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Controlled Rough Paths on Manifolds

A dissertation submitted in partial satisfaction of the
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in

Mathematics

by

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2015

The dissertation of Jeremy Sean Semko is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California, San Diego

2015

EPIGRAPH

Hey guys!

Oh Big Gulps huh? Alright!

Welp, see you later!

—Lloyd Christmas

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Portions of Chapters 1, 2, 3, 4, and 5 and Appendices A and B are adapted

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ABSTRACT OF THE DISSERTATION

Controlled Rough Paths on Manifolds

by

Jeremy Sean Semko

Doctor of Philosophy in Mathematics

University of California, San Diego, 2015

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We build the foundation for a theory of controlled rough paths on manifolds. A number of natural candidates for the definition of manifold valued controlled rough paths are developed and shown to be equivalent. The theory of controlled rough one-forms along such a controlled path and their resulting integrals are then defined. This general integration theory does require the introduction of an additional geometric structure on the manifold which we refer to as a "parallelism." The transformation properties of the theory under change of parallelisms is explored. Using these transformation properties, it is shown that the integration of a smooth one-form along a manifold valued controlled rough path is in fact well defined independent of any additional geometric structures. We present a theory of push-forwards and show how it is compatible with our integration theory. We give a number of characterizations for solving a rough differential equation when the solution is interpreted as a controlled rough path on a manifold and then show such solutions exist and are unique. We develop the notion of parallel translation along a controlled rough path. This lends itself to a theory of rolling and unrolling maps for not only controlled rough paths but controlled rough one-forms.

Chapter 1

Introduction

In a series of papers [20–22], Terry Lyons introduced and developed the far reaching theory of rough path analysis. This theory allows one to solve (deterministically) differential equations driven by rough signals at the expense of "enhancing" the rough signal with some additional information. Lyons' theory has found numerous applications to stochastic calculus and stochastic differential equations, for example see [4], [5], [6], [8], and the references therein. For some more recent applications, see [1], [19], [18], [9] , and [2].

The rough path theory mentioned above has been almost exclusively developed in the context of state spaces being either finite or infinite dimensional Banach spaces with the two exceptions of [7] and [3]. In [7], a version of manifold valued rough paths is developed in the context of "currents," while in [3] the authors develop a more concrete theory by working with embedded submanifolds.

The purpose of the dissertation is to define and develop a third interpretation of rough paths on manifolds based on Gubinelli's [14] notions of "controlled" rough paths. As Gubinelli's perspective has proved extremely useful in the flat case (most notably see Hairer [15]), it is expected such a theory of controlled rough paths on manifolds can give new insights as well as applications to the existing literature. In the following section, for the readers convenience, we will provide stated results with their respective numbers as they appear in the following chapters.

1.1 Main Results of the Dissertation

This section provides the main results of the dissertation while avoiding most of the technical details. Let

$$\mathbf{X}_{s,t} := 1 + x_{s,t} + \mathbb{X}_{s,t} \in \mathbb{R} \oplus W \oplus W^{\otimes 2}$$

be a weak-geometric rough path in $W := \mathbb{R}^k$ with $1 \leq p < 3$ (See Definition 2.6 below for the definition of a rough path) . Generally speaking, one can think of the term $\mathbb{X}_{s,t}$ by the "identity"

$$\mathbb{X}_{s,t} \text{ ``}=\text{''} \int_s^t x_{s,\tau} \otimes dx_\tau. \tag{1.1}$$

When $x_{s,t}$ is not regular enough, the right hand side of Eq. (1.1) is not uniquely defined (the reader can refer to Section 2.1 below to see why this is the case); in this situation, it is necessary to decree what the value of "$\int_s^t x_{s,\tau} \otimes dx_\tau$" is. In stochastic settings, one often constructs this "enhancement" using probability tools.

1.1.1 Main Results of Chapter 3

Let M^d be a d – dimensional manifold. A *rough path controlled by* $\mathbf{X}$ on M (see Definition 3.24) is a pair of continuous functions $y : [0, T] \to M$, and $y^\dagger : [0, T] \to L(W, TM)$ such that (somewhat imprecisely speaking) for all

$0 \le s \le t \le T$;

$$
\begin{aligned}
&1)\ y_s^\dagger : W \to T_{y_s}M,\\
&2)\ \psi\left(y_s, y_t\right) = y_s^\dagger x_{s,t} + O\left(\left|x_{s,t}\right|^2\right),\\
&3)\ U\left(y_s, y_t\right) y_t^\dagger - y_s^\dagger = O\left(\left|x_{s,t}\right|\right),
\end{aligned}
$$

where ψ is a "logarithm" on M and U is a "parallelism" on M. Loosely, a "logarithm" $\psi : M \times M \to TM$ is a function that locally behaves like subtraction, i.e.

$$\psi\left(y_s, y_t\right) \approx y_t - y_s.$$

Of course, $y_t - y_s$ does not make sense on a manifold, and, instead, the output is a tangent vector based at y_s (see Definition 3.4 below for the precise definition). Likewise, a "parallelism" is a type of transport from one tangent space to another that locally looks like the identity, i.e.

$$U\left(y_s, y_t\right) \approx I_{T_{y_s}M \leftarrow T_{y_t}M}.$$

Again, the right hand side does not make sense on a general manifold and so the reader is referred to Definition 3.5. A pairing of a logarithm and parallelism, $\mathcal{G} := (\psi, U)$, is called a gauge. As a sanity check, we note that when $M = \mathbb{R}^d$, one identifies all tangent spaces in which case one typically takes $U\left(m, n\right) = I$ and $\psi\left(m, n\right) = n - m$. In this case, the definition of a rough path controlled by $\mathbf{X}$ reduces to

$$
\begin{aligned}
&1)\ y_s^\dagger : W \to \mathbb{R}^d,\\
&2)\ y_t - y_s = y_s^\dagger x_{s,t} + O\left(\left|x_{s,t}\right|^2\right),\\
&3)\ y_t^\dagger - y_s^\dagger = O\left(\left|x_{s,t}\right|\right).
\end{aligned}
$$

which is precisely the definition of a controlled rough path on Euclidean space (see Section 2.4 for motivation and details).

Alternatively one can define controlled rough paths locally via a chart ϕ by requiring (see Definition 3.29)

$$\phi(y_t) - \phi(y_s) - d\phi \circ y_s^\dagger x_{s,t} = O\left(|x_{s,t}|^2\right) \text{ and } d\phi \circ y_t^\dagger - d\phi \circ y_s^\dagger = O\left(|x_{s,t}|\right).$$

The main content of Chapter 3 is proving that these two notions are the same:

Theorem 3.33 *Let $\mathbf{y} := \left(y, y^\dagger\right)$ be a pair of continuous functions as in Notation 3.23, M be a manifold, and $\mathcal{G} = (\psi, U)$ be any gauge on M. Then $\mathbf{y}$ is a chart controlled rough path (Definition 3.29) if and only if it is a (ψ, U)-controlled rough path (Definition 3.24).*

The set of pairs $\mathbf{y} = \left(y, y^\dagger\right)$ satisfying any of the equivalent conditions is denoted $CRP_{\mathbf{X}}(M)$. Moreover, these manifold-valued rough paths may also be characterized as pairs $\mathbf{y} = \left(y, y^\dagger\right)$ whose "push-forwards" under smooth real-valued functions are controlled rough paths on $\mathbb{R}$:

Theorem 3.48 $\mathbf{y} = \left(y, y^\dagger\right) \in CRP_{\mathbf{X}}(M)$ *if and only if for every* $f \in C^\infty(M)$,

$$f_* \mathbf{y} = \left(f(y), df \circ y^\dagger\right) \in CRP_{\mathbf{X}}(\mathbb{R}).$$

Two natural examples of manifold valued controlled rough paths are as follows (and are explained in more detail in Section 3.4).

1. If M^d is an embedded submanifold and the path $x_s \in W$ happens to lie in M (i.e. $x_s \in M$ for all s in $[0, T]$), then $(x_s, P(x_s))$ is an $M-$valued rough path controlled by $\mathbf{X}$ where $P(m)$ is orthogonal projection onto $T_m M$. This is Example 3.45 below. In fact, any projection will work.

2. If $f : W \to M^d \subseteq \mathbb{R}^{\tilde{k}}$ is smooth, then $(f(x_s), f'(x_s))$ is a rough path controlled by $\mathbf{X}$. This is Example 3.47 below.

1.1.2 Main Results of Integration Theory in Chapter 4

Let $\mathcal{G} = (\psi, U)$ be a gauge, V be a Banach space, and $\mathbf{y} = \left(y, y^{\dagger}\right)$ be an M – valued controlled rough path as above. A pair of continuous functions $\alpha : [0,T] \to L(TM, V)$ and $\alpha^{\dagger} : [0,T] \to L(W \otimes TM, V)$ is a $U-$*controlled (rough) one-form along* y with values in a Banach space V provided (see Definition 4.1 for details);

1. $\alpha_s : T_{y_s}M \to V$ for all s,
2. $\alpha_s^{\dagger} : W \otimes T_{y_s}M \to V$ for all s,
3. $\alpha_t \circ U(y_t, y_s) - \alpha_s - \alpha_s^{\dagger}(x_{s,t} \otimes (\cdot)) = O\left(|x_{s,t}|^2\right)$, and
4. $\alpha_t^{\dagger} \circ (I \otimes U(y_t, y_s)) - \alpha_s^{\dagger} = O(|x_{s,t}|)$.

To abbreviate notation we write $\boldsymbol{\alpha}_s = \left(\alpha_s, \alpha_s^{\dagger}\right)$. We denote $CRP_y^U(M,V)$ as the space of $U-$controlled one-forms along y.

As an example, if $\alpha \in \Omega^1(M,V)$ is a smooth one-form on M and U is a parallelism, the following proposition shows how to construct $\alpha_s^{\dagger U}$ so that $\boldsymbol{\alpha}_s^U = \left(\alpha_s := \alpha|_{T_{y_s}M}, \alpha_s^{\dagger U}\right)$ is a $U-$controlled (rough) one-form along y. The covariant derivative ∇^U (see Remark 3.9) in the statement of the proposition is is defined by

$$\nabla_{v_m}^U(Y) := \frac{d}{dt}|_0 U(m, \sigma_t) Y(\sigma_t)$$

where σ is any curve in M with $\dot{\sigma}_0 = v_m$.

Proposition 4.34 *Suppose that $\alpha \in \Omega^1(M,V)$ is a V – valued one-form and U*

is a parallelism on M, *then*

$$\boldsymbol{\alpha}_s^{(y,U)} := \left(\alpha_{y_s}, \alpha_s^{\dagger(y,U)}\right) := \left(\alpha|_{T_{y_s}M}, \nabla^U_{y_s^\dagger(\cdot)}\alpha\right) \in CRP_y^U\left(M,V\right).$$

Given a controlled rough path $\mathbf{y} = \left(y, y^\dagger\right) \in CRP_{\mathbf{X}}\left(M\right)$ and an $\boldsymbol{\alpha}_s = \left(\alpha_s, \alpha_s^\dagger\right) \in CRP_y^U\left(M,V\right)$, Theorem 4.21 proves the existence and uniqueness of the integral $\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}}\right\rangle$:

Theorem 4.21 *Let* $\mathcal{G} := \left(\psi, U\right)$ *be a gauge,* $\boldsymbol{\alpha} \in CRP_y^U\left(M,V\right)$, *and* $\tilde{z}_{s,t}$ *be as in Definition 4.20. Then there exists a unique* $\mathbf{z} = \left(z, z^\dagger\right) \in CRP_{\mathbf{X}}\left(V\right)$ *such that* $z_0 = 0$, $z_{s,t} \underset{3}{\approx} \tilde{z}_{s,t}$, *and* $z_s^\dagger = \alpha_s \circ y_s^\dagger$. *We denote this unique controlled rough path by* $\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}}\right\rangle$, *i.e.*

$$\int_s^t \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}}\right\rangle := \left[\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}}\right\rangle\right]_{s,t}^1 \underset{3}{\approx} \left\langle \boldsymbol{\alpha}_s, \mathbf{y}_{s,t}^{\mathcal{G}}\right\rangle \text{ and } \left[\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}}\right\rangle\right]_s^\dagger = \alpha_s \circ y_s^\dagger.$$

The integral $\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}}\right\rangle$, a standard flat $V-$valued controlled rough path along $\mathbf{X}$, satisfies a basic but useful associativity property; Theorem 4.24 (also see Proposition 4.6) makes this idea precise. A reduced version of this property is proved in Theorem 4.41 in the case of one-forms is used to prove some results relating to rough parallel translation in Chapter 6.

The integral $\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}}\right\rangle$, as the notation suggests, a priori depends on a choice of gauge, $\mathcal{G} = \left(\psi, U\right)$. However, the following corollary shows that the integral actually only depends on the parallelism, U.

Corollary 4.31 *The integral,* $\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}}\right\rangle$ *only depends on the choice of parallelism* U *and not on the logarithm used to make the gauge* $\mathcal{G} = \left(\psi, U\right)$.

Although the integral $\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}}\right\rangle$ does depend on the choice of parallelism U, Theorem 4.32 shows how, by using "natural" transformations relating all of the relevant structures, one can preserve $\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}}\right\rangle$ under change of parallelism, $U \to \tilde{U}$:

Theorem 4.32 *The map*

$$\boldsymbol{\alpha}_s = \left(\alpha_s, \alpha_s^{\dagger}\right) \longrightarrow \tilde{\boldsymbol{\alpha}}_s := \left(\tilde{\alpha}_s, \tilde{\alpha}_s^{\dagger}\right) := \left(\alpha_s, \alpha_s^{\dagger} + \alpha_s S_{y_s}^{\tilde{U},U} y_s^{\dagger} \otimes I\right)$$

is a bijection from $CRP_y^U (M, V)$ *to* $CRP_y^{\tilde{U}} (M, V)$ *such that*

$$\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}} \right\rangle = \int \left\langle \tilde{\boldsymbol{\alpha}}, d\mathbf{y}^{\tilde{\mathcal{G}}} \right\rangle.$$

By combining all of the previous results, we can uniquely define and compute the integral $\int \alpha (d\mathbf{y})$ when α is a one-form on M without needing to invoke a gauge (see Theorem 4.35). In fact, a gauge independent formula by using charts is given in Corollary 4.39.

Corollary 4.39 *Let* ϕ *be a chart on* M. *For all* $a, b \in [0, T]$ *such that* $y [a, b] \subset D (\phi)$, *we have the approximation*

$$\left[\int \alpha (d\mathbf{y})\right]_{s,t}^{1} \underset{3}{\approx} \alpha_{y_s} \left((d\phi_{y_s})^{-1} [\phi (y_t) - \phi (y_s)]\right) + d\left(\alpha_{(\cdot)} \circ \left(d\phi_{(\cdot)}\right)^{-1} d\phi_{y_s}\right)_{y_s} \circ y_s^{\dagger \otimes 2} \mathbb{X}_{s,t}$$

holds for all $s < t \in [a, b]$.

One of the niceties of the integral $\int \alpha (d\mathbf{y})$ is that it shares many of the properties that would hold were $\mathbf{y}$ a smooth path. For instance, if we denote $f_* \mathbf{y}$ as the "push-forward" of $\mathbf{y}$ by f (see Definition 4.43 for more details), we have an expected "Push me-Pull me" property.

Theorem 4.47 *Let* $f : M \to \tilde{M}$, *let* $\mathbf{y}_s = \left(y_s, y_s^{\dagger}\right) \in CRP_{\mathbf{X}} (M)$ *and let* $\tilde{\alpha} \in \Omega^1 \left(\tilde{M}, V\right)$. *Then*

$$\left[\int f^* \alpha (d\mathbf{y})\right]^1 = \left[\int \alpha (d (f_* \mathbf{y}))\right]^1.$$

Moreover

$$\int f^* \alpha (d\mathbf{y}) = \int \alpha (d (f_* \mathbf{y})).$$

1.1.3 Main Results of RDE Theory in Chapter 5

In Section 5.2, we discuss the notion of a controlled rough path $\mathbf{y} = \left(y, y^{\dagger}\right)$ solving the rough differential equation (RDE)

$$d\mathbf{y}_t = F_{d\mathbf{X}}\left(y_t\right) \quad \text{with} \quad y_0 = \bar{y}_0$$

when $F : W \to \Gamma\left(TM\right)$. Essentially $\mathbf{y}$ will solve such an equation if the path y, when pushed forward by any smooth function f, has the correct "Taylor expansion" and $y^{\dagger}$ is the correct derivative, i.e. $y_s^{\dagger} = F_{(\cdot)}\left(y_s\right)$. We note that if F is linear with its range in the algebra of differential operators, we can extend it uniquely to $\mathcal{F}$ which acts on the tensor algebra $T\left(\mathbb{R}^n\right)$. With this notation in mind, we provide the full-fledged definition here.

Definition 5.2 *$\mathbf{y} = \left(y, y^{\dagger}\right)$ on $I_0 = [0, T]$ or $[0, T)$ solves $d\mathbf{y}_t = F_{d\mathbf{X}_t}\left(y_t\right)$ if $y_s^{\dagger} = F_{(\cdot)}\left(y_s\right)$ and for every $f \in C^{\infty}\left(M\right)$ and $[a, b] \subseteq I_0$, the approximation*

$$f\left(y_t\right) - f\left(y_s\right) \underset{3}{\approx} \left(\mathcal{F}_{\mathbf{X}_{s,t}} f\right)\left(y_s\right)$$

holds for $a \leq s \leq t \leq b$. If in addition $y_0 = \bar{y}_0$, we say $\mathbf{y}$ solves $d\mathbf{y}_t = F_{d\mathbf{X}_t}\left(y_t\right)$ with initial condition $y_0 = \bar{y}_0$.

This definition is not the only way to state that an RDE is solved; three equivalent characterizations are given in the following theorem.

Theorem 5.3 *Let y be a path in M on I_0 with $y_s^{\dagger} = F_{\cdot}\left(y_s\right)$. Let $\mathbf{y} = \left(y, y^{\dagger}\right) \in CRP_{\mathbf{X}}\left(M\right)$. The following are equivalent.*

1. *For every chart ϕ with $a, b \in I_0$ such that $y\left([a, b]\right) \subseteq D\left(\phi\right)$ the approximation*

$$\phi\left(y_t\right) \underset{3}{\approx} \phi\left(y_s\right) + d\phi \circ F_{x_{s,t}}\left(y_s\right) + F_w\left(y_s\right)\left[d\phi \circ F_{\tilde{w}}\right]|_{w \otimes \tilde{w} = \mathbb{X}_{s,t}}$$

holds $a \le s \le t \le b$; that is

$$\phi(y_t) - \phi(y_s) = \int_s^t \left\langle \left(\left[d\phi \circ F_{(\cdot)}\right]_* \mathbf{y}\right)_\tau, d\mathbf{X}_\tau \right\rangle$$

for $a \le s \le t \le b$.

2. *If V is a Banach space, $\alpha \in \Omega^1(M, V)$, and $[a,b]$ is such that $[a,b] \subseteq I_0$ then*

$$\int_s^t \alpha(d\mathbf{y}) \underset{3}{\approx} \alpha\left(F_{x_{s,t}}(y_s)\right) + F_w(y_s)\left[\alpha \circ F_{\tilde{w}}\right]|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}}$$

for $a \le s \le t \le b$; that is

$$\int_s^t \alpha(d\mathbf{y}) = \int_s^t \left\langle \left(\left[\alpha \circ F_{(\cdot)}\right]_* \mathbf{y}\right)_\tau, d\mathbf{X}_\tau \right\rangle$$

for $a \le s \le t \le b$.

3. $\mathbf{y}$ *solves $d\mathbf{y}_t = F_{d\mathbf{X}_t}(y_t)$; that is*

$$f(y_t) - f(y_s) = \int_s^t \left\langle \left(\left[df \circ F_{(\cdot)}\right]_* \mathbf{y}\right)_\tau, d\mathbf{X}_\tau \right\rangle$$

for every $f \in C^\infty(M)$.

With any differential equation theory, it is necessary to understand existence and uniqueness. These are provided in tandem in Theorem 5.4 and Theorem 5.5 below:

Theorem 5.4 *Let $F : W \to \Gamma(TM)$ be linear and let $\bar{y}_0$ be a point in M. There exists a local in time solution to the differential equation $d\mathbf{y}_t = F_{d\mathbf{X}_t}(y_t)$ with initial condition $y_0 = \bar{y}_0$.*

Theorem 5.5 *Let $T > 0$. There is unique solution $\mathbf{y}_t \in CRP_{\mathbf{X}}(M)$ to $d\mathbf{y}_t = F_{d\mathbf{X}_t}(y_t)$ with initial condition $y_0 = \bar{y}_0$ existing either on all of $[0,T]$ or on $[0,\tau)$ for some $\tau < T$ such that the closure of $\{y_t : 0 \le t < \tau\}$ is not compact.*

With these results proved along with a few others in Chapter 5, we have the tools necessary to develop the notions of parallel translation, rolling, and unrolling.

1.1.4 Main Results of Parallel Translation Theory in Chapter 6

If a manifold M is equipped with a covariant derivative then, given an initial frame and a smooth path, it is classical to develop parallel translation along the path. Thus, for a smooth path y, we can transport a tangent vector $v_{y_0} \in T_{y_0}M$ to another tangent vector in $T_{y_t}M$ for any $t \in [0, T]$ in a natural way.

The concept of parallel translation can be generalized to any principal bundle $G \to P \xrightarrow{\pi} M$ with a connection ω. To see more about these concepts, we refer the reader to Appendix C. In this setting, the analog of parallel translation is horizontal lifting. Just as one can lift a smooth path y_t in M to a path in P such that $\omega(\dot{y}_t) = 0$, one can lift a controlled rough path in M to a controlled rough path in P with similar characteristics:

Theorem 6.8 (Existence of Horizontal Lifts) *Let $G \to P \xrightarrow{\pi} M$ be a principal bundle with connection ω, $\mathbf{y} = \left(y, y^{\dagger}\right) \in CRP_{\mathbf{X}}(M)$, and $\bar{u}_0 \in P_{y_0}$. Then there exists a unique horizontal lift $\mathbf{u} = \left(u, u^{\dagger}\right) \in CRP_{\mathbf{X}}(P)$ above $\mathbf{y}$ such that $u_0 = \bar{u}_0$.*

Once we have horizontal lifts, it is easy to specialize to the case of parallel translation, where we examine the situation when $P = GL(M)$. This gives us a definition of an object which we now know exists:

Definition 6.10 *Let $GL(M)$ be the frame bundle above M with structure group $GL(d)$, let $\mathbf{y} = \left(y, y^{\dagger}\right) \in CRP_{\mathbf{X}}(M)$, and let ∇ be a covariant derivative on TM. Further, let $\bar{u}_0 \in GL(M)_{y_0}$. **Parallel translation along $\mathbf{y}$ starting at $\bar{u}_0$** is the unique $\mathbf{u}$ which is an element of $CRP_{\mathbf{y}}^{H,\nabla}(GL(M)) := CRP_{\mathbf{y}}^{H,\omega^{\nabla}}(GL(M))$ such that*

1. $\pi_* \mathbf{u} = \mathbf{y}$

2. $\int \omega^{\nabla}(d\mathbf{u}) \equiv \mathbf{0}$

where ω^{∇} *is the connection form associated to* ∇ *(see Eq. (C.1)).*

Once we have parallel translation capabilities for controlled rough paths, we can used it to create a correspondence elements in $CRP_{\mathbf{X}}(M)$ and $CRP_{\mathbf{X}}(GL(M))$. This idea is what is developed in Theorem 6.14. We can then use this theorem to understand a correspondence of elements in $CRP_{\mathbf{X}}(M)$ and $CRP_{\mathbf{X}}(\mathbb{R}^d)$. This rough version of Cartan's rolling and unrolling maps is the content of Corollary 6.16 below.

Corollary 6.16 *Let* ∇ *be a covariant derivative on* M, $o \in M$, *and* $u_o \in GL(M)_o$. *There is a one-to-one map from* $CRP_{\mathbf{X}}(M)$ *starting at* o *defined on* $[0,T]$ *and* $CRP_{\mathbf{X}}(\mathbb{R}^d)$ *starting at* 0 *defined on* $[0,T]$ *given by*

$$\begin{array}{ccccc} CRP_{\mathbf{X}}(M) & \longrightarrow & CRP_{\mathbf{y}}^{H,\nabla}(GL(M)) & \longrightarrow & CRP_{\mathbf{X}}(\mathbb{R}^d) \\ \mathbf{y} & \longrightarrow & \mathbf{h}(\mathbf{y}, u_o) & \longrightarrow & \int \hat{\theta}(d\mathbf{h}(\mathbf{y}, u_o)) \end{array}$$

where $\hat{\theta}$ *is the canonical one-form.*

Just as one can roll and unroll controlled rough paths, one can also do the same rough one-forms along a path. For example, if $\boldsymbol{\alpha} = (\alpha, \alpha^{\dagger}) \in CRP_y^{U^{\nabla}}(M, V)$, one can unroll it into an element in $CRP_{\mathbf{X}}(L(\mathbb{R}^d, V))$ simply by precomposing it with u_s where $\mathbf{u}_s = (u_s, u_s^{\dagger})$ is parallel translation along y. Details are given in Proposition 6.18 along with the inverse rolling map.

Lastly, with the ability to roll and unroll both controlled rough paths and controlled rough one forms, one can integrate either on the manifold or in Euclidean space. It turns out that the answer is the same independent of the perspective taken:

Theorem 6.21 *Let* $\mathbf{y} = \left(y, y^{\dagger}\right)$ *be an element of* $CRP_{\mathbf{X}}(M)$*, let* ∇ *a covariant derivative on* TM*, and let* $u_{y_0} \in GL(M)$*. Further let* $\boldsymbol{\alpha} \in CRP_y^{U^{\nabla}}(M, V)$*, let* $\tilde{\boldsymbol{\alpha}}^{\nabla} := \left(\tilde{\alpha}^{\nabla}, \left(\tilde{\alpha}^{\dagger}\right)^{\nabla}\right) \in CRP_{\mathbf{X}}\left(L\left(\mathbb{R}^d, V\right)\right)$ *be the unrolled rough one-form, and let* $\tilde{\mathbf{y}} := \int \hat{\theta}\left(d\mathbf{h}\left(\mathbf{y}, u_{y_0}\right)\right) \in CRP_{\mathbf{X}}\left(\mathbb{R}^d\right)$ *be the unrolled path. If* ψ *is a logarithm and* $\mathcal{G} = \left(\psi, U^{\nabla}\right)$ *we have*

$$\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}} \right\rangle = \int \left\langle \tilde{\boldsymbol{\alpha}}^{\nabla}, d\tilde{\mathbf{y}} \right\rangle.$$

Portions of Chapter 1 are adapted from material awaiting publication as Driver, B.K.; Semko, J.S., "Controlled Rough Paths on Manifolds I," submitted, *Revista Matemática Iberoamericana*, 2015. The dissertation author was the primary author of this paper.

Chapter 2

Background: Rough Path Theory

2.1 Why Rough Paths are Needed

Let $T > 0$ and let $W = \mathbb{R}^k$. In standard calculus, it is often the case that a path $x : [0, T] \rightarrow W$ is regular enough that it has finite length, i.e. its variation

$$V_1(x) := \sup_{\Pi \in \mathcal{P}(0,T)} \left(\sum_{i=1}^{n} \left| x_{t_i} - x_{t_{i-1}} \right| \right) \tag{2.1}$$

is finite. Here, $\mathcal{P}(0, T)$ is the set of all partitions $\Pi = \{t_i\}_{0 \leq i \leq n}$ such that $0 = t_0 < t_1 < \ldots < t_n = T$ (note that the n and t_i appearing in Eq. (2.1) depend on the partition Π).

Example 2.1 *One scenario in which $V_1(x)$ is easy to compute is when $x \in C^1([0, T])$; in this case we have*

$$V_1(x) = \int_0^T |\dot{x}_\tau| \, d\tau.$$

As one moves toward situations in which typical paths are "rougher", the case of a path x having $V_1(x) = \infty$ is common. While all paths x with $V_1(x) = \infty$ are not equally rough, it is necessary to generalize the notion of variation to encompass and describe a larger set of paths. Let $p \geq 1$; we define the $p-$variation as

$$V_p(x) := \sup_{\Pi \in \mathcal{P}(0,T)} \left(\sum_{i=1}^{n} \left| x_{t_i} - x_{t_{i-1}} \right|^p \right)^{1/p}.$$

It is not hard to see that if $p \leq q$ then we have the inclusion

$$\{x : V_p(x) < \infty\} \subseteq \{x : V_q(x) < \infty\}.$$

This is a simple consequence of the fact that, if $\{a_i\}_{i=1}^n$ is a set of positive numbers and, restricting p to $[1, \infty)$, the function

$$p \longrightarrow \left(\sum_{i=1}^{n} a_i^p \right)^{1/p}$$

is decreasing.

Proof. Let $q \geq p$ and denote $r := q - p \geq 0$. Then

$$\begin{aligned} \left(\sum_{i=1}^{n} a_i^q \right) &= \left(\sum_{i=1}^{n} a_i^{p+r} \right) \\ &\leq \left(\max_{1 \leq i \leq n} a_i \right)^r \left(\sum_{i=1}^{n} a_i^p \right) \\ &\leq \left(\sum_{i=1}^{n} a_i^p \right)^{r/p} \left(\sum_{i=1}^{n} a_i^p \right) \\ &= \left(\sum_{i=1}^{n} a_i^p \right)^{q/p}. \end{aligned}$$

Exponentiating by $1/q$ shows that

$$\left(\sum_{i=1}^{n} a_i^q \right)^{1/q} \leq \left(\sum_{i=1}^{n} a_i^p \right)^{1/p}.$$

■

One issue with paths x such that $V_1(x) = \infty$ is that there is not always a canonical notion of integration. One attempt at doing so leads us to a definition which is formally integration by parts known as the Young integral.

Definition 2.2 (Young Integral) *Let $p, q > 0$ such that $\theta := \frac{1}{p} + \frac{1}{q} > 1$. Suppose x is a path in $\mathbb{R}$ such that $x_0 = 0$ and $V_p(x) < \infty$ and let f be an element of $C^1([0,T])$. Defining*

$$\int_0^T f(\tau)\,dx_\tau := -\int_0^T \dot{f}(\tau)\,x_\tau d\tau + f(t)\,x_t\big|_0^T$$

we have that

$$\left|\int_0^T f(\tau)\,dx_\tau\right| \leq C(\theta)\left(\|f\|_\infty + V_q(f)\right)V_p(x)$$

where $C(\theta)$ is a constant depending on θ.

Thus, we may extend the integral to those $f \in C([0,T])$ such that $V_q(f) < \infty$.

Definition 2.2 lets us extend an integral to a large class of paths; if $p < 2$ and $V_p(x) < \infty$, we may define the integral

$$\int_0^T x_\tau \otimes dx_\tau.$$

Here, the i, j component of $\int_0^T x_\tau \otimes dx_\tau$ can be computed as

$$\int_0^T x_\tau^i dx_\tau^j$$

using the Young integral above. In some sense, this definition provides the only meaningful way to compute $\int_0^T x_\tau \otimes dx_\tau$ such that the analytic properties of x are preserved by by the integral (see Theorem 2.2.1 of [22]).

Unfortunately, those familiar with stochastic calculus and properties of Brownian motion understand the shortcoming of Definition 2.2: A sample path of $k-$dimensional Brownian motion B_t, with probability 1, has the property that $V_p(B) = \infty$ if $p \leq 2$. Thus, one cannot use the above technology to try to make sense of the pathwise integral

$$\int_0^T B_t \otimes dB_t. \tag{2.2}$$

Hope is not completely lost, however, as (multiple) meanings have been given to the expression in Eq. (2.2) in a probabilistic sense. While this is sufficient in the case of Brownian motion, Rough Path theory attempts to extend rigorously an integration theory in multiple dimensions which is immune to the shortcomings described in this section.

2.2 From $p-$variation to Controls

Section 2.1 demonstrated how integration can break down when the paths involved have finite $p-$variation only for $p \geq 2$. While everything from now on could be developed using the $p-$variation norms V_p, it will be more fluid to develop the notion of a control. We define

$$\Delta_{[S,T]} = \{(s,t) : S \leq s \leq t \leq T\}. \tag{2.3}$$

Definition 2.3 *A function* $f : \Delta_{[0,T]} \to \mathbb{R}_+$ *is* ***superadditive*** *if* $f(s,t) + f(t,u) \leq f(s,u)$ *for all* $0 \leq s \leq t \leq u \leq T$.

Definition 2.4 *A* ***control*** ω *is a continuous function* $\omega : \Delta_{[0,T]} \to \mathbb{R}_+$ *which is superadditive and such that* $\omega(s,s) = 0$ *for all* $s \in [0,T]$.

For the remainder of the dissertation, we will continue to denote

$$W := \mathbb{R}^k$$

for some $k > 0$ and denote the increment $x_{s,t} := x_t - x_s$. The following fact lets us leave behind the $p-$variation notion in favor of controls:

Fact 2.5 *A path $x \in C([0,T], W)$ has finite $p-$variation (i.e. $V_p(x) < \infty$) if and only if there exists a control ω such that*

$$|x_{s,t}| \leq \omega(s,t)^{1/p} \tag{2.4}$$

for all $s, t \in \Delta_{[S,T]}$.

Some remarks are in order about this fact. If x has finite $p-$variation, it is straightforward, yet somewhat tedious to show that the function

$$\omega_{X,p}(s,t) := \sup_{\Pi \in \mathcal{P}(s,t)} \left(\sum_{i=1}^{n} \left| x_{t_i} - x_{t_{i-1}} \right|^p \right)$$

is a control such that

$$|x_{s,t}| \leq \omega_{X,p}(s,t)^{1/p}.$$

On the other hand, if Eq. (2.4) holds, then for any partition Π of $[0,T]$, we have

$$\begin{aligned} \left(\sum_{i=1}^{n} \left| x_{t_i} - x_{t_{i-1}} \right|^p \right)^{1/p} &\leq \left(\sum_{i=1}^{n} \omega(t_i, t_{i-1}) \right)^{1/p} \\ &\leq \omega(0,T)^{1/p} \\ &< \infty. \end{aligned}$$

While there exists a full theory for all $p \geq 1$ (for example, see [12] or [13]), this dissertation will focus only on the case when $1 \leq p < 3$.

2.3 Rough Paths

As discussed in Section 2.1, we do not have a canonical way to define the quantity

$$\int_s^t x_{s,\tau} \otimes dx_\tau \in W \otimes W \tag{2.5}$$

when x has finite $p-$variation only for $p \geq 2$. However, if we work formally with $\int_s^t x_{s,\tau} \otimes dx_\tau$, we can develop a few properties that "should" hold: Let $s < t < u$:

1. $\int_s^u x_{s,\tau} \otimes dx_\tau$ should satisfy

$$\begin{aligned}\int_s^u x_{s,\tau} \otimes dx_\tau &= \int_s^t x_{s,\tau} \otimes dx_\tau + \int_t^u x_{s,\tau} \otimes dx_\tau \\ &= \int_s^t x_{s,\tau} \otimes dx_\tau + \int_t^u (x_{t,\tau} - x_{s,t}) \otimes dx_\tau \\ &= \int_s^t x_{s,\tau} \otimes dx_\tau + \int_t^u x_{t,\tau} \otimes dx_\tau + x_{s,t} \otimes x_{t,u}\end{aligned}$$

That is, denoting $\mathbb{X}_{s,t} := \int_s^t x_{s,\tau} \otimes dx_\tau$, we should have

$$\mathbb{X}_{s,u} = \mathbb{X}_{s,t} + \mathbb{X}_{t,u} + x_{s,t} \otimes x_{t,u}$$

2. Suppose x is such that $|x_{s,t}| \leq \omega(s,t)^{1/p}$ and suppose for now that $p < 2$. Young's Inequality (see [23]) along with an integral bound gives

$$\begin{aligned}\left|\int_s^t x_{s,\tau} \otimes dx_\tau\right| &\leq \int_s^t |x_{s,\tau}|\,|dx_\tau| \\ &\leq C(2/p)\left(V_p\left(x|_{[s,t]}\right)\right)^2 \\ &\leq C(2/p)\,\omega(s,t)^{2/p}\end{aligned}$$

where $1 < C(2/p) < \infty$. By replacing ω with $C(2/p)^{p/2}\,\omega$ if necessary, we may assume that

$$\left| \int_s^t x_{s,\tau} \otimes dx_\tau \right| \leq \omega (s,t)^{2/p}.$$

Extending this bound to $p \geq 2$, we should have

$$|\mathbb{X}_{s,t}| \leq \omega (s,t)^{2/p}.$$

3. The components of $\int_s^t x_{s,\tau} \otimes dx_\tau$ should satisfy (using integration by parts)

$$\begin{aligned} &\left[\int_s^t x_{s,\tau} \otimes dx_\tau \right]^{i,j} + \left[\int_s^t x_{s,\tau} \otimes dx_\tau \right]^{j,i} \\ &= \int_s^t x_{s,\tau}^i dx_\tau^j + \int_s^t x_{s,\tau}^j dx_\tau^i \\ &= x_{s,t}^i x_{s,t}^j \end{aligned}$$

That is, the symmetric part of $\mathbb{X}_{s,t}$ $(\mathrm{sym}(\mathbb{X}_{s,t}))$ should satisfy

$$\mathrm{sym}(\mathbb{X}_{s,t}) = \frac{1}{2} x_{s,t} \otimes x_{s,t}.$$

These formal properties help motivate the definition of a rough path:

Definition 2.6 *Let* $\mathbf{X} = (x, \mathbb{X})$ *where*

$$x : [0,T] \to W \qquad \text{and} \qquad \mathbb{X} : \Delta_{[0,T]} \to W \otimes W$$

and are continuous. Then $\mathbf{X}$ *is a* p***-rough path with control*** ω *if*

1. *The Chen identity holds:*

$$\mathbb{X}_{s,u} = \mathbb{X}_{s,t} + \mathbb{X}_{t,u} + x_{s,t} \otimes x_{t,u} \tag{2.6}$$

for all $0 \leq s \leq t \leq u \leq T$.

2. For all $0 \le s \le t \le T$*,*

$$|x_{s,t}| \le \omega(s,t)^{1/p} \qquad \text{and} \qquad |\mathbb{X}_{s,t}| \le \omega(s,t)^{2/p}. \tag{2.7}$$

Further, we say that $\mathbf{X}$ *is* ***weak-geometric*** *if the symmetric part of* $\mathbb{X}_{s,t}$ *(*$\text{sym}(\mathbb{X}_{s,t})$*) satisfies the relation*

$$\text{sym}(\mathbb{X}_{s,t}) = \frac{1}{2} x_{s,t} \otimes x_{s,t}.$$

Note that the point of defining a rough path is not attempt to compute an iterated integral $\int_s^t x_{s,\tau} \otimes dx_\tau$. Instead, it is to stipulate what these quantities are and to ensure they act "enough" like integrals. In the smooth case, we have an obvious candidate for $\mathbb{X}_{s,t}$. Before presenting it, we will note that, in this dissertation, V,$\tilde{V}$, and $\hat{V}$ will denote Banach spaces, and $L\left(V, \tilde{V}\right)$ will denote the bounded linear transformations from V to $\tilde{V}$.

Example 2.7 *If* $x_t \in C^\infty([0,T], V)$ *is a smooth curve in* V *and*

$$\mathbb{X}_{s,t} = \int_{s \le \sigma \le \tau \le t} dx_\sigma \otimes dx_\tau = \int_s^t x_{s,\tau} \otimes dx_\tau, \tag{2.8}$$

then $\mathbf{X} = (x, \mathbb{X})$ *is a weak-geometric rough path. In this example we could take* $p = 1$*.*

Once a rough path $\mathbf{X} \in W \oplus W^{\otimes 2}$ is given, one can do integration against $\mathbf{X}$ and solve differential equations driven by $\mathbf{X}$. For instance, if $F : W \to L(W, V)$ is a smooth function, then by Theorem 3.3.1 of [22], there exists a unique integral $\int_0^t \langle F(x_\tau), d\mathbf{X}_\tau \rangle$ such that the increments satisfy

$$\left| \int_s^t \langle F(x_\tau), d\mathbf{X}_\tau \rangle - F(x_s) x_{s,t} - F'(x_s) \mathbb{X}_{s,t} \right| \le C\omega(s,t)^{3/p} \tag{2.9}$$

for some $C > 0$. For example, one such way to compute the integral is to take a limit of Riemann sums:

$$\int_s^t \langle F(x_\tau), d\mathbf{X}_\tau \rangle = \lim_{|\mathcal{P}(s,t)|\to 0} \sum_{t_i \in \Pi} \left[F(x_s)\, x_{s,t} + F'(x_s)\, \mathbb{X}_{s,t} \right]$$

where $|\mathcal{P}(s,t)|$ is the mesh of the partition and Π is any partition with this mesh size.

2.4 Controlled Rough Paths

By examining the terms which approximate the integral $\int_s^t \langle F(x_\tau), d\mathbf{X}_\tau \rangle$ in Eq. (2.9), we can make some observations regarding the terms $F(x_s)$ and $F'(x_s)$. Namely by Taylor's Theorem we have

$$F(x_t) - F(x_s) - F'(x_s)\, x_{s,t} = O\left(|x_{s,t}|^2\right)$$

and

$$F'(x_t) - F'(x_s) = O(|x_{s,t}|).$$

It turns out, as Gubinelli discovered in [14], that these properties are all that are necessary to develop an integration theory. This leads us to the definition of Controlled Rough Paths:

Definition 2.8 *Let* $\mathbf{X}$ *be a p-rough path on* $W \oplus W^{\otimes 2}$ *with control* ω*. The continuous pair* $\mathbf{y} := \left(y, y^\dagger\right) \in C([a,b], V) \times C([a,b], L(W,V))$ *is a* V *– valued* ***rough path controlled by*** $\mathbf{X}$ *(denoted* $\mathbf{y} \in CRP_{\mathbf{X}}([a,b], V)$*) if there exists a* C *such that*

1. $\left|y_t - y_s - y_s^\dagger x_{s,t}\right| \leq C\omega(s,t)^{2/p}$*, and*
2. $\left|y_t^\dagger - y_s^\dagger\right| \leq C\omega(s,t)^{1/p}$ *for all* $s \leq t$ *in* $[0,T]$*.*

We denote $CRP_{\mathbf{X}}(V) := CRP_{\mathbf{X}}([0,T],V)$ *for some fixed* $T < \infty$.

The approximations in Definition 2.8 are statements which only need to hold locally due to the following (easy) sewing lemma.

Lemma 2.9 (Sewing Lemma) *Let*

$$\mathbf{y} := \left(y, y^{\dagger}\right) \in C([0,T],V) \times C([0,T],L(W,V))$$

and let $0 = t_0 < t_1 < \ldots < t_l = T$ *be a partition of* $[0,T]$ *such that* $\mathbf{y}|_{[t_i,t_{i+i}]}$ *is a rough path controlled by* $\mathbf{X}|_{[t_i,t_{i+1}]} := \left(x|_{[t_i,t_{i+1}]}, \mathbb{X}|_{\Delta_{[t_i,t_{i+1}]}}\right)$ *for all* $0 \leq i \leq l-1$. *Then* $\mathbf{y}$ *is a rough path controlled by* $\mathbf{X}$.

Proof. Let C_i with $0 \leq i \leq l-1$ be such that

$$\left|y_t - y_s - y_s^{\dagger} x_{s,t}\right| \leq C_i \omega(s,t)^{2/p} \quad \text{and} \quad \left|y_t^{\dagger} - y_s^{\dagger}\right| \leq C_i \omega(s,t)^{1/p}$$

whenever $(s,t) \in \Delta_{[t_i,t_{i+1}]}$. Let $\tilde{C} := \sum_{i=0}^{l-1} C_i$. Then by a telescoping series argument and the fact that ω is increasing (because it is superadditive), it is clear that

$$\left|y_t^{\dagger} - y_s^{\dagger}\right| \leq \tilde{C}\omega(s,t)^{1/p} \;\; \forall \;\; (s,t) \in \Delta_{[0,T]}.$$

Now let $C = (2l-1)\tilde{C}$. If $(s,t) \in \Delta_{[0,T]}$ then there exists j and j^* such that $s \in [t_j, t_{j+1}]$ and $t \in [t_{j^*}, t_{j^*+1}]$ with $j \leq j^*$. If $j = j^*$ then

$$\left|y_t - y_s - y_s^{\dagger} x_{s,t}\right| \leq C\omega(s,t)^{2/p}$$

trivially. Otherwise, we have

$$\begin{aligned} y_t - y_s - y_s^\dagger x_{s,t} &= \left(y_t - y_{t_{j^*}}\right) + \left(y_{t_{j+1}} - y_s\right) + \sum_{i=j+1}^{j^*-1} \left(y_{t_{i+1}} - y_{t_i}\right) \\ &\quad - y_s^\dagger x_{s,t_{j+1}} - y_s^\dagger x_{t_{j^*},t} - \sum_{i=j+1}^{j^*-1} y_s^\dagger x_{t_i,t_{i+1}} \\ &= \left(y_t - y_{t_{j^*}} - y_{t_{j^*}}^\dagger x_{t_{j^*},t}\right) + \left(y_{t_{j+1}} - y_s - y_s^\dagger x_{s,t_{j+1}}\right) \\ &\quad + \left[y_{t_{j^*}}^\dagger - y_s^\dagger\right] x_{t_{j^*},t} + \sum_{i=j+1}^{j^*-1} \left(y_{t_{i+1}} - y_{t_i} - y_{t_i}^\dagger x_{t_i,t_{i+1}}\right) \\ &\quad + \sum_{i=j+1}^{j^*-1} \left[y_s^\dagger - y_{t_i}^\dagger\right] x_{t_i,t_{i+1}}. \end{aligned}$$

Taking absolute values and using the fact that ω is superadditive, we have that the absolute value of each term on the right (including those within the summations) is bounded by $\tilde{C}\omega(s,t)^{2/p}$. Thus

$$\begin{aligned} \left|y_t - y_s - y_s^\dagger x_{s,t}\right| &\leq (2l-1)\,\tilde{C}\omega(s,t)^{2/p} \\ &= C\omega(s,t)^{2/p} \end{aligned}$$

■

In [14, Theorem 1], the following generalization of Theorem 3.3.1 of [22] is proved.

Theorem 2.10 *Let* $\mathbf{X}$ *be a* p*-rough path on* $W \oplus W^{\otimes 2}$ *with control* ω *and let* $\left(y, y^\dagger\right)$ *be an* $L(W,V)$ *– valued rough path controlled by* $\mathbf{X}$*. Then there exists a* $z \in C([0,T], V)$ *with* $z_0 = 0$ *and a* $C \geq 0$ *such that*

$$\left|z_t - z_s - y_s x_{s,t} - y_s^\dagger \mathbb{X}_{s,t}\right| \leq C\omega(s,t)^{3/p} \tag{2.10}$$

for all $s \leq t$ *in* $[0,T]$.

We will more commonly refer to the path z_t as $\int_0^t \langle \mathbf{y}_\tau, d\mathbf{X}_\tau \rangle$ and its increment, $z_{s,t} := z_t - z_s$, as $\int_s^t \langle \mathbf{y}_\tau, d\mathbf{X}_\tau \rangle$. Theorem 2.12 below is a generalization of Theorem 2.10, but before we state it, we will make a remark about certain identifications of spaces.

Remark 2.11 *If $V, \tilde{V}$, and $\hat{V}$ are vector spaces, we can make the identification*

$$L\left(V, L\left(\tilde{V}, \hat{V}\right)\right) \cong L\left(V \otimes \tilde{V}, \hat{V}\right)$$

via the map $\Xi : L\left(V, L\left(\tilde{V}, \hat{V}\right)\right) \longrightarrow L\left(V \otimes \tilde{V}, \hat{V}\right)$ given by

$$\Xi(\alpha)[v \otimes \tilde{v}] = \alpha \langle v \rangle \langle \tilde{v} \rangle .$$

if $\alpha \in L\left(V, L\left(\tilde{V}, \hat{V}\right)\right)$.

The proof of the following theorem (modulo a reparameterization) may be found in [14] or [12, Remark 4.11].

Theorem 2.12 *Let $\mathbf{X}$ be a p-rough path on $W \oplus W^{\otimes 2}$ with control ω, let $\left(y, y^\dagger\right)$ be an V – valued rough path controlled by $\mathbf{X}$ and let $\boldsymbol{\alpha} = \left(\alpha, \alpha^\dagger\right)$ be an $L\left(V, \tilde{V}\right)$ – valued rough path controlled by $\mathbf{X}$ where $\alpha_s^\dagger \in L\left(W, L\left(V, \tilde{V}\right)\right) \cong L\left(W \otimes V, \tilde{V}\right)$. Then there exists a $z \in C([0,T], V)$ with $z_0 = 0$ and a $C > 0$ such that*

$$\left| z_t - z_s - \alpha_s (y_t - y_s) - \alpha_s^\dagger \left(I \otimes y_s^\dagger\right) \mathbb{X}_{s,t} \right| \leq C\omega(s,t)^{3/p} \tag{2.11}$$

for all $s \leq t$ in $[0,T]$. Moreover if we let $z_s^\dagger := \alpha_s \circ y_s^\dagger$, then $\mathbf{z}_s := \left(z_s, z_s^\dagger\right)$ is a $\tilde{V}$ – valued controlled rough path.

The path z_t in this case will be denoted $\int_0^t \langle \boldsymbol{\alpha}_\tau, d\mathbf{y}_\tau \rangle$.

Notation 2.13 *Let $F_{s,t}$ and $G_{s,t}$ be a pair of functions into a normed space. When it is not important to keep careful track of constants we will often write*

$F_{s,t} \underset{i}{\approx} G_{s,t}$ (for any $i \in \mathbb{N}$) to indicate that there exists $C < \infty$ and $\delta > 0$ such that$|F_{s,t} - G_{s,t}| \le C\omega(s,t)^{i/p}$ for all $0 \le s \le t \le T$ with $|t-s| \le \delta$.

We will typically summarize Inequality (2.11) by writing

$$\int_s^t \langle \boldsymbol{\alpha}_\tau, d\mathbf{y}_\tau \rangle \underset{3}{\approx} \langle \boldsymbol{\alpha}_s, \mathbf{y}^{\mathbb{X}}_{s,t} \rangle := \alpha_s y_{s,t} + \alpha^\dagger_s \left(I \otimes y^\dagger_s\right) \mathbb{X}_{s,t} \tag{2.12}$$

wherein we let $\mathbf{y}^{\mathbb{X}}_{s,t}$ be the increment process defined by,

$$\mathbf{y}^{\mathbb{X}}_{s,t} := \left(y_{s,t}, \left(I \otimes y^\dagger_s\right) \mathbb{X}_{s,t}\right). \tag{2.13}$$

Notice that Theorem 2.10 does indeed follow from Theorem 2.12 upon replacing $(\alpha, \alpha^\dagger)$ by $(y, y^\dagger)$ and $(y, y^\dagger)$ by (x, I_W) in Inequality (2.11).

Remark 2.14 (Motivations) *In order to develop some intuition for the expression appearing on the right side of Eq. (2.12), suppose for the moment that all functions $\mathbf{X}$, $(y, y^\dagger)$, and $(\alpha, \alpha^\dagger)$ are smooth so that $\mathbb{X}$ is given by Eq. (2.8). In this case we want $z_{s,t}$ to be the usual integral $\int_s^t \alpha_\tau \dot{y}_\tau d\tau$ and to arrive at the expression in Inequality (2.11) we look for an appropriate second order approximation to the integral. Since $p = 1$ now we may conclude*

$$\alpha_{s,\tau} = \alpha^\dagger_s x_{s,\tau} + O\left((\tau - s)^2\right)$$

and

$$y_t - y_\tau = y^\dagger_\tau (x_t - x_\tau) + O\left((t-\tau)^2\right) \implies \dot{y}_\tau = y^\dagger_\tau \dot{x}_\tau.$$

We have the identity;

$$\int_s^t \alpha_\tau dy_\tau = \int_s^t [\alpha_s + \alpha_{s,\tau}] \dot{y}_\tau d\tau = \alpha_s y_{s,t} + \int_s^t \alpha_{s,\tau} \dot{y}_\tau d\tau. \tag{2.14}$$

The last term on the right hand side is approximated up to an error of size

$O\left((t-s)^3\right)$ *as follows,*

$$\begin{aligned}\int_s^t \alpha_{s,\tau}\dot{y}_\tau d\tau &= \int_s^t \alpha_{s,\tau} y_\tau^\dagger \dot{x}_\tau d\tau \\ &= \int_s^t \alpha_s^\dagger x_{s,\tau} y_\tau^\dagger \dot{x}_\tau d\tau + O\left((t-s)^3\right) \\ &= \int_s^t \alpha_s^\dagger x_{s,\tau} y_s^\dagger \dot{x}_\tau d\tau + O\left((t-s)^3\right) \\ &= \alpha_s^\dagger \left(I\otimes y_s^\dagger\right)\int_s^t x_{s,\tau}\otimes \dot{x}_\tau d\tau + O\left((t-s)^3\right) \\ &= \alpha_s^\dagger \left(I\otimes y_s^\dagger\right)\mathbb{X}_{s,t} + O\left((t-s)^3\right).\end{aligned} \tag{2.15}$$

Combining Eq. (2.14) and Eq. (2.15) gives the approximate equality,

$$\int_s^t \alpha_\tau dy_\tau = \alpha_s y_{s,t} + \alpha_s^\dagger \left(I\otimes y_s^\dagger\right)\mathbb{X}_{s,t} + O\left((t-s)^3\right).$$

Controlled rough paths are also useful in interpreting solutions to rough differential equations. Let $F : V \to L(W,V)$ be smooth where we will write $F(a)\,w$ as $F_w(a)$. We can then make sense of the rough differential equation

$$d\mathbf{y}_t = F_{d\mathbf{X}_t}(y_t) \tag{2.16}$$

with initial condition $y_0 = \bar{y}_0$. We will need a bit of notation regarding tensor products before we say what it means to solve such an equation.

Notation 2.15 *If* $\Xi : W\times W \to V$ *is a bilinear form into a vector space* V*, by the universal property of tensor products,* Ξ *factors through a unique linear function* $\Xi^\otimes$ *on* $W\otimes W$ *such that* $\Xi^\otimes(w\otimes\tilde{w}) = \Xi(w,\tilde{w})$ *for a simple tensor* $w\otimes\tilde{w}$*. If* $\mathbb{W}\in W\otimes W$ *we will abuse notation and write*

$$\Xi(w,\tilde{w})\,|_{w\otimes\tilde{w}=\mathbb{W}} = \Xi(w\otimes\tilde{w})\,|_{w\otimes\tilde{w}=\mathbb{W}} = \Xi^\otimes(\mathbb{W}),$$

where, to be precise, if $\mathbb{W} = \sum w_i \otimes \tilde{w}_i$ *then*

$$\Xi^{\otimes}(\mathbb{W}) = \sum \Xi(w_i, \tilde{w}_i).$$

We say the controlled rough path $\mathbf{y} = \left(y, y^{\dagger}\right)$ defined on[1] $I_0 = [0, T)$ or $I_0 = [0, T]$ solves Eq. (2.16) if for every $[0, b] \subseteq I_0$, we have

$$\begin{aligned} y_{s,t} &\underset{3}{\approx} F_{x_{s,t}}(y_s) + \left(\partial_{F_w(y_s)} F_w\right)(y_s)|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}} \\ y_s^{\dagger} &= F_{\cdot}(y_s) \end{aligned}$$

for all $s, t \in [0, b]$. If in addition $y_0 = \bar{y}_0$, we say $\mathbf{y}$ solves Eq. (2.16) with initial condition $y_0 = \bar{y}_0$.

The existence and uniqueness of solutions (at least of the path y_s) to these differential equations (provided F is sufficiently regular) is due to Lyons [22]. Clearly if y_s is given, then $y_s^{\dagger}$ exists and is uniquely determined by $y_s^{\dagger} = F_{\cdot}(y_s)$. One may refer to Subsection B in the Appendix for more results regarding rough differential equations on Euclidean space.

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[1] Here we allow that $\mathbf{y} \in CRP_{\mathbf{X}}(I_0, V)$ if is in an element of $CRP_{\mathbf{X}}(K, V)$ for every compact interval $K \in I_0$.

Chapter 3

Manifold Rough Path Theory

3.1 Some Differential Geometric Notions with Examples

Let $M = M^d$ be a d-dimensional manifold, TM be its tangent space, and $\pi := \pi_{TM} : TM \to M$ be the natural projection map. Throughout, let $\mathbf{X} = (x, \mathbb{X})$ be a weak-geometric p-rough path on $[0, T]$ with with values in $W \oplus W^{\otimes 2}$ and control ω.

The letters x and y will appear in the dissertation generally as paths, but occasionally they will refer to arbitrary points in Euclidean space. The context will allow the reader to identify their proper usage.

Notation 3.1 *When $M = \mathbb{R}^d$ we will identify $T\mathbb{R}^d$ with $\mathbb{R}^d \times \mathbb{R}^d$ via*

$$\mathbb{R}^d \times \mathbb{R}^d \ni (m, v) \to v_m := \frac{d}{dt}|_0 (m + tv) \in T_m\mathbb{R}^d$$

and, by abuse of notation, we let $|v_m| = |v|$ when $|\cdot|$ is the standard Euclidean norm.

Notation 3.2 *Whenever ϕ is a map, let $D(\phi)$ and $R(\phi)$ denote the domain and range of ϕ respectively. If $\phi \in C^\infty\left(M, \mathbb{R}^{d'}\right)$ has open domain, let $d\phi : TD(\phi) \longrightarrow$*

$\mathbb{R}^{d'}$ *be defined by*

$$d\phi\left(v_m\right) := \frac{d}{dt}|_0\phi\left(\sigma\left(t\right)\right) \in \mathbb{R}^{d'} \tag{3.1}$$

where σ *is such that* $\sigma\left(0\right) = m \in D\left(\phi\right)$ *and* $\dot{\sigma}\left(0\right) = v_m \in T_mM$. *Denote* $d\phi_m := d\phi|_{T_mM}$. *If* $f \in C^\infty\left(M, \tilde{M}\right)$ *where* $\tilde{M}$ *is another manifold, we let* f_* *be the push-forward of* f *so that* $f_* : TD\left(f\right) \longrightarrow T\tilde{M}$ *is defined by*

$$f_*\left(v_m\right) := \frac{d}{dt}|_0 f\left(\sigma\left(t\right)\right) \in T_{f(m)}\tilde{M}$$

where again $\dot{\sigma}\left(0\right) = v_m$. *Analogously we let* $f_{*m} = f_*|_{T_mM}$. *Note that* $\phi_*\left(v_m\right) = \left(\phi\left(m\right), d\phi\left(v_m\right)\right) = \left[d\phi\left(v_m\right)\right]_{\phi(m)}$.

3.1.1 Gauges

Definition 3.3 *Let* $\mathcal{U}$ *be an open set on* M. *An open set* $\mathcal{D}^{\mathcal{U}} \subseteq M \times M$ *is a* $\mathcal{U}$ *–* ***diagonal domain*** *if it contains the diagonal of* $\mathcal{U}$, *that is* $\Delta^{\mathcal{U}} := \bigcup_{m\in\mathcal{U}}\left(m, m\right) \subseteq \mathcal{D}^{\mathcal{U}}$. *A* ***local diagonal domain*** *is a* $\mathcal{V}$ *– diagonal domain for some nonempty open* $\mathcal{V} \subseteq M$.

If $\mathcal{U} = M$ *we write* $\mathcal{D} := \mathcal{D}^M$ *and refer to* $\mathcal{D}$ *simply as a diagonal domain.*

Throughout the dissertation, $\mathcal{D}$ will always denote a diagonal domain.

Definition 3.4 *A smooth function* $\psi : \mathcal{D} \to TM$ *is called a* ***logarithm*** *if:*

1. $\psi\left(m, n\right) \in T_mM$

2. $\psi\left(m, m\right) = 0_m$

3. $\psi\left(m, \cdot\right)_*|_{T_mM} = I_m$

We also write ψ_m *for* $\psi\left(m, \cdot\right)$.

If the above holds for ψ *defined on a local diagonal domain, we may refer to* ψ *as a* ***local logarithm***.

If E is a any vector bundle, we will denote the smooth sections of E by $\Gamma(E)$. We define $L(TM, TM)$ as the vector bundle $\tilde{E}$ over the manifold $M \times M$ such that $\tilde{E}_{(n,m)} = L(T_m M, T_n M)$ and

$$\tilde{E} = \bigcup \left\{ \tilde{E}_{(n,m)} : n, m \in M \right\}.$$

Definition 3.5 *A smooth section $U \in \Gamma(L(TM, TM))$ with domain $\mathcal{D}$ (i.e. for each $(n,m) \in \mathcal{D}$, $U(n,m)$ is an element of $L(T_m M, T_n M)$) is called a* ***parallelism*** *if $U(m,m) = I_m$. If U is only defined on a local diagonal domain, we refer to U as a* ***local parallelism****.*

Definition 3.6 *We call the pair $\mathcal{G} := (\psi, U)$ (where ψ and U have common domain $\mathcal{D}$) a* ***gauge*** *on the manifold M. If $\mathcal{D}$ is replaced by a local diagonal domain, we call $\mathcal{G}$ a* ***local gauge****.*

Example 3.7 *If $M = \mathbb{R}^d$, the maps $\psi(x,y) = [y - x]_x$ and $U_{x,y} v_y = v_x$ form the* ***standard gauge*** *on $\mathbb{R}^d$.*

Example 3.8 *One natural example of a gauge comes from any covariant derivative ∇ on TM. The construction is as follows. Choose an arbitrary Riemannian metric g on M. If $m, n \in M$ are "close enough", there is a unique vector v_m with minimum length such that $n = \exp_m^\nabla(v_m)$. We denote this vector by $\psi^\nabla(m,n) := \left(\exp_m^\nabla\right)^{-1}(n)$ or by $\exp_m^{-1}(n)$ if ∇ is clear from the context. We further let*

$$U^\nabla(n,m) := //_1 \left(t \to \exp_m \left(t \exp_m^{-1}(n)\right)\right),$$

where, for any smooth curve $\sigma : [0,1] \to M$, we let $//_s(\sigma) = //_s^\nabla(\sigma) : T_{\sigma(0)} M \to T_{\sigma(s)} M$ denote parallel translation along σ up to time $s \in [0,1]$. It is shown in Corollary 3.22 that there is a diagonal domain $\mathcal{D} \subseteq M \times M$ such that $\left(\psi^\nabla, U^\nabla\right)$ so defined is a gauge on $\mathcal{D}$.

Remark 3.9 *We can also get a covariant derivative from a parallelism. If U is a parallelism, then we can define covariant derivative ∇^U on TM by*

$$\nabla^U_{v_m}(Y) := \frac{d}{dt}|_0 U(m,\sigma_t)\,Y(\sigma_t),$$

where $\dot{\sigma}(0) = v_m$ and Y is a vector field on M.

Remark 3.10 *Although the definition of a gauge includes stipulating a U, if we have just ψ, we can define $U^\psi(n,m) := \psi(n,\cdot)_{*m}$ and set $\mathcal{G}^\psi := \left(\psi, U^\psi\right)$.*

Remark 3.11 *We may make a local gauge out of a chart ϕ. Indeed, we pull back the flat gauge in Example 3.7 to M to define*

$$\begin{aligned}\psi^\phi(m,n) &:= (d\phi_m)^{-1}\left[\phi(n) - \phi(m)\right]\\ U^\phi(n,m) &:= (d\phi_n)^{-1}\, d\phi_m.\end{aligned}$$

This is a gauge which is also consistent with Remark 3.10 and $D\left(\psi^\phi\right) = D\left(U^\phi\right) = D(\phi)\times D(\phi)$.

Before moving on to controlled rough paths on manifolds, let us record the structure of the general gauge on $\mathbb{R}^d$.

Notation 3.12 *If (ψ, U) is a local gauge on $\mathbb{R}^d$, then we write $\left(\bar{\psi}, \bar{U}\right)$ to mean the functions determined by the relations*

$$\psi(x,y) = \left[\bar{\psi}(x,y)\right]_x \quad \textit{and} \quad U(x,y)(v_y) = \left[\bar{U}(x,y)\,v\right]_x$$

so that $\bar{\psi}(x,y) \in \mathbb{R}^d$ and $\bar{U}(x,y) \in \mathrm{End}\left(\mathbb{R}^d\right)$.

Theorem 3.13 *If $\mathcal{G} = (\psi, U)$ is a local gauge on $\mathbb{R}^d$, for every open convex subset $\mathcal{V} \subseteq \mathbb{R}^d$ such that $\mathcal{V}\times\mathcal{V} \subseteq D(\mathcal{G})$, there exists smoothly varying functions*

$A(x,y) \in L\left(\left(\mathbb{R}^d\right)^{\otimes 2}, \mathbb{R}^d\right)$ and $B(x,y) \in L\left(\mathbb{R}^d, \mathrm{End}\left(\mathbb{R}^d\right)\right)$ defined for $(x,y) \in \mathcal{V} \times \mathcal{V}$ such that

$$\bar{U}(x,y) = I + B(x,y)(y-x), \tag{3.2}$$

$$\bar{\psi}(x,y) = y - x + A(x,y)(y-x)^{\otimes 2}, \tag{3.3}$$

$$B(x,x) = D_2\bar{U}(x,x), \text{ and } A(x,x) = \frac{1}{2}\left(D_2^2\bar{\psi}\right)(x,x). \tag{3.4}$$

The converse holds as well.

Furthermore, we can find a smoothly varying function C defined on $\mathcal{V} \times \mathcal{V}$ such that $C(x,y) \in L\left(\left(\mathbb{R}^d\right)^{\otimes 3}, \mathbb{R}^d\right)$ and

$$C(x,x) = \frac{1}{6}\left(D_2^3\bar{\psi}\right)(x,x), \text{ and} \tag{3.5}$$

$$\bar{\psi}(x,y) = y - x + \frac{1}{2}\left(D_2^2\bar{\psi}\right)(x,x)(y-x)^{\otimes 2} + C(x,y)(y-x)^{\otimes 3}. \tag{3.6}$$

Proof. Let x, y be points in $\mathcal{V}$. Taylor's theorem with integral remainder applied to the second variable with x fixed gives,

$$\bar{U}(x,y) = I + \int_0^1 \left(D_2\bar{U}\right)(x, x + t(y-x))(y-x)\,dt$$

and

$$\bar{\psi}(x,y) = 0 + \left(D_2\bar{\psi}\right)(x,x)(y-x) + \int_0^1 \left(D_2^2\bar{\psi}\right)(x, x+t(y-x))(y-x)^{\otimes 2}(1-t)\,dt$$

from which Eqs. (3.2) – (3.4) follow with

$$B(x,y) = \int_0^1 \left(D_2\bar{U}\right)(x, x+t(y-x))\,dt \text{ and}$$
$$A(x,y) = \int_0^1 \left(D_2^2\bar{\psi}\right)(x, x+t(y-x))(1-t)\,dt.$$

The converse statement is easy to verify. The proof of Eqs. (3.5) and (3.6) also follow by Taylor's theorem (now to third order) in which case,

$$C(x,y) = \frac{1}{2}\int_0^1 \left(D_2^3\bar{\psi}\right)(x, x + t(y-x))(y-x)^{\otimes 2}(1-t)^2\,dt.$$

■

Let $B_r(x) \subseteq \mathbb{R}^d$ be the open ball of radius r centered at x.

Remark 3.14 *If ψ and $\tilde{\psi}$ are local logarithms on $\mathbb{R}^d$, it is easy to check using Theorem 3.13 that for all $\tilde{x} \in \mathbb{R}^d$, there exists an $r > 0$ and $C > 0$ such that $|\psi(x,y)| \leq C\left|\tilde{\psi}(x,y)\right|$ for all $x, y \in B_r(\tilde{x})$.*

We now wish to transfer these local results to the manifold setting. In order to do this we need to develop some notation for stating that two objects on a manifold are "close" up to some order. Let g be any smooth Riemannian metric on M.

Notation 3.15 *We write d_g for the metric associated to g and define $|v_m|_g := \sqrt{g_m(v_m, v_m)}$ $\forall$ $v_m \in TM$. Further, we let $|\cdot|_{g,op}$ be the operator "norm" induced by $|\cdot|_g$ on $L(TM, V)$, i.e. if $f_m \in L(T_mM, V)$, then*

$$|f_m|_{g,op} := \sup\left\{|f_m\langle v_m\rangle| : |v_m|_g = 1\right\}.$$

Definition 3.16 *Let F, G be smooth TM [respectively $L(TM,TM)$] valued functions with $\mathcal{W}$ – diagonal domains. The expression*

$$F(m,n) =_k G(m,n) \;\; \text{on } \mathcal{W} \tag{3.7}$$

indicates that for every point in $w \in \mathcal{W}$, there exists an open $\mathcal{O}_w \subseteq M$ containing

w such that $\mathcal{O}_w \times \mathcal{O}_w \subseteq D(F) \cap D(G)$ and a $C > 0$ such that

$$|F(m,n) - G(m,n)|_{g,[g,op]} \leq C\,(d_g(m,n))^k \tag{3.8}$$

for all $m, n \in \mathcal{O}_w$.

Occasionally we will omit the reference to $\mathcal{W}$ in which case it we mean the condition (3.8) holds where it makes sense to hold.

Note that in (3.7), the reference to g is not explicit. In fact, the definition does not depend on the choice of g as all Riemannian metrics are locally equivalent. [See Corollary A.4 in the Appendix for precise statement and proof of this standard fact.]

We may also use the $=_k$ notation to make statements in regards to other measures of distance:

Corollary 3.17 *Let $\mathcal{W}$ be an open subset of M and g and $\tilde{g}$ be any two Riemannian metrics on M. If $F(m,n) =_k G(m,n)$ on $\mathcal{W}$ (so that F and G have $\mathcal{W}$-diagonal domains), then for every local logarithm ψ and $w \in \mathcal{W}$ such that $(w,w) \in D(\psi)$, there exists an open $\mathcal{O}_w \subseteq \mathcal{W}$ containing w and $C > 0$ such that*

$$|F(m,n) - G(m,n)|_{g,[g,op]} \leq C\,|\psi(m,n)|_{\tilde{g}}^k \ \ \forall\ m,n \in \mathcal{O}_w.$$

In particular, using the local logarithm $\psi(m,n) = (d\phi_m)^{-1}[\phi(n) - \phi(m)]$, we have that if $w \in D(\phi) \cap \mathcal{W}$, then there exists an $\mathcal{O}_w \subseteq D(\phi) \cap \mathcal{W}$ and a $C > 0$ such that

$$|F(m,n) - G(m,n)|_{g,[g,op]} \leq C\,|\phi(n) - \phi(m)|^k \ \ \forall\ m,n \in \mathcal{O}_w.$$

Proof. The proof of the Corollary will use Remark 3.14 and the local equivalence of any two Riemannian metrics, Corollary A.4 in the Appendix. First we simplify matters by assuming that we are working in Euclidean space which may be accomplished by pushing the metric and functions forward using charts. Assuming this, we now derive a local inequality that holds for any two logarithms ψ and $\tilde{\psi}$ when $(w,w) \in D(\psi) \cap D\left(\tilde{\psi}\right)$. Namely, there exist an open neighborhood, $\mathcal{O}_w$, of w such that

$$\left|\tilde{\psi}(m,n)\right|_g \leq C_1 \left|\tilde{\psi}(m,n)\right| \leq C_2 C_1 \left|\psi(m,n)\right| \leq C_3 C_2 C_1 \left|\psi(m,n)\right|_{\tilde{g}}$$

for every $(m,n) \in \mathcal{O}_w \times \mathcal{O}_w$. The first and third inequality above follow from Corollary A.4 with one metric being the standard Euclidean metric and the other metric being g or $\tilde{g}$ respectively, and the second inequality is true by Remark 3.14. Thus, there exists a $\tilde{C}$ such that

$$\left|\tilde{\psi}(m,n)\right|_g \leq \tilde{C} \left|\psi(m,n)\right|_{\tilde{g}}$$

Now let ∇^g be the Levi-Civita covariant derivative associated to g. By setting $\tilde{\psi}(m,n) = \left(\exp_m^{\nabla^g}\right)^{-1}(n)$ and shrinking $\mathcal{O}_w$ if necessary to ensure that $\left(\exp_{(\cdot)}^{\nabla^g}\right)^{-1}(\cdot)$ is defined and injective on $\mathcal{O}_w \times \mathcal{O}_w$, we have that

$$\left|\left(\exp_m^{\nabla^g}\right)^{-1}(n)\right|_g \leq \tilde{C} \left|\psi(m,n)\right|_{\tilde{g}}.$$

In this setting, $d_g(m,n) = \left|\left(\exp_m^{\nabla^g}\right)^{-1}(n)\right|_g$, and since $F(m,n) =_k G(m,n)$ on $\mathcal{W}$ (by shrinking $\mathcal{O}_w$ if necessary), we have

$$\left|F(m,n) - G(m,n)\right|_{g,[g,op]} \leq \hat{C}\left(d_g(m,n)\right)^k \quad \forall\, m,n \in \mathcal{O}_w$$

for some $\hat{C}$. Thus, we have

$$\left|F(m,n) - G(m,n)\right|_{g,[g,op]} \leq \hat{C}\left(\tilde{C}\right)^k \left|\psi(m,n)\right|_{\tilde{g}}^k.$$

which is the statement of the Corollary with $C := \hat{C}\left(\tilde{C}\right)^k$. ■

In the sequel, Corollary 3.17 will typically be used without further reference in order reduce the proof of showing $F(m,n) =_k G(m,n)$ in the manifold setting to a local statement about functions on convex neighborhoods in $\mathbb{R}^d$ equipped with the standard Euclidean flat metric structures. The first example of this strategy will already occur in the proof of Corollary 3.18 below. For a general parallelism it is not true that $U(n,m)^{-1} = U(m,n)$, yet $U(m,n)$ is always a very good approximation to $U(n,m)^{-1}$.

Corollary 3.18 *If U is a parallelism on a manifold, M, then*

$$U(n,m)^{-1} =_2 U(m,n).$$

Proof. This is a local statement so we may use Corollary 3.17 to reduce to the case that M is a convex open subset of $\mathbb{R}^d$. We then may use Theorem 3.13 to learn

$$\bar{U}(n,m)^{-1} = (I + [B(n,m)(m-n)])^{-1} = I + [B(n,m)(n-m)] + O\left(|n-m|^2\right)$$

while

$$\bar{U}(m,n) = (I + [B(m,n)(n-m)]).$$

Subtracting these two equations shows,

$$\begin{aligned}\bar{U}(n,m)^{-1} - \bar{U}(m,n) &= [B(n,m) - B(m,n)](n-m) + O\left(|n-m|^2\right) \\ &= O\left(|n-m|^2\right)\end{aligned}$$

wherein we have used $B(n,m) - B(m,n)$ vanishes for $m = n$ and therefore is of order $|m-n|$. ■

3.1.2 A Covariant Derivative Gives Rise to a Gauge

Let ∇ be a covariant derivative on TM, and g be any fixed Riemannian metric on M. Let $G : TM \to M \times M$ be the function on TM defined by

$$G(v_m) := \left(m, \exp_m^\nabla(v_m)\right) \text{ for all } v_m \in D(G), \tag{3.9}$$

where $D(G)$ is the domain of G defined by

$$D(G) := \left\{v_m \in TM : t \to \exp_m^\nabla(tv_m) \text{ exists for } 0 \le t \le 1\right\}.$$

We will now develop a subset of $D(G)$ for which G is injective. For each $m \in M$, let Λ_m denote the set of $r > 0$ so that $B_r(0_m) \subseteq D(G)$, $\exp_m^\nabla(B_r(0_m))$ is an open neighborhood of m in M, and $\exp_m^\nabla : B_r(0_m) \to \exp_m^\nabla(B_r(0_m))$ is a diffeomorphism (here $B_r(0_m)$ is the open ball in T_mM centered at 0_m with radius r). The fact that Λ_m is not empty is a consequence of the inverse function theorem and the fact that $\left(\exp_m^\nabla\right)_{*0_m} = I_{T_mM}$ is invertible. We now define $r_m := \sup \Lambda_m$ where $r_m = \infty$ is possible and allowed. A little thought shows that $\exp_m^\nabla(B_{r_m}(0_m))$ is open and $\exp_m^\nabla : B_{r_m}(0_m) \to \exp_m^\nabla(B_{r_m}(0_m))$ is a diffeomorphism, i.e. either $r_m = \infty$ or $r_m \in \Lambda_m$.

Let us now set $\mathcal{C}^* := \cup_{m \in M} B_{r_m}(0_m) \subseteq TM$ and let $G^* : \mathcal{C}^* \to M \times M$ be the map defined by

$$G^*(v_m) := \left(m, \exp_m^\nabla(v_m)\right) \text{ for all } v_m \in \mathcal{C}^*.$$

It is easy to verify that G^* is injective.

We will now build our domain $\mathcal{C}$ for which $G|_{\mathcal{C}}$ is diffeomorphic onto its range. First we need a simple local invertibility proposition.

Proposition 3.19 *Let G be the function defined in Eq. (3.9). Then for each $m \in$*

M, there exists open subsets $\mathcal{V}_m \subseteq TM$ and $\mathcal{W}_m \subseteq M$ such that $0_m \in \mathcal{V}_m,\ m \in \mathcal{W}_m$, and $G|_{\mathcal{V}_m} : \mathcal{V}_m \to \mathcal{W}_m \times \mathcal{W}_m$ is a diffeomorphism.

Proof. As this a local result we may assume that $M = \mathbb{R}^d$ and identify TM with $M \times M$. The function $G : TM \to M \times M$ then takes on the form $G(x,v) = \left(x, \bar{G}(x,v)\right)$ where $\bar{G}(x,0) = x$ and $\left(D_2\bar{G}\right)(x,0) = I_M$ for all $x \in M$. A simple computation then shows

$$G'(x,0) = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \text{ for all } x \in M.$$

The result now follows by an application of the inverse function theorem. ■

Notation 3.20 *If $\mathcal{W}$ is an open subset of M and $\epsilon > 0$, let $\mathcal{U}(\mathcal{W},\epsilon)$ be the open subset of TM defined by*

$$\mathcal{U}(\mathcal{W},\epsilon) := \left\{v \in \pi^{-1}(\mathcal{W}) \subseteq TM : |v|_g < \epsilon\right\}.$$

Theorem 3.21 *Let $\mathcal{C} := \bigcup \mathcal{U}(\mathcal{W},\epsilon)$ where the union is taken over all open subsets $\mathcal{W} \subseteq M$ and $\epsilon > 0$ such that $\mathcal{U}(\mathcal{W},\epsilon) \subseteq D(G)$ and $G|_{\mathcal{U}(\mathcal{W},\epsilon)} : \mathcal{U}(\mathcal{W},\epsilon) \to G(\mathcal{U}(\mathcal{W},\epsilon))$ is a diffeomorphism. Then $\mathcal{C}$ is an open subset of TM such that $\mathcal{D} := G(\mathcal{C})$ is open in $M \times M$, $G : \mathcal{C} \to \mathcal{D}$ is a diffeomorphism,*

$$\{0_m : m \in M\} \subseteq \mathcal{C} \subseteq \mathcal{C}^* \quad \text{and} \quad \Delta^M = \{(m,m) : m \in M\} \subseteq \mathcal{D}.$$

Proof. According to Proposition 3.19, for each $m \in M$ there exists an open neighborhood $\mathcal{W}$ of $m \in M$ and $\epsilon > 0$ so that $\mathcal{U}(\mathcal{W},\epsilon) \subseteq D(G)$ and $G : \mathcal{U}(\mathcal{W},\epsilon) \to G(\mathcal{U}(\mathcal{W},\epsilon))$ is a diffeomorphism. From this it follows that $\{0_m : m \in \mathcal{W}\} \subseteq \mathcal{C}$ and $\mathcal{U}(\mathcal{W},\epsilon) \subseteq \mathcal{C}^*$. As $m \in M$ was arbitrary we may conclude $\{0_m : m \in M\} \subseteq \mathcal{C} \subseteq \mathcal{C}^*$. It is now easily verified that $G(\mathcal{C}) = \cup G(\mathcal{U}(\mathcal{W},\epsilon))$ is open, $G : \mathcal{C} \to G(\mathcal{C})$ is a surjective local diffeomorphism and hence is a diffeomorphism as $G|_{\mathcal{C}}$ is injective (since $G|_{\mathcal{C}^*}$ is injective). ■

Corollary 3.22 *Continuing the notation used in Theorem 3.21, we have $\mathcal{D}$ is a diagonal domain and $\psi := G|_{\mathcal{C}}^{-1} : \mathcal{D} \to \mathcal{C} \subseteq TM$ is a logarithm. Moreover, if we define*

$$U(m,n) := //_1 \left(t \longrightarrow \exp^{\nabla}(t\psi(m,n))\right)^{-1} : T_nM \to T_mM$$

for all $(m,n) \in \mathcal{D}$, then U is a parallelism on M.

Proof. The only thing that remains to be proven is that $U(m,n)$ is smoothly varying. This is a consequence of the fact that solutions to ordinary differential equations depend smoothly on their starting points and parameter in the vector fields. To be more explicit in this case, for $a \in \mathbb{R}^d$ let $B_a^{\nabla}(\mu) = \dot{u}(0)$ where $u(t) = //_t \left(\exp^{\nabla}((\cdot)\mu a)\right)\mu$ for μ in the frame bundle $GL(M)$ over M, so that B_a^{∇} are the ∇ – horizontal vector fields. Now suppose that $w \in M$ is given and $O(m) : \mathbb{R}^d \to T_mM$ is a local frame defined for m in an open neighborhood $\mathcal{W}$ of w. For $v \in \pi^{-1}(\mathcal{W}) \cap \mathcal{C}$ let $\gamma(t) = \exp^{\nabla}(tv)$ and $u(t) := //_t(\gamma) O(\pi(v))$. We then have

$$\dot{\gamma}(t) = //_t(\gamma)v = u(t) O(\pi(v))^{-1} v \text{ and}$$
$$\frac{\nabla u}{dt} = 0 \text{ with } u(0) = O(\pi(v)).$$

These equations are equivalent to solving

$$\dot{u}(t) = B^{\nabla}_{O(\pi(v))^{-1}v}(u(t)) \text{ with } u(0) = O(\pi(v)) \tag{3.10}$$

in which case $\gamma(t) = \pi_{O(M)}(u(t))$ where $\pi_{O(M)}$ is the projection map from $O(M)$ to M. We now define $F(v) := u(1)$ provided $v \in \pi^{-1}(\mathcal{W}) \cap \mathcal{C}$. It then follows that $F : \pi^{-1}(\mathcal{W}) \cap \mathcal{C} \to GL(M)$ is smooth as the solutions to Eq. (3.10) depend smoothly on its starting point and parameter. From this we learn for $(m,n) \in G\left(\pi^{-1}\left(\mathcal{W}\right) \cap \mathcal{C}\right)$ that

$$U\left(n,m\right) = F\left(\psi\left(m,n\right)\right) O\left(m\right)^{-1}$$

is a smooth function of (m,n). ■

3.2 Definitions of Controlled Rough Paths

Notation 3.23 *Throughout the remainder of the dissertation,* $\mathbf{y} := \left(y, y^{\dagger}\right)$ *denotes a pair of continuous functions,* $y \in C\left(\left[0,T\right], M\right)$ *and* $y^{\dagger} \in C\left(\left[0,T\right], L\left(W, TM\right)\right)$, *such that* $y_s^{\dagger} \in L\left(W, T_{y_s}M\right)$ *for all* s.

Definition 3.24 *Let* (ψ, U) *be a gauge. The pair* $\left(y_s, y_s^{\dagger}\right)$ *is* (ψ, U) −***rough path controlled by*** $\mathbf{X}$ *if there exists a* $C > 0$ *and* $\delta > 0$ *such that*

1.
$$\left|\psi\left(y_s, y_t\right) - y_s^{\dagger} x_{s,t}\right|_g \leq C\omega\left(s,t\right)^{2/p} \tag{3.11}$$

and

2.
$$\left|U\left(y_s, y_t\right) y_t^{\dagger} - y_s^{\dagger}\right|_g \leq C\omega\left(s,t\right)^{1/p} \tag{3.12}$$

hold whenever $0 \leq s \leq t \leq T$ *and* $|t-s| \leq \delta$. *Occasionally we will refer to* y_s *as the path and* $y_s^{\dagger}$ *as the derivative process (or Gubinelli derivative).*

Remark 3.25 *In Definition 3.24 and in the definitions that follow, we use the convention that the* δ *is small enough to ensure that all of the expressions are well defined (in particular here it is small enough to ensure* $(y_s, y_t) \in \mathcal{D}$*).*

Remark 3.26 *Any path* z_s *in Euclidean space naturally gives rise to a two-parameter "increment process," namely* $z_{s,t} = z_t - z_s$. *If* φ *is any function such* *that $\varphi(z,\tilde{z}) \approx \tilde{z} - z$, then it makes sense to define $z^{\varphi}_{s,t} := \varphi(z_s, z_t)$. This serves as motivation for the following notation.*

Notation 3.27 *Given a gauge, $\mathcal{G} = (\psi, U)$, let $y^{\psi}_{s,t} := \psi(y_s, y_t)$ and $\left(y^{\dagger}\right)^{U}_{s,t} := U(y_s, y_t)\, y^{\dagger}_t - y^{\dagger}_s$. These will be referred to as the $\mathcal{G}-$**local increment processes of** $\left(y, y^{\dagger}\right)$.*

Remark 3.28 *With Notation 3.27, (3.11) becomes $\left|y^{\psi}_{s,t} - y^{\dagger}_s x_{s,t}\right| \leq C\omega(s,t)^{2/p}$ and (3.12) becomes $\left|\left(y^{\dagger}\right)^{U}_{s,t}\right| \leq C\omega(s,t)^{1/p}$.*

Definition 3.24 gives one possible notion of a controlled rough path on a manifold. We can also define such an object without having to provide a metric or gauge by using charts on the manifold.

Definition 3.29 *The pair $\mathbf{y}_s = \left(y_s, y^{\dagger}_s\right)$ is a **chart-rough path controlled by** $\mathbf{X}$ if for every chart ϕ on M and every $[a,b]$ such that $y([a,b]) \subseteq D(\phi)$ we have the existence of a $C_{\phi,a,b} \geq 0$ such that, for all $a \leq s \leq t \leq b$,*

1.
$$\left|\phi(y_t) - \phi(y_s) - d\phi \circ y^{\dagger}_s x_{s,t}\right| \leq C_{\phi,a,b}\,\omega(s,t)^{2/p} \tag{3.13}$$

and

2.
$$\left|d\phi \circ y^{\dagger}_t - d\phi \circ y^{\dagger}_s\right| \leq C_{\phi,a,b}\,\omega(s,t)^{1/p} \tag{3.14}$$

We will denote $C_{\phi,a,b}$ by C_{ϕ} when no confusion is likely to arise.

Notation 3.30 *If $\left(y_s, y^{\dagger}_s\right)$ is a chart rough path and ϕ is a chart as in Definition 3.29, we will write $\phi_* \mathbf{y}_s$ to mean*

$$\phi_* \mathbf{y}_s := \phi_*\left(y_s, y^{\dagger}_s\right) := \left(\phi \circ y_s, d\phi \circ y^{\dagger}_s\right).$$

Note that as long as y remains away from the boundary of $D(\phi)$, then $\phi_* \mathbf{y}_s$ is a controlled rough path on $\mathbb{R}^d$. Another way to think of this is that a chart controlled rough path is one which pushes forward to a controlled rough path in $\mathbb{R}^d$.

Before moving on, we'll make a few remarks.

Remark 3.31 *If $y^\dagger$ is any function satisfying the conditions in either of Definitions 3.24 or 3.29, then $s \to y_s^\dagger$ is automatically continuous. For example, if $\left(y_s, y_s^\dagger\right)$ satisfies the conditions of a (ψ, U) −rough path in Definition 3.24, then the function $t \to U(y_s, y_t)\, y_t^\dagger$ is a continuous at s and therefore $t \to y_t^\dagger = U(y_s, y_t)^{-1} U(y_s, y_t)\, y_t^\dagger$ is continuous at s.*

Remark 3.32 *If $M = \mathbb{R}^d$ and $\phi = I$ then the chart Definition 3.29 reduces to the usual Definition 2.8 of controlled rough paths. In this case, we identify all the tangent spaces with $\mathbb{R}^d$ and forget the base point in the derivative process.*

3.3 Chart and Gauge CRP Definitions are Equivalent

Theorem 3.33 *Let $\mathbf{y} := \left(y, y^\dagger\right)$ be a pair of continuous functions as in Notation 3.23, M be a manifold, and $\mathcal{G} = (\psi, U)$ be any gauge on M. Then $\mathbf{y}$ is a chart controlled rough path (Definition 3.29) if and only if it is a (ψ, U)-controlled rough path (Definition 3.24).*

Corollary 3.34 *We have the equality of sets*

$$\{(\psi, U) - \textit{rough paths}\} = \left\{\left(\tilde{\psi}, \tilde{U}\right) - \textit{rough paths}\right\}$$

for any gauges (ψ, U) and $\left(\tilde{\psi}, \tilde{U}\right)$ on M.

Notation 3.35 *Let $CRP_{\mathbf{X}}(M)$ be the collection of* ***controlled rough paths in*** M*, i.e. pairs of functions $\mathbf{y} = \left(y, y^{\dagger}\right)$ as in Notation 3.23 which satisfy either (and hence both) of Definitions 3.24 or 3.29.*

We will prove Theorem 3.33 after assembling a number of preliminary results that will be needed in the proof and in the rest of the dissertation.

3.3.1 Results Used in Proof of Theorem 3.33

Our first result is a local version of Theorem 3.33.

Theorem 3.36 *Let $\mathcal{G} = (\psi, U)$ be a gauge on $\mathbb{R}^d$, $\mathbf{z} = \left(z, z^{\dagger}\right) \in C\left([a,b], \mathbb{R}^d\right) \times C\left([a,b], L\left(W, \mathbb{R}^d\right)\right)$, and $\mathcal{W}$ be an open convex set such that $z([a,b]) \subseteq \mathcal{W}$ and $\mathcal{W} \times \mathcal{W} \subseteq D(\mathcal{G})$. Then $\mathbf{z} \in CRP_{\mathbf{X}}\left(\mathbb{R}^d\right)$ iff $\mathbf{z}$ is a $(\psi, U)-$rough path controlled by $\mathbf{X}$ with the choice $\delta := b - a$.*

Proof. Suppose $\mathbf{z} \in CRP_{\mathbf{X}}\left(\mathbb{R}^d\right)$. By Theorem 3.13,

$$\bar{\psi}(x,y) = y - x + A(x,y)(y-x)^{\otimes 2} \;\; \forall \; x, y \in \mathcal{W}.$$

Clearly A is bounded if it is restricted to x, y in the convex hull of $z([a,b])$ (which is compact and contained in $\mathcal{W}$). Thus, for all such points, we have there exists a C_1 such that

$$\left|\bar{\psi}(x,y) - (y-x)\right| \leq C_1 |y-x|^2 . \tag{3.15}$$

Taking $y = z_t$ and $x = z_s$ in this inequality shows

$$\left|\bar{\psi}(z_s, z_t) - z_{s,t}\right| \leq C_1 |z_t - z_s|^2 . \tag{3.16}$$

Since $\mathbf{z} \in CRP_{\mathbf{X}}\left(\mathbb{R}^d\right)$, there exists a C_2 such that

$$\left|z_{s,t} - z_s^{\dagger} x_{s,t}\right| \leq C_2 \omega\left(s,t\right)^{2/p} \tag{3.17}$$

$$\left|z_{s,t}^{\dagger}\right| \leq C_2 \omega\left(s,t\right)^{1/p}. \tag{3.18}$$

By enlarging C_2 if necessary we may further conclude,

$$\left|z_{s,t}\right| \leq C_2 \omega\left(s,t\right)^{1/p}. \tag{3.19}$$

Using Eqs. (3.17) and (3.19) in Eq. (3.16) gives the existence of a $C_3 < \infty$ such that

$$\left|\bar{\psi}\left(z_s, z_t\right) - z_s^{\dagger} x_{s,t}\right| \leq C_3 \omega\left(s,t\right)^{2/p}.$$

By Theorem 3.13 once more, we have

$$\bar{U}\left(x,y\right) = I + B\left(x,y\right)\left(y-x\right). \tag{3.20}$$

As was the case for A, B is bounded on the convex hull of $z\left([a,b]\right)$ so that there exists a C_4 such that

$$\begin{aligned}\left|\bar{U}\left(z_s, z_t\right) z_t^{\dagger} - z_s^{\dagger}\right| &\leq \left|z_{s,t}^{\dagger}\right| + C_4 \left|z_{s,t}\right| \\ &\leq \left(C_2 + C_4 C_2\right) \omega\left(s,t\right)^{1/p}.\end{aligned}$$

Thus $\mathbf{z}$ is a $\left(\psi, U\right)-$rough path controlled by $\mathbf{X}$ with the choice $\delta := b - a$ where our $C := \max\left\{C_1, C_2\left(1 + C_4\right)\right\}$.

For the converse direction, suppose $\mathbf{z}$ is a $\left(\psi, U\right)-$rough path controlled by $\mathbf{X}$ with the choice $\delta := b-a$ as in Definition 3.24. From Eq. (3.15) and the triangle inequality we have

$$\left|y - x\right| \leq C_1 \left|y - x\right|^2 + \left|\bar{\psi}\left(x,y\right)\right|.$$

Taking $x = z_s$ and $y = z_t$ in this inequality and using Definition 3.24 we may find $C_2 < \infty$ such that

$$\begin{aligned} |z_{s,t}| &\le C_1 |z_{s,t}|^2 + |\psi(z_s, z_t)| \\ &\le C_1 |z_{s,t}|^2 + C_2 \omega(s,t)^{1/p} \end{aligned}$$

for all $s \le t$ in $[a,b]$. By the uniform continuity of z on $[a,b]$, there exists $\epsilon > 0$ such that $C_1 |z_{s,t}| \le \frac{1}{2}$ when $|t-s| \le \epsilon$ which combined with the previous inequality implies

$$|z_{s,t}| \le 2C_2 \omega(s,t)^{1/p} \text{ when } |t-s| \le \epsilon.$$

For general $a \le s \le t \le b$ we may write $z_{s,t}$ as a sum of at most $n \le (b-a)/\epsilon$ increments whose norms are bounded by $2C_2\omega(s,t)^{1/p}$ wherein we have repeatedly used the estimate above along with the monotonicity of ω resulting from superactivity. Thus we conclude, with $C_3 := 2C_2(b-a)/\epsilon < \infty$, that

$$|z_{s,t}| \le C_3 \omega(s,t)^{1/p} \;\; \forall\, s,t \in [a,b].$$

This estimate along with the inequality in Eq. (3.15) gives,

$$\left|\bar{\psi}(z_s, z_t) - z_{s,t}\right| \le C_1 |z_{s,t}|^2 \le C_1 C_3^2 \omega(s,t)^{2/p} \;\; \forall\, s,t \in [a,b].$$

The previous inequality along with the assumption that $\mathbf{z}$ is a (ψ, U) $-$rough path shows there exists $C_4 < \infty$ such that

$$\left|z_{s,t} - z_s^\dagger x_{s,t}\right| \le \left|z_{s,t} - \bar{\psi}(z_s, z_t)\right| + \left|\bar{\psi}(z_s, z_t) - z_s^\dagger x_{s,t}\right| \le C_4 \omega(s,t)^{2/p}.$$

From Eq. (3.20), there exists a C_5 such that

$$\left|z_{s,t}^\dagger\right| \le \left|U(z_s, z_t)\, z_t^\dagger - z_s^\dagger\right| + C_5 |z_{s,t}|.$$

This inequality along with the assumption that $\mathbf{z}$ is a (ψ, U) −rough path shows there exists $C_6 < \infty$ such that $\left|z^{\dagger}_{s,t}\right| \leq C_6\omega(s,t)^{1/p}$ for all $a \leq s \leq t \leq b$. Thus we have shown $\mathbf{z} \in CRP_{\mathbf{X}}\left(\mathbb{R}^d\right)$. ■

The rest of this section is now devoted to a number of "stitching" arguments which will be used to piece together a number of local versions of Theorem 3.33 over subintervals as described in Theorem 3.36 into the full global version as stated in Theorem 3.33. For the rest of this section let $\mathcal{X}$ be a topological space and $0 \leq S < T < \infty$.

Lemma 3.37 *If $y : [S,T] \to \mathcal{X}$ is continuous and $y([S,T]) \subseteq \bigcup_{\alpha\in A} \mathcal{O}_\alpha$ where $\{\mathcal{O}_\alpha\}_{\alpha\in A}$ is a collection of open subsets of $\mathcal{X}$, then there exists a partition of $[S,T]$, $S = t_0 < t_1 < \ldots < t_l = T$, and $\alpha_i \in A$ such that for all i less than l, we have*

$$y([t_i, t_{i+1}]) \subseteq \mathcal{O}_{\alpha_i}$$

Proof. Define

$$T^* := \sup\{t : S \leq t \leq T, \text{ the conclusion of the theorem holds for } [S,t]\}.$$

Note that trivially $T^* > S$. For sake of contradiction, suppose $T^* < T$. Then there exists an $\epsilon > 0$ such that $T^* + \epsilon < T$, $T^* - \epsilon > S$ and $y(T^* - \epsilon, T^* + \epsilon) \subset \mathcal{O}_{\alpha^*}$ for some α^*. But the condition of the theorem holds for $T^* - \epsilon$ for some partition P. By appending P with $T^* + \lambda\epsilon$ with $\lambda \in (-1, 1]$ we have that $T^* \geq T^* + \epsilon$ which is absurd. Thus, we must have that $T^* = T$. ■

Definition 3.38 *The set $\{a_i, b_i\}_{i=0}^{l} \subset [S,T]$ is an **interlaced cover of** $[S,T]$ if $S = a_0 < a_1 < b_0 < a_2 < b_1 < a_3 < b_2 < \ldots < a_l < b_{l-1} < b_l = T$. Let $y : [S,T] \to \mathcal{X}$. The set $\{a_i, b_i\}_{i=0}^{l}$ is an **interlaced cover for** y if $\{a_i, b_i\}_{i=0}^{l}$ is an interlaced cover of $[S,T]$ and $y(a_{i+1}) \neq y(b_i)$ for all i less than l.*

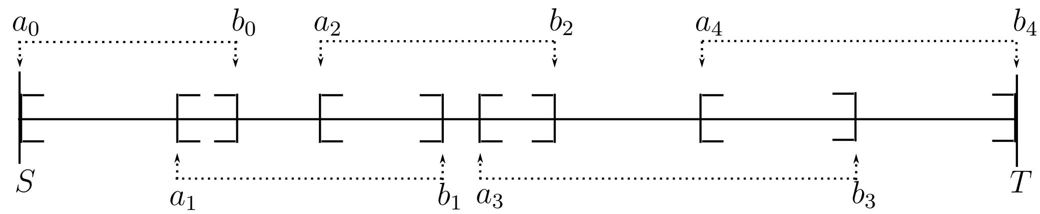

Figure 3.1: An interlaced cover of $[S,T]$

Corollary 3.39 *Suppose* $y : [S,T] \to \mathcal{X}$ *is continuous and* $y([S,T]) \subseteq \bigcup_{\alpha\in A} \mathcal{O}_\alpha$ *where* $\{\mathcal{O}_\alpha\}_{\alpha\in A}$ *is a collection of open sets* $\mathcal{O}_\alpha$*. There exists an interlaced cover for* y*,* $\{a_i, b_i\}_{i=0}^{l}$ *such that* $y([a_i, b_i]) \subseteq \mathcal{O}_{\alpha_i}$*. Note that for such a setup, this implies* $y([a_{i+1}, b_i]) \subseteq \mathcal{O}_{\alpha_i} \cap \mathcal{O}_{\alpha_{i+1}}$

Proof. The first step will be a technical one to get rid of unnecessary endpoints. Let t_i' and α_i' be as given in Lemma 3.37. Then clearly $y(t_i') \in \mathcal{O}_{\alpha_{i-1}'} \cap \mathcal{O}_{\alpha_i'}$ for all $1 \le i < l'$. Starting with t_1', we check if $y([t_0', t_1']) \subseteq \mathcal{O}_{\alpha_1}$. In the case it is, we may renumber our partition after removing t_1' and $\mathcal{O}_{\alpha_0'}$ to get a new set of t_j' and α_j' which still satisfy the result of the lemma. Continuing this process inductively, we may assume that we have such a set $\{t_i, \alpha_i\}_{i=0}^{l}$ such that $y([t_i, t_{i+1}])$ is not contained in $\mathcal{O}_{\alpha_{i+1}}$.

To construct the desired interlaced cover, we define $b_i := t_{i+1}$ for all $i \le l := l' - 1$ and $a_0 := S$. Note for now that this means $y([b_{i-1}, b_i]) \subseteq \mathcal{O}_{\alpha_i}$. Then we define the "lower end" stopping time T_i for all $i > 0$ by the formula

$$T_i := \inf\left\{t < b_i : y([t, b_i]) \subseteq \mathcal{O}_{\alpha_{i+1}}\right\}.$$

By construction and because we refined our partition, $b_{i-1} \le T_i < b_i$. It is clear that $y(T_i) \ne y(b_i)$ by the continuity of y. Thus, there exists a time T_i^* such that $T_i < T_i^*$ and $y(T_i^*) \ne y(b_i)$. Define

$$a_{i+i} := T_i^*$$

for all $0 < i < n$. Since $y([b_{i-1}, b_i]) \subseteq \mathcal{O}_{\alpha_i}$ and $a_i > b_{i-1}$, we have that $y([a_i, b_i]) \subseteq \mathcal{O}_{\alpha_i}$. ■

Since the following patching trick will be used multiple times in later proofs, we will prove it here in more generality to avoid too much indexing notation later.

Lemma 3.40 *Let ω be a control and $\{a_i, b_i\}_{i=0}^{l}$ be an interlaced cover of $[S,T]$ such that $\omega(a_{i+1}, b_i) > 0$ for all $i < n$. Let $\theta > 0$ and $F : D \to [0,\infty)$ be a bounded function such that $D \subseteq \Delta_{[S,T]}$ and for each $1 \leq i \leq l$ there exists $C_i < \infty$ such that*

$$F(s,t) \leq C_i \omega(s,t)^{\theta} \text{ for all } (s,t) \in \Delta_{[a_i,b_i]} \cap D.$$

Then there exists a $\tilde{C} < \infty$ such that

$$F(s,t) \leq \tilde{C}\omega(s,t)^{\theta} \ \forall \ (s,t) \in D. \tag{3.21}$$

Proof. Let

$$\begin{aligned} m &:= \min\left\{\omega(a_{i+1}, b_i)^{\theta} : 0 \leq i < n\right\}, \\ C &:= \max\{C_i : 0 \leq i \leq n\}, \text{ and} \\ M &:= \sup\{F(s,t) : (s,t) \in D\} < \infty \end{aligned}$$

and then define $\tilde{C} := \max\left\{\frac{M}{m}, C\right\}$. We claim that Inequality (3.21) holds.

If there exists an i such that $s,t \in [a_i, b_i] \cap D$, then (3.21) holds trivially. Otherwise, let i^* be the largest i such that $s \in [a_i, b_i]$. Then $s < a_{i^*+1}$ and $t > b_{i^*}$. However this says that $[s,t] \supset [a_{i^*+1}, b_{i^*}]$ so that

$$F(s,t) \leq M = \frac{M}{m} m \leq \tilde{C}\omega(a_{i^*+1}, b_{i^*})^{\theta} \leq \tilde{C}\omega(s,t)^{\theta}.$$

■

3.3.2 Proof of Theorem 3.33

The recurring strategy here will be localize appropriately to work in the $\mathbb{R}^d$ case so that we may apply Theorem 3.36. We must choose these localizations carefully so that we may patch the estimates together (with two different strategies) using the lemmas above. One method of patching is a bit more involved than the other; therefore we will present it more formally:

Remark 3.41 (Proof Strategy) *Let $y : [a,b] \to M$ be the first component of $\left(y, y^{\dagger}\right)$ where $\left(y, y^{\dagger}\right)$ is either a $(\psi, U)-$ controlled rough path or chart controlled rough path. Also suppose for each $m \in y\left([a,b]\right)$, we are given an open neighborhood, $\mathcal{W}_m \subseteq M$, of m. By Corollary 3.39, there exists an interlaced cover for y, $\{a_i, b_i\}_{i=1}^{l}$ and $\{m_i\}_{i=1}^{l}$ such that $y\left([a_i, b_i]\right) \subseteq \mathcal{W}_{m_i}$ and $\omega\left(a_{i+1}, b_i\right) > 0$. Thus, if $F : D \to [0, \infty)$ is a bounded function such that $D \subseteq \Delta_{[a,b]}$, then in order to prove that*

$$F\left(s,t\right) \leq C\omega\left(s,t\right)^{\theta} \;\; \forall \;\; \left(s,t\right) \in D, \tag{3.22}$$

it suffices to prove; for each $1 \leq i \leq l$ there exists $C_i < \infty$ such that

$$F\left(s,t\right) \leq C_i\omega\left(s,t\right)^{\theta} \text{ for all } \left(s,t\right) \in \Delta_{[a_i,b_i]} \cap D.$$

Therefore in attempting to prove an assertion in the form of Inequality (3.22), we may assume, without loss of generality, that $y\left([a,b]\right) \subseteq \mathcal{W}$ where the $\mathcal{W}$ will have nice properties dependent on our setting.

The proof of Theorem 3.33 will consist of two steps:

1. If gauge conditions of (3.11) and (3.12) hold for some $C > 0$ and $\delta > 0$, then the chart conditions of (3.13) and (3.14) hold. We will reduce this to the $\mathbb{R}^d$ case immediately, then use Lemma 2.9 to patch the estimates together.

2. If the chart condition of (3.13) and (3.14) hold, then gauge condition of (3.11) and (3.12) hold for an appropriately chosen δ. Here we will first show which local estimates we need to satisfy to use Remark 3.41 and then reduce to the $\mathbb{R}^d$ case.

In simple terms, step 1 is "localize then patch" and step 2 is "cut nicely, localize, then patch".

Proof of Theorem 3.33. Step 1: Definition 3.24 $\implies$ Definition 3.29.

We'll first assume that the gauge definition holds, i.e. that there exists a $\delta > 0$ and a $C_1 > 0$ such that

$$\left|\psi\left(y_s, y_t\right) - y_s^{\dagger} x_{s,t}\right|_g \leq C_1 \omega\left(s,t\right)^{2/p} \tag{3.23}$$

and

$$\left|U\left(y_s, y_t\right) y_t^{\dagger} - y_s^{\dagger}\right|_g \leq C_1 \omega\left(s,t\right)^{1/p}$$

hold for all $0 \leq s \leq t \leq T$ such that $|t-s| \leq \delta$. Let ϕ be a chart on M and let $[a,b]$ be such that $y\left([a,b]\right) \subseteq D\left(\phi\right)$. If we define

$$\begin{aligned}
\psi^{\phi}\left(x,y\right) &:= \phi_* \psi\left(\phi^{-1}\left(x\right), \phi^{-1}\left(y\right)\right) \\
U^{\phi}\left(x,y\right) &:= \phi_* U\left(\phi^{-1}\left(x\right), \phi^{-1}\left(y\right)\right) \circ \left(\phi_*^{-1}\right)_{\phi(y)} \\
\left(z_s, z_s^{\dagger}\right) &:= \phi_*\left(\mathbf{y}_s\right) = \left(\phi\left(y_s\right), d\phi \circ y_s^{\dagger}\right)
\end{aligned}$$

then it is clear that there exists a $C_2 = C_2\left(\phi_*\right)$ such that

$$\left|\bar{\psi}^{\phi}\left(z_s, z_t\right) - z_s^{\dagger} x_{s,t}\right| \leq C_2 \omega\left(s,t\right)^{2/p} \tag{3.24}$$

$$\left|\bar{U}^{\phi}\left(z_s, z_t\right) z_t^{\dagger} - z_s^{\dagger}\right| \leq C_2 \omega\left(s,t\right)^{1/p} \tag{3.25}$$

for all $a \leq s \leq t \leq b$ such that $t - s \leq \delta$ where $\left(\psi^{\phi}, U^{\phi}\right)$ is a local gauge on $\mathbb{R}^d$ and $\left(\bar{\psi}^{\phi}, \bar{U}^{\phi}\right)$ is consistent with Notation 3.12. Thus $\left(z, z^{\dagger}\right)$ is a $\left(\psi^{\phi}, U^{\phi}\right)$ −rough path controlled by $\mathbf{X}$. Finally we need to use this information to show there exists a $C_{\phi,a,b}$ such that

$$\left|z_t - z_s - z_s^\dagger x_{s,t}\right| \le C_{\phi,a,b}\,\omega\left(s,t\right)^{2/p}. \tag{3.26}$$

and

$$\left|z_t^\dagger - z_s^\dagger\right| \le C_{\phi,a,b}\,\omega\left(s,t\right)^{1/p} \tag{3.27}$$

for all s,t such that $a \le s \le t \le b$.

In light of the Sewing Lemma 2.9 and Lemma 3.37, we only need to show that for each $u \in [a,b]$, the inequalities (3.26) and (3.27) hold with $C_{\phi,a,b}$ replaced with C_u for all $s,t \in (u-\delta_u, u+\delta_u) \cap [a,b]$ such that $s \le t$ for some $\delta_u > 0$.

For any $u \in [a,b]$, let $\mathcal{W}_u$ be an open convex set of z_u such that $\mathcal{W}_u \times \mathcal{W}_u \subseteq D\left(\psi^\phi\right)$. We then choose $\delta_u > 0$ to be such that $z\left([u-\delta_u, u+\delta_u] \cap [a,b]\right) \subseteq \mathcal{W}_u$ and $2\delta_u \le \delta$. However, now we are in the setting of Theorem 3.36 and are therefore finished with this step.

Step 2: Definition 3.29 $\implies$ Definition 3.24

Suppose that the chart item (3.13) holds. We must prove that there exists a $\delta, C > 0$ such that

$$\begin{aligned} \left|\psi\left(y_s, y_t\right) - y_s^\dagger x_{s,t}\right|_g &\le C\omega\left(s,t\right)^{2/p} \\ \left|U\left(y_s, y_t\right) y_t^\dagger - y_s^\dagger\right|_g &\le C\omega\left(s,t\right)^{1/p} \end{aligned}$$

for all $s \le t$ such that $|t-s| \le \delta$.

We choose δ such that $|t-s| \le \delta$ for $0 \le s \le t \le T$ implies that both $\left|\psi\left(y_s, y_t\right)\right|_g$ and $\left|U\left(y_s, y_t\right)\right|_g$ make sense and are bounded. Around every point m of $y\left([0,T]\right)$, there exists an open $\mathcal{O}_m$ containing m and such that $\mathcal{O}_m \times \mathcal{O}_m \subseteq \mathcal{D}$. Additionally there exists a chart ϕ^m such that $m \in D\left(\phi^m\right)$. By considering an open ball around $\phi^m\left(m\right)$ in $R\left(\phi^m\right)$ and shrinking the radius, we may assume that $\mathcal{V}_m := D\left(\phi^m\right) \subseteq \mathcal{O}_m$ and the range, $\mathcal{W}_m := \phi\left(\mathcal{V}_m\right)$, of ϕ^m is convex. Since

$\{\mathcal{V}_m\}_{m\in y([0,T])}$ is an open cover of $y([0,T])$, we may use this cover along with $D = \{(s,t) : 0 \le s \le t \le T \text{ and } |t-s| \le \delta\}$ to employ the proof strategy in Remark 3.41. We will do this twice, with $F(s,t) = \left|\psi(y_s, y_t) - y_s^{\dagger} x_{s,t}\right|_g$ in the first iteration and $F(s,t) = \left|U(y_s, y_t)\, y_t^{\dagger} - y_s^{\dagger}\right|_g$ in the second; this will reduce us to considering the case where there exists a single chart ϕ such that $y([0,T]) \subseteq D(\phi)$, $D(\phi) \times D(\phi) \subseteq \mathcal{D}$ and $\mathcal{W} = R(\phi)$ is convex.

Now that we have reduced to a single chart ϕ, we may define (ψ^{ϕ}, U^{ϕ}) and the path $(z, z^{\dagger})$ as in Step 1. Then $z([0,T]) \subseteq \mathcal{W}$ and $\mathcal{W} \times \mathcal{W} \subseteq D(\psi^{\phi}) = D(U^{\phi})$. However, by Theorem 3.36 we have that the proper estimates hold because $\mathbf{z}$ is a (ψ^{ϕ}, U^{ϕ}) −rough path controlled by $\mathbf{X}$. Therefore, we are finished by patching using Remark 3.41. ■

Remark 3.42 *In the proof of Theorem 3.33, we would have been able to show (and did so somewhat indirectly) that Inequality (3.13) implies Inequality (3.11) for some $\delta > 0$. However, it is not true in general that, for a fixed δ, Inequality (3.11) implies Inequality (3.13). See Example 3.43 below for a counterexample.*

Example 3.43 *Let x_s and y_s be the $C([0,2], \mathbb{R})$ paths defined by*

$$y_s = x_s = \begin{cases} 0 & \text{if } 0 \le s \le 1 \\ s^{1/p} - 1 & \text{if } 1 \le s \le 2 \end{cases}$$

and the control $\omega(s,t)$ be defined by

$$\omega(s,t) = \begin{cases} 0 & \text{if } t \le 1 \\ t - (s \vee 1) & \text{if } t \ge 1 \end{cases}.$$

Then it is easy to check that

$$|x_{s,t}| \le \omega(s,t)^{1/p}$$

Let

$$y_s^{\dagger} = \begin{cases} 2-2s & \text{if} \quad 0 \le s \le \frac{1}{2} \\ 1 & \text{else} \quad \frac{1}{2} \le s \le 2 \end{cases}.$$

Then if $t - s \le 1/2$, $y_{s,t} - y_s^{\dagger} x_{s,t} = 0$ *so that* $\left(y, y^{\dagger}\right)$ *satisfies Inequality (3.11) with* $\delta = 1/2$ *and* $\psi(x, y) = y - x$. *On the other hand if* $s = 0$ *and* $t = 1 + \epsilon$, *then*

$$y_{s,t} - y_s^{\dagger} x_{s,t} = \epsilon^{1/p} - 2\epsilon^{1/p} = -\epsilon^{1/p}.$$

Thus

$$\frac{\left|y_{0,1+\epsilon} - y_0^{\dagger} x_{0,1+\epsilon}\right|}{\omega(0, 1+\epsilon)^{2/p}} = \frac{1}{\epsilon^{1/p}}$$

so that $\left(y, y^{\dagger}\right)$ *does not satisfy Inequality (3.13) with the identity chart.*

In situations in which we are given a covariant derivative ∇ on a manifold, by Example 3.8, we have an equivalent definition:

Example 3.44 *The pair* $\left(y_s, y_s^{\dagger}\right)$ *is an element of* $CRP_{\mathbf{X}}(M)$ *if and only if there exists a* C *such that*

1.

$$\left|\left(\exp_{y_s}^{\nabla}\right)^{-1}(y_t) - y_s^{\dagger} x_{s,t}\right|_g \le C\omega(s,t)^{2/p} \tag{3.28}$$

2.

$$\left|U_{y_s,y_t}^{\nabla} y_t^{\dagger} - y_s^{\dagger}\right|_g \le C\omega(s,t)^{1/p} \tag{3.29}$$

where $\left(\exp_m^{\nabla}\right)^{-1}$ *and* $U_{n,m}^{\nabla}$ *are defined as in Example 3.8 and the inequalities hold when* (y_s, y_t) *are in the domain* $\mathcal{D}$ *as given in Theorem 3.21. In particular, on a Riemannian manifold we can use this definition with the Levi-Civita covariant derivative.*

Before providing yet another equivalent definition of controlled rough paths on manifolds, we will present some examples.

3.4 Examples of Controlled Rough Paths

Recall $\mathbf{X} = (x, \mathbb{X})$ is a weak-geometric rough path with values in $W \oplus W^{\otimes 2}$ where $W = \mathbb{R}^k$. The results here will rely on basic approximations found in the Appendix, Section A.1.

Example 3.45 *Let $M^d \subseteq W$ be an embedded submanifold and for every $m \in M^d$, let $P(m)$ be the orthogonal projection onto the tangent space $T_m M$. Suppose $x_s \in M^d$ for all s in $[0, T]$. Then $(x_s, P(x_s)) \in CRP_{\mathbf{X}}(M)$.*

Proof. We will use the gauge as given in Example 3.44 where the ∇ is the Levi-Civita covariant derivative from the induced metric from Euclidean space. Verifying that $P(x_s)$ lives in the correct space is trivial.

Next, to show Inequality 3.28 is satisfied, we use item 1 of Lemma A.2 which says

$$\exp_m^{-1}(\tilde{m}) = P(m)(\tilde{m} - m) + O\left(|\tilde{m} - m|^3\right) \text{ for all } m \in M^d.$$

Letting $m = x_s$ and $\tilde{m} = x_t$, we are done.

Inequality (3.29) is also satisfied as a result of Lemma A.2 which says that $U^{\nabla}_{\tilde{m},m} = P(m) + O(|\tilde{m} - m|)$. Thus

$$\begin{aligned} P(x_t) - U^{\nabla}_{x_t, x_s} P(x_s) &\underset{1}{\approx} P(x_t) - P(x_s) P(x_s) \\ &= P(x_t) - P(x_s) \\ &\underset{1}{\approx} 0 \end{aligned}$$

■

Remark 3.46 *The $P(m)$ in Example 3.45 can actually be any projection. The necessary approximations can be deduced by the proof of Lemma A.2 without using any orthogonality assumptions.*

The next example will be proved in more generality in Section 4.2.2. However, we find it instructive to prove it without charts and in the embedded context where the reader may be more comfortable.

Example 3.47 *Let f be a smooth function from W to an embedded manifold $\tilde{M}^d \subseteq \mathbb{R}^{\tilde{k}}$. Then $(f(x_s), f'(x_s)) \in CRP_{\mathbf{X}}\left(\tilde{M}\right)$.*

Proof. Again we will use the Levi-Civita covariant derivative $\tilde{\nabla}$ from the embedded metric. First we note that $f'(x_s)$ lives in the correct space as $R(f) \subseteq \tilde{M}^d$.

To show Inequality (3.28) holds one can use the fact that $(f((x_s), f'(x_s))$ is a controlled rough path in the embedded space or Taylor's Theorem to see that

$$f(x_t) - f(x_s) - f'(x_s)(x_t - x_s) \underset{2}{\approx} 0$$

which easily implies

$$P(f(x_s))[f(x_t) - f(x_s) - f'(x_s)(x_t - x_s)] \underset{2}{\approx} 0.$$

But again by Lemma A.2

$$\begin{aligned} &P(f(x_s))[f(x_t) - f(x_s) - f'(x_s)(x_t - x_s)] \\ &\quad = P(f(x_s))[f(x_t) - f(x_s)] - f'(x_s)(x_t - x_s) \\ &\quad \underset{2}{\approx} \left(\exp^{\tilde{\nabla}}_{f(x_s)}\right)^{-1}(f(x_t)) - f'(x_s)(x_t - x_s). \end{aligned}$$

Thus

$$\left(\exp^{\tilde{\nabla}}_{f(x_s)}\right)^{-1}(f(x_t)) - f'(x_s)(x_t - x_s) \underset{2}{\approx} 0.$$

Lastly to show Inequality (3.29), we have

$$f'(x_t) - f'(x_s) \underset{1}{\approx} 0$$

and therefore

$$\begin{aligned} 0 &\underset{1}{\approx} P\left(f\left(x_t\right)\right)\left[f'\left(x_t\right)-f'\left(x_s\right)\right] \\ &= f'\left(x_t\right)-P\left(f\left(x_t\right)\right)f'\left(x_s\right) \\ &\underset{1}{\approx} f'\left(x_t\right)-U^{\tilde{\nabla}}_{f(x_t),f(x_s)}f'\left(x_s\right), \end{aligned}$$

wherein we have used $P\left(f\left(x_t\right)\right)f'\left(x_t\right)=f'\left(x_t\right)$ in the second line and Lemma A.2 in the last. Thus $\left(f\left(x_s\right),f'\left(x_s\right)\right)\in CRP_{\mathbf{X}}\left(\tilde{M}\right)$ ■

3.5 Smooth Function Definition of CRP

In the spirit of semi-martingales on manifolds [see for example [11, Chapter III] or [10, 16, 17]], we can define controlled rough paths on manifolds as elements which, when composed with any smooth function, give rise to a one-dimensional controlled rough path on flat space. More precisely we have the following theorem.

Theorem 3.48 $\mathbf{y}=\left(y,y^{\dagger}\right)\in CRP_{\mathbf{X}}\left(M\right)$ *if and only if for every* $f\in C^{\infty}\left(M\right)$,

$$f_*\mathbf{y}=\left(f\left(y\right),df\circ y^{\dagger}\right)\in CRP_{\mathbf{X}}\left(\mathbb{R}\right).$$

Proof. The proof that $\mathbf{y}\in CRP_{\mathbf{X}}\left(M\right)$ implies that $f_*\mathbf{y}\in CRP_{\mathbf{X}}\left(\mathbb{R}\right)$ for every $f\in C^{\infty}\left(M\right)$ will be deferred to the more general case proved in Proposition 4.42 (in which case we consider the codomain of f to be a manifold $\tilde{M}$).

To prove the converse, let ϕ be a chart and $0\leq a<b\leq T$ be such that $y\left(\left[a,b\right]\right)\subseteq D\left(\phi\right)$ and let $\mathcal{O}\subset M$ be an open set such that $\bar{\mathcal{O}}$ is compact and

$$y\left(\left[a,b\right]\right)\subseteq\mathcal{O}\subseteq\bar{\mathcal{O}}\subseteq D\left(\phi\right).$$

Then by using a cutoff function we can manufacture global functions $f^i\in C^{\infty}\left(M\right)$

which agree with the coordinates ϕ^i on $\mathcal{O}$. The assumption that $f^i_* \mathbf{y}$ is a controlled rough path for $1 \le i \le d$ then shows the inequalities in Eqs. (3.13) and (3.14) of Definition 3.29 hold. ■

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Chapter 4

Integration Theory

4.1 Integration of Controlled One-Forms

In the flat case, a controlled rough path with values in an appropriate Euclidean spaces can be integrated against another controlled rough path (see Theorem 2.12) provided their controlling rough path $\mathbf{X}$ is the same. The integral in this case is another rough path controlled by $\mathbf{X}$. We can do something similar on manifolds, though it will be necessary to add some extra structure. As usual let $\mathbf{y}_s = \left(y_s, y_s^\dagger\right)$ be a controlled rough path on M controlled by $\mathbf{X} = (x, \mathbb{X}) \in W \oplus W^{\otimes 2}$. Let V be a Banach space.

4.1.1 Controlled One-Forms Along a Rough Path

Let U be a parallelism on M.

Definition 4.1 *The pair $\left(\alpha_s, \alpha_s^\dagger\right)$ is a V – valued $U-$controlled (rough) one-form along y_s if*

1. $\alpha_s \in L\left(T_{y_s}M, V\right)$

2. $\alpha_s^\dagger \in L\left(W \otimes T_{y_s}M, V\right)$

3. $\alpha_t \circ U(y_t, y_s) - \alpha_s - \alpha_s^\dagger (x_{s,t} \otimes (\cdot)) \underset{2}{\approx} 0$

4. $\alpha_t^\dagger \circ (I \otimes U(y_t, y_s)) - \alpha_s^\dagger \underset{1}{\approx} 0$

By items 3 and 4, we mean these hold if $|t - s| < \delta$ *for some* $\delta > 0$ *to ensure the expressions make sense.*

Remark 4.2 *For the sake of clarity, by item 3 of Definition 4.1, we mean that if* s, t *are close, then there exists a* C *such that*

$$\left|\alpha_t \circ U(y_t, y_s) - \alpha_s - \alpha_s^\dagger (x_{s,t} \otimes (\cdot))\right|_{g,op} \leq C\omega(s,t)^{2/p}.$$

For item 4, we mean for s, t *close, there exists a* C *such that*

$$\left|\alpha_t^\dagger \circ (w \otimes U(y_t, y_s)) - \alpha_s^\dagger (w \otimes (\cdot))\right|_{g,op} \leq C|w|\,\omega(s,t)^{1/p}$$

for all $w \in W$*. By Corollary A.4, it does not matter which Riemannian metric* g *we choose here.*

Notation 4.3 *Let* $CRP_y^U(M, V)$ *denote those* $\boldsymbol{\alpha}_s := \left(\alpha_s, \alpha_s^\dagger\right)$ *satisfying Definition 4.1. We refer to* $CRP_y^U(M, V)$ *as a* ***space of*** $U-$***controlled one-forms along*** $\mathbf{y}$.

Remark 4.4 *If* $M = \mathbb{R}^d$ *and* $U = I$ *and we identify* $T_{y_s}M$ *with* $\mathbb{R}^d$ *then Definition 4.1 reduces to the flat case definition of a* $L\left(\mathbb{R}^d, V\right)$ *– valued rough path controlled by* $\mathbf{X}$.

Remark 4.5 *Note that 3 and 4, of Definition 4.1 force continuity of both* α_s *and* $\alpha_s^\dagger$.

We can take linear combinations of elements of $CRP_y^U(M, V)$ to form other elements in $CRP_y^U(M, V)$. The following proposition, whose simple proof is left to the reader, shows how to construct more non-trivial examples of elements in $CRP_y^U(M,V)$.

Proposition 4.6 *If V and $\tilde{V}$ are Banach spaces, $\boldsymbol{\alpha} \in CRP_y^U(M,V)$ and*

$$\mathbf{f} = \left(f, f^\dagger\right) \in CRP_{\mathbf{X}}\left(\mathrm{Hom}\left(V, \tilde{V}\right)\right),$$

then

$$(\mathbf{f}\boldsymbol{\alpha})_s := \left(f_s\alpha_s, f_s^\dagger\alpha_s + f_s\alpha_s^\dagger\right) \in CRP_y^U\left(M, \tilde{V}\right).$$

where by $f_s^\dagger\alpha_s$ we mean $f_s^\dagger((\cdot)\otimes\alpha_s(\cdot))$.

Our next goal is to define an integral of $\boldsymbol{\alpha}_s$ along $\mathbf{y}_s$. However, this integral will depend on a choice of parallelism and for this reason we need to introduce the "compatibility tensor" which measures the difference between two parallelisms.

4.1.2 The Compatibility Tensors

Definition 4.7 *The **compatibility tensor,** $S^{\tilde{U},U} \in \Gamma(T^*M \otimes T^*M \otimes TM)$, of two parallelisms $\tilde{U}$ and U on M is the defined by*

$$S_m^{\tilde{U},U} := d\left[U(\cdot,m)^{-1}\tilde{U}(\cdot,m)\right]_m.$$

In more detail if $v_m, w_m \in T_mM$, then

$$S_m^{\tilde{U},U}[v_m \otimes w_m] = v_m\left[x \to U(x,m)^{-1}\tilde{U}(x,m)\,w_m\right].$$

Remark 4.8 *There are actually multiple ways to define $S_m^{\tilde{U},U}$. For example, we* *have on simple tensors*

$$\begin{aligned} S_m^{\tilde{U},U}(v_m \otimes w_m) &= d\left[U(m,\cdot)\,\tilde{U}(m,\cdot)^{-1} w_m\right]_m v_m \\ &= \left(\nabla_{v_m}\left[\tilde{U}(\cdot,m) - U(\cdot,m)\right]\right) w_m \\ &= \left(\nabla_{v_m}\left[U(m,\cdot) - \tilde{U}(m,\cdot)\right]\right) w_m \end{aligned} \tag{4.1}$$

where ∇ is any covariant derivative on M. Similar to the proofs of Corollary 3.18 above and Theorem 4.15 below, the identities in Eq. (4.1) are straightforward to prove by employing charts to reduce them to Euclidean space identities.

Example 4.9 *If ∇ and $\tilde{\nabla}$ are two covariant derivatives on TM, $U = U^{\nabla}$, $\tilde{U} = U^{\tilde{\nabla}}$, and $A \in \Omega^1(\mathrm{End}(TM))$ such that $\nabla = \tilde{\nabla} + A$, then*

$$S_m^{\tilde{U},U}(v_m \otimes w_m) = A(v_m)\, w_m \in T_m M.$$

Indeed,

$$\begin{aligned} v_m\left[U(\cdot,m)^{-1}\,\tilde{U}(\cdot,m)\, w_m\right] &= \nabla_{v_m}\left[\tilde{U}(\cdot,m)\, w_m\right] \\ &= \tilde{\nabla}_{v_m}\left[\tilde{U}(\cdot,m)\, w_m\right] + A(v_m)\,\tilde{U}(m,m)\, w_m \\ &= 0 + A(v_m)\, w_m = A(v_m)\, w_m. \end{aligned}$$

Example 4.10 (Converse of Example 4.9) *If U and $\tilde{U}$ are two parallelisms on M and $\nabla = \nabla^U$ and $\tilde{\nabla} = \nabla^{\tilde{U}}$ are the corresponding covariant derivatives on TM (as in Remark 3.9), then*

$$\nabla_{v_m} = \tilde{\nabla}_{v_m} + S_m^{\tilde{U},U}(v_m \otimes (\cdot)) \;\; \forall\; v_m \in T_m M.$$

The verification is as follows. If Y is a vector-field on M and σ_t is such that

$\dot{o}_0 = v_m$, *we have*

$$\nabla_{v_m} Y - \tilde{\nabla}_{v_m} Y := \frac{d}{dt}|_0 \left[U(m, \sigma_t) - \tilde{U}(m, \sigma_t)\right] Y(\sigma_t)$$
$$= \left(\nabla_{v_m} \left[U(m, \cdot) - \tilde{U}(m, \cdot)\right]\right) Y(m) + 0 \cdot \nabla_{v_m} Y$$
$$= S_m^{\tilde{U},U}(v_m \otimes Y(m))$$

wherein we have used Eq. (4.1) for the last equality.

Lemma 4.11 *If* $U, \tilde{U}$, *and* $\hat{U}$ *are three parallelisms, then*

$$S^{\hat{U},U} = S^{\hat{U},\tilde{U}} + S^{\tilde{U},U} \text{ and } S^{\tilde{U},U} = -S^{U,\tilde{U}}.$$

Proof. For $v_m, w_m \in T_m M$, an application of the product rules shows

$$S_m^{\hat{U},U}(v_m \otimes w_m) = v_m \left[U(\cdot, m)^{-1} \hat{U}(\cdot, m) w_m\right]$$
$$= v_m \left[\left[U(\cdot, m)^{-1} \tilde{U}(\cdot, m)\right] \left[\tilde{U}(\cdot, m)^{-1} \hat{U}(\cdot, m)\right] w_m\right]$$
$$= S_m^{\hat{U},\tilde{U}}(v_m \otimes w_m) + S_m^{\tilde{U},U}(v_m \otimes w_m).$$

Similarly,

$$S^{U,\tilde{U}}[v_m \otimes (\cdot)] = v_m \left[\tilde{U}(\cdot, m)^{-1} U(\cdot, m)\right]$$
$$= v_m \left[U(\cdot, m)^{-1} \tilde{U}(\cdot, m)\right]^{-1}$$
$$= -v_m \left[U(\cdot, m)^{-1} \tilde{U}(\cdot, m)\right]$$
$$= -S^{\tilde{U},U}[v_m \otimes (\cdot)].$$

■

Notation 4.12 *If* $\mathcal{G} := (\psi, U)$ *is a gauge, we let* $S^{\mathcal{G}} := S^{\psi_*,U}$ *be the compatibility tensor between* U^{ψ} *and* U, *where* $U^{\psi}(m, n) := \psi(m, \cdot)_{*n}$ *as in Remark 3.10.*

If we have a covariant derivative ∇ on M, then as in Example 3.8 we have the choice of gauge $\mathcal{G} = (\psi, U) = \left(\left(\exp^{\nabla}\right)^{-1}, U^{\nabla} \right)$. In this case, the tensor $S_m^{\mathcal{G}}$ is a more familiar object.

Lemma 4.13 *If* $\psi = \left(\exp^{\nabla}\right)^{-1}$ *and* $U = U^{\nabla}$*, then*

$$S_m^{\mathcal{G}} = \frac{1}{2} T_m^{\nabla}$$

where T^{∇} *is the Torsion tensor of* ∇*.*

Proof. By transferring the covariant derivative and functions using charts, we may assume we are working on Euclidean space. In this case, by Eq. (A.13) and Corollary A.6, we have

$$\begin{aligned}
S_m^{\mathcal{G}}\left((m,v)\otimes(m,w)\right) &= \left(\nabla_{(m,v)}\left[U_{m,\cdot}^{\nabla} - \left(\exp_m^{\nabla}\right)_{*\cdot}^{-1}\right]\right) w \\
&= \left[\partial_{(m,v)} + A_m\langle v\rangle\right]\left[U_{m,\cdot}^{\nabla} - \left(\exp_m^{\nabla}\right)_{*\cdot}^{-1}\right] w \\
&= \left(U_{m,\cdot}^{\nabla}\right)'(m)\left[v\otimes w\right] - \left(\left(\exp_m^{\nabla}\right)^{-1}\right)''(m)\left[v\otimes w\right] \\
&\quad + A_m\langle v\rangle\langle w\rangle - A_m\langle v\rangle\langle w\rangle \\
&= A_m\langle v\rangle\langle w\rangle - \frac{1}{2}A_m\langle v\rangle\langle w\rangle - \frac{1}{2}A_m\langle w\rangle\langle v\rangle \\
&= \frac{1}{2}\left[A_m\langle v\rangle\langle w\rangle - A_m\langle w\rangle\langle v\rangle\right] \\
&= \frac{1}{2}T_m^{\nabla}\left((m,v)\otimes(m,w)\right).
\end{aligned}$$

■

Here is one last example of a gauge and its compatibility tensor.

Proposition 4.14 *Let* G *be a Lie group and* ∇ *be the left covariant derivative on* TG *uniquely determined by requiring the left invariant vector fields to be covariantly* *constant, i.e.* $\nabla \tilde{A} = 0$ *for all* $A \in \mathfrak{g}$*. Then for* g *near* k*,*

$$U^{\nabla}(g,k) = //(k \to g) = L_{gk^{-1}*}, \tag{4.2}$$

and

$$\psi^{\nabla}(k,g) = \left(\exp_k^{\nabla}\right)^{-1}(g) = k \cdot \log\left(k^{-1}g\right) \tag{4.3}$$

where $L_g : G \to G$ *is left multiplication by* $g \in G$ *and* $\log$ *is the local inverse of the map* $A \to e^A$*. Moreover the compatibility tensor for this gauge is given by*

$$S(\xi_g, \eta_g) = -\frac{1}{2} L_{g*}\left[\theta(\xi_g), \theta(\eta_g)\right] \quad \text{for all } \xi_g, \eta_g \in T_g G \tag{4.4}$$

where θ *is the Maurer-Cartan form on* G *defined by* $\theta(\xi) := L_{g^{-1}*}\xi \in \mathfrak{g} := T_e G$ *for all* $\xi \in T_g G$*.*

Proof. The torsion of ∇ is given by

$$T\left(\tilde{A}, \tilde{B}\right) = \nabla_{\tilde{A}} \tilde{B} - \nabla_{\tilde{B}} \tilde{A} - \left[\tilde{A}, \tilde{B}\right] = -\widetilde{[A,B]}$$

or equivalently as

$$T(\xi_g, \eta_g) = -L_{g*}\left[\theta(\xi_g), \theta(\eta_g)\right] \text{ for all } \xi_g, \eta_g \in T_g G.$$

Eq. (4.4) follows from the above formula along with the result in Lemma 4.13.

If $\xi(t)$ is a path TG above $\sigma(t) \in G$ it may be written as $\xi(t) = L_{\sigma(t)*}\theta(\xi(t))$. Since $L_{\sigma(t)*}$ is parallel translation, it follows that

$$\frac{\nabla \xi(t)}{dt} = L_{\sigma(t)*} \frac{d}{dt} \theta(\xi(t)).$$

Thus $\xi(t) \in TG$ is parallel iff $\theta(\xi(t))$ is constant for all t. If σ is a general curve in

G, we may conclude

$$//\left(\sigma|_{[s,t]}\right) = L_{\sigma(t)*}L_{\sigma(s)^{-1}*} = L_{\sigma(t)\sigma(s)^{-1}*}$$

and therefore U^{∇} is given as in Eq. (4.2).

A curve $\sigma(t) \in G$ is a geodesic iff $\dot{\sigma}(t)$ is parallel iff $\theta(\dot{\sigma}(t)) = A$ for some $A \in \mathfrak{g}$. That is $\dot{\sigma}(t) = \tilde{A}(\sigma(t))$ with $\sigma(0) = k \in G$. The solution to this equation is $\sigma(t) = ke^{tA}$ and hence we have shown that $\exp_k^{\nabla}(k \cdot A) = ke^{A}$. So setting $g = ke^{A}$ and solving for A gives $A = \log(k^{-1}g)$ and the formula for ψ^{∇} in Eq. (4.3) now follows. ■

The last three results of this subsection show how the compatibility tensor allows us to compare two different parallelisms and two different logarithms on M.

Theorem 4.15 *Suppose that U and $\tilde{U}$ are two parallelisms on M and ψ is a logarithm on M, then*

$$U(m,n)\tilde{U}(m,n)^{-1} =_2 I + S_m^{\tilde{U},U}(\psi(m,n) \otimes (\cdot)). \tag{4.5}$$

Proof. By using charts it suffices to prove the theorem when $M = \mathbb{R}^d$. By Taylor's theorem (see Theorem 3.13),

$$\begin{aligned} U(m,n) &=_2 I + [(D_2U)(m,m)(n-m)] \text{ and} \\ \tilde{U}(m,n) &=_2 I + \left[\left(D_2\tilde{U}\right)(m,m)(n-m)\right] \end{aligned}$$

and therefore

$$\begin{aligned} &U(m,n)\tilde{U}(m,n)^{-1} \\ &\quad =_2 (I + [(D_2U)(m,m)(n-m)])\left(I - \left[\left(D_2\tilde{U}\right)(m,m)(n-m)\right]\right) &(4.6) \\ &\quad =_2 I + \left[\left((D_2U)(m,m) - \left(D_2\tilde{U}\right)(m,m)\right)(n-m)\right]. &(4.7) \end{aligned}$$

However, by Eq. (4.1) we have

$$S_m^{\tilde{U},U} = (D_2 U)(m,m) - \left(D_2 \tilde{U}\right)(m,m). \tag{4.8}$$

Using this identity back in Eq. (4.7) shows

$$U(m,n)\,\tilde{U}(m,n)^{-1} =_2 I + S_m^{\tilde{U},U}\left([n-m]_m \otimes (\cdot)\right)$$

from which Eq. (4.5) follows because $\psi(m,n) =_2 [n-m]_m$. ■

Corollary 4.16 *If $\mathcal{G} = (\psi, U)$ is a gauge on M, then*

$$\psi(n,\cdot)_{*m} =_2 U(n,m)\left[I + S_m^{\mathcal{G}}\left(\psi(m,n) \otimes (\cdot)\right)\right]. \tag{4.9}$$

In particular

$$\psi(y_t,\cdot)_{*y_s} \underset{2}{\approx} U(y_t,y_s)\left[I + S_{y_s}^{\mathcal{G}}\left(\psi(y_s,y_t) \otimes (\cdot)\right)\right]. \tag{4.10}$$

Proof. Theorem 4.15 implies

$$U(m,n)\,\psi(m,\cdot)_{*n}^{-1} =_2 I + S_m^{\mathcal{G}}\left(\psi(m,n) \otimes (\cdot)\right)$$

while Corollary 3.18 shows,

$$U(m,n)^{-1} =_2 U(n,m) \text{ and } \psi(m,\cdot)_{*n}^{-1} =_2 \psi(n,\cdot)_{*m}.$$

Eq. (4.9) now easily follows from the last two displayed equations. The second statement follows by patching. ■

Lastly we may use the compatibility tensor to compare two logarithms.

Proposition 4.17 *Suppose that ψ and $\tilde{\psi}$ are two logarithms on a manifold M.*

Then the compatibility tensor, $S^{\psi_*,\tilde{\psi}_*}$ *is symmetric and*

$$\psi(m,n) - \tilde{\psi}(m,n) =_3 \frac{1}{2} S_m^{\tilde{\psi}_*,\psi_*,} (\psi(m,n) \otimes \psi(m,n)). \tag{4.11}$$

Proof. As usual it suffices to prove this result when $M = \mathbb{R}^d$ in which case we omit the base points of tangent vectors. From Eq. (4.8) with $U(x,y) = \psi_x'(y)$ and $\tilde{U}(x,y) = \tilde{\psi}_x'(y)$, we see that

$$S_x^{\tilde{\psi}_*,\psi_*} = \psi_x''(x) - \tilde{\psi}_x''(x) \tag{4.12}$$

which is symmetric since mixed partial derivatives commute. Then by Taylor's theorem and Eq. (4.12),

$$\begin{aligned} \psi(x,y) - \tilde{\psi}(x,y) &= \frac{1}{2}\left[\psi_x''(x) - \tilde{\psi}_x''(x)\right](y-x)^{\otimes 2} + O\left(|y-x|^3\right) \\ &= \frac{1}{2} S_x^{\tilde{\psi}_*,\psi_*}\left(\psi(x,y)^{\otimes 2}\right) + O\left(|y-x|^3\right), \end{aligned}$$

wherein we have also used $(y-x)^{\otimes 2} =_3 \psi(x,y)^{\otimes 2}$. ■

Remark 4.18 *If* ∇ *is any covariant derivative on* TM, *then*

$$S_m^{\tilde{\psi}_*,\psi_*} = \left[\nabla d\left(\psi(m,\cdot) - \tilde{\psi}(m,\cdot)\right)\right]_m = \mathrm{Hess}_m^{\nabla}\left(\psi_m - \tilde{\psi}_m\right)$$

where $\mathrm{Hess}_m^{\nabla} f := [\nabla df]_m$. *By choosing* ∇ *to be Torsion free we again see that* $S_m^{\tilde{\psi}_*,\psi_*}$ *is a symmetric tensor.*

4.1.3 U − Controlled Rough Integration

Our next goal is to construct "the" integral, $\int \langle \boldsymbol{\alpha}, d\mathbf{y}\rangle$, where $\mathbf{y} \in CRP_{\mathbf{X}}(M)$ and $\boldsymbol{\alpha} \in CRP_y^U(M,V)$. We begin with the following proposition in the smooth category which is meant to motivate the definitions to come.

Proposition 4.19 *Assume (in this proposition only) that all functions,* $\mathbf{y}_s$, $\boldsymbol{\alpha}_s$, *and* x_s *are smooth,* $p = 1$, *and* $\omega(s,t) = |t-s|$. *Further assume* $\mathbf{y}$ *(respectively* $\boldsymbol{\alpha}$*) still satisfy the estimates of being controlled rough path (along* $\mathbf{y}$*). Then*

$$\int_s^t \alpha_\tau \dot{y}_\tau d\tau = \alpha_s \left[\psi(y_s, y_t) + S_{y_s}^{\mathcal{G}}\left(y_s^\dagger \otimes y_s^\dagger \mathbb{X}_{s,t}\right)\right] + \alpha_s^\dagger \left(I \otimes y_s^\dagger\right)\mathbb{X}_{s,t} + O\left((t-s)^3\right). \tag{4.13}$$

Proof. Our assumptions give,

$$\begin{aligned}
&\psi(y_s, y_t) = y_s^\dagger x_{s,t} + O\left((t-s)^2\right) \implies \dot{y}_s = y_s^\dagger \dot{x}_s, \\
&\alpha_t U(y_t, y_s) = \alpha_s + \alpha_s^\dagger x_{s,t} + O\left((t-s)^2\right), \\
&U(y_s, y_t)\, y_t^\dagger = y_s + O(t-s), \text{ and} \\
&\alpha_t^\dagger (I \otimes U(y_t, y_s)) = \alpha_s^\dagger + O(t-s).
\end{aligned}$$

We start with the identity,

$$\begin{aligned}
\int_s^t \alpha_\tau \dot{y}_\tau d\tau &= \int_s^t \alpha_\tau U(y_\tau, y_s)\, U(y_\tau, y_s)^{-1} \dot{y}_\tau d\tau \\
&= \int_s^t \left[\alpha_s + \alpha_s^\dagger x_{s,\tau} + O\left((\tau-s)^2\right)\right] U(y_\tau, y_s)^{-1} \dot{y}_\tau d\tau \\
&= \int_s^t \alpha_s U(y_\tau, y_s)^{-1} \dot{y}_\tau d\tau + \int_s^t \alpha_s^\dagger x_{s,\tau} U(y_\tau, y_s)^{-1} \dot{y}_\tau d\tau + O\left((t-s)^3\right) \\
&= \int_s^t \alpha_s U(y_s, y_\tau)\, \dot{y}_\tau d\tau + \int_s^t \alpha_s^\dagger x_{s,\tau} U(y_s, y_\tau)\, \dot{y}_\tau d\tau + O\left((t-s)^3\right).
\end{aligned} \tag{4.14}$$

$$=: A + B + O\left((t-s)^3\right) \tag{4.15}$$

wherein we have used Corollary 3.18 in order to show it is permissible to replace

$U(y_\tau, y_s)^{-1}$ by $U(y_s, y_\tau)$ above. The B term is then easily estimated as

$$
\begin{aligned}
B &= \int_s^t \alpha_s^\dagger x_{s,\tau} U(y_s, y_\tau)\, \dot{y}_\tau d\tau = \int_s^t \alpha_s^\dagger x_{s,\tau} U(y_s, y_\tau)\, y_\tau^\dagger \dot{x}_\tau d\tau \\
&= \int_s^t \alpha_s^\dagger x_{s,\tau} y_s^\dagger \dot{x}_\tau d\tau + O\left((t-s)^3\right) = \alpha_s^\dagger \left(I \otimes y_s^\dagger\right) \mathbb{X}_{s,t} + O\left((t-s)^3\right).
\end{aligned}
$$

The estimate of the A term to order $O\left((t-s)^3\right)$ requires more care. For this term we use

$$
\frac{d}{dt}\psi(y_s, y_t) = \psi(y_s, \cdot)_{*y_t}\, \dot{y}_t \implies \dot{y}_t = \psi(y_s, \cdot)_{*y_t}^{-1} \frac{d}{dt}\psi(y_s, y_t)
$$

and (from Theorem 4.15) that

$$
U(y_s, y_\tau)\, \psi(y_s, \cdot)_{*y_\tau}^{-1} =_2 I + S_{y_s}^{\mathcal{G}}\left(\psi(y_s, y_\tau) \otimes (\cdot)\right)
$$

in order to conclude,

$$
\begin{aligned}
A &:= \int_s^t \alpha_s U(y_s, y_\tau)\, \dot{y}_\tau d\tau = \int_s^t \alpha_s U(y_s, y_\tau)\, \psi(y_s, \cdot)_{*y_\tau}^{-1} \frac{d}{d\tau}\psi(y_s, y_\tau)\, d\tau \\
&= \int_s^t \alpha_s \left[I + S_{y_s}^{\mathcal{G}}\left(\psi(y_s, y_\tau) \otimes (\cdot)\right)\right] \frac{d}{d\tau}\psi(y_s, y_\tau)\, d\tau + O\left(|t-s|^3\right) \\
&= \alpha_s\left(\psi(y_s, y_t)\right) + \alpha_s \int_s^t S_{y_s}^{\mathcal{G}}\left(\psi(y_s, y_\tau) \otimes \frac{d}{d\tau}\psi(y_s, y_\tau)\right) d\tau + O\left(|t-s|^3\right) \\
&= \alpha_s\left(\psi(y_s, y_t)\right) + \alpha_s \int_s^t S_{y_s}^{\mathcal{G}}\left(y_s^\dagger x_{s,\tau} \otimes y_s^\dagger \dot{x}_\tau\right) d\tau + O\left(|t-s|^3\right) \\
&= \alpha_s\left(\psi(y_s, y_t)\right) + \alpha_s S_{y_s}^{\mathcal{G}}\left(y_s^\dagger \otimes y_s^\dagger \mathbb{X}_{s,t}\right) + O\left(|t-s|^3\right).
\end{aligned}
$$

Putting this all together proves Eq. (4.13). ■

The following definition is motivated by the right hand side of Eq. (4.13).

Definition 4.20 ($(\mathcal{G}, \mathbf{y})-$ integrator) *Given a gauge $\mathcal{G} := (\psi, U)$ and a path $\mathbf{y} \in CRP_{\mathbf{X}}(M)$, the $(\mathcal{G}, \mathbf{y})-$* ***integrator*** *is the increment process;*

$$
\mathbf{y}_{s,t}^{\mathcal{G}} := \left(\psi(y_s, y_t) + S_{y_s}^{\mathcal{G}}\left(y_s^{\dagger\otimes 2}\mathbb{X}_{s,t}\right), \left(I \otimes y_s^\dagger\right)\mathbb{X}_{s,t}\right) \in T_{y_s}M \times \left[W \otimes T_{y_s}M\right].
$$

Moreover, for $\boldsymbol{\alpha} \in CRP_y^U(M,V)$ *(see Notation 4.3) let*

$$\tilde{z}_{s,t} := \left\langle \boldsymbol{\alpha}_s, \mathbf{y}^{\mathcal{G}}_{s,t} \right\rangle = \alpha_s \left(\psi(y_s, y_t) + S^{\mathcal{G}}_{y_s} \left(y_s^{\dagger \otimes 2} \mathbb{X}_{s,t} \right) \right) + \alpha_s^{\dagger} \left(I \otimes y_s^{\dagger} \right) \mathbb{X}_{s,t} \qquad (4.16)$$

which is defined for $(s,t) \in \Delta_{[0,T]}$ *with* $|t-s| < \delta$ *for some sufficiently small* $\delta > 0$.

Recall that a two-parameter function $F : \Delta_{[0,T]} \longrightarrow V$ is an almost additive functional if there exists a $\theta > 1$, a control $\tilde{\omega}(s,t)$ and a $C > 0$ such that

$$|F_{s,u} - F_{s,t} - F_{t,u}| \leq C \tilde{\omega}(s,t)^{\theta} \qquad (4.17)$$

for all $0 \leq s \leq t \leq u \leq T$.

Theorem 4.21 *Let* $\mathcal{G} := (\psi, U)$ *be a gauge,* $\boldsymbol{\alpha} \in CRP_y^U(M,V)$, *and* $\tilde{z}_{s,t}$ *be as in Definition 4.20. Then there exists a unique* $\mathbf{z} = \left(z, z^{\dagger}\right) \in CRP_{\mathbf{X}}(V)$ *such that* $z_0 = 0$, $z_{s,t} \underset{3}{\approx} \tilde{z}_{s,t}$, *and* $z_s^{\dagger} = \alpha_s \circ y_s^{\dagger}$. *We denote this unique controlled rough path by* $\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}} \right\rangle$, *i.e.*

$$\int_s^t \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}} \right\rangle := \left[\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}} \right\rangle \right]^1_{s,t} \underset{3}{\approx} \left\langle \boldsymbol{\alpha}_s, \mathbf{y}^{\mathcal{G}}_{s,t} \right\rangle \text{ and } \left[\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}} \right\rangle \right]^{\dagger}_{s} = \alpha_s \circ y_s^{\dagger}.$$

Proof. By Theorem 4.26 below, $\tilde{z}_{s,t} := \left\langle \boldsymbol{\alpha}_s, \mathbf{y}^{\mathcal{G}}_{s,t} \right\rangle$ is an almost additive functional and therefore by Lyons [22, Theorem 3.3.1] there exists a unique additive functional $z_{s,t}$ such that $z_{s,t} \underset{3}{\approx} \tilde{z}_{s,t}$. Moreover,

$$z_{s,t} \underset{3}{\approx} \tilde{z}_{s,t} \underset{2}{\approx} \alpha_s \left(\psi(y_s, y_t) \right) \underset{2}{\approx} \alpha_s \left(y_s^{\dagger} x_{s,t} \right)$$

which shows that $\mathbf{z}_s := \left(z_s, \alpha_s \circ y_s^{\dagger} \right)$ is indeed a controlled rough path with values in V. ∎

Example 4.22 *In the case that* $U = U^{\psi}$ *so that*

$$\alpha_t \circ (\psi_{y_t})_{*y_s} - \alpha_s - \alpha_s^{\dagger}(x_{s,t} \otimes (\cdot)) \underset{2}{\approx} 0$$

we have that $\mathbf{y}^{\mathcal{G}}_{s,t} := \left(\psi(y_s, y_t), \left(I \otimes y_s^{\dagger}\right)\mathbb{X}_{s,t}\right)$ *and so*

$$\int_s^t \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}^{\psi}} \right\rangle \underset{3}{\approx} \alpha_s(\psi(y_s, y_t)) + \alpha_s^{\dagger}\left(I \otimes y_s^{\dagger}\right)\mathbb{X}_{s,t}.$$

Example 4.23 *If* $\mathcal{G}^{\nabla} = \left(\left(\exp^{\nabla}\right)^{-1}, U^{\nabla}\right)$*, then by Lemma 4.13, we have that*

$$\int_s^t \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}^{\nabla}} \right\rangle \underset{3}{\approx} \alpha_s\left(\exp_{y_s}^{-1}(y_t)\right) + \alpha_s^{\dagger}\left(I \otimes y_s^{\dagger}\right)\mathbb{X}_{s,t} + \alpha_s\left(\frac{1}{2}T_{y_s}^{\nabla} \circ y_s^{\dagger\otimes 2}\mathbb{X}_{s,t}\right).$$

If $\mathbf{f}, \boldsymbol{\alpha}$, and $\mathbf{f}\boldsymbol{\alpha} \in CRP_y^U\left(M, \tilde{V}\right)$ are as in Proposition 4.6, then the following expected associativity property holds.

Theorem 4.24 (Associativity Theorem I) *Let us continue the notation in Theorem 4.21. If* $\mathbf{f}$ *and* $\mathbf{f}\boldsymbol{\alpha} := \left(f_s\alpha_s, f_s^{\dagger}(I \otimes \alpha_s) + f_s\alpha_s^{\dagger}\right)$ *are as in Proposition 4.6 and* $\mathbf{z} = \left(z, z^{\dagger}\right) = \int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}} \right\rangle$*, then*

$$\int \langle \mathbf{f}, d\mathbf{z} \rangle = \int \left\langle \mathbf{f}\boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}} \right\rangle,$$

or in other words,

$$\int \left\langle \mathbf{f}, d\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}} \right\rangle \right\rangle = \int \left\langle \mathbf{f}\boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}} \right\rangle.$$

Proof. We have the approximations

$$\begin{aligned}\left[\int \left\langle \mathbf{f}\boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}}\right\rangle\right]^1_{s,t} &\underset{3}{\approx} f_s\alpha_s\left(\psi\left(y_s,y_t\right)+\mathcal{S}^{\mathcal{G}}_{y_s}\left(y_s^{\dagger\otimes 2}\mathbb{X}_{s,t}\right)\right)\\ &\quad+\left[\left(f_s^{\dagger}\left(I\otimes\alpha_s\right)+f_s\alpha_s^{\dagger}\right)\right]\left(I\otimes y_s^{\dagger}\right)\mathbb{X}_{s,t}\\ &= f_s\left(\alpha_s\left(\psi\left(y_s,y_t\right)+\mathcal{S}^{\mathcal{G}}_{y_s}\left(y_s^{\dagger\otimes 2}\mathbb{X}_{s,t}\right)\right)+\alpha_s^{\dagger}\left(I\otimes y_s^{\dagger}\right)\mathbb{X}_{s,t}\right)\\ &\quad+f_s^{\dagger}\left(I\otimes\alpha_s y_s^{\dagger}\right)\mathbb{X}_{s,t}\\ &\underset{3}{\approx} f_s\left(z_{s,t}\right)+f_s^{\dagger}\left(I\otimes z_s^{\dagger}\right)\mathbb{X}_{s,t}\\ &\underset{3}{\approx}\left[\int\left\langle \mathbf{f}, d\mathbf{z}\right\rangle\right]^1_{s,t}.\end{aligned}$$

As the first and last terms of this equation are additive functionals, they must be equal.

Secondly

$$\left[\int \left\langle \mathbf{f}\boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}}\right\rangle\right]^{\dagger}_{s} = f_s\alpha_s\left(y_s^{\dagger}\right) = f_s z_s^{\dagger} = \left[\int\left\langle \mathbf{f}, d\mathbf{z}\right\rangle\right]^{\dagger}_{s}.$$

Thus, the two controlled rough paths are equal. ■

Remark 4.25 *The $(\mathcal{G},\mathbf{y})-$ integrator $\mathbf{y}^{\mathcal{G}}_{s,t}$ is helpful in easing notation so that the integral is simply written $\int_s^t \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}}\right\rangle$. A more honest notation for this integral would be*

$$\int_s^t \left\langle\left(\alpha,\alpha^{\dagger}\right), d\left(y^{\psi},\mathbb{X}\right)\right\rangle_{\mathcal{S}^{\mathcal{G}}_{y^{\dagger}}}$$

where $\mathcal{S}^{\mathcal{G}}_{y^{\dagger}}(s)$ is the block matrix defined by

$$\mathcal{S}^{\mathcal{G}}_{y^{\dagger}}(s) := \begin{pmatrix} I & S^{\mathcal{G}}_{y_s}\circ\left(y_s^{\dagger}\right)^{\otimes 2}\\ 0 & I\otimes y_s^{\dagger}\end{pmatrix}$$

and $\langle\cdot,\cdot\rangle_{\mathcal{S}^{\mathcal{G}}_{y^{\dagger}}}$ is the “inner product” given by the matrix $\mathcal{S}^{\mathcal{G}}_{y^{\dagger}}$. When s is close to t, *we have*

$$\int_s^t \left\langle \left(\alpha, \alpha^\dagger\right), d\left(y^\psi, \mathbb{X}\right)\right\rangle_{\mathcal{S}^{\mathcal{G}}_{y^\dagger}} \approx \left(\alpha_s, \alpha_s^\dagger\right) \begin{pmatrix} I & S^{\mathcal{G}}_{y_s} \circ \left(y_s^\dagger\right)^{\otimes 2} \\ 0 & I \otimes y_s^\dagger \end{pmatrix} \begin{pmatrix} y^\psi_{s,t} \\ \mathbb{X}_{s,t} \end{pmatrix}$$

$$= \alpha_s \left(\psi\left(y_s, y_t\right) + S^{\mathcal{G}}_{y_s}\left(y_s^{\dagger \otimes 2}\mathbb{X}_{s,t}\right)\right) + \alpha_s^\dagger \left(I \otimes y_s^\dagger\right)\mathbb{X}_{s,t}.$$

4.1.4 Almost Additivity Result

The following theorem was the key ingredient in the proof of Theorem 4.21 on the existence of rough path integration in the manifold setting.

Theorem 4.26 (Almost Additivity) *If $\mathcal{G} := (\psi, U)$ is a gauge and $\boldsymbol{\alpha}$ is an element of $CRP_y^U(M, V)$, then $\tilde{z}_{s,t} \in V$ defined as in Definition 4.20 is an almost additive functional.*

The proof of Theorem 4.26 will be given after Corollary 4.29 which states that logarithms are "almost additive." We first need a couple of lemmas. Recall from Definition 3.4 that $\psi_x = \psi(x, \cdot)$.

Lemma 4.27 *If $U, \tilde{U}$ are two parallelisms on M, then*

$$S^{\tilde{U},U}_{y_t} \circ U\left(y_t, y_s\right)^{\otimes 2} \underset{1}{\approx} U\left(y_t, y_s\right) \circ S^{\tilde{U},U}_{y_s}.$$

Proof. By the usual patching arguments it suffices to prove this lemma for $M = \mathbb{R}^d$. In the Euclidean space setting the identity is trivial to prove since $U(n, m) =_1 I$ and $S^{\tilde{U},U}_n =_1 S^{\tilde{U},U}_m$. ■

Lemma 4.28 *Let K be a compact, convex set in $\mathbb{R}^d$. If ψ is a logarithm with domain $\mathcal{D}$ and $K \times K \subseteq \mathcal{D}$, then there exists a C_K such that*

$$\left|\psi_y'(x)\psi(x,y) + \psi(y,z) - \psi_y'(x)\psi(x,z)\right|$$
$$\leq C_K \max\left\{\left|\psi(x,y)\right|, \left|\psi(y,z)\right|, \left|\psi(x,z)\right|\right\}^3$$

for all $x, y, z \in K$*.*

Proof. We will use the notation $|x,y,z| := \max\{|y-x|, |z-y|, |z-x|\}$ and write $f(x,y,z) =_k g(x,y,z)$ iff $f(x,y,z) = g(x,y,z) + O\left(|x,y,z|^k\right)$. Since ψ is zero on the diagonal and $\psi'_y(y) = id$ for all y, it follow from Taylor's theorem (or see Theorem 3.13) that

$$\begin{aligned}\psi'_y(x) &=_2 id + \psi''_y(y)(x-y) \text{ and}\\ \psi(x,y) &=_3 (y-x) + \frac{1}{2}\psi''_x(x)(y-x)^{\otimes 2}\\ &=_3 (y-x) + \frac{1}{2}\psi''_y(y)(y-x)^{\otimes 2}. \end{aligned} \tag{4.18}$$

from these approximations we learn,

$$\psi(x,y) - \psi(x,z) =_3 y - z + \frac{1}{2}\psi''_y(y)\left[(y-x)^{\otimes 2} - (z-x)^{\otimes 2}\right]$$

and

$$\begin{aligned}&\psi'_y(x)\psi(x,y) - \psi'_y(x)\psi(x,z)\\ &\qquad =_3 \left[id + \psi''_y(y)(x-y)\otimes(\cdot)\right](\psi(x,y) - \psi(x,z))\\ &\qquad =_3 y - z + \frac{1}{2}\psi''_y(y)\left[(y-x)^{\otimes 2} - (z-x)^{\otimes 2}\right]\\ &\qquad\quad + \psi''_y(y)\left[(x-y)\otimes(y-z)\right].\end{aligned}$$

As simple calculation now shows, with $a = y - x$ and $b = y - z$, that

$$\frac{1}{2}\left[(y-x)^{\otimes 2} - (z-x)^{\otimes 2}\right] + (x-y)\otimes(y-z) = -\frac{1}{2}\left[b^{\otimes 2} + b\otimes a - a\otimes b\right].$$

Since $\psi''_y(y)\, a\otimes b = \psi''_y(y)\, b\otimes a$ (mixed partial derivatives commute), the last two displayed equations give

$$\psi_y'(x)\psi(x,y) - \psi_y'(x)\psi(x,z) =_3 y - z - \frac{1}{2}\psi_y''(y)\,b^{\otimes 2}$$
$$= -\left[(z-y) + \frac{1}{2}\psi_y''(y)(z-y)^{\otimes 2}\right] =_3 -\psi(y,z).$$

The bounds derived above are uniform over a compact set K. Because of Eq. (4.18), we may replace $O\left(|x,y,z|^3\right)$ with $O\left(\max\{|\psi(x,y)|, |\psi(y,z)|, |\psi(x,z)|\}^3\right)$. ■

Corollary 4.29 *If $\left(y_s, y_s^\dagger\right)$ is a controlled rough path and ψ is a logarithm, there exists C_ψ, $\delta_\psi > 0$ such that if $0 \le s \le t \le u \le T$ and $u - s \le \delta_\psi$, then*

$$\left|\psi(y_t, y_u) - \psi(y_t, \cdot)_{*y_s}[\psi(y_s, y_u) - \psi(y_s, y_t)]\right|_g \le C_\psi \omega(s,u)^{3/p}$$

Proof. Around every point in $y([0,T])$, using our usual techniques, we can find a neighborhood $\mathcal{W}$ such that $\mathcal{W} \times \mathcal{W} \subseteq \mathcal{D}$ and maps to a convex open set by a chart. We can then use Remark 3.41 with a slightly modified version (which includes three variables instead of two) of Lemma 3.40 to create a global estimate. We can then choose a δ such that $u - s \le \delta$ forces the path to lie within one of these sets $\mathcal{W}$. Therefore, it suffices to prove the estimate locally. However, we can push forward the metric and ψ to a convex set on Euclidean space. The rest follows from the Lemma 4.28 and the fact that $|\psi(y_s, y_t)| \le C\omega(s,t)^{1/p}$ for all $|t-s| \le \delta$ for some $C < \infty$ and $\delta > 0$. ■

4.1.5 Proof of Theorem 4.26

Proof of Theorem 4.26. Let $0 \le s \le t \le u \le T$. Throughout this proof, we will use the notation $\underset{i}{\approx}$ with respect to the times s and u. To prove the statement, we need to show $\tilde{z}_{s,t} + \tilde{z}_{t,u} \underset{3}{\approx} \tilde{z}_{s,u}$. We begin by working on the three terms for $\tilde{z}_{t,u}$

in the following equation

$$\tilde{z}_{t,u} = \alpha_t\left(\psi\left(y_t, y_u\right)\right) + \alpha_t^{\dagger}\left(I \otimes y_t^{\dagger}\right)\mathbb{X}_{t,u} + \alpha_t\left(S_{y_t}^{\mathcal{G}} \circ y_t^{\dagger\otimes 2}\mathbb{X}_{t,u}\right). \tag{4.19}$$

Using Corollary 4.29 followed by Corollary 4.16 we find

$$\begin{aligned}
\alpha_t\left(\psi\left(y_t, y_u\right)\right) &\underset{3}{\approx} \alpha_t \psi\left(y_t, \cdot\right)_{*y_s}\left[\psi\left(y_s, y_u\right) - \psi\left(y_s, y_t\right)\right] \\
&\underset{3}{\approx} \alpha_t U\left(y_t, y_s\right)\left[I + S_{y_s}^{\mathcal{G}}\left(\psi\left(y_s, y_t\right) \otimes (\cdot)\right)\right]\left[\psi\left(y_s, y_u\right) - \psi\left(y_s, y_t\right)\right] \\
&\underset{3}{\approx} \left[\alpha_s + \alpha_s^{\dagger} x_{s,t} \otimes (\cdot)\right]\left[I + S_{y_s}^{\mathcal{G}}\left(\psi\left(y_s, y_t\right) \otimes (\cdot)\right)\right]\left[\psi\left(y_s, y_u\right) - \psi\left(y_s, y_t\right)\right] \\
&\underset{3}{\approx} \alpha_s\left[I + S_{y_s}^{\mathcal{G}}\left(\psi\left(y_s, y_t\right) \otimes (\cdot)\right)\right]\left[\psi\left(y_s, y_u\right) - \psi\left(y_s, y_t\right)\right] \\
&\quad + \alpha_s^{\dagger} x_{s,t} \otimes \left[\psi\left(y_s, y_u\right) - \psi\left(y_s, y_t\right)\right].
\end{aligned}$$

Combining this equation with the estimates

$$\psi\left(y_s, y_t\right) \underset{2}{\approx} y_s^{\dagger} x_{s,t} \text{ and } \psi\left(y_s, y_u\right) - \psi\left(y_s, y_t\right) \underset{2}{\approx} y_s^{\dagger}\left[x_{s,u} - x_{s,t}\right] = y_s^{\dagger} x_{t,u},$$

then shows,

$$\alpha_t\left(\psi\left(y_t, y_u\right)\right) \underset{3}{\approx} \alpha_s\left[\psi\left(y_s, y_u\right) - \psi\left(y_s, y_t\right)\right] + \alpha_s\left(y_s^{\dagger}\right)^{\otimes 2} x_{s,t} \otimes x_{t,u} + \alpha_s^{\dagger}\left(I \otimes y_s^{\dagger}\right) x_{s,t} \otimes x_{t,u}. \tag{4.20}$$

By the definitions of $CRP_{\mathbf{X}}(M)$ and $CRP_y^U(M, V)$ we have

$$\begin{aligned}
\alpha_t^{\dagger}\left(I \otimes y_t^{\dagger}\right)\mathbb{X}_{t,u} &\underset{3}{\approx} \alpha_t^{\dagger}\left(I \otimes U\left(y_t, y_s\right) y_s^{\dagger}\right)\mathbb{X}_{t,u} \\
&= \alpha_t^{\dagger}\left(I \otimes U\left(y_t, y_s\right)\right)\left(I \otimes y_s^{\dagger}\right)\mathbb{X}_{t,u} \underset{3}{\approx} \alpha_s^{\dagger}\left(I \otimes y_s^{\dagger}\right)\mathbb{X}_{t,u}.
\end{aligned} \tag{4.21}$$

Lastly by the definitions of $CRP_{\mathbf{X}}(M)$ and $CRP_y^U(M,V)$ along with Lemma 4.27 with $\tilde{U}(m,n) = (\psi_m)_{*n}$, we have and

$$\alpha_t\left(S_{y_t}^{\mathcal{G}} \circ y_t^{\dagger\otimes 2}\mathbb{X}_{t,u}\right) \underset{3}{\approx} \alpha_t\left(S_{y_t}^{\mathcal{G}} \circ U(y_t,y_s)^{\otimes 2} \circ y_s^{\dagger\otimes 2}\mathbb{X}_{t,u}\right)$$
$$\underset{3}{\approx} \alpha_t\left(U(y_t,y_s) \circ S_{y_s}^{\mathcal{G}} \circ y_s^{\dagger\otimes 2}\mathbb{X}_{t,u}\right) \underset{3}{\approx} \alpha_s\left(S_{y_s}^{\mathcal{G}} \circ y_s^{\dagger\otimes 2}\mathbb{X}_{t,u}\right). \quad (4.22)$$

Adding together Eqs. (4.20) – (4.22) to

$$\tilde{z}_{s,t} = \alpha_s(\psi(y_s,y_t)) + \alpha_s^\dagger\left(I \otimes y_s^\dagger\right)\mathbb{X}_{s,t} + \alpha_s\left(S_{y_s}^{\mathcal{G}} \circ y_s^{\dagger\otimes 2}\mathbb{X}_{s,t}\right)$$

while making use Chen's identity in Eq. (2.6) shows

$$\tilde{z}_{s,t} + \tilde{z}_{t,u} \underset{3}{\approx} \alpha_s(\psi(y_s,y_u)) + \alpha_s^\dagger\left(I \otimes y_s^\dagger\right)\mathbb{X}_{s,u} + \alpha_s\left(S_{y_s}^{\mathcal{G}} \circ y_s^{\dagger\otimes 2}\mathbb{X}_{s,u}\right) = \tilde{z}_{s,u}.$$

■

4.1.6 A Map from $CRP_y^U(M,V)$ to $CRP_y^{\tilde{U}}(M,V)$

Suppose that $\mathcal{G} = (\psi, U)$ and $\tilde{\mathcal{G}} = \left(\tilde{\psi}, \tilde{U}\right)$ are two gauges on M. Generally, if $\boldsymbol{\alpha} := \left(\alpha, \alpha^\dagger\right) \in CRP_y^U(M,V)$, there is no reason to expect it also to be an element of $CRP_y^{\tilde{U}}(M,V)$. However, the main theorem [Theorem 4.32] of this section shows there is a "natural" bijection between $CRP_y^U(M,V)$ and $CRP_y^{\tilde{U}}(M,V)$ which preserves the notions of integration. The following proposition is needed in the proof of Theorem 4.32 and moreover motivates the statement of the theorem.

Proposition 4.30 *If $\mathcal{G} = (\psi, U)$ and $\tilde{\mathcal{G}} = \left(\tilde{\psi}, \tilde{U}\right)$ are two gauges on M and $\mathbf{y} = \left(y, y^\dagger\right) \in CRP_{\mathbf{X}}(M)$, then*

$$\mathbf{y}_{s,t}^{\mathcal{G}} \underset{3}{\approx} \mathbf{y}_{s,t}^{\tilde{\mathcal{G}}} + \left(S_{y_s}^{\tilde{U},U}\left(\left(y_s^\dagger\right)^{\otimes 2}\mathbb{X}_{s,t}\right), 0\right), \quad (4.23)$$

where $\mathbf{y}^{\mathcal{G}}_{s,t}$ *and* $\mathbf{y}^{\tilde{\mathcal{G}}}_{s,t}$ *are as in Definition 4.20.*

Proof. From Proposition 4.17,

$$\begin{aligned}\psi(y_s, y_t) - \tilde{\psi}(y_s, y_t) &\underset{3}{\approx} \frac{1}{2} S^{\tilde{\psi}_*, \psi_*}_{y_s} \left(\psi(y_s, y_t) \otimes \psi(y_s, y_t)\right) \\ &\underset{3}{\approx} \frac{1}{2} S^{\tilde{\psi}_*, \psi_*}_{y_s} \left(\left(y_s^\dagger \otimes y_s^\dagger\right)[x_{s,t} \otimes x_{s,t}]\right) = S^{\tilde{\psi}_*, \psi_*}_{y_s} \left(\left(y_s^\dagger\right)^{\otimes 2} \mathbb{X}_{s,t}\right)\end{aligned}$$

wherein we have used $S^{\tilde{\psi}_*, \psi_*}_{y_s}$ is symmetric and $\mathbf{X} = (x, \mathbb{X})$ is a weak-geometric rough path for the last equality. Making use of this estimate it now follows that

$$\begin{aligned}\mathbf{y}^{\mathcal{G}}_{s,t} - \mathbf{y}^{\tilde{\mathcal{G}}}_{s,t} &= \left(\psi(y_s, y_t) - \tilde{\psi}(y_s, y_t) + \left(S^{\mathcal{G}}_{y_s} - S^{\tilde{\mathcal{G}}}_{y_s}\right)\left(\left(y_s^\dagger\right)^{\otimes 2} \mathbb{X}_{s,t}\right), 0\right) \\ &\underset{3}{\approx} \left(\left(S^{\tilde{\psi}_*, \psi_*}_{y_s} + S^{\mathcal{G}}_{y_s} - S^{\tilde{\mathcal{G}}}_{y_s}\right)\left(\left(y_s^\dagger\right)^{\otimes 2} \mathbb{X}_{s,t}\right), 0\right). \end{aligned} \tag{4.24}$$

On the other hand, by Lemma 4.11,

$$\begin{aligned}S^{\tilde{\psi}_*, \psi_*} &= S^{\tilde{\psi}_*, \tilde{U}} + S^{\tilde{U}, \psi_*} = S^{\tilde{\psi}_*, \tilde{U}} + S^{\tilde{U}, U} + S^{U, \psi_*} \\ &= S^{\tilde{\mathcal{G}}} - S^{\mathcal{G}} + S^{\tilde{U}, U}\end{aligned}$$

which combined with Eq. (4.24) gives Eq. (4.23). ■

Corollary 4.31 *The integral,* $\int \langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}} \rangle$ *only depends on the choice of parallelism* U *and not on the logarithm used to make the gauge* $\mathcal{G} = (\psi, U)$*.*

Proof. From Proposition 4.30 with $U = \tilde{U}$, it follows that

$$\int_s^t \langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}} \rangle \underset{3}{\approx} \langle \boldsymbol{\alpha}_s, \mathbf{y}^{\mathcal{G}}_{s,t} \rangle \underset{3}{\approx} \left\langle \boldsymbol{\alpha}_s, \mathbf{y}^{\tilde{\mathcal{G}}}_{s,t} \right\rangle \underset{3}{\approx} \int_s^t \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\tilde{\mathcal{G}}} \right\rangle$$

from which it follows that the two additive functionals, $\int \langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}} \rangle$ and $\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\tilde{\mathcal{G}}} \right\rangle$, must be equal. ■

If $\boldsymbol{\alpha} = \left(\alpha, \alpha^{\dagger}\right) \in CRP_{y}^{U}\left(M, V\right)$ and $U \neq \tilde{U}$, then

$$\left\langle \boldsymbol{\alpha}_{s}, \mathbf{y}_{s,t}^{\mathcal{G}} \right\rangle \underset{3}{\approx} \left\langle \boldsymbol{\alpha}_{s}, \mathbf{y}_{s,t}^{\tilde{\mathcal{G}}} + \left(S_{y_s}^{\tilde{U},U} \left(\left(y_s^{\dagger}\right)^{\otimes 2} \mathbb{X}_{s,t} \right), 0 \right) \right\rangle = \left\langle \tilde{\boldsymbol{\alpha}}_{s}, \mathbf{y}_{s,t}^{\tilde{\mathcal{G}}} \right\rangle \tag{4.25}$$

where $\tilde{\boldsymbol{\alpha}}_s$ is defined in Eq. (4.26) below. The identity in Eq. (4.25) suggests the following theorem.

Theorem 4.32 *The map*

$$\boldsymbol{\alpha}_s = \left(\alpha_s, \alpha_s^{\dagger}\right) \longrightarrow \tilde{\boldsymbol{\alpha}}_s := \left(\tilde{\alpha}_s, \tilde{\alpha}_s^{\dagger}\right) := \left(\alpha_s, \alpha_s^{\dagger} + \alpha_s S_{y_s}^{\tilde{U},U} y_s^{\dagger} \otimes I\right) \tag{4.26}$$

is a bijection from $CRP_{y}^{U}\left(M, V\right)$ *to* $CRP_{y}^{\tilde{U}}\left(M, V\right)$ *such that*

$$\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}} \right\rangle = \int \left\langle \tilde{\boldsymbol{\alpha}}, d\mathbf{y}^{\tilde{\mathcal{G}}} \right\rangle . \tag{4.27}$$

Proof. The only thing that is really left to prove here is the assertion that $\tilde{\boldsymbol{\alpha}} \in CRP_{y}^{\tilde{U}}\left(M, V\right)$. First we prove that item 3 of Definition 4.1 holds for $\tilde{\boldsymbol{\alpha}}$.

From Theorem 4.15 with $m = y_s$ and $n = y_t$, we find

$$U\left(y_s, y_t\right) \tilde{U}\left(y_s, y_t\right)^{-1} \underset{2}{\approx} I + S_{y_s}^{\tilde{U},U}\left(\psi\left(y_s, y_t\right) \otimes (\cdot)\right)$$

and then combining this result with Corollary 3.18 shows

$$\tilde{U}\left(y_t, y_s\right) \underset{2}{\approx} U\left(y_t, y_s\right) \left[I + S_{y_s}^{\tilde{U},U}\left(\psi\left(y_s, y_t\right) \otimes (\cdot)\right) \right] . \tag{4.28}$$

From this equation and the fact that $\boldsymbol{\alpha}\in CRP_y^U(M,V)$, we learn

$$\begin{aligned}\alpha_t\tilde{U}(y_t,y_s)-\alpha_s &\underset{2}{\approx} \alpha_t U(y_t,y_s)\left[I+S_{y_s}^{\tilde{U},U}(\psi(y_s,y_t)\otimes(\cdot))\right]-\alpha_s \\ &\underset{2}{\approx} \left(\alpha_s+\alpha_s^\dagger x_{s,t}\right)\left[I+S_{y_s}^{\tilde{U},U}(\psi(y_s,y_t)\otimes(\cdot))\right]-\alpha_s \\ &\underset{2}{\approx} \alpha_s^\dagger x_{s,t}+\alpha_s S_{y_s}^{\tilde{U},U}\left(y_s^\dagger x_{s,t}\otimes(\cdot)\right)=\tilde{\alpha}_s^\dagger(x_{s,t}\otimes(\cdot))\end{aligned}$$

as desired.

Next we check item 4 of Definition 4.1. We are given

$$\begin{aligned}0 &\underset{1}{\approx} \alpha_t^\dagger\circ(I\otimes U(y_t,y_s))-\alpha_s^\dagger \\ &= \tilde{\alpha}_t^\dagger\circ\left(I\otimes\tilde{U}(y_t,y_s)\right)-\tilde{\alpha}_s^\dagger \\ &\quad -\alpha_t\circ S_{y_t}^{\tilde{U},U}\circ\left(y_t^\dagger\otimes U(y_t,y_s)\right)+\alpha_s\circ S_{y_s}^{\tilde{U},U}\circ\left(y_s^\dagger\otimes I\right)\end{aligned}$$

wherein we have used that $U(y_s,y_t)\underset{1}{\approx}\tilde{U}(y_s,y_t)$ (for example, see Eq. (4.28)). We therefore must show the last line is approximately 0. However, by Lemma 4.27, we have $S_{y_t}^{\tilde{U},U}\circ U(y_t,y_s)^{\otimes 2}\underset{1}{\approx}U(y_t,y_s)\circ S_{y_t}^{\tilde{U},U}$. Thus

$$\begin{aligned}&\alpha_t\circ S_{y_t}^{\tilde{U},U}\circ\left(y_t^\dagger\otimes U(y_t,y_s)\right)-\alpha_s\circ S_{y_s}^{\tilde{U},U}\circ\left(y_s^\dagger\otimes I\right) \\ &\quad\underset{1}{\approx} \alpha_t\circ S_{y_t}^{\tilde{U},U}\circ\left(U(y_t,y_s)\,y_s^\dagger\otimes U(y_t,y_s)\right)-\alpha_s\circ S_{y_s}^{\tilde{U},U}\circ\left(y_s^\dagger\otimes I\right) \\ &\quad\underset{1}{\approx} [\alpha_t\circ U(y_t,y_s)-\alpha_s]\left[S_{y_s}^{\tilde{U},U}\circ\left(y_s^\dagger\otimes I\right)\right]\underset{1}{\approx}0.\end{aligned}$$

■

4.2 Integrating One-Forms Along a CRP

Lemma 4.33 *Let V be a Banach space and U be a parallelism on M. If $\alpha \in \Omega^1(M,V)$ is a V – valued smooth one-form on M, then*

$$\alpha_n \circ U(n,m) - \alpha_m =_2 \nabla^U_{\psi(m,n)}\alpha$$

where ∇^U is the covariant derivative defined in Remark 3.9.

Proof. By definition, $\nabla^U_{v_m}\alpha$ is determined by the product rule,

$$v_m[\alpha(Y)] = \left(\nabla^U_{v_m}\alpha\right)(Y(m)) + \alpha_m\left(\nabla^U_{v_m}Y\right). \tag{4.29}$$

However, we may also write

$$\begin{aligned} v_m[\alpha(Y)] &= \frac{d}{dt}|_0\alpha\left(U(m,\sigma_t)^{-1}U(m,\sigma_t)Y(\sigma_t)\right) \\ &= \frac{d}{dt}|_0\alpha\left(U(m,\sigma_t)^{-1}Y(m)\right) + \alpha_m\left(\nabla^U_{v_m}Y\right) \end{aligned}$$

where σ_t is such that $\dot{\sigma}_0 = v_m$. Combining the last two facts shows that

$$\nabla^U_{v_m}\alpha = \frac{d}{dt}|_0\left[\alpha \circ U(m,\sigma_t)^{-1}\right]. \tag{4.30}$$

By Corollary 3.18, we may alternatively write Eq. (4.30) as

$$\nabla^U_{v_m}\alpha = \frac{d}{dt}|_0\left[\alpha \circ U(\sigma_t,m)\right].$$

To prove the lemma, we note this is a local result and we therefore may assume $M=\mathbb{R}^d$. Then by Taylor's theorem,

$$\begin{aligned}\alpha_n\circ U(n,m) &= \alpha_m + D\left[\alpha_{(\cdot)}\circ U(\cdot,m)\right](m)(n-m)+O\left(|n-m|^2\right)\\ &= \alpha_m+\nabla^U_{(n-m)_m}\alpha+O\left(|n-m|^2\right)\\ &= \alpha_m+\nabla^U_{\psi(m,n)}\alpha+O\left(|\psi(m,n)|^2\right).\end{aligned}$$

■

Suppose that $\alpha\in\Omega^1(M,V)$ is a V – valued one-form and U is a parallelism on M. We wish to take $\alpha^U_s=\alpha_{y_s}:=\alpha|_{T_{y_s}M}$. Making use of Lemma 4.33, we find

$$\alpha^U_t\circ U(y_t,y_s)-\alpha_s \underset{2}{\approx} \nabla^U_{\psi(y_s,y_t)}\alpha \underset{2}{\approx} \nabla^U_{y_s^\dagger x_{s,t}}\alpha \tag{4.31}$$

and this computation suggests the following proposition.

Proposition 4.34 *Suppose that $\alpha\in\Omega^1(M,V)$ is a V – valued one-form and U is a parallelism on M, then*

$$\boldsymbol{\alpha}^{(y,U)}_s:=\left(\alpha_{y_s},\alpha^{\dagger(y,U)}_s\right):=\left(\alpha|_{T_{y_s}M},\nabla^U_{y_s^\dagger(\cdot)}\alpha\right)\in CRP^U_y(M,V).$$

Proof. In light of how $\boldsymbol{\alpha}^{y,U}_s$ has been defined and of Eq. (4.31), we need only verify Item 4 in Definition 4.1 is satisfied. To this end, suppose that $w\in W$, then

$$\begin{aligned}\alpha^{\dagger(y,U)}_t\circ(I\otimes U(y_t,y_s))(w\otimes(\cdot)) &= \left(\nabla^U_{y_t^\dagger w}\alpha\right)U(y_t,y_s)\\ &\underset{1}{\approx}\left(\nabla^U_{U(y_t,y_s)y_s^\dagger w}\alpha\right)U(y_t,y_s)\end{aligned} \tag{4.32}$$

wherein we have used Inequality (3.12) along with Corollary 3.18 in the last line. Since for $v_m\in T_mM$ the function $F(n):=\left(\nabla^U_{U(n,m)v_m}\alpha\right)U(n,m)\in L(T_mM,V)$ is smooth, it follows by Taylor's theorem that $F(n)=_1 F(m)$ which translates to

$$\left(\nabla^U_{U(n,m)v_m}\alpha\right)U(n,m)=_1\nabla^U_{v_m}\alpha.$$

Taking $m = y_s$, $n = y_t$, and $v_m = y_s^\dagger w$ in this estimates shows

$$\left(\nabla^U_{U(y_t,y_s)y_s^\dagger w}\alpha\right) U(y_t, y_s) \underset{1}{\approx} \nabla^U_{y_s^\dagger w}\alpha$$

which combined with Eq. (4.32) completes the proof. ■

Theorem 4.35 *If* $\alpha \in \Omega^1(M,V)$ *is a* V *– valued one-form, then the integral* $\int \left\langle \boldsymbol{\alpha}^{(y,U)}, d\mathbf{y}^{\mathcal{G}}\right\rangle$ *is independent of any choice of gauge* $\mathcal{G} = (\psi, U)$ *on* M*. In the future we denote this integral more simply as* $\int \langle \boldsymbol{\alpha}, d\mathbf{y}\rangle$ *.*

Proof. Suppose that U and $\tilde{U}$ are two parallelisms. According to Theorem 4.32 it suffices to show

$$\alpha_s^{\dagger\left(y,\tilde{U}\right)} = \alpha_s^{\dagger(y,U)} + \alpha_{y_s} S_{y_s}^{\tilde{U},U}\left[y_s^\dagger \otimes I\right]. \tag{4.33}$$

We will see that Eq. (4.33) is a fairly direct consequence of Example 4.10 which, when translated to the language of forms (see Eq. (4.29)), states

$$\nabla_{v_m}\alpha = \tilde{\nabla}_{v_m}\alpha - \alpha \circ S_m^{\tilde{U},U}\left(v_m \otimes (\cdot)\right). \tag{4.34}$$

So for $w \in W$, we have

$$\begin{aligned}\alpha_s^{\dagger\left(y,\tilde{U}\right)} w = \tilde{\nabla}_{y_s^\dagger w}\alpha &= \nabla_{y_s^\dagger w}\alpha + \alpha_{y_s} S_m^{\tilde{U},U}\left(y_s^\dagger w \otimes (\cdot)\right) \\ &= \alpha_s^{\dagger(y,U)} w + \alpha_{y_s} S_m^{\tilde{U},U}\left(y_s^\dagger w \otimes (\cdot)\right)\end{aligned}$$

which proves Eq. (4.33). ■

Let us now record a number of possible different expressions for computing $\int_s^t \alpha(d\mathbf{y})$ depending on the choice of gauge we make.

Proposition 4.36 *Let* $\mathcal{G} = (\psi, U)$ *be a gauge. There exists a* $\delta > 0$ *such that for*

$s<t$ and $t-s<\delta$, the approximation

$$\left[\int \alpha\left(d\mathbf{y}\right)\right]_{s,t}^{1} \underset{3}{\approx} \alpha_{y_s}\left(\psi\left(y_s, y_t\right)\right) + \left[\left(\nabla_{(\cdot)}^{U}\alpha\right)_{y_s} + \alpha_{y_s} \circ S_{y_s}^{\mathcal{G}}\right] \circ y_s^{\dagger\otimes 2}\mathbb{X}_{s,t}$$

holds.

In the case that we take $U = U^{\psi}$, we get a slightly simpler formula.

Corollary 4.37 *Let ψ be a logarithm. There exists a $\delta>0$ such that for $s<t$ and $t-s<\delta$, the approximation*

$$\left[\int \alpha\left(d\mathbf{y}\right)\right]_{s,t}^{1} \underset{3}{\approx} \alpha_{y_s}\left(\psi\left(y_s, y_t\right)\right) + d\left(\alpha_{(\cdot)} \circ \left(\psi_{(\cdot)}\right)_{*y_s}\right)_{y_s} \circ y_s^{\dagger\otimes 2}\mathbb{X}_{s,t}$$

holds.

Example 4.38 *Let ∇ be a covariant derivative on M. There exists a $\delta>0$ such that for $s<t$ and $t-s<\delta$, the approximation*

$$\left[\int \alpha\left(d\mathbf{y}\right)\right]_{s,t}^{1} \underset{3}{\approx} \alpha_{y_s}\left(\left(\exp_{y_s}^{\nabla}\right)^{-1}\left(y_t\right)\right) + \left[\left(\nabla\alpha\right)_{y_s} + \frac{1}{2}\alpha_{y_s} \circ T_{y_s}^{\nabla}\right] \circ y_s^{\dagger\otimes 2}\mathbb{X}_{s,t}$$

holds. Indeed this follows immediately from Proposition 4.36, Lemma 4.13, and the fact that

$$\begin{aligned}\left(\nabla\alpha\right)_{y_s}\left(v_m, w_m\right) &:= v_m\left[\alpha\left(\boldsymbol{W}\right)\right] - \alpha\left(\nabla_{v_m}\boldsymbol{W}\right)\\ &= d\left(\alpha_{(\cdot)} \circ \boldsymbol{W}\left(\cdot\right)\right)_{y_s}\left(v_m\right) - \alpha\left(\nabla_{v_m}\boldsymbol{W}\right)\end{aligned}$$

where $\boldsymbol{W}$ is any vector field such that $\boldsymbol{W}(m) = w_m$. Choosing $\boldsymbol{W} = U^{\nabla}(\cdot, m)\, w_m$, we have

$$\nabla_{v_m}\boldsymbol{W} = \nabla_{v_m}U^{\nabla}\left(\cdot, m\right) w_m = 0$$

by the definition of parallel translation.

4.2.1 Integration of a One-Form Using Charts

It is easy to see that by independence of gauges, the integral of a one-form along $\left(y_s, y_s^\dagger\right)$ is an object which we only need to compute locally. As mentioned in Remark 3.11 we have an example of a local gauge by using a chart. Plugging this formula into the integral approximation from Corollary 4.37, we get the following.

Corollary 4.39 *Let ϕ be a chart on M. For all $a, b \in [0,T]$ such that $y([a,b]) \subset D(\phi)$, we have the approximation*

$$\left[\int \alpha\left(d\mathbf{y}\right)\right]_{s,t}^{1} \underset{3}{\approx} \alpha_{y_s}\left(\left(d\phi_{y_s}\right)^{-1}\left[\phi\left(y_t\right)-\phi\left(y_s\right)\right]\right)+d\left(\alpha_{(\cdot)}\circ\left(d\phi_{(\cdot)}\right)^{-1}d\phi_{y_s}\right)_{y_s}\circ y_s^{\dagger\otimes 2}\mathbb{X}_{s,t} \tag{4.35}$$

holds for all $s < t \in [a,b]$.

Although this formula looks a bit complicated, it may be reduced to something that makes more sense. First, note that

$$\alpha_m \circ \left(d\phi_m\right)^{-1} = \left[\left(\phi^{-1}\right)^* \alpha\right]_{\phi(m)}.$$

Thus we can reduce the right hand side Eq. (4.35) to

$$\begin{aligned}&\left[\left(\phi^{-1}\right)^*\alpha\right]_{\phi(y_s)}\left(\phi\left(y_t\right)-\phi\left(y_s\right)\right)+d\left(\left[\left(\phi^{-1}\right)^*\alpha\right]_{\phi(\cdot)}d\phi_{y_s}\right)_{y_s}\circ y_s^{\dagger\otimes 2}\mathbb{X}_{s,t}\\&=\left[\left(\phi^{-1}\right)^*\alpha\right]_{\phi(y_s)}\left(\phi\left(y_t\right)-\phi\left(y_s\right)\right)+\left[\left(\phi^{-1}\right)^*\alpha\right]_{\phi(y_s)}'\left[d\phi_{y_s}\circ y_s^\dagger\right]^{\otimes 2}\mathbb{X}_{s,t}.\end{aligned}$$

Now, if we recall Notation 3.30, we see that this is approximately equal to another rough integral. More precisely

$$\left[\int \alpha\left(d\mathbf{y}\right)\right]_{s,t}^{1} \underset{3}{\approx} \left[\int \left(\phi^{-1}\right)^*\alpha\left(d\phi_*\mathbf{y}\right)\right]_{s,t}^{1}.$$

However, additive functionals are unique up to this order, so in fact

$$\left[\int \alpha\left(d\mathbf{y}\right)\right]_{s,t}^{1} = \left[\int \left(\phi^{-1}\right)^{*} \alpha\left(d\phi_{*}\mathbf{y}\right)\right]_{s,t}^{1}$$

which is a relation which should hold under any reasonable integral. This is summarized in the following theorem which gives us an alternative way of defining this integral.

Theorem 4.40 *The integral, $\int \alpha\left(d\mathbf{y}\right)$, is the unique V – valued rough path controlled by $\mathbf{X}$ on $[0,T]$ starting at 0 determined by*

1. $\left[\int \alpha\left(d\mathbf{y}\right)\right]_{s,t}^{1} = \left[\int \left(\left(\phi^{-1}\right)^{*} \alpha\right)\left(d\phi_{*}\mathbf{y}\right)\right]_{s,t}^{1}$ *for any chart and $s < t \in [0,T]$ such that $y\left([s,t]\right) \subset D\left(\phi\right)$*

2. $\left[\int \alpha\left(d\mathbf{y}\right)\right]_{s}^{\dagger} = \alpha_{y_s} \circ y_s^{\dagger}$.

[See Theorem 4.47 below for a more general version of this theorem.]

The next theorem with our current toolset can now be proved in two different ways. We can reduce the result to a special case of Theorem 4.24 or, by using the chart definitions of integration along a one-form, can reduce it to its validity in the flat case. The first method is quick but may hide the concept of what is happening. We therefore provide both proofs.

Theorem 4.41 (Associativity Theorem II) *Suppose that $y \in CRP\left(M\right)$, $\alpha \in \Omega^{1}\left(M,V\right)$, and $K : M \rightarrow L\left(V,\tilde{V}\right)$ is a smooth function so that $K\alpha \in \Omega^{1}\left(M,\tilde{V}\right)$. If $\mathbf{z} = \int \alpha\left(d\mathbf{y}\right) \in CRP\left(V\right)$, then*

$$\int \left(K\alpha\right)\left(d\mathbf{y}\right) = \int \left\langle K_{*}\left(\mathbf{y}\right), d\mathbf{z}\right\rangle \quad \left(=: \int \left\langle K_{*}\left(\mathbf{y}\right), d\int \alpha\left(d\mathbf{y}\right)\right\rangle\right),$$

where $K_{*}\left(\mathbf{y}\right) = \left(K\left(y\right), K_{*y}y^{\dagger}\right) \in CRP_{\mathbf{X}}\left(\mathrm{Hom}\left(V,V'\right)\right)$.

Proof. Method 1: Letting $\mathcal{G} = (\psi, U)$ be any gauge, we define $\mathbf{f} := \left(f, f^{\dagger}\right) \in CRP_{\mathbf{X}}\left(\operatorname{Hom}\left(V, \tilde{V}\right)\right)$ by the formula

$$f_s := K\left(y_s\right) \quad \text{and} \quad f_s^{\dagger} := K_{*y_s} y_s^{\dagger}$$

and $\boldsymbol{\alpha}^{(y,U)}$ as in Proposition 4.34 (see Proposition 4.42 below to see why $\mathbf{f} \in CRP_{\mathbf{X}}\left(\operatorname{Hom}\left(V, \tilde{V}\right)\right)$). Then by Theorem 4.24, we have

$$\int \left\langle \mathbf{f}\boldsymbol{\alpha}^{(y,U)}, d\mathbf{y}^{\mathcal{G}} \right\rangle = \int \left\langle \mathbf{f}, d\mathbf{z} \right\rangle \tag{4.36}$$

where $\mathbf{z} = \int \left\langle \boldsymbol{\alpha}^{(y,U)}, d\mathbf{y}^{\mathcal{G}} \right\rangle = \int \alpha\left(d\mathbf{y}\right)$. The right hand side in Equation (4.36) is simply $\int \left\langle K_*\left(\mathbf{y}\right), d\mathbf{z} \right\rangle$ while the $\mathbf{f}\boldsymbol{\alpha}^{(y,U)}$ term on the left hand side can be recognized as $\left(K\alpha\right)^{(y,U)}$. Indeed, by the product rule with ∇^U, we have

$$\begin{aligned}
\left(K\alpha\right)_s^{(y,U)} &= \left(K\left(y_s\right)\alpha|_{T_{y_s}M}, \nabla^U_{y_s^{\dagger}(\cdot)}\left[K\left(\cdot\right)\alpha\right]\right) \\
&= \left(K\alpha|_{T_{y_s}M}, K_{*y_s}y_s^{\dagger}\alpha + K\left(y_s\right)\nabla^U_{y_s^{\dagger}(\cdot)}\alpha\right) \\
&= \left(f_s\alpha_s, f_s^{\dagger}\alpha + f_s\alpha_s^{\dagger(y,U)}\right) \\
&= \mathbf{f}\boldsymbol{\alpha}^{(y,U)}.
\end{aligned}$$

Thus

$$\int \left(K\alpha\right)\left(d\mathbf{y}\right) := \int \left\langle \left(K\alpha\right)_s^{(y,U)}, d\mathbf{y}^{\mathcal{G}} \right\rangle = \int \left\langle \mathbf{f}\boldsymbol{\alpha}^{(y,U)}, d\mathbf{y}^{\mathcal{G}} \right\rangle = \int \left\langle K_*\left(\mathbf{y}\right), d\mathbf{z} \right\rangle.$$

Method 2: By a simple patching argument, this is really a local result and hence using the chart definitions of integration it suffices to check this result in the case M is an open subset of $\mathbb{R}^d$. First we check the derivative processes. From the definitions we have

$$z_s^\dagger = \alpha_{y_s} \circ y_s^\dagger \quad \text{and} \quad \left[\int (K\alpha)(d\mathbf{y})\right]_s^\dagger = (K\alpha)_{y_s} \circ y_s^\dagger = K(y_s)\,\alpha_{y_s} \circ y_s^\dagger = K(y_s)\,z_s^\dagger.$$

Thus

$$\left[\int (K\alpha)(d\mathbf{y})\right]_s^\dagger = K(y_s)\,z_s^\dagger.$$

On the other hand

$$\left[\int \langle K_*(\mathbf{y}), d\mathbf{z}\rangle\right]_s^\dagger = [K(y)]_s\,z_s^\dagger = K(y_s)\,z_s^\dagger$$

Similarly for the paths

$$z_{s,t} \underset{3}{\approx} \alpha(y_{s,t}) + \alpha'_{y_s} y_s^{\dagger\otimes 2}\mathbb{X}_{s,t}.$$

and so

$$\begin{aligned}\left[\int (K\alpha)(d\mathbf{y})\right]_{s,t}^1 &\underset{3}{\approx} (K\alpha)_{y_s} y_{s,t} + (K\alpha)'_{y_s} y_s^{\dagger\otimes 2}\mathbb{X}_{s,t} \\ &= K(y_s)\,\alpha_{y_s} y_{s,t} + K(y_s)\,\alpha'_{y_s} y_s^{\dagger\otimes 2}\mathbb{X}_{s,t} \\ &\quad + \left[K'_{y_s}\left(y_s^\dagger(\cdot) \otimes \alpha y_s^\dagger(\cdot)\right)\right]\mathbb{X}_{s,t} \\ &\underset{3}{\approx} K(y_s)\,z_{s,t} + K'_{y_s}\left(y_s^\dagger \otimes z_s^\dagger\right)\mathbb{X}_{s,t}.\end{aligned}$$

On the other hand

$$\begin{aligned}\left[\int \langle K_*(\mathbf{y}), d\mathbf{z}\rangle\right]_{s,t}^1 &\underset{3}{\approx} K(y_s)\,z_{s,t} + [K_*(\mathbf{y})]_s^\dagger\, z_s^\dagger \mathbb{X}_{s,t} \\ &= K(y_s)\,z_{s,t} + K'_{y_s}\left(y_s^\dagger \otimes z_s^\dagger\right)\mathbb{X}_{s,t}.\end{aligned}$$

Comparing these expressions completes the proof. ■

4.2.2 Push-forwards of Controlled Rough Paths

Let $M = M^d$ and $\tilde{M} = \tilde{M}^{\tilde{d}'}$ be manifolds. Let $f : M \to \tilde{M}$ be smooth and suppose $\mathbf{y}_s = \left(y_s, y_s^\dagger\right) \in CRP_{\mathbf{X}}(M)$. In Definition 4.43 below, we are going to give a definition of the push-forward of $\mathbf{y}$ by f which generalizes Example 3.47.

Proposition 4.42 *The pair* $\left(f(y_s), f_* \circ y_s^\dagger\right)$ *is an element of* $CRP_{\mathbf{X}}\left(\tilde{M}\right)$.

Proof. Suppose $\tilde{\phi}$ is a chart on $\tilde{M}$ such that $f \circ y([a,b]) \subseteq D\left(\tilde{\phi}\right)$. We must show that

$$\left|\tilde{\phi} \circ f(y_t) - \tilde{\phi} \circ f(y_s) - d\tilde{\phi} \circ f_* y_s^\dagger x_{s,t}\right| \leq C_{\tilde{\phi},a,b}\, \omega(s,t)^{2/p} \tag{4.37}$$

and

$$\left|d\tilde{\phi} \circ f_* y_t^\dagger - d\tilde{\phi} \circ f_* y_s^\dagger\right| \leq C_{\tilde{\phi},a,b}\, \omega(s,t)^{1/p} \tag{4.38}$$

hold for some $C_{\tilde{\phi},a,b}$ for all $s \leq t$ in $[a,b]$. We can again use our proof strategy outlined in Remark 3.41 to treat this problem in nice neighborhoods. We leave it to the reader to follow the pattern of earlier proofs to see that we can assume without loss of generality that there is a chart ϕ on M such that $y([a,b]) \subseteq D(\phi)$ and $R(\phi)$ is convex. Which these simplifications, we note that $\left(z_s, z_s^\dagger\right) := \left(\phi(y_s), d\phi \circ y_s^\dagger\right)$ is a controlled rough path on $R(\phi)$ and the function $F := \tilde{\phi} \circ f \circ \phi^{-1} : R(\phi) \to R\left(\tilde{\phi}\right)$ is a map between Euclidean spaces. Therefore Inequalities (4.37) and (4.38) reduce to the fact that the pair $\left(F(z_s), F'(z_s) \circ z_s^\dagger\right)$ is a controlled rough path in $\mathbb{R}^{\tilde{d}}$ (which is trivial by applying Taylor's theorem after we check that we get the correct terms); indeed, by a simple computation, we have

$$\begin{aligned}
F'(z_s) \circ z_s^\dagger &= d\tilde{\phi} \circ f_* \circ \left(d\phi^{-1}\right)_{z_s} \circ d\phi_{y_s} \circ y_s^\dagger \\
&= d\tilde{\phi} \circ f_* \circ \left(d\phi_{y_s}\right)^{-1} \circ d\phi_{y_s} \circ y_s^\dagger \\
&= d\tilde{\phi} \circ f_* y_s^\dagger
\end{aligned}$$

and clearly $F(z_s) = \tilde{\phi} \circ f(y_s)$. ■

Definition 4.43 *The **push-forward** of* $\mathbf{y}$ *denoted by* $f_*\mathbf{y}$ *or* $f_*\left(y, y^\dagger\right)$ *is the rough path controlled by* $\mathbf{X}$ *with path* $f(y_s)$ *and derivative process* $f_* \circ y_s^\dagger$*. If* $\tilde{M} = \mathbb{R}^{\tilde{d}}$*, we will abuse notation an write* $f_*\mathbf{y}_s$ *to mean* $\left(f(y_s), df \circ y_s^\dagger\right)$ *(i.e. we forget the base point on the derivative process).*

Remark 4.44 *The push-forward operation on elements in* $CRP_{\mathbf{X}}(M)$ *is clearly covariant, i.e. if* $f : M \to N$ *and* $g : N \to P$ *are two smooth maps of manifolds,* $M, N,$ *and* P*, then* $(g \circ f)_*(\mathbf{y}) = g_*(f_*(\mathbf{y}))$*.*

This definition is consistent with how we defined the integral of a one-form along a controlled rough path in the sense that we have a fundamental theorem of calculus. Let V be a Banach space.

Theorem 4.45 *Let* $\mathbf{y}_s = \left(y_s, y_s^\dagger\right) \in CRP_{\mathbf{X}}(M)$ *and* f *be a smooth function from* M *to* V*. Then*

$$f(y_s) - f(y_0) = \left[\int df\,[d\mathbf{y}]\right]^1_{0,s}$$

where df *is interpreted as a one-form. Since we have* $df \circ y_s^\dagger = \left[\int df\,[d\mathbf{y}]\right]^\dagger_s$ *we have the equality*

$$f_*\left(y, y^\dagger\right) - (f(y_0), 0) = \int df\,(d\mathbf{y})\,.$$

Proof. Although there are ways to do this proof without much machinery, we find it more instructive to work on a Riemannian manifold with the Levi-Civita covariant derivative. Since we have proved that the integral is independent of choice of metric, it does not matter which one we pick. With this in mind, we have the approximation

$$\left[\int df\,[d\mathbf{y}]\right]^1_{s,t} \underset{3}{\approx} df_{y_s}\left(\exp^{-1}_{y_s}(y_t)\right) + (\nabla df)_{y_s}\left[y_s^{\dagger \otimes 2}\mathbb{X}_{s,t}\right]$$

and as ∇df is symmetric, it follows that

$$\begin{aligned}\left[\int df\,[d\mathbf{y}]\right]^1_{s,t} &\underset{3}{\approx} df_{y_s}\left(\exp^{-1}_{y_s}(y_t)\right) + \frac{1}{2}(\nabla df)_{y_s}\left[y_s^{\dagger\otimes 2}(x_{s,t}\otimes x_{s,t})\right] \\ &\underset{3}{\approx} df_{y_s}\left(\exp^{-1}_{y_s}(y_t)\right) + \frac{1}{2}(\nabla df)_{y_s}\left[\exp^{-1}_{y_s}(y_t)^{\otimes 2}\right] \\ &\underset{3}{\approx} f(y_t) - f(y_s).\end{aligned}$$

The last approximation above follows from Taylor's Theorem on manifolds (Theorem A.1 in the Appendix). Note here that $f(y_t) - f(y_s)$ is additive so that

$$\left[\int df\,[d\mathbf{y}]\right]^1_{s,t} = f(y_t) - f(y_s).$$

■

Remark 4.46 *If $M \subseteq$ is an embedded submanifold of $W = \mathbb{R}^k$, $\left(y_s, y_s^\dagger\right)$ is an element of $CRP_{\mathbf{X}}(M)$, $I : M \to W$ denotes the identity (or embedding) map, and $\left(z_s, z_s^\dagger\right) := I_*\left(y_s, y_s^\dagger\right)$, then we have*

$$z_s = y_s \quad \textit{and} \quad z_s^\dagger = \pi_2 \circ y_s^\dagger$$

where π_2 is the projection of the tangent vector component (i.e. it forgets the base point). We can associate to it a unique rough path $(y, \mathbb{Y})$ in W such that

$$\left(z_s^\dagger \otimes z_s^\dagger\right)\mathbb{X}_{s,t} \underset{3}{\approx} \mathbb{Y}_{s,t}.$$

In this case, this is a rough path in the embedded sense (See [3]) since

$$[I(y_s) \otimes Q(y_s)]\,[\mathbb{Y}]_{s,t} \underset{3}{\approx} [I(y_s) \otimes Q(y_s)]\left[z_s^\dagger \otimes z_s^\dagger\right]\mathbb{X}_{s,t} = 0$$

as $Q(y_s) \circ z_s^\dagger = 0$ where $Q = I - P$ and $P(x)$ is orthogonal projection onto the *tangent space at x.*

Lastly, we have a relation between push-forwards of paths and pull-backs of one-forms.

Theorem 4.47 (Push me-Pull me) *Let $f : M \to \tilde{M}$, let $\mathbf{y}_s = \left(y_s, y_s^\dagger\right) \in CRP_{\mathbf{X}}(M)$ and let $\tilde{\alpha} \in \Omega^1\left(\tilde{M}, V\right)$. Then*

$$\left[\int f^*\alpha\left(d\mathbf{y}\right)\right]^1 = \left[\int \alpha\left(d\left(f_*\mathbf{y}\right)\right)\right]^1. \tag{4.39}$$

Moreover

$$\int f^*\alpha\left(d\mathbf{y}\right) = \int \alpha\left(d\left(f_*\mathbf{y}\right)\right).$$

Proof. This is a statement we only have to prove locally. Indeed for each $s \in [0,T]$, there are charts ϕ^s and $\tilde{\phi}^s$ on M and $\tilde{M}$ respectively such that $y_s \in D\left(\phi^s\right)$ and $f\left(y_s\right) \in D\left(\tilde{\phi}^s\right)$ which are open. We take $\mathcal{U}_s := f^{-1}\left(D\left(\tilde{\phi}^s\right)\right) \cap D\left(\phi^s\right)$ and shrink it if necessary so that $\mathcal{V}_s = \phi\left(\mathcal{U}_s\right)$ is convex. Thus if we can prove that Eq. (4.39) holds whenever $y\left([a,b]\right) \subseteq \mathcal{U}$ such that $\phi\left(\mathcal{U}\right)$ is convex and such that $f\left(y\left([a,b]\right)\right) \subseteq D\left(\tilde{\phi}\right)$, we will be done. We do this now:

By Theorem 4.40, the fact that pull-backs are contravariant, and that push-forwards are covariant, we have

$$\begin{aligned}
\left[\int f^*\alpha\left(d\mathbf{y}\right)\right]^1_{s,t} &= \left[\int \left(\phi^{-1}\right)^* f^*\alpha\left(d\phi_*\mathbf{y}\right)\right]^1_{s,t} \\
&= \left[\int \left(f\circ\phi^{-1}\right)^* \alpha\left(d\phi_*\mathbf{y}\right)\right]^1_{s,t} \\
&= \left[\int \left(\tilde{\phi}^{-1}\circ\tilde{\phi}\circ f\circ\phi^{-1}\right)^* \alpha\left(d\phi_*\mathbf{y}\right)\right]^1_{s,t} \\
&= \left[\int \left(\tilde{\phi}\circ f\circ\phi^{-1}\right)^* \left(\left(\tilde{\phi}^{-1}\right)^*\alpha\right)\left(d\phi_*\mathbf{y}\right)\right]^1_{s,t} \\
&= \left[\int \left(\tilde{\phi}^{-1}\right)^* \alpha\left(d\left(\left(\tilde{\phi}\circ f\circ\phi^{-1}\right)_*\phi_*\mathbf{y}\right)\right)\right]^1_{s,t}
\end{aligned}$$

where the last step is just Eq. (4.39) on Euclidean space. This is a simple computation (for example, see the appendix of [3]). Thus, we have

$$\begin{aligned}\left[\int f^{*}\alpha\left(d\mathbf{y}\right)\right]_{s,t}^{1} &= \left[\int\left(\tilde{\phi}^{-1}\right)^{*}\alpha\left(d\left(\left(\tilde{\phi}\circ f\circ\phi^{-1}\right)_{*}\phi_{*}\mathbf{y}\right)\right)\right]_{s,t}^{1} \\ &= \left[\int\left(\tilde{\phi}^{-1}\right)^{*}\alpha\left(d\left(\tilde{\phi}_{*}\left(f_{*}\mathbf{y}\right)\right)\right)\right]_{s,t}^{1} \\ &= \left[\int\alpha\left(d\left(f_{*}\mathbf{y}\right)\right)\right]_{s,t}^{1}.\end{aligned}$$

The fact that

$$\left[\int f^{*}\alpha\left(d\mathbf{y}\right)\right]^{\dagger} = \left[\int\alpha\left(d\left(f_{*}\mathbf{y}\right)\right)\right]^{\dagger}$$

is trivial. ■

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Chapter 5

Rough Differential Equations

5.1 A Flat Case Result

Before discussing rough differential equations on a manifold, we will give an equivalent condition for a controlled rough path $\mathbf{z} \in CRP_{\mathbf{X}}\left(\mathbb{R}^d\right)$ to satisfy the RDE approximation on a compact interval in the flat case using logarithms.

For the next proposition, let ψ be a logarithm on $\mathbb{R}^d$ such that $\psi(x, y) = \left(x, \bar{\psi}(x, y)\right)$.

Proposition 5.1 *Let $z : [a, b] \to \mathbb{R}^d$ be a path and let $\mathcal{W} \subseteq \mathbb{R}^d$ be an open convex set such that $z([a, b]) \subseteq \mathcal{W}$ and $\mathcal{W} \times \mathcal{W} \subseteq D(\psi)$. Then*

$$z_{s,t} \underset{3}{\approx} F_{x_{s,t}}(z_s) + \left(\partial_{F_w(z_s)} F_{\tilde{w}}\right)(z_s)|_{w \otimes \tilde{w} = \mathbb{X}_{s,t}} \tag{5.1}$$

if and only if

$$\bar{\psi}(z_s, z_t) \underset{3}{\approx} F_{x_{s,t}}(z_s) + \left(\partial_{F_w(z_s)}\left[\bar{\psi}'_{z_s}(\cdot)\, F_{\tilde{w}}(\cdot)\right]\right)(z_s)|_{w \otimes \tilde{w} = \mathbb{X}_{s,t}} \tag{5.2}$$

Proof. If $z_\cdot$ satisfies Eq. (5.1), then from Eq. (3.6) of Theorem 3.13 with $y = z_t$

and $x = z_s$ we find,

$$\bar{\psi}(z_s, z_t) = z_{s,t} + \frac{1}{2}\bar{\psi}''_x(x)(z_{s,t})^{\otimes 2} + C(z_s, z_t)(z_{s,t})^{\otimes 3} \tag{5.3}$$

$$\underset{3}{\approx} F_{x_{s,t}}(z_s) + \left(\partial_{F_w(z_s)} F_{\tilde{w}}\right)(z_s)|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}} + \frac{1}{2}\bar{\psi}''_{z_s}(z_s)\left[F_{x_{s,t}}(z_s)\right]^{\otimes 2}, \tag{5.4}$$

wherein C is a smooth function and we have made use of the fact that $z_{s,t} \underset{1}{\approx} 0$. By the product rule and the fact that ψ is a logarithm it follows that

$$\left(\partial_{F_w(z_s)}\left[\bar{\psi}'_{z_s}(\cdot) F_{\tilde{w}}(\cdot)\right]\right)(z_s) = \bar{\psi}''_{z_s}(z_s) F_w(z_s) \otimes F_{\tilde{w}}(z_s) + \bar{\psi}'_{z_s}(z_s)\left(\partial_{F_w(z_s)} F_{\tilde{w}}\right)(z_s)$$

$$= \bar{\psi}''_{z_s}(z_s) F_w(z_s) \otimes F_{\tilde{w}}(z_s) + \left(\partial_{F_w(z_s)} F_{\tilde{w}}\right)(z_s). \tag{5.5}$$

Since $\mathbf{X}$ is a weak-geometric rough path and $\bar{\psi}''_{z_s}(z_s)$ is symmetric, we also have,

$$\bar{\psi}''_{z_s}(z_s) F_w(z_s) \otimes F_{\tilde{w}}(z_s)|_{|w\otimes\tilde{w}=\mathbb{X}_{s,t}} = \frac{1}{2}\bar{\psi}''_{z_s}(z_s)\left[F_{x_{s,t}}(z_s)\right]^{\otimes 2},$$

which combined with Eq. (5.5) shows,

$$\left(\partial_{F_w(z_s)}\left[\bar{\psi}'_{z_s}(\cdot) F_{\tilde{w}}(\cdot)\right]\right)(z_s)|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}} \tag{5.6}$$

$$= \left(\partial_{F_w(z_s)} F_{\tilde{w}}\right)(z_s)|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}} + \frac{1}{2}\bar{\psi}''_{z_s}(z_s)\left[F_{x_{s,t}}(z_s)\right]^{\otimes 2}. \tag{5.7}$$

Equation (5.2) now follows directly from Eqs. (5.4) and (5.6).

Conversely, now assume that Eq. (5.2) holds. From Eq. (5.2) and the fact that $\mathbf{X}$ is a rough path there exists $C_1 < \infty$ such that $\left|\bar{\psi}(z_s, z_t)\right| \le C_1\omega(s,t)^{1/p}$. Combining this observation with Eq. (5.3) easily implies $z_{s,t} \underset{1}{\approx} 0$. Indeed, by uniform continuity, there exists a $\delta > 0$ such that if $|t-s| \le \delta$, we have

$$|z_{s,t}| \le \left|\bar{\psi}(z_s, z_t)\right| + \left|\frac{1}{2}\psi''_{z_s}(z_s)(z_{s,t})^{\otimes 2} + C(z_s, z_t)(z_{s,t})^{\otimes 3}\right|$$

$$\le C_1\omega(s,t)^{1/p} + \frac{1}{2}|z_{s,t}|.$$

By using an argument similar to the proof of Theorem 3.36 we can bootstrap these local inequalities to prove the existence of a $C_2 < \infty$ such that $|z_{s,t}| \leq C_2 \omega(s,t)^{1/p}$ for $a \leq s \leq t \leq b$.

From Eqs. (5.3) and (5.2),

$$\begin{aligned} z_{s,t} &= \bar{\psi}(z_s, z_t) - \frac{1}{2} \bar{\psi}''_{z_s}(z_s) (\psi(z_s, z_t))^{\otimes 2} + C(z_s, z_t)(z_{s,t})^{\otimes 3} \\ &\underset{3}{\approx} F_{x_{s,t}}(z_s) + \left(\partial_{F_w(z_s)} \left[\bar{\psi}'_{z_s}(\cdot) F_{\tilde{w}}(\cdot)\right]\right)(z_s)|_{w \otimes \tilde{w} = \mathbb{X}_{s,t}} - \frac{1}{2} \bar{\psi}''_{z_s}(z_s) \left(F_{x_{s,t}}(z_s)\right)^{\otimes 2} \\ &= F_{x_{s,t}}(z_s) + \left(\partial_{F_w(z_s)} F_{\tilde{w}}\right)(z_s), \end{aligned}$$

wherein we have used Eq. (5.6) for the last equality. ■

5.2 RDEs on a Manifold

We now move to the manifold case. Let $F : M \to L(W, TM)$ be smooth such that $F(m) \in L(W, T_m M)$. Alternatively we can think of $F : W \to \Gamma(TM)$ where the map $w \to F_w(\cdot)$ is linear. We wish to give meaning to the differential equation

$$d\mathbf{y}_t = F_{d\mathbf{X}_t}(y_t) \tag{5.8}$$

with initial condition $y_0 = \bar{y}_0$. To do this, first recall that any vector field can be transferred to Euclidean space by using charts. If $\mathcal{U} \subseteq D(\phi)$ where ϕ is a chart and $\mathcal{V} := \phi(\mathcal{U})$ then

$$F^\phi := d\phi \circ \left(F \circ \phi^{-1}\right)$$

is a vector field on $\mathcal{V}$ (which does not carry the base point). If $\mathbf{y}_t$ is to "solve" (5.8) then $\mathbf{z}_t := \phi_* \mathbf{y}_t$ should solve the differential equation

$$d\mathbf{z}_t = F^\phi_{d\mathbf{X}_t}(z_t). \tag{5.9}$$

In the Euclidean case, Equation (5.9) is satisfied if

$$z_t \underset{3}{\approx} z_s + F^{\phi}_{x_{s,t}}(z_s) + \left(\partial_{F^{\phi}_{w}(z_s)} F^{\phi}_{\tilde{w}}\right)(z_s)\,|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}} \tag{5.10}$$
$$z^{\dagger}_s = F^{\phi}_{(\cdot)}(z_s)$$

By writing out Equation (5.10) we have

$$\phi(y_t) \underset{3}{\approx} \phi(y_s) + d\phi \circ F_{x_{s,t}}(y_s) + \left(\partial_{d\phi\circ F_w(y_s)} d\phi \circ \left(F_{\tilde{w}} \circ \phi^{-1}\right)\right)(\phi(y_s))\,|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}}$$
$$= \phi(y_s) + d\phi \circ F_{x_{s,t}}(y_s) + F_w(y_s)\left[d\phi \circ F_{\tilde{w}}\right]|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}}. \tag{5.11}$$

As a reminder, if F is linear with its range in the algebra of differential operators, we can extend it uniquely to $\mathcal{F}$ which acts on the tensor algebra $T(\mathbb{R}^n)$. In that case, we may write (5.11) more concisely as

$$\phi(y_t) \underset{3}{\approx} \phi(y_s) + \left(\mathcal{F}_{\mathbf{X}_{s,t}}\phi\right)(y_s). \tag{5.12}$$

This approximation will be satisfied for our solution to a rough differential equation on a manifold. However, we will opt to define our solution in a coordinate-free but equivalent way:

Definition 5.2 $\mathbf{y} = (y, y^{\dagger})$ *on* $I_0 = [0,T]$ *or* $[0,T)$ *solves (5.8) if* $y^{\dagger}_s = F_{(\cdot)}(y_s)$ *and for every* $f \in C^{\infty}(M)$ *and* $[a,b] \subseteq I_0$*, the approximation*

$$f(y_t) - f(y_s) \underset{3}{\approx} \left(\mathcal{F}_{\mathbf{X}_{s,t}} f\right)(y_s)$$

holds for $a \leq s \leq t \leq b$.

If in addition $y_0 = \bar{y}_0$*, we say* $\mathbf{y}$ *solves (5.8) with initial condition* $y_0 = \bar{y}_0$.

While this is an intuitive definition, there are many workable characterizations of solving a rough differential equation. Before presenting a few more, we note that if $\alpha \in \Omega^1(M,V)$ and $F : M \to L(W,TM)$ is smooth, then the composition $\alpha \circ F_{(\cdot)}$ is a smooth map from M to V. Given $\mathbf{y} \in CRP_{\mathbf{X}}(M)$, we can then define the push-forward $\left[\alpha \circ F_{(\cdot)}\right]_* \mathbf{y} \in CRP_{\mathbf{X}}(L(W,V))$. Recall from Theorem 2.10 that we can define the integral increment

$$\int_s^t \left\langle \left(\left[\alpha \circ F_{(\cdot)}\right]_* \mathbf{y}\right)_\tau, d\mathbf{X}_\tau \right\rangle.$$

With this idea in mind, we now give other characterizations of solving Eq. (5.8).

Theorem 5.3 *Let y be a path in M on I_0 with $y_s^\dagger = F_\cdot(y_s)$. Let $\mathbf{y} = \left(y, y^\dagger\right) \in CRP_{\mathbf{X}}(M)$. The following are equivalent.*

1. *For every chart ϕ with $a, b \in I_0$ such that $y([a,b]) \subseteq D(\phi)$ the approximation*

$$\phi(y_t) \underset{3}{\approx} \phi(y_s) + d\phi \circ F_{x_{s,t}}(y_s) + F_w(y_s)\left[d\phi \circ F_{\tilde{w}}\right]|_{w \otimes \tilde{w} = \mathbb{X}_{s,t}} \tag{5.13}$$

holds $a \le s \le t \le b$; that is

$$\phi(y_t) - \phi(y_s) = \int_s^t \left\langle \left(\left[d\phi \circ F_{(\cdot)}\right]_* \mathbf{y}\right)_\tau, d\mathbf{X}_\tau \right\rangle$$

for $a \le s \le t \le b$.

2. *If V is a Banach space, $\alpha \in \Omega^1(M,V)$, and $[a,b]$ is such that $[a,b] \subseteq I_0$ then*

$$\int_s^t \alpha(d\mathbf{y}) \underset{3}{\approx} \alpha\left(F_{x_{s,t}}(y_s)\right) + F_w(y_s)\left[\alpha \circ F_{\tilde{w}}\right]|_{w \otimes \tilde{w} = \mathbb{X}_{s,t}}$$

for $a \le s \le t \le b$; that is

$$\int_s^t \alpha(d\mathbf{y}) = \int_s^t \left\langle \left(\left[\alpha \circ F_{(\cdot)}\right]_* \mathbf{y}\right)_\tau, d\mathbf{X}_\tau \right\rangle$$

for $a \le s \le t \le b$.

3. $\mathbf{y}$ *solves (5.8); that is*

$$f(y_t) - f(y_s) = \int_s^t \left\langle \left(\left[df \circ F_{(\cdot)} \right]_* \mathbf{y} \right)_\tau, d\mathbf{X}_\tau \right\rangle$$

for every $f \in C^\infty(M)$.

Proof. We will only prove the approximations in each case, that is the first statement of each item. The second statements are immediate from the definitions.

($1 \implies 2$) We assume that $\mathbf{y}$ satisfies the approximation in Eq. (5.13) for any chart. Let $[a,b] \subseteq I_0$ be given. For every $m \in y([a,b])$, we have there exists a chart ϕ^m with open domain $\mathcal{V}_m := D(\phi^m)$ containing m whose range $R(\phi^m)$ is convex. We may now use our patching strategy outlined in Remark 3.41 with the cover $\{V_m\}_{m \in y([a,b])}$ applied to the function

$$(s,t) \longrightarrow \int_s^t \alpha(d\mathbf{y}) - \alpha\left(F_{x_{s,t}}(y_s)\right) - F_w(y_s)\left[\alpha \circ F_{\tilde{w}}\right]|_{w \otimes \tilde{w} = \mathbb{X}_{s,t}}$$

to reduce to the case where $y([a,b])$ is contained in the domain of a single chart.

With this reduction, we can further reduce to the flat case by defining $\mathbf{z}_t := (\phi(y_t), F_\cdot(y_s))$ and $F^\phi := d\phi(F \circ \phi^{-1})$ and showing

$$\begin{aligned} &\int_s^t \alpha(d\mathbf{y}) - \alpha\left(F_{x_{s,t}}(y_s)\right) - F_w(y_s)\left[\alpha \circ F_{\tilde{w}}\right]|_{w \otimes \tilde{w} = \mathbb{X}_{s,t}} \\ &= \int_s^t \left(\left(\phi^{-1}\right)^* \alpha\right)(d\mathbf{z}) - \left(\left(\phi^{-1}\right)^* \alpha\right)_{z_s}\left(F^\phi_{x_{s,t}}(z_s)\right) \\ &\quad - \left(\partial_{F^\phi_w(z_s)}\left[\left(\phi^{-1}\right)^* \alpha \circ F^\phi_{\tilde{w}}\right]\right)(z_s)|_{w \otimes \tilde{w} = \mathbb{X}_{s,t}}. \end{aligned}$$

The above equality is true due to the following three identities:

$$\int_s^t \alpha\left(d\mathbf{y}\right) = \int_s^t \left(\left(\phi^{-1}\right)^* \alpha\right)\left(d\mathbf{z}\right), \tag{5.14}$$

$$\alpha\left(F_{x_{s,t}}\left(y_s\right)\right) = \left(\left(\phi^{-1}\right)^* \alpha\right)_{z_s}\left(F^{\phi}_{x_{s,t}}\left(z_s\right)\right), \text{ and} \tag{5.15}$$

$$F_w\left(y_s\right)\left[\alpha \circ F_{\tilde{w}}\right] = \left(\partial_{F^{\phi}_w(z_s)}\left[\left(\phi^{-1}\right)^* \alpha \circ F^{\phi}_{\tilde{w}}\right]\right)\left(z_s\right). \tag{5.16}$$

Equation (5.14) is true by Theorem 4.47. The differential geometric identities in Eqs. (5.15) and (5.16) are simply a matter of unwinding the definitions.

($2 \implies 3$) By letting $\alpha = df$ and using Theorem 4.45, we have

$$\begin{aligned} f\left(y_t\right) - f\left(y_s\right) &= \int_s^t df\left(d\mathbf{y}\right) \\ &\underset{3}{\approx} df\left(F_{x_{s,t}}\left(y_s\right)\right) + F_w\left(y_s\right)\left[df \circ F_{\tilde{w}}\right]|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}} \\ &= \left(\mathcal{F}_{\mathbf{X}_{s,t}} f\right)\left(y_s\right) \end{aligned}$$

($3 \implies 1$) We leave it to the reader to work through the details of this step which follow exactly as in the proof of Theorem 3.48 by letting f^i be the coordinates of ϕ. ■

By Theorem B.2 in the Appendix, we see that a solution to a rough differential equation in flat space does actually satisfy Eq. (5.8). Moreover, we immediately get local existence of solutions:

Theorem 5.4 *Let $F : W \to \Gamma\left(TM\right)$ be linear and let $\bar{y}_0$ be a point in M. There exists a local in time solution to the differential Eq. (5.8) with initial condition $y_0 = \bar{y}_0$.*

Proof. Let ϕ be any chart such that $\bar{y}_0 \in D\left(\phi\right)$. Then there exists a solution on some time interval $[0, \tau]$ in $R\left(\phi\right)$ to the differential equation

$$d\mathbf{z}_t = F^{\phi}_{d\mathbf{X}_t}\left(z_t\right)$$

with initial condition $z_0 = \phi(\bar{y}_0)$. If $\tilde{\phi}$ is any other chart such that $[a,b] \subseteq [0,\tau]$ and $y([a,b]) \subseteq D\left(\tilde{\phi}\right)$, then the transition map $\tilde{\phi} \circ \phi^{-1}$ has a domain containing $z([a,b])$. It is easy to check that

$$F^{\tilde{\phi}} = \left(F^{\phi}\right)^{\tilde{\phi} \circ \phi^{-1}}$$

and by Corollary B.5, after unraveling the notation, we have

$$\tilde{\phi}(y_t) \underset{3}{\approx} \tilde{\phi}(y_s) + d\tilde{\phi} \circ F_{x_{s,t}}(y_s) + F_w(y_s)\left[d\tilde{\phi} \circ F_{\tilde{w}}\right]|_{w \otimes \tilde{w} = \mathbb{X}_{s,t}}.$$

Thus satisfying the rough differential equation approximation in one chart is sufficient prove that it hold in all charts. ■

Solutions to rough differential equations will be unique on the intersection of their time domain up to some possible explosion time. This is stated more precisely in the following theorem.

Theorem 5.5 *Let $T > 0$. There is unique solution $\mathbf{y}_t \in CRP_{\mathbf{X}}(M)$ to $d\mathbf{y}_t = F_{d\mathbf{X}_t}(y_t)$ with initial condition $y_0 = \bar{y}_0$ existing either on all of $[0,T]$ or on $[0,\tau)$ for some $\tau < T$ such that the closure of $\{y_t : 0 \leq t < \tau\}$ is not compact.*

Proof. This proof follows the strategy of the proof of Theorem 4.2 in [3]. First we will show that we can always concatenate a solution $\mathbf{y}$ provided it has not exploded yet:

Suppose there exists a $\mathbf{y}$ solving $d\mathbf{y}_t = F_{d\mathbf{X}_t}(y_t)$ with initial condition $y_0 = \bar{y}_0$ on $[0,\tau)$. If there exists a compact $K \subseteq M$ such that $\{y_t : 0 \leq t < \tau\} \subseteq K$, then there is a sequence of increasing times $t_n \in [0,\tau)$ such that $t_n \to \tau$ and $y_\infty := \lim_{n\to\infty} y(t_n)$ exists and is in K. We can now choose a chart ϕ such that the closure of $D(\phi)$ is compact and such that $y_\infty \in D(\phi)$. Let $\mathbf{z}_t$ and a be such that $\mathbf{z}_t := \phi_* \mathbf{y}$ on some time interval $[a,\tau)$ such that $y([a,\tau)) \subseteq D(\phi)$. By appealing to Lemma B.1 in the Appendix, there exists an $\epsilon > 0$ and a $U \subseteq D(\phi)$ containing y_∞

such that for any $s \in [\tau - \epsilon, \tau]$ and $\bar{z} \in U$, there exists $\tilde{\mathbf{z}} \in CRP_{\mathbf{X}}\left(\mathbb{R}^d\right)$ defined on $[s, \tau + \epsilon]$ which solves

$$d\tilde{\mathbf{z}}_t = F^{\phi}_{x_{s,t}}\left(\tilde{z}_t\right) \quad \text{with} \quad \tilde{z}_s = \bar{z}.$$

Letting n be sufficiently large, we have that $t_n \in [\tau - \epsilon, \tau]$ and we let $\tilde{\mathbf{z}}$ be the solution to $d\tilde{\mathbf{z}}_t = F^{\phi}_{x_{s,t}}\left(\tilde{z}_t\right)$ with initial condition $\tilde{z}_s = z\left(t_n\right)$. Then we can concatenate $\mathbf{z}$ and $\tilde{\mathbf{z}}$ in the sense of Lemma 2.9. By pulling these back to the manifold by ϕ^{-1}, we now have a solution $\tilde{\mathbf{y}}$ on M which is defined on $[0, \tau + \epsilon]$.

With the preceding fact shown, we may now prove the theorem. We define

$$\tau := \sup\left\{T_0 \in (0, T) : \exists \mathbf{y} \text{ solving } d\mathbf{y}_t = F_{d\mathbf{X}_t}\left(y_t\right) \text{ with } y_0 = \bar{y}_0\right\}.$$

We can then for any $t < \tau$ define $\mathbf{y}_t := \hat{\mathbf{y}}_t$ where $\hat{\mathbf{y}}_t$ is any solution to $d\mathbf{y}_t = F_{d\mathbf{X}_t}\left(y_t\right)$ with initial condition $y_0 = \bar{y}_0$. By the uniqueness of solutions to rough differential equations on flat space and the fact that we can cover any portion of the path with the domain of a chart, we know that $\mathbf{y}_t$ is well defined, and in fact satisfies $d\mathbf{y}_t = F_{d\mathbf{X}_t}\left(y_t\right)$ on all of $[0, \tau)$. If the closure of $\{y_t : 0 \leq t < \tau\}$ is compact, then from what we showed above, we can produce a solution $\tilde{\mathbf{y}}$ which is defined on $[0, \tau + \epsilon]$ for some $\epsilon > 0$. In this case, τ must be T and $\tilde{\mathbf{y}}|_{[0,T]}$ is a solution defined on all of $[0, T]$. ■

Definition 5.6 *Let $f : M \to N$ be a smooth map between manifolds. Let $F : W \to \Gamma(TM)$ and $\tilde{F} : W \to \Gamma(TN)$ be linear. We say F and $\tilde{F}$ are $f-$* ***related dynamical systems*** *if*

$$f_* F_w = \tilde{F}_w \circ f \textit{ for all } w \in W.$$

As in the flat case and shown in the Appendix in Theorem B.4, we have a relation between dynamical systems. The proof is no different in the manifold case, and so we omit it.

Theorem 5.7 *Suppose $f : M \to N$ is a smooth map between manifolds and let $F : W \to \Gamma(TM)$ and $\tilde{F} : W \to \Gamma(TN)$ be $f-$related dynamical systems. If $\mathbf{y}$ solves the initial value problem Eq. (5.8), then $\tilde{\mathbf{y}}_t := \left(\tilde{y}_t, \tilde{y}_s^\dagger\right) := f_* \tilde{\mathbf{y}}_t$ solves*

$$d\tilde{\mathbf{y}}_t = \tilde{F}_{d\mathbf{X}_t}(\tilde{y}_t) \quad \text{with} \quad \tilde{y}_0 = f(\bar{y}_0).$$

5.2.1 RDEs from the Gauge Perspective

Following the theme of Theorem 3.33, we also have a way to view a solution to a differential equation using the gauge perspective. Let ψ be a logarithm on M with diagonal domain $\mathcal{D}$.

Theorem 5.8 *Let y be a path in M on I_0 with $y_s^\dagger = F_\cdot(y_s)$. Let $\mathbf{y} = \left(y, y^\dagger\right)$. Then $\mathbf{y}$ solves Equation (5.8) if and only if for every a, b such that $[a, b] \subseteq I_0$, there exists a $\delta > 0$ such that*

$$\psi(y_s, y_t) \underset{3}{\approx} F_{x_{s,t}}(y_s) + F_w(y_s)\left[(\psi_{y_s})_* F_{\tilde{w}}\right]|_{w \otimes \tilde{w} = \mathbb{X}_{s,t}}. \tag{5.17}$$

provided $a \leq s \leq t \leq b$ and $t - s < \delta$.

Proof. This proof will be similar to the proof of Theorem 3.33.

First we show the condition of Theorem 5.8 implies that $\mathbf{y}$ solves Equation (5.8). Let ϕ be a chart and let $[a, b]$ be such that $y([a, b]) \subseteq D(\phi)$. By defining

$$z_s := \phi(y_s)$$
$$\psi^\phi(x, y) := \phi_* \psi\left(\phi^{-1}(x), \phi^{-1}(y)\right)$$
$$F_w^\phi(x) := d\phi\left(F_w\left(\phi^{-1}(x)\right)\right)$$

and denoting $\psi^{\phi}(x,y) = \left(x, \bar{\psi}^{\phi}(x,y)\right)$, Eq. (5.17), once pushed forward by ϕ, can be written as

$$\bar{\psi}^{\phi}(z_s, z_t) \underset{3}{\approx} F^{\phi}_{x_{s,t}}(z_s) + \left(\partial_{F^{\phi}_w(z_s)}\left[\bar{\psi}^{\phi\prime}_{z_s}(\cdot)\, F^{\phi}_{\tilde{w}}(\cdot)\right]\right)(z_s)\,|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}}$$

provided $a \leq s \leq t \leq b$ and $t - s < \delta$. We then must prove that z solves Eq. (5.10) for all $a \leq s \leq t \leq b$. However, by appealing to Lemma 3.37 and Lemma B.6 of the Appendix, we only need to prove Eq. (5.10) holds for every u in $[a,b]$ for $s \leq t$ in $(u - \delta_u, u + \delta_u) \cap [a,b]$ for some δ_u. We do this now:

For any $u \in [a,b]$, let $\mathcal{W}_u$ be an open convex set of z_u such that $\mathcal{W}_u \times \mathcal{W}_u \subseteq D\left(\psi^{\phi}\right)$. We then choose $\delta_u > 0$ such that $z\left([u-\delta_u, u+\delta_u] \cap [a,b]\right) \subseteq \mathcal{W}_u$ and $2\delta_u \leq \delta$. We are now in the setting of Proposition 5.1 and have therefore shown $\mathbf{y}$ solves Eq. (5.8).

For the reverse implication, let $[a,b] \subseteq I_0$ be given. Choose $\delta > 0$ such that $|t-s| \leq \delta$ for $a \leq s \leq t \leq b$ implies that $\left|\psi(y_s, y_t)\right|_g$ is bounded. Around every point m of $y([a,b])$, there exists an open $\mathcal{O}_m$ containing m such that $\mathcal{O}_m \times \mathcal{O}_m \subseteq \mathcal{D}$. Additionally for each m there exists a chart ϕ^m such that $m \in D(\phi^m)$, $D(\phi^m) \subseteq \mathcal{O}_m$, and $\mathcal{W}_m := R(\phi^m)$ is convex. We may now use Remark 3.41 with the cover $\{\mathcal{V}_m\}_{m \in y([a,b])}$ and $D = \{(s,t) : a \leq s \leq t \leq b \text{ and } |t-s| \leq \delta\}$ with the function

$$(s,t) \longrightarrow \psi(y_s, y_t) - F_{x_{s,t}}(y_s) - F_w(y_s)\left[(\psi_{y_s})_* \circ F_{\tilde{w}}\right]|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}}.$$

Doing this, we have reduced to considering the case of our path being contained in the domain of a single chart ϕ such that $D(\phi) \times D(\phi) \subseteq \mathcal{D}$ and $R(\phi)$ is convex. By using the same definitions above for z_s, F^{ϕ}, and ψ^{ϕ}, we reduce proving

$$\psi(y_s, y_t) \underset{3}{\approx} F_{x_{s,t}}(y_s) + F_w(y_s)\left[(\psi_{y_s})_* \circ F_{\tilde{w}}\right]|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}}$$

to the flat case

$$\bar{\psi}^{\phi}(z_s, z_t) \underset{3}{\approx} F^{\phi}_{x_{s,t}}(z_s) + \left(\partial_{F^{\phi}_w(z_s)}\left[\bar{\psi}^{\phi\prime}_{z_s}(\cdot)\, F^{\phi}_{\tilde{w}}(\cdot)\right]\right)(z_s)\,|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}}.$$

This is now in the setting of Proposition 5.1 and hence we are finished. ■

Akin to the integral formulas, there is also a characterization of solving a differential equation which involves a gauge (ψ, U).

Theorem 5.9 $\mathbf{y} = \left(y, y^{\dagger}\right)$ *on* I_0 *solves (5.8) if and only if* $y^{\dagger}_s = F_{(\cdot)}(y_s)$ *and for all* $[a,b] \subseteq I_0$*, there exists a* $\delta > 0$ *such that* $|t-s| \leq \delta$*, and* $a \leq s \leq t \leq b$ *implies*

$$\psi(y_s, y_t) \underset{3}{\approx} F_{x_{s,t}}(y_s) + \left(-S^{\psi_*,U}_{y_s}\left[F_w(y_s)\otimes F_{\tilde{w}}(y_s)\right] + F_w(y_s)\left[U(y_s,\cdot)\, F_{\tilde{w}}\right]\right)|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}}.$$

Proof. This follows immediately from the product rule:

$$\begin{aligned} F_w(y_s)\left[(\psi_{y_s})_*\, F_{\tilde{w}}\right] &= F_w(y_s)\left[(\psi_{y_s})_{*(\cdot)}\, U(y_s,\cdot)^{-1}\, U(y_s,\cdot)\, F_{\tilde{w}}\right] \\ &= -S^{\psi_*,U}_{y_s}\left[F_w(y_s)\otimes F_{\tilde{w}}(y_s)\right] + F_w(y_s)\left[U(y_s,\cdot)\, F_{\tilde{w}}\right] \end{aligned}$$

■

Example 5.10 *If* ∇ *is a covariant derivative, then* $\mathbf{y}$ *on* I_0 *solves (5.8) if and only if* $y^{\dagger}_s = F(y_s)$ *and*

$$\exp^{-1}_{y_s}(y_t) \underset{3}{\approx} F_{x_{s,t}}(y_s) + \left(\nabla_{F_w(y_s)} F_{\tilde{w}}\right) - \frac{1}{2} T^{\nabla}\left[F_w(y_s)\otimes F_{\tilde{w}}(y_s)\right]|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}}$$

for s *and* t *close.*

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Chapter 6

Parallel Translation and Related Topics

6.1 Some Auxiliary Results

This section gathers some further results which are needed below to discuss rough horizontal lifts.

Lemma 6.1 *Let M and N be two manifolds and let π^M and π^N be the projection maps from $M \times N$ to M and N respectively. The map*

$$\left(\pi_*^M, \pi_*^N\right) : CRP_{\mathbf{X}}\left(M \times N\right) \to CRP_{\mathbf{X}}\left(M\right) \times CRP_{\mathbf{X}}\left(N\right) \tag{6.1}$$

is a bijection.

Proof. Let $\mathbf{u} = \left(u, u^{\dagger}\right) \in CRP_{\mathbf{X}}\left(M \times N\right), \mathbf{y} = \left(y, y^{\dagger}\right) := \pi_*^M\left(\mathbf{u}\right) \in CRP_{\mathbf{X}}\left(M\right)$ and $\mathbf{z} = \left(z, z^{\dagger}\right) := \pi_*^N\left(\mathbf{u}\right) \in CRP_{\mathbf{X}}\left(N\right).$ Then $u_t = \left(y_t, z_t\right)$ is uniquely determined by y and z. Similarly, since

$$\left(\pi_*^M, \pi_*^N\right) : T_{u_t}\left(M \times N\right) \longrightarrow T_{y_t}M \times T_{z_t}N$$

is an isomorphism it follows that $u_t^\dagger : W \to T_{u_t}(M \times N)$ is uniquely determined by $\left(\pi_*^M, \pi_*^N\right) u_t^\dagger = \left(y_t^\dagger, z_t^\dagger\right)$. Thus, the only real content of the lemma is that the map in Eq. (6.1) is surjective.

Suppose that $\mathbf{y} = \left(y, y^\dagger\right) \in CRP_{\mathbf{X}}(M)$ and $\mathbf{z} = \left(z, z^\dagger\right) \in CRP_{\mathbf{X}}(N)$ are given and define $\mathbf{u} = \left(u, u^\dagger\right)$ so that $u = (y, z)$ and $\left(\pi_*^M, \pi_*^N\right) u_t^\dagger = \left(y_t^\dagger, z_t^\dagger\right)$. To finish the proof we need to verify that $\mathbf{u} \in CRP_{\mathbf{X}}(M \times N)$. To this end suppose that ψ^M and ψ^N are logarithms on M and N respectively. We then let

$$\psi\left((m, n), (\tilde{m}, \tilde{n})\right) := \left(\psi^M(m, \tilde{m}), \psi^N(n, \tilde{n})\right) \in T_{(m,n)}[M \times N]$$

where we are now using $\left(\pi_*^M, \pi_*^N\right)$ to identify $T[M \times N]$ with $TM \times TN$. The map ψ is now a logarithm on $M \times N$. The parallelism U^ψ associated to this logarithm satisfies

$$U^\psi\left((m, n), (\tilde{m}, \tilde{n})\right) = U^{\psi^M}(m, \tilde{m}) \times U^{\psi^N}(n, \tilde{n}).$$

Therefore we have

$$\psi\left((y_s, z_s), (y_t, z_t)\right) = \left(\psi^M(y_s, y_t), \psi^N(z_s, z_t)\right) \underset{2}{\approx} \left(y_s^\dagger x_{s,t}, z_s^\dagger x_{s,t}\right) = u_s^\dagger x_{s,t}$$

and

$$\begin{aligned} U^\psi\left((y_s, z_s), (y_t, z_t)\right) u_t^\dagger &= \left(U^{\psi^M}(y_s, y_t) \times U^{\psi^N}(z_s, z_t)\right) \left(y_t^\dagger, z_t^\dagger\right) \\ &= \left(U^{\psi^M}(y_s, y_t)\, y_t^\dagger, U^{\psi^N}(z_s, z_t)\, z_t^\dagger\right) \\ &\underset{1}{\approx} \left(y_s^\dagger, z_s^\dagger\right) = u_s^\dagger \end{aligned}$$

from which it follows that $\mathbf{u} \in CRP_{\mathbf{X}}(M \times N)$. ■

In this section, we will use heavily results from the RDE theory in Chapter 5. We will also add a slightly more general interpretation of solving such a differential equation as explained in the following notation.

Notation 6.2 *Let* $\mathbf{z} = \left(z, z^{\dagger}\right) \in CRP_{\mathbf{X}}\left(\mathbb{R}^{\tilde{k}}\right)$ *and suppose* $F : \mathbb{R}^{\tilde{k}} \to \Gamma(TM)$ *is linear. Let* $\mathbf{Z} = (z, \mathbb{Z})$ *be the rough path associated to* $\mathbf{z}$ *such that*

$$\mathbb{Z}_{s,t} \underset{3}{\approx} \left(z_s^{\dagger} \otimes z_s^{\dagger}\right) \mathbb{X}_{s,t}$$

We say $\mathbf{y} = \left(y, y^{\dagger}\right) \in CRP_{\mathbf{X}}(M)$ *on* $I_0 = [0, T]$ *or* $[0, T)$ *solves the differential equation*

$$d\mathbf{y}_t = F_{d\mathbf{z}_t}(y_t)$$

if $y_s^{\dagger} = F_{z_s^{\dagger}(\cdot)}(y_s)$ *and for every* $f \in C^{\infty}(M)$ *and* $[a, b] \subseteq I_0$*, the approximation*

$$f(y_t) - f(y_s) \underset{3}{\approx} \left(\mathcal{F}_{\mathbf{Z}_{s,t}} f\right)(y_s)$$

holds for $a \le s \le t \le b$*.*

Note that this is equivalent to solving the differential equation

$$d\mathbf{y}_t = F_{d\mathbf{Z}_t}(y_t)$$

except that we demand $y_s^{\dagger} = F_{z_s^{\dagger}(\cdot)}(y_s)$ *instead of* $y_s^{\dagger} = F_{(\cdot)}(y_s)$ *so that* $\mathbf{y}$ *is indeed controlled by* $\mathbf{X}$*. That* $\mathbf{y} \in CRP_{\mathbf{X}}(M)$ *is a consequence of the simple fact (left to the reader) that if* $\mathbf{y} = \left(y, y^{\dagger}\right) \in CRP_{\mathbf{Z}}(M)$ *and* $\mathbf{z} = \left(z, z^{\dagger}\right) \in CRP_{\mathbf{X}}\left(\mathbb{R}^{\tilde{k}}\right)$*, then*

$$\left(y, y^{\dagger} z^{\dagger}\right) \in CRP_{\mathbf{X}}(M).$$

Theorem 6.3 *Let* G *be a Lie group and set* $F_A(g) := -A \cdot g = -R_{g*}A$ *for all* $A \in \mathfrak{g}$*. Let* $\mathbf{z} \in CRP_{\mathbf{X}}(\mathfrak{g})$ *be defined on* $[0, T]$*. There exists a unique global solution* $\mathbf{g} \in CRP_{\mathbf{X}}(G)$ *solving*

$$d\mathbf{g}_t = F_{d\mathbf{z}_t}(g_t) \tag{6.2}$$

with initial condition

$$g_0 = e \in G.$$

Proof. This proof is adapted from Theorem 4.20 in [3]. By Theorem 5.5, we know we only have to rule out the case that $\mathbf{g}$ exists maximally on $[0,\tau)$ for some $\tau \leq T$ where $\{g_t : 0 \leq t < \tau\}$ does not have compact closure.

By Lemma B.1, there exists an $\epsilon > 0$ such that for any $t_0 \in [0,T]$ there is a solution $\mathbf{h}$ defined on $[t_0, (t_0+\epsilon)\wedge T]$ solving $d\mathbf{h} = F_{d\mathbf{z}_t}(h_t)$ with initial condition $h_{t_0} = e$. If right multiplication map is given by $R_{g_2}(g_1) = g_1 g_2$, then it is easy to see that F is R_g-related to itself so that $\mathbf{k} = (R_{\bar{g}})_* \mathbf{h}$ on $[t_0, (t_0+\epsilon)\wedge T]$ solves

$$d\mathbf{k}_t = F_{d\mathbf{z}_t}(k_t)$$

with initial condition

$$k_{t_0} = \bar{g}.$$

Choosing $t_0 \in (0 \vee (\tau - \epsilon/2), \tau)$, we can concatenate $\mathbf{g}$ and $\mathbf{k}$ where we start $\mathbf{k}$ at t_0 with initial condition $k_{t_0} = g(t_0)$. This gives us a solution defined on an interval strictly larger than $[0,\tau)$ and thus shows that the second case mentioned above cannot occur. ■

Theorem 6.4 *Let G be a Lie group and $\theta := \theta_r \in \Omega^1(G,\mathfrak{g})$ be the right – Maurer-Cartan form on G, i.e.*

$$\theta(\xi_g) = \xi_g \cdot g^{-1} := R_{g^{-1}*}\xi_g.$$

Further set $F_A(g) := -A \cdot g = -R_{g*}A$ for all $A \in \mathfrak{g}$. If $\mathbf{z} := \left(z, z^\dagger\right) \in CRP_{\mathbf{X}}(\mathfrak{g})$ is a $\mathfrak{g}$ – valued rough path, then $\mathbf{g} = \left(g, g^\dagger\right) \in CRP_{\mathbf{X}}(G)$ satisfies

$$d\mathbf{g} = F_{d\mathbf{z}}(g) \tag{6.3}$$

iff

$$\int \theta \left(d\mathbf{g} \right) = -\mathbf{z}. \tag{6.4}$$

Proof. First we note that

$$\theta \left(F_b \left(g \right) \right) = -b \quad \text{for all} \quad b \in \mathfrak{g}, g \in G \tag{6.5}$$

and

$$F_{\theta(\xi_g)} \left(g \right) = -\xi_g \quad \text{for all} \quad \xi_g \in T_g G. \tag{6.6}$$

Assume Eq. (6.3) holds. Using Theorem 5.3 we learn that

$$\left[\int \theta \left(d\mathbf{g} \right) \right]_{s,t}^{1} \underset{3}{\approx} \theta \left(F_{z_{s,t}} \left(g_s \right) \right) + F_A \left(g_s \right) \theta \left(F_B \right) |_{A \otimes B = z_s^\dagger \otimes z_s^\dagger \mathbb{X}_{s,t}} = -z_{s,t}$$

wherein we have used Eq. (6.5) along with the fact that

$$F_A \left(g_s \right) \theta \left(F_B \right) = F_A \left(g_s \right) \left(-B \right) = 0$$

which also follows from Eq. (6.5). Moreover we have

$$\left[\int \theta \left(d\mathbf{g} \right) \right]_{s}^{\dagger} = \theta \left(F_{z_s^\dagger(\cdot)} \left(g_s \right) \right) = -z_s^\dagger$$

and hence we have shown that Eq. (6.4) holds.

Conversely, let us suppose that Eq. (6.4) holds and $\alpha \in \Omega^1 \left(G, V \right)$ is a smooth one-form on G. We will show that

$$\int_s^t \alpha \left(d\mathbf{g} \right) \underset{3}{\approx} \alpha \left(F_{z_{s,t}} \left(g_s \right) \right) + F_A \left(g_s \right) \alpha \left(F_B \right) |_{A \otimes B = z_s^\dagger \otimes z_s^\dagger \mathbb{X}_{s,t}} \tag{6.7}$$

and

$$g_s^\dagger = F_{z_s^\dagger(\cdot)} \left(g_s \right) \tag{6.8}$$

To prove Eq. (6.8), we note that by Eq. (6.5)

$$g_s^\dagger = -F_{\theta\left(g_s^\dagger(\cdot)\right)}\left(g_s\right) = F_{z_s^\dagger(\cdot)}\left(g_s\right)$$

where the second equality follows from the fact that

$$\theta\left(g_s^\dagger\left(\cdot\right)\right) = -z_s^\dagger.$$

To prove Eq. (6.7), we will first write α as the composition of two functions. Given $\xi_g \in T_gG$ we have $\xi_g = \theta\left(\xi_g\right) \cdot g = R_{g*}\theta\left(\xi_g\right)$ and therefore

$$\alpha\left(\xi_g\right) = \alpha\left(R_{g*}\theta\left(\xi_g\right)\right) = \left(R_g^*\alpha\right)\theta\left(\xi_g\right).$$

This shows $\alpha = K\theta$ where $K : G \to \mathrm{End}\left(\mathfrak{g}, V\right)$ is the function defined by

$$K\left(g\right) := R_g^*\alpha = \alpha_g \circ R_{g*}.$$

Applying Theorem 4.41 with $\mathbf{y}$ replaced by $\mathbf{g}$ shows,

$$\int \alpha\left(d\mathbf{g}\right) = \int \left(K\theta\right)\left(d\mathbf{g}\right) = -\int K_*\left(\mathbf{g}\right)d\mathbf{z}.$$

So, according to Theorem 5.3, it only remains to show

$$-\left[\int K_*\left(\mathbf{g}\right)d\mathbf{z}\right]_{s,t}^1 \underset{3}{\approx} \alpha\left(F_{z_{s,t}}\left(g_s\right)\right) + F_A\left(g_s\right)\alpha\left(F_B\right)|_{A\otimes B = z_s^\dagger \otimes z_s^\dagger \mathbb{X}_{s,t}}. \tag{6.9}$$

In order to work out the left side of Eq. (6.9) we need to expand out $K_{*g_s}g_s^\dagger$

which we now do. If $\xi_g = \dot{g}(0) \in T_g G$ and $A \in \mathfrak{g}$, then

$$\begin{aligned}(K_* \xi_g) A &= \frac{d}{dt}|_0 K(g(t)) A = \frac{d}{dt}|_0 \left(R^*_{g(t)} \alpha\right) A \\ &= \frac{d}{dt}|_0 \alpha\left(R_{g(t)*} A\right) = -\frac{d}{dt}|_0 \alpha\left(F_A(g(t))\right) \\ &= -\xi_g \alpha(F_A).\end{aligned}$$

Therefore for $A, B \in \mathfrak{g}$,

$$\left(K_{*g_s} g_s^\dagger A\right) B = -\left(g_s^\dagger A\right) \alpha(F_B). \tag{6.10}$$

As mentioned above, we have

$$g_s^\dagger A = F_{z_s^\dagger A}(g_s)$$

which combined with Eq. (6.10) shows

$$\left(K_{*g_s} g_s^\dagger A\right) B = -F_{z_s^\dagger A}(g_s) \alpha(F_B).$$

Using this result it now follows that

$$\begin{aligned}-\left[\int K_*(\mathbf{g})\, d\mathbf{z}\right]^1_{s,t} &\underset{3}{\approx} -K(g_s) z_{s,t} - \left(K_{*g_s} g_s^\dagger\right)\left(I \otimes z_s^\dagger\right) \mathbb{X}_{s,t} \\ &= -\alpha_{g_s} R_{g_s *} z_{s,t} + F_{z_s^\dagger A}(g_s) \alpha(F_B)|_{A\otimes B = \left(I \otimes z_s^\dagger\right)\mathbb{X}_{s,t}} \\ &= \alpha\left(F_{z_{s,t}}(g_s)\right) + F_A(g_s) \alpha(F_B)|_{A \otimes B = z_s^\dagger \otimes z_s^\dagger \mathbb{X}_{s,t}}\end{aligned}$$

which is the desired result. ■

6.2 Controlled Rough Path Horizontal Lifts

Here we show that rough horizontal lifts always exist and are unique. Given all of the preparation, this section is now fairly straightforward and clean. The reader may refer to Appendix C for the definition of a connection one-form and other concepts related to principal bundles.

Definition 6.5 (Rough Horiontal Lifts) *Let* $\left(G \to P \overset{\pi}{\to} M, \omega\right)$ *be a principal bundle with connection* ω *and* $\mathbf{y} = \left(y, y^{\dagger}\right) \in CRP_{\mathbf{X}}(M)$. *A controlled rough path* $\mathbf{u} = \left(u, u^{\dagger}\right) \in CRP_{\mathbf{X}}(P)$ *is said to be a* ***horizontal lift*** *of* $\mathbf{y}$ *if*

$$\pi_*(\mathbf{u}) = \mathbf{y} \quad \textit{and} \quad \int \omega(d\mathbf{u}) \equiv \mathbf{0} \tag{6.11}$$

where $\mathbf{0}$ *is the controlled rough path whose path and derivative process are identically* 0. *We will write* $\mathbf{u} \in CRP_{\mathbf{y}}^{H,\omega}(P)$ *for such lifts.*

Remark 6.6 *One might ask that Definition 6.5 should include the requirement that*

$$u^{\dagger} = \mathcal{B}_u^{\omega} y^{\dagger}$$

or more explicitly $u_t^{\dagger} w = \mathcal{B}_{u_t}^{\omega} y_t^{\dagger} w$ *for all* t *and* $w \in W$ *(see Notation C.9). However, this is redundant as* $\pi_*(\mathbf{u}) = \mathbf{y}$ *implies that*

$$\pi_* u_s^{\dagger} = y_s^{\dagger} \tag{6.12}$$

while $\int \omega(d\mathbf{u}) \equiv 0$ *implies that*

$$\omega\left(u_s^{\dagger}\right) = 0.$$

Eq. (6.12) and Eq. (6.13) together imply that $u^{\dagger} = \mathcal{B}_u^{\omega} y^{\dagger}$.

Proposition 6.7 *Suppose $G \to \tilde{P} \overset{\tilde{\pi}}{\to} \tilde{M}$ is another principal bundle with connection $\tilde{\omega}$. Further suppose that $f : M \to \tilde{M}$ is a smooth map, $F : P \to \tilde{P}$ is a bundle map above f, and $\omega = F^{*}\tilde{\omega}$. If $\mathbf{u} = \left(u, u^{\dagger}\right) \in CRP_{\mathbf{X}}\left(P\right)$ is a horizontal lift of $\mathbf{y} \in CRP_{\mathbf{X}}\left(M\right)$, then $\mathbf{v} := F_{*}\left(\mathbf{u}\right) \in CRP_{\mathbf{X}}\left(\tilde{P}\right)$ is a a horizontal lift of $f_{*}\left(\mathbf{y}\right) \in CRP_{\mathbf{X}}\left(\tilde{M}\right)$.*

Proof. By Remark 4.44 and the fact that $\mathbf{u}$ is a lift of $\mathbf{y}$ we have

$$\tilde{\pi}_{*}F_{*}\left(\mathbf{u}\right) = \left(\tilde{\pi} \circ F\right)_{*}\left(\mathbf{u}\right) = \left(f \circ \pi\right)_{*}\left(\mathbf{u}\right) = f_{*}\pi_{*}\left(\mathbf{u}\right) = f_{*}\left(\mathbf{y}\right)$$

and hence $F_{*}\left(\mathbf{u}\right)$ is a lift of $f_{*}\left(\mathbf{y}\right)$. Secondly we observe from the push-me pull-me Theorem 4.47 that

$$\int \tilde{\omega}\left(d\mathbf{v}\right) = \int \tilde{\omega}\left(dF_{*}\mathbf{u}\right) = \int \left(F^{*}\tilde{\omega}\right)\left(d\mathbf{u}\right) = \int \omega\left(d\mathbf{u}\right) \equiv \mathbf{0}.$$

■

Theorem 6.8 (Existence of Horizontal Lifts) *Let $G \to P \overset{\pi}{\to} M$ be a principal bundle with connection ω, $\mathbf{y} = \left(y, y^{\dagger}\right) \in CRP_{\mathbf{X}}\left(M\right)$, and $\bar{u}_0 \in P_{y_0}$. Then there exists a unique horizontal lift $\mathbf{u} = \left(u, u^{\dagger}\right) \in CRP_{\mathbf{X}}\left(P\right)$ above $\mathbf{y}$ such that $u_0 = \bar{u}_0$.*

Proof. The proof follows the lines of the smooth case. Because of Proposition 6.7 and simple patching arguments we may reduce to considering the case that $P = M \times G$ is a trivial bundle where M is now an open subset of $\mathbb{R}^d$. In light of Lemma 6.1, the desired horizontal lift may be expressed in the form, $\mathbf{u}_s = \left(\mathbf{y}_s, \mathbf{g}_s\right)$ for some $\mathbf{g} = \left(g, g^{\dagger}\right) \in CRP_{\mathbf{X}}\left(G\right)$ which is to be determined. We now find the equations that $\mathbf{g}$ has to solve in order for $\mathbf{u}$ to be horizontal.

Let $\pi_G : P = M \times G \to G$ and $\pi_M : P \to M$ be the natural projection maps. Making use of Theorem 4.41, we may deduce that

$$\int \omega\left(d\mathbf{u}\right) \equiv \mathbf{0} \iff \int \left(Ad_{\pi_G}\omega\right)\left(d\mathbf{u}\right) \equiv \mathbf{0}. \tag{6.13}$$

where by $Ad_{\pi_G}\omega$, we mean the one-form given by

$$Ad_{\pi_G}\omega\left(v_m, \xi_g\right) = Ad_g\omega\left(v_m, \xi_g\right).$$

Indeed if $\int \omega\left(d\mathbf{u}\right) \equiv \mathbf{0}$, then

$$\int Ad_{\pi_G}\omega\left(d\mathbf{u}\right) = \int \left(Ad_{\pi_G}\right)_*\left(\mathbf{u}\right)\left(\int \omega\left(d\mathbf{u}\right)\right) \equiv \mathbf{0}$$

and if $\int \left(Ad_{\pi_G}\omega\right)\left(d\mathbf{u}\right) \equiv 0$

$$\begin{aligned}\int \omega\left(d\mathbf{u}\right) &= \int \left(Ad_{\pi_G^{-1}} Ad_{\pi_G}\omega\right)\left(d\mathbf{u}\right)\\ &= \int Ad_{\pi_G^{-1}}\left(\mathbf{u}\right)\left(\int Ad_{\pi_G}\omega\left(d\mathbf{u}\right)\right) \equiv \mathbf{0}.\end{aligned}$$

On the other hand

$$\begin{aligned}\left(Ad_{\pi_G}\omega\right)\left(\left(v_m, \xi_g\right)\right) &= Ad_g\left[\theta_l\left(\xi_g\right) + Ad_{g^{-1}}\Gamma\left(v_m\right)\right]\\ &= \theta_r\left(\xi_g\right) + \Gamma\left(v_m\right)\end{aligned}$$

from which we deduce

$$Ad_{\pi_G}\omega = \pi_G^*\theta_r + \pi_M^*\Gamma.$$

An application of the push-me pull-me Theorem 4.47 then shows

$$\begin{aligned}\int \left(Ad_{\pi_G}\omega\right)\left(d\mathbf{u}\right) &= \int \left(\pi_G^*\theta_r\right)\left(d\mathbf{u}\right) + \int \left(\pi_M^*\Gamma\right)\left(d\mathbf{u}\right)\\ &= \int \theta_r\left(d\mathbf{g}\right) + \int \Gamma\left(d\mathbf{y}\right).\end{aligned}$$

Combining these statements shows $\int \omega\left(d\mathbf{u}\right) \equiv \mathbf{0}$ is equivalent to $\mathbf{u}_s = \left(\mathbf{y}_s, \mathbf{g}_s\right)$ satisfying

$$\int \theta_r\left(d\mathbf{g}\right) = -\int \Gamma\left(d\mathbf{y}\right) \tag{6.14}$$

which according to Theorem 6.4 is equivalent to $\mathbf{g}$ satisfying

$$d\mathbf{g} = F_{d\mathbf{z}}(g) \tag{6.15}$$

where $F_A(g) = -R_{g*}A$ for all $A \in \mathfrak{g}$ and $\mathbf{z} := \int \Gamma(d\mathbf{y})$. It is now know by Theorem 6.3 that (given initial conditions for g_0) that Eq. (6.15) has global unique solutions. ■

Notation 6.9 *If $\mathbf{y} \in CRP_{\mathbf{X}}(M)$ and $u_{y_0} \in P_{y_0}$, we write $\mathbf{h}(\mathbf{y}, u_{y_0}) \in CRP_{\mathbf{X}}(P)$ to denote the horizontal lift that exists by Theorem 6.8.*

We can now specialize the above to define parallel translation.

Definition 6.10 *Let $GL(M)$ be the frame bundle above M with structure group $GL(d)$, let $\mathbf{y} = (y, y^\dagger) \in CRP_{\mathbf{X}}(M)$, and let ∇ be a covariant derivative on TM. Further, let $\bar{u}_0 \in GL(M)_{y_0}$.* ***Parallel translation along $\mathbf{y}$ starting at $\bar{u}_0$*** *is the unique $\mathbf{u}$ which is an element of $CRP_{\mathbf{y}}^{H,\nabla}(GL(M)) := CRP_{\mathbf{y}}^{H,\omega^\nabla}(GL(M))$ such that*

1. $\pi_* \mathbf{u} = \mathbf{y}$
2. $\int \omega^\nabla(d\mathbf{u}) \equiv \mathbf{0}$

where ω^∇ is the connection form associated to ∇ (see Eq. (C.1)).

6.3 Rough Rolling and Unrolling

We will first introduce some terminology which will be useful for this section.

Definition 6.11 *A manifold M is* ***parallelizable*** *if there exists a linear map, $Y : \mathbb{R}^d \to \Gamma(TM)$ such that, for each $m \in M$, the map*

$$\mathbb{R}^d \ni a \longrightarrow Y_a(m) \in T_m M$$

is an isomorphism.

For every Y that parallelizes M, there exists an $\mathbb{R}^d-$ valued one-form given by

$$\theta^Y(v_m) := \left[Y_{(\cdot)}(m)\right]^{-1} v_m.$$

Example 6.12 *Let M be a manifold and let ∇ be a covariant derivative on TM. Then $GL(M)$ is parallelizable with $T_gGL(M) \cong \mathbb{R}^d \times gl(d)$ where $gl(d)$ is the set of $d \times d$ matrices. One choice of $Y^{GL(M)}$ in this case is defined by*

$$Y^{GL(M)}(u)(a, A) := B_a^{\nabla}(u) + \tilde{A}(u)$$

where B_a^{∇} is the horizontal vector field defined by

$$B_a^{\nabla}(u) = \dot{\mu}(0) \quad \text{where} \quad \mu(t) := //_t\left(\exp^{\nabla}((\cdot)\, ua)\right)u$$

and where $\tilde{A}$ is the vertical vector field given by

$$\tilde{A}(u) := \frac{d}{dt}|_0 ue^{tA} = u \cdot A.$$

Moreover, we have the associated $\mathbb{R}^d \times gl(d)-$valued one-form $\theta^{Y^{GL(M)}} := \left(\hat{\theta}, \omega^{\nabla}\right)$ which is constructed such that

$$\left(\hat{\theta}, \omega^{\nabla}\right)\left(B_a^{\nabla}(u) + \tilde{A}(u)\right) = (a, A)$$

for all $(a, A) \in \mathbb{R}^d \times gl(d)$ and $u \in GL(M)$ where $\hat{\theta}$ is the canonical one-form given by

$$\hat{\theta}(\dot{u}_0) = u_0^{-1}(\pi_* \dot{u}_0).$$

6.3.1 Rolling and Unrolling of paths

Here we have our main theorem about rolling and unrolling of paths.

Theorem 6.13 *Let M be a parallelizable manifold (by Y) and let θ^Y the associated one-form. Fix some point $o \in M$. Then every $\mathbf{y} \in CRP_{\mathbf{X}}(M)$ with $y_0 = o$ on the interval $[0,T]$ determines a path $\mathbf{z} \in CRP_{\mathbf{X}}\left(\mathbb{R}^d\right)$ with $z_0 = 0$ on the interval $[0,T]$ by the map*

$$\mathbf{y} \longrightarrow \mathbf{z} := \int \theta^Y(d\mathbf{y}).$$

such that

$$d\mathbf{y}_t = Y_{d\mathbf{z}_t}(y_t) \quad \text{and} \quad y_0 = o.$$

Alternatively, suppose that $\mathbf{z} \in CRP_{\mathbf{X}}\left(\mathbb{R}^d\right)$ with $z_0 = 0$ on the interval $[0,T]$ and let $\mathbf{y}$ be the solution to

$$d\mathbf{y}_t = Y_{d\mathbf{z}_t}(y_t)$$

with initial condition $y_0 = o$ with possible explosion time τ. Then over $[0,\tau)$ we have

$$\int \theta^Y(d\mathbf{y}) = \mathbf{z}.$$

Proof. This proof follows nearly word for word the proof of Theorem 6.4 if we replace F with $-Y$. Though this earlier proof was specialized to the Lie group case (which could not explode in finite time), the only fact we used about F and θ was how they interacted with each other. ■

We can now use Theorem 6.13 with Example 6.12 to specialize to a correspondence of paths of $GL(M)$ and those of $\mathbb{R}^d \times gl(d)$.

Theorem 6.14 *Fix some point $o \in M$ and u_o a frame at o. Every path $\mathbf{u} \in CRP_{\mathbf{X}}(GL(M))$ with $u_0 = u_o$ on the interval $[0,T]$ gives rise to a path $\mathbf{z} \in CRP_{\mathbf{X}}\left(\mathbb{R}^d \times gl(d)\right)$ with $z_0 = (0,0)$ on the interval $[0,T]$ via the map*

$$\mathbf{u} \longrightarrow \mathbf{z} := \int \theta^{Y^{GL(M)}}(d\mathbf{u}).$$

such that

$$d\mathbf{u}_t = Y^{GL(M)}_{d\mathbf{z}_t}(\mathbf{u}_t) \quad \textit{and} \quad u_0 = u_o$$

Alternatively, every $\mathbf{z} \in CRP_{\mathbf{X}}\left(\mathbb{R}^d \times gl(d)\right)$ *with* $z_0 = (0,0)$ *on the interval* $[0,T]$ *gives rise to* $\mathbf{u} \in CRP_{\mathbf{X}}(GL(M))$ *with* $u_0 = u_o$ *on with possible explosion time* τ *via the differential equation*

$$d\mathbf{u}_t = Y^{GL(M)}_{d\mathbf{z}_t}(\mathbf{u}_t) \quad \textit{and} \quad u_0 = u_o.$$

In this case, over $[0,\tau)$*, we have*

$$\mathbf{z} := \int \theta^{Y^{GL(M)}}(d\mathbf{u}).$$

We can now use Theorem 6.14 to give an alternative characterization of parallel translation.

Theorem 6.15 *Let* $\mathbf{u} \in CRP_{\mathbf{X}}(GL(M))$ *such that* $\pi_*\mathbf{u} = \mathbf{y}$*. Then* $\mathbf{u}$ *is an element of* $CRP^{H,\nabla}_{\mathbf{y}}(GL(M))$ *if and only if there exists an* $\mathbb{R}^d$*-valued controlled rough path* $\mathbf{a} = \left(a, a^\dagger\right)$ *such that*

$$d\mathbf{u}_t = B^{\nabla}_{d\mathbf{a}_t}(u_t)$$

where B^∇ *are the horizontal vector fields introduced in Example 6.12.*

Proof. If $d\mathbf{u}_t = B^{\nabla}_{d\mathbf{a}_t}(u_t)$, then by Theorem 5.3 we have

$$\begin{aligned}\int_s^t \omega^\nabla(d\mathbf{u}) &\underset{3}{\approx} \omega^\nabla\left(B^\nabla_{a_{s,t}}(u_s)\right) + B_{a_s^\dagger w}(u_s)\left[\omega^\nabla \circ B_{a_s^\dagger \tilde{w}}\right]|_{w\otimes\tilde{w} = \mathbb{X}_{s,t}} \\ &= 0\end{aligned}$$

as $\omega^\nabla \circ B^\nabla_a = 0$ for all $a \in \mathbb{R}^d$. Additionally $\left[\int \omega^\nabla(d\mathbf{u})\right]^\dagger_s = \omega^\nabla\left(B_{a_s^\dagger(\cdot)}(u_s)\right) = 0.$

Conversely, if $\mathbf{u} \in CRP_{\mathbf{y}}^{H,\nabla}(GL(M))$, then by Theorem 6.14, we have the existence of $\mathbf{z}$ given by

$$\mathbf{z} := \int \theta^{Y^{GL(M)}}(d\mathbf{u})$$

where $\theta^{Y^{GL(M)}} = \left(\hat{\theta}, \omega^{\nabla}\right)$. In this case, we also have

$$d\mathbf{u}_t = Y_{d\mathbf{z}_t}^{GL(M)}(u_t).$$

Let $\mathbf{a} = \pi_{1*}\mathbf{z}$ where $\pi_1 : \mathbb{R}^d \times gl(d) \to \mathbb{R}^d$ is projection onto the first component and note that this means

$$\mathbf{z} = (\mathbf{a}, \mathbf{0}).$$

However, we have

$$\begin{aligned} Y_{(a,0)}^{GL(M)}(u) &= B_a^{\nabla}(u) + \tilde{0}(u) \\ &= B_a^{\nabla}(u) \end{aligned}$$

so that

$$d\mathbf{u}_t = Y_{d(\mathbf{a},\mathbf{0})_t}^{GL(M)}(u_t) \iff d\mathbf{u}_t = B_{d\mathbf{a}_t}^{\nabla}(u_t).$$

Thus we have proved the theorem. ■

We can now put parallel translation and Theorem 6.14 together to get a correspondence of paths between $CRP_{\mathbf{X}}(M)$ and $CRP_{\mathbf{X}}\left(\mathbb{R}^d\right)$.

Corollary 6.16 *Let ∇ be a covariant derivative on M, $o \in M$, and $u_o \in GL(M)_o$. There is a one-to-one map from $CRP_{\mathbf{X}}(M)$ starting at o defined on $[0,T]$ and $CRP_{\mathbf{X}}\left(\mathbb{R}^d\right)$ starting at 0 defined on $[0,T]$ given by*

$$\begin{array}{ccccc} CRP_{\mathbf{X}}(M) & \longrightarrow & CRP_{\mathbf{y}}^{H,\nabla}(GL(M)) & \longrightarrow & CRP_{\mathbf{X}}\left(\mathbb{R}^d\right) \\ \mathbf{y} & \longrightarrow & \mathbf{h}(\mathbf{y}, u_o) & \longrightarrow & \int \hat{\theta}\left(d\mathbf{h}(\mathbf{y}, u_o)\right) \end{array}$$

where $\hat{\theta}$ is the canonical one-form.

6.3.2 Rolling and Unrolling of rough one-forms

Let ∇ be a covariant derivative.

Lemma 6.17 *Let $\mathbf{y} = \left(y, y^{\dagger}\right) \in CRP_{\mathbf{X}}(M)$, let U^{∇} be the parallelism associated to ∇ and let $\mathbf{u} = \left(u, u^{\dagger}\right)$ be parallel translation along $\mathbf{y}$ with any initial frame. We have the approximation*

$$u_t - U^{\nabla}_{y_t,y_s} u_s \underset{2}{\approx} 0. \tag{6.16}$$

Proof. These are all local statements, thus we may assume that we are working in $\mathbb{R}^d \times GL(d)$. In that case, we have $\nabla_{(m,v_m)} = \partial_{(m,v_m)} + A_m \langle v_m \rangle$ and may write the left hand side of Eq. (6.16) as $(y_t, g_t) - U^{\nabla}_{y_t,y_s}(y_s, g_s)$ where $U^{\nabla}_{y_t,y_s}(y_s, g) = \left(y_t, \bar{U}^{\nabla}_{y_t,y_s} g\right)$ for some $\bar{U} : \left(\mathbb{R}^d\right)^2 \to \operatorname{Aut}(GL(d))$. Thus

$$\begin{aligned} u_t - U^{\nabla}_{y_t,y_s} u_s &= (y_t, g_t) - U^{\nabla}_{y_t,y_s}(y_s, g_s) \\ &= \left(0, g_t - \bar{U}^{\nabla}_{y_t,y_s} g_s\right). \end{aligned}$$

Therefore we just need to show

$$g_t - \bar{U}^{\nabla}_{y_t,y_s} g_s \underset{2}{\approx} 0.$$

By Lemma A.5, we have $\bar{U}^{\nabla}_{y_t,y_s} \underset{2}{\approx} I - A_{y_s}\langle y_t - y_s \rangle$ and therefore

$$g_t - \bar{U}^{\nabla}_{y_t,y_s} g_s \underset{2}{\approx} g_t - \left(I - A_{y_s}\langle y_t - y_s \rangle\right) g_s$$

On the other hand, by the proof of Theorem 6.8, we have that

$$\theta_r\left(\left[g_t - g_s\right]_{g_s}\right) \underset{2}{\approx} -\Gamma_{y_s}\langle y_t - y_s \rangle \tag{6.17}$$

where $\theta_r\left([g_t - g_s]_{g_s}\right) = g_t g_s^{-1} - I$. Thus multiplying Eq. (6.17) on the right by g_s yields the approximation

$$g_t \underset{2}{\approx} g_s - \left(\Gamma_{y_s}\langle y_t - y_s\rangle\right) g_s.$$

Working through the definition of Γ and ω^∇, it is easy to see that $A = \Gamma$ in this setting and therefore

$$\begin{aligned} g_t - \bar{U}^\nabla_{y_t, y_s} g_s &\underset{2}{\approx} g_t - \left(I - \Gamma_{y_s}\langle y_t - y_s\rangle\right) g_s \\ &\underset{2}{\approx} g_s - \left(\Gamma_{y_s}\langle y_t - y_s\rangle\right) g_s - g_s + \Gamma_{y_s}\langle y_t - y_s\rangle g_s \\ &= 0. \end{aligned}$$

■

Given Lemma 6.17, we have the following.

Proposition 6.18 *Let* $\mathbf{u} = \left(u_t, u_t^\dagger\right) = \mathbf{h}\left(\mathbf{y}, u_{y_0}\right) \in CRP_{\mathbf{X}}\left(GL\left(M\right)\right)$ *be parallel translation started at* u_{y_0} *along* $\mathbf{y} := \left(y, y^\dagger\right)$ *with respect to* ∇.

1. *Let* $\tilde{\boldsymbol{\alpha}} := \left(\tilde{\alpha}, \tilde{\alpha}^\dagger\right) \in CRP_{\mathbf{X}}\left(L\left(\mathbb{R}^d, V\right)\right)$. *Then* $\boldsymbol{\alpha}^\nabla := \left(\alpha^\nabla, \left(\alpha^\dagger\right)^\nabla\right)$ *defined by*

$$\begin{aligned} \alpha_s^\nabla &:= \tilde{\alpha}_s \circ u_s^{-1} \\ \left(\alpha_s^\dagger\right)^\nabla &:= \tilde{\alpha}_s^\dagger \circ \left(I \otimes u_s^{-1}\right) \end{aligned}$$

is an element of $CRP_y^{U^\nabla}\left(M, V\right)$.

2. *Let* $\boldsymbol{\alpha} \in CRP_y^{U^\nabla}\left(M, V\right)$. *Then* $\tilde{\boldsymbol{\alpha}}^\nabla := \left(\tilde{\alpha}^\nabla, \left(\tilde{\alpha}^\dagger\right)^\nabla\right)$ *defined by*

$$\begin{aligned} \tilde{\alpha}_s^\nabla &:= \alpha_s \circ u_s \\ \left(\tilde{\alpha}_s^\dagger\right)^\nabla &:= \alpha_s^\dagger\left(I \otimes u_s\right) \end{aligned}$$

is an element of $CRP_{\mathbf{X}}\left(L\left(\mathbb{R}^d, V\right)\right)$.

Proof. For item 1 (suppressing the ∇), we have

$$\begin{aligned}
&\alpha_t \circ U_{y_t,y_s} - \alpha_s - \alpha_s^\dagger\left(x_{s,t} \otimes (\cdot)\right) \\
&\quad = \tilde{\alpha}_t \circ u_t^{-1} \circ U_{y_t,y_s} - \tilde{\alpha}_s \circ u_s^{-1} - \alpha_s^\dagger\left(x_{s,t} \otimes u_s^{-1}(\cdot)\right) \\
&\quad = \left[\tilde{\alpha}_t - \tilde{\alpha}_s - \alpha_s^\dagger\left(x_{s,t} \otimes (\cdot)\right)\right] u_s^{-1} + \tilde{\alpha}_t\left(u_t^{-1} \circ U_{y_t,y_s} - u_s^{-1}\right) \\
&\quad \underset{2}{\approx} \tilde{\alpha}_t\left(u_t^{-1} \circ U_{y_t,y_s} - u_s^{-1}\right) \\
&\quad \underset{2}{\approx} 0
\end{aligned}$$

where the last step follows from Lemma 6.17. Additionally, we have

$$\begin{aligned}
\alpha_t^\dagger \circ \left(I \otimes U_{y_t,y_s}\right) - \alpha_s^\dagger &= \tilde{\alpha}_t^\dagger \circ \left(I \otimes u_t^{-1} U_{y_t,y_s}\right) - \tilde{\alpha}_s^\dagger \circ \left(I \otimes u_s^{-1}\right) \\
&\underset{1}{\approx} \left(\tilde{\alpha}_t^\dagger - \tilde{\alpha}_s^\dagger\right) \circ \left(I \otimes u_s^{-1}\right) + \tilde{\alpha}_t^\dagger\left(I \otimes \left(u_t^{-1} U_{y_t,y_s} - u_s^{-1}\right)\right) \\
&\underset{1}{\approx} 0
\end{aligned}$$

Item 2 is similar and also reduces to the validity of the lemma. ■

Given a covariant derivative, we have a way to unroll both a path and a rough one-form onto flat space. Theorem 6.21 below shows that we get the same answer if we integrate on the manifold as we get if we unroll both the path and the rough one-form onto flat space and integrate there. Before presenting it, we will provide another lemma which approximates an identity which holds in the smooth case.

Remark 6.19 *The results of Lemma 6.20 below are analogues to equation determining the unrolled map in the smooth category; in this case if* y_s *is a smooth path on* M *and* $u_s \in GL(M)$ *is parallel translation along* y_s*, then* $\tilde{y}_s$ *is the path on* $\mathbb{R}^d$

starting at 0 *determined by the differential equation*

$$\tilde{y}'_s = u_s^{-1} y'_s.$$

Lemma 6.20 *Let* $\mathbf{y} = \left(y, y^{\dagger}\right) \in CRP_{\mathbf{X}}(M)$*,* ∇ *a covariant derivative on* TM*,* $u_{y_0} \in GL(M)_{y_0}$ *,* $\mathbf{u} = \mathbf{h}(\mathbf{y}, u_{y_0})$ *the lift of* $\mathbf{y}$ *into* $GL(M)$ *which exists by Theorem 6.8, and* $\tilde{\mathbf{y}} := \int \hat{\theta}\left(d\mathbf{h}(\mathbf{y}, u_{y_0})\right) \in CRP_{\mathbf{X}}\left(\mathbb{R}^d\right)$ *the unrolled path. Additionally let* ψ *be any logarithm on* M *and* U^{∇} *be the parallelism associated to* ∇*, and further denote* $\mathcal{G} := \left(\psi, U^{\nabla}\right)$*. Then*

1. $$\tilde{y}_{s,t} \underset{3}{\approx} u_s^{-1}\left[\psi(y_s, y_t) + \mathcal{S}^{\mathcal{G}} y_s^{\dagger \otimes 2} \mathbb{X}_{s,t}\right] \tag{6.18}$$

   and

2. $$\tilde{y}^{\dagger}_s = u_s^{-1} y^{\dagger}_s. \tag{6.19}$$

Proof. As usual, because these results are purely local, by using a trivialization, we may assume we are working on $\mathbb{R}^d \times GL(d)$ where we will write $\mathbf{u} = (\mathbf{y}, \mathbf{g})$ and write $\nabla_{(m,v_m)} = \partial_{(m,v_m)} + \Gamma_m \langle v_m \rangle$. We let $\nabla^{GL(d)}$ be any covariant derivative on $GL(d)$ and $\psi^{Gl(d)}$ be any logarithm on $GL(d)$. Thus, $\mathcal{G}^{GL(d)} := \left(\psi^{GL(d)}, U^{\nabla^{GL(d)}}\right)$ is a gauge on $GL(d)$ while $\left(\nabla, \nabla^{GL(d)}\right)$ is a covariant derivative on $T\left[\mathbb{R}^d \times GL(d)\right] = \mathbb{R}^d \times gl(d)$ and $\left(\mathcal{G}, \mathcal{G}^{GL(d)}\right)$ is a gauge on $\mathbb{R}^d \times GL(d)$. With these geometrical objects in place, we can use the definition of $\tilde{\mathbf{y}}$ via to find an approximation via Proposition 4.36:

We have

$$\begin{aligned}\tilde{y}_{s,t} \underset{3}{\approx} \hat{\theta}\Big(&\left(\psi, \psi^{GL(d)}\right)\left((y_s, g_s), (y_t, g_t)\right) + \mathcal{S}^{\left(\mathcal{G}, \mathcal{G}^{GL(d)}\right)}\left(y_s^{\dagger}, g_s^{\dagger}\right)^{\otimes 2} \mathbb{X}_{s,t}\Big) \\ &+ \left[\left(\nabla, \nabla^{GL(d)}\right)\hat{\theta}\right]\left(y_s^{\dagger}, g_s^{\dagger}\right)^{\otimes 2} \mathbb{X}_{s,t}\end{aligned}$$

where $\hat{\theta}$ is the canonical one form defined in Example 6.12. Note that in this trivial bundle setting,

$$\hat{\theta}\left(v_m, h_g\right) = g^{-1}v.$$

We claim that the second summand $\left[\left(\nabla, \nabla^{GL(d)}\right)\hat{\theta}\right]\left(y_s^\dagger, g_s^\dagger\right)^{\otimes 2}\mathbb{X}_{s,t}$ above is zero. Assuming this claim, we have

$$\begin{aligned}
\tilde{y}_{s,t} &\underset{3}{\approx} \hat{\theta}\left(\left(\psi, \psi^{GL(d)}\right)\left(\left(y_s, g_s\right), \left(y_t, g_t\right)\right) + \mathcal{S}^{\left(\mathcal{G}, \mathcal{G}^{GL(d)}\right)}\left(y_s^\dagger, g_s^\dagger\right)^{\otimes 2}\mathbb{X}_{s,t}\right) \\
&= g_s^{-1}\psi\left(y_s, y_t\right) + \left(\mathcal{S}^{\mathcal{G}} y_s^{\dagger\otimes 2}, \mathcal{S}^{\mathcal{G}^{GL(d)}} g_s^{\dagger\otimes 2}\right)\mathbb{X}_{s,t} \\
&= g_s^{-1}\psi\left(y_s, y_t\right) + g_s^{-1}\mathcal{S}^{\mathcal{G}} y_s^{\dagger\otimes 2}\mathbb{X}_{s,t} \\
&= g_s^{-1}\left[\psi\left(y_s, y_t\right) + \mathcal{S}^{\mathcal{G}} y_s^{\dagger\otimes 2}\mathbb{X}_{s,t}\right]
\end{aligned}$$

which is Eq. (6.18).

To prove the claim, we will prove the more general fact that

$$\left[\left(\nabla, \nabla^{GL(d)}\right)\hat{\theta}\right]\left(y_s^\dagger, g_s^\dagger\right)^{\otimes 2}\left(w \otimes \tilde{w}\right) = 0$$

for every $w, \tilde{w} \in W$. To see this, we first let $\left(\mathbf{Y}_s^{\tilde{w}}, \mathbf{G}_s^{\tilde{w}}\right)$ be the (local) vector field given by

$$\mathbf{Y}_s^{\tilde{w}} := U^{\nabla}\left(\cdot, y_s\right) y_s^\dagger \tilde{w} \quad \text{and} \quad \mathbf{G}_s^{\tilde{w}} := U^{\nabla^{GL(d)}}\left(\cdot, g_s\right) g_s^\dagger \tilde{w}.$$

Note that we have defined $\left(\mathbf{Y}_s^{\tilde{w}}, \mathbf{G}_s^{\tilde{w}}\right)$ in such a way to write

$$\left[\left(\nabla, \nabla^{GL(d)}\right)\hat{\theta}\right]\left(y_s^\dagger, g_s^\dagger\right)^{\otimes 2}\left(w \otimes \tilde{w}\right) = \left[y_s^\dagger w, g_s^\dagger w\right]\left[\hat{\theta}\left(\mathbf{Y}^{\tilde{w}}, \mathbf{G}^{\tilde{w}}\right)\right].$$

Secondly, we recall that $\left(g, g^\dagger\right)$ is such that

$$g_s^\dagger = -\Gamma_{y_s}\left\langle y_s^\dagger\left(\cdot\right)\right\rangle g_s$$

(this follows directly from examining the second order part of Eq. (6.14) in the proof of Theorem 6.8). Thirdly, we have by differentiating Eq. (A.14), we have $v_{y_s}\left[\mathbf{Y}_s^{\tilde{w}}\right] = -\Gamma_{y_s}\langle v\rangle\left\langle y_s^{\dagger}\tilde{w}\right\rangle$ for any $v_{y_s}\in T_{y_s}\mathbb{R}^d$. Putting these facts together, we have

$$\begin{aligned}
&\left[\left(\nabla,\nabla^{GL(d)}\right)\hat{\theta}\right]\left(y_s^{\dagger},g_s^{\dagger}\right)^{\otimes 2}(w\otimes\tilde{w})\\
&=\left[y_s^{\dagger}w,g_s^{\dagger}w\right]\left[\hat{\theta}\left(\mathbf{Y}_s^{\tilde{w}},\mathbf{G}_s^{\tilde{w}}\right)\right]\\
&=\left[y_s^{\dagger}w,g_s^{\dagger}w\right]\left[(y,g)\longrightarrow g^{-1}\mathbf{Y}_s^{\tilde{w}}(y)\right]\\
&=-g_s^{-1}\left(g_s^{\dagger}w\right)g_s^{-1}y_s^{\dagger}\tilde{w}-g_s^{-1}\Gamma_{y_s}\left\langle y_s^{\dagger}w\right\rangle\left\langle y_s^{\dagger}\tilde{w}\right\rangle\\
&=g_s^{-1}\Gamma_{y_s}\left\langle y_s^{\dagger}w\right\rangle\left\langle g_sg_s^{-1}y_s^{\dagger}\tilde{w}\right\rangle-g_s^{-1}\Gamma_{y_s}\left\langle y_s^{\dagger}w\right\rangle\left\langle y_s^{\dagger}\tilde{w}\right\rangle\\
&=0
\end{aligned}$$

where the third equality is true as

$$h_g\left[g\longrightarrow g^{-1}\right]=-g^{-1}h_gg^{-1}\quad\forall h_g\in T_gGL(d).$$

Thus, the claim is proved and Eq. (6.18) holds.

Equation (6.19) holds directly from the definition of $y_s^{\dagger}$:

$$\begin{aligned}
\tilde{y}_s^{\dagger}&=\hat{\theta}\left(y_s^{\dagger},g_s^{\dagger}\right)\\
&=g_s^{-1}y_s^{\dagger}.
\end{aligned}$$

■

With Lemma 6.20 in place, the following theorem is nearly immediate.

Theorem 6.21 *Let* $\mathbf{y}=\left(y,y^{\dagger}\right)$ *be an element of* $CRP_{\mathbf{X}}(M)$*, let* ∇ *a covariant derivative on* TM*, and let* $u_{y_0}\in GL(M)$*. Further let* $\boldsymbol{\alpha}\in CRP_y^{U^{\nabla}}(M,V)$*, let* $\tilde{\boldsymbol{\alpha}}^{\nabla}:=\left(\tilde{\alpha}^{\nabla},\left(\tilde{\alpha}^{\dagger}\right)^{\nabla}\right)\in CRP_{\mathbf{X}}\left(L\left(\mathbb{R}^d,V\right)\right)$ *be the unrolled rough one-form, and let* $\tilde{\mathbf{y}}:=\int\hat{\theta}\left(d\mathbf{h}\left(\mathbf{y},u_{y_0}\right)\right)\in CRP_{\mathbf{X}}\left(\mathbb{R}^d\right)$ *be the unrolled path. If* ψ *is a logarithm and*

$\mathcal{G} = \left(\psi, U^{\nabla}\right)$ *we have*

$$\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}} \right\rangle = \int \left\langle \tilde{\boldsymbol{\alpha}}^{\nabla}, d\tilde{\mathbf{y}} \right\rangle. \tag{6.20}$$

Proof. The right hand side of Eq. (6.20) is approximated by

$$\int_s^t \left\langle \tilde{\boldsymbol{\alpha}}^{\nabla}, d\tilde{\mathbf{y}} \right\rangle \underset{3}{\approx} \tilde{\alpha}_s^{\nabla}\left(\tilde{y}_{s,t}\right) + \left(\tilde{\alpha}_s^{\dagger}\right)^{\nabla}\left(I \otimes \tilde{y}_s^{\dagger}\right)\mathbb{X}_{s,t} \tag{6.21}$$

$$= \alpha_s\left(u_s \tilde{y}_{s,t}\right) + \alpha_s^{\dagger}\left(I \otimes u_s \tilde{y}_s^{\dagger}\right)\mathbb{X}_{s,t}. \tag{6.22}$$

Combining Lemma 6.20 with Eq. (6.22), we have

$$\begin{aligned}
&\int_s^t \left\langle \tilde{\boldsymbol{\alpha}}^{\nabla}, d\tilde{\mathbf{y}} \right\rangle \\
&\quad \underset{3}{\approx} \alpha_s\left(\psi\left(y_s, y_t\right) + \mathcal{S}^{\mathcal{G}} y_s^{\dagger \otimes 2}\mathbb{X}_{s,t}\right) + \alpha_s^{\dagger}\left(I \otimes y_s^{\dagger}\right)\mathbb{X}_{s,t} \\
&\quad \underset{3}{\approx} \int_s^t \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}} \right\rangle.
\end{aligned}$$

Secondly, we have

$$\begin{aligned}
\left[\int \left\langle \tilde{\boldsymbol{\alpha}}^{\nabla}, d\tilde{\mathbf{y}} \right\rangle\right]_s^{\dagger} &= \tilde{\alpha}^{\nabla}\tilde{y}_s^{\dagger} \\
&= \alpha_s u_s u_s^{-1} y_s^{\dagger} \\
&= \alpha_s y_s^{\dagger} \\
&= \left[\int \left\langle \boldsymbol{\alpha}, d\mathbf{y}^{\mathcal{G}} \right\rangle\right]_s^{\dagger}.
\end{aligned}$$

■

Appendix A

Riemannian Manifolds

A.1 Taylor Expansion

Let (M, g) be a Riemannian manifold, ∇ be the Levi-Civita covariant derivative, $\exp(tv)$ be the geodesic flow, and $//_t(\sigma)$ denote parallel translation relative to ∇. Recall that Taylor's formula with integral remainder states for any smooth function g on $[0, 1]$, that

$$G(1) = \sum_{k=0}^{n} \frac{1}{n!} G^{(k)}(0) + \frac{1}{n!} \int_0^1 G^{(n+1)}(t)(1-t)^n \, dt. \tag{A.1}$$

We now apply this result to $G(t) := f(\exp_m(tv))$ where $f \in C^\infty(M)$, $v \in T_m M$ and $m \in M$. To this end let $\sigma(t) := \exp(tv)$ so that $\nabla \dot{\sigma}(t)/dt = 0$. It then follows that

$$\begin{aligned}
\dot{G}(t) &= df(\dot{\sigma}(t)) = df_{\sigma(t)}(\dot{\sigma}(t)), \\
\ddot{G}(t) &= \frac{d}{dt}df_{\sigma(t)}(\dot{\sigma}(t)) = \left(\nabla_{\dot{\sigma}(t)}df\right)(\dot{\sigma}(t)) + df_{\sigma(t)}\left(\frac{\nabla}{dt}\dot{\sigma}(t)\right) \\
&= \left(\nabla_{\dot{\sigma}(t)}df\right)(\dot{\sigma}(t)) = (\nabla df)(\dot{\sigma}(t)\otimes\dot{\sigma}(t)) \\
&\vdots \\
G^{(k)}(t) &= \left(\nabla^{k-1}df\right)\left(\dot{\sigma}(t)^{\otimes k}\right) = \left(\nabla^{k-1}df\right)\left(\overbrace{\dot{\sigma}(t)\otimes\cdots\otimes\dot{\sigma}(t)}^{k\text{ times}}\right). \qquad \text{(A.2)}
\end{aligned}$$

Therefore we may conclude that

$$\begin{aligned}
f(\exp_m(v)) = G(1) &= \sum_{k=0}^{n}\frac{1}{n!}G^{(k)}(0) \\
&= f(x) + \sum_{k=1}^{n}\frac{1}{k!}\left(\nabla^{k-1}df\right)\left(v^{\otimes k}\right) \qquad \text{(A.3)} \\
&\quad + \frac{1}{n!}\int_0^1 (\nabla^n df)\left(\dot{\sigma}(t)^{\otimes(n+1)}\right)(1-t)^n\,dt. \qquad \text{(A.4)}
\end{aligned}$$

Letting $n = \exp_m(v)$ in this formula then gives the following version of Taylor's theorem on a manifold.

Theorem A.1 *Let $f \in C^\infty(M)$ and $m, n \in M$ with $d_g(m,n)$ sufficiently small so that there exists a unique $v \in T_mM$ such that $|v|_{g_m} \leq d(m,n)$ and $n = \exp_m(v)$.*

Then we have

$$f(n) = f(m) + \sum_{k=1}^{n} \frac{1}{k!} \left(\nabla^{k-1} df\right) \left(v^{\otimes k}\right)$$
$$+ \frac{1}{n!} \int_0^1 \left(\nabla^n df\right) \left(\dot{\sigma}(t)^{\otimes(n+1)}\right) (1-t)^n \, dt \tag{A.5}$$
$$= f(m) + \sum_{k=1}^{n} \frac{1}{k!} \left(\nabla^{k-1} df\right) \left(\left[\exp_m^{-1}(n)\right]^{\otimes k}\right) \tag{A.6}$$
$$+ \frac{1}{n!} \int_0^1 \left(\nabla^n df\right) \left(\dot{\sigma}(t)^{\otimes(n+1)}\right) (1-t)^n \, dt \tag{A.7}$$

where $\sigma(t) = \exp_m(tv)$. *In particular since* $|\dot{\sigma}(t)|_g = |v|_g = d_g(m,n)$ *it follows that*

$$f(n) = f(m) + \sum_{k=1}^{n} \frac{1}{k!} \left(\nabla^{k-1} df\right) \left(\left[\exp_m^{-1}(n)\right]^{\otimes k}\right) + O\left(d(m,n)^{n+1}\right). \tag{A.8}$$

Lemma A.2 *Let* M *be an embedded submanifold of* $W = \mathbb{R}^k$ *and* $P(m) : W \to T_m M$ *be orthogonal projection onto the tangent space. If* $m, n \in M$ *are close, then;*

1. $P(m)\left[\exp_m^{-1}(n) - (n-m)\right] = O\left(|n-m|^3\right).$

   Moreover, $\exp_m^{-1}(n) - (n-m) = O\left(|n-m|^2\right)$

2. $U^{\nabla}(n,m) = P(m) + dP\left(\exp_m^{-1}(n)\right) + O\left(|n-m|^2\right) = P(n) + O\left(|n-m|^2\right)$

3. $P(n) - P(m) = dP\left(\exp_m^{-1}(n)\right) + O\left(|n-m|^2\right).$

   Here $U^{\nabla}(n,m)$ *refers to the parallelism defined in Example 3.8.*

Proof. We will denote $v := \exp_m^{-1}(n) \in T_m M$ and $\sigma(t) = \exp_m(tv)$.

For 1, we have by Taylor expansion on manifolds (Theorem A.1) that

$$G(n) = G(m) + dG(v) + \frac{1}{2}(\nabla dG)(v \otimes v) + \frac{1}{2} \int_0^1 \left(\nabla^2 dG\right) \left(\dot{\sigma}(t)^{\otimes 3}\right) (1-t)^2 \, dt$$

where $G \in C^{\infty}(M, W)$. Letting $G(m) = m$ as a function into W, we have

$$n = m + \exp_m^{-1}(n) + \frac{1}{2}(\nabla P)(v \otimes v) + O\left(|v|_g^3\right).$$

Rearranging, we have

$$\exp_m^{-1}(n) - (n - m) = -\frac{1}{2}(\nabla P)(v \otimes v) + O\left(|v|_g^3\right) \tag{A.9}$$

so that

$$P(m)\left[\exp_m^{-1}(n) - (n - m)\right] = -\frac{1}{2}P(m)(\nabla P)(v \otimes v) + O\left(|v|_g^3\right).$$

Note that $(\nabla P)(v \otimes v) = dP(v)\, v = dP(v)\, P(m)\, v$. Using the identities $dPQ - PdQ = 0$ and $dP = -dQ$, where $Q = I - P$, we get that $PdPP = 0$. Thus we have

$$P(m)\left[\exp_m^{-1}(n) - (n - m)\right] = O\left(|v|^3\right).$$

Lastly, in a small neighborhood around m, $|v|_g = |m - n| + o(|m - n|)$ so that

$$P(m)\left[\exp_m^{-1}(n) - (n - m)\right] = O\left(|n - m|^3\right)$$

The fact that $\exp_m^{-1}(n) - (n - m) = O\left(|n - m|^2\right)$ is immediate from Eq. (A.9).

For 3, we use Taylor's theorem again this time with G defined by $G(n) := P(n)$ to see that

$$P(n) - P(m) = dP\left(\exp_m^{-1}(n)\right) + O\left(|v|^2\right).$$

As before, this is equivalent to $P(n) - P(m) = dP\left(\exp_m^{-1}(n)\right) + O\left(|m - n|^2\right)$.

Lastly for 2, Taylor applied to $G_m : M \to L\left(T_m M, \mathbb{R}^N\right)$ defined by $G_m(n) =$

$U^{\nabla}(n,m)$ gives

$$U^{\nabla}(n,m) - P(m) = dG_m\left(\exp_m^{-1}(n)\right) + O\left(|m-n|^2\right).$$

But

$$\begin{aligned} dG_m\left(\exp_m^{-1}(n)\right) &= \frac{d}{dt}|_0 U(\sigma(t), m) \\ &= -dQ(\dot{\sigma}(t))|_0 \\ &= -dQ\left(\exp_m^{-1}(n)\right) \\ &= dP\left(\exp_m^{-1}(n)\right). \end{aligned}$$

Thus we have

$$U^{\nabla}(n,m) = P(m) + dP\left(\exp_m^{-1}(n)\right) + O\left(|m-n|^2\right)$$

which is the first equality of 2. The second equality follows trivially from this and 3. ■

A.2 Equivalence of Riemannian Metrics on Compact Sets

Proposition A.3 *Let $\pi : E \to N$ be a real rank $d < \infty$ vector bundle over a finite dimensional manifold N. Further suppose that E is equipped with smoothly varying fiber inner product g and let $S_g := \{\xi \in E : g(\xi,\xi) = 1\}$ be a sub-bundle of E. Then for any compact $K \subseteq N$, $\pi^{-1}(K) \cap S_g$ is a compact sets.*

Proof. We wish to show that every sequence $\{\xi_l\}_{l=1}^{\infty} \subset \pi^{-1}(K) \cap S_g$ has a convergent subsequence. Since $\{\pi(\xi_l)\}_{l=1}^{\infty}$ is a sequence in K, by passing to a subsequence if necessary we may assume that $m := \lim_{l\to\infty} \pi(\xi_l)$ exists in K. By passing to a further subsequence if necessary we may assume that $\{\xi_l\}_{l=1}^{\infty} \in \pi^{-1}(K_0) \cap S_g$ where K_0 is a compact neighborhood of m which is contained in an open neighborhood U over which E is trivializable and hence we may now assume that $\pi^{-1}(U) = U \times \mathbb{R}^d$ and that $\xi_l = (n_l, v_l)$ where $\lim_{l\to\infty} n_l = m \in K_0$.

Let S^{d-1} denote the standard Euclidean unit sphere inside of $\mathbb{R}^d$. The function, $F : U \times S^{d-1} \to (0, \infty)$ defined by $F(n, v) = g((n, v), (n, v))$ is smooth and hence has a minimum $c > 0$ and a maximum, $C < \infty$ on the compact set, $K \times S^{d-1}$. Therefore by a simple scaling argument we conclude that

$$c\,|v|^2 \le g((n, v), (n, v)) \le C\,|v|^2 \;\; \forall\, n \in K \text{ and } v \in \mathbb{R}^d. \tag{A.10}$$

From the lower bound in Inequality (A.10) and the assumption that $1 = g(\xi_l, \xi_l)$ it follows that $|v_l|_{\mathbb{R}^d} \le 1/\sqrt{c}$ for all l and therefore has a convergent sub-sequence $\{v_{l_k}\}_{k=1}^{\infty}$. This completes the proof as $\{\xi_{l_k} = (n_{l_k}, v_{l_k})\}_{k=1}^{\infty}$ is convergent as well. ■

Corollary A.4 *If $g, \tilde{g}$ are two Riemannian metrics on TM, $K \subseteq M$ is compact, then there exists $0 < c_K, C_K < \infty$ such that*

$$c_K\,|v|_{\tilde{g}_m} \le |v|_{g_m} \le C_K\,|v|_{\tilde{g}_m} \;\; \forall\, v \in \pi^{-1}(K). \tag{A.11}$$

In other words, all Riemannian metrics are equivalent when restricted to compact subsets, $K \subset M$.

Proof. The function, $F : TM \to [0, \infty)$, defined by $F(v) := g(v, v)$ is smooth and positive when restricted to $S_{\tilde{g}} \cap \pi^{-1}(K)$ which is compact by Proposition A.3. Therefore there exists $0 < c_K < C_K < \infty$ such that $c_K^2 \le g(v, v) \le C_K^2$ for all $v \in S_{\tilde{g}} \cap \pi^{-1}(K)$ from which Inequality (A.11) follows by a simple scaling argument. ■

A.2.1 Covariant Derivatives on Euclidean Space

On $\mathbb{R}^d$ every covariant derivative takes the form $\nabla_{(x,v)} = \partial_v + A_x \langle v \rangle$ where $A : \mathbb{R}^d \to L\left(\mathbb{R}^d, L\left(\mathbb{R}^d, \mathbb{R}^d\right)\right)$. If $\sigma_x^v(t) = \exp_x(tv)$ where $\exp = \exp^\nabla$, we have by definition

$$\begin{aligned}
\partial_{\dot\sigma_x^v(t)} \dot\sigma_x^v &= -A_{\sigma_x^v(t)} \langle \dot\sigma_x^v(t) \rangle \dot\sigma_x^v(t) \\
\dot\sigma_x^v(0) &= v \\
\sigma_x^v(0) &= x
\end{aligned}$$

In particular if $f_x = \exp_x(\cdot)$ plugging in at $t = 0$ we get

$$f_x''(0)[v \otimes v] = -A_x \langle v \rangle v.$$

Now if we denote $G_x := \exp_x^{-1}(\cdot)$ and by differentiating $f_x \circ G_x$ twice, we get that

$$G_x''(x)[v \otimes v] = A_x \langle v \rangle v.$$

Indeed we have

$$\begin{aligned}
0 &= (f_x \circ G_x)''(x) \\
&= \left[f_x'(G_x(x))\, G_x'(x)\right]' \\
&= f_x''(G_x(x))\left[G_x'(x) \otimes G_x'(x)\right] + f_x'(G_x(x))\, G_x''(x).
\end{aligned}$$

Since $G_x(x) = 0$, $G_x'(x) = I$, and $f_x'(0) = I$ we have

$$f_x''(0) = -G_x''(x).$$

Parallel translation $U^{\nabla}\left(\sigma_x^v\left(t\right),x\right)$ solves

$$\begin{aligned}\frac{d}{dt}U^{\nabla}\left(\sigma_x^v\left(t\right),x\right) &= -A_{\sigma_x^v(t)}\left\langle\dot{\sigma}_x^v\left(t\right)\right\rangle U^{\nabla}\left(\sigma_x^v\left(t\right),x\right)\\ U^{\nabla}\left(x,x\right) &= I\end{aligned}$$

Again, using $t=0$ we have that if $\tilde{G}_x = U^{\nabla}\left(\cdot,x\right)$ then

$$\tilde{G}_x'\left(x\right)v = -A_x\left\langle v\right\rangle.$$

To summarize, we have

$$\left(\exp_x^{-1}\right)''\left(x\right)\left[v\otimes v\right] = A_x\left\langle v\right\rangle v \tag{A.12}$$

and

$$\left(U^{\nabla}\left(\cdot,x\right)\right)'\left(x\right)v = -A_x\left\langle v\right\rangle.$$

Since $\left(\exp_x^{-1}\right)''\left(x\right)$ is symmetric, we have that

$$\begin{aligned}\left(\exp_x^{-1}\right)''\left(x\right)\left[v\otimes w\right] &= \frac{1}{2}\left(\exp_x^{-1}\right)''\left(x\right)\left(v\otimes w + w\otimes v\right)\\ &\quad+\frac{1}{2}\left(\exp_x^{-1}\right)''\left(x\right)\left(v\otimes w - w\otimes v\right)\\ &= \frac{1}{2}\left(\exp_x^{-1}\right)''\left(x\right)\left(v\otimes w + w\otimes v\right)\\ &= \frac{1}{2}A_x\left(v\otimes w + w\otimes v\right)\\ &= \frac{1}{2}\left(A_x\left\langle v\right\rangle w + A_x\left\langle w\right\rangle v\right)\end{aligned} \tag{A.13}$$

Another way of saying this is that $\left(\exp_x^{-1}\right)''\left(x\right)$ equals the symmetric part of A_x. By using this fact and Taylor's theorem, we get the following result.

Lemma A.5 *If $\nabla_{(x,v)} = \partial_v + A_x \langle v \rangle$ is a covariant derivative on $\mathbb{R}^d$, then*

$$\left(\exp_x^{\nabla}\right)^{-1}(y) - (y - x) - \frac{1}{2} A_x \langle y - x \rangle \langle y - x \rangle = O\left(|y - x|^3\right)$$

$$U^{\nabla}(y, x) - I + A_x \langle y - x \rangle = O\left(|y - x|^2\right) \tag{A.14}$$

where $|x - y|$ is small enough for these terms to make sense.

Corollary A.6 *If $\nabla_{(x,v)} = \partial_v + A_x \langle v \rangle$ is a covariant derivative on $\mathbb{R}^d$, then*

$$U^{\nabla}(y, x) - I - A_y \langle x - y \rangle = O\left(|y - x|^2\right)$$

where $|x - y|$ is small enough for these terms to make sense. In particular, we have

$$\left(U^{\nabla}(x, \cdot)\right)'(x)\, v = A_x \langle v \rangle$$

Proof. This is immediate after expanding $A_{(\cdot)}$ about x in the direction $y - x$ in Eq. (A.14) with Taylor's theorem. ■

Portions of Appendix A are adapted from material awaiting publication as Driver, B.K.; Semko, J.S., "Controlled Rough Paths on Manifolds I," submitted, *Revista Matemática Iberoamericana*, 2015. The dissertation author was the primary author of this paper.

Appendix B

Rough Differential Equation Results in Euclidean Space

The following lemma (which is Corollary 2.17 in [3] and was proved using Theorem 10.14 of [13]) proves useful in the manifold case.

Lemma B.1 *Let $U \subseteq \mathbb{R}^d$ be an open set and U_1 be a precompact open set whose closure is contained in U. There exists a $\delta > 0$ such that for all $(\bar{z}_0, t_0) \in U_1 \times [0, T]$, the rough differential equation*

$$d\mathbf{z}_t = F_{d\mathbf{X}_t}(z_t) \quad \textit{with} \quad z_{t_0} = \bar{z}_0$$

has a unique solution $\mathbf{z} \in CRP_{\mathbf{X}}(\mathbb{R}^d)$ which is defined on $[t_0, t_0 + \delta \wedge T]$ with $z_t \in U$ for all $t \in [t_0, t_0 + \delta \wedge T]$.

We now state an equivalent condition for the path $\mathbf{z}$ to solve Eq. (2.16).

Theorem B.2 *Let $U \subseteq \mathbb{R}^d$ be open such and $\mathbf{z} = (z, z^\dagger) \in CRP_{\mathbf{X}}(\mathbb{R}^d)$ defined on I_0 such that $z(I_0) \subseteq U$. Then $\mathbf{z}$ solves Eq. (2.16) if and only if $z_s^\dagger = F_\cdot(z_s)$ and* *for every* $[a,b] \subseteq I_0$, *Banach space* V, *and* $\alpha \in \Omega^1(U,V)$, *the approximation*

$$\int_s^t \alpha(d\mathbf{z}) \underset{3}{\approx} \alpha_{z_s}\left(F_{x_{s,t}}(z_s)\right) + \left(\partial_{F_w(z_s)}\left[\alpha \circ F_{\tilde{w}}\right]\right)(z_s)\,|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}}$$

holds.

Proof. This is proved in [3] [Theorem 4.5 by letting $M = U$] but included here for completeness. To prove the "if" direction, it suffices to let $\alpha = d(I_U)$ and notice that

$$\int_s^t d(I_U)(d\mathbf{z}) = z_t - z_s$$

by Theorem 4.45 and that $d(I_U)_u(\tilde{u}) = \tilde{u}$ so that

$$d(I_U)_{z_s}\left(F_{x_{s,t}}(z_s)\right) = F_{x_{s,t}}(z_s)$$

and

$$\left(\partial_{F_w(z_s)}\left[d(I_U) \circ F_{\tilde{w}}\right]\right)(z_s) = \left(\partial_{F_w(z_s)}F_{\tilde{w}}\right)(z_s).$$

To prove the "only if" direction, by definition we have

$$z_{s,t} \underset{3}{\approx} F_{x_{s,t}}(z_s) + \left(\partial_{F_w(z_s)}F_{\tilde{w}}\right)(z_s)\,|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}}$$

and

$$\int_s^t \alpha(d\mathbf{z}) \underset{3}{\approx} \alpha_{z_s}(z_{s,t}) + \alpha'_{z_s}\left(F_\cdot(z_s) \otimes F_\cdot(z_s)\,\mathbb{X}_{s,t}\right).$$

Combining these approximations, we have

$$\begin{aligned}\int_s^t \alpha(d\mathbf{z}) &\underset{3}{\approx} \alpha_{z_s}(z_{s,t}) + \alpha'_{z_s}\left(F_\cdot(z_s) \otimes F_\cdot(z_s)\,\mathbb{X}_{s,t}\right)\\ &\underset{3}{\approx} \alpha_{z_s}\left(F_{x_{s,t}}(z_s) + \left(\partial_{F_w(z_s)}F_{\tilde{w}}\right)(z_s)\right) + \alpha'_{z_s}\left(F_w(z_s) \otimes F_{\tilde{w}}(z_s)\right)|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}}\\ &= \alpha_{z_s}\left(F_{x_{s,t}}(z_s)\right) + \left(\partial_{F_w(z_s)}\left[\alpha \circ F_{\tilde{w}}\right]\right)(z_s)\,|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}}\end{aligned}$$

where the last equality follows from the calculation

$$\left(\partial_{F_w(z_s)}\left[\alpha\circ F_{\tilde{w}}\right]\right)(z_s)=\left(\partial_{F_w(z_s)}\left[\alpha_{z_s}\circ F_{\tilde{w}}(\cdot)\right]\right)(z_s)+\left(\partial_{F_w(z_s)}\alpha_{(\cdot)}\circ F_{\tilde{w}}(z_s)\right)(z_s)$$
$$=\alpha_{z_s}\left(\left(\partial_{F_w(z_s)}F_{\tilde{w}}\right)(z_s)\right)+\alpha'_{z_s}\left(F_w(z_s)\otimes F_{\tilde{w}}(z_s)\right)$$

■

Theorem B.4 below is useful in showing that a solution to an RDE in the flat case satisfies our manifold Definition 5.2. Let U and $\tilde{U}$ be open sets for the remainder of this subsection.

Definition B.3 *Let $f : U \subseteq \mathbb{R}^d \to \tilde{U} \subseteq \mathbb{R}^{\tilde{d}}$ be a smooth map. Let $F : U \to L\left(W, \mathbb{R}^d\right)$ and $\tilde{F} : \tilde{U} \to L\left(W, \mathbb{R}^{\tilde{d}}\right)$ be smooth. We say F and $\tilde{F}$ are $f-$* ***related dynamical systems*** *if*

$$f'(x)\,F_w(x)=\tilde{F}_w\circ f(x)\ \text{ for all } w\in W.$$

Theorem B.4 *Suppose $f : U \subseteq \mathbb{R}^d \to \tilde{U} \subseteq \mathbb{R}^{\tilde{d}}$ is a smooth map and let $F : U \to L\left(W, \mathbb{R}^d\right)$ and $\tilde{F} : \tilde{U} \to L\left(W, \mathbb{R}^{\tilde{d}}\right)$ be $f-$related dynamical systems. If $\mathbf{z}$ solves*

$$d\mathbf{z}_t=F_{d\mathbf{X}_t}(z_t)$$

with initial condition $z_0=\bar{z}_0$, then $\tilde{\mathbf{z}}_t := \left(\tilde{z}_t, \tilde{z}_s^{\dagger}\right) := f_*\mathbf{z}_t$ solves

$$d\tilde{\mathbf{z}}_t=\tilde{F}_{d\mathbf{X}_t}(\tilde{z}_t)$$

with initial condition $\tilde{z}_0 = f(\bar{z}_0)$.

Proof. We have by letting $\alpha := df$ in Theorem B.2

$$\begin{aligned}
\tilde{z}_{s,t} &= f(z_t) - f(z_s) \\
&\underset{3}{\approx} f'(z_s) F_{x_{s,t}}(z_s) + \partial_{F_w(z_s)} [f'(\cdot) F_{\tilde{w}}(\cdot)](z_s) |_{w\otimes\tilde{w}=\mathbb{X}_{s,t}} \\
&\underset{3}{\approx} \tilde{F}_{x_{s,t}}(\tilde{z}_s) + \left(\partial_{F_w(z_s)} \tilde{F}_{\tilde{w}} \circ f\right)(z_s) |_{w\otimes\tilde{w}=\mathbb{X}_{s,t}} \\
&\underset{3}{\approx} \tilde{F}_{x_{s,t}}(\tilde{z}_s) + \tilde{F}'_{\tilde{w}}(f(z_s)) f'(z_s) F_w(z_s) |_{w\otimes\tilde{w}=\mathbb{X}_{s,t}} \\
&\underset{3}{\approx} \tilde{F}_{x_{s,t}}(\tilde{z}_s) + \tilde{F}'_{\tilde{w}}(f(z_s)) \tilde{F}_w \circ f(z_s) |_{w\otimes\tilde{w}=\mathbb{X}_{s,t}} \\
&\underset{3}{\approx} \tilde{F}_{x_{s,t}}(\tilde{z}_s) + \left(\partial_{\tilde{F}_w(\tilde{z}_s)} \tilde{F}_{\tilde{w}}\right)(\tilde{z}_s) |_{w\otimes\tilde{w}=\mathbb{X}_{s,t}}.
\end{aligned}$$

Additionally

$$\tilde{z}_t^\dagger = f'(z_t) z_t^\dagger = f'(z_t) F_{(\cdot)}(z_t) = \tilde{F}_{(\cdot)}(\tilde{z}_t).$$

■

Corollary B.5 *Let $\phi : U \subseteq \mathbb{R}^d \to \tilde{U} \subseteq \mathbb{R}^d$ be a diffeomorphism with $\phi(z(I_0)) \subseteq U$. Then $\mathbf{z}$ on I_0 solves*

$$d\mathbf{z}_t = F_{d\mathbf{X}_t}(z_t)$$

with initial condition $z_0 = \bar{z}_0$ if and only if $\tilde{\mathbf{z}} := \phi_* \mathbf{z}$ on I_0 solves

$$d\tilde{\mathbf{z}}_t = F^{\phi}_{d\mathbf{X}_t}(\tilde{z}_t)$$

with initial condition $\tilde{z}_0 = \phi(\bar{z}_0)$ where $F^\phi := d\phi \circ (F \circ \phi^{-1})$.

Proof. This follows from Theorem B.4 by seeing that F is $\phi-$related to F^ϕ. ■

This last lemma helps patch solutions in the manifold case.

Lemma B.6 *Let $z \in C([0,T], V)$ and let $0 = t_0 < t_1 < \ldots < t_l = T$ be a partition of $[0,T]$. If*

$$z_{s,t} \underset{3}{\approx} F_{x_{s,t}}(z_s) + \left(\partial_{F_w(z_s)} F_{\tilde{w}}\right)(z_s) |_{w\otimes\tilde{w}=\mathbb{X}_{s,t}} \tag{B.1}$$

holds for all $t_i \leq s \leq t \leq t_{i+1}$ *and* $0 \leq i < l$ *then Eq. (B.1) holds for* $0 \leq s \leq t \leq T$.

In particular, if $\mathbf{z}_t$ *solves* $d\mathbf{z}_t = F_{d\mathbf{X}_t}(z_t)$ *with* $z_0 = \bar{z}_0$ *on* $[0,\tau]$ *and* $\tilde{\mathbf{z}}_t$ *solves* $d\tilde{\mathbf{z}}_t = F_{d\mathbf{X}_t}(\tilde{z}_t)$ *with* $\tilde{z}_\tau = z_\tau$ *on* $[\tau, T]$*, then the concatenation of* $\mathbf{z}_t$ *and* $\tilde{\mathbf{z}}_t$ *in the sense of Lemma 2.9 solves* $d\mathbf{z}_t = F_{d\mathbf{X}_t}(z_t)$ *with* $z_0 = \bar{z}_0$ *on* $[0,T]$.

Proof. This proof is identical from [3] [Lemma A.2], adapted here with different notation. We will only prove it in the case of two subintervals. First note that

$$F_w(y) = F_w(x) + F_w'(x)(y-x) + O\left(|w|\,|y-x|^2\right)$$

and

$$\left(\partial_{F_w(y)} F_{\tilde{w}}\right)(y) = \left(\partial_{F_w(x)} F_{\tilde{w}}\right)(x) + O\left(|w|\,|\tilde{w}|\,|y-x|\right)$$

by Taylor's theorem and the fact that $w \to F_w$ is linear. Using these facts, we have

$$\begin{aligned}
z_{s,t} &= z_{s,\tau} + z_{\tau,t} \\
&\underset{3}{\approx} F_{x_{s,\tau}}(z_s) + \left(\partial_{F_w(z_s)} F_{\tilde{w}}\right)(z_s)\,|_{w\otimes\tilde{w}=\mathbb{X}_{s,\tau}} + F_{x_{\tau,t}}(z_\tau) + \left(\partial_{F_w(z_\tau)} F_{\tilde{w}}\right)(z_\tau)\,|_{w\otimes\tilde{w}=\mathbb{X}_{\tau,t}} \\
&\underset{3}{\approx} F_{x_{s,t}}(z_s) + F'_{x_{\tau,t}}(z_s)(z_{s,\tau}) \\
&\quad + \left(\partial_{F_w(z_s)} F_{\tilde{w}}\right)(z_s)\,|_{w\otimes\tilde{w}=\mathbb{X}_{s,\tau}} + \left(\partial_{F_w(z_s)} F_{\tilde{w}}\right)(z_s)\,|_{w\otimes\tilde{w}=\mathbb{X}_{\tau,t}} \\
&\underset{3}{\approx} F_{x_{s,t}}(z_s) + F'_{x_{\tau,t}}(z_s)\left(F_{x_{s,\tau}}(z_s)\right) + \left(\partial_{F_w(z_s)} F_{\tilde{w}}\right)(z_s)\,|_{w\otimes\tilde{w}=\mathbb{X}_{s,\tau}+\mathbb{X}_{\tau,t}} \\
&= F_{x_{s,t}}(z_s) + \left(\partial_{F_w(z_s)} F_{\tilde{w}}\right)(z_s)\,|_{w\otimes\tilde{w}=\mathbb{X}_{s,\tau}+\mathbb{X}_{\tau,t}+x_{s,\tau}\otimes x_{\tau,t}} \\
&= F_{x_{s,t}}(z_s) + \left(\partial_{F_w(z_s)} F_{\tilde{w}}\right)(z_s)\,|_{w\otimes\tilde{w}=\mathbb{X}_{s,t}}.
\end{aligned}$$

■

Portions of Appendix B are adapted from material awaiting publication as Driver, B.K.; Semko, J.S., "Controlled Rough Paths on Manifolds I," submitted, *Revista Matemática Iberoamericana*, 2015. The dissertation author was the primary author of this paper.

Appendix C

Smooth Horizontal Lifting

C.1 Connections

This section will develop the motivation for a connection one-form. Let $E \to M$ be a vector bundle with fiber V and let $G = \mathrm{Aut}(V)$. Further let P be the associated principal bundle to E, i.e. $P \to M$ is a fiber bundle with fibers, $P_m := GL(V, E_m)$ for each $m \in M$. Notice that these fibers are diffeomorphic to G, G acts on the right of P by composition, so that $u_m g = u_m \circ g$ for all $u_m \in P_m$ and $g \in G$.

If E is equipped with a covariant derivative, ∇, we may construct a $\mathfrak{g} := \mathrm{Lie}(G) = \mathrm{End}(V)$ – valued one-form $\omega = \omega^\nabla$ on P by

$$\omega^\nabla(\dot{u}(0)) := u(0)^{-1} \frac{\nabla}{dt}|_{t=0} u(t) \tag{C.1}$$

for all smooth paths in P. This one-form has the following properties;

1. If $u(t) = u_0 e^{tA}$ for some $u_o \in P$ and $A \in \mathfrak{g}$ then

$$\omega^\nabla(\dot{u}(0)) := u_o^{-1} \frac{\nabla}{dt}|_{t=0} u_0 e^{tA} = u_o^{-1} u_0 A = A.$$

We denote $\dot{u}(0)$ in this example by $\tilde{A}(u_0)$ or simply by $u_0 A$.

2. If $g \in G$, then

$$\begin{aligned}\left(R_g^* \omega^\nabla\right)(\dot{u}(0)) &= \omega^\nabla\left(\left(R_g\right)_* \dot{u}(0)\right) = \omega^\nabla\left(\frac{d}{dt}|_0 [u(t) g]\right) \\ &= [u(0) g]^{-1} \frac{\nabla}{dt}|_{t=0} [u(t) g] = g^{-1}\left[u(0)^{-1} \frac{\nabla}{dt}|_{t=0} u(t)\right] g \\ &= Ad_{g^{-1}}\left[\omega^\nabla(\dot{u}(0))\right].\end{aligned}$$

This shows that every covariant derivatives gives rise to connection one-form ω^∇ as in Definition C.1 below on $P(E)$.

Definition C.1 *Let G be a Lie group, $P \to M$ be a principal bundle with structure group, G, and $\mathfrak{g} := \mathrm{Lie}(G) := T_e G$. We write $G \to P \xrightarrow{\pi} M$ to denote that P is a principal bundle over M with structure group G and projection map π. A $\mathfrak{g}$ – valued one-form, ω, on P is a connection one-form provided;*

1. *$\omega\left(\tilde{A}(\cdot)\right) = A$ for all $A \in \mathfrak{g}$ where $\tilde{A}(u_0) := \frac{d}{dt}|_0 u_o e^{tA}$ – which is a typical "vertical" vector in TP (see Notation C.7 below).*

2. *$R_g^* \omega = Ad_{g^{-1}} \omega$ for all $g \in G$, i.e.*

$$\left(R_g^* \omega\right)(\xi_u) = Ad_{g^{-1}} \omega(\xi_u) \ \forall \ g \in G \text{ and } \xi_u \in TP.$$

Example C.2 (Trivial Bundle Case) *Suppose that $P = M \times G$ is a trivial principal bundle and ω is a connection form on P. In this case we may associate to ω a one-form on M with values in $\mathfrak{g}$ by setting*

$$\Gamma(v_m) := \omega((v_m, 0_e)) \in \mathfrak{g} \ \forall \ v_m \in T_m M. \tag{C.2}$$

Furthermore we may reconstruct ω *from* Γ *as follows. Let* $v_m \in T_mM$, $A \in \mathfrak{g}$, *and* $g \in G$ *so that* $\tilde{A}(g)$ *is the generic element of* T_gG. *We then have*

$$\begin{aligned}\omega\left(v_m, \tilde{A}(g)\right) &= \omega(v_m, 0_g) + \omega\left(0_m, \tilde{A}(g)\right) \\ &= \omega\left((R_g)_*(v_m, 0_e)\right) + A \\ &= \left(R_g^*\omega\right)((v_m, 0_e)) + A \\ &= Ad_{g^{-1}}[\omega((v_m, 0_e))] + A \\ &= A + Ad_{g^{-1}}\Gamma(v_m).\end{aligned}$$

In this way we see that connections on $M \times G$ *are in one to one correspondence with* $\mathfrak{g}$ *– valued one-forms on* M.

Before finishing this example let us compute $\omega(\dot{u}(t))$ *where we write* $u(t) = (y(t), g(t))$ *for any smooth curve* u *in* P. *The key point is to observe that* $\dot{g}(t) = \tilde{A}(g(t))$ *where* $A{:=}L_{g(t)^{-1}*}\dot{g}(t)$ *and therefore,*

$$\omega(\dot{u}(t)) = \omega((\dot{y}(t), \dot{g}(t))) = L_{g(t)^{-1}*}\dot{g}(t) + Ad_{g(t)^{-1}}\Gamma(\dot{y}(t)). \tag{C.3}$$

Alternatively stated, if $(v_m, \xi_g) \in T_{(m,g)}(M \times G) \cong T_mM \times T_gG$, *then*

$$\omega((v_m, \xi_g)) = \theta(\xi_g) + Ad_{g^{-1}}\Gamma(v_m), \tag{C.4}$$

where

$$\theta(\xi_g) := L_{g^{-1}*}\xi_g \in \mathfrak{g} \tag{C.5}$$

is the left ***Maurer–Cartan*** *form on* G.

C.2 Horizontal Lifts

Definition C.3 (Smooth Horiontal Lifts) *Let* $\left(G \to P \overset{\pi}{\to} M, \omega\right)$ *be a principal bundle with connection,* ω*, and* $y(t)$ *be a smooth curve in* M*. We say that* $t \to u(t) \in P$ *is a horizontal lift of* y *provided; i) it is a lift, i.e.* $\pi \circ u = y$ *(or equivalently* $u(t) \in P_{y(t)}$ *for all* t*) and ii) it is horizontal, i.e.* $\omega(\dot{u}(t)) = 0$ *for all* t*.*

Example C.4 (Trivial Bundle Case II) *Let us continue the notation in Example C.2 and suppose that* $y(t) \in M$ *is a smooth curve. Any lift of* y *is of the form* $u(t) = (y(t), g(t))$ *for some smooth curve,* $t \to g(t) \in G$*. From Eq. (C.3) it follows that* u *is horizontal iff*

$$0 = L_{g(t)^{-1}*}\dot{g}(t) + Ad_{g(t)^{-1}}\Gamma(\dot{y}(t))$$

or equivalently by applying $Ad_{g(t)*}$ *to both sides of this equation iff*

$$0 = R_{g(t)^{-1}*}\dot{g}(t) + \Gamma(\dot{y}(t)).$$

In the matrix group case this is equivalent to solving,

$$\dot{g}(t) + \Gamma(\dot{y}(t))\, g(t) = 0.$$

These differential equations have global unique solutions once we specify $g(0) = g_0$ *for some* $g_0 \in G$*. Hence for trivial bundles we have shown that to each* $u_0 \in P_{y(0)}$ *there exists a unique horizontal lift,* $u(\cdot)$*, of* y *such that* $u(0) = u_0$*.*

Before ending this section, let us consider what happens to all of these structures under pull backs.

Example C.5 *Suppose* $G \to \tilde{P} \overset{\tilde{\pi}}{\to} \tilde{M}$ *is another principal bundle with the same structure group,* $f : M \to \tilde{M}$ *is a smooth map, and* $F : P \to \tilde{P}$ *is a bundle map* *above f, i.e. $\tilde{\pi} \circ F = f \circ \pi$ and $F(ug) = F(u)\, g$ for all $u \in P$ and $g \in G$. This last statement may be written as $F \circ R_g = R_g \circ F$.*

A connection $\tilde{\omega}$ on $\tilde{P}$ pulls back to a $\mathfrak{g}$ – valued one-form, $\omega := F^{*}\tilde{\omega}$ on P. This one-form is again a connection on P since;

$$\begin{aligned}
\omega(u \cdot A) &= (F^{*}\tilde{\omega})\left(\frac{d}{dt}|_0 ue^{tA}\right) = \tilde{\omega}\left(F_{*}\frac{d}{dt}|_0 ue^{tA}\right) \\
&= \tilde{\omega}\left(\frac{d}{dt}|_0 F\left(ue^{tA}\right)\right) = \tilde{\omega}\left(\frac{d}{dt}|_0 F(u)\, e^{tA}\right) \\
&= \tilde{\omega}(F(u) \cdot A) = A
\end{aligned}$$

and

$$\begin{aligned}
R_g^{*}\omega &= R_g^{*}F^{*}\tilde{\omega} = (F \circ R_g)^{*}\tilde{\omega} = (R_g \circ F)^{*}\tilde{\omega} \\
&= F^{*}R_g^{*}\tilde{\omega} = F^{*}\left(Ad_{g^{-1}}\tilde{\omega}\right) = Ad_{g^{-1}}F^{*}\tilde{\omega} = Ad_{g^{-1}}\omega.
\end{aligned}$$

Moreover, if $y(t)$ is a smooth curve in M and $u(t)$ is a horizontal lift of y, then $F \circ u$ is a horizontal lift of $f \circ y$. To see this is the case we have $\tilde{\pi} \circ F \circ u = f \circ \pi \circ u = f \circ y$ so that $F \circ u$ is a lift of $f \circ y$. Moreover,

$$\tilde{\omega}\left(\frac{d}{dt}F(u(t))\right) = (F^{*}\tilde{\omega})(\dot{u}(t)) = \omega(\dot{u}(t)) = 0.$$

As a consequence of these examples we may easily prove the following theorem.

Theorem C.6 (Existence of Horizontal Lifts) *Let $G \to P \overset{\pi}{\to} M$ be a principal bundle with connection ω, y be a smooth curve in M, and $u_0 \in P_{y(0)}$. Then there exists a unique horizontal lift $u(t)$ above y such that $u(0) = u_0$.*

Proof. Let us first prove local existence and uniqueness. We choose an open neighborhood, $U \subseteq M$ of $y(0)$ such that $\pi^{-1}(U)$ may be trivialized, i.e. there exists a bundle isomorphism $F : U \times G \to \pi^{-1}(U)$ such that $\pi \circ F = \tilde{\pi}$ where $\tilde{\pi}$ is projection onto the first factor of $U \times G$. We then let $\tilde{\omega} := F^* \omega$ and $g_0 \in G$ be determined by $(y(0), g_0) = F^{-1}(u_0)$.

From Example C.5 we know that $\tilde{\omega}$ is a connection on $U \times G$. Now choose $\tau > 0$ so that $y([0,\tau]) \subseteq U$. We may then use Example C.4 to conclude there exists a unique $g(t) \in G$ for $0 \leq t \leq \tau$ such that $\tilde{u}(t) := (y(t), g(t))$ is the $\tilde{\omega}$ – horizontal lift of y starting at $\tilde{u}(0) = (y(0), g_0)$. It then follows from Example C.5 that $u(t) = F(\tilde{u}(t))$ is the unique horizontal lift of $y(t)$ for $0 \leq t \leq \tau$ starting at u_0. By a finite covering argument we may continue this horizontal lift to the time interval for which y is defined. The uniqueness is also easily proved at the same time. ■

There is one last horizontal lifting proposition we should record here.

Notation C.7 *Let $G \to P \overset{\pi}{\to} M$ be a principal bundle with connection ω. The **vertical subspace** at $u \in P$ is defined by*

$$\mathcal{V}_u := \{\xi \in T_u P : \pi_* \xi = 0\} \subseteq T_u P$$

*and the **horizontal subspace** at $u \in P$ is defined by*

$$\mathcal{H}_u^{\omega} := \{\xi \in T_u P : \omega(\xi) = 0\} \subseteq T_u P.$$

Proposition C.8 (Infinitesimal Horizontal Lifting) *If $G \to P \overset{\pi}{\to} M$ is a principal bundle with connection ω, then*

1. $\mathcal{V}_u = \{u \cdot A : A \in \mathfrak{g}\}$,
2. $T_u P = \mathcal{V}_u \oplus \mathcal{H}_u^{\omega}$ *for all $u \in P$, and*
3. $\pi_* : \mathcal{H}_u^{\omega} \to T_{\pi(u)} M$ *is a linear isomorphism.*

Proof. The results of this proposition are purely local and hence we may assume that P is the trivial bundle $M \times G$. In this model with $u = (m, g)$,

$$\begin{aligned} \mathcal{V}_u &= \{(0_m, \xi_g) : \xi_g \in T_g G\} = \left\{\left(0_m, \tilde{A}(g)\right) : A \in \mathfrak{g}\right\} \\ &= \{(0_m, L_{g*}A) : A \in \mathfrak{g}\} = \{u \cdot A : A \in \mathfrak{g}\} \end{aligned}$$

which proves item 1. Letting $\Gamma(v_m) := \omega((v_m, 0_e))$ as in Example C.2, we have

$$\omega\left(v_m, \tilde{A}(g)\right) = A + Ad_{g^{-1}}\Gamma(v_m)$$

and so $\left(v_m, \tilde{A}(g)\right)$ is horizontal iff $A = -Ad_{g^{-1}}\Gamma(v_m)$. Therefore it follows that

$$\begin{aligned} \mathcal{H}_u^\omega &= \{(v_m, -L_{g*} \cdot Ad_{g^{-1}}\Gamma(v_m)) : v_m \in T_m M\} \\ &= \{(v_m, -R_{g*}\Gamma(v_m)) : v_m \in T_m M\}. \end{aligned}$$

From these descriptions of $\mathcal{V}_u$ and $\mathcal{H}_u^\omega$ it is easily seen that $\mathcal{V}_u \cap \mathcal{H}_u^\omega = \{0\}$ and $T_u P = \mathcal{V}_u + \mathcal{H}_u^\omega$ and hence item 2. is proved. Item 3. is also now trivial to check since it is clear that $\pi_*(v_m, -R_{g*}\Gamma(v_m)) = v_m$ defines an isomorphism from $\mathcal{H}_u^\omega$ onto $T_m M$. ■

Notation C.9 *If $G \to P \overset{\pi}{\to} M$ be a principal bundle with connection ω, let $\mathcal{B}_u^\omega : T_{\pi(u)}M \to \mathcal{H}_u^\omega$ be the inverse of $\pi_{*u}|_{\mathcal{H}_u^\omega}$. Thus if $v \in T_{\pi(u)}M$, then $\xi = \mathcal{B}_u^\omega v$ iff $\pi_*\xi = v$ and $\omega(\xi) = 0$. We refer to $\mathcal{B}_u^\omega v$ as the* ***horizontal lift*** *of v to $T_u P$.*

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