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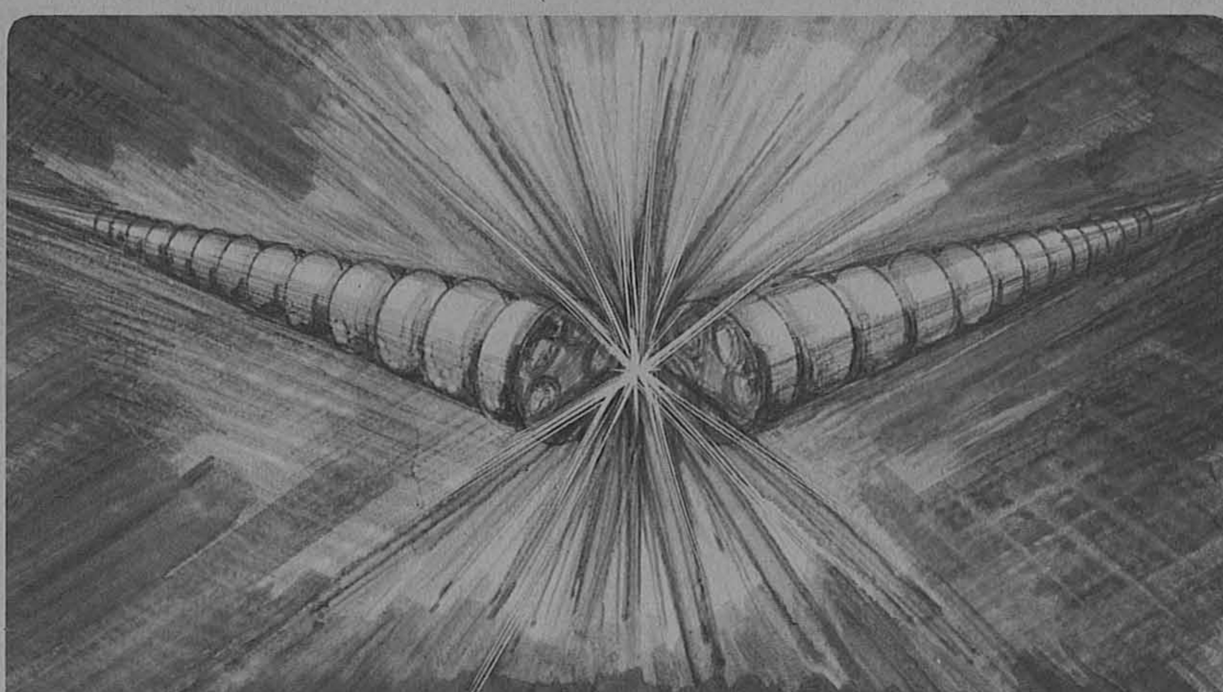

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Abstract

The spatial properties of the optical field and hence the performance of a free electron laser depend on the fact that the electron beam, which acts as both an amplifying and a refractive medium, is transversely nonuniform. Under certain circumstances, optical guiding may be realized, where the optical field is stably confined near the electron beam and amplified along the beam over many Rayleigh ranges. We show that the three-dimensional evolution of the optical field through the interaction region can be determined by a guided mode expansion before saturation. Optical guiding occurs when the fundamental growing mode becomes dominant. The guided mode expansion is made possible by implementing the biorthogonality of the eigenmodes of the coupled electron-beam--optical-wave system. The eigenmodes are found to be of vectorial form with three components; one specifies the guided optical mode and the other two describe the density and the energy modulations of the electron beam.

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I. Introduction

The new generation of high power, short wavelength free electron lasers (FELs) are expected to operate in the high gain regime (exponential growth of power in a single pass). To achieve such a high gain, a high quality electron beam and a long wiggler are necessary. However, apart from fabrication issues, the useful wiggler length is limited by diffractive spreading of the optical field. The diffraction problem has been given much attention in recent years and it is found that under certain circumstances the transverse inhomogeneity of the electron beam may lead to a phenomenon known as "optical guiding" in that the optical field is confined near the electron beam with a stable transverse profile and amplified along the beam over many Rayleigh ranges. In this paper we present a general formalism for the determination of three-dimensional (3D) propagation of the optical field through an unsaturated FEL amplifier. Our formalism is especially useful for the understanding and analysis of optical guiding in high gain FELs.

FEL theory was originally developed in a one-dimensional (1D) model in which the finiteness of the electron beam and laser mode area was taken into account by means of a filling factor computed from the spatial overlap of the electron beam and laser mode. This 1D model, though useful as a first approximation, is valid only under the conditions of weak diffraction and low single pass gain where the spatial structure of the input laser is barely altered by the electron beam and the 3D effects thus can be taken as higher order corrections[1].

Various approaches have been undertaken to investigate 3D effects in FELs since the early 1980s. Numerical simulations, though capable of dealing with most realistic system conditions, lack physical insight. Analytical methods have largely resorted to the mode analysis technique. In the later approach the optical field is expressed as an expansion in a complete set of transverse modes and the evolution of the field is then determined by dynamically solving for the expansion coefficients.

Elleaume and Deacon[2] extended the 1D FEL equations into 3D using the complete set of free-space modes. They showed that the propagation and amplification of an input wave (usually a free-space mode) results in the excitation of additional modes. Since the free-space modes are not eigenmodes of the interacting beam-wave system, cross excitation occurs even in the linear regime. The amplitudes of the modes vary as the wave propagates due to the evolving modulation of the electrons. This theory, though applicable to quite a broad range of situations in principle, is useful in practice only at low gain where the cross-coupling is weak and involves only a small number of low order modes.

The breakdown of the free-space mode expansion at high gain is due to the poor match between the expansion basis and the spatial structure of the stimulated emission. To improve the efficiency of the technique, a more physically thoughtful "source dependent expansion" was proposed by Sprangle et al[3]. In this method the expansion basis as well as the coefficients are updated according to the local conditions of the driving source. At each position along the direction of propagation, the basis is chosen such that the minimum number of modes are needed to describe the local optical field accurately. The source dependent method as a numerical approach is useful particularly for treating more irregular electron beam conditions such as bending and focussing, but the local adaptive procedures introduce difficulties to the overall system optimization.

Unlike the free-space and source-dependent mode methods, our eigenmode approach uses orthogonal self-consistent solutions of the system as the basis. The wave propagation, once determined by a mode expansion at the entrance of the amplifier, is known throughout the whole interaction region without any further calculation. The key issues in this approach are, first of all how to find the eigenmodes, and secondly how to describe the evolution of the system in terms of these modes; in other words, how to solve the initial value problem.

Several authors[4,6,9,10,11] have identified a set of guided optical modes in the linear regime before saturation. It was found that these modes form both a discrete and a continuous spectrum and they can either grow, decay or propagate with constant amplitude. It then appears natural and advantageous to make use of these modes in the analysis of optical guiding. However, unlike guided modes in fiber optics, these modes are not orthogonal. To solve the initial value problem, approaches other than direct mode expansion must be used. A Laplace transform method was employed by Moore[5], leading to a Green's function with each simple pole corresponding to a discrete growing guided mode. Moore applied this method to obtain the coupling coefficient from an input wave into the fundamental growing mode. However, although this technique would apply to the other discrete growing modes, it is unclear how it could be adapted to handle the decaying and the continuous modes. In a modified version of Moore's method, Krinsky and Yu[10], instead of evaluating the Green's function at each pole, expanded the Green's function in a particular orthonormal set of functions. As with Moore's solutions their method applies only to the discrete growing modes.

The next step was taken by Kim[11]. In Kim's theory, the evolution of the optical field is treated together with the coupled phase space distribution of electrons, which allows the initial value problem to be solved by Van Kampen's eigenmode expansion technique[12]. It is implicit in this approach that the guided optical mode and the induced wave in the electron beam can be taken together to form biorthogonal vectors. Indeed we argue on physical grounds that the lack of orthogonality of the optical modes taken separately arises from an incomplete description of the coupled system. In this paper, we develop explicitly the multi-component eigenmode vectors by a much more simple method, demonstrate their biorthogonality, and show how they can be used to solve the complete initial value problem.

Our approach starts from the single particle formalism, and each eigenmode is constructed as a vector with three components corresponding, respectively, to the guided optical mode, the density modulation and the energy modulation of the electron beam. By explicitly including these three physical components of the coupled beam-wave system, we obtain eigenmodes which are biorthogonal in the transverse coordinate space (instead of electron's phase space as with Van Kampen's method). While Kim's method is also capable of treating self-amplified spontaneous emission, our approach is simpler and much easier to work with. Our formulation also provides a useful link between FELs and traveling wave tubes.

The organization of this paper is outlined as follows: In Section II, we derive a 3D linear wave equation for FELs taking electron dynamics into account and assuming small signal conditions. We then reexpress the wave equation in vector form in order to construct the proper eigenmodes for the coupled system. Based on the new formalism, the energy transfer process is discussed and energy conservation is verified in Section III. Section IV describes the solution of the initial value problem, which includes the proof of the biorthogonality of the eigenmodes and the determination of the expansion coefficients.

II. Evolution Equation

The paraxial wave equation for the FEL can be derived[13,14] assuming: (1) a monochromatic radiation field of the form: $\mathbf{E}(\mathbf{r},t)\exp(kz-\omega t)$, where $\mathbf{E}(\mathbf{r},t) = E(\mathbf{r},t)\exp[i\phi(\mathbf{r},t)]$ defines the slowly varying amplitude and phase of the radiation field; (2) a transverse electron beam density profile independent of the longitudinal distance along a uniform planar wiggler; and (3) a long beam of highly relativistic electrons, thus neglecting the short pulse effects and electrostatic force between the electrons. In CGS units, the equation reads

$$\left[\nabla_\perp^2 + 2ik\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\right]\mathbf{E}(\mathbf{r},t) = -i\frac{4\pi eK[JJ]k}{\gamma}n(\mathbf{r}_\perp) < \exp(-i\zeta)>_{av} , \tag{1}$$

where $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the transverse Laplacian, $n(\mathbf{r}_\perp) = n_0 u(\mathbf{r}_\perp)$ is the transverse density distribution function of the electron beam with on-axis density per unit volume n_0, and $K = eB\lambda_w/2\pi mc^2$ is the wiggler parameter characterizing the strength of a wiggler with peak magnetic field B and period λ_w. The shorthand expression [JJ] is (for planar wigglers) defined as the difference of the two Bessel functions: $J_0[K^2/(4+2K^2)] - J_1[K^2/(4+2K^2)]$. Each electron is characterized by a pair of dynamic variables: the phase of the electron's transverse motion with respect to the optical carrier wave: $\zeta \equiv (k + k_w)z - \omega t$, where $k_w = 2\pi/\lambda_w$, and associated phase velocity or energy detuning: $\nu = \partial\zeta/\partial\tau$, where $\tau = z/L$ with L being the wiggler length. The symbol $< >_{av}$ indicates an ensemble average in electron's longitudinal phase space.

Equation (1) can be simplified to

$$\left(\nabla_\perp^2 + i\frac{\partial}{\partial \tau}\right)a = -2iPu(\bar{\mathbf{r}}_\perp) < \exp(-i\zeta)>_{av} \tag{2}$$

with the aid of the following definitions

$$\bar{z} = z - ct(z), \quad \bar{\mathbf{r}}_\perp = \sqrt{\frac{2k}{L}}\,\mathbf{r}_\perp , \tag{3}$$

$$a \equiv \frac{2\pi e L N K[JJ]}{mc^2\gamma^2} E \exp(i\phi) , \tag{4}$$

$$P \equiv \frac{2\pi^2 e^2 K^2 [JJ]^2 L^2 N n_0}{mc^2\gamma^3} , \tag{5}$$

where N is the number of wiggler periods. The dynamic behavior of each individual electron in the combined field of the radiation and wiggler magnet is governed by the well-known pendulum equation [13]

$$\frac{\partial^2 \zeta}{\partial \tau^2} = |a| \cos(\zeta + \phi) . \tag{6}$$

In general, the coupled nonlinear equations (2) and (6) can be solved only by numerical procedures. However, significant analytical progress is possible in the small signal regime where Eq.(2) and (6) can be linearized in the field amplitude a. Under weak field condition $|a| << 1$, one may solve the pendulum equation for ζ to the first order in a:

$$\zeta \approx \zeta_0 + v_0\tau + \int_0^{\tau}\int_0^{\tau'} a \cos(\zeta_0 + v_0\tau'' + \phi)\, d\tau'' d\tau', \tag{7}$$

where ζ_0 and v_0 are the electron's initial phase and energy detuning. Substituting ζ into the RHS of Eq.(2), keeping only the lowest order term in a and averaging over a uniform initial phase distribution of monoenergetic electrons, one obtains a linear wave equation (for brevity, from here on we will drop the overbar on the scaled transverse coordinates and denote the initial detuning parameter v_0 simply by v)

$$\left(\nabla_\perp^2 + i \frac{\partial}{\partial \tau} \right) a = - P u(\mathbf{r}_\perp) \exp(-iv\tau) \int_0^{\tau}\int_0^{\tau'} a \exp(iv\tau'') d\tau'' d\tau' . \tag{8}$$

Equation (8) describes the 3D evolution of the radiation field self-consistently and applies to both the low and high gain regimes.

It is worthwhile at this point to examine some important features of Eq.(8) by contrasting it with a paraxial wave equation of a familiar form:

$$(\nabla_\perp^2 + i\frac{\partial}{\partial\tau})\, a = F(\mathbf{r}_\perp,\tau)\, a\,, \tag{9}$$

where F, characterizing a local medium response, can be interpreted as either a spatially distributed gain or an index of refraction or a mixture of both. In special cases, Eq.(9) has well-known solutions. If F is a constant, Eq.(9) describes the propagation of waves in free space. One type of solutions is a complete set of discrete Laguerre-Gaussian (or Hermite-Gaussian) functions, whereas another type includes a continuum of self-similar radial Bessel functions (or plane waves, depending on the chosen coordinate system). The Bessel function solutions are also complete and thus mathematically equivalent to the discrete ones. Another situation occurs when F depends only on the transverse spatial variables. The solutions in this case are self-similar and consist of both a discrete set and a continuum. The discrete solutions, unlike the self-similar Bessel function solutions in free space, are bound and physically correspond to guided modes. In both cases the solutions are orthogonal (but not necessarily power orthogonal) in transverse coordinate space.

As seen from Eq.(8) and (9), the medium response in the FEL (the equivalent of F in Eq.(9)) is field dependent and nonlocal. In a FEL the electrons which constitute the gain medium drift through the amplifier. They can be bunched or trapped in the optical field or otherwise modulated in a way that depends upon the upstream conditions. Essentially, it is these features and consequently various special effects associated with wave propagation in FELs that form the subject of this paper.

For convenience, we define:

$$I = \exp(-i\nu\tau)\int_0^{\tau}\int_0^{\tau'} a\,\exp(i\nu\tau'')\,d\tau''\,d\tau'\,, \tag{10}$$

and

$$\tilde{a} \equiv a\,\exp(i\nu\,\tau),\quad \tilde{I} \equiv I\,\exp(i\nu\tau)\ , \tag{11}$$

noting

$$\tilde{I}(\tau = 0) = 0\ , \quad \frac{\partial \tilde{I}}{\partial \tau}(\tau = 0) = 0\ . \tag{12}$$

Eq.(8) then becomes:

$$(\nabla_\perp^2 + i\frac{\partial}{\partial \tau} + \nu)\,\tilde{a} = -Pu(\mathbf{r}_\perp)\,\tilde{I}\ . \tag{13}$$

Taking the second derivative of Eq.(13) with respect to τ, one obtains:

$$\frac{\partial^2}{\partial \tau^2}(\nabla_\perp^2 \tilde{a}) + i\frac{\partial^3}{\partial \tau^3}\tilde{a} + \nu\frac{\partial^2}{\partial \tau^2}\tilde{a} = -Pu(\mathbf{r}_\perp)\,\tilde{a}\ . \tag{14}$$

Equation (14) along with the initial conditions Eq.(12) is equivalent to the original Eq.(8). The new equation, Eq.(14), now admits the guided mode solutions of the form

$$\tilde{a} = g(\mathbf{r}_\perp)\exp(-i\lambda\tau)\ , \tag{15}$$

where λ is the propagation constant, generally complex, and g is the transverse profile of the mode which can be determined by inserting Eq.(15) into Eq.(14), yielding:

$$\nabla_\perp^2 g + [\,\lambda + \nu - Pu(\mathbf{r}_\perp)/\lambda^2]\,g = 0\ . \tag{16}$$

Equation (16) is same as that originally discussed by Moore[4]. We shall from here on refer it as the guided mode equation, or simply as the mode equation.

Clearly, the mode equation defines an eigenvalue problem, but not of the ordinary Sturm-Liouville type because of the nonlinear term in $1/\lambda^2$. As a consequence, the eigenfunctions of Eq.(16) are not orthogonal to each other. However, this difficulty can be formally removed by breaking Eq.(14), a third order differential equation with respect to τ, into three first order ones. Introducing three auxiliary functions X, Y and Z relating to each other by the following equations:

$$i\frac{\partial}{\partial \tau}X = Y\ , \tag{17}$$

$$i\frac{\partial}{\partial\tau}Y = Z - L_1 Y\,, \tag{18}$$

$$i\frac{\partial}{\partial\tau}Z = L_2 X\,, \tag{19}$$

where $L_1 = \nabla_\perp^2 + \nu$ and $L_2 = Pu(\mathbf{r}_\perp)$, we obtain a set of equations (17)-(19) equivalent to Eq.(14) if we let $Y = \tilde{a}$. Putting Eqs.(17)-(19) into a more compact form, one obtains a Schrödinger equation

$$i\frac{\partial}{\partial\tau}\psi = H\psi\,, \tag{20}$$

with non-Hermitian Hamiltonian

$$H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -L_1 & 1 \\ L_2 & 0 & 0 \end{bmatrix}, \tag{21}$$

and vector solution: $\psi = [\,X, Y, Z\,]$.

Although the exact form of the transformation from Eq.(14) to Eq.(20) is somewhat arbitrary, the definition of the functions X, Y and Z was made to highlight the physical significance of these new functions as well as to emphasize the mathematical elegance of the resulting formalism. With the aid of Eqs.(10), (11), (17)-(19), and the definition of Y, it can be shown that:

$$\psi = \begin{bmatrix} -i\dfrac{\partial\tilde{I}}{\partial\tau} \\ \tilde{a} \\ -L_2\tilde{I} \end{bmatrix}. \tag{22}$$

The three components in the vector of Eq.(22) can be interpreted as, respectively, the energy modulation of the electron beam, the amplitude of the radiation field and the density modulation of the electron beam. Accordingly, from Eq.(12) and (22) the initial conditions can be expressed as:

$$\psi(0) = \begin{bmatrix} 0 \\ \tilde{a}\,(0) \\ 0 \end{bmatrix} . \tag{23}$$

Equation (20) is the governing equation for the evolution of both the optical field and the modulation of the electron beam. It defines a linear eigenvalue problem by

$$HV = \lambda V . \tag{24}$$

Each solution of Eq.(24) corresponds to an eigenmode which, by definitions (15) and (22), can be expressed as

$$\psi_n = V_n \exp(-i\lambda_n \tau) \,, \quad V_n = \begin{bmatrix} \dfrac{1}{\lambda_n} \\ 1 \\ \dfrac{Pu(\mathbf{r}_\perp)}{\lambda_n^2} \end{bmatrix} g_n \,, \tag{25}$$

where λ_n and g_n are the eigenvalue and eigenfunction of the mode equation (16).

Because the Hamiltonian in Eq.(21) is real, there are only three types of eigenmodes defining, respectively, the longitudinally growing, decaying and oscillatory modes. The growing and the decaying modes are symmetrical in the sense that if λ and V are eigensolutions, so are their complex conjugates λ^* and V^*. Restricted by the boundary conditions, the growing and decaying modes are bound and discrete. The oscillatory modes with real eigenvalues are not bound and their propagation constants can take any value within a certain range, as shown in Appendix B.

It should be noted that amplification in all practical FELs is a multi-eigenmode process. Due to the limitations of electron accelerators, the initial state of an FEL amplifier can not be prepared as an eigenmode, even if the initial optical mode has the profile corresponding to an eigenmode, because the initial electrons are more or less uniformly distributed in phase. Since the electron beam can not generally be prepared in a form matching the eigenmode solutions, a combination of many discrete and continuous eigenmodes is necessary to characterize the full behavior of the system near the entrance.

Before proceeding to solve the initial value problem, it is helpful to examine further the general physical contexts of the new formalism and verify the consistency of the linear model.

III. Energy Equation

An FEL is essentially a device which transfers energy from an electron beam to an optical wave. In the process of energy transfer the electrons are modulated by a coherent wave through the coupling provided by the wiggler field, and then radiate coherently thereby amplifying the wave. Depending on the initial conditions, an inverse process may also happen in which the wave is damped. In either case, the total energy of the electrons and the wave should be a conserved quantity.

Along with our fundamental Eq.(20), it is useful to consider an adjoint equation which can be obtained by applying a unitary adjoint operator A, defined by

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{26}$$

to the original Eq.(20). With Eq.(20) and it's adjoint one may derive

$$i\frac{\partial}{\partial\tau}(\psi\tilde{\psi}) = \tilde{\psi}H\psi - \psi\tilde{H}\tilde{\psi}, \tag{27}$$

where $\widetilde{H} = (H^A)^*$ and $\widetilde{\psi} = (\psi^A)^*$. The asterisks indicate complex conjugates, and the superscripts A indicate adjoint quantities, explicitly we note $H^A = AHA^{-1}$ and $\psi^A = A\psi$.

Using definition (22), an energy equation follows from Eq.(27):

$$\frac{\partial \mu}{\partial \tau} + \nabla_{\perp} \cdot \mathbf{S} = 0 , \tag{28}$$

where

$$\mu = \psi\widetilde{\psi} = a a^* + iPu(\mathbf{r}_{\perp})\left(I^* \frac{\partial I}{\partial \tau} - I \frac{\partial I^*}{\partial \tau} + i2\nu I I^* \right) \tag{29}$$

and

$$\mathbf{S} = i (a^* \nabla_{\perp} a - a \nabla_{\perp} a^*) . \tag{30}$$

The quantity μ is the energy density of the system which consists of the radiation energy

$$\mu_r = aa^* \tag{31}$$

and the energy associated with the modulation of the electron beam

$$\mu_e = iPu(\mathbf{r}_{\perp})(I^* \frac{\partial I}{\partial \tau} - I \frac{\partial I^*}{\partial \tau} + i2\nu I I^*) . \tag{32}$$

The quantity **S** is the radial Poynting vector of the radiation field. Integrating Eq.(28) over the transverse coordinate plane and indicating this operation with angle brackets we derive the energy conservation for a bound field:

$$\frac{d\langle\mu\rangle}{d\tau} = 0 . \tag{33}$$

In particular, for a bound eigenmode, using Eqs.(25) and (29) the total energy can be expressed as:

$$<\mu> = \left\{<|g|^2> + \frac{2P\mathrm{Re}(\lambda)}{|\lambda|^4} <u(\mathbf{r}_\perp)|g|^2>\right\} \exp\left[2\mathrm{Im}(\lambda)\tau\right] . \tag{34}$$

For both the growing and the decaying mode $\mathrm{Im}(\lambda) \neq 0$, the energy conservation Eq.(33) thus requires the quantity within the braces on the RHS of Eq.(34) to be zero. In each bound eigenmode the energy of the radiation field is positive and the energy associated with the modulation of the electron beam is negative. The two parts are equal in magnitude and make the net energy carried by that mode zero. For an eigenmode with $\mathrm{Im}(\lambda) > 0$ the energy in the radiation field grows exponentially, whereas the energy of the electrons decays at the same rate. The inverse process occurs in an eigenmode with $\mathrm{Im}(\lambda) < 0$. It is interesting to note that the same situation is encountered in microwave tubes [15] where the electron beam is coupled to the cavity modes which are, of course, the guided modes of the radiation field.

IV. Initial Value Problem

We have shown that the evolution of the coupled system can be solved as an initial value problem defined by Eq.(20) along with the initial condition (23). In addition, a linear eigenvalue problem (24) is recovered with a full spectrum of self-similar eigenmodes (25). We are now ready to see that the orthogonality indeed exists in the eigenmodes. With these orthogonal modes the solution to the initial value problem becomes straightforward.

A. Biorthogonality

The eigenmodes of Eq.(24) are not orthogonal to each other since the Hamiltonian (21) is not Hermitian, but rather they are orthogonal to the eigenmodes of the adjoint equation. This property is known as biorthogonality[16]. Denoting a solution of Eq.(24) by λ_n and V_n, and a solution of the equation adjoint to Eq.(24) by λ_m and V_m^A, an integral relation can be formed:

$$< V_m^A\, H\, V_n - V_n\, H^A\, V_m^A > = (\lambda_n - \lambda_m) < V_m^A\, V_n >. \tag{35}$$

Using definitions (21) and (25), the LHS of Eq.(35) can be reduced to a line integral at infinitely large radial distance

$$\int_{\infty} (g_n \nabla_\perp g_m - g_m \nabla_\perp g_n)\, dl \,, \tag{36}$$

where g_m and g_n are solutions of the mode equation.

There are two cases which need separate attention in evaluating the integral (36). If at least one mode is bound, the integral vanishes since it can be shown from Eq.(16) that the bound modes decay exponentially far outside the electron beam. In this case, we have:

$$< V_m^A \, V_n > \; = N_n \, \delta_{nm} \,, \tag{37}$$

where N_n is the normalization constant, which can be evaluated as follows:

$$N_n = < V_n^A \, V_n > = < (1 + 2Pu(\mathbf{r}_\perp)/\lambda_n^3)\, g_n^2 > . \tag{38}$$

Another situation occurs when both modes are in the continuum. As shown in Appendix A, these modes reduce to the free-space modes far outside the electron beam, and thus have the same orthogonality and normalization as the corresponding free-space modes:

$$< V_{\lambda'}^A \, V_\lambda > \; = \; 4\pi\, \delta(\lambda' - \lambda) \,, \tag{39}$$

because the integral (36) depends only on the behavior of the radiation field at large radial distance.

In summary, each bound mode is biorthogonal to every other bound mode and to all the modes in the continuum. The same rule also applies to the continuous modes, with a difference in the normalization.

B. Guided Mode Expansion

A general solution to the initial value problem can be expressed as a linear superposition of all the eigenmodes, including a summation of the discrete bound modes and an integration of the continuous oscillatory modes:

$$\psi = \sum_n C_n V_n \exp(-i\lambda_n \tau) + \int C_\lambda V_\lambda \exp(-i\lambda\tau)\, d\lambda\,, \tag{40}$$

where the expansion coefficients are constant scalars. By definitions (22) and (25), the solution (40) has three parts, one for each component in the vector ψ. The component for the radiation field is:

$$\tilde{a} = \sum_n C_n g_n \exp(-i\lambda_n \tau) + \int C_\lambda g_\lambda \exp(-i\lambda\tau)\, d\lambda\,. \tag{41}$$

With the aid of Eq.(37) and the initial condition Eq.(23), the expansion coefficient for a discrete mode is:

$$C_n = \frac{< V_n^A\, \psi(0) >}{< V_n^A\, V_n >} = \frac{1}{N_n} < \tilde{a}\,(0)\, g_n >. \tag{42}$$

This result is identical to what Moore[5] obtained for the fundamental mode using the Laplace Transform method; here we have shown that Eq.(42) is valid for all the discrete modes, including the decaying ones. The expansion coefficient for a continuous mode is, noting Eq.(39)

$$C_\lambda = \frac{1}{4\pi} < \tilde{a}(0)\, g_\lambda > . \tag{43}$$

The first term on the RHS of Eq.(41) represents the contributions of the discrete bound modes. This term as a whole describes those aspects of FEL operation associated with energy exchange. The power launched into each individual mode is either amplified or damped, but remains transversely confined indefinitely within the wiggler length. Of course, the bound modes are not power orthogonal, so in general the full mode expansion is required to solve for the power flow within the optical wave. It is interesting to note that the expansion coefficient Eq.(42) is not simply an overlap integral between an input field and the mode, but is modified by a factor depending on the mode itself. Consequently, the mode is optimally excited by an input field which is the complex conjugate of that mode. Consider the power carried by a bound mode in the expansion (41), normalized by the power of the input field $<|a(0)|^2>$:

$$G = \frac{<|C_n g_n \exp(-i\lambda_n \tau)|^2>}{<|\tilde{a}(0)|^2>} = G_0 \exp\{2\mathrm{Im}(\lambda_n)\tau\} \ . \tag{44}$$

G has two parts, an exponential growth factor determined by the eigenvalue λ_n of the mode and an input power coupling coefficient:

$$G_0 = \frac{|C_n|^2 <|g_n|^2>}{<|\tilde{a}(0)|^2>} \ , \tag{45}$$

which depends on the input field as well as the mode. While the eigenvalues λ_n for a given system are fixed, the power coupling G_0 varies according to the profile of the input field. The maximal value of G_0 is reached with $\tilde{a}(0) = g_n^*$. This property will be referred to as the conjugate input coupling condition and it's profound effects in the design and optimization of high gain FEL amplifiers and resonators will be discussed in other publications[7,8].

The continuous modes play a quite different role in the expansion Eq.(41). By themselves these modes are not physically meaningful since they are not bound. However, the integral over a continuum of these modes describes the behavior expected from diffraction. In the absence of an electron beam, these modes reduce to the exact free-space modes in the form of self-similar Bessel functions. For this reason the continuous modes may also be called the diffraction modes. The integral on the RHS of Eq.(41) has constant energy and is power orthogonal to the sum of the discrete modes in the first term.

The guided mode expansion Eq.(41) provides a full description of the 3D evolution of an arbitrary input field through the entire interaction region. It is valid at arbitrary gain and at radiation powers up to saturation. In particular, it is most useful for the analysis of optical guiding in a high gain system. To understand optical guiding in terms of the mode analysis, consider a single pass saturated high gain amplifier. The interaction region (the wiggler length) can be divided into three parts. The first part is a transition region which starts from the beginning of the amplifier where many eigenmodes are excited by an input field. The excited modes, each having different transverse profiles as well as propagation constants and growth rates, then propagate independently (a unique property of the eigenmode). The transition region is characterized by an evolving transverse field profile as the multiple optical modes propagate and interfere. If the eigenmodes are nondegenerate, in particular, if the growth rate of the fundamental mode is sufficiently larger than that of the other modes, an exponential growth region can be reached before the end of the wiggler. Here, the fundamental mode dominates, thus the field grows exponentially without modification of the spatial profile. This region extends to the point where the field grows beyond the limits of the small signal approximation. In the final, saturated region of the amplifier, our eigenmode approach breaks down, analysis of the 3D beam-wave interaction becomes considerably more complicated due to the nonlinearity, and normally one has to pursue numerical solutions.

According to the mode propagation picture outlined above, optical guiding is a stable evolution process in which an input field approaches a dominant eigenmode of the system. More important, guiding will inevitably be simultaneously achieved with exponential gain and full transverse coherence. For guiding to occur, the wiggler length L must be larger than the gain length L_g of the fundamental guided mode, and the gain length must be larger than the Rayleigh range L_r corresponding to the electron beam size and the radiation wavelength (assuming reasonably well chosen input profile):

$$\begin{aligned} & L > Lg \, , \\ & Lg > Lr \, . \end{aligned} \tag{46}$$

The first condition guarantees high gain, whereas the second condition assures nondegeneracy of the eigenmodes. In the presence of guiding, the narrow electron beam, required by the second condition, plays the major role of discriminating between different modes by diffraction. Quantitative verification of the conditions (46) is presented elsewhere[7,8]

We remark that the completeness of the eigenmodes in expansion (40) is assumed without proof. This does not seem to be a problem, at least from the physical point of view because, first of all, the self-similar form of the solutions assumed in Eq.(25) does not limit the generality of the solutions to the evolution equation (20) since this form is a consequence of the longitudinal invariance of the Hamiltonian (21). Secondly, there are only three possible types of self-similar solutions to Eq.(24), and the expansion (40) includes all of them. Finally, the guided mode expansion (41) has been compared with the numerical simulations for a high gain FEL and excellent agreement is obtained[7,8].

V. Conclusions

In this paper we have explored the physical and mathematical basis for the analysis of the 3D propagation of the optical field in a free electron laser with a spatially confined gain medium. We showed that the optical field in an FEL amplifier can be expressed as an expansion in the guided optical modes despite the lack of orthogonality of these modes. Most important, it is found that the guided optical mode is only one of three physical components of the eigenmode of the beam-wave system. Each eigenmode specifies the state of the energy and density modulation of the electron beam as well as the optical field. The three-component eigenmodes are biorthogonal whereas the optical modes themselves, do not possess this useful property. The biorthogonal eigenmodes make possible an eigenmode expansion describing the evolution of the system and hence a complete solution to the initial value problem. While several of the results of this analysis were anticipated in the previous research of Moore[4,5], Kim[11] and Krinsky et al.[10], and by earlier studies by Pierce[15] of amplification in microwave traveling wave tubes, our three-component eigenmode expansion provides a new unified and systematic approach to the analysis of mode formation and propagation in unsaturated FEL amplifiers and oscillators.

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Appendix A: Orthogonality and Normalization for the Continuous Modes

In this appendix we prove the orthogonality and normalization of the eigenmodes in the continuum. It is noted that the orthogonality of the eigenmodes depends on a line integral Eq.(36) at infinitely large radial distance. Far outside of the electron beam, the mode equation Eq.(16) is the same as the following:

$$\nabla_{\perp}^2 g + (\lambda + \nu) g = 0 . \tag{A1}$$

Denoting by λ, f and λ', f ' two solutions of the above equation, an integral relation can be formed as:

$$\int_{\infty} (f_{\lambda} \nabla_{\perp} f_{\lambda'} - f_{\lambda'} \nabla_{\perp} f_{\lambda}) dl = (\lambda - \lambda') < f_{\lambda} f_{\lambda'} > . \tag{A2}$$

The LHS of Eq.(A2) is a line integral at infinite radial distance which should be identical to Eq.(36). Then by Eq.(35), we have:

$$< V_{\lambda'}^{A} V_{\lambda} > = < f_{\lambda} f_{\lambda'} > . \tag{A3}$$

Equation (A1) has well-known Bessel function solutions, thus we can evaluate the RHS of Eq.(A3), yielding:

$$< f_{\lambda} f_{\lambda'} > = \int_0^{2\pi} \int_0^{\infty} J_0(\kappa r) \, J_0(\kappa' r) \, r \, dr \, d\varphi = \frac{2\pi}{\kappa} \delta(\kappa - \kappa') , \tag{A4}$$

where $\kappa^2 = \lambda + \nu$. It is easy to verify

$$\frac{\delta(\kappa - \kappa')}{\kappa} = 2\delta(\lambda - \lambda') . \tag{A5}$$

Therefore the orthonormalization for the continuous modes is:

$$< V_{\lambda'}^{A} V_{\lambda} > = 4\pi \, \delta(\lambda' - \lambda) . \tag{A6}$$

Appendix B: The Continuous Mode Are Not Bound

It is shown in this appendix that a mode with a real propagation constant can not be bound. We prove this by first assuming the mode is confined, and then showing the contradiction the assumption would lead to.

Multiplying Eq.(16) by g* and integrating over the transverse plane, we have:

$$\lambda^3 + \nu\lambda^2 - \Omega\lambda^2 - \Sigma = 0 , \tag{B1}$$

where

$$\Omega = \frac{<|\nabla_\perp g|^2>}{<|g|^2>} \tag{B2}$$

and

$$\Sigma = \frac{<Pu(\mathbf{r}_\perp)|g|^2>}{<|g|^2>} . \tag{B3}$$

Equation (B1) can be written as

$$\lambda + \nu = \frac{\Sigma}{\lambda^2} + \Omega . \tag{B4}$$

It follows from Eq.(B4) that $\lambda + \nu$ must be a positive quantity, since $\Sigma > 0$, $\Omega > 0$ and λ is real.

On the other hand, the mode equation (16) has an asymptotic solution far outside the electron beam

$$g \approx \sqrt{2/\pi\kappa r}\, \exp(i\kappa r - i\pi/4) \tag{B5}$$

where $\kappa = \sqrt{\lambda + \nu}$. The mode is confined only if it decays exponentially at large radial distances. For real λ, this requires $\lambda + \nu < 0$ as seen from Eq.(B5). This requirement is not consistent with Eq.(B4).

We thus conclude that a mode with a real propagation constant can not be bound, and that the possible values of λ fall into a continuous range due to the relaxation of the boundary condition on g at $r = \infty$.

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