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# Probabilistic Analysis for Scheduling with Conflicts

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**Abstract**

In this paper, we consider the scheduling of jobs that may be competing for mutually exclusive resources. We model the conflicts between

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jobs with a *conflict graph*, so that all concurrently running jobs must form an independent set in the graph. This model is natural and general enough to have applications in a variety of settings; however, we are motivated by the following two specific applications: traffic intersection control and session scheduling in high speed local area networks with spatial reuse. Our goal is to bound the maximum response time of any job in the system. It has been previously shown [13] that the best competitive ratio achievable by any online algorithm for the maximum response time on interval or bipartite graphs is $\Omega(n)$, where $n$ is the number of nodes in the conflict graph. As a result, we study scheduling with conflicts under probabilistic assumptions about the input. Each node $i$ has a value $p_i$ such that a job arrives at node $i$ in any given time unit with probability $p_i$. Arrivals at different nodes and during different time periods are independent. Under reasonable assumptions on the input sequence, we are able to obtain a bounded competitive ratio for an arbitrary conflict graph. In addition, if the conflict graph is a perfect graph, we give an algorithm whose competitive ratio converges to 1.

# 1 Introduction

In this paper, we consider scheduling jobs which are competing for limited resources. Jobs arrive in the system through time and require a certain set of resources to be completed. Any two jobs which require the same resource can not be executed simultaneously. We model the conflicts between jobs by a *conflict graph* where each node in the graph represents a type of job. Jobs of the same type have the same requirements. If two types of jobs demand a common resource, there is an edge between those nodes in the graph. Thus, at all times, the set of jobs currently being executed must belong to nodes which form an independent set in the graph. Note that if there are two jobs of the same type in the system, one must wait until the other is completed.

We were motivated by the following two specific applications:

**Traffic Intersection Control** ([4, 5, 6, 8, 9, 10, 11, 17, 19, 20, 21, 22, 24, 25, 26]). Today's traffic intersection controllers are based on thirty year old signal phasing strategies. Signal phasings are optimized offline with historical data, downloaded into the controller and triggered by the presence of vehicles. Even the state of the art in adaptive traffic signal control only extend the optimization to a few seconds before every phase change. However, one expected consequence of an effective advanced traveler information system (ATIS) [1, 2, 3, 15] is the rerouting of congested traffic to streets and arterials that may either temporarily be under-utilized or which normally operate below capacity. Under such conditions, signal settings which have been determined based on recurrent traffic demand will not, in general, be "tuned" to accommodate the transient demand generated by the real-time driver information. As a result, system performance (as well as the effectiveness of the ATIS) is limited by the capacity of the signal system to adapt to transient traffic demand. Better strategies will be necessary for many of the proposed Intelligent Transportation Systems.

A traffic intersection is depicted in Figure 1. As all drivers know, the traffic on 1 is typically not allowed to proceed with the traffic on 2, 3, 4, 7, or 8. The complete conflict graph for the traffic intersection is also depicted in Figure 1. The intersection controller must schedule the vehicles through the intersection so as to avoid any conflicts. We consider a 'job' to be a platoon or closely spaced line of cars which must pass through the intersection.

**Scheduling in high-speed local-area networks with spatial reuse** ([7]). Local area networks with spatial reuse allow the concurrent access and

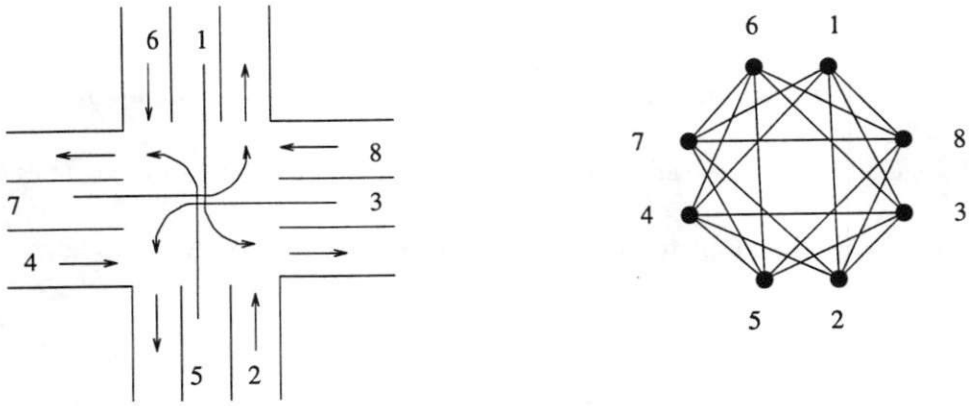


Figure 1: The graph depicting a traffic intersection.

transmission of user data with no intermediate buffering of packets. If some node $s$ has to send data to some other node $t$, a session is established between the $s$ and $t$. A session typically lasts for much longer than its data transmission time and can be active only if it has exclusive use of all the links in its route from $s$ to $t$. Therefore, sessions whose routes share at least one link are in conflict. Data transmissions among sessions must be scheduled so as to avoid these conflicts. We examine the problem of scheduling connections on a bus network where there is exactly one possible route between any two pairs of points. Thus, if connections are defined by the two nodes which must be connected, it is determined whether a given pair of connections will conflict with each other.

In the first application and for small networks in the second application, it is reasonable to assume that each job requires roughly the same amount of time to execute. Thus, we adopt a discrete model of time and assume that each job requires one time unit to be completed once it is started. At the beginning of a time unit, jobs may arrive on any subset of the nodes in $G$. The algorithm then chooses any independent set of nodes from which to schedule a job. At the end of the time unit, the scheduled jobs are gone from the graph. Then at the beginning of the next time unit, another set of jobs may arrive.

There are two natural optimization problems that arise from this model. The first is to minimize the *total response time* of all jobs in the system. The second is to minimize the *maximum response time* of any job which enters the system. We focus on bounding the maximum response time of any job which is the maximum, over all jobs $j$, of $d_j - a_j$, where $d_j$ is the time when $j$ departs and $a_j$ is the time when $j$ arrives. In both applications we consider, it is important to guarantee the best turnaround time to any job entering the system.

## 1.1 Our Results.

In a previous paper [13], we have shown that deterministic online algorithms have substantial limitations in this model. Specifically, we showed that on a path of $n$ nodes, the competitive ratio can be as bad as $\Omega(n)$. Furthermore, there is no known competitive ratio which is bounded by any function of $n$ for bipartite or interval graphs. As a result, we are lead to consider probabilistic assumptions over input sequences. The class of distributions that we consider can be defined by a vector $D = (p_1, p_2, \ldots, p_n)$. At each time unit, a job arrives on node $i$ with probability $p_i$. Arrivals in successive time units and on different nodes are completely independent. Any such vector $P$ induces a distribution over arrival sequences which we will call $\mathcal{D}(P)$. For a given algorithm $A$, we will be interested in finding an algorithm which minimizes $E_{\sigma\in\mathcal{D}(P)}[\mathtt{cost}_A(\sigma)]$, where $\mathtt{cost}_A(\sigma)$ is the maximum response time of any job when algorithm $A$ schedules input sequence $\sigma$. We are also interested in determining how good our algorithm is in comparison to the optimal algorithm (which knows $\sigma$ in advance). In the style of analyzing online algorithms against a *diffuse adversary* as defined in [16], we determine

$$\frac{E_{\sigma\in\mathcal{D}(P)}[\mathtt{cost}_A(\sigma)]}{E_{\sigma\in\mathcal{D}(P)}[\mathtt{cost}_{OPT}(\sigma)]}.$$

In all cases, we are able to obtain the same bound (to within additive lower order terms) for

$$E_{\sigma \in \mathcal{D}(P)} \left[ \frac{\texttt{cost}_A(\sigma)}{\texttt{cost}_{OPT}(\sigma)} \right].$$

The first set of bounds apply to general conflict graphs. Then we show how to significantly improve those bounds in the case that the conflict graph is a perfect graph. Note that the class of perfect graphs includes both of the applications mentioned earlier.

It is reasonable to restrict the set of distributions to stable distributions where it is possible to schedule jobs in such a way that the number of jobs in the system returns to 0 with probability 1. Let $G^l$ denote the *extended graph* induced by the job arrivals in the first $l$ time units. This graph is obtained by replacing each node in the graph by a clique whose size is the number of jobs at that node. If two nodes are adjacent in the conflict graph, then the two corresponding cliques are completely connected. The chromatic number of $G^l$ is the number of time units necessary to schedule the set of jobs arriving in the first $l$ time units. Let $\epsilon_l = l - E[\chi(G^l)]$. Certainly if $\epsilon_l < 0$ for all $l$, then even the optimal algorithm will accumulate a continually growing backlog of jobs. Thus, the only distributions of interest are those where there exists an $l$

such that $\epsilon_l > 0$. In section 3, we will focus our attention on a more restricted class of distributions where $\epsilon_1 > 0$. In section 4, we will focus our attention on the more general class of distributions where $\epsilon_l > 0$.

In all cases, the algorithm that we analyze is the simple algorithm which for a given $l$, gathers all the jobs that arrive in each block of $l$ consecutive time units and optimally schedules them before any of the jobs in the next block of $l$ time units. Note that we are assuming that the conflict graph is small or simple enough that it is feasible to color any induced extended graph either in real time or with some pre-processing. Let $G_i^l$ be the extended subgraph induced by the jobs that arrive in the $i^{th}$ block of $l$ consecutive time units.

**Algorithm l-block:** Optimally schedule the jobs from the $i^{th}$ $l$-block starting at the first time unit after the $i^{th}$ $l$-block finishes and after all jobs from the $i-1^{st}$ $l$-block have been scheduled.

The first theorem bounds the competitive ratio of $l$-block on any conflict graph:

**Theorem 1** *Let $G$ be any conflict graph and $P$ be any distribution vector. For edge $(a,b)$ such that $p_a p_b$ is maximized, if $\epsilon_l > 0$,*

$$\frac{E[\mathtt{cost}_{1-block}]}{E[\mathtt{cost}_{OPT}]} = \frac{l\chi \ln\left(\frac{1}{p_a p_b}\right)}{\epsilon_l}(1+o(1)),$$

*where* $m$ *is the length of the input sequence and* $\chi$ *is the chromatic number of the conflict graph.*

In the case of $\epsilon_1 > 0$, we obtain the following bound

**Theorem 2** *Consider any conflict graph* $G = (V, E)$ *and distribution vector* $P$. *If* $\epsilon_1 > 0$, *then*

$$\frac{E[\mathtt{cost}_{1-block}]}{E[\mathtt{cost}_{OPT}]} \leq \frac{\chi \ln(2|E|)}{\epsilon_1}(1 + o(1)).$$

It is reasonable to think that by combining jobs from consecutive $l$-blocks (i.e. considering larger $l$), some improvement in the performance can be gained. It is possible that $E[\chi(G_1^2)] < E[\chi(G_1^1)] + E[\chi(G_2^1)]$, in which case, the 2-block algorithm will perform better than the 1-block algorithm. If this is not the case, then even the adversary can not combine jobs from consecutive blocks which should lead to a stronger lower bound on the optimal cost. We are able to formalize this intuition in the case where the conflict graph is a perfect graph. In this case, we can prove that as $l$ grows, the competitive ratio of $l$-block converges to 1. That is, the performance of $l$-block converges to the optimal offline algorithm.

**Theorem 3** *Let* $G$ *be a conflict graph which is a perfect graph and let* $P$ *be the distribution vector. Suppose that* $\epsilon_1 > 0$. *Then there is an algorithm* $A$

*such that*

$$\frac{E[\texttt{cost}_A]}{E[\texttt{cost}_{OPT}]} = 1 + o(1),$$

*where the* $o(1)$ *is a function which tends towards* 0 *as the length of the sequence grows.*

The algorithm $A$ uses the $l$-block algorithm, periodically increasing $l$. The proof generalizes easily to the case where $E[\chi(G^r)] < r$ for some $r$. However, the algorithm must use a block length which is an integer multiple of $r$.

## 1.2 Previous Work.

Minimizing the maximum response time for general conflict graphs is NP-hard: even when all jobs arrive in a single time unit, the problem is equivalent to graph coloring [14]. Even approximating the minimum maximum response time to within a fixed polynomial factor is NP-complete [18]. In [13], we focus on the more traditional worst-case analysis. Our results focused on two special classes of graphs motivated by our applications: interval graphs and bipartite graphs. We argued that the problem of scheduling with the traffic intersection conflict graph depicted in Figure 1 is equivalent to scheduling on a $K_{2,2}$. We described a simple algorithm and proved that it obtains a competitive ratio of 4 on a $K_{2,2}$. This result was then generalized for arbitrary bipartite and interval graphs. Although the algorithms for bipartite and intervals graphs are quite different, the bounds they achieve are the same: we proved that for any sequence of jobs, the algorithm can complete every job in time $O(n^3 A^2)$, where $A$ is the maximum response time over all jobs in the optimal schedule and $n$ is the number of nodes in the conflict graph. Note that to achieve a bound on the competitive ratio, we would have had to upper bound the cost of the algorithm by a function which is linear in $A$. We obtained a lower bound of $\Omega(n)$ on the competitive ratio of any algorithm on an $n$-node path. Since a path is both bipartite and an interval graph, this gives a lower bound for both classes.

# 2 A Useful Result from Queuing Theory

All of our bounds on the maximum response time make use of the following result from queuing theory which follows from a result due to Iglehart [12].

**Lemma 4** *Suppose we have a sequence of identical independantly distributed (i.i.d.) random variables* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *such that* $\alpha_i \in \mathcal{Z}$, $E[\alpha_i] < 0$, *and* $\alpha_i \leq c$ *for some constant* $c$. *We will denote the* $Prob[\alpha_i = x]$ *by* $p_x$. *Then let*

*$\gamma$ be the constant greater than 1 which satisfies:*

$$\sum_{x \in \mathcal{Z}} \gamma^x p_x = 1.$$

*Define the sequence $\hat{\alpha}_i = \max\{0, \alpha_i + \hat{\alpha}_{i-1}\}$, and let*

$$M_m = \max_{1 \leq i \leq m} \hat{\alpha}_i.$$

*Then*

$$\frac{\ln m}{\ln \gamma}(1 - o(1)) \leq E[M_m] \leq \frac{\ln m}{\ln \gamma}(1 + o(1)),$$

*where the $o(1)$ is taken to be a function of $m$.*

Iglehart's theorem is proven for continuous time arrivals. We use a result from Spitzer [23] to determine the distribution for discrete time arrivals. Iglehart's result gives a very tight characterization of the distribution over $M_m$. This allows us to derive an upper bound for the expectation of the ratio of the online and offline costs as well as a bound on the ratio of the expectations. See the Appendix for details.

We will frequently consider equations of the following form:

$$\sum_x p_x \gamma^x = 1.$$

As long as

$$\sum_x p_x = 1,$$

the equation will have exactly two solutions: 1 and some value greater than 1. When we refer to the *solution* to such an equation, we will always be referring to the unique solution greater than 1.

# 3 $E[\chi(G_i^1)] < 1$

Throughout this section, we will assume that $E[\chi(G_i^1)] < 1$ and analyze the 1-block algorithm.

## 3.1 A Bound for the Expected Cost

In order to find an upper bound for the algorithm, we will use Lemma 4 with $\alpha_i = \chi(G_i^1) - 1$. Let $r_j$ be the probability that $\chi(G_i^1) = j$. Define $a_j = Prob[\alpha_i = j] = r_{j+1}$ and let $\gamma_{alg}$ be the solution to the following equation:

$$\sum_{x \in \mathcal{Z}} (\gamma_{alg})^x a_x = 1.$$

Let $\epsilon = 1 - E[\chi(G_i^1)]$. Lemma 4 tells us the cost of the algorithm as a function of $\gamma_{alg}$. It is just a matter now of determining a bound for $\gamma_{alg}$.

**Lemma 5**

$$\gamma_{alg} \geq \left(1 + \frac{\epsilon}{1 - r_1 - \epsilon}\right)^{\frac{1}{\chi}},$$

*where $\chi$ is the maximum value that $\chi(G_i^1)$ can reach.*

**Proof.** Since the $\alpha_i$'s are i.i.d., we will omit the subscript $i$. We know that $\alpha$ is an integer which is at least $-1$ and is at most $\chi$, the chromatic number of the conflict graph. Thus, $\gamma_{alg}$ is the solution to

$$\sum_{-1 \leq j \leq \chi} a_j \gamma^j = 1.$$

This is the same value which satisfies:

$$\sum_{0 < j \leq \chi} \left( \frac{a_j}{1 - a_0} \right) \gamma^j + \frac{a_{-1}}{(1 - a_0)\gamma} = 1.$$

Thus, we can just consider the random variable $\alpha'$ which is $j$ with probability

$$a'_j = \frac{a_j}{1 - a_0},$$

for $j = -1, 1, 2, \ldots, \chi$. Define

$$E[\alpha'] = -\epsilon' = \frac{-\epsilon}{1 - a_0}.$$

For every $j > 0$, let $x_j$ be the random variable which can only take on the values $j$ or $-1$ and whose expectation is $-\epsilon'$. $\hat{a}_j = Prob[x_j = j]$ is fixed and can be easily determined as a function of $\epsilon'$ and $j$. Now suppose we generated $\alpha'$ in the following way: Pick $j \in \{1, 2, \ldots, \chi\}$ with probability $p_j$ such that

$p_j \cdot \hat{a}_j = a'_j$. Then generate $x_j$, and let $\alpha'$ take on the value of $x_j$. It can be verified that this is exactly the distribution of $\alpha'$ and that the $p_j$'s sum to 1.

For $j > 0$, let $\gamma_j$ be the solution to

$$\hat{a}_j(\gamma)^j + \frac{1 - \hat{a}_j}{\gamma} = 1.$$

Now let

$$\gamma_{min} = \min_{1 \leq j \leq \chi} \gamma_j.$$

For all $j$,

$$\hat{a}_j(\gamma_{min})^j + \frac{1 - \hat{a}_j}{\gamma_{min}} \leq 1.$$

Thus,

$$\begin{aligned} \sum_{0<j\leq\chi} \left(\frac{a_j}{1 - a_0}\right)(\gamma_{min})^j + \frac{a_{-1}}{(1 - a_0)\gamma_{min}} &= \sum_{0<j\leq\chi} a'_j(\gamma_{min})^j + \frac{a'_{-1}}{\gamma_{min}} \\ &= \sum_{0<j\leq\chi} p_j \left[\hat{a}_j(\gamma_{min})^j + \frac{1 - \hat{a}_j}{\gamma_{min}}\right] \\ &\leq 1. \end{aligned}$$

Note that if we plot

$$\sum_{0<j\leq\chi} a'_j\gamma^j + \frac{a'_{-1}}{\gamma}$$

as a function of $\gamma$, it is below 1 only for those values of $\gamma$ between 1 and $\gamma_{alg}$. Thus, $\gamma_{alg} \geq \gamma_{min}$.

Solving for $\hat{a}_j$ and plugging it in, we get that $\gamma_j$ is the solution to the following equation:

$$\left(\frac{1-\epsilon'}{j+1}\right)\gamma^j + \left(\frac{j+\epsilon'}{j+1}\right)\frac{1}{\gamma} = 1. \tag{1}$$

It can be verified, that if we let

$$\gamma \leftarrow \left(\frac{1}{1-\epsilon'}\right)^{\frac{1}{j}},$$

then

$$\left(\frac{1-\epsilon'}{j+1}\right)\gamma^j + \left(\frac{j+\epsilon'}{j+1}\right)\frac{1}{\gamma} \leq 1.$$

This tells us that

$$\gamma_j \geq \left(\frac{1}{1-\epsilon'}\right)^{\frac{1}{j}},$$

and

$$\begin{aligned}
\gamma_{alg} &\geq \gamma_{min} \\
&= \min_{1\leq j\leq \chi} \gamma_j \\
&\geq \min_{1\leq j\leq \chi} \left(\frac{1}{1-\epsilon'}\right)^{\frac{1}{j}} \\
&= \left(1+\frac{\epsilon}{1-\epsilon-r_1}\right)^{\frac{1}{\chi}}.
\end{aligned}$$

□

Using the lower bound for $\gamma_{alg}$, we just have to plug the expression into the bound from Lemma 4 and simplify.

**Theorem 6**

$$E[\mathtt{cost}_{1-block}] \leq \frac{\chi \ln m}{\epsilon}(1 + o(1)),$$

*where $m$ is the length of the sequence.*

**Proof.** Using Lemma 4, we know that

$$\begin{aligned} E[\mathtt{cost}_{1-block}] &\leq \frac{\ln m}{\ln \gamma_{alg}}(1 + o(1)) \\ &\leq \frac{\chi \ln m}{\ln\left(1 + \frac{\epsilon}{1 - r_1 - \epsilon}\right)}(1 + o(1)) \\ &\leq \frac{\chi \ln m}{\epsilon}(1 + o(1)). \end{aligned}$$

□

## 3.2 Bounding the Competitive Ratio

In order to bound the ratio of the online to the offline, we need now a lower bound for the optimal. For the lower bound on the optimal, we will determine the backlog of jobs which accumulate on a single edge. We pick the edge $(a, b)$ such that $p_a p_b$ is maximized. Then we invoke Lemma 4 with $\alpha_i$ defined as follows:

$$\alpha_i = \begin{cases} +1 & \text{with probability } p_a p_b \\ 0 & \text{with probability } p_a(1-p_b) + p_b(1-p_a) \\ -1 & \text{with probability } (1-p_a)(1-p_b). \end{cases}$$

Then $\gamma_{opt}$ is chosen so that $\gamma_{opt} > 1$ and

$$\gamma_{opt}[p_a p_b] + [p_a(1-p_b) + p_b(1-p_a)] + \frac{[(1-p_a)(1-p_b)]}{\gamma_{opt}} = 1. \qquad (2)$$

There are two solutions to (2). The first is 1 and the second is less than

$$\frac{1}{p_a p_b}.$$

**Lemma 7**

$$\gamma_{opt} \leq \frac{2|E|}{1-\epsilon-r_1},$$

*where $|E|$ is the number of edges in the conflict graph.*

**Proof.** In order to find an upper bound for

$$\frac{1}{p_a p_b},$$

consider the graph obtained from the jobs that arrive during the $t^{th}$ time unit. Let $n_t$ be the number of nodes which are part of connected components of size more than one in this graph.

$$\begin{aligned} E[\chi(G_t^1)] &\leq r_1 + E[n_t] \\ &\leq r_1 + 2 \sum_{(i,j)\in E(G)} p_i p_j \\ &\leq r_1 + 2|E|(p_a p_b). \end{aligned}$$

Thus,

$$E[\chi(G_i^1)] - r_1 = 1 - \epsilon - r_1 \leq 2|E|(p_a p_b).$$

The latter inequality holds since edge $(a, b)$ was chosen to maximize $p_a p_b$. Regrouping, we get that

$$\frac{1}{p_a p_b} \leq \frac{2|E|}{1 - \epsilon - r_1}.$$

Thus,

$$\gamma_{opt} \leq \frac{2|E|}{1 - \epsilon - r_1}.$$

□

Using the lower bound on the optimal and the upper bound on the algorithm, we can upper bound the ratio of the expected cost of the algorithm over the expected cost of the optimal.

**Lemma 8**

$$\frac{E[\mathtt{cost}_{1-block}]}{E[\mathtt{cost}_{OPT}]} \leq \frac{\chi(G)\ln(2|E|)}{\epsilon}(1 + o(1)).$$

**Proof.** Using Lemma 4, we know that

$$\begin{aligned}
\frac{E[\mathtt{cost}_{1-block}]}{E[\mathtt{cost}_{OPT}]} &\leq \frac{\ln \gamma_{OPT}}{\ln \gamma_{alg}}(1+o(1)) \\
&\leq \frac{\chi(G)\ln\left(\frac{2|E|}{1-\epsilon-r_1}\right)}{\ln\left(1+\frac{\epsilon}{1-r_1-\epsilon}\right)}(1+o(1)) \\
&\leq \frac{\chi(G)(1-r_1-\epsilon)\ln\left(\frac{2|E|}{1-\epsilon-r_1}\right)}{\epsilon}(1+o(1)) \\
&\leq \frac{\chi(G)\ln(2|E|)}{\epsilon}(1+o(1)).
\end{aligned}$$

□

# 4 $E[\chi(G_i^l)] < l$

We will now consider the case where $l-1 < E[\chi(G_i^l)] < l$ and examine the performance of the $l$-block algorithm.

## 4.1 A Bound for the Expected Cost

In order to find an upper bound for the algorithm, we will be using Lemma 4 with $\alpha_i = \chi(G_i^l) - l$. Let $r_j$ now be the probability that $\chi(G_i^l) = j$. We now define $a_j = Prob[\alpha_i = j] = r_{j+l}$. Now let $\gamma_{alg}$ be the value greater than 1 such that

$$\sum_{x \in \mathcal{Z}} (\gamma_{alg})^x a_x = 1.$$

Now let $\epsilon = l - E[\chi(G_i^l)]$.

**Lemma 9**

$$\gamma_{alg} \geq \left(1 + \frac{(lc+1)\epsilon}{lc(1 - r_l - \epsilon)}\right)^{\frac{1}{l(c+1)}},$$

*where $c$ is the maximum value that $\chi(G_i^1)$ can reach.*

**Proof.** Define

$$\epsilon' = \frac{\epsilon}{1 - r_l} = \frac{\epsilon}{1 - a_0}.$$

For $1 \leq j \leq lc$ and $1 \leq k \leq l$, it is possible to find values $\bar{a}_j \leq a_j$ and $\bar{a}_{-k} \leq a_{-k}$ so that if we let

$$p_{j,k} = \frac{\bar{a}_j}{\bar{a}_j + \bar{a}_{-k}},$$

then

$$jp_{j,k} - k(1 - p_{j,k}) = \frac{-\epsilon}{1 - a_0} = -\epsilon'.$$

Now if we let $\gamma_{j,k}$ be the value greater than 1 which satisfies

$$p_{j,k}\gamma_{j,k}^j + \frac{1 - p_{j,k}}{\gamma_{j,k}^k} = 1,$$

then $\gamma_{alg}$ is a weighted average of the $\gamma_{j,k}$'s. We will prove that for all $j$ and $k$,

$$\gamma_{j,k} \geq \left(1 + \frac{(lc+1)\epsilon}{lc(1 - r_l - \epsilon)}\right)^{\frac{1}{l(c+1)}}.$$

We first observe that

$$\left(1 + \frac{(lc+1)\epsilon}{lc(1 - r_l - \epsilon)}\right) \leq \left(1 + \frac{(j+k)\epsilon'}{j(k - \epsilon')}\right) = \frac{k(1 - p_{j,k})}{j p_{j,k}}.$$

We will show that

$$\gamma_{j,k} \geq \left(\frac{k(1 - p_{j,k})}{j p_{j,k}}\right)^{\frac{1}{j+k}}.$$

Alternatively, we will show that

$$p_{j,k}\left(\frac{k(1 - p_{j,k})}{j p_{j,k}}\right)^{\frac{j}{j+k}} + (1 - p_{j,k})\left(\frac{j p_{j,k}}{k(1 - p_{j,k})}\right)^{\frac{k}{j+k}} \leq 1.$$

This inequality follows from the fact that $\epsilon' > 0$ and hence

$$p_{j,k} < \frac{k}{j+k}.$$

When

$$p_{j,k} = \frac{k}{j+k},$$

the above inequality is an equality, and the left hand side decreases with $p_{j,k}$.

□

## 4.2 Bounding the Competitive Ratio

For the lower bound on the optimal, we will again determine the backlog of jobs which accumulate on a single edge. We again pick the edge $(a, b)$ such that $p_a p_b$ is maximized. Then we invoke Lemma 4 with $\alpha_i$ defined as follows:

$$\alpha_i = \begin{cases} +l & \text{with probability } [p_a p_b]^l \\ +l-1 & \text{with probability } l(p_a p_b)^{l-1}[p_a(1-p_b)+p_b(1-p_a)] \\ \vdots & \\ -l & \text{with probability } [(1-p_a)(1-p_b)]^l. \end{cases}$$

Then $\gamma_{opt}$ is chosen so that $\gamma_{opt} > 1$ and

$$\gamma_{opt}^l [p_a p_b]^l + \gamma_{opt}^{l-1} l(p_a p_b)^{l-1}[p_a(1-p_b)+p_b(1-p_a)] + \cdots + \frac{[(1-p_a)(1-p_b)]^l}{\gamma_{opt}^l} = 1. \tag{3}$$

**Lemma 10**

$$\gamma_{opt} \leq \frac{1}{p_a p_b}.$$

**Proof.** There are two solutions to (3). The first is 1 and the second is less than

$$\frac{1}{p_a p_b}.$$

Thus

$$\gamma_{opt} \leq \frac{1}{p_a p_b}.$$

□

Using the lower bound on the optimal and the upper bound on the algorithm, we can upper bound the ratio of the expected cost of the algorithm over the expected cost of the optimal.

**Lemma 11**

$$\frac{E[\mathtt{cost}_{l-block}]}{E[\mathtt{cost}_{OPT}]} \leq \frac{l(\chi(G)) \ln\left(\frac{1}{p_a p_b}\right)}{\epsilon}(1+o(1)).$$

**Proof.** Using Lemma 4, we know that

$$\begin{aligned}
\frac{E[\mathtt{cost}_{l-block}]}{E[\mathtt{cost}_{OPT}]} &\leq \frac{\ln \gamma_{OPT}}{\ln \gamma_{alg}}(1+o(1)) \\
&\leq \frac{l(\chi(G)+1) \ln\left(\frac{1}{p_a p_b}\right)}{\ln\left(1+\frac{(\chi(G)+1)\epsilon}{\chi(G)(1-r_l-\epsilon)}\right)}(1+o(1)) \\
&\leq \frac{l(\chi(G)+1)\chi(G)(1-r_l-\epsilon) \ln\left(\frac{1}{p_a p_b}\right)}{(\chi(G)+1)\epsilon}(1+o(1)) \\
&\leq \frac{l\chi(G) \ln\left(\frac{1}{p_a p_b}\right)}{\epsilon}(1+o(1)).
\end{aligned}$$

□

# 5 Perfect Graphs

We now turn to the special case where the conflict graph is a perfect graph. We are assuming that it is the case that $E[\chi(G^1)] < 1$, although the proof goes through under the weaker condition that for some $r$, $E[\chi(G^r)] < r$ as long as the block size is chosen to be a multiple of $r$. We will consider the $l$-block algorithm. Let $\gamma_{on}(l) > 1$ satisfy

$$\sum_x Prob[\chi(G^l) - l = x](\gamma_{on}(l))^x = 1.$$

Let $C$ be a clique in the graph. Let $C_i^1$ be the number of jobs arriving in the clique in the $i^{th}$ time unit. Let $C_i^l$ be the number of jobs arriving in the $i^{th}$ $l$-block. The same $\gamma_C > 1$ satisfies both of the following equations:

$$\begin{aligned} 1 &= \sum_x (\gamma_C)^x Prob[C^1 - 1 = x] \\ 1 &= \sum_x (\gamma_C)^x Prob[C^l - l = x] \end{aligned}$$

We prove the following theorem:

**Theorem 12** *There is a clique $C$ such that*

$$\gamma_{on}(l) \geq \gamma_C^{1-o(1)}.$$

The $o(1)$ above is taken to be as $m$ grows. All other parameters of the system are constants.

The proof of Theorem 3 is straightforward once Theorem 12 has been established:

**Proof of Theorem 3.** The algorithm implements the $l$-block algorithm, periodically increasing $l$ so that $l$ grows with the length of the sequence and the additional cost of $l$ grows more slowly than the cost of the algorithm. One way to achieve this is to double $l$ every time the cost of the algorithm increases by a factor of 4. In this case,

$$\begin{aligned}\frac{E[\texttt{cost}_{online}]}{E[\texttt{cost}_{OPT}]} &= \frac{\ln \gamma_{opt}}{\ln \gamma_{on}(l)}(1+o(1)) \\ &\leq \frac{\ln \gamma_C}{\ln \gamma_C^{1-o(1)}}(1+o(1)) \\ &= (1+o(1)).\end{aligned}$$

□

Before proving Theorem 12, we require some definitions. Fix a value for $l$. For each clique $C$, let

$$\begin{aligned} p_{C,x} &= Prob[C^l - l = x] \\ \mu_C &= 1 - E[C^1] \end{aligned}$$

$$\mu_{min} = \min_C \mu_C$$
$$B = \frac{-\mu_{min} l}{2}$$
$$\alpha = \frac{1}{6\chi^2}$$

As defined above, $\gamma_C$ is the solution to

$$\sum_x p_{C,x}(\gamma_C)^x = 1.$$

Define $\hat{\gamma}_C$ to be the solution to

$$\left[1 - N \sum_{x>B} p_{C,x}\right] (\hat{\gamma}_C)^B + N \sum_{x>B} p_{C,x}(\hat{\gamma}_C)^x = 1,$$

where $N$ is the number of cliques in the graph.

**Lemma 13** *If*

$$\mu_C \leq \frac{\mu_{min}}{\alpha},$$

*then* $(\gamma_C)^{1-o(1)} \leq \hat{\gamma}_C$, *where the* $o(1)$ *tends towards* 0 *as* $l$ *gets large.*

We first give the proof of the theorem modulo Lemma 13.

**Proof of Theorem 12.** Consider the solutions to the following series of equations. In each step, weight is either added to the coefficients on the right or weight is shifted from smaller values of $x$ to larger values of $x$. In either case, the value of $\gamma$ which satisfies the equation does not increase.

$$\begin{aligned}
1 &= \sum_x Prob[\chi(G^l) - l = x]\gamma^x \\
1 &= \sum_x Prob[\max_C C^l - l = x]\gamma^x \\
1 &= \sum_{x \le B} Prob[\max_C C^l - l = x]\gamma^B + \sum_{x > B} Prob[\max_C C^l - l = x]\gamma^x \\
1 &= \sum_{x \le B} \sum_C p_{C,x}\gamma^B + \sum_{x > B} \sum_C p_{C,x}\gamma^x
\end{aligned}$$

Let $\bar{\gamma}$ be the solution to the last equation. Notice that

$$\sum_{x \le B} \sum_C p_{C,x}\gamma^B + \sum_{x > B} \sum_C p_{C,x}\gamma^x$$

is the average of

$$\sum_{x \le B} N p_{C,x}\gamma^B + \sum_{x > B} N p_{C,x}\gamma^x,$$

for all cliques. Thus, if we let $D$ be the clique with the smallest $\hat{\gamma}_D$, we know that

$$\gamma_{on}(l) \ge \bar{\gamma} \ge \hat{\gamma}_D.$$

If we can show that

$$\mu_D \le \frac{\mu_{min}}{\alpha},$$

then we can invoke Lemma 13 and the theorem follows. Suppose that for some $C$,

$$\mu_C \geq \frac{\mu_{min}}{\alpha}.$$

Let $F$ be the clique which realizes $\mu_{min}$. We will show that $\hat{\gamma}_F \leq \gamma_F \leq \hat{\gamma}_C$ which establishes that $C$ can not be the clique with the smallest $\hat{\gamma}_C$. $\gamma_F$ is maximized when $[F^1 - 1]$ is $+1$ with probability $p$ and $-1$ with probability $1-p$ where $p$ is chosen so that $p-(1-p) = \mu_{min}$. It can be verified that in this case, $\gamma_F \leq 1 + \mu_{min} \leq 1 + \alpha\mu_C$. Thus, all we must do is to show that

$$(1+\mu_{min})^B + N \sum_{x \geq B} p_{C,x} (1+\alpha\mu_C)^x \leq 1.$$

Since

$$B = \frac{-\mu_{min} l}{2},$$

the first term is

$$(1+\mu_{min})^{\frac{-\mu_{min} l}{2}} \leq e^{\frac{-l\mu_{min}^2}{2}}.$$

As $l$ grows, this term becomes very small.

The remainder of the left hand side is maximized when $[C^1 - 1]$ is $+\chi$ with probability $p$ and $-\chi$ with probability $1-p$ where $p$ is chosen so that

$E[1 - C^1] = \mu_C$. Since $-\mu_C = \chi p - \chi(1-p)$, we get that

$$p = \frac{\chi - \mu_C}{2\chi}.$$

$p_{C,x}$ is only non-zero with probability when $x$ is a multiple of $\chi$ in which case

$$Prob[C^l - l = y\chi] = \binom{l}{\frac{l+y}{2}} \left(\frac{1}{2} - \frac{\mu_C}{2\chi}\right)^{\frac{l+y}{2}} \left(\frac{1}{2} + \frac{\mu_C}{2\chi}\right)^{\frac{l-y}{2}}.$$

For ease of notation, we make the substitution

$$\beta \leftarrow \frac{\mu_C}{\chi}.$$

$$\begin{aligned}
p_{C,\chi y} &= \binom{l}{\frac{l+y}{2}} \frac{1}{2^l} (1-\beta)^{\frac{l+y}{2}} (1+\beta)^{\frac{l-y}{2}} \\
&\leq \binom{l}{\frac{l+y}{2}} \frac{1}{2^l} \left(1 - \frac{\beta}{2}\right)^{\frac{l+y}{2}} \left(1 + \frac{\beta}{2}\right)^{\frac{l-y}{2}} \left[\frac{1-\beta}{1-\frac{\beta}{2}} \frac{1+\beta}{1+\frac{\beta}{2}}\right]^{\frac{l}{2}} \left[\frac{1-\beta}{1-\frac{\beta}{2}} \frac{1+\frac{\beta}{2}}{1+\beta}\right]^{\frac{y}{2}} \\
&\leq \binom{l}{\frac{l+y}{2}} \frac{1}{2^l} \left(1 - \frac{\beta}{2}\right)^{\frac{l+y}{2}} \left(1 + \frac{\beta}{2}\right)^{\frac{l-y}{2}} \left[\frac{1 - \frac{\beta}{2} - \frac{\beta^2}{2}}{1 + \frac{\beta}{2} - \frac{\beta^2}{2}}\right]^{\frac{y}{2}} \\
&\leq \binom{l}{\frac{l+y}{2}} \frac{1}{2^l} \left(1 - \frac{\beta}{2}\right)^{\frac{l+y}{2}} \left(1 + \frac{\beta}{2}\right)^{\frac{l-y}{2}} \left(1 - \frac{\beta}{3}\right)^{y}
\end{aligned}$$

Now to bound the second term in the sum:

$$\begin{aligned}
& N \sum_{x \geq B} p_{C,x} \left(1 + \alpha\mu_C\right)^x \\
\leq\ & N \sum_{y \geq \frac{B}{\chi}} \binom{l}{\frac{l+y}{2}} \left(\frac{1}{2} - \frac{\mu_C}{4\chi}\right)^{\frac{l+y}{2}} \left(\frac{1}{2} + \frac{\mu_C}{4\chi}\right)^{\frac{l-y}{2}} \left(1 - \frac{\mu_C}{3\chi}\right)^{y} \left(1 + \frac{\mu_C}{6\chi^2}\right)^{y\chi}
\end{aligned}$$

$$\leq N \sum_{y \geq \frac{B}{\chi}} \binom{l}{\frac{l+y}{2}} \left(\frac{1}{2} - \frac{\mu_C}{4\chi}\right)^{\frac{l+y}{2}} \left(\frac{1}{2} + \frac{\mu_C}{4\chi}\right)^{\frac{l-y}{2}} \left(1 - \frac{\mu_C}{3\chi}\right)^{y} \left(1 + \frac{\mu_C}{3\chi}\right)^{y}$$

$$\leq N \sum_{y \geq \frac{B}{\chi}} \binom{l}{\frac{l+y}{2}} \left(\frac{1}{2} - \frac{\mu_C}{4\chi}\right)^{\frac{l+y}{2}} \left(\frac{1}{2} + \frac{\mu_C}{4\chi}\right)^{\frac{l-y}{2}}$$

The latter summation is simply the probability that the sum of $l$ i.i.d. random variables whose expectation is

$$\frac{-\mu_C}{2}$$

is at least

$$B = \frac{-\mu_{min} l}{2} > \frac{-\mu_C l}{4}.$$

This value is exponentially small in $l$ and for large enough $l$, it will be much less than

$$\frac{1}{N}.$$

□

**Proof of Lemma 13.** Throughout this proof, we refer to one clique $C$, so we drop the subscript $C$ from the parameters. The bound is proven by a series of steps. In each step we consider the solution to a new equation. The solution to the first equation is $\gamma_C$ and the solution to the last equation is $\hat{\gamma}_C$. At each step we prove that the equation changes by a sufficiently small amount that the solution decreases by at most a power of $1 - o(1)$. Let

$$
\begin{aligned}
B^{+} &= \frac{-\mu l \alpha}{2} \\
B^{-} &= -\mu l \left(1 + \frac{\alpha}{2}\right) \\
P^{+} &= Prob[C^{l} - l \geq B^{+}] \leq e^{\frac{-\mu^{2}\left(1-\frac{\alpha}{2}\right)^{2} l}{2\chi^{2}}} \\
P^{-} &= Prob[C^{l} - l \leq B^{-}] \leq e^{\frac{-\mu^{2}\alpha^{2} l}{8\chi^{2}}}
\end{aligned}
$$

To derive the inequalities above, observe that $C^{l} - l$ is the sum of $l$ i.i.d. random variables. The expectation of $C^{1} - 1$ is $\mu$ and its maximum value is $\chi$. Then apply a standard Chernoff inequality.

Now consider the following four equations:

$$
\begin{aligned}
\gamma_0 \text{ satisfies } 1 &= \sum_{x} p_x \gamma^{x} \\
\gamma_1 \text{ satisfies } 1 &= \sum_{x \leq B^{-}} p_x \gamma^{B^{+}} + \sum_{x > B^{-}} p_x \gamma^{x} \\
\gamma_2 \text{ satisfies } 1 &= (1 - P^{+})\gamma^{B^{+}} + \sum_{x > B^{+}} p_x \gamma^{x} \\
\gamma_3 \text{ satisfies } 1 &= (1 - NP^{+})\gamma^{B^{+}} + \sum_{x > B^{+}} N \gamma^{x} p_x
\end{aligned}
$$

The following lemma which is proven in the Appendix will be very useful.

**Lemma 14** *Suppose that $\hat{\gamma} > 1$ and satisfies the following inequality*

$$\sum_{s \leq x \leq b} p_x \hat{\gamma}^{x} \leq 1 + A.$$

*Suppose that $\bar{\gamma}$ is the solution to*

$$\sum_{s\leq x\leq b} p_x \bar{\gamma}^x = 1.$$

*Then $\bar{\gamma} \geq \hat{\gamma}^{1-\epsilon}$ as long as there exists $X < 0$ and $P$ such that the following conditions are satisfied.*

*1.*

$$\sum_{s\leq x\leq b} p_x = 1.$$

*2.*

$$\epsilon \geq \frac{\ln\left(1+\frac{3A}{1-P}\right)}{\ln\hat{\gamma}}.$$

*3.*

$$\sum_{s\leq x\leq X} p_x \geq 1-P.$$

*4.*

$$\hat{\gamma}^{-X} \geq -6s.$$

*5.*

$$\frac{A}{1-P} \leq \frac{1}{6}.$$

**Step 1:** We know that

$$\sum_{x \leq B^-} p_x(\gamma_0)^{B^+} + \sum_{x \geq B^-} p_x(\gamma_0)^x \leq 1 + P^-.$$

We must establish that for sufficiently large $l$, all the conditions of Lemma 14 are satisfied in order to lower bound $\gamma_1$. We will let $A \leftarrow P^-$, $P \leftarrow P^+$, $X \leftarrow B^+$, $s \leftarrow -l$, and $b \leftarrow \chi l$.

Condition 1 follows from the fact that

$$\sum_{s \leq x \leq b} p_x = \sum_x p_x = 1.$$

Condition 3 follows from the fact that a fraction of at least $1 - P^+$ of the weight is on values which are at most $B^+$. To establish condition 4, we must prove that $\gamma_0^{-B^+} \geq 6l$. $\gamma_0$ is minimized when $C^1 - 1$ is $+\chi$ with probability $p$ and $-\chi$ with probability $1 - p$, where $p$ is chosen so that $\chi p - (1-p)\chi = \mu$. In this case,

$$\gamma_0 \geq \left(1 + \frac{\mu}{\chi}\right)^{\frac{1}{\chi+1}}.$$

Thus,

$$(\gamma_0)^{-B^+} \geq \left(1 + \frac{\mu}{\chi}\right)^{\frac{l\mu\alpha}{2(\chi+1)}} \geq e^{\frac{l\alpha\mu^2}{2\chi(\chi+1)}}.$$

As long as

$$\frac{l}{\log l} \geq \frac{4\chi(\chi+1)}{\alpha\mu^2},$$

we will have that $\gamma_0^{-B^+} \geq 6l$. Condition 5 is satisfied since $P^-$ is exponentially small in $l$, so there is an $l$ large enough so that

$$\frac{P^-}{1-P^+} \leq \frac{1}{6}.$$

Thus, by Lemma 14, $\gamma_1 \geq \gamma_0^{1-\epsilon_1}$ for

$$\epsilon_1 \geq \frac{\ln\left(1+\frac{3P^-}{(1-P^+)}\right)}{\ln \gamma_0}.$$

**Step 2:** We know that

$$\sum_{x \leq B^+} p_x(\gamma_1)^{B^+} + \sum_{x>B^+} p_x(\gamma_1)^x \leq 1+Y,$$

where

$$Y \leq \gamma_1^{B^+} - \gamma_1^{B^-} \leq \gamma_1^{B^+} = \gamma_1^{\frac{-\mu\alpha l}{2}}.$$

In a similar manner to Step 1, letting $A \leftarrow Y$, $P \leftarrow P^+$, $X \leftarrow B^+$, $s \leftarrow -l$, and $b \leftarrow \chi l$, we can show that the conditions for Lemma 14 are satisfied which will give us that $\gamma_2 \geq \gamma_1^{1-\epsilon_2}$ for

$$\epsilon_2 \geq \frac{\ln\left(1+\frac{3Y}{(1-P^+)}\right)}{\ln \gamma_1}.$$

Using the fact that

$$\gamma_1 \geq \left(1+\frac{\mu}{2\chi}\right)^{\frac{1}{\chi+1}},$$

$$\gamma_1^{\frac{-\mu l \alpha}{2}}$$

is exponentially small in $l$. Thus, $l$ can be chosen large enough so that

$$\frac{Y}{1 - P^+} \leq \frac{1}{6}.$$

Again, using the fact that

$$\gamma_1 \geq \left(1 + \frac{\mu}{2\chi}\right)^{\frac{1}{\chi+1}},$$

an almost identical argument as in Step 1 can be used to show that for $l$ large enough, $\gamma_1^{B^+} \geq 6l$. Thus, by Lemma 14, $\gamma_2 \geq \gamma_1^{1-\epsilon_2}$ for

$$\epsilon_2 \geq \frac{\ln\left(1 + \frac{3Y}{(1-P^+)}\right)}{\ln \hat{\gamma}}.$$

**Step 3:** Let $d = \ln l$, and pick the largest $S$ so that $(\gamma_2)^S \leq N^d$. We know that

$$\left[1 - \sum_{B^+ \leq x \leq S} N p_x - \sum_{x \geq S} p_x\right] (\gamma_2)^{B^+} + \sum_{B^+ \leq x \leq S} N(\gamma_2)^x p_x + \sum_{x \geq S} p_x \gamma_2^x$$
$$\leq \ (N^{d+1} P^+) + 1.$$

Since $P^+$ is exponentially small in $l$ and $N$ is a constant which depends only on the graph, when $l$ is large enough, $N^{d+1}P^+$ will be at most

$$\frac{1}{6},$$

and we can apply Lemma 14 as in Steps 1 and 2. Thus, we have some $\gamma_{2a}$ which satisfies

$$(\gamma_{2a})^{B^+} + \sum_{B^+ \leq x \leq S} N(\gamma_{2a})^x p_x + \sum_{x \geq S} p_x(\gamma_{2a})^x = 1,$$

where $\gamma_{2a} \geq \gamma_2^{1-o(1)}$. Now let $\gamma_{2b} = (\gamma_{2a})^{1-\frac{1}{d}}$. When $x \geq S$, we know that

$$(\gamma_{2b})^x \leq \frac{(\gamma_{2a})^x}{N}.$$

Thus,

$$(\gamma_{2b})^{B^+} + \sum_{x \geq B^+} N p_x(\gamma_{2b})^x \leq 1 + (\gamma_{2b})^{B^+}.$$

Since $\gamma_{2b}^{B^+}$ is very small, we can apply Lemma 14 again to get that $\gamma_3 \geq \gamma_{2b}^{1-o(1)}$.

□

For perfect graphs, the following lemma shows that if there exists $r$ such that $E[\chi(G^r)] < r$, then for all $l > r$, $E[\chi(G^l)] < l$.

**Lemma 15** *If $G$ is a perfect graph, then*

$$\frac{E[\chi(G_i^l)]}{E[\chi(G_i^{l+1})]} \geq \frac{l}{l+1}.$$

**Proof.** Let $G_{n_1,n_2,\ldots,n_k}$ be the extended graph formed by the arrival of $n_i$ jobs on node $i$. Let $q_i = 1 - p_i$.

$$E[\chi(G_i^l)] = \sum_{n_1=0}^{l} \sum_{n_2=0}^{l} \cdots \sum_{n_k=0}^{l} \chi(G_{n_1,n_2,\ldots,n_k}) \prod_{i=1}^{k} \binom{l}{n_i} {p_i}^{n_i} {q_i}^{l-n_i}.$$

$$E[\chi(G_i^{l+1})] = \sum_{n_1=0}^{l+1} \sum_{n_2=0}^{l+1} \cdots \sum_{n_k=0}^{l+1} \chi(G_{n_1,n_2,\ldots,n_k}) \prod_{i=1}^{k} \binom{l+1}{n_i} {p_i}^{n_i} {q_i}^{l+1-n_i}.$$

Each term

$$\chi(G_{n_1,n_2,\ldots,n_k}) \prod_{i=1}^{k} \binom{l}{n_i} {p_i}^{n_i} {q_i}^{l-n_i} \tag{4}$$

in $E[\chi(G_i^l)]$ is equivalent to

$$\sum_{u_1=n_1}^{n_1+1} \sum_{u_2=n_2}^{n_2+1} \cdots \sum_{u_k=n_k}^{n_k+1} \chi(G_{n_1,n_2,\ldots,n_k}) \prod_{i=1}^{k} \binom{l}{n_i} {p_i}^{u_i} {q_i}^{l+1-u_i}. \tag{5}$$

The equivalence can be verified by observing that there are $2^k$ terms in the summation in (5), each one represented by a subset $S$ of $\{1, 2, \ldots k\}$. For $i \in S$, we take $u_i = n_i + 1$ and for $i \notin S$, we take $u_i = n_i$. The equivalence follows using the identity

$$\sum_{S \subseteq \{1,2,\ldots,k\}} \prod_{i \in S} p_i \prod_{i \notin S} q_i = 1.$$

Now we can express $E[\chi(G_i^l)]$ by substituting the term in (5) for the term in (4) and regrouping. Let $\hat{n}_i = \max\{0, n_i - 1\}$ and $\check{n}_i = \min\{l, n_i\}$. All the terms in $E[\chi(G_i^l)]$ with the factor

$$\prod_{i=1}^{k} {p_i}^{n_i} {q_i}^{l+1-n_i}$$

in its equivalent representation from (5) sum up to

$$\sum_{u_1=\hat{n}_1}^{\check{n}_1} \sum_{u_2=\hat{n}_2}^{\check{n}_2} \cdots \sum_{u_k=\hat{n}_k}^{\check{n}_k} \chi(G_{u_1,u_2,\ldots,u_k}) \prod_{i=1}^{k} \begin{pmatrix} l \\ u_i \end{pmatrix} {p_i}^{n_i} {q_i}^{l+1-n_i}. \tag{6}$$

$E[\chi(G_i^l)]$ can be obtained by summing the above expression over all combinations for the $n_i$'s ranging from 0 to $l+1$. The term in $E[\chi(G_i^{l+1})]$ with the factor

$$\prod_{i=1}^{k} {p_i}^{n_i} {q_i}^{l+1-n_i}$$

is

$$\chi(G_{n_1,n_2,\ldots,n_k}) \prod_{i=1}^{k} \begin{pmatrix} l+1 \\ n_i \end{pmatrix} {p_i}^{n_i} {q_i}^{l+1-n_i}. \tag{7}$$

We wish to show that $(l+1) \cdot$ (Expression (6)) is at least $l \cdot$ (Expression (7)). This is the same as showing

$$\frac{1}{l} \left[ \sum_{u_1=\hat{n}_1}^{\check{n}_1} \sum_{u_2=\hat{n}_2}^{\check{n}_2} \cdots \sum_{u_k=\hat{n}_k}^{\check{n}_k} \chi(G_{u_1,u_2,\ldots,u_k}) \prod_{i=1}^{k} \begin{pmatrix} l \\ u_i \end{pmatrix} \right] \tag{8}$$

$$\geq \frac{1}{l+1} \left[ \chi(G_{n_1,n_2,\ldots,n_k}) \prod_{i=1}^{k} \begin{pmatrix} l+1 \\ n_i \end{pmatrix} \right]. \tag{9}$$

Let each term

$$\chi(G_{u_1,u_2,\ldots,u_k})\prod_{i=1}^{k}\left(\begin{array}{c} l \\ \\ u_i \end{array}\right)$$

in (8) be a vertex in a graph $H$. Let there be an edge between any two of these terms that differ in only one of their $u_i$'s. $H$ is a hypercube of dimension $d$ equal to the number of $i$'s such that $\hat{n}_i \neq \check{n}_i$.

Since $G_{n_1,n_2,\ldots,n_k}$ is a perfect graph, there is a subset $S$ such that

$$\chi(G_{n_1,n_2,\ldots,n_k}) = \sum_{j\in S} n_j.$$

Note that we never need to include $j$ in $S$ if $n_j = 0$. If we substitute

$$\sum_{j\in S} n_j$$

into (9), we get

$$\frac{1}{l+1}\sum_{j\in S} n_j\left[\prod_{i=1}^{k}\binom{l+1}{n_i}\right] = \frac{1}{l+1}\sum_{j\in S}(l+1)\binom{l}{n_j-1}\prod_{i\neq j,1\leq j\leq k}\binom{l+1}{n_i}.$$

We know that

$$\chi(G_{u_1,u_2,\ldots,u_k}) \geq \sum_{j\in S} u_j.$$

Thus, (8) is at least

$$\begin{aligned} & \frac{1}{l}\sum_{u_1=\hat{n}_1}^{\check{n}_1}\sum_{u_2=\hat{n}_2}^{\check{n}_2}\cdots\sum_{u_k=\hat{n}_k}^{\check{n}_k}\sum_{j\in S} u_j\prod_{i=1}^{k}\binom{l}{u_i} \\ = \; & \frac{1}{l}\sum_{j\in S}\sum_{u_1=\hat{n}_1}^{\check{n}_1}\sum_{u_2=\hat{n}_2}^{\check{n}_2}\cdots\sum_{u_k=\hat{n}_k}^{\check{n}_k} l\binom{l-1}{u_j-1}\prod_{i\neq j,1\leq j\leq k}\binom{l}{u_i}. \end{aligned}$$

The theorem follows from the identity:

$$\binom{l}{n_j - 1} \prod_{i \neq j, 1 \leq j \leq k} \binom{l+1}{n_i} = \sum_{u_1=\hat{n}_1}^{\check{n}_1} \sum_{u_2=\hat{n}_2}^{\check{n}_2} \cdots \sum_{u_k=\hat{n}_k}^{\check{n}_k} \binom{l-1}{u_j - 1} \prod_{i \neq j, 1 \leq j \leq k} \binom{l}{u_i}.$$

This identity can be proven by induction on $k$ using the fact that for $n_i = l+1$ or $n_i = 0$, then $\hat{n}_i = \check{n}_i$, and

$$\binom{l+1}{n_i} = \binom{l}{\hat{n}_i}.$$

In the case of $i = j$, we use the fact that

$$\binom{l}{n_i - 1} = \binom{l-1}{\hat{n}_i - 1}.$$

(Recall that it will never be the case that $n_j = 0$ for any $j \in S$.) Then for $1 \leq n_i \leq l$, we use the fact that

$$\binom{l+1}{n_i} = \binom{l}{\hat{n}_i} + \binom{l}{\check{n}_i}.$$

In the case of $i = j$, we use the fact that

$$\binom{l}{n_i - 1} = \binom{l-1}{\hat{n}_i - 1} + \binom{l-1}{\check{n}_i - 1}.$$

□

# 6 Open Problems

We would like to extend the analysis of the $l$-block algorithm to non-perfect graphs and slowly varying loads $p_i$ at node $i$.

# References

[1] J.L. Adler, W.W. Recker and M.G. McNally, Using interactive simulation to model driver behavior under ATIS. in *Proceedings for the 4th International Conference on Microcomputers in Transportation* (J. Chow, ed.). ASCE. New York, 1993, pp. 344–355.

[2] H. Al-Deek, S. Ishak and A.E. Radwan, The potential impact of advanced traveler information systems (ATIS) on accident rates in an urban transportation network. in *Proceedings for the 4th Vehicle Navigation and Information Systems Conference.* IEEE. Piscataway, NJ, 1993, pp. 633–636.

[3] H. Al-Deek and A. Kanafani, Modeling the benefits of advanced traveler information systems in corridors with incidents. *Transportation Res. Part C*, **1** (1993), 303–324.

[4] M. Bell, *Future directions in traffic signal control*, Transportation Research Part A, 26 (1992), pp. 303–313.

[5] R.D. Bretherton and G.T. Bowen, Recent enhancements to SCOOT–SCOOT version 2.4, in *Proceedings for the 3rd International Conference on Road Traffic Control.* IEE, London, 1990, pp. 95–98.

[6] D. Bullock and C. Hendrickson, *Roadway traffic control software*, IEEE Transactions on Control Systems Technology, 2 (1994), pp. 255-264.

[7] J. Chen, I. Cidon and Y. Ofek, *A local fairness algorithm for gigabit LANs/MANs with spatial reuse*, IEEE Journal on Selected Areas in Communications, 11 (1993), pp. 1183-1192.

[8] S. Chiu and S. Chand, Adaptive traffic signal control using fuzzy logic, in *Proceedings for the 2nd IEEE International Conference on Fuzzy Systems.* IEEE, New York, 1993, pp. 1371–1376.

[9] R.E. Fenton, *IVHS/AHS: Driving into the future*, IEEE Control Systems Magazine, 14:6 (December 1994), pp. 13-20.

[10] N.H. Gartner, OPAC: strategy for demand-responsive decentralized traffic signal control, in *Control, Computers, Communications in Transportation* (J.P. Perrin, ed.). Pergamon. Oxford, UK, 1990, pp. 241–244.

[11] J.J. Henry and J.L. Farges, PRODYN in *Control, Computers, Communications in Transportation* (J.P. Perrin, ed.). Pergamon. Oxford, UK, 1990, pp. 253–255.

[12] D.L. Iglehart, *Extreme values in the GI/G/1 queue*, The Annals of Mathematical Statistics, 43 (1972), pp. 627-635.

[13] S. Irani and V. Leung, *Scheduling with conflicts, and applications to traffic signal control*, in Proceedings of the Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, 1996.

[14] R.M. Karp, *Reducibility among combinatorial problems*, in Complexity of Computer Computations, Plenum Press, New York, 1972.

[15] A.J. Khattak, J.L. Schofer and F.S. Koppelman, Commuters' enroute diversion and return decisions: analysis and implications for advanced traveler information systems. *Transportation Res. Part A*, **27** (1993), 101–111.

[16] E. Koutsoupias and C.H. Papadimitriou, *Beyond competitive analysis*, in 35th Symposium on Foundation of Computer Science, IEEE Computer Society Press, 1994.

[17] P.R. Lowrie, *SCATS, Sydney Co-Ordinated Adaptive Traffic System: a Traffic Responsive Method of Controlling Urban Traffic*, Roads and Traffic Authority, Darlinghurst, NSW, Australia, 1990.

[18] C. Lund and M. Yannakakis, *On the hardness of approximating minimization problems*, in Proceedings of the Twenty-Fifth Annual ACM Symposium on Theory of Computing, 1993.

[19] V. Mauro and C. Di Taranto, UTOPIA in *Control, Computers, Communications in Transportation* (J.P. Perrin, ed.). Pergamon. Oxford, UK, 1990, pp. 245–252.

[20] C. Pappis and E. Mamdani, A fuzzy logic controller for a traffic junction, *IEEE Trans. Systems Man Cybernet.*, **7** (1977), 707–717.

[21] F.S. Roberts, *Discrete Mathematical Models with Applications to Social, Biological, and Environmental Problems*, Prentice Hall, Englewood Cliffs, NJ, 1976.

[22] D.I. Robertson and R.D. Bretherton, Optimizing networks of traffic signals in real time–the SCOOT method, *IEEE Trans. on Vehicular Tech.*, **40** (1991), 11–15.

[23] F. Spitzer, *Principles of Random Walk*, Springer-Verlag, New York, 1964.

[24] K.E. Stoffers, *Scheduling of traffic lights - a new approach*, Transportation Research, 2 (1968), pp. 199-234.

[25] F.V. Webster, *Traffic Signal Settings*, Her Majesty's Stationery Office, London, 1958.

[26] R.L. Wilshire, *Traffic signals*, in Traffic Engineering Handbook, Prentice Hall, Englewood Cliffs, NJ, 1992.

# A Appendix: Proofs of Queuing Theory Results

## A.1 Iglehart's Lemma

Iglehart proves a continuous time version of our lemma where arrival and departure rates from a queue are i.i.d. random variables. In order to prove the discrete time version which we use here, we use the following lemma which can be found in Spitzer, p. 218 [23].

**Lemma 16** *Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables. The $X_i$'s are integers and we will denote the $Prob[X_i = j]$ by $p_j$. We will consider the random walk defined by*

$$S_n = \sum_{i=1}^{n} X_i.$$

*Suppose that $X_i$ is a-periodic,*

$$\sum_{j=1}^{\infty} j p_j < \infty$$

*and $E[X_i] = \mu < 0$. Let $\gamma$ be the positive number satisfying*

$$\sum_{x \in Z} \gamma^x p_x = 1.$$

*Suppose further that*

$$0 < \sum_{x \in Z} x \gamma^x p_x < \infty.$$

*If we define $M = \max\{S_n \mid n \geq 0\}$, then there is a constant $k_1$ such that*

$$\lim_{x \to \infty} Prob[M \geq x] \to k_1 \gamma^{-x}.$$

With this lemma, Iglehart's proof goes through, practically unchanged to yield the following tight characterization of the distribution over $M_m$.

**Lemma 17** *Suppose we have a sequence of identical independantly distributed random variables $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\alpha_i \in \mathcal{Z}$, $E[\alpha_i] < 0$, and $\alpha_i \leq c$ for some constant c. We will denote the $Prob[\alpha_i = x]$ by $p_x$. Then let $\gamma$ be the positive constant which satisfies*

$$\sum_{x \in \mathcal{Z}} \gamma^x p_x = 1.$$

*Define the sequence $\hat{\alpha}_i = \max\{0, \alpha_i + \hat{\alpha}_{i-1}\}$, and let*

$$M_m = \max_{1 \leq i \leq m} \hat{\alpha}_i.$$

*Then*

$$\lim_{m \to \infty} Pr[\ln \gamma M_m - \log km \leq x] = \Lambda^{\frac{1}{\phi}}(x),$$

*where $\Lambda(x) = e^{-e^{-x}}$, $k$ is a constant which depends only on $\alpha_i$, and $\phi = E[\min\{j \mid j > 0, \hat{\alpha}_j = 0\}]$.*

## A.2 Proof of Lemma 4

Let $\gamma$,$\phi$ and $M_m$ be defined as in §A.1. We will show that there is a constant $k$ such that for large enough $m$,

$$\frac{\ln km - \ln \phi}{\ln \gamma} - \frac{1}{\gamma - 1} \leq E[M_m] \leq \frac{\ln km}{\ln \gamma} + \frac{2}{\phi(\gamma - 1)}.$$

Using Lemma 17, for $m$ large enough we have that

$$\begin{aligned} Prob\left[M_m - \frac{\ln km}{\ln \gamma} \geq y\right] &= 1 - \frac{1}{e^{\frac{e^{-y\ln\gamma}}{\phi}}} \\ &\leq 1 - \frac{1}{1 + \frac{2}{\phi \cdot e^{y \ln \gamma}}} \\ &\leq \frac{2}{\phi \cdot e^{y\ln\gamma}}. \end{aligned}$$

Thus,

$$\begin{aligned} E[M_m] - \frac{\ln km}{\ln\gamma} &\leq \sum_{y=1}^{\infty} \frac{2 \cdot e^{-y\ln\gamma}}{\phi} \\ &= \frac{2}{\phi}\sum_{y=1}^{\infty}\gamma^{-y} = \frac{2}{\phi(\gamma-1)}. \end{aligned}$$

For the lower bound, we will use the fact that for $m$ large enough,

$$Prob\left[\frac{\ln km - \ln\phi}{\ln\gamma} - M_m \geq y\right] = e^{-\gamma^y}.$$

Thus,

$$\begin{aligned} \frac{\ln km - \ln\phi}{\ln\gamma} - E[M_m] &\leq \sum_{y=1}^{\infty}\frac{1}{e^{\gamma^y}} \\ &\leq \sum_{y=1}^{\infty}\frac{1}{\gamma^y} = \frac{1}{\gamma - 1}. \end{aligned}$$

□

## A.3 Bounding the Expectation of the Ratio

Iglehart's Lemma gives a very tight characterization of $M_m$ which allows us to prove a bound on the ratio of the expected maximum achieved by two sequences. In order to upper bound the ratio, we need to establish that the maxima of the two sequences are positively correlated. In all of our problems, the maximum accumulation of jobs on an edge or clique in the graph is positively correlated with the performance of the $l$-block algorithm, so the following lemma implies that our bounds hold for the expectation of the ratio as well as the ratio of the expectation.

**Lemma 18** *Suppose we have two random variables $M_n$ and $\bar{M}_n$, each defined by its own random walk as described above. $\gamma$ and $\phi$ are as defined above for $M_n$, and $\bar{\gamma}$ and $\bar{\phi}$ are the corresponding values for $\bar{M}_n$. Suppose that $M_n$ and $\bar{M}_n$ are positively correlated so that for all $x$ and $y$ such that $x \geq y$,*

$$E[M_n \mid \bar{M}_n = x] \geq E[M_n \mid \bar{M}_n = y].$$

*Furthermore, suppose that $\bar{M}_n = 0$ only if $M_n = 0$. Then*

$$\lim_{n\to\infty} E\left[\frac{M_n}{\bar{M}_n}\right] \leq \frac{\ln \bar{\gamma}}{\ln \gamma}(1 + o(1)).$$

**Proof.** From the positive correlation assumption, we have that

$$E\left[\frac{M_n}{\bar{M}_n}\right] \leq E[M_n]E\left[\frac{1}{\bar{M}_n}\right].$$

Let

$$A = \frac{\ln kn - \ln \bar{\phi}}{\ln \bar{\gamma}}.$$

We know that

$$Pr[\bar{M}_n \leq A - x] = e^{-\bar{\gamma}^x} \text{ for } 1 \leq x \leq A - 1.$$

Let

$$X = \sum_{x=1}^{A-1} e^{-\bar{\gamma}^x}.$$

Then $Prob[\bar{M}_n \geq A] \geq 1 - X$. Thus,

$$\begin{aligned} E\left[\frac{1}{\bar{M}_n}\right] &\leq \frac{1-X}{A} + \sum_{x=1}^{A-1} e^{-\bar{\gamma}^x}\left(\frac{1}{A-x}\right) \\ &= \frac{1-X}{A} + \sum_{x=1}^{A-1} e^{-\bar{\gamma}^x}\left(\frac{1}{A}\left(1 + \frac{x}{A-x}\right)\right) \\ &= \frac{1}{A}\left(1 + \sum_{x=1}^{A-1} \frac{xe^{-\bar{\gamma}^x}}{A-x}\right). \end{aligned}$$

We must now establish that

$$\sum_{x=1}^{A-1} \frac{xe^{-\bar{\gamma}^x}}{A-x} = o(1),$$

using the fact that $A$ grows with respect to $n$ while all the other variables are constant. We divide the sum into two parts and bound each part separately:

$$\sum_{x=1}^{A-1} \frac{xe^{-\bar{\gamma}^x}}{A-x} = \sum_{x=1}^{\frac{A}{2}} \frac{xe^{-\bar{\gamma}^x}}{A-x} + \sum_{x=1}^{\frac{A}{2}} \frac{(A-x)e^{-\bar{\gamma}^{A-x}}}{x}.$$

The first summation can be bounded as follows:

$$\sum_{x=1}^{\frac{A}{2}} \frac{xe^{-\bar{\gamma}^x}}{A-x} \leq \frac{2}{A} \sum_{x=1}^{\frac{A}{2}} \frac{x}{\bar{\gamma}^x} \leq \frac{2}{A(\bar{\gamma}-1)^2} = O\left(\frac{1}{A}\right).$$

The second summation can be bounded as follows:

$$\begin{aligned} \sum_{x=1}^{\frac{A}{2}} \frac{(A-x)e^{-\bar{\gamma}^{A-x}}}{x} &\leq \frac{A}{\bar{\gamma}^A} \sum_{x=1}^{\frac{A}{2}} \frac{\bar{\gamma}^x}{x} \\ &\leq \frac{A}{\bar{\gamma}^{\frac{A}{2}}} \sum_{x=1}^{\frac{A}{2}} \frac{1}{x} \\ &\leq \frac{A \ln A}{\bar{\gamma}^{\frac{A}{2}}} = o(1). \end{aligned}$$

Thus, using the result from Lemma 4, we can conclude that

$$E\left[\frac{M_n}{\bar{M}_n}\right] \leq \frac{E[M_n]}{A}(1+o(1)) \leq \frac{\ln \bar{\gamma}}{\ln \gamma}(1+o(1)).$$

□

# B Proof of Lemma 14

In order to prove Lemma 14 we need the following lemma which we state without proof.

**Lemma 19** *Consider $\gamma$ which satisfies*

$$\sum_x p_x \gamma^x = c.$$

*Suppose that*

$$\sum_{-s \leq x \leq b} p_x = \sum_x p_x = 1.$$

*For each pair $(i,j) \in [1,\ldots,s] \times [1,\ldots,b]$, we can find $\alpha_{ij}$ and $\beta_{ij}$ such that*

$$\begin{aligned} \alpha_{ij}\gamma^{-i} + \beta_{ij}\gamma^{j} &= c(\alpha_{ij} + \beta_{ij}) \\ \sum_j \alpha_{ij} &= p_{-i} \\ \sum_i \beta_{ij} &= p_j \end{aligned}$$

We use Lemma 19 to get a set of $\alpha$'s and $\beta$'s such that for any pair $(a,c)$,

$$\alpha_{ac}\hat{\gamma}^{-a} + \beta_{ac}\hat{\gamma}^{c} = (1+A)(\alpha_{ac} + \beta_{ac}).$$

After normalizing, we get that for

$$p = \frac{\beta_{ac}}{\alpha_{ac} + \beta_{ac}},$$

$$(1-p)\hat{\gamma}^{-a} + p\hat{\gamma}^{c} = (1+A).$$

We would like to find a $\bar{\gamma}$ such that for any $a \geq -X$,

$$[(1-p)\hat{\gamma}^{-a} + p\hat{\gamma}^{c}] - [(1-p)\bar{\gamma}^{-a} + p\bar{\gamma}^{c}] \geq \frac{A}{1-P}. \tag{10}$$

If we succeed, then we will have

$$\begin{aligned}
& \sum_i \sum_j [\alpha_{ij}\bar{\gamma}^{-i} + \beta_{ij}\bar{\gamma}^{j}] \\
\leq\ & \sum_{i \geq -X} \sum_j [\alpha_{ij}\bar{\gamma}^{-i} + \beta_{ij}\bar{\gamma}^{j}] + \sum_{i \leq -X} \sum_j [\alpha_{ij}\bar{\gamma}^{-i} + \beta_{ij}\bar{\gamma}^{j}] \\
\leq\ & \sum_{i \geq -X} \sum_j \left(1 + A - \frac{A}{1-P}\right)(\alpha_{ij} + \beta_{ij}) + \sum_{i \leq -X} \sum_j (1+A)(\alpha_{ij} + \beta_{ij}) \\
\leq\ & \sum_{i \geq -X} p_{-i}\left(1 + A - \frac{A}{1-P}\right) + \sum_j (1+A)p_j + \sum_{i \leq -X} p_{-i}(1+A) \\
\leq\ & (1-P)\left(1 + A - \frac{A}{1-P}\right) + (1+A)P \\
=\ & 1.
\end{aligned}$$

Using $(1-p)\hat{\gamma}^{-a} + p\hat{\gamma}^{c} = (1+A)$ to solve for $p$, we get that

$$p = \frac{1 + A - \hat{\gamma}^{-a}}{\hat{\gamma}^{c} - \hat{\gamma}^{-a}}.$$

We use this value to plug into (10). We also make the substitutions $\bar{\gamma} \leftarrow \hat{\gamma}^{1-\epsilon}$, where

$$\epsilon = \frac{\ln(1+3A')}{c \ln \hat{\gamma}} \leq \frac{\ln(1+3A')}{\ln \hat{\gamma}}$$

and

$$A' = \frac{A}{1-P}.$$

This results in the following inequality which we would like to establish:

$$\left(\frac{\hat{\gamma}^{c-a} - \hat{\gamma}^{-a}(1+A)}{\hat{\gamma}^{c} - \hat{\gamma}^{-a}}\right)[1-(1+3A')^{\frac{a}{c}}] + \left(\frac{(1+A)\hat{\gamma}^{c} - \hat{\gamma}^{c-a}}{\hat{\gamma}^{c} - \hat{\gamma}^{-a}}\right)\left(\frac{3A'}{1+3A'}\right) \geq A'.$$

We will make the substitution $\beta = \hat{\gamma}^a$ and

$$k = \frac{c}{a}.$$

Keep in mind the assumptions of the lemma which tell us that

$$\beta = \hat{\gamma}^a \geq \hat{\gamma}^{-X} \geq -6s \geq \frac{6a}{c} = \frac{6}{k}.$$

We want to prove the following:

$$\left(\frac{\beta^{k-1} - \beta^{-1}(1+A)}{\beta^{k} - \beta^{-1}}\right)[1-(1+3A')^{\frac{1}{k}}] + \left(\frac{(1+A)\beta^{k} - \beta^{k-1}}{\beta^{k} - \beta^{-1}}\right)\left(\frac{3A'}{1+3A'}\right) \geq A'.$$

If

$$A' \leq \frac{1}{6},$$

this is true as long as

$$-\frac{3A'}{k}\left(\frac{\beta^{k-1} - \beta^{-1}(1+A)}{\beta^{k} - \beta^{-1}}\right) + 2A'\left(\frac{(1+A)\beta^{k} - \beta^{k-1}}{\beta^{k} - \beta^{-1}}\right) \geq A'.$$

It is sufficient to show that

$$\frac{3}{k\beta}(1-\beta^k)+\beta^k-2\beta^{k-1}+\frac{1}{\beta}\geq 0.$$

Since $k\beta \geq 6$, it is sufficient to verify that

$$\frac{1}{2}(1-\beta^k)+\beta^k-2\beta^{k-1}+\frac{1}{\beta}\geq 0.$$

Since $\beta \geq 4$, the inequality holds. □