Title

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Journal

Canadian Geographies / Géographies canadiennes, 19(4)

ISSN

0008-3658

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Publication Date

1975-12-01

DOI

10.1111/j.1541-0064.1975.tb01873.x

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Peer reviewed

GENERALIZATION OF THE FIRST HORTON LAW

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THE CONCEPT of stream numbers, first introduced by Horton and later modified by Strahler, underlies one of the most widely used methods in the description of channel networks (Horton 1945, p. 281; Strahler 1952, p. 1120). Not only does it provide a numerical (but incomplete) characterization of a network's topology, but it has also led to the observation and formulation of empirical regularities (e.g., the now famous "laws" established by Horton) and their formal explanation by means of theory construction (Shreve 1966, 1969; Smart 1968; Werner 1970). Nevertheless, the stream number concept is less than satisfactory for at least two reasons:

1. The disaggregation of channel networks into sets of streams is not based on any geologic or morphologic reasoning, nor does the physiography of a drainage basin and its network lend itself to this type of analysis. Rather, the stream concept as defined by Strahler is a strictly mathematical notion from graph theory (Riordan 1958, p. 135). It is therefore not surprising that it is strongly related only to those concepts which are derived from it, i.e., to the length, slopes, and drainage areas of the streams. Conversely, it exhibits little relation to most of the geomorphic and hydrologic parameters traditionally used in drainage basin and network description (e.g., Anderson 1949; Morisawa 1962; Werner and Smart 1973, p. 293).

2. In determining the stream numbers of a given channel network, each individual stream has to be identified. From these raw data only the stream numbers are obtained, and the remaining information is not utilized. In particular, the stream numbers allow one only to conclude how many streams of a given order merge to form streams of the next higher order; most of the information contained in the data (e.g., how many streams of order i merge with streams of order j when $i \neq j$, and from which side in each case) is not used at all, although it holds the key to various generalizations of the Horton Laws and, possibly, to other topologic/morphometric relationships in drainage basins.

This paper addresses the second of the two shortcomings of the stream number concept as described above. It offers a partial solution to the problem of data utilization and derives several interesting conclusions. More specifically:

1. The use of the raw data is increased by establishing a more detailed description of the network under consideration.

2. Based on this detailed description, the paper gives additional insight into the mathematical structure of channel network topology and provides a new network classification scheme.

3. The First Horton Law is shown to be a special case of the realtionships characterizing the patterns of merging streams in topologically random channel networks.

THE STREAM MERGER MATRIX OF A CHANNEL NETWORK

For the purpose of this paper we define the *stream merger matrix*, MM, of a channel network of order m as the matrix $(n_{ij} \mid i, j = 1,2, ..., m)$, where n_{ij} is defined as the

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Stream Number
Sequence:

32 9 3 1

Stream Merger
Matrix:

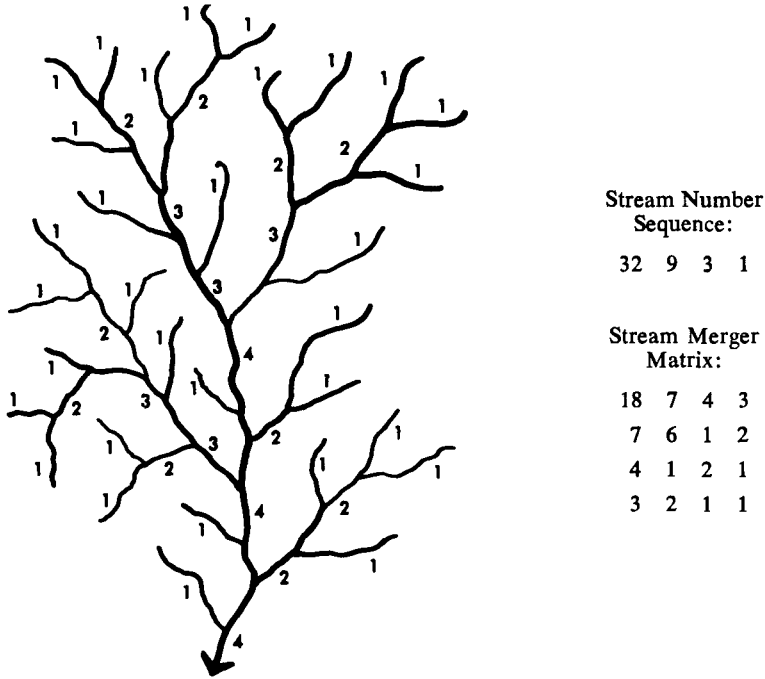| 18	7	4	3
7	6	1	2
4	1	2	1
3	2	1	1

FIGURE 1. Channel network with stream numbers and stream merger matrix showing the number of mergers among streams of all orders.

number of (Strahler) streams of order i merging with streams of order j. We further define $n_{mm} = 1$. Figure 1 shows a channel network of magnitude 32 as well as its stream number sequence (n_i), and its merger matrix $MM = (n_{ij})$. Since the matrix contains the frequencies of mergers among streams of equal order i, where $i = 1, ..., m - 1$, it obviously includes the information given by the sequence (n_i). The matrix MM has a number of noteworthy properties, most of which will be listed here without proof because they are evident or can easily be verified.

Let C be a network of magnitude n and order m with stream numbers $n_1, n_2, ..., n_m$ where $n_1 = n$ and $n_m = 1$, and let $MM = (n_{ij} \mid i, j = 1, ..., m)$ be the associated merger matrix. MM is symmetric; thus, all of the following statements which refer to rows (columns) apply to columns (rows) as well. By definition, $n_{mm} = 1$. The remaining elements of the main diagonal are equal to the stream numbers n_i multiplied by 2:

(1) $n_{ii} = 2n_{i+1}$,   where $i < m$.

The network magnitude n is equal to the sum of the elements of the first row. The sum of the ith row is equal to the number of links of all streams of order i, where $i = 1, ..., m$. Thus, the sum of all elements of MM is equal to the total number of network links, i.e.,

(2) $$\sum_{i=1}^{m} \sum_{j=1}^{m} n_{ij} = 2n - 1.$$

If the network is reduced by eliminating all streams of order 1 through i, then the merger matrix of the remaining network is the submatrix of MM created by cancelling

the first i rows and columns. The sum of the elements of the submatrix is equal to the preceding diagonal element minus 1, or:

$$(3) \qquad n_{ii} = \left(\sum_{k=i+1}^{m} \sum_{j=i+1}^{m} n_{kj} \right) + 1 = 2 \cdot \sum_{j=i+1}^{m} n_{i+1,j}, \qquad \text{where } i < m.$$

The list of relationships among streams of different orders given above can easily be extended and has some convenient applications in the analysis of network topology. For example, the average number $\bar{L}(i)$ of links for streams of order i, using our matrix notation, is

$$(4) \qquad \bar{L}(i) = \frac{2}{n_{i-1,i-1}} \sum_{j=1}^{m} n_{ij}, \qquad \text{where } 1 < i \leqslant m.$$

Notice that this average number of links for streams of order i has been derived from the matrix elements although the matrix refers only to streams and no information on the number of links per stream has been collected.

A more important property of the stream merger matrix as a possible research tool is probably its discriminating power. The grouping of networks by stream numbers has led to weak or inconclusive results regarding possible relations between sets of stream numbers and morphologic-hydrologic variables. This may be due in part to the relatively small number of classes produced by the stream number classification. In contrast, if networks are grouped by defining a class as consisting of all networks which have the same merger matrix, the result will be a much finer classification in which the stream numbers constitute a subclassification.

Table I shows the number of classes generated by the stream number as well as the matrix classification for varying network magnitude n.

Table II shows the disaggregation of the three stream number classes for networks of magnitude 7 when the matrix classification is used. The figures in brackets are the numbers of topologically distinct networks in each class.

The mathematical expressions for the respective class frequencies can be established by a few combinatorial manipulations. Equation (5) below specifies, for any set of networks having the same stream number sequence, the number of classes when these networks are classified according to their merger matrices. Equation (7) gives the number of topologically different networks having the same merger matrix. Following are the proofs of the two equations:

(a) Let n_i ($i = 1, ..., m$) be a sequence of stream numbers. The number of streams of order i which merge with streams of higher order is $u = n_i - 2n_{i+1}$. There are $v = m - i$ stream order classes of higher order. Therefore the number of ways in which the u streams of order i can be assigned to the streams of the v higher-order classes is the binomial coefficient $C(u + v - 1, u)$. Hence, the number f of different

TABLE I

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
(n_i)	1	1	1	2	2	3	3	5	5	7	7	10	10	13	13
(n_{ij})	1	1	1	2	3	5	7	11	16	25	37	57	84	125	179

TABLE II

Stream numbers

Order: I	II	III	Class size	Corresponding stream merger matrices with class sizes			
7	2	1	[80]	$\begin{pmatrix} 4 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{pmatrix}$ [8]	$\begin{pmatrix} 4 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ [32]	$\begin{pmatrix} 4 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ [24]	$\begin{pmatrix} 4 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ [16]
7	3	1	[20]	$\begin{pmatrix} 6 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ [8]	$\begin{pmatrix} 6 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ [12]		
7		1	[32]	$\begin{pmatrix} 2 & 5 \\ 5 & 1 \end{pmatrix}$ [32]			

merger matrices corresponding to the stream number sequence $(n_i \mid i = 1, ..., m)$ is

$$(5) \qquad f(n_i \mid i = 1, ... , m) = \prod_{i=1}^{m-2} \binom{n_i - 2n_{i+1} + m - i - 1}{n_i - 2n_{i+1}} \qquad (m > 2).$$

It is easy to see that, for $m \le 2$, f will always be one.

(b) Consider a merger matrix $MM = (n_{ij} \mid i, j = 1, ..., m)$. The streams of order j are subdivided by the streams of order k, $j > k > i$, into

$$(6) \qquad w = \sum_{k=i+1}^{m} n_{kj}$$

segments. The number of ways in which the n_{ij} streams of order i can be attached to either side of these segments of the order j streams is therefore $C(n_{ij} + w - 1, n_{ij})$ times $2^{n_{ij}}$. Since the matrix is symmetrical and the elements on the diagonal are fixed once the off-diagonal elements are determined, the number of topologically different networks having the same stream merger matrix (n_{ij}) is

$$(7) \qquad F(n_{ij} \mid i, j = 1, ... , m) = \prod_{i=1}^{m-1} \prod_{j=i+1}^{m} \binom{\sum_{k=i}^{m} n_{kj} - 1}{n_{ij}} 2^{n_{ij}}.$$

The information content of the merger matrix MM can be increased, and the classification can be further refined, by defining n_{ij} as the number of streams of order i joining streams of order j from the left side (n_{ji} for mergers from the right). Although the merger matrix is usually no longer symmetrical, most of the above results and considerations apply once a few obvious adjustments are made.

THE EXPECTED STREAM MERGER MATRIX FOR TOPOLOGICALLY RANDOM CHANNEL NETWORKS

The statements made so far hold, without restriction, for any dendritic network or set of networks. In this section, we shall first establish the expected merger matrix for networks of a given order when chosen at random and then deduce several relation-

ships among the matrix elements. These relationships subsume the First Horton Law as a special case.

Let us assume that in the absence of geologic control randomly selected channel networks can be treated as finite subnetworks of infinite topologically random channel networks (Shreve 1969, p. 398). This specifically implies that networks of the same magnitude but different topology are equally likely. The assumption seems justified for at least two reasons. Typically, natural channel networks investigated in geomorphology are embedded in networks of very large magnitude. Furthermore, theoretical implications of the assumption have by and large been verified by empirical tests, and the notion of topological randomness of natural channel networks is generally accepted as a good approximation (Smart 1972, p. 341). The following two theorems hold for finite subnetworks selected at random from infinite topologically random networks (Shreve 1969, pp. 399–400). The expected magnitude $E_m(n)$ for a stream of order m is

$$(8) \qquad\qquad E_m(n) = (2^{2m-1} + 1)/3,$$

i.e., the expected stream numbers of a channel network of order i are, starting with the stream of highest order, 1, 3, 11, 43, 171, ..., $E_i(n)$. The sequence quickly approximates a geometric progression with constant factor 4 and represents a theoretical explanation of the First Horton Law. Furthermore, the average number of streams of order j which merge with a stream of order i, where $i > j$, is 2^{i-j-1}. Using these two relationships we obtain, after some algebraic manipulation, the following equations for the expected stream merger matrix $E_m(MM)$ of networks of order m:

$$(9) \qquad\qquad E_m(n_{ii}) = (2^{2m-2i} + 2)/3 \qquad (i = 1, ... , m),$$

$$(10) \qquad E_m(n_{ij}) = (2^{2m-i-j} + 2^{|i-j|-1})/3 \qquad (i, j = 1, ... , m; \ i \neq j).$$

Table III shows the expected merger matrix $E_5(MM)$ for networks of order 5. Note that the elements of the main diagonal do not satisfy equation (10), i.e., the numbers of mergers among streams of equal order do not fall "in line" with the numbers of mergers among streams of unequal order. Only for m approaching infinity, for i, j finite, and after converting frequencies into probabilities does equation (9) become a special case of equation (10). (The matrix corresponding to these conditions consists of the probabilities of the mergers of streams of order i, j in topologically random networks of infinite magnitude. The values of these probabilities are $2^{-(i+j)}, i, j = 1, 2, ...$)

The elements of the matrix shown in Table IV are the mean values for eight observed channelled networks of order 5 as sampled from USGS 1:24,000 maps (mostly Pennsylvania and Kentucky; data kindly provided by Dr S. Smart, IBM Watson Research Center, New York). They are presented here for illustrative purposes only; the sample size does not permit a statistical analysis which would allow any conclusions as to whether the observed values of Table IV differ significantly from the expected figures of Table III. Besides, the underlying assumption of random network topology as a first approximation for natural channel networks has been extensively tested in the literature; thus the emphasis of this section of the paper is not on another test of that assumption but rather on some of its implications.

TABLE III

EXPECTED MERGER MATRIX FOR
NETWORKS OF ORDER 5

86	43	22	12	8
43	22	11	6	4
22	11	6	3	2
12	6	3	2	1
8	4	2	1	1

TABLE IV

MEAN MERGER MATRIX FOR EIGHT
OBSERVED NETWORKS OF ORDER 5

75.1	41.4	30.5	17.9	11.1
41.4	19.3	7.9	6.1	4.4
30.5	7.9	5.3	1.8	2.5
17.9	6.1	1.8	2.0	0.6
11.1	4.4	2.5	0.6	1

The elements of the first row (column) of the expected merger matrix $E(MM)$ have simple relationships to the remaining matrix elements:

$$(11) \qquad n_{11} = \left(\sum_{i=2}^{m} n_{i1} \right) + 1; \qquad n_{1j} = \sum_{i=2}^{m} n_{ij} \quad \text{for } j > 1.$$

Hence $E(MM)$ can be built stepwise from the one element matrix [1] by repeating simple addition of the row and column elements. The above relationships show that the determinant of any $E(MM)$ is equal to 1, but it is not clear whether this has any specific meaning in a morphologic context.

Since the elements of the main diagonal of the expected merger matrix are the expected stream numbers multiplied by 2, they satisfy the Horton Law. Do the expected frequencies of mergers among streams of unequal order display a similar regularity? The following statements, which can be easily deduced, provide interesting answers to this question.

(a) The elements of each diagonal parallel to the main diagonal of the matrix approximate a geometric progression with constant factor 4. They have the same sequence of bifurcation ratios as the elements of the main diagonal, i.e., as the expected stream numbers. Let R_1 denote that sequence of bifurcation ratios and $E(n_i)$ the expected number of streams of order i. The above statement can then be expressed mathematically as

$$(12) \qquad R_1 = \frac{E(n_{i-1,j-1})}{E(n_{ij})} = \frac{E(n_{i-1})}{E(n_i)} = \frac{2^{2m-2i+3} + 1}{2^{2m-2i+1} + 1} = 3/1, \ 11/3, \ 43/11, \ldots ,$$

where $i = m, m - 1, \ldots, j + 1$ and $1 < j < m$ and $i - j$ is a positive constant c. Since the matrix is symmetric, exchanging i and j will take care of all diagonals to the right of the main diagonal. As an example, consider the three diagonals adjacent to the main diagonal in Table III. If we extend the matrix towards the upper left (i.e., if we consider the expected numbers of stream mergers in a network of higher than fifth order) their respective figures are 1, 3, 11, 43; 2, 6, 22, (86); and 4, 12, (44), (172). It is easy to see that all these sequences of expected stream merger frequencies

have the same sequence of bifurcation ratios (which is that of the main diagonal if we exclude n_{55}, and therefore that of the First Horton Law).

(b) On either side of the main diagonal, each series of elements parallel to the minor diagonal approximates a geometric progression with constant factor 1, and they all have the same sequence (or subsequence) R_2 of bifurcation ratios. In other words, any series of matrix elements parallel to the minor diagonal will approximate a constant value as it progresses towards the main diagonal. The sequence of bifurcation ratios R_2 is given by

$$(13) \qquad R_2 = \frac{E(n_{i-1,j+1})}{E(n_{ij})} = 1 - \frac{3}{2^{2m-2i+3} + 4} = 3/4, \; 11/12, \; 43/44, \; ... \; ,$$

where $i = m, m - 1, ...,$ and $i \geq j$ and $i + j$ is some constant c. (For $j = 1, 2, ...$ and $i \geq j$ and $i + j = c$ we obtain subsequences of R_2 which correspond to series of matrix elements starting in the first column rather than the last row; the equivalent relations on the other side of the main diagonal are again described by the symmetry of the matrix.) The minor diagonal in Table III is too short to demonstrate how the elements gradually approximate a constant and, therefore, the bifurcation ratios approximate the value 1. For a channel network of order 10 the values of the minor diagonal are, respectively, 256, 192, 176, 172, ..., and of the diagonal adjacent and parallel to it 128, 96, 88, 86, ... Again it is obvious that the sequences have the same bifurcation ratio figures.

(c) The two bifurcation sequences R_1 and R_2 which characterize the matrix elements located either parallel to the main diagonal or orthogonal to it differ only by a constant factor:

$$(14) \qquad\qquad\qquad R_1 = 4R_2.$$

Combining the three statements, we can now formulate a generalization of the First Horton Law.

The expected frequencies of mergers among streams of order i, j with $i - j = c$ (constant) approximate a geometric progression with constant factor 4, i.e., they satisfy the First Horton Law. For the special case $i - j = 0$ these frequencies correspond to the network stream numbers for which this law was originally formulated. Moreover, the various progressions corresponding to different values of c have the same sequence of bifurcation ratios.

The expected frequencies of mergers among streams of orders i, j with $i + j$ constant approximate a geometric progression with constant factor 1. Again the various progressions corresponding to different values of c have the same sequence (or subsequence) of bifurcation ratios, and this sequence is related to the first one by a constant factor 4. Thus, the Horton Law is again satisfied, except that the bifurcation ratios approximate the value unity rather than 4.

Figure 2 shows a histogram-type illustration of the expected merger matrix $E(MM)$ for channel networks of order 5. The base of the diagram is formed by the pairs i, j $(1 \leq i, j \leq 5)$ and the third dimension represents the expected frequencies $E(n_{ij})$ of mergers among streams of orders i and j. The main diagonal (i.e., the heavy dotted line connecting A and B) is the graphic representation of the original Horton Law as derived by Shreve (1969, p. 399). Notice that the various diagonals
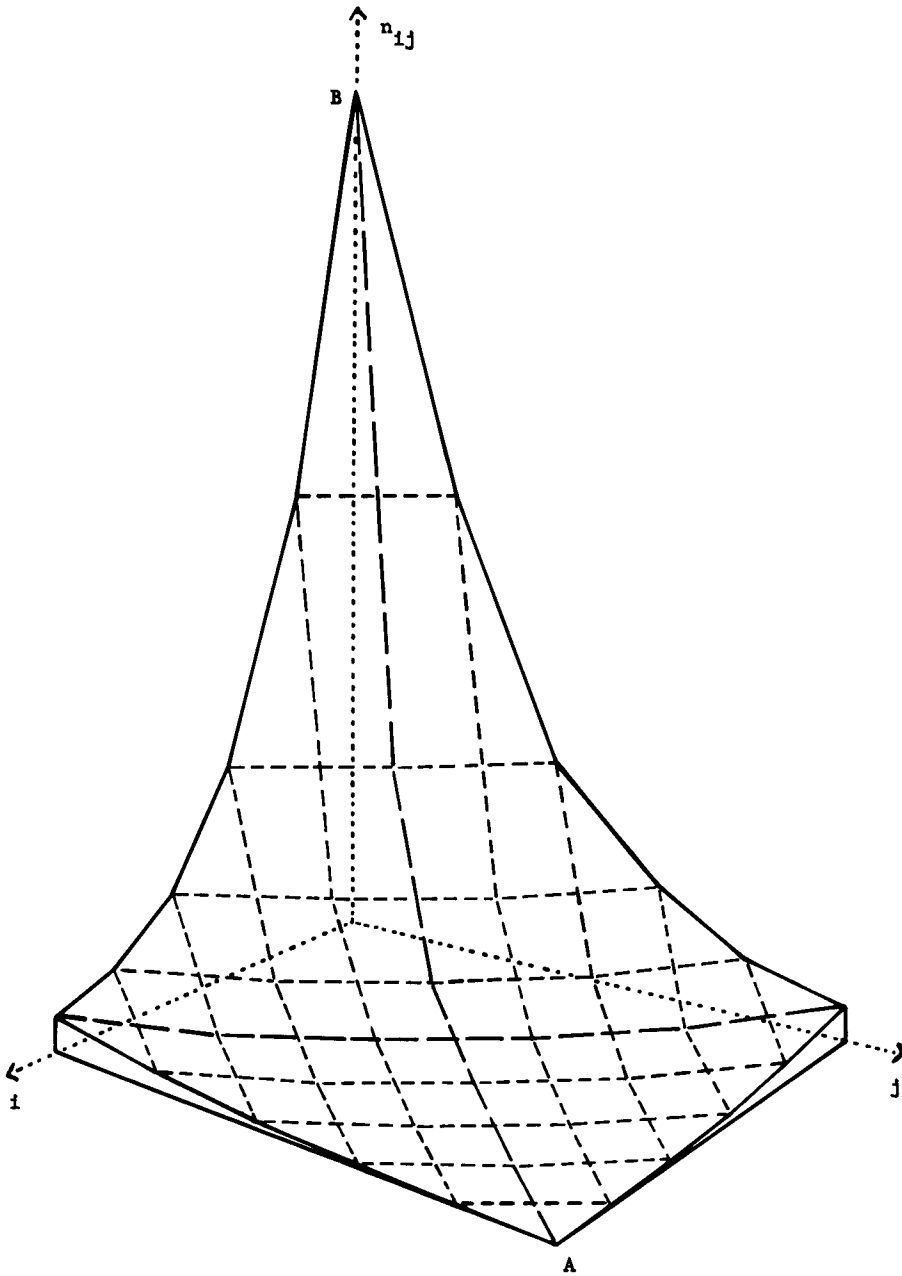
FIGURE 2. Histogram showing the three-dimensional frequency distribution of the expected number of mergers among streams of orders i and j in a channel network of order five. The main diagonal on the surface (connecting A and B) corresponds to the First Horton Law; notice that the diagonals parallel to it show the same slope distribution. The histogram also shows that the minor diagonal and those parallel to it approximate horizontal lines.

parallel to the main diagonal tend to have the same slope distribution, and that the diagonals parallel to the minor diagonal approximate horizontal lines. The third feature of the generalized Horton Law as presented in this paper, namely the similar slope distribution of the two types of diagonals (equation (14)), cannot as readily be recognized in this illustration.

SOME CONCLUDING COMMENTS

There are many papers which deal with the mathematics of channel network topology, and adding yet another one makes it increasingly important to reflect on the merits of such theoretical studies. The following statements are made to identify the position which this and similar papers occupy, in the author's judgment, within the field of geomorphology.

1. One could argue that, by definition, any subject matter becomes a part of a discipline if, among the members of the corresponding profession(s), there are researchers who want to explore it, editors who are prepared to publish the results, and readers who are interested in reading about it. In this case, the question as to whether graph-theoretical research on network topology should become part of geomorphology has already been answered by the course of events since Horton first introduced (semi-)topologic concepts and observations into the geomorphic literature.

2. The above argument is unsatisfactory to the extent that every discipline should try to establish a rational base for its activities, rather than see itself only as a partially interrelated sequence of historical accidents. If the field of geomorphology is to qualify as a scientific discipline its research has to emphasize, by definition, scientific methodology including the construction of theory. The work of Shreve and others has generated such a theory, namely that of the random topology of channel networks. This theory not only represents the mathematical analysis of certain abstract branching patterns which may or may not have real world counterparts, but it is also a geomorphologically relevant theory because it provides theoretical explanations for observed geomorphic phenomena. Examples are: (a) distribution of bifurcation ratios in channel networks (Shreve 1966); (b) distribution of the ratios of stream lengths in channel networks (Smart 1968; Shreve 1967, 1969); (c) the 0.6 power relationship between main stream length and basin area in channel networks (Shreve 1970; Werner and Smart 1973); (d) the relationship between the Langbein area/distance distribution parameter and basin area (Langbein et al. 1947; Werner and Smart 1973); (e) several other relationships between morphometric basin and network variables (Werner 1975). It should be emphasized that some of these examples show the applicability of theorems derived from the theory of topologic randomness to morpho*metric* parameters, thereby tying topologic concepts and relationships into the larger context of geomorphic research in general.

3. The theory of topologic randomness of channel networks permits the derivation of a virtually infinite number of theorems, each of which could become a geomorphic "law" once it has been established under which real world conditions natural channel networks actually satisfy the specific behaviour predicted by the theorem. (It is at least questionable, however, if every empirical regularity backed up by theory is

therefore of sufficient interest.) Alternatively, the lack of correspondence between theoretical prediction and real world observation draws attention to peculiarities of natural channel networks which otherwise would not have been noticed. An example is the discovery by Shreve and Krumbein that the distribution of *cis* and *trans* links in natural channel networks is far from random and shows a large and systematic bias in favour of *cis* links.

4. Within the framework of these rather general comments, the contribution of this paper is twofold:

(a) The paper deduces several new theorems including, specifically, a generalization of what has become known as the "First Horton Law." In this respect, the paper is as "relevant" as Horton's Law (the relevance of which, however, may be considered as only historic, inasmuch as it led to the graph-theoretical investigations of channel networks).

(b) The concept of stream numbers has so far resisted being linked to other geomorphic parameters, either because such relationships simply do not exist, or because stream numbers are too crude a concept to possess any significant discriminating power. This paper provides a more detailed description and analysis of the stream patterns of channel networks and will therefore allow a more sensitive testing regarding the geomorphic relevance or irrelevance of the stream number concept.

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