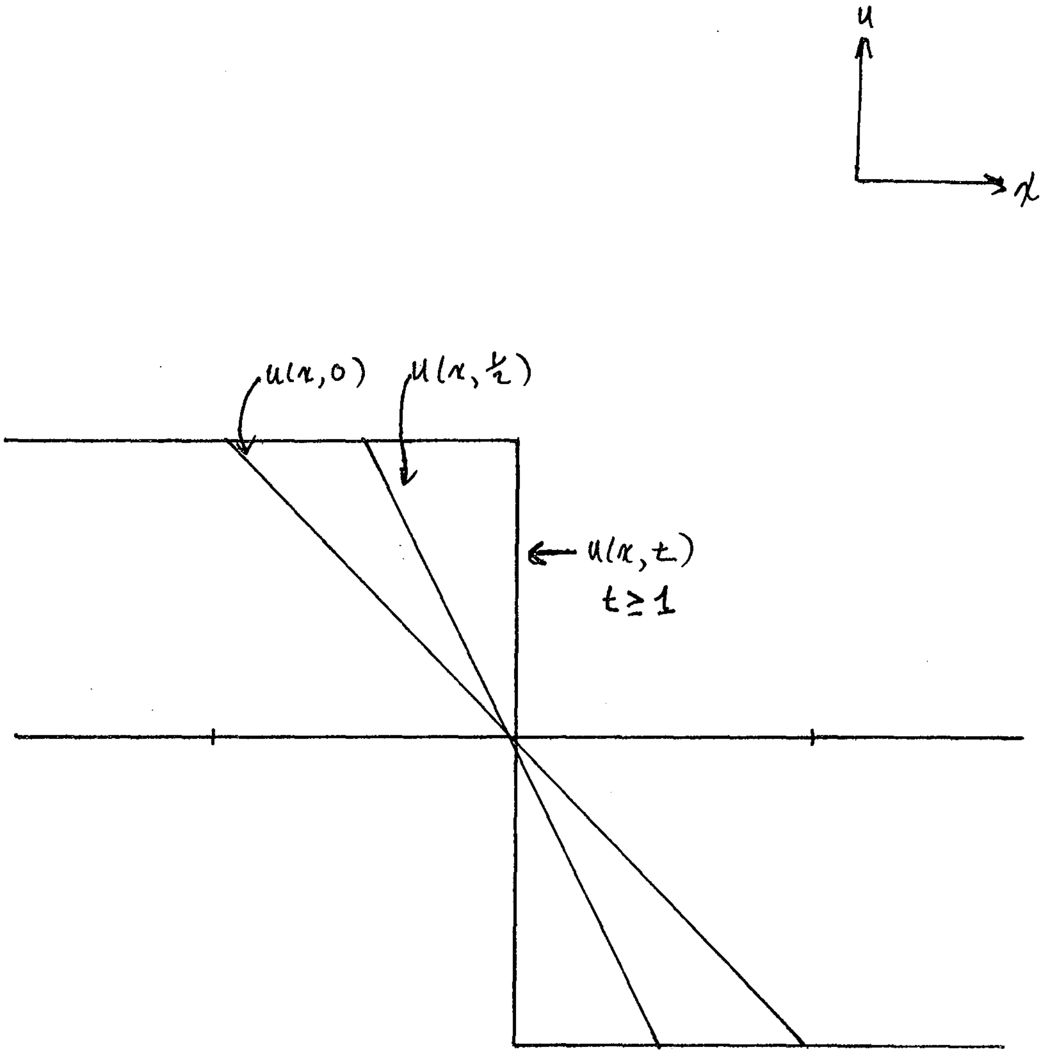


Figure 2.9 Compression Wave Solution to the Inviscid Burger's Equation

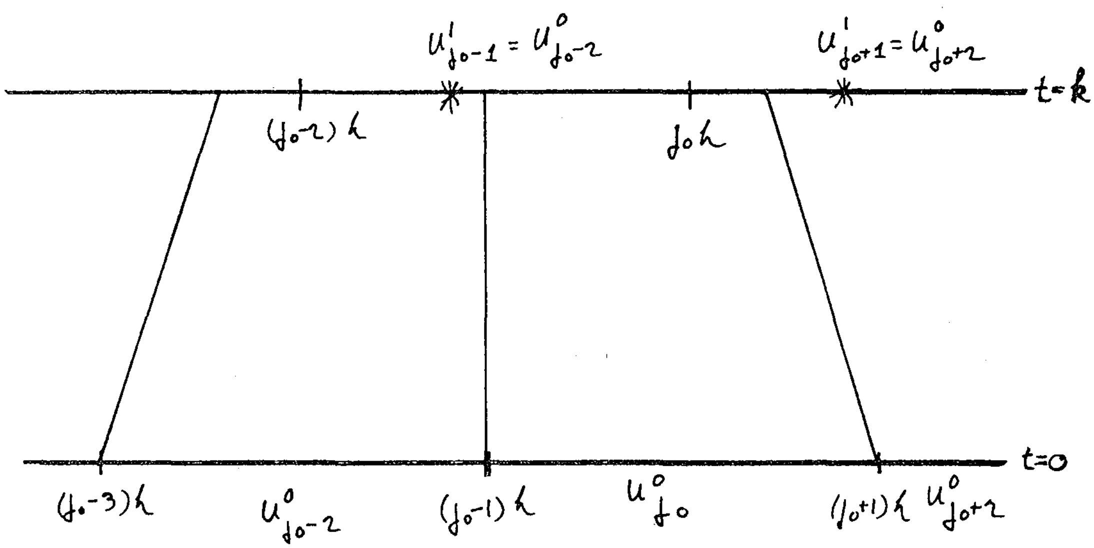


Figure 2.10 Sampling an Approximate Compression Wave

(* Denotes Sampled Point)

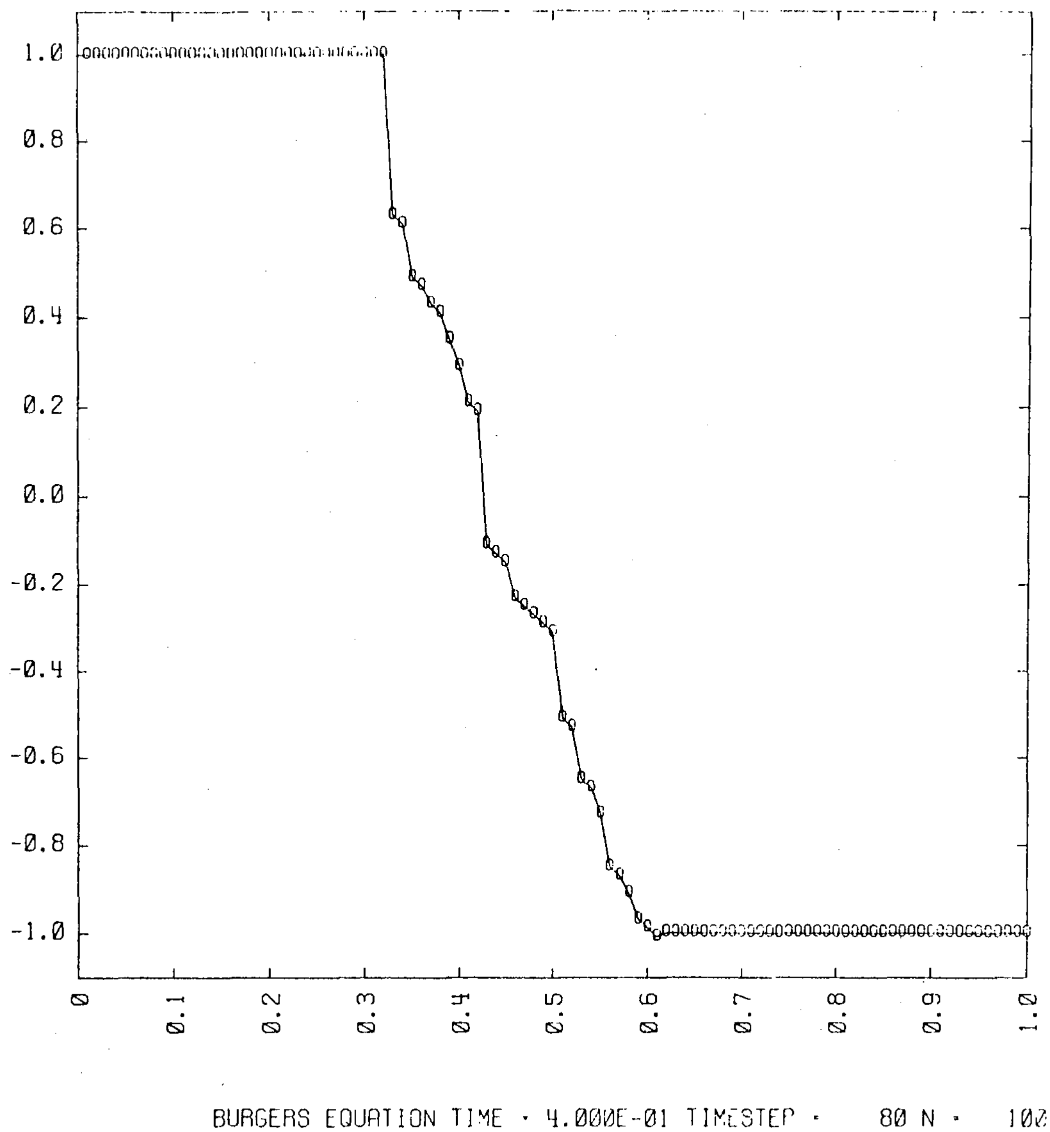


Figure 2.11 Inviscid Burgers' Equation, Compression Wave

Random Sampling

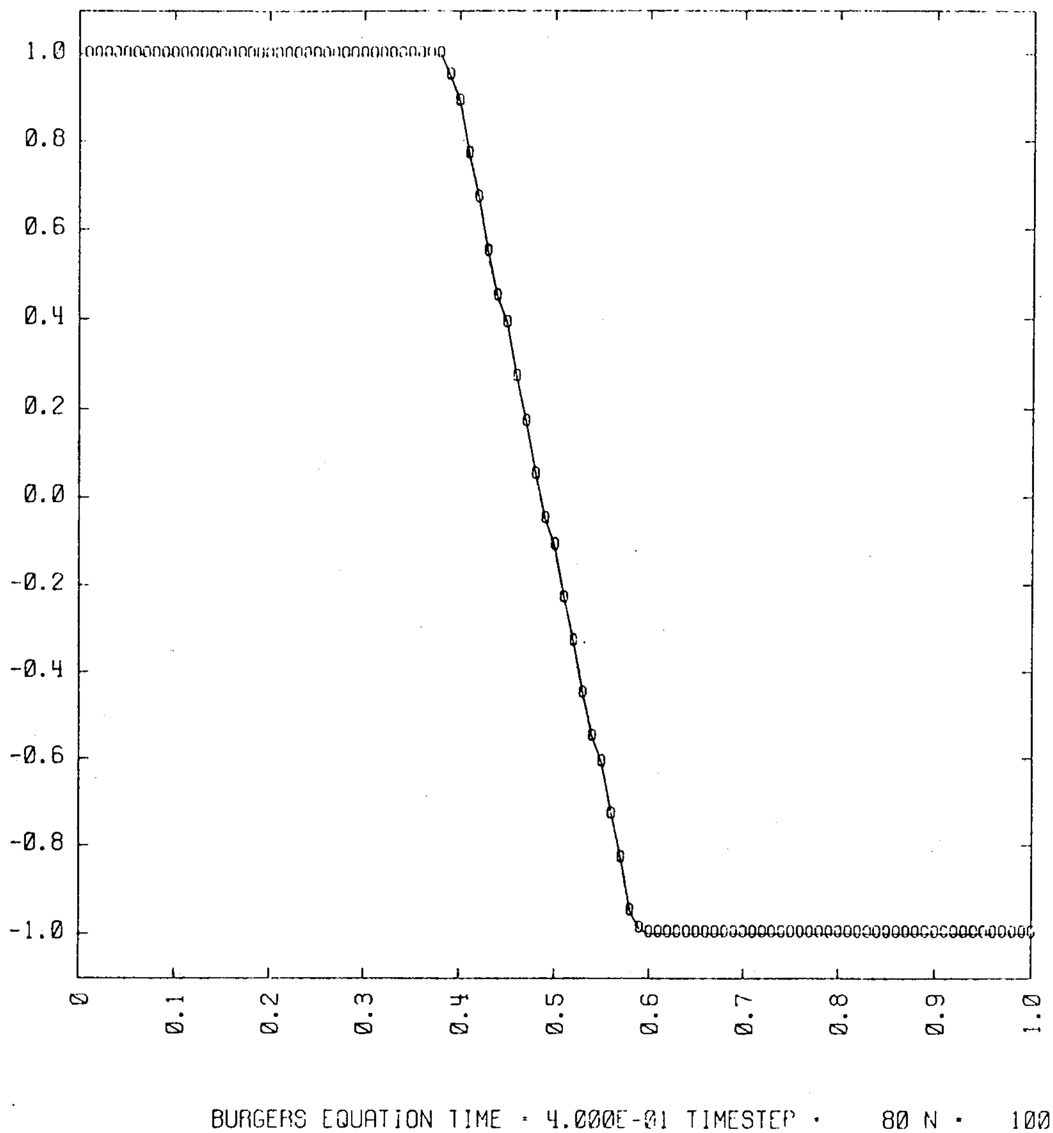


Figure 2.12 Inviscid Burgers' Equation, Compression Wave van der Corput Sampling

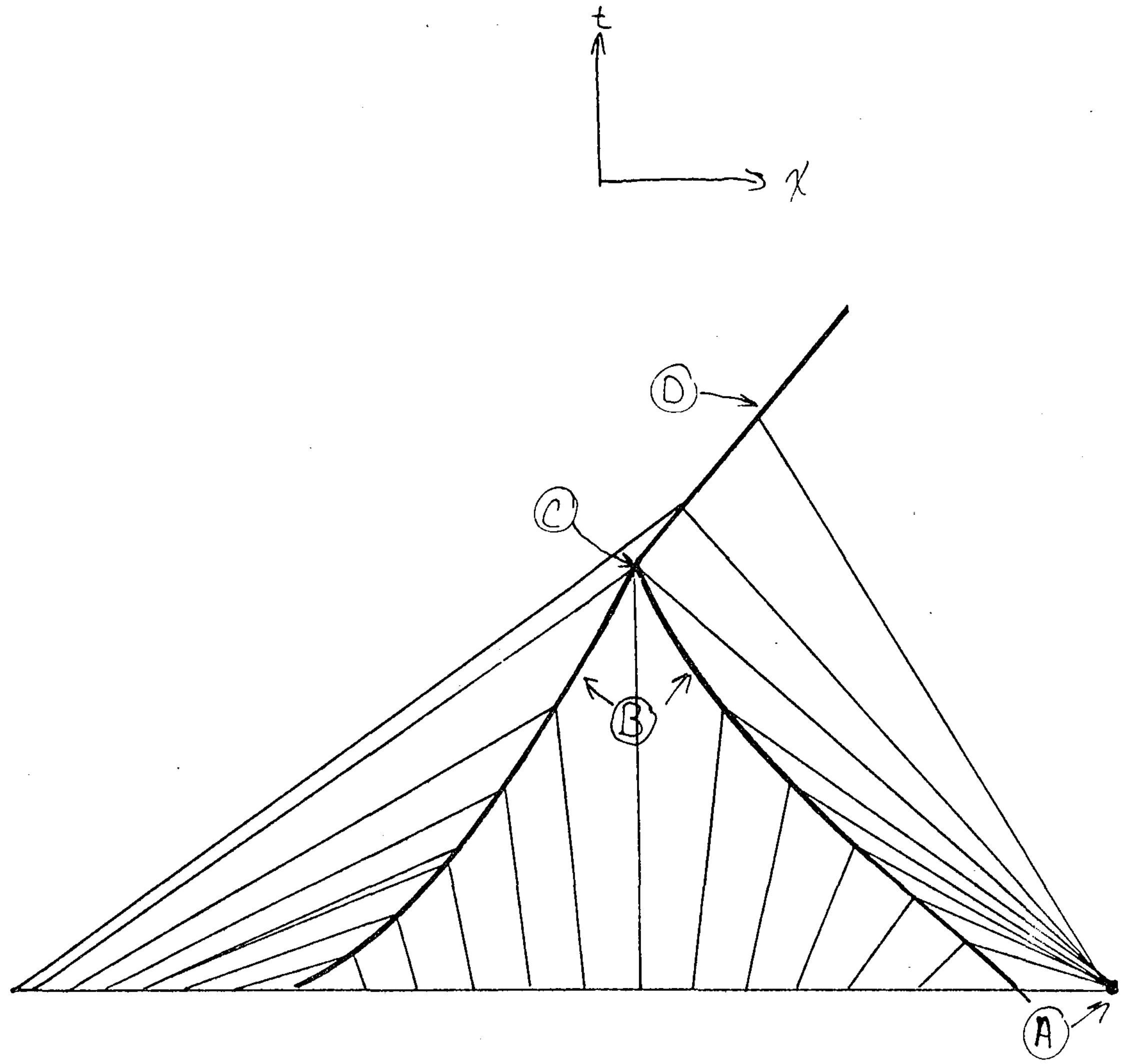


Figure 2.13

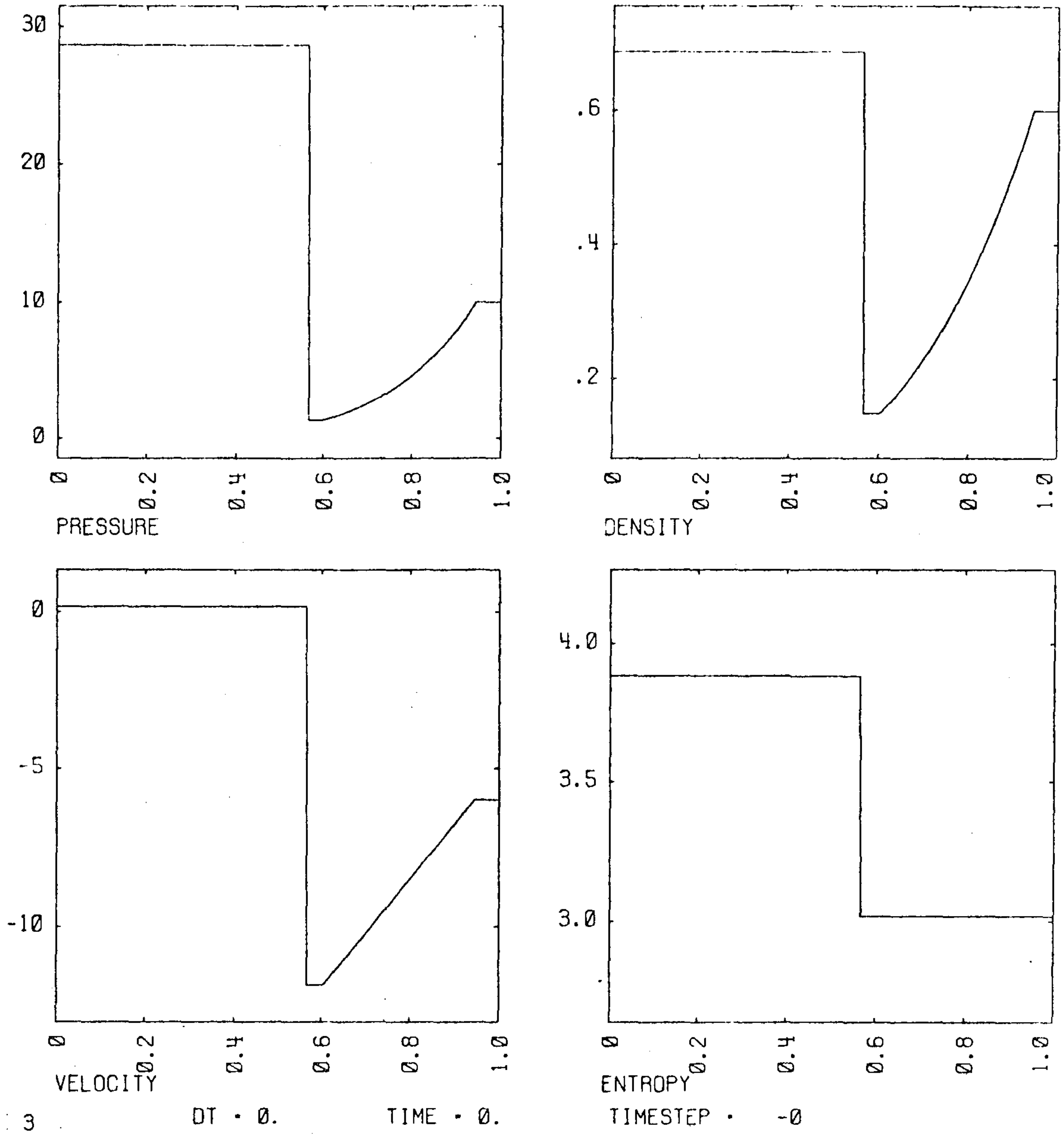


Figure 2.14

Initial Conditions for One-Dimensional Test Problem

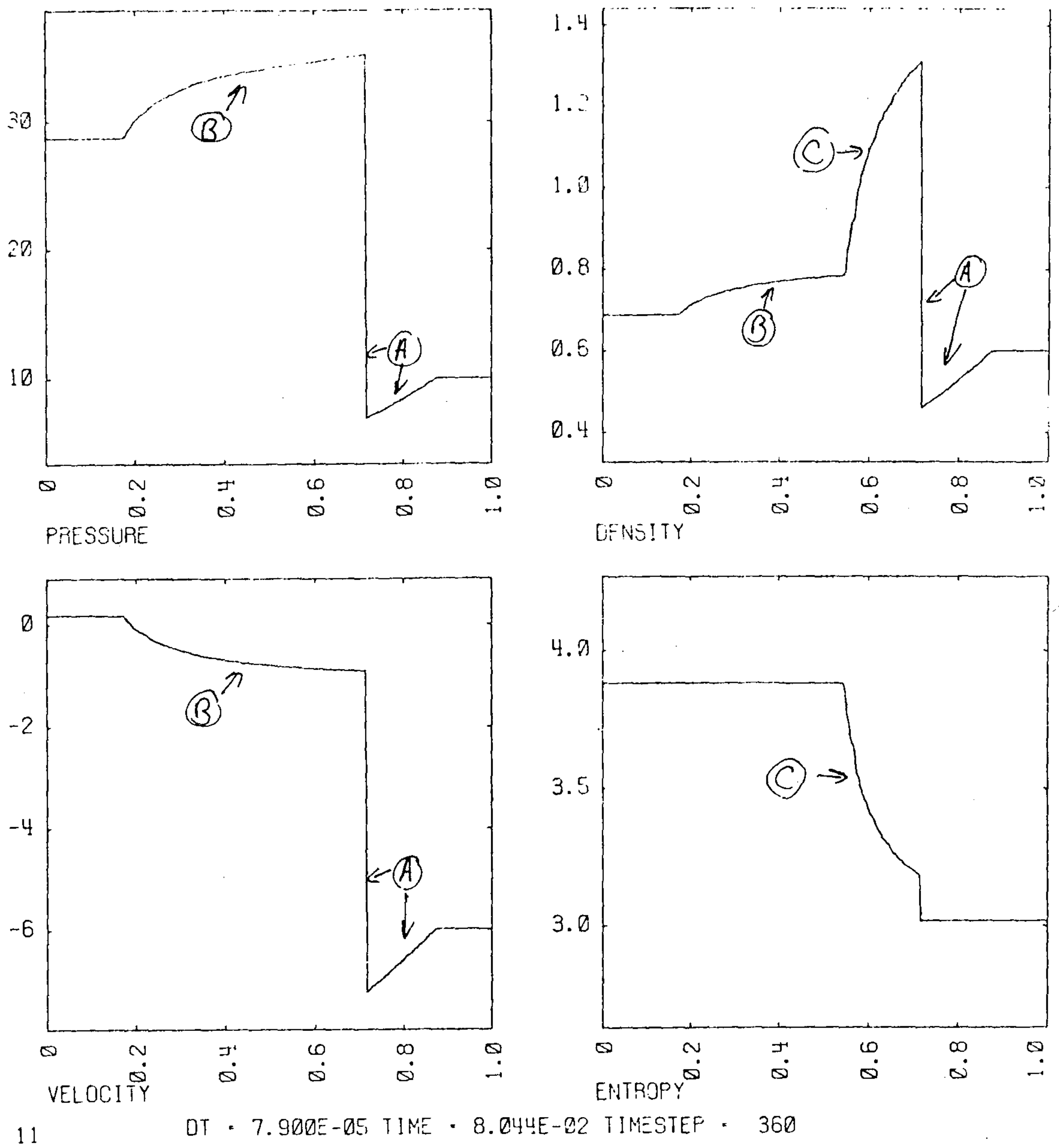


Figure 2.15 Computed Solution to the One-Dimensional Test Problem, van der Corput Sampling

$\Delta x = .0025$

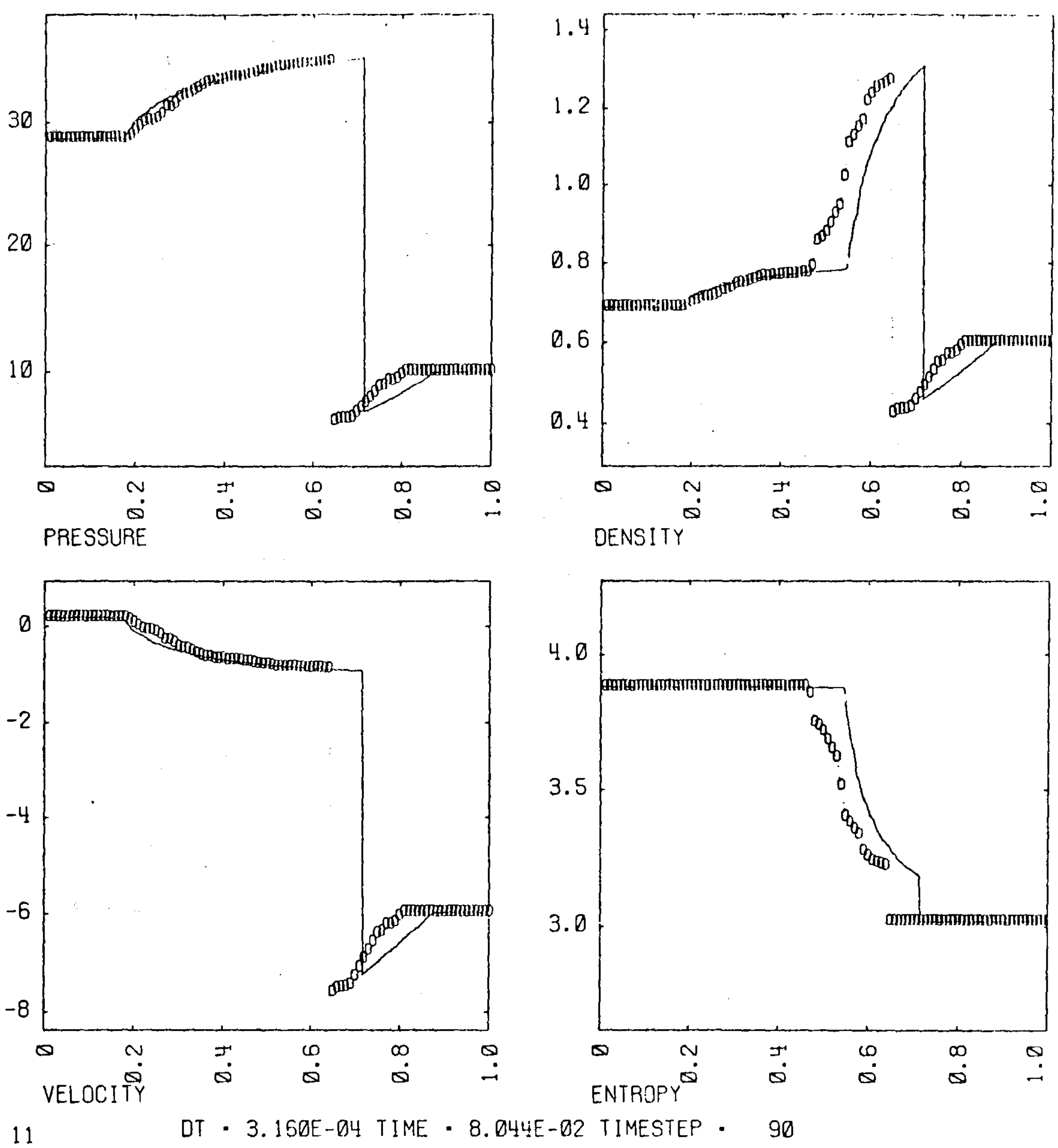


Figure 2.16

Computed Solution to One-Dimensional Test Problem

Random Sampling, $\Delta x = .01$

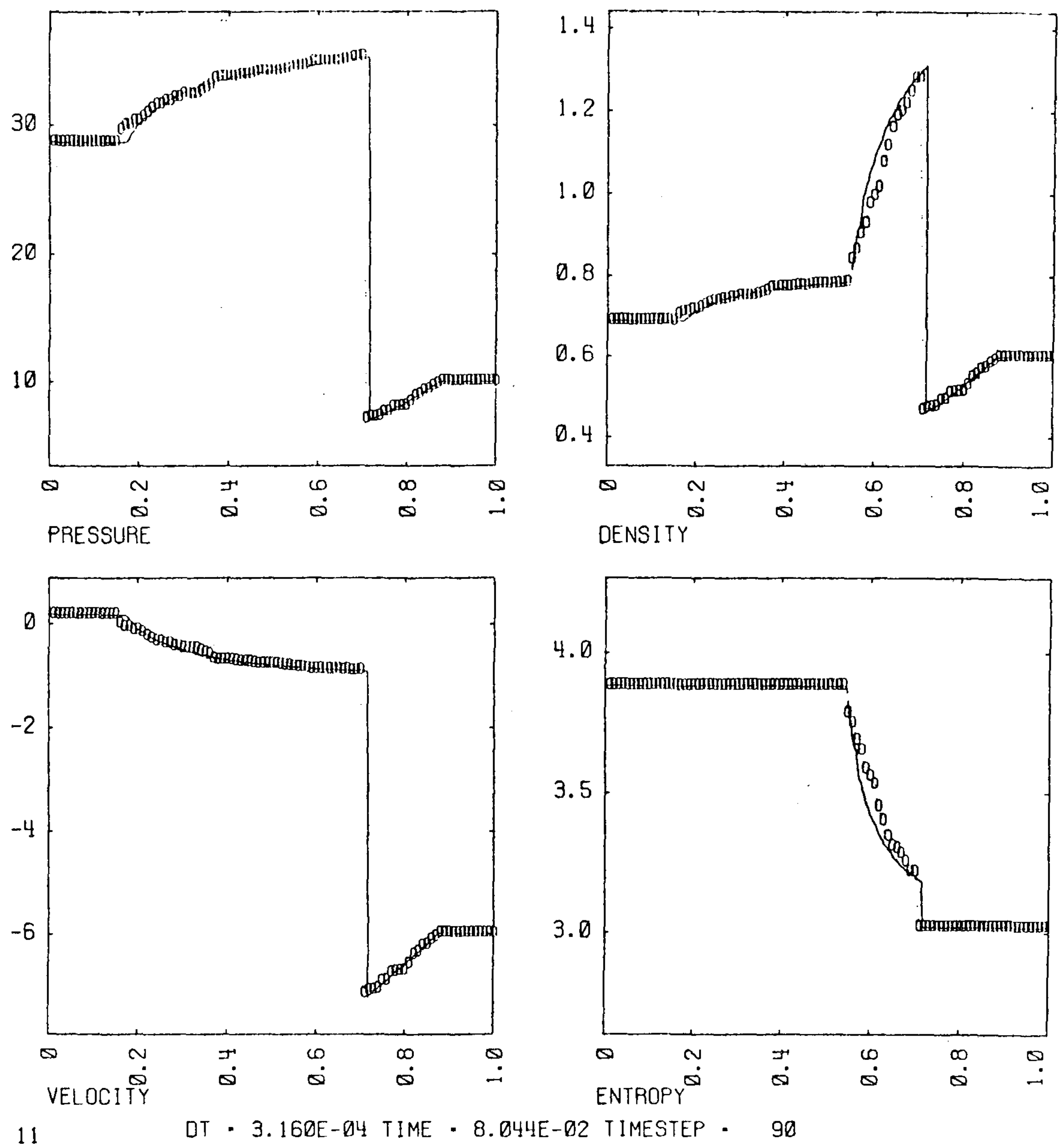


Figure 2.17

Computed Solution to One-Dimensional Test Problem

(7,3) Stratified Random Sampling, $\Delta x = .01$

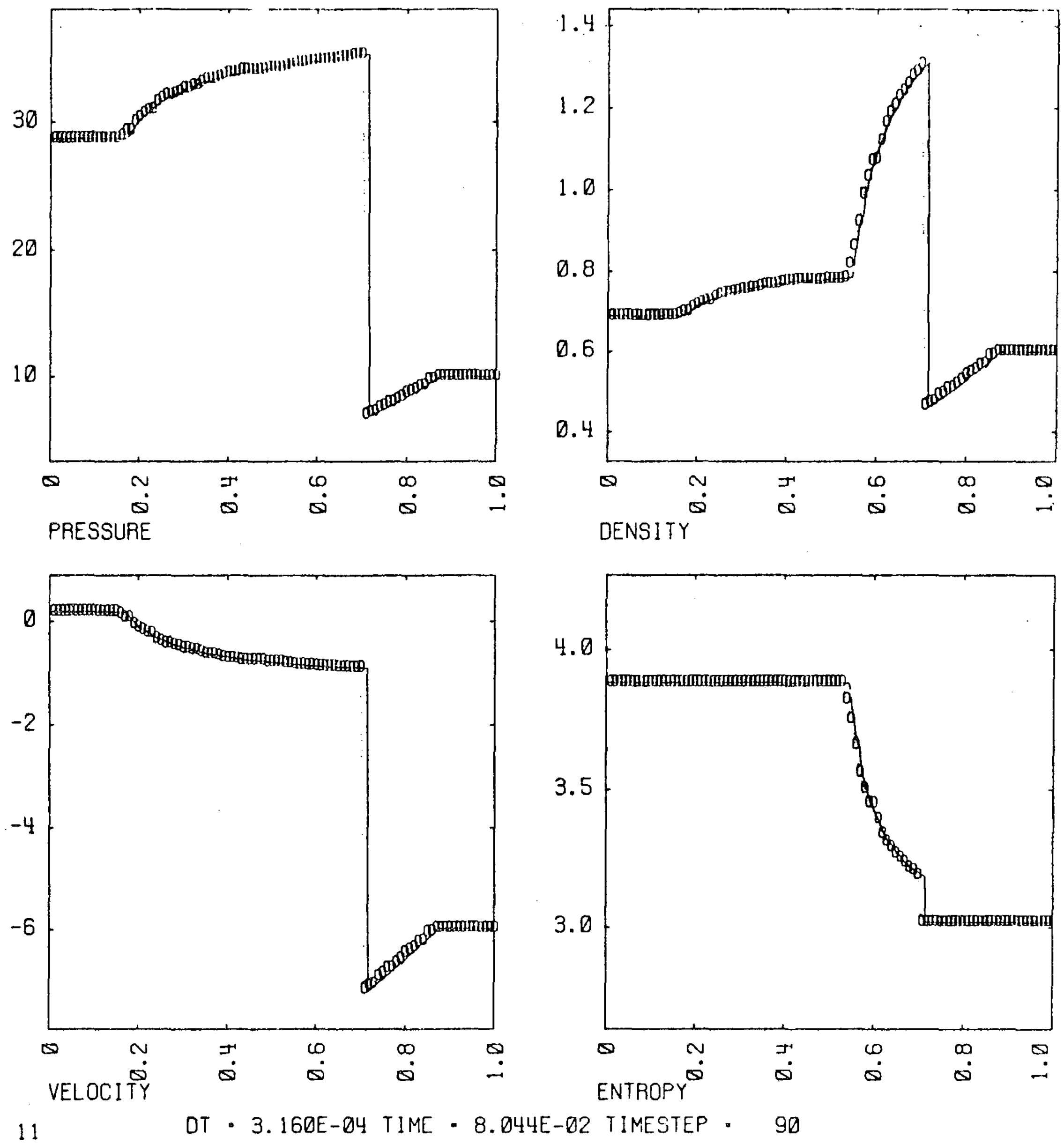


Figure 2.18

Computed Solution to One -Dimensional Test Problem

van der Corput Sampling, $\Delta x = .01$

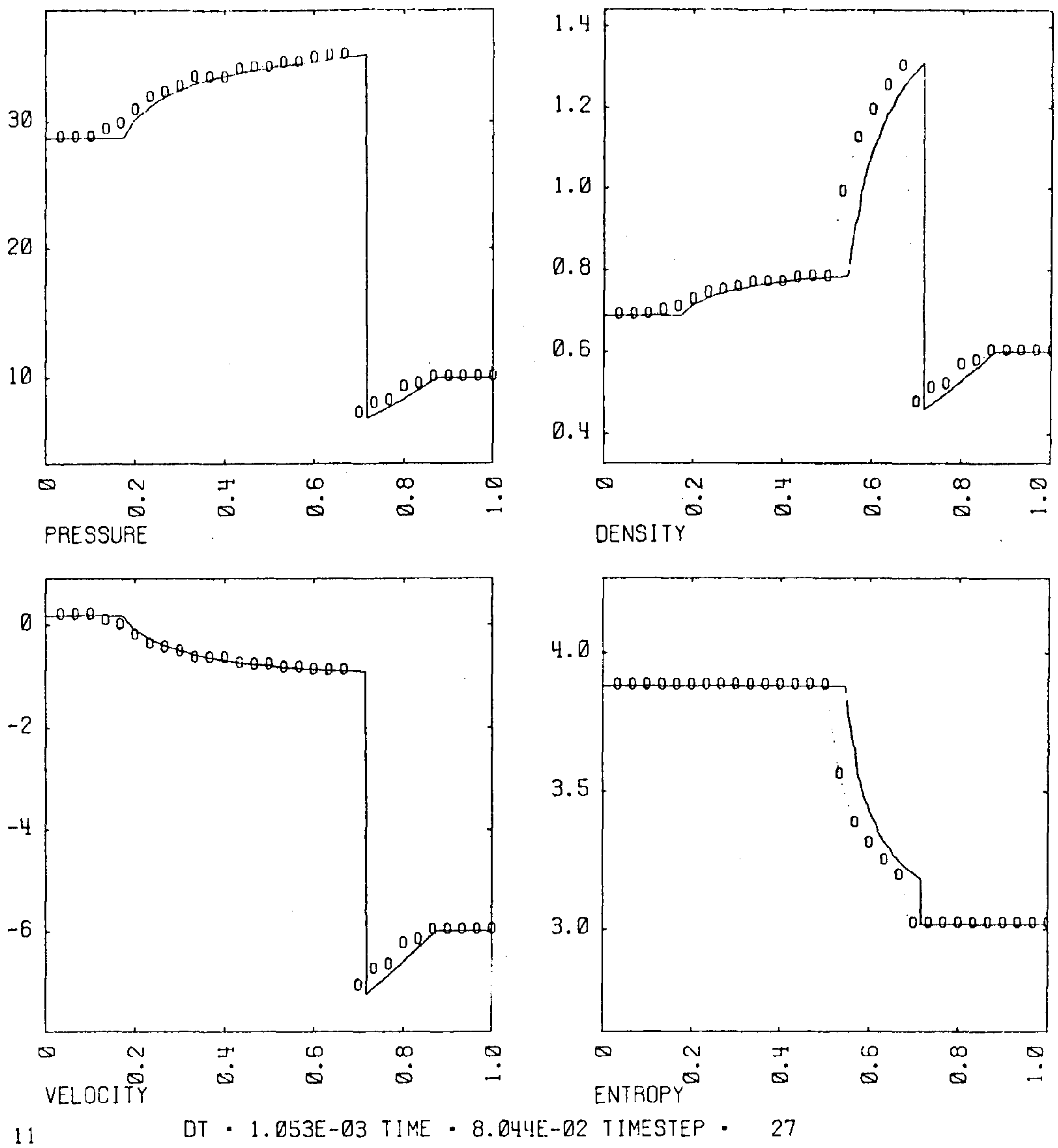


Figure 2.19

Computed Solution to One-Dimensional Test Problem

van der Corput Sampling, $\Delta x = 1/30$

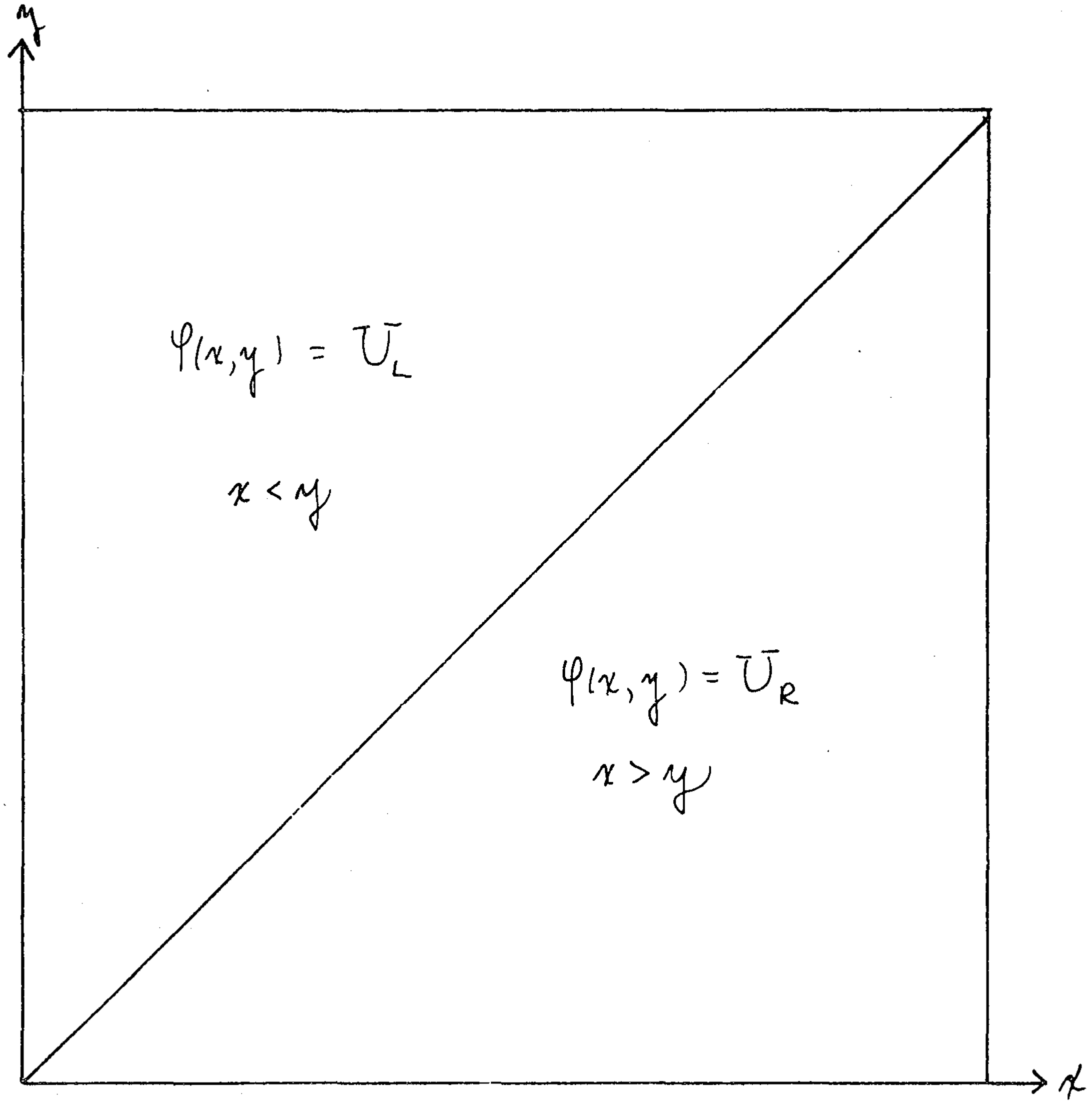


Figure 3.1 Computational Domain and Initial Conditions for Two-Dimensional Riemann Problem

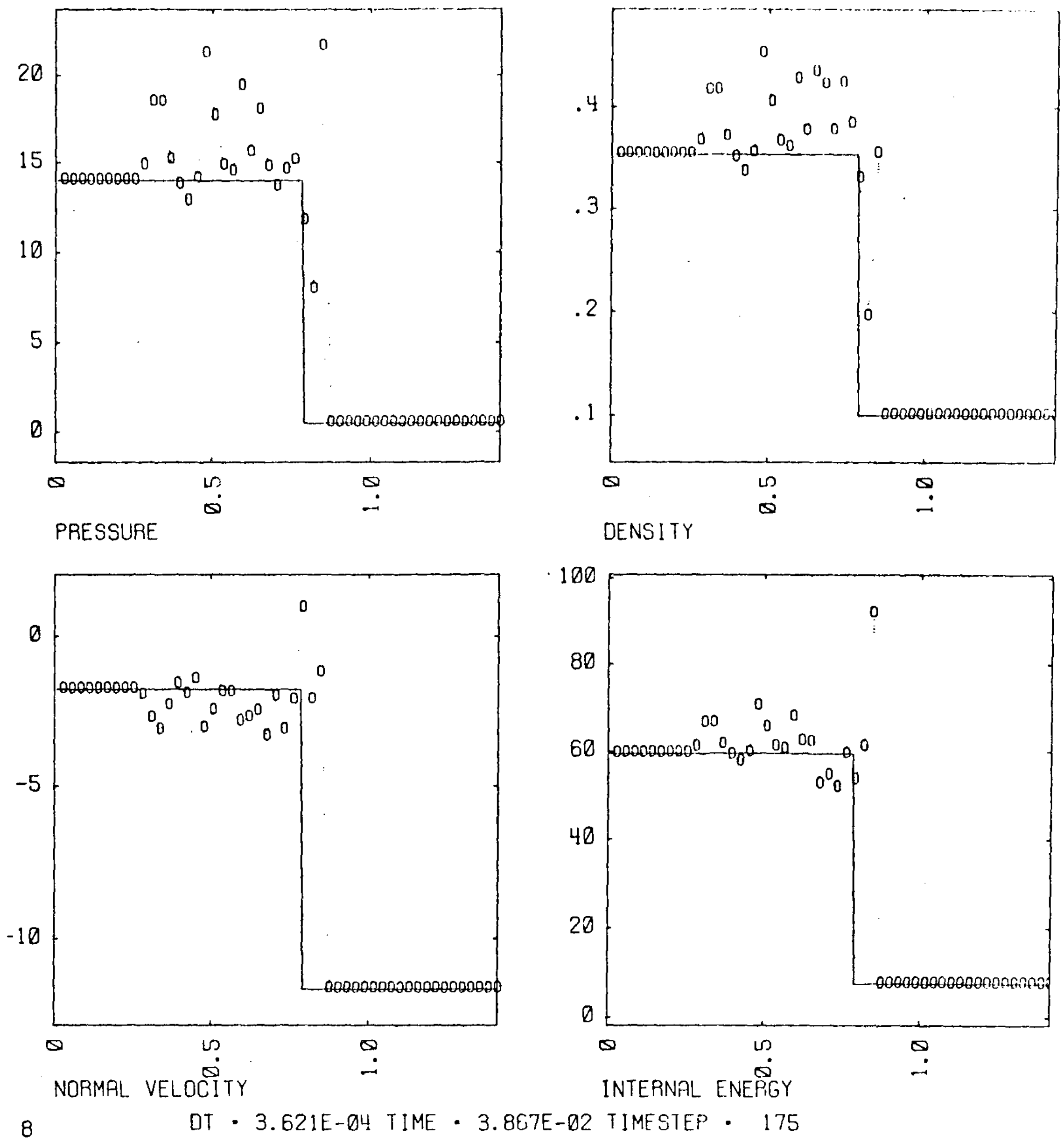


Figure 3.2

Computed Solution for the Two=Dimensional Riemann Problem

$p_L = 14 \qquad p_R = .5$

$\rho_L = .353 \qquad \rho_R = .1$

$v_{N,L} = -1.78 \qquad v_{N,R} = -11.6 \qquad \gamma = 1.667, \quad \Delta x = \Delta y = .02$

No Artificial Viscosity

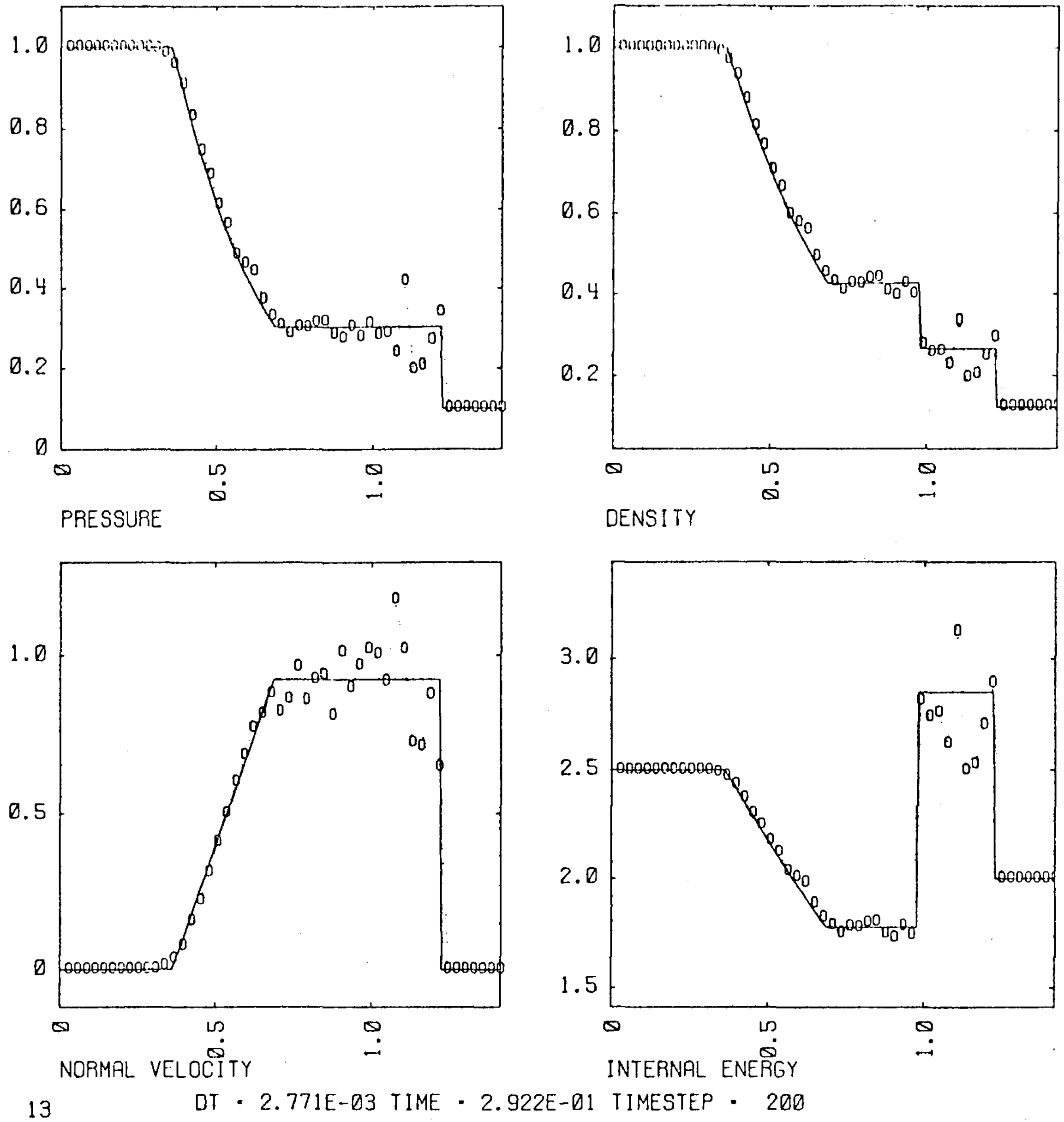


Figure 3.3

Computed Solution to the Two-Dimensional Riemann Problem

$p_L = 1.0$ $p_R = 0.1$

$\rho_L = 1.0$ $\rho_R = 0.125$

$v_{N,L} = 0$ $v_{N,R} = 0$ $\gamma = 1.4$ $\Delta x = \Delta y = .02$

No Artificial Viscosity

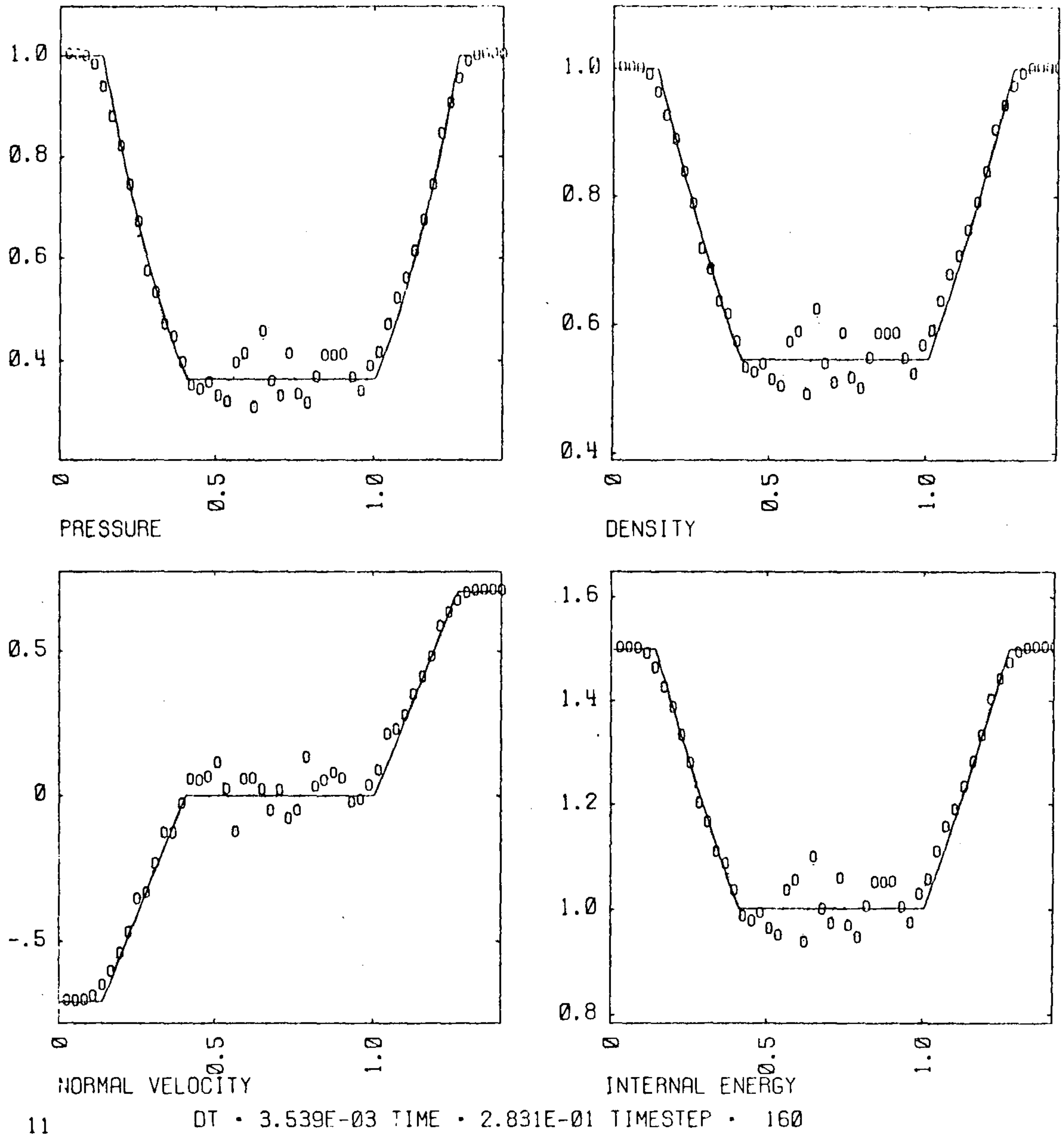


Figure 3.4

Computed Solution to the Two-Dimensional Riemann Problem

$p_L = 1.0 \qquad p_R = 1.0$

$\rho_L = 1.0 \qquad \rho_R = 1.0$

$v_{N,L} = -.707 \qquad v_{N,R} = .707 \qquad \gamma = 1.667, \Delta x = \Delta y = .02$

No Artificial Viscosity

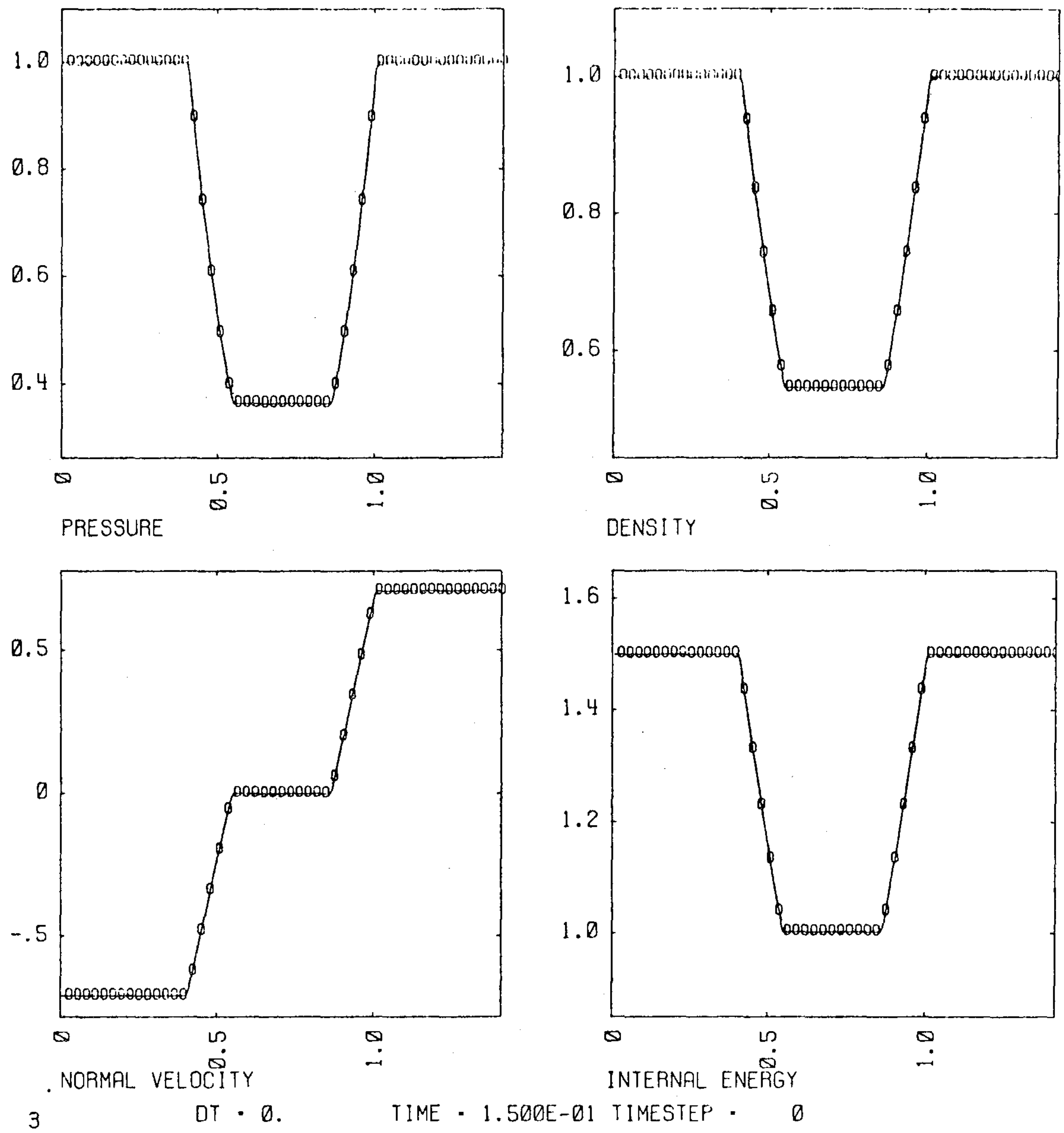


Figure 3.5

Initial Conditions for Continuous Two-Dimensional Test Problem

$\varphi(x,y) = \varphi(x',y')$, $x - x' = y - y'$

Shown here are profiles of $\varphi(x, 1-x)$.

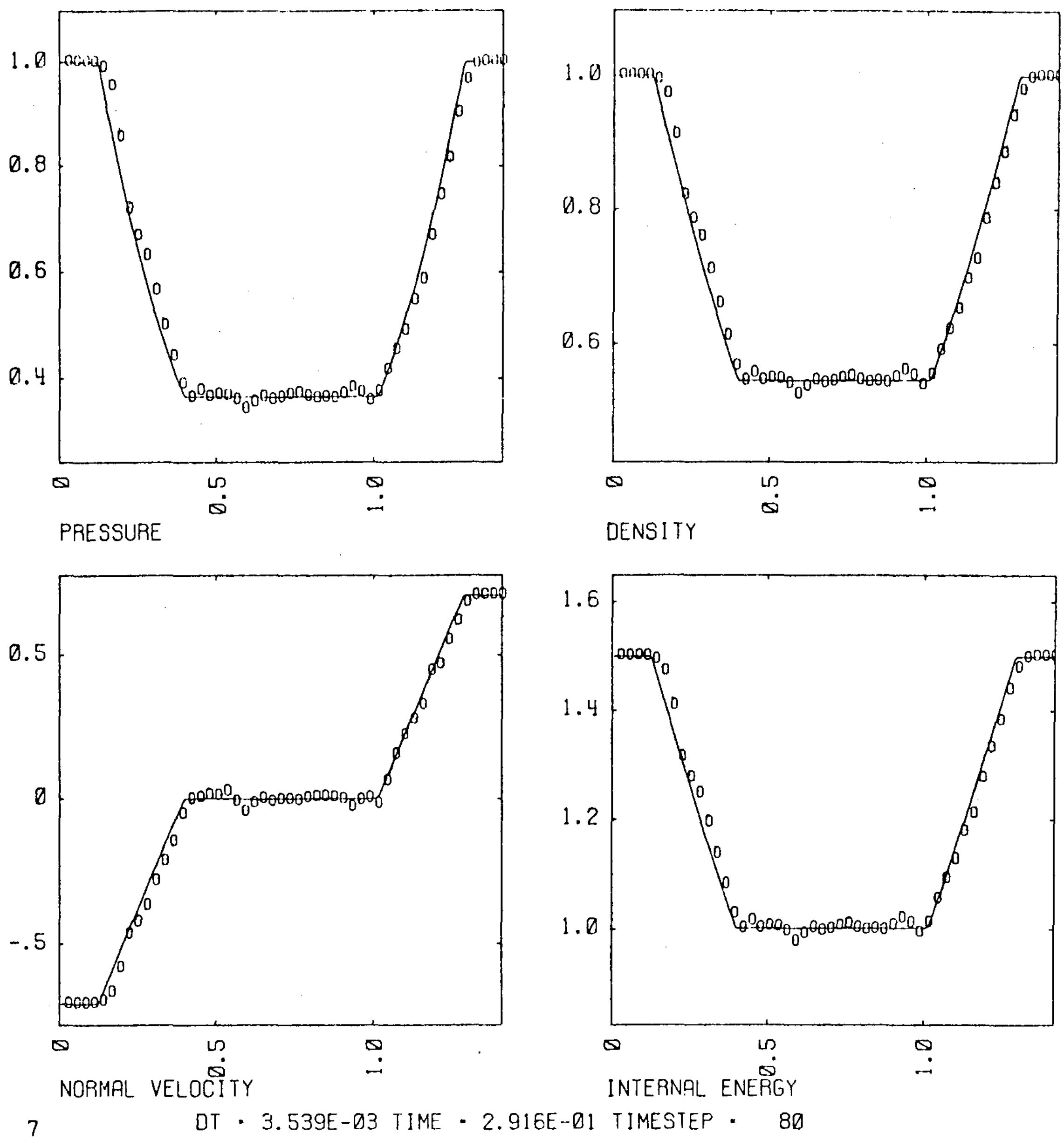


Figure 3.6

Computed Solution at Later Time for Initial Conditions in Figure 3.5

No Artificial Viscosity

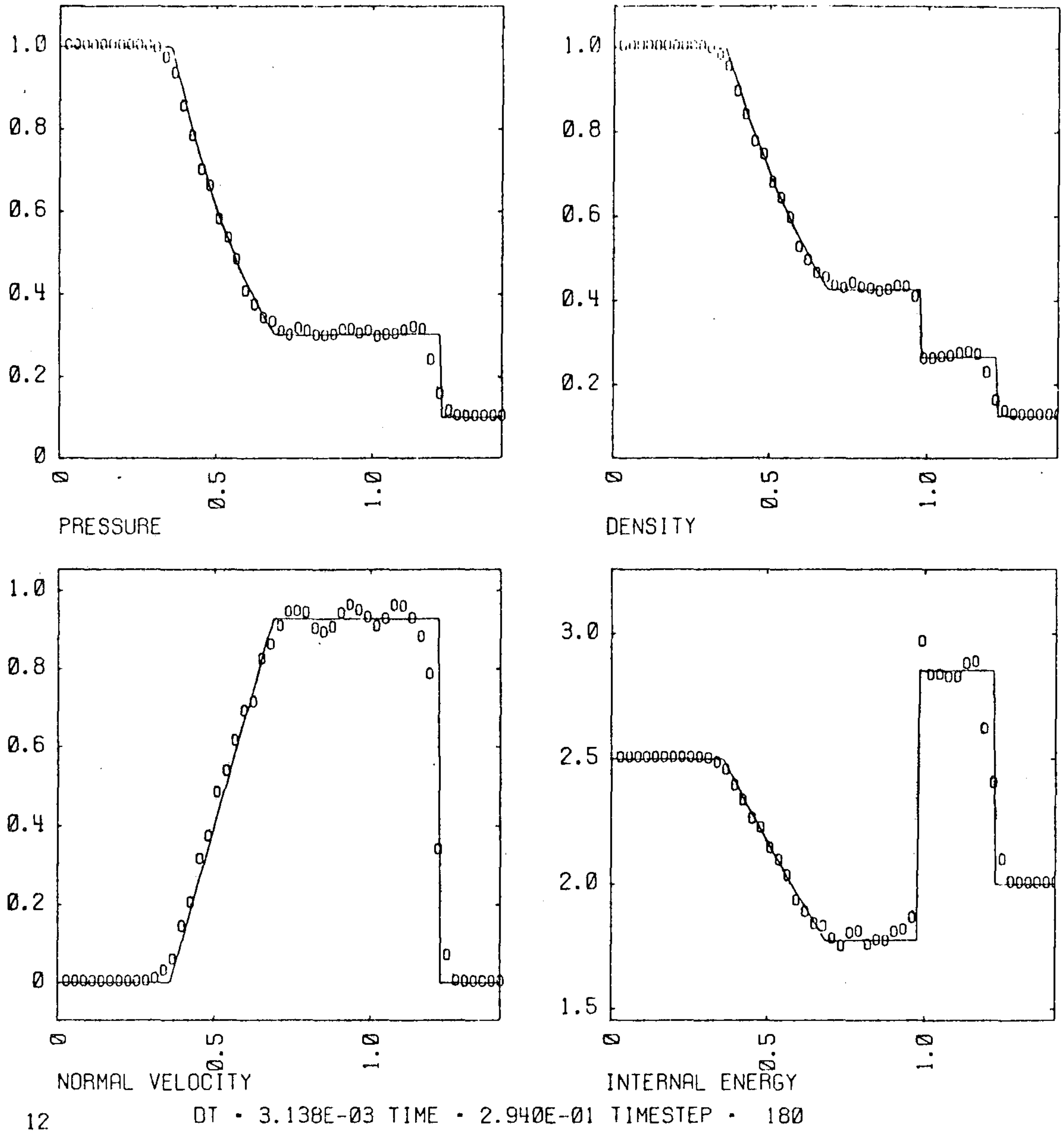


Figure 3.7

Computed Solution to the Two-Dimensional Riemann Problem

$p_L = 1.0 \qquad p_R = 0.1$

$\rho_L = 1.0 \qquad \rho_R = 0.125$

$v_{N,L} = 0 \qquad v_{N,R} = 0 \qquad \gamma = 1.4 \quad \Delta x = \Delta y = .02$

Artificial Viscosity Used

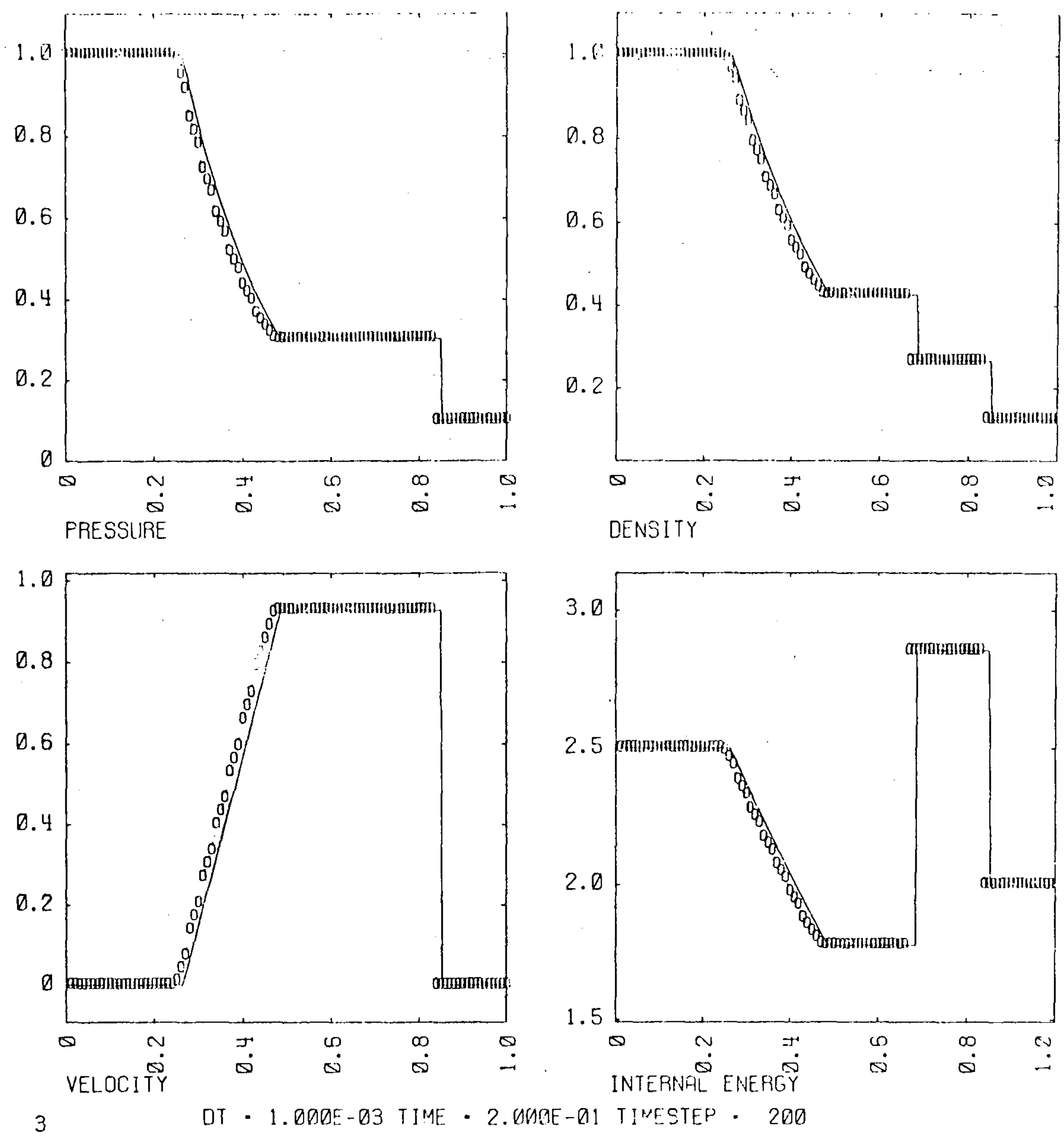


Figure 4.1 Computed Solution to the One-Dimensional Shock Tube van der Corput Sampling, $\Delta x = .01$

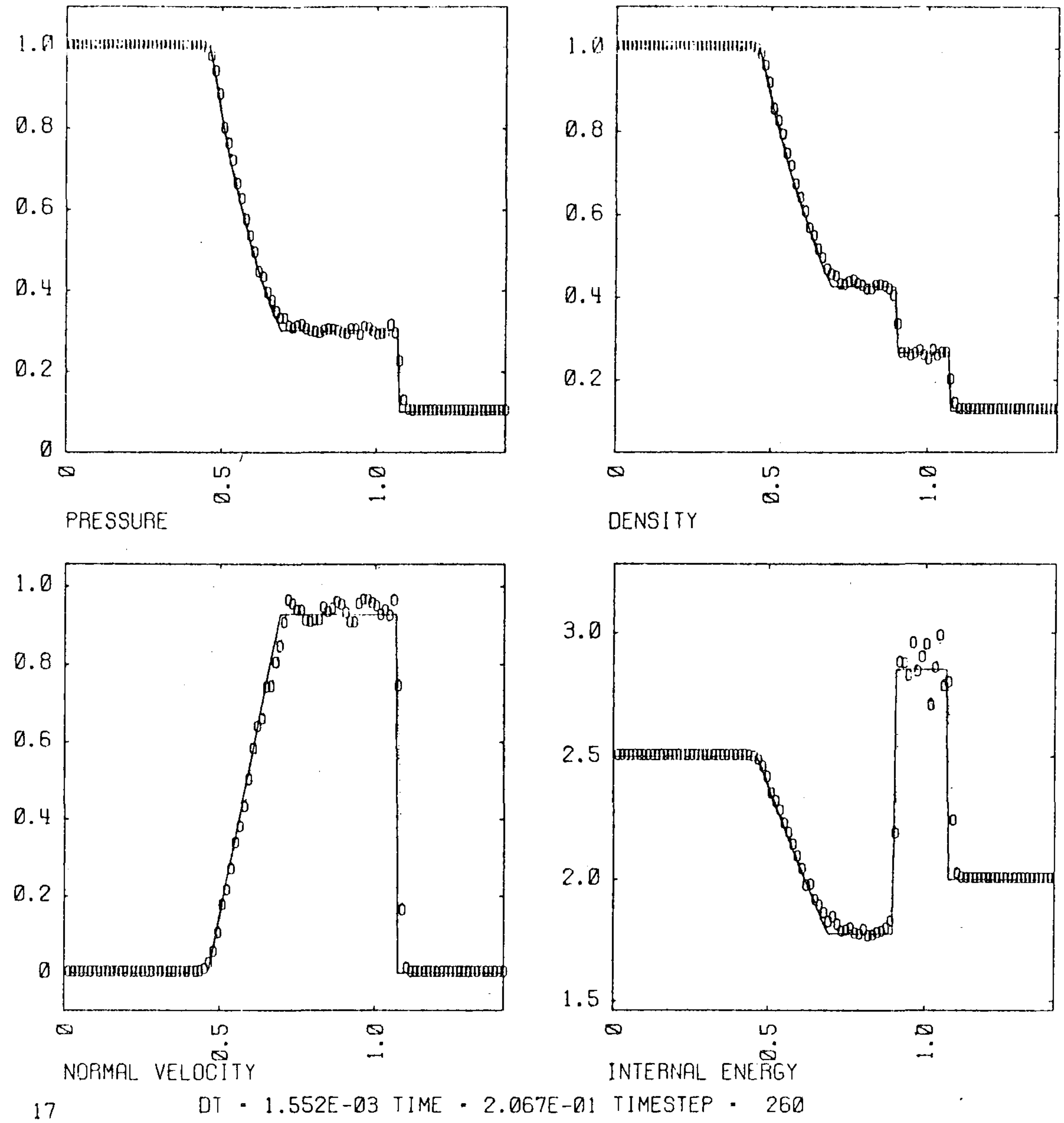


Figure 4.2

Computed Solution to Two-Dimensional Riemann Problem

$p_L = 1.0 \qquad p_R = 0.1$

$\rho_L = 1.0 \qquad \rho_R = 0.125$

$v_{N,L} = 0 \qquad v_{N,R} = 0 \qquad \gamma = 1.4 \qquad \Delta x = \Delta y = .01$

Artificial Viscosity Used

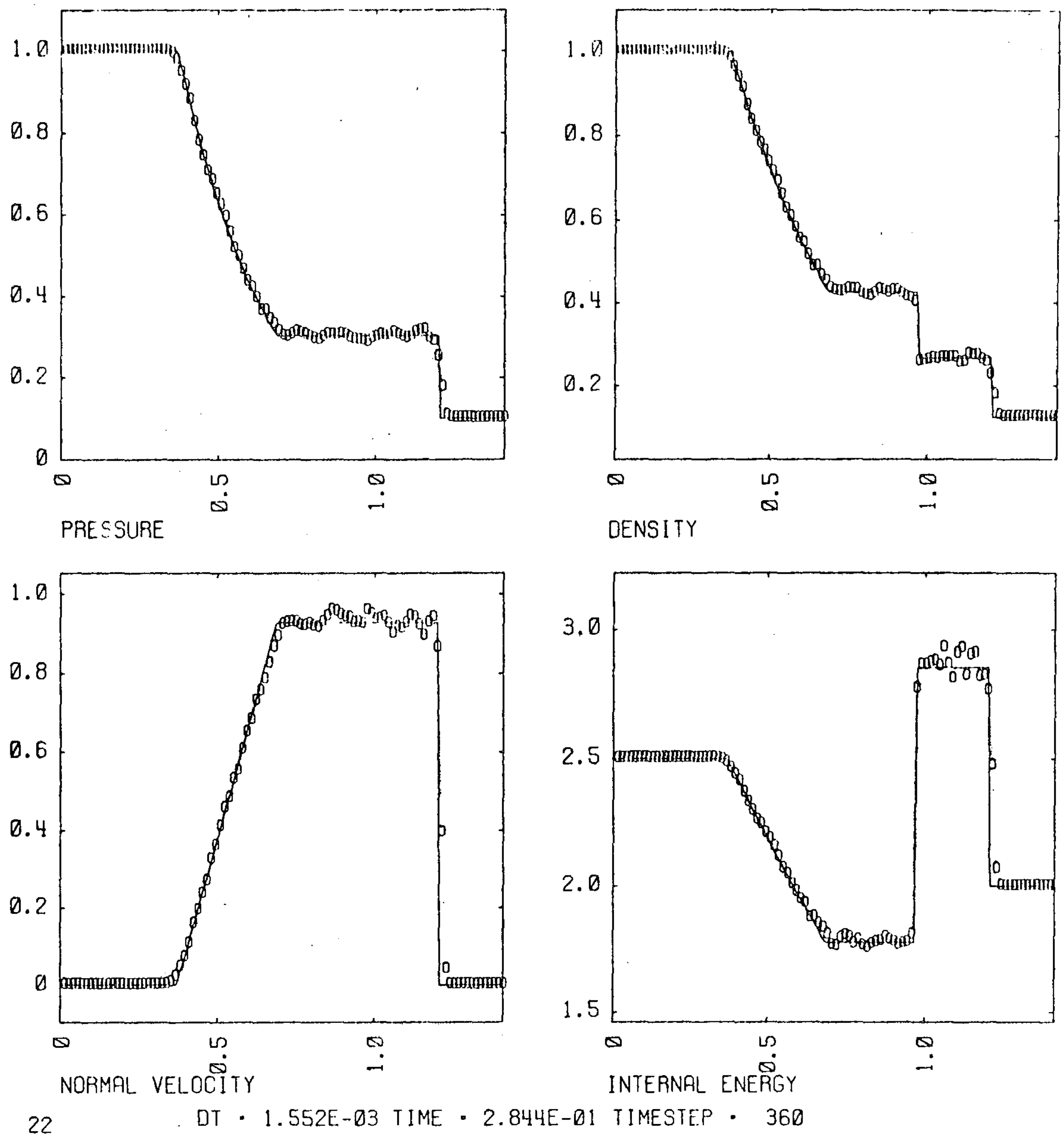


Figure 4.3

Same as Figure 4.2, but for Later Time

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