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# **CENTER FOR REAL ESTATE AND URBAN ECONOMICS**

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**A FRAMEWORK FOR ANALYZING THE  
SIMULTANEOUS DEFAULT AND PREPAYMENT  
OPTIONS FOR FIXED RATE MORTGAGES**

By

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DANIEL C. LEE

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A FRAMEWORK FOR ANALYZING THE SIMULTANEOUS DEFAULT AND PREPAYMENT  
OPTIONS FOR FIXED RATE MORTGAGES †

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WORKING PAPER NO. 93-212

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## Abstract

This paper develops a theoretical model for evaluating simultaneously default and prepayment options for the Fixed Rate Mortgage (FRM). Default is a put option by which the borrower can "put" his house to the lender at the current market value of the mortgage. Prepayment is a call by which the borrower can buy back his mortgage debt at its current par value. Our research finds that an FRM contains a hidden exchangeability feature between the default put option and the prepayment call option. The borrower does not own the embedded call option; instead, the borrower owns an exchange compound option which entitles its holder to exchange the put for the call. Our findings also indicate that in order to evaluate and price properly FRM's, one needs to simultaneously take into account the inter-relationship between the default and prepayment options. Our analysis is consistent with previous simulation studies, such as Kau et al. (1986, 1990), that find the value of the simultaneous put-call options in the FRM is considerably different from the value of the sum of the independent put plus the independent call.

## I. INTRODUCTION

A burgeoning body of literature has emerged to explain default and prepayment behavior for the Fixed Rate Mortgage (FRM). Researchers recognize that a standard FRM contract can be viewed as a stream of risk-free cash flows plus a portfolio of embedded options. Default is a put option giving the borrower the right to sell the house to the lender at a price equal to the market value of the mortgage; and prepayment is a call option giving the borrower the right to buy back his mortgage debt at its current par value.

Most early research analyzes the embedded constituent options of an FRM by examining the simplified contracts that have some, but not all, of the options found in a standard FRM. As noted by Kau, Keenan, Muller and Epperson (KKME) (1986), the more common approaches concentrate on the right to prepay a mortgage while assuming away the possibility of default, or consider default but exclude the possibility of prepayment. In other words, either the put or the call is ignored. Simulation results presented by KKME (1986, 1990) confirm that the marginal value contribution of either option in the standard FRM is not equal to the value of the same option in the mortgage where only that option is available to the borrower.

This paper develops the theoretical foundations for evaluating simultaneously the prepayment and the default options contained in the FRM. Our analysis finds that an FRM contains a "hidden" exchangeability feature between the default put and the prepayment call options. This finding helps explain simulation results, such as in KKME (1986, 1990) and Schwartz and Torous (ST) (1992), and avers the importance of the simultaneous option approach to modelling mortgage pricing. With the aid of a formal analytical model and a graphical presentation, we will

examine the interaction of the two options and demonstrate how the interactions between these constituent options affect each other's value and borrower's behavior.

The plan of this paper is as follows. In Section II, we discuss the idiosyncratic nature of the exercise prices of the options embedded in the FRM. In Sections III and IV, we provide a conceptual framework for decomposing the joint put/call option, and analyzing the structure of exchangeability between the put and the call options. In Section V, we present an intuitively appealing graphical analog to the theory outlined in Sections III and IV. The paper concludes with a summary.

## II. THE CHANGING STRIKING PRICES OF THE CALL AND THE PUT OPTIONS

Unlike a stock option, whose striking price is usually constant, the striking price of the call embedded in an FRM is a function of the coupon rate and time. It follows a combination of continuous and jump processes. The deterministic nature of the changing striking price is a result of the accumulated interest and the periodic payments, including reduction of principal along with interest due over the life of the loan. The striking price (i.e., the unpaid book balance plus the accrued interest since the last payment date) increases until the payment date. On that date, when the debt service payment is made, the striking price is reduced abruptly by an amount equal to the pre-determined debt-service payment.

The "jump" complicates the valuation of the call, but can be solved by using Geske's (1977) compound option approach, as adapted by Epperson, Kau, Keenan, and Muller (EKKM) (1985), for valuing the default put. The FRM mortgage contract fits into a European option framework when the prepayment possibility is excluded. Following Geske (1977), puttable mortgages are compound options, because at each payment date

prior to the last one, the borrower either defaults or, by making the scheduled payment, purchases a new option to default at the next payment date.

Although the default option the borrower owns is American, as EKKM (1985) and KKME (1986) point out, in the absence of non-financial motivation, default will only occur at the next scheduled payment date when debt service is due, since the borrower can enjoy the free housing services at the expense of the lender until such time.<sup>1</sup> As a result, both the amortization and default characteristics of an FRM are captured by treating it as a European compound option, where the payoff to an expiring option is a further set of options covering the next period.

The prepayment characteristic of an FRM can be treated in the same way. The only difference is that the call provision is treated as an American compound option, because, unlike default, prepayment may occur at any time. The borrower does not necessarily exercise the prepayment option only on the scheduled payment date. A rational borrower will balance the rising accrued interest (at the contract rate) during the entire time interval between payment dates against the continuously changing market interest rate and house price to determine the optimal time to prepay.

Since the current value of the mortgage is affected by options to default or prepay in the future, the analysis of value requires working backwards with the value of later options feeding recursively into the earlier ones. KKME (1986) solves the pricing partial differential equation (PDE) using an explicit finite difference method subject to the

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<sup>1</sup> Legally and technically, default takes place only when the borrower fails to meet his/her financial obligation, i.e., fails to make the payment no later than the payment date.



boundary conditions.<sup>2</sup> This is necessary because the possibility of a prepayment converts a simple valuation problem into a complicated optimal stopping problem.

Since the borrower's optimal mortgage termination strategy can be implemented by a dynamic programming procedure, this approach is used to represent the borrower's optimizing behavior. The calculation of the explicit finite difference method (and the binomial model) starts at the maturity date of the FRM,  $T$ , and works backwards through the time grid (tree).<sup>3</sup> The values of the constituent options corresponding to any state of nature at time  $T$  (i.e., the terminal conditions) are assumed to be known.

The option values at time  $t$  capture not only the effect of early exercise possibilities at time  $t$ , but also the effect of early exercise possibilities at subsequent times.<sup>4</sup> This process gives the borrower a way to take into account the opportunity costs, which include delaying the exercise of the call and suffering for paying the interest based on the coupon rate instead of the current lower market rate if the call is

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<sup>2</sup> It is also possible to use the binomial approach to solve the problem. The extension of the binomial approach has been developed to show that, in the limit, it converges to continuous time multivariate stochastic models, as in Evnine (1983). The discrete time recurrence relation provides an elegant numerical solution to the PDE produced by a continuous model such as ours.

<sup>3</sup> As illustrated by Hull (1989, p.242), the procedure for calculating partial derivatives using finite difference methods is similar to the procedure using the lattice approach (the binomial model). In this study, we will refer to both the explicit finite difference method and the lattice approach as a dynamic programming procedure.

<sup>4</sup> That is, the holding values of the embedded options.

otherwise exercised and the mortgage is refinanced. The process examining the boundary conditions at each point in time determines if early exercise is optimal. Since finding the optimal exercise strategy is an integral part of the valuation problem, the borrower is assumed to be monitoring the boundary conditions constantly so as to decide whether to terminate or to continue to hold the FRM. This process is a full recursive procedure of a dynamic programming model. In our subsequent discussion we use the following notation:

$H_t$  -- the market value of the collateral (the house) at calendar time  $t$ ;  
 $B_t(B_0, T, i_{c,0})$  -- the unpaid book balance of the mortgage liability plus the accrued interest at time  $t$ , assuming an original loan amount of  $B_0$ , an original term to maturity of  $T$  payment-periods (in months for monthly payment FRM), and an original contract rate of  $i_{c,0}$ ;

$kB_0$  -- periodic debt service, where  $k$  is the mortgage constant;

$M_t^{PC}(r, T, i_{c,0})$  -- the market value of the puttable and callable mortgage  $B_t(B_0, T, i_{c,0})$  at time  $t$ , where  $r$  is the one-period risk-free interest rate observed at time  $t$ ;

$M_t(r, T, i_{c,0})$  -- the value of an otherwise comparable non-callable and non-puttable (i.e., option-free) mortgage at time  $t$  that has a contract rate equal to the rate on the existing mortgage,  $M_t^{PC}(r, T, i_{c,0})$ ;

$O_{t,t^*}(\cdot)$  -- the present value of the  $t^*$ th option (either the put, the call or the exchange option that is "current" at time  $t$ ) of the  $T$  consecutive one-payment-period-life compound option  $O$ 's when the current time is  $t$ ; to simplify notation, when  $t$  is any point in time within the  $t^*$ th payment-period (i.e., between any payment date  $t-1^*$  and  $t^*$ ) of the FRM contract life ( $T$  payment-periods), the second subscript " $t^*$ " will be dropped where there is no ambiguity as to their value; e.g.,  $O_{1.5,2}$  is the value of the second compound option at calendar time  $t = 1.5$ ;

$O_t^E(\cdot)$  -- the intrinsic value from exercising the option  $O_t(\cdot)$  at time  $t$ ;

$O_t^H(\cdot)$  -- the holding value of the option  $O_t(\cdot)$  at time  $t$ ;

$C_t(r, T, i_{c,0})$  -- the value of the  $t^*$ th one-payment-period-life simple prepayment (call) option to the borrower at time  $t$ ;

$D_t(B_0, T, i_{c,0})$  -- the value of the  $t^*$ th one-payment-period-life simple default (put) option to the borrower at time  $t$ .

All the options above are compound options, except for the put,  $D_t$ , and the call,  $C_t$ . To avoid potential confusion, a superscript "s" will be added to indicate the non-compound options. For simplification, we will drop these functional dependencies where no ambiguity exists about their value. For any variable  $V$ , we use  $V^-$  and  $V^+$  to indicate the variable's value before and after the debt service payment occurs.

### III. GENERAL ARBITRAGE RELATIONSHIPS<sup>5</sup>

An FRM is a long-term contract between a borrower and a lender. At each payment date the borrower actually has three choices. One choice is to make the payment and retain the house until the next payment is due. A second choice is to default on the loan and lose the house. A final choice is to replace the existing loan with a new one. The borrower analyzes each of these choices and takes the action that maximizes his wealth position.

Define  $J_t(B_0, T, i_{c,0})$  as the joint (combined) value (net costs) of a series of the default and the prepayment options (i.e.,  $D_{t,t^*}$ ,  $C_{t,t^*}$ , ...,  $D_{t,T}$ ,  $C_{t,T}$ ) to the borrower at time  $t$ . It is the current joint market value of the two constituent options at time  $t$  under the optimal exercise strategy. The borrower maximizes his wealth position by constantly monitoring the values of  $J_t^H$ ,  $C_t^E$  and  $D_t^E$  at each  $t$  in order to decide if the FRM should be kept alive by making the payment when it

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<sup>5</sup> The arbitrage proofs in this paper do not require distributional assumptions. The derived values of embedded options depend on the elimination of risk-free arbitrage opportunities. Hence, the results provide a set of conditions that must be satisfied by any reasonable option pricing theory in a perfect market setting.

is due or be terminated by either defaulting or prepaying.

At any point in time  $t$  between any payment date  $t-1^*$  and  $t^*$  (where  $t^* < T$ ), the borrower holds a corresponding position

$$H_t + J_t - M_t$$

where

$$J_t = \text{Max}\{ J_{t,t+1^*}, \text{Max}\{ J_t^H, \text{Max}\{ C_t, D_t \} \} \} \quad (1)$$

$$C_t = \text{Max}\{ C_t^H, M_t - B_t \} \quad (2)$$

$$D_t = \text{Max}\{ D_t^H, M_t - H_t \} \quad (3)$$

$$M_t^{\text{pc}} = M_t - J_t.$$

The default put and the prepayment call are the simple form options, while the joint option  $J_t$  is the complex form. The holding value of the option,  $J_t^H$ , takes into account the present value of all the future  $J_{t,T^*}$ 's and their costs (where  $t^* < T^* \leq T$ ). That is, the dynamic programming solution for  $J_{t,t^*}^H$  has incorporated the element that has the largest present value among the series of  $\{ J_{t,t+1^*}, J_{t,t+2^*}, \dots, J_{t,T-1}, J_{t,T} \}$  and the present value of all the future debt service payments (directly and indirectly).<sup>6</sup>

The creation of  $J_t$  in this model captures the interdependence of the lives and values of both options. If the call and the put options were independent (un-intersected),<sup>7</sup> then at any time  $t$  when the exercise

<sup>6</sup> In other words, the holding values used are the global maximums of the options of interest with respect to the remaining life of the FRM.

<sup>7</sup> If one option alters the value of the other option within a portfolio, we define the values of the two options as intersected or interdependent. That is, independence requires that the value decomposition principle (i.e., the parts sum to the whole) holds for any portfolio that consists of the two options. In our case, the two

values of both options dominate their corresponding holding values,  $C_t + D_t = J_t^H$  would be a correct specification.<sup>8</sup> Unfortunately, this is usually not true, because when one option is exercised, the other will be "terminated" automatically. Moreover, the flexibility of allowing the borrower to choose either of the two options engenders a premium that would be contained in  $J_t^H$ . As a result, an optimal borrower's strategy is to exercise the option only when it offers the highest financial benefit (i.e.,  $J_t$  is worth more dead than alive).

The position of the borrower at time T before making the last payment is

$$H_T + J_T^- - kB_0$$

where

$$J_T^- = \text{Max}\{ C_T^-, D_T^- \}$$

$$C_T^- = 0$$

$$D_T^- = D_T^{E-} = \text{Max}\{ 0, kB_0 - H_T \}.$$

The borrower holds the house and has an obligation to make the debt service payment  $kB_0$ ; he simultaneously has a put option  $D_T$  on the house allowing him to sell the house to the lender for  $kB_0$  if he wishes. A rational borrower will sell only if  $kB_0 \geq H_T$ . The prepayment option has no value at date T, since no outstanding payment will exist beyond date T. Conceptually, after the  $T-1_{th}$  payment is made, we can treat the last payment-period's  $J_t$  (i.e.,  $J_{t,T}$ ) as a simple American joint put/call

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options are not only intersected but also perfectly correlated because both options share the same underlying asset and, hence, the same Ito term.

<sup>8</sup> In other words, the value decomposition principle can be used to analyze the value of each individual constituent option.

option of an FRM,<sup>9</sup> where  $T-1 \leq t \leq T$  and  $T$  is the maturity date of the joint option.<sup>10</sup>

At any earlier payment date  $t^*$  before the payment is made, the borrower holds a corresponding position<sup>11</sup>

$$H_{t^*} + J_{t^*}^- - M_{t^*}^-$$

where

$$M_{t^*}^- = M_{t^*}^+ + kB_0$$

$$J_{t^*}^- \equiv J_{t^*, t^*}^- = \text{Max}\{ J_{t^*, t+1^*}^-, \text{Max}\{ C_{t^*}^-, D_{t^*}^- \} \}$$

$$C_{t^*}^- = \text{Max}\{ C_{t^*}^{H^-}, C_{t^*}^{E^-} \}$$

$$C_{t^*}^{H^-} = 0$$

$$C_{t^*}^{E^-} = \text{Max}\{ 0, M_{t^*}^- - B_{t^*}^- \}$$

$$D_{t^*}^- = \text{Max}\{ D_{t^*}^{H^-}, D_{t^*}^{E^-} \}$$

$$D_{t^*}^{H^-} = 0$$

$$D_{t^*}^{E^-} = \text{Max}\{ 0, M_{t^*}^- - H_{t^*} \}$$

$$M_{t^*}^{pc^-} = M_{t^*}^- - J_{t^*}^-.$$

Since both the default put and the prepayment call are simple form options having only one payment-period of life, their holding values at maturity are all zero.  $J_t$  is treated as a compound option with the maturity equal to the time interval between the scheduled payment dates, and the payment can be treated as the price to purchase the next compound option. This is very similar to the way EKKM (1985) and KKME (1986, 1990) treat putable mortgages as compound European put options.

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<sup>9</sup> It is a simple option since no more options exist to be purchased.

<sup>10</sup> This property of  $J_{t,T}$  (i.e.,  ${}^s J_t$ ) is presented in Corollary 1.

<sup>11</sup> For any variable  $V_{t^*, t^*}$ , we use  $V_{t^*}$  to simplify the notation.

If an isolated trading strategy (using the underlying assets) provides exactly the same payoffs as the related derivative asset, then, if there are to be no arbitrage opportunities, the current cost of establishing the positions (the duplicating portfolios) required by the trading strategy must equal the current value of the related derivative asset.<sup>12</sup> In other words, when a portfolio of options and scheduled payments has identical cash flows with the option of interest,  $J_t$ , it must also have the same value. This insight justifies the use of the simpler existing underlying assets to manufacture the complex or currently non-existent derivative assets by following derived replicating trading strategies.

Applying Roll's (1977) and Ross's (1978) valuation by portfolio duplication technique, we can solve the complex option,  $J_t$ , by designing a portfolio of options that exactly duplicates its boundary conditions. Define a simple American option  ${}^S X_t^{C,D}$  that gives the holder the right to exchange asset D for asset C prior to the maturity date,  $t^*$ . As Margrabe (1978) demonstrates, the owner will exercise this type of option if and only if it brings in a positive return. This implies the terminal condition

$${}^S X_{t^*}^{C,D} = \text{Max}\{ 0, C - D \}. \quad (4)$$

The option value is non-negative with a maximum of C. If assets C and D are worth at least zero, then  $0 \leq {}^S X_{t^*}^{C,D} \leq C$ . Since  ${}^S X_{t^*}^{C,D} = \text{Max}\{ 0, C - D \} = \text{Max}\{ D, C \} - D$ , we know that  $\text{Max}\{ D, C \} = D + {}^S X_{t^*}^{C,D}$  is true. If we let D be any American put option  $D_t$  and C be any American call option  $C_t$ , it follows that Proposition I will hold:

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<sup>12</sup> See Rubinstein (1987).

PROPOSITION I. In a perfect market, a simple American exchange option  ${}^sX_t^{C,D}$  will not be exercised until it is optimal to exercise  $C_t$ .

Proof: Let us consider two portfolios.

Portfolio A consists of

- (a) the short sale of an American put option  $D_t$ ,
- (b) the purchase of an American call option  $C_t$ .

Portfolio B is

- (c) the purchase of a simple American exchange option  ${}^sX_t^{C,D}$ .

All the three options have the same maturity date,  $t^*$ . The relationship between the terminal values for the two portfolios is:

		On the maturity date, $t^*$ .	
Portfolio	Current value	$C_{t^*} < D_{t^*}$	$C_{t^*} \geq D_{t^*}$
A	$C_t - D_t$	$C_{t^*} - D_{t^*}$	$C_{t^*} - D_{t^*}$
B	${}^sX_t^{C,D}$	0	$C_{t^*} - D_{t^*}$
Comparison of portfolios' value		$V_B > V_A$	$V_B = V_A$

In any state of nature, portfolio B pays an amount greater than or equal to portfolio A. Therefore, portfolio B must have a higher price than portfolio A. That is,  ${}^sX_t^{C,D} \geq C_t - D_t$ . Since  $C_t - D_t$  is the exercise value of  ${}^sX_t^{C,D}$  at any point in time before maturity, and is dominated by  ${}^sX_t^{C,D}$ , keeping  ${}^sX_t^{C,D}$  alive until it is optimal to exercise  $C_t$  provides the option holder higher return. *Q.E.D.*

This important result implies that no separated effort is needed for determining the optimal conversion timing between the put and the call. The only optimal time to exercise  ${}^sX_t^{C,D}$  is when immediate exercise of  $C_t$  is optimal. As a result,  $C_t$  and  ${}^sX_t^{C,D}$  have synchronized effective lives.



As a related consequence, Proposition II will also be true:

**PROPOSITION II.** The value of a simple American joint put/call option of a zero-coupon bullet FRM,  ${}^S J_t$  (where  $0 \leq t \leq t^*$  and  $t^*$  is the maturity date for all four options,  ${}^S J_t$ ,  $C_t$ ,  $D_t$  and  ${}^S X_t^{C,D}$ ), is the sum of the values of the following two independent (un-intersected) embedded options:

- (a) a simple European put option  $D_t$ ,
- (b) plus a simple American exchange option  ${}^S X_t^{C,D}$ .

**Proof:** To show that  ${}^S J_t = D_t + {}^S X_t^{C,D}$ , consider the following two portfolios.

Portfolio A represents

- (a) a simple European put option  $D_t$ ,
- (b) plus a simple American exchange option  ${}^S X_t^{C,D}$ .

Portfolio B is

(c) the purchase of a simple American joint option  ${}^S J_t$ , which can be expressed as Choice{  $C_t$ ,  $D_t$  } to represent the borrower's optimal exercise strategy.

At time,  $t$ , before and on the maturity date, the cash receipts and net position are as follows.

When it is advantageous at any time  $t$  to exercise:<sup>13</sup>

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<sup>13</sup> Proposition I has guaranteed that the timing for exercising  ${}^S X_t^{C,D}$  will occur simultaneously with the exercise of  $C_t$ .

$C_t$	$D_t$ (Exercisable only when $t=t^*$ )
<hr/>	
Portfolio positions are	Portfolio positions are
From (a) traded-in	From (a) Exercised
(b) $D_t$ is exchanged for	(b) Expired
$C_t$ 's immediate exercise	
<hr/>	
Total A: $C_t^E$	Total A: $D_t^E$
From (c) Choose & Exercise $C_t$	From (c) Choose & Exercise $D_t$
<hr/>	
Total B: $C_t^E$	Total B: $D_t^E$

Since both portfolios have the same cash inflow under the optimal exercising strategy at any point in time, they must have the same value. That is, Choice{  $C_t, D_t$  }  $\equiv$   ${}^S J_t = D_t + {}^S X_t^{C,D}$ . Q.E.D.

This proposition signifies that, for a zero-coupon bullet FRM, the value of its embedded put/call joint option is the sum of a simple European put option  $D_t$  plus a simple American exchange option  ${}^S X_t^{C,D}$ .

**COROLLARY 1.** The value of  $J_{t,T}$  (the last payment-period's  $J_t$  after the  $T-1_{th}$  payment is made) embedded in a standard FRM is the sum of the values of the following two independent embedded options:

- (a) a simple European put option  $D_t$ ,
- (b) plus a simple American exchange option  ${}^S X_t^{C,D}$ ,

where  $T-1 \leq t \leq T$  and  $T$  is the maturity date of all four options.

**Proof:** This is a direct result from Proposition II. After the  $T-1_{th}$  payment is made, the last payment-period's  $J_t$  (i.e.,  $J_{t,T}$ ) is a simple American joint put/call option of an FRM,  ${}^S J_t$ . Q.E.D.

It is clear that the financial benefits provided by both constituent put and call options embedded in the mortgage are mutually exclusive when either is exercised. In an efficient market, it would be incorrect to price the joint option  $J_t$  by determining the value of each option separately and independently and adding them up. The exchange option  ${}^s X_t^{C,D}$  clearly corrects this "double-counting" problem.

#### IV. THE HIDDEN FEATURE OF EXCHANGEABILITY BETWEEN THE EMBEDDED DEFAULT PUT AND THE EMBEDDED PREPAYMENT CALL

To extend and modify the model developed in the last section, we define  ${}^{2m} X_t^{C,D}$  as an American compound exchange option with uncertain but limited (nominal) maturity. The "2m" superscript indicates the existence of two possible maturity dates: the "nominal" maturity date, which is the usual option maturity date, and the "conditional" maturity date, which is conditional upon (synchronized with) the maturity date of  $D_t$ . The "conditional" maturity date specifies that if  $D_t$  is exercised, option  ${}^{2m} X_t^{C,D}$  will be terminated immediately and automatically. This feature resembles the relationship between the  $C_t$  and the  $D_t$  in the joint option of the FRM. The holder of  ${}^{2m} X_t^{C,D}$  is entitled either to exchange  $D_{t,t^*}$  for  $C_{t,t^*}$  when  $t \leq t^*$  or to purchase portfolio  $S_{t^*,t+1^*}$  when  $t = t^*$ . The portfolio  $S_{t^*,t+1^*}$  consists of a new European simple option  $D_{t^*,t+1^*}$  to default on the next payment date ( $t+1^*$ ) and a new American compound exchange option  ${}^{2m} X_{t^*,t+1^*}^{C,D}$ .<sup>14</sup>

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<sup>14</sup> The exchange options are compound options except for the  $T_{th}$  one. The  $T_{th}$  exchange option of the  $T$  consecutive exchange options (during the  $T_{th}$  payment-period of the loan) can be expressed as:

$${}^{2m} X_{t,T}^{C,D} = \text{Max} \{ {}^{2m} X_{t,T}^{C,D|H}, C_{t,T}^E - D_{t,T}^H \},$$

Since  $J_t^H$  is the envelope of  $J_{t,T^*}$ 's,  $C_t^H$  and  $D_t^H$  (where  $t^* < T^* \leq T$ ), equation (1) can be rewritten as  $J_t = \text{Max}\{ J_t^H, \text{Max}\{ C_t^E, D_t^E \} \} = \text{Max}\{ J_t^H, C_t^E, D_t^E \}$ .<sup>15</sup> Without loss of generality, we have dropped all the max operators in the second level, because, as long as the action taken among the three choices gives the borrower the largest payoff, he achieves his optimal strategy. It requires that the  $J_t$  holder constantly compare all three choices to determine how and when to terminate the mortgage.

**PROPOSITION III.** The value of an American compound joint put/call option of an FRM,  $J_t$  (where  $0 \leq t \leq T$ ), is the sum of the values of:

- (a) an American compound exchange option,  ${}^{2m}X_t^{C,D}$ ,
- (b) plus a simple European put option,  $D_t$ .

**Proof:** Construct two portfolios. Portfolio A represents

- (a) an American compound exchange option,  ${}^{2m}X_t^{C,D}$ , and
- (b) a simple European put option,  $D_t$ .

Portfolio B consists of

- (c) the purchase of an American compound joint option  $J_t$ .

To prove that both portfolios generate identical cash flows in any state of nature at each point of time, we examine the relationship between the

where  ${}^{2m}X_{t,T}^{C,D|H}$  is the holding value of the simple exchange option  ${}^{2m}X_{t,T}^{C,D}$ . It is a simple option since this is the last one in the series and no more options exist to be purchased.

<sup>15</sup> As we have mentioned before, the flexibility of allowing the borrower to choose either of the two options carries a premium. This premium is captured by  $J_t^H$ . That is,  $J_t^H \geq C_t^H$  and  $J_t^H \geq D_t^H$ . However, once  $J_t$  is exercised, it will capture only the exercise value of the constituent call or put. That is,  $J_t^E = C_t^E$  or  $J_t^E = D_t^E$ .

boundary conditions and terminal values for the two portfolios through time and demonstrate that for optimal exercise strategies identical results will occur. Since both portfolios are compound options, we investigate the pattern of cash flows generated by the consecutive option portfolios purchased between the end of that period and the maturity date of the FRM (as a result of the optimal exercising strategy of both portfolios A and B)

Proceeding inductively by a dynamic-programming-like technique and working backwards, we show that portfolio A can duplicate portfolio B's results not only on any payment date but also at every point in time between any payment dates. As a result, the value of portfolio A must be no less than that of portfolio B to satisfy arbitrage conditions; that is,  ${}^{2m}X_{0,1}^{C,D} + D_{0,1} \geq J_{0,1}$ .

Following the same procedure, we also demonstrate that portfolio B can duplicate all the optimal cash flows generated by portfolio A under all possible states over all possible lives of the FRM. That is,  ${}^{2m}X_{0,1}^{C,D} + D_{0,1} \leq J_{0,1}$ . Since  ${}^{2m}X_{0,1}^{C,D} + D_{0,1} \geq J_{0,1}$  and  ${}^{2m}X_{0,1}^{C,D} + D_{0,1} \leq J_{0,1}$ , it must be that  ${}^{2m}X_{0,1}^{C,D} + D_{0,1} = J_{0,1}$ . Hence, both portfolios must be identical since each is able to act as the other's envelope under the circumstance that each follows its own optimal exercise strategy from the date the FRM is originated until the maturity date.

Moreover, the recursive nature of the series of the options and the borrower's decision making process guarantee that for any  $t$  the value of the joint option  $J_t$  must equal the value of the portfolio consisting of  ${}^{2m}X_t^{C,D}$  and  $D_t$ . That is,  ${}^{2m}X_t^{C,D} + D_t = J_t$  where  $0 \leq t \leq T$ . A complete proof is provided in Appendix I. *Q.E.D.*

The intuition behind these results is straightforward. The

implicit existence of the exchange option embedded in the FRM provides the borrower a safety net. If the borrower is doing well with the put option alone, the "service" provided by the call option is no longer needed. However, if the environment is working against the borrower and the call option is needed, the "insurance policy"--the option to exchange the put (which the borrower owns) for the call (which the borrower needs but does not own)--will bail the borrower out. As a result, the pricing mechanism with the explicit use of the exchange option will reflect and adjust for this contingent usage. The "double-counting" problem is thereby avoided.

## **V. GRAPHICAL EXPOSITION**

Since the value of a mortgage and the action of the borrower depend on the possible future course of mortgage interest rates and house values, it is useful to illustrate the state space of such paths and to identify the regions that correspond to the borrower's mutually exclusive optimal actions, which include prepayment, default, or continuation. Because a picture is allegedly worth a thousand words, we have included a set of graphs showing how state variables affect the borrower's exercise strategy. For the graphical example-presentation, we will adapt the simulation approach used in KKME (1986) with assumed similar key parameters. The valuation equation and the values of the parameters and variables are summarized in Appendix II.

### **A. VALUE SENSITIVITY OF THE OPTION-FREE, THE CALLABLE, AND THE PUTABLE MORTGAGE TO INTEREST RATES AND HOUSE PRICES**

Figure 1 constructs a three-dimensional graph for the values of a corresponding option-free (i.e., default-free and non-prepayable) mortgage under different interest rates and house prices. Since the

units for both axes are nonlinear, the graph of the slopes for the value changes between the different points in the  $(H,r)$  plane has been distorted. However, it provides a complete graph of the mortgage values under all possible combinations of interest rate and house price. The option-free mortgage can be viewed as a continuous-payment annuity that depends only on its coupon rate, time to maturity, and the pattern of market interest rates.

As observed by Hendershott and Van Order (1987), the values derived under different interest rates have an expected-present-value interpretation. The (undistorted version of the) value curve should be convex to the origin, and the house value plays no role in this case. In other words, for a constant interest rate, the value of the option-free mortgage is the same for all possible house prices.

Figure 2 constructs a three-dimensional graph for the values of a callable mortgage under different interest rates and house prices. Figure 3 shows the snapshot of borrower's optimal action in the state space  $(H,r)$  after one chronological period (i.e.,  $t=1$ ; a time-step in the computer program after the mortgage origination date ( $t=0$ )). In other words, the values calculated in Figure 2 are the expected values of the mortgage at  $t=0$  assuming the borrower's optimal actions at  $t \geq 1$ . A value of 0.75 in the state space indicates the prepayment region, while a zero value indicates the continuation region. For the 0.75 region, each of the value contours for the interest rate axis are "negative convex."<sup>16</sup>

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<sup>16</sup> The prices reflect the possible optimal call exercise even when the interest rate is not close to the optimal call rate. The optimal call rate is defined as the interest rate at which exercising the call option would minimize the borrower's liability.

The value curve for  $H=\$1.0000$  indicates the value of the callable mortgage when the house is worth one dollar. Since homogeneity allows us to interpret a change in the loan-to-value-ratio (LTVR) as a change in the house price for a given loan, the house price of  $\$1.0$  is equivalent to an LTVR of 0.85. Since in this case we rule out the default possibility, all the value curves along the interest rate dimension are identical in Figure 2. In other words, house prices have no effect on the value of a default-free, prepayable mortgage.

Figure 4 constructs a three-dimensional graph for the values of a putable mortgage for different interest rates and house prices. Figure 5 shows the snapshot of borrower's optimal action in the state space  $(H,r)$  after one chronological period (i.e.,  $t=1$ ). A value of 0.5 in the state space indicates the default region, while a zero value indicates the continuation region.

The value curves associated with low house prices along the interest rate axis in Figure 4 also show the feature of "negative convexity" even though the mortgage has no embedded prepayment option. The default put now acts like a prepayment call. For example, the mortgage value for a house value of  $\$1$  is capped by  $\$1$ ; while for the house value of  $\$0.6$  it is capped by  $\$0.6$ . Once the interest rate reaches a certain level, it will be optimal to exchange relief from the mortgage liability for the mortgaged house. In other words, the borrower will use the willful default strategy to extinguish (prepay) the mortgage. This portion of the value contours reflects the prices of the putable mortgage anticipating the put to act as the surrogate of the call.

Figure 6 combines the decision state spaces in Figures 3 and 5. The area indicated by 1.25 represents the region where the individual



put in a puttable mortgage and the individual call in a callable mortgage could substitute for each other's role. In other words, when only the prepayment call is available, the borrower will exercise the call in this area to minimize the liability. If only the default put is available to the borrower, the borrower will use the willful default strategy to "prepay" the mortgage. Although both options can achieve the same "prepayment" effect, the exercise values are different because of the different exercise prices of these two options even if no social or reputational cost difference between default and prepayment is assumed. In the puttable mortgage case, the exercise price is  $H$ , whereas in the callable mortgage case, the exercise price is the book value of the loan.

#### **B. VALUE SENSITIVITY OF THE PUTTABLE AND CALLABLE MORTGAGE TO INTEREST RATES AND HOUSE PRICES**

Figure 7 constructs a three-dimensional graph for the values of a puttable and callable ("standard") FRM under different interest rates and house prices. All the value curves along the interest rate dimension are "negative convex." Figure 8 shows the snapshot of the borrower's optimal action in the state space  $(H,r)$  after one chronological time-step ( $t=1$ ) after the mortgage origination date ( $t=0$ ;  $yr=15$ ). "Yr" in the figures indicates the remaining life of the mortgage. A value of 0.5 in the state space indicates the default region. A value of 0.75 indicates the prepayment region, while a zero value indicates the continuation region.

Figures 9 to 13 illustrate the same snapshot for every three year interval until the maturity date. We can easily observe that the regions are changing over time. Since the mortgage is amortized at an accelerated speed over time, as time passes, lower house prices will be

required to trigger the use of the default option, thereby shrinking the default regions over time. On the other hand, the prepayment region is expanding over time, since it becomes more feasible to use the prepayment call to replace the default put as a result of mortgage amortization.

The expansion of the prepayment region is not only growing toward the default region but also toward the continuation region over time. As the value of the default put decreases, the exercise value of the prepayment call tends to dominate the holding value of the put. When the mortgage is young (and associated with high LTVR), it is easier for the holding value of the default option to dominate the exercise value of the prepayment call, especially in the area where the three regions meet.

However, for the very high house prices (relative to the par value of the FRM), the exercise value of the prepayment call tends to dominate the holding value of the joint option over time. When it comes to the maturity date before the debt service payment is made, the prepayment region disappears. In other words, the call option has no value. On the other hand, the default region is reduced to the area where the house values are less than the last payment. This observation indicates the importance of taking into account the default put value in the mortgage pricing calculation when the LTVR is large. When the loan to value ratio is not large, the value of a callable mortgage could be a good approximation for the value of a standard mortgage. As a result, in pricing a seasoned mortgage pool, ignoring the existence of the default put is less likely to distort the pricing of the pool. However, for a young mortgage pool (associated with relatively high LTVR), ignoring the existence of the default put is likely to engender a

significant overestimate of the pool value.

## VI. CONCLUSION

This paper examines the pricing of inseparable and synchronized contingent claims contained implicitly in the FRM. Since both the cash flow emanating from the mortgage and the value of the collateral (i.e., house) are risky, we use the generalized Black-Scholes (1973) model as extended by Margrabe (1978) to analyze the options of exchanging one risky asset for another. Moreover, the periodic payment pattern that generates "jumps" in the exercise prices of the call (the book value of the mortgage) and the market value of the underlying mortgage asset suggests extending our model along the lines developed in Geske's (1979) work with compound options. By combining these approaches, we are able to prove that the American joint option (i.e., the co-existence of the default put and the prepayment call) embedded in the FRM contract can be viewed as a portfolio of options consisting of a European default put and an American option for exchanging the European default put for the American prepayment call. The "traditional" view of the borrower purchasing a call and a put from the lender is conceptually misleading, since the borrower never actually owns a "complete" call option.

Without loss of generality, we are able to use the exercise and the holding values of the constituent options to construct analytically a series of contour maps that demonstrate the interdependency and the dynamic interaction between the two options. This approach has the major advantage of being intuitive and easy to understand. Our approach not only lays out all possible scenarios, but also analyzes the interaction between the embedded put and call options.

## APPENDIX I

This appendix sketches out a proof of Proposition III.

PROPOSITION III.<sup>17</sup> The value of an American compound joint put/call option of an FRM,  $J_t$  (where  $0 \leq t \leq T$ ), is the sum of the values of:

- (a) an American compound exchange option,  ${}^{2m}X_t^{C,D}$ ,
- (b) plus a simple European put option,  $D_t$ .

Proof: Consider two portfolios.

Portfolio A represents

- (a) an American compound exchange option,  ${}^{2m}X_t^{C,D}$ , and
- (b) a simple European put option,  $D_t$ .

Portfolio B consists of

- (c) the purchase of an American compound joint option  $J_t$ .

To prove that both portfolios generate identical cash flows in any state of nature at any point of time, we examine the relationship between the boundary conditions and terminal values for the two portfolios through time and demonstrate that identical results will occur if optimal exercise strategies are adopted. Since both portfolios are compound options, we also investigate the pattern of cash flows generated by the consecutive option portfolios purchased between the end of that period and the maturity date of the FRM (as a result of the optimal exercising strategy of both portfolio A and B). To show that portfolio A can duplicate all the cash flows generated by portfolio B for all possible states, do not exercise any option in portfolio A until the holder of portfolio B exercises either the constituent call or the constituent put

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<sup>17</sup> For sake of concreteness, we use one month as the maturity for all the four options,  ${}^{2m}X_t^{C,D}$ ,  $C_t$ ,  $D_t$  and  $J_t$ . However, the analysis applies to any type of FRM payment schedule.

following an optimal exercise strategy. In the following Tables, B's actions represent all of the optimal strategies for B.

On the maturity date T before the payment is made, the cash receipts and net position are as follows.

(I)  
For  $C_T^{E-} = C_T^{H-} = D_T^{E-} = D_T^{H-} = 0$

Portfolio positions are

From (a) Expired

(b) Expired

---

Total A: Make the payment due

From (c) Expired

---

Total B: Make the payment due

(II)  
For  $D_T^{E-} > D_T^{H-} = C_T^{E-} = C_T^{H-} = 0$

Portfolio positions are

From (a) Expired

(b) Exercised

---

Total A:  $kB_0 - H_T$  in cash

From (c) Exercised (with put)

---

Total B:  $kB_0 - H_T$  in cash

When the payment is made, the borrower is free from the mortgage liability and owns the mortgaged property outright. Both portfolios clearly generate identical cash flows under all the possible terminal states, (I) and (II).

At any point in time t between T-1 and T, the cash receipts and net position are as follows.

(III) For  $J_t^H \geq D_t^E$  ( $\geq C_t$ )

Portfolio positions are

From (a) Open

(b) Open

---

Total A: Open

From (c) Open

---

Total B: Open

Unlike prepayment, default is not a rational action between scheduled

payment-dates.<sup>18</sup> In other words,  $D_t^H$  dominates  $D_t^E$ . As a consequence, keeping all the options alive is an optimal strategy for B.

(IV) For  $C_t = C_t^E \geq D_t$  and  $C_t^E \geq J_t^H$  (V) For  $C_t = C_t^H \geq D_t$  (and  $C_t^E < J_t^H$ )

Cash receipts are

Portfolio positions are

From (a) Exercised

From (a) Open

(b)  $D_t$  is exchanged for  $C_t$ 's

(b) Open

immediate exercise

---

Total A:  $M_t - B_t$  in cash

---

Total A: Open

From (c) Exercised (with call)

From (c) Open

---

Total B:  $M_t - B_t$  in cash

---

Total B: Open

Table (IV) and (V) show that, at any time before the final payment date, the holder of portfolio B will exercise the joint option only when the constituent call dominates the constituent put and the exercise value of the constituent call is greater than or equal to the holding value of the joint option. The optimal time for B to exercise is when  $C_t = C_t^E \geq D_t$  and  $C_t^E \geq J_t^H$  as in condition (IV). This is the moment that  $C_t^E$  dominates  $C_t^H$ ,  $D_t^E$ ,  $D_t^H$  and  $J_t^H$ . When the constituent call is exercised, by definition the joint option is terminated. On the other hand, if condition (V) takes place, it will be optimal for B to maintain status quo.

Since portfolio A is required to mimic what portfolio B does, when (and only when) the constituent call in portfolio B is exercised, the

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<sup>18</sup> The put will never be exercised pre-maturely since it is, in effect, a European put due to the "free housing services" argument. Moreover, the relationship  $D_t^H \geq D_t^{E-} > D_t^{E+}$  always holds not only for  $T-1 < t < T$  but also for all  $(t-1)^* < t < t^*$  where  $t^*$  and  $(t-1)^*$  are any payment date and its previous payment date, respectively. Also, the value of  $D_t^H$  is captured by  $J_t^H$  here. That is,  $J_t^H \geq D_t^H$ .

exchange option in portfolio A will be exercised. As a result, both constituent call and constituent put in portfolio A are terminated simultaneously. When following portfolio B's optimal action, the holder of portfolio A is able to duplicate portfolio B's results in (III) to (V). Both portfolios in cases (III) through (V) generate the same cash flows. The same reasoning can be generalized to the boundary condition analysis at any point in time,  $t$ , between any payment date  $t-1^*$  and  $t^*$ , where  $1 \leq t^* \leq T$ .

On the payment date  $T-1$  before the payment is made, the cash receipts and net position are as follows:

(VI)  
 For  $C_{T-1}^{E-} > D_{T-1}^{E-}$  &  $C_{T-1}^{E-} \geq J_{T-1}^{H-}$   
 Cash receipts are  
 From (a) Exercised  
 (b)  $D_{T-1}$  is exchanged for  
 $C_{T-1}$ 's immediate exercise

---

Total A:  $M_{T-1}^- - B_{T-1}^-$  in cash  
 From (c) Exercised (with call)

---

Total B:  $M_{T-1}^- - B_{T-1}^-$  in cash<sup>20</sup>

(VII)  
 For  $D_{T-1}^{E-} \geq C_{T-1}^{E-}$  &  $D_{T-1}^{E-} \geq J_{T-1}^{H-}$   
 Cash receipts are  
 From (a) Terminated<sup>19</sup>  
 (b) Exercised

---

Total A:  $M_{T-1}^- - H_{T-1}$  in cash  
 From (c) Exercised (with put)

---

Total B:  $M_{T-1}^- - H_{T-1}$  in cash

---

<sup>19</sup> Recall that the "conditional" maturity feature specifies that if  $D_t$  is exercised, option  ${}^{2m}X_t^{C,D}$  will be terminated immediately.

<sup>20</sup> Since  $M_{T-1}^- - B_{T-1}^- = (M_{T-1}^- - kB_0) - (B_{T-1}^- - kB_0) = M_{T-1}^+ - B_{T-1}^+$ , this table is also applicable for  $C_{T-1}^{E+} > D_{T-1}^{E+}$  &  $C_{T-1}^{E+} \geq J_{T-1}^{H+}$ .

(VIII) For  $J_{T-1}^{H-} \geq C_{T-1}^{E-}$  &  $J_{T-1}^{H-} \geq D_{T-1}^{E-}$

Portfolio positions are

From (a) Exercised to purchase next portfolio,  ${}^{2m}X_{T-1,T}^{C,D}$  and  $D_{T-1,T}$

(b) Expired

---

Total A: Buy next portfolio,  ${}^{2m}X_{T-1,T}^{C,D}$  and  $D_{T-1,T}$ . (Make the payment due)

From (c) Exercised to purchase next  $J_t$

---

Total B: Buy next  $J_t$  (i.e.,  $J_{T-1,T}$ ) (Make the payment due)

All the three candidate options,  $C_{T-1}^{E-}$ ,  $D_{T-1}^{E-}$  and  $J_{T-1}^{H-}$ , are explicitly evaluated at the same time. When the exercise value of a constituent option dominates the rest of the "candidates," the mortgage will be terminated either by default or by prepayment.

When the exercise value of the constituent put dominates the exercise value of the constituent call and dominates or is equal to the net present value of all future joint options (represented by  $J_{T-1}^{H-}$ ), the holder of portfolio B will exercise the constituent put and ignore the payment due. The FRM is defaulted. In this case, to mimic B's optimal action, A's holder will exercise the put option and let the exchange option expire.

In contrast, when the exercise value of the constituent call dominates the exercise value of the constituent put and dominates or is equal to the net present value of all future joint options (represented by  $J_{T-1}^{H-}$ ), portfolio B's holder will exercise the constituent call immediately so as to "receive" the capital gains of  $M_{T-1} - B_{T-1}$ . Thus, the FRM is prepaid. Again, to mimic B's optimal action, portfolio A's holder will exercise the exchange option by using the holding put to exchange for the call and will exercise this call immediately to



"receive" the capital gains of  $M_{T-1}^- - B_{T-1}^-$ . As a result, all three options are terminated.

When none of the above scenarios takes place, it is optimal for the joint option holder, B, to continue to make the payment to keep the FRM alive (i.e., to purchase the next compound joint option,  $J_t$ , consisting of similar constituent options with new strike prices  $M_{T-1}^+$  and  $B_{T-1}^+$ ). To mimic this action, A's holder will buy the next portfolio  $S_{T-1,T}$  (i.e.,  ${}^{2m}X_{T-1,T}^{C,D}$  and  $D_{T-1,T}$ ). Tables (VI) through (VIII) above clearly show that each portfolio will receive the same amount of cash as its counterpart does in every possible state. Since the value of  $D_{T-1}^{H-}$  and  $C_{T-1}^{H-}$  are captured by  $J_{T-1}^{H-}$ , these actions are clearly optimal strategies for B.<sup>21</sup>

The same reasoning can be generalized to the boundary condition analysis to any payment date  $t^*$  where  $1 \leq t^* \leq T-1$ .<sup>22</sup> Notice that when calculating the present value of all future options, the periodic payments have been taken into account; e.g.,  $B_{t^*}^-$  was adjusted to  $B_{t^*}^+$ . Moreover, even if portfolio B does not follow an optimal exercise strategy, portfolio A can still follow B's action and duplicate the same cash flows.

Proceeding inductively by working backwards through this dynamic-programming technique, we have shown that portfolio A's holder could make the same payment on any payment date for buying a consecutive set of options to be used in the next period and thereby generate cash flows identical to those produced by portfolio B over all possible lives and

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<sup>21</sup> That is,  $J_{T-1}^{H-} \geq D_{T-1}^{H-}$  and  $J_{T-1}^{H-} \geq C_{T-1}^{H-}$ .

<sup>22</sup> The only adjustment needed is the substitution of  $t^*$  for all of the subscripts  $T-1$ . The rest of the proof is identical.

states of the FRM. On the FRM origination date  $t = 0$ , both portfolios have the same cash inflow  $B_0$  with certainty while portfolio A's holder receives  ${}^{2m}X_{0,1}^{C,D}$  and  $D_{0,1}$  and portfolio B's holder receives  $J_{0,1}$ . Since portfolio A can duplicate portfolio B's results not only on any payment date but also at any time between payment dates, the value of portfolio A must be no less than that of portfolio B to satisfy arbitrage condition; that is,  ${}^{2m}X_{0,1}^{C,D} + D_{0,1} \geq J_{0,1}$ .

To prove that  ${}^{2m}X_{0,1}^{C,D} + D_{0,1} \leq J_{0,1}$ , the holder of portfolio B does not exercise any option until the holder of portfolio A exercises either of the constituent options following an optimal exercise strategy. At any time,  $t$ , at the non-payment date, the cash receipts and net position are as follows.

(IX) When it is optimal for A's holder to exercise  $C_t$ :<sup>23</sup>

---

Portfolio positions are

From (a) Exercised

(b)  $D_t$  is exchanged for

$C_t$ 's immediate exercise

---

Total A:  $C_t^E$  (i.e.,  $M_t - B_t$  in cash)

From (c) Exercise  $C_t$

---

Total B:  $C_t^E$  (i.e.,  $M_t - B_t$  in cash)

---

<sup>23</sup> Since  $D_t$  is a simple European put option, we need not consider it.

(X) When it is optimal for A's holder to keep the FRM alive:

Portfolio positions are

From (a) Open

(b) Open

Total A: Open

From (c) Open

Total B: Open

At any payment date  $t^*$  where  $1 \leq t^* \leq T-1$  before the payment is made, the cash receipts and net position are as follows (we will use  $T-1$  to illustrate here).

When it is advantageous for A's holder to exercise:

(XI)  $C_{T-1}$

(XII)  $D_{T-1}$

Cash receipts are

Cash receipts are

From (a) Exercised

From (a) Terminated

(b)  $D_{T-1}$  is exchanged for

(b) Exercised

$C_{T-1}$ 's immediate exercise

Total A:  $M_{T-1} - B_{T-1}$  in cash

Total A:  $M_{T-1} - H_{T-1}$  in cash

From (c) Exercised (with Call)

From (c) Exercised (with Put)

Total B:  $M_{T-1} - B_{T-1}$  in cash

Total B:  $M_{T-1} - H_{T-1}$  in cash

(XIII) When it is advantageous for A's holder to keep the FRM alive:

Portfolio positions are

From (a) Exercised to purchase next portfolio,  ${}^{2m}X_t^{C,D}$  and  $D_t$

(b) Expired

Total A: Buy next portfolio,  ${}^{2m}X_{t,T}^{C,D}$  and  $D_{t,T}$ . (Make the payment due)

From (c) Exercised to purchase next  $J_t$ .

---

Total B: Buy next  $J_t$  (i.e.,  $J_{t,T}$ ) (Make the payment due)

On the maturity date T before the payment is made, the cash receipts and net position are as follows.

(XIV) When it is advantageous for A's holder to exercise  $D_T$ :

Cash receipts are

From (a) Expired

(b) Exercised

---

Total A:  $M_T^- - H_T$  in cash

From (c) Exercised (with Put)

---

Total B:  $M_T^- - H_T$  in cash

(XV) When it is advantageous for A's holder not to exercise the put,  $D_T$ :

Portfolio positions are

From (a) Expired

(b) Expired

---

Total A: Make the payment due

From (c) Expired

---

Total B: Make the payment due

Following the same dynamic-programming procedure as before, we have demonstrated that portfolio B can duplicate all the optimal cash flows generated by portfolio A for all possible states over all possible lives of the FRM. That is, we have proved that  ${}^{2m}X_{0,1}^{C,D} + D_{0,1} \leq J_{0,1}$ . To satisfy both conditions,  ${}^{2m}X_{0,1}^{C,D} + D_{0,1} \geq J_{0,1}$  and  ${}^{2m}X_{0,1}^{C,D} + D_{0,1} \leq J_{0,1}$ , it must be that  ${}^{2m}X_{0,1}^{C,D} + D_{0,1} = J_{0,1}$ . Both portfolios must be identical

since each is able to act as the other's envelope under the circumstance that each follows its own optimal exercise strategy from the date the FRM is originated until the maturity date.

Moreover, the recursive nature (the compound form) of the series of the options also guarantees that at any time  $t$  the value of the joint option  $J_t$  must equal the value of the portfolio consisting of  ${}^{2m}X_t^{C,D}$  and  $D_t$ . That is,  ${}^{2m}X_t^{C,D} + D_t = J_t$  where  $0 \leq t \leq T$ . Since for any time,  $t$ , the series from the arbitrage portfolio  $\{ {}^{2m}X_{t,t^*}^{C,D}, D_{t,t^*}, {}^{2m}X_{t^*,t+1^*}^{C,D}, D_{t^*,t+1^*}, \dots, {}^{2m}X_{T-1,T}^{C,D}, D_{T-1,T} \}$  duplicate the boundary conditions of the series of joint options  $\{ J_{t,t^*}, J_{t^*,t+1^*}, \dots, J_{T-1,T} \}$ , the total value of the embedded constituent call and constituent put options is equal to the total value of the embedded put and exchange options. *Q.E.D.*

APPENDIX II

VALUES FOR PARAMETERS AND VARIABLES USED IN KKME 1986'S PAPER

$k = 0.25$	$m = 0.1$	$\phi = 0$	$r_0 = 0.1$
$\sigma_r = 0.15$	$b = 0.05$	$c = 0.124$	$T = 15$
$\sigma_H = 0.15$	$\rho = 0$	$H_0 = 1.0$	$B_0 = 0.85$
$r_{scale} = 0.1$	$pmicoverage = 0.25$		

where  $k$  is the speed of adjustment coefficient;  $m$  is the long term mean instantaneous riskless rate;  $b$  is the housing payout rate (i.e. housing service flow);  $(\mu - b)$  is the mean appreciation rate of the house value;  $\sigma_r$  is the instantaneous standard deviation of the risk-free interest rate;  $\sigma_H$  is the instantaneous standard deviation of the value of the mortgaged house;  $\rho$  is the instantaneous correlation coefficient between the increments to the standardized Wiener processes  $dz_r$  and  $dz_H$ ;  $\phi$  is the market price of interest rate risk;  $c$  is the coupon rate of the mortgage;  $C$  is the continuous rate of mortgage payment;  $r_0$ ,  $H_0$  and  $B_0$  are initial short term interest rate, mortgaged house value and loan amount respectively;  $T$  is the initial term to maturity;  $r_{scale}$  is the scale factor for nonlinear mapping in  $r$  dimension;  $pmicoverage$  is the PMI coverage ratio; and the standard second-order partial differential equation for valuation of the FRM,  $V(H,r,t)$ , takes the form:

$$\frac{1}{2}(\sigma_H H)^2 V_{HH} + \frac{1}{2}(\sigma_r \sqrt{r})^2 V_{rr} + \rho \sigma_r \sqrt{r} \sigma_H H V_{rH} + (k(m - r) + \phi r)V_r + (r - b)H V_H - rV + V_t + C = 0$$

if the interest rate dynamics can be expressed as:

$$dr = k(m - r)dt + \sigma_r \sqrt{r} dz_r$$

and the mortgaged house value dynamics can be given by:

$$dH = (\mu - b)Hdt + \sigma_H H dz_H$$

with

$$(dz_r)(dz_H) = \rho dt.$$

## REFERENCES

- Black, F. and M. Scholes "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, Vol.81, No.3, May-June 1973.
- Epperson, J.F., J.B. Kau, D.C. Keenan and W.J. Muller "Pricing Default Risk in Mortgages." *AREUEA Journal*, Fall 1985.
- Evnine, Jeremy "Three Essays in the Use of Options Pricing Theory." Ph.D. Thesis, University of California at Berkeley, 1983.
- Geske, Robert "The Valuation of Corporate Liabilities as Compound Options." *Journal of Financial and Quantitative Analysis*, November 1977.
- Geske, Robert "The Valuation of Compound Options." *Journal of Financial Economics*, Vol.7, No.1, March 1979.
- Hendershott, P.H. and R. Van Order "Pricing Mortgages: An Interpretation of the Models and Results." *Journal of Financial Services Research*, Vol.1, No.1, September 1987.
- Hull, John "Options, Futures, and Other Derivative Securities." Englewood Cliffs, NJ: Prentice-Hall, Inc. 1989.
- Kau, J., D. Keenan, W. Muller, and J. Epperson "Option Theory and Fixed Rate Mortgages." University of Georgia, Working Paper, 1986.
- Kau, J., D. Keenan, W. Muller, and J. Epperson "Pricing Commercial Mortgages and Their Mortgage-Backed Securities." *Journal of Real Estate Finance and Economics*, Vol.3, No.4, December, 1990.
- Margrabe, William "The Value of an Option to Exchange one Asset for Another." *The Journal of Finance*, Vol.33, No.1, March 1978.
- Merton, Robert C. "The Theory of Rational Option Pricing." *The Bell Journal of Economics and Management Science*, Vol.4, No.1, Spring 1973.
- Mulherin, J. Harold and Walter J. Muller, III "Volatile Interest Rates and the Divergence of Incentives in Mortgage Contracts." *Journal of Law, Economics and Organization*, Vol.3, No.1, Spring 1987.
- Roll, Richard "An Analytical Valuation Formula for Unprotected American Call Options on Stocks with Known Dividends." *Journal of Financial Economics*, Vol.5, August 1977.
- Ross, Stephen A. "A Simple Approach to the Valuation of Risky Streams." *The Journal of Business*, Vol.51, No.3, July 1978.
- Rubinstein, Mark "Derivative Assets Analysis." *Journal of Economic Perspectives*, Vol.1, No.2, Fall 1987.
- Schwartz, Eduardo S. and Walter N. Torous "Prepayment, Default and the Valuation of Mortgage Pass-Through Securities." *The Journal of Business*, Vol.65, No.2, April 1992.

# 2-STATE VRBL-BORROWER'S VALUE(RHO=+0.0)

OPTION FREE MORTGAGE

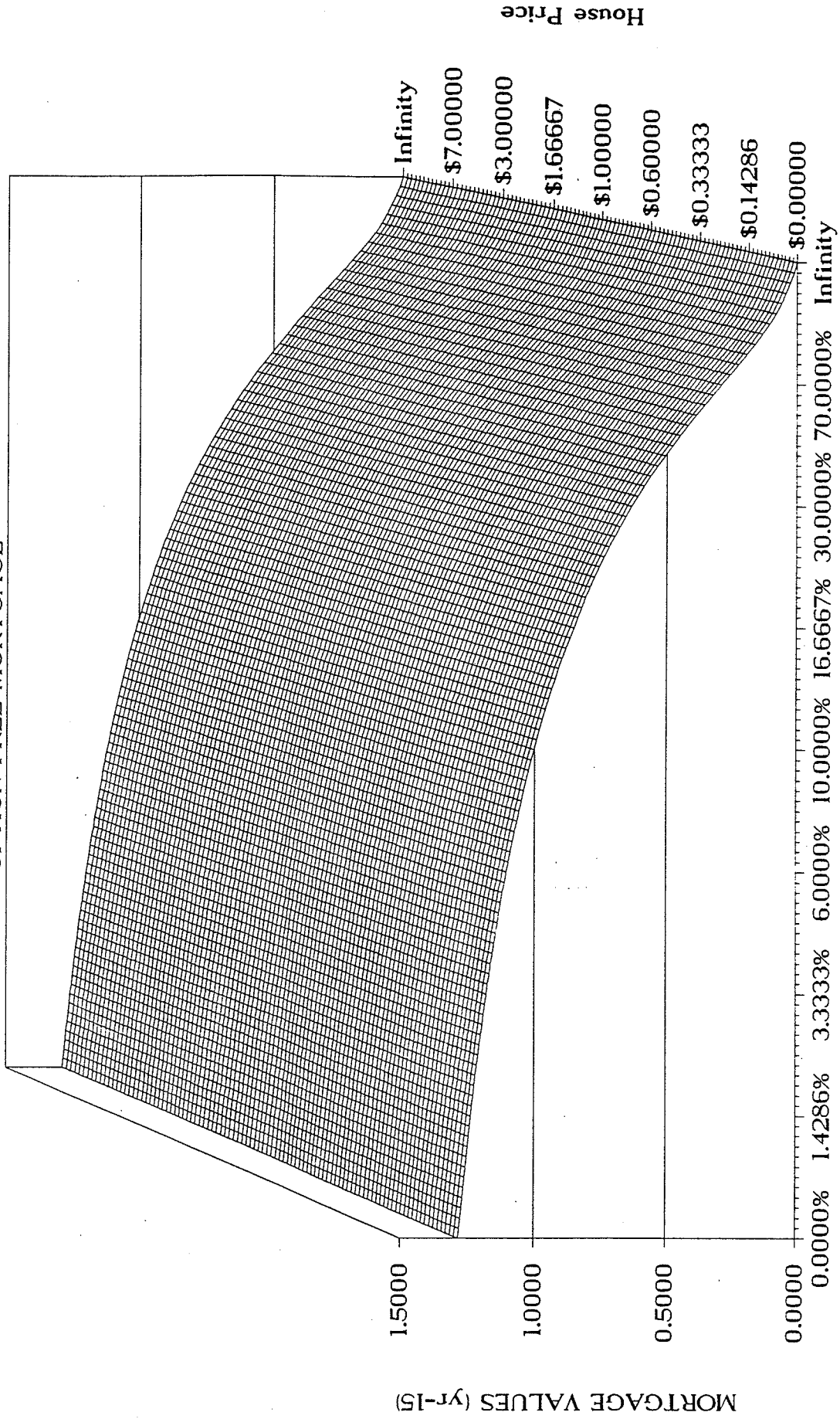


Figure 1 Interest Rate



# 2-STATE VRBL-BORROWER'S VALUE(RHO=+0.0)

INSURED MORTGAGE WITH PREPAYMENT OPTION

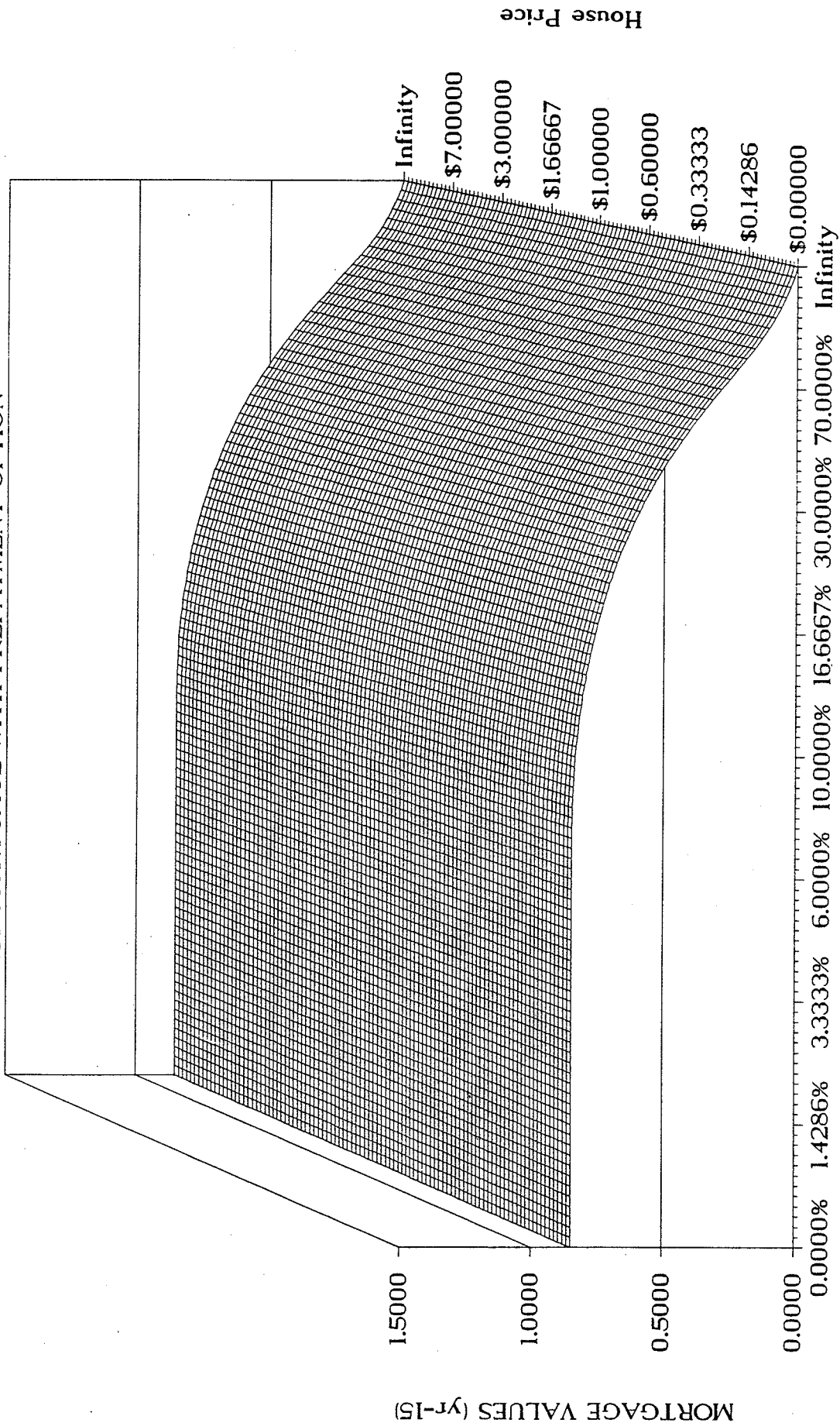


Figure 2 Interest Rate

# 2-STATE VRBL-BORROWER'S VALUE (RHO=+0.0)

INSURED MORTGAGE WITH PREPAYMENT OPTION

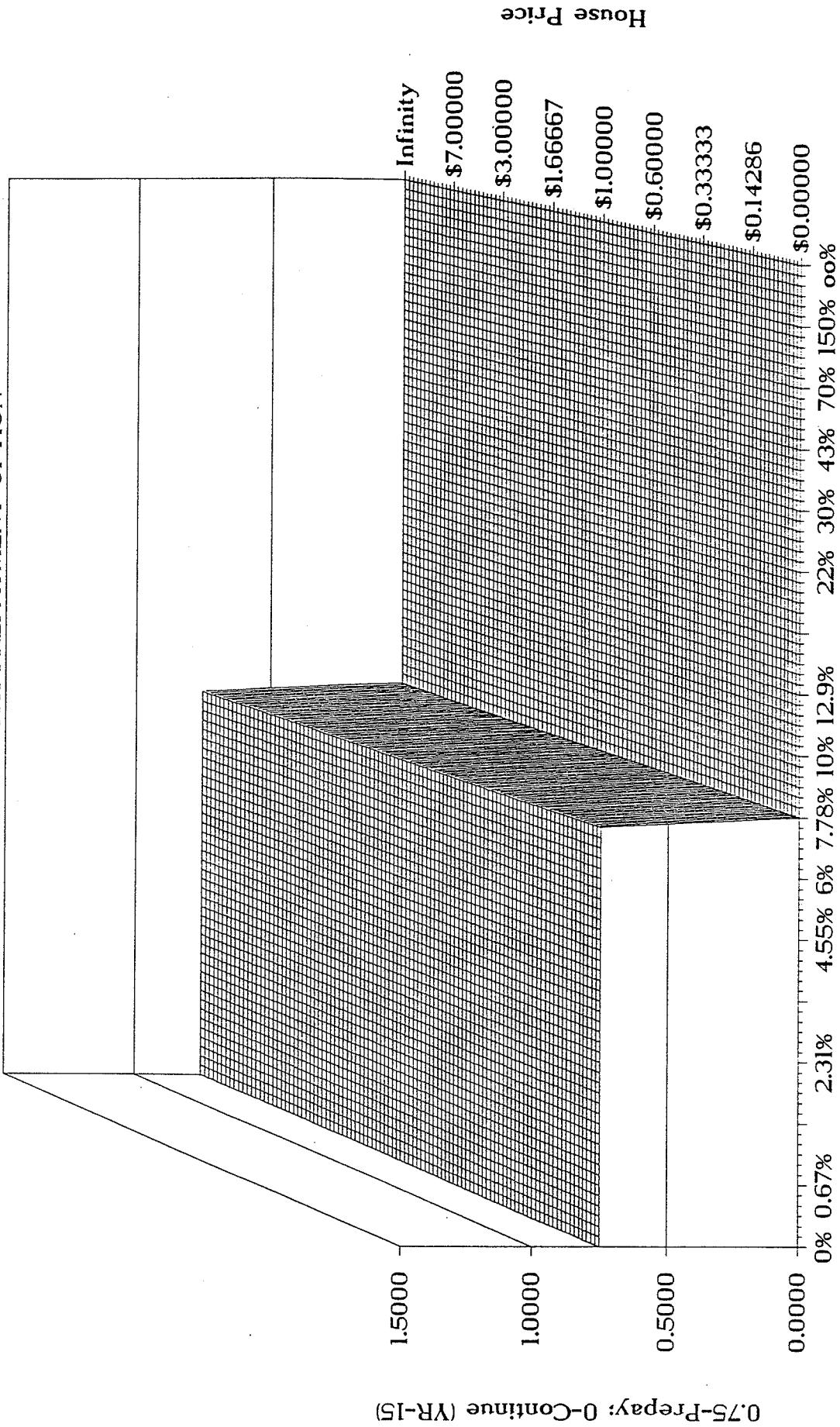


Figure 3

# 2-STATE VRBL-BORROWER'S VALUE(RHO=+0.0)

INSURED MORTGAGE WITH DEFAULT OPTION

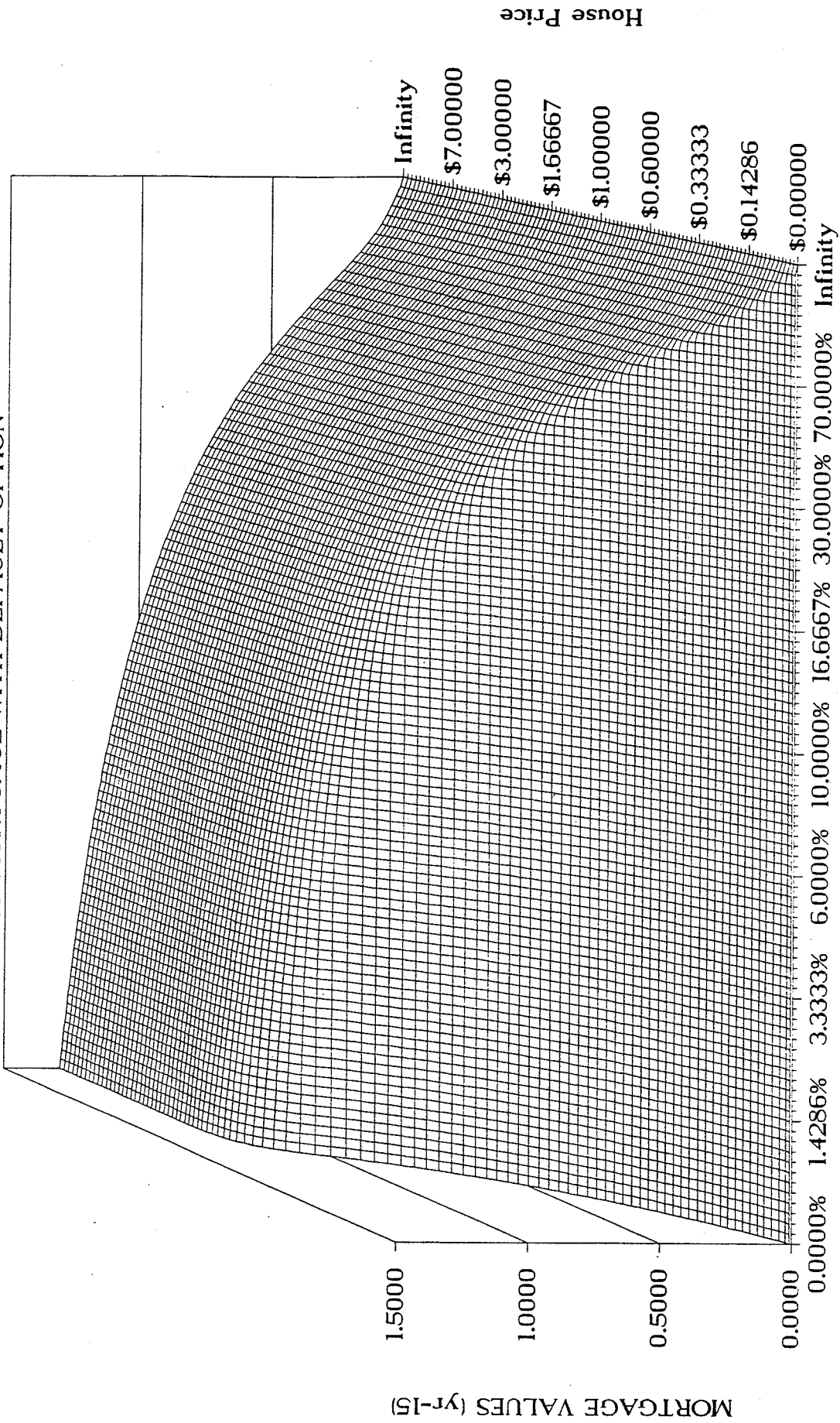


Figure 4

# 2-STATE VRBL-BORROWER'S VALUE(RHO=+0.0)

INSURED MORTGAGE WITH DEFAULT OPTION

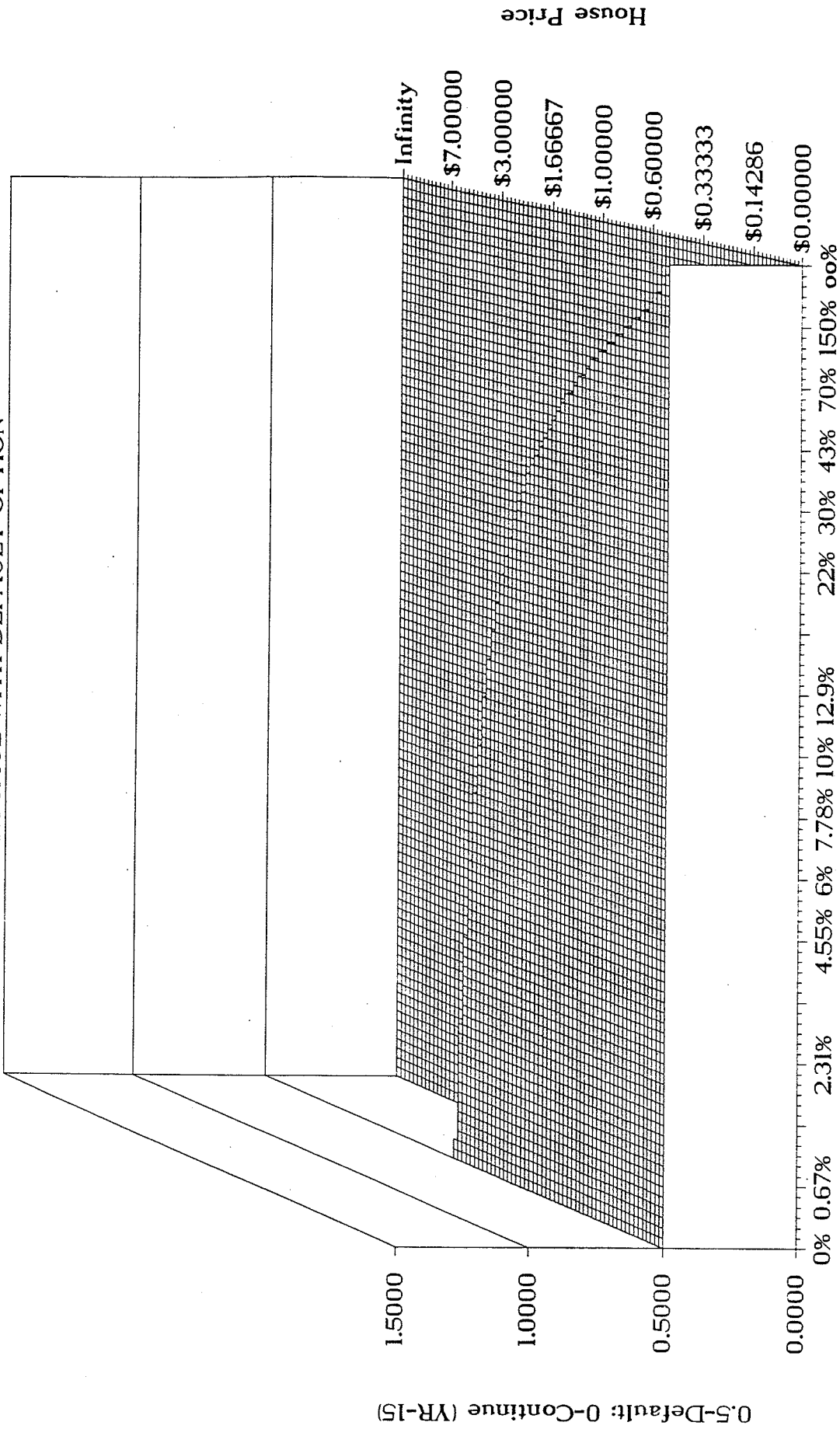


Figure 5 Interest Rate

# 2-STATE VRBL-BORROWER'S VALUE(RHO=0.0)

ADDING PUTABLE & CALLABLE MTGES(YR-15)

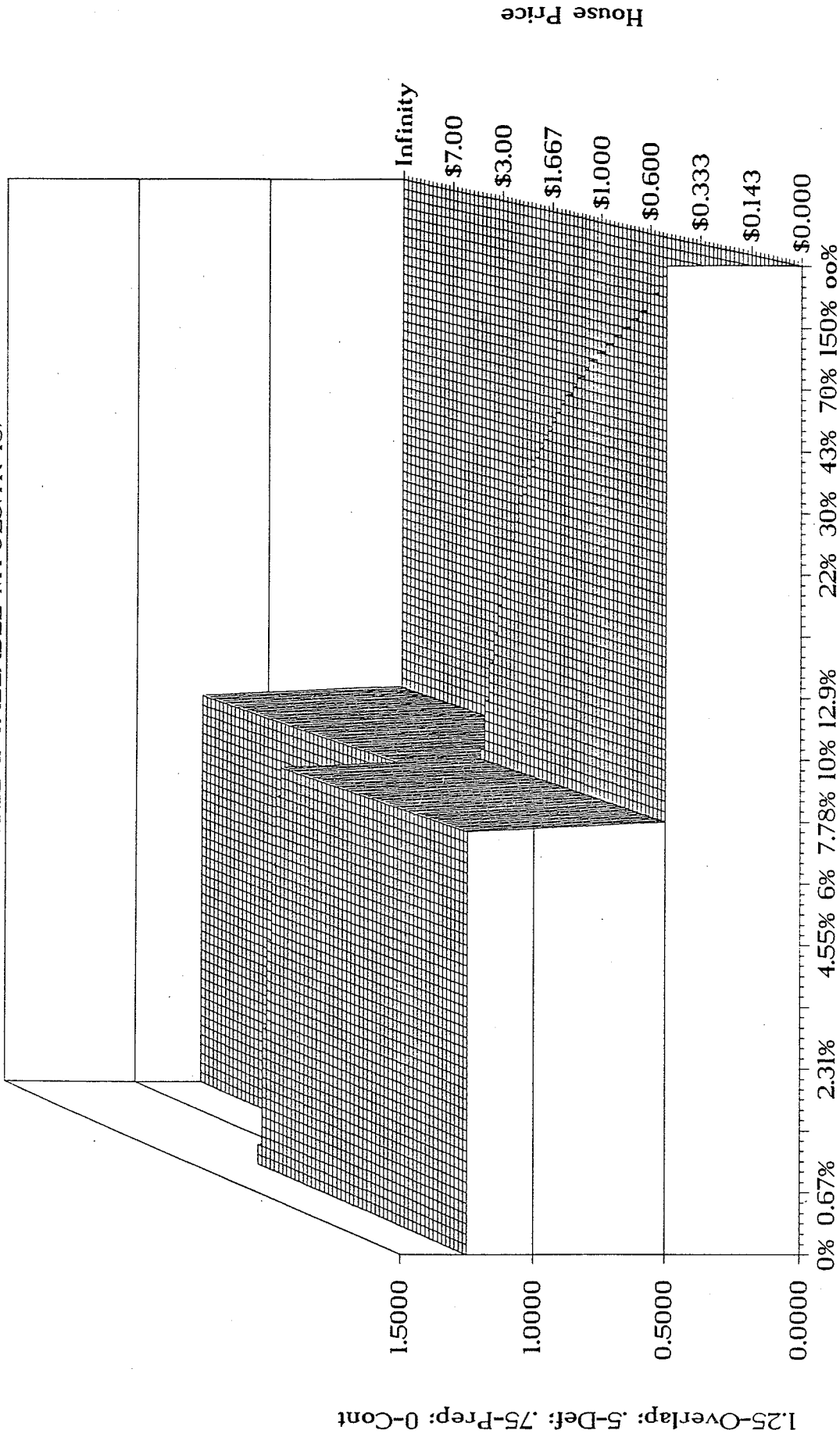


Figure 6 Interest Rate

# 2-STATE VRBL-BORROWER'S VALUE(RHO=+0.0)

WITH DEFAULT, PREPAYMENT OPTIONS & PMI

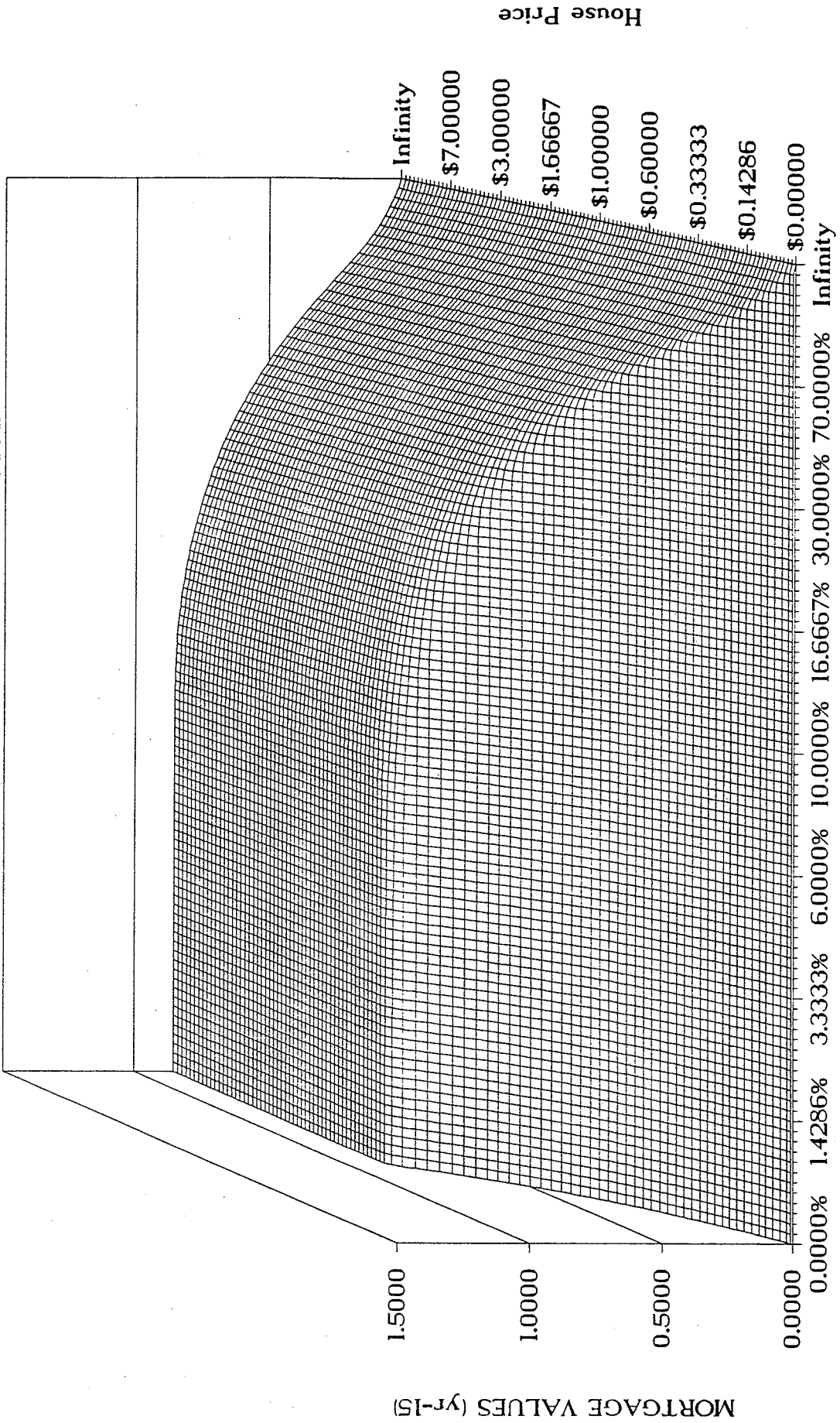


Figure 7 Interest Rate

# 2-STATE VRBL-BORROWER'S VALUE(RHO=+0.0)

WITH DEFAULT, PREPAYMENT OPTIONS & PMI

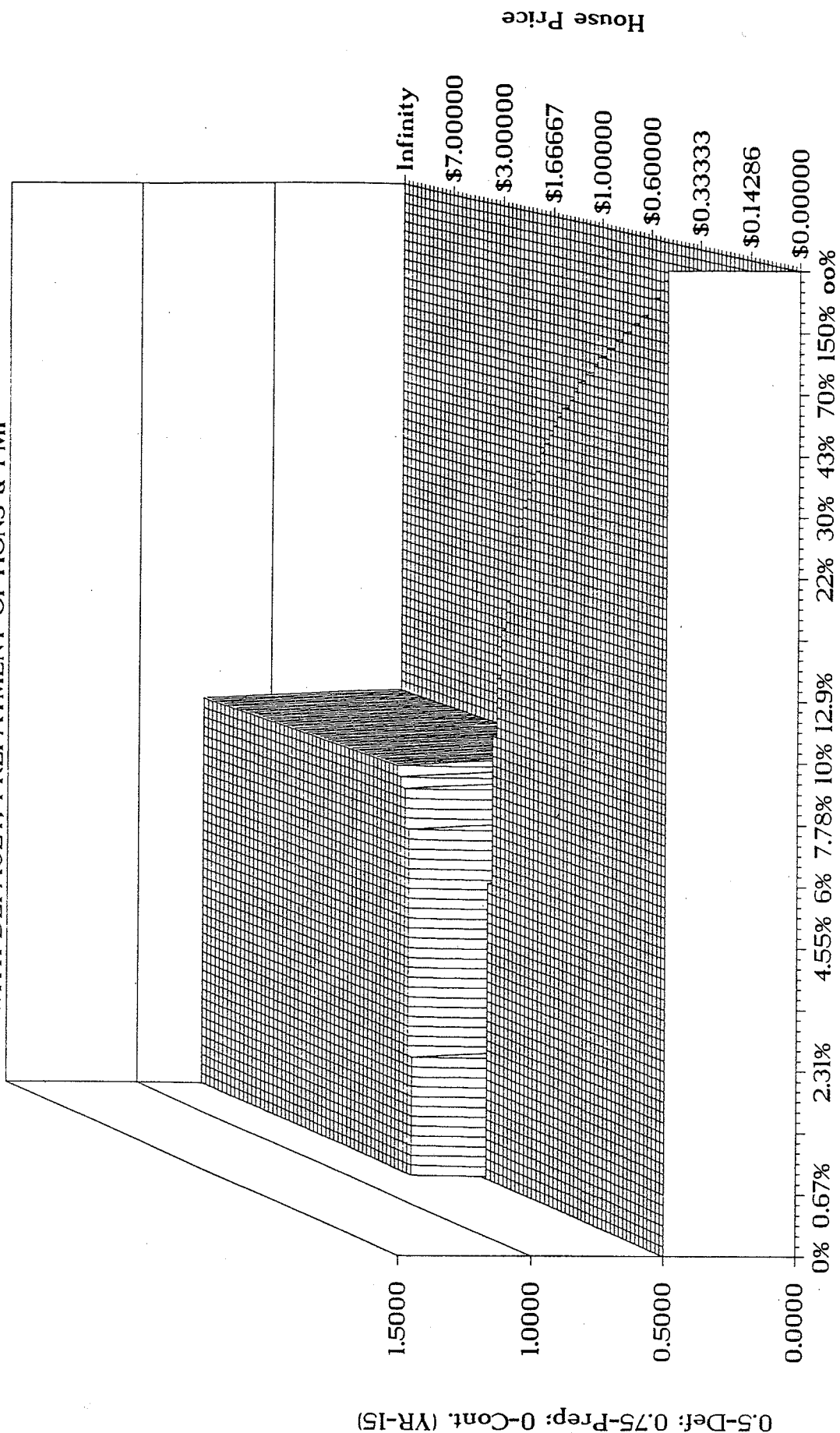


Figure 8 Interest Rate

# 2-STATE VRBL-BORROWER'S VALUE(RHO=+0.0)

WITH DEFAULT, PREPAYMENT OPTIONS & PMI

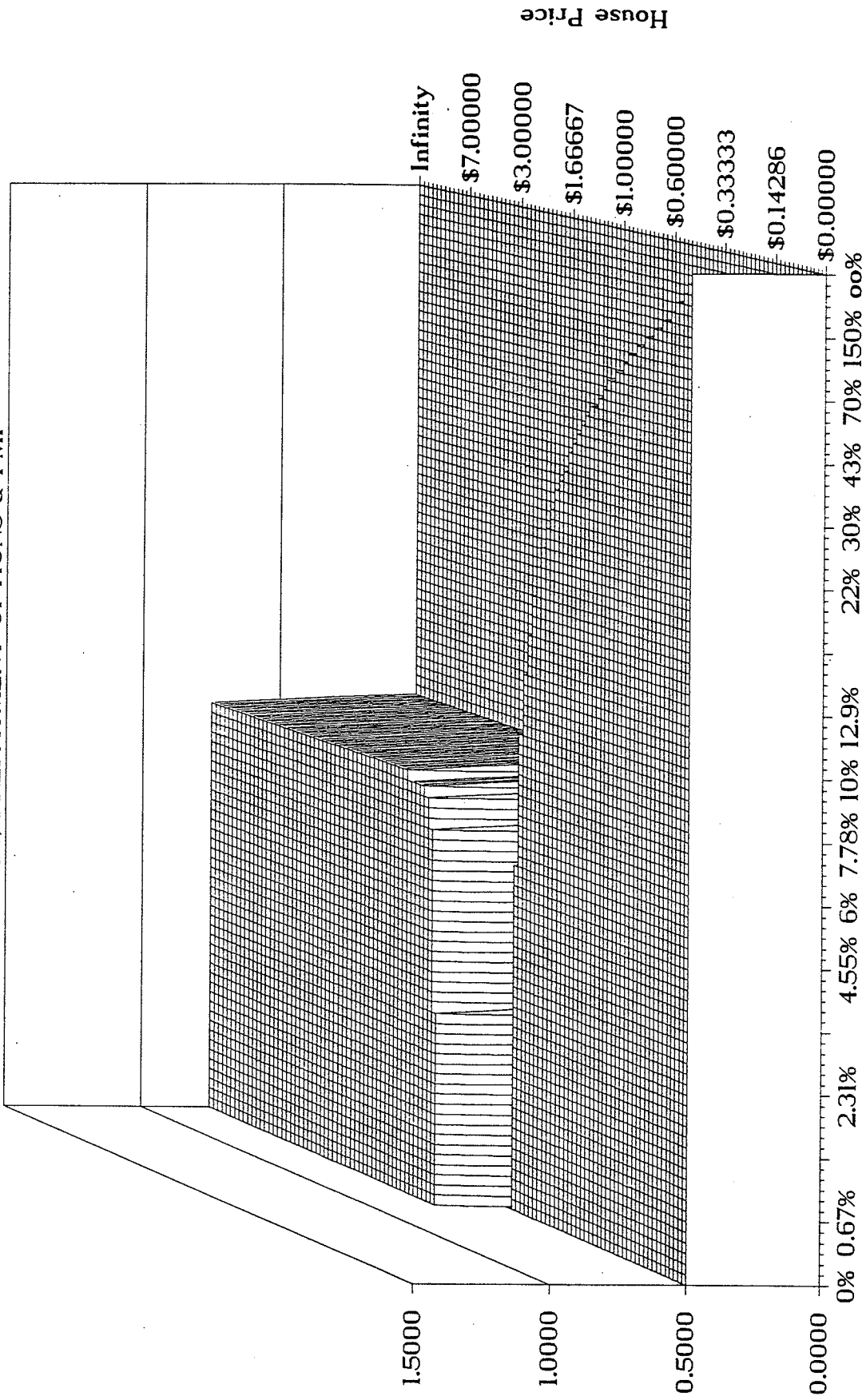


Figure 9 Interest Rate



# 2-STATE VRBL-BORROWER'S VALUE(RHO=+0.0)

WITH DEFAULT, PREPAYMENT OPTIONS & PMI

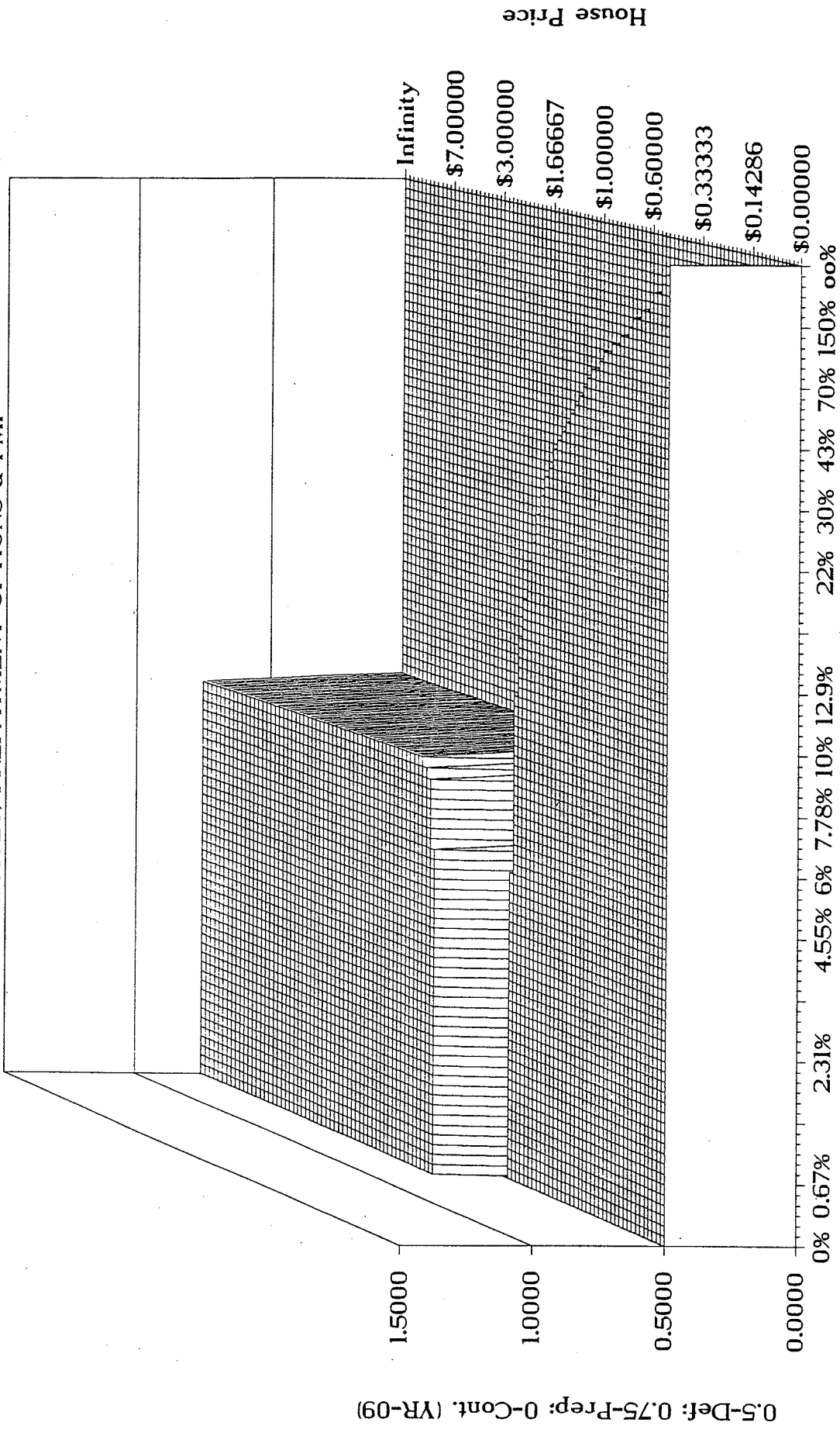


Figure 10

Interest Rate

# 2-STATE VRBL-BORROWER'S VALUE(RHO=+0.0)

WITH DEFAULT, PREPAYMENT OPTIONS & PMI

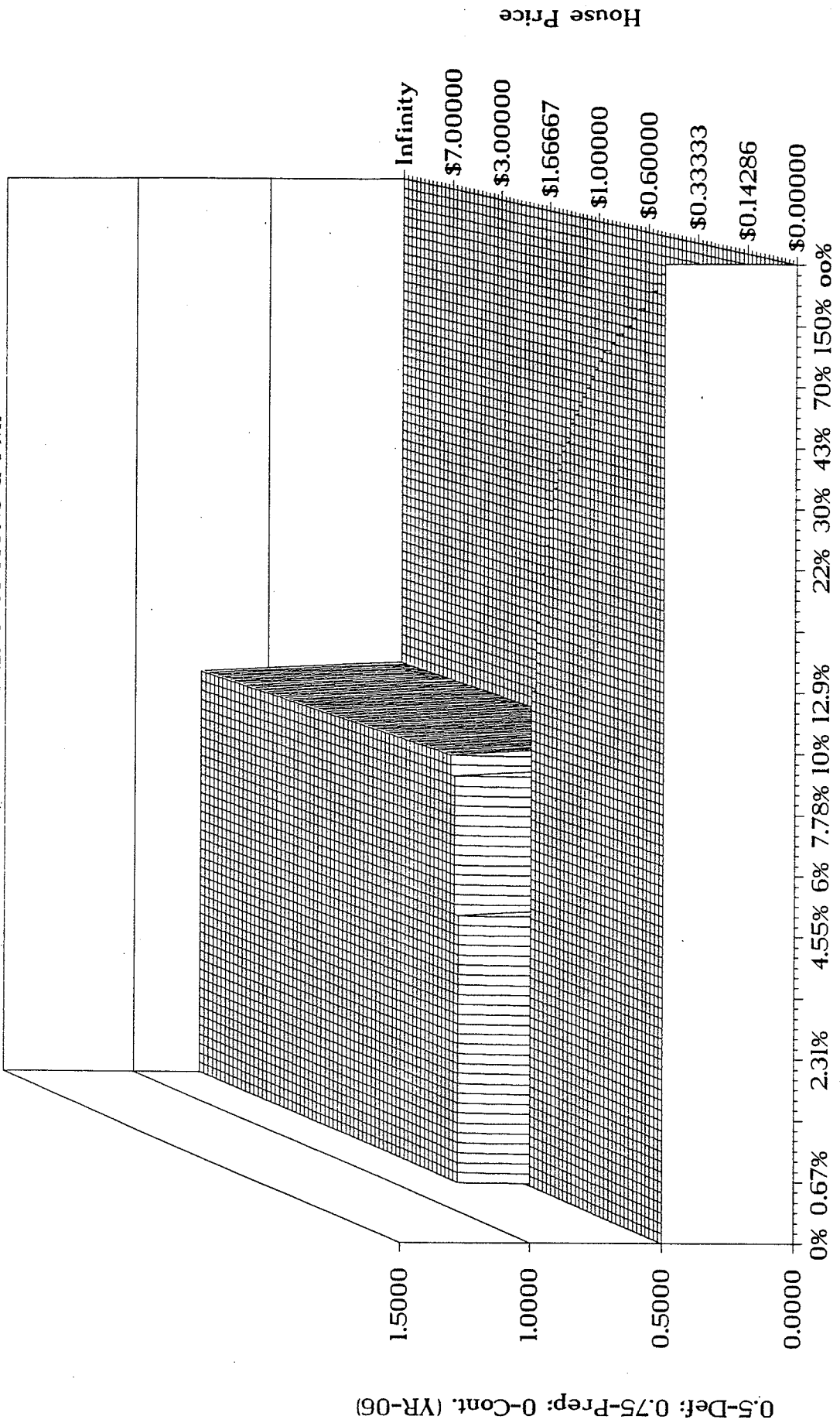


Figure 11 Interest Rate

# 2-STATE VRBL-BORROWER'S VALUE(RHO=+0.0)

WITH DEFAULT, PREPAYMENT OPTIONS & PMI

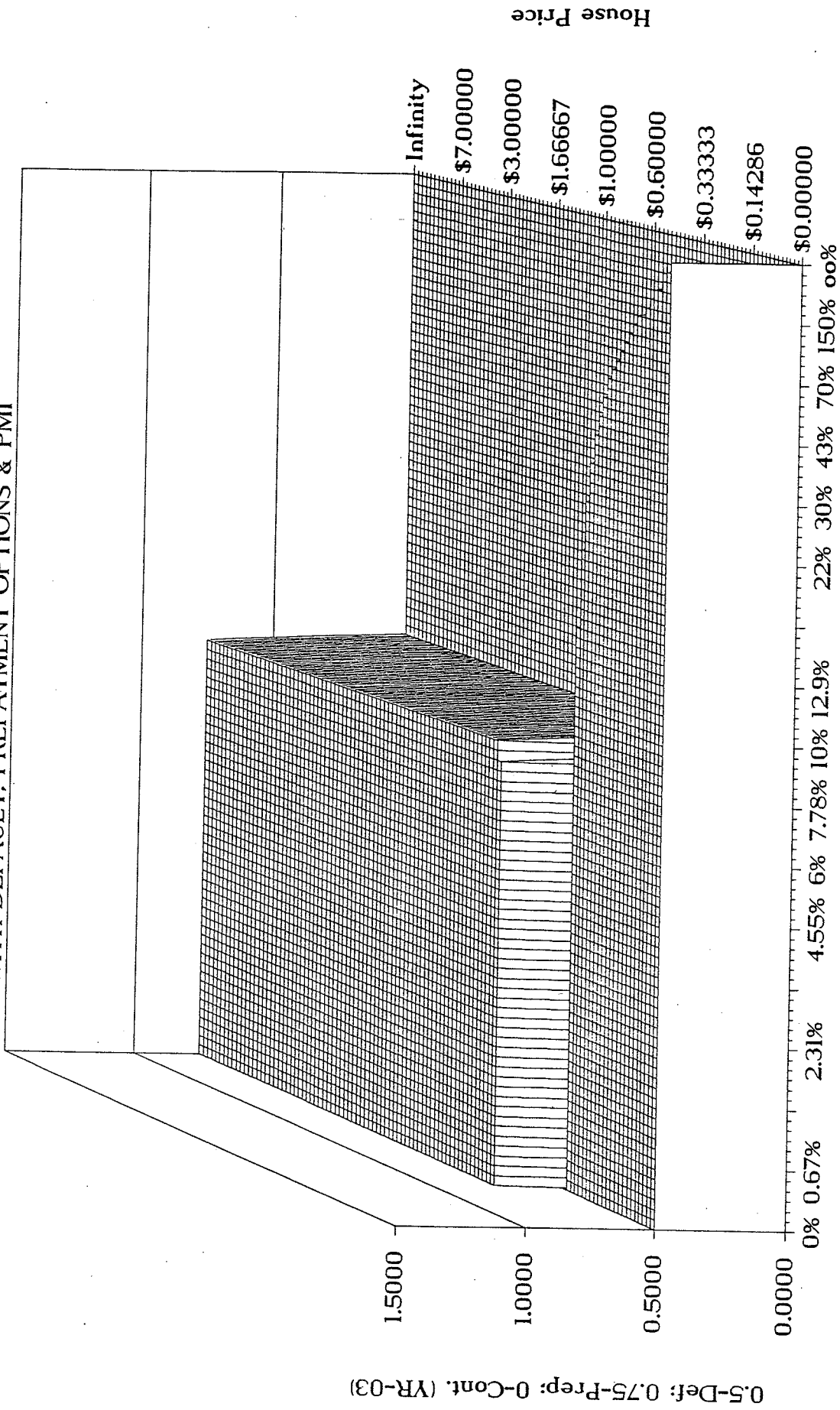


Figure 12

Interest Rate

# 2-STATE VRBL-BORROWER'S VALUE(RHO=+0.0)

WITH DEFAULT, PREPAYMENT OPTIONS & PMI

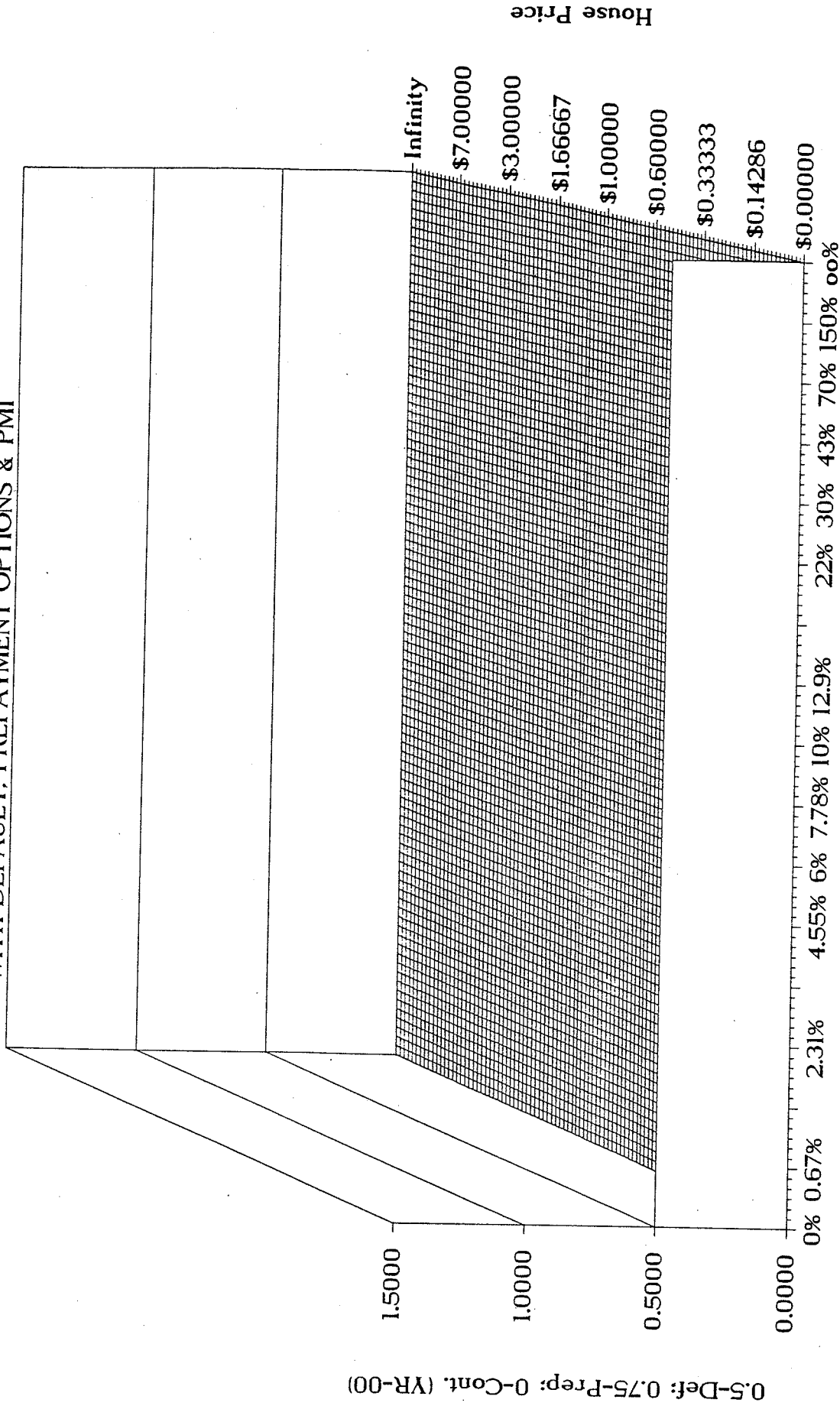


Figure 13

Interest Rate