Title

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Journal

Physical Review D, 55(12)

ISSN

2470-0010

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Publication Date

1997-06-15

DOI

10.1103/physrevd.55.7960

Peer reviewed

Neutrino oscillations in curved spacetime: A heuristic treatment

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(Received 28 October 1996)

We discuss neutrino oscillations in curved spacetime. Our heuristic approach can accommodate matter effects and gravitational contributions to neutrino spin precession in the presence of a magnetic field. By way of illustration, we perform explicit calculations in the Schwarzschild geometry. In this case, gravitational effects on neutrino oscillations are intimately related to the redshift. We discuss how spacetime curvature could affect the resonance position and adiabaticity of matter-enhanced neutrino flavor conversion.
[S0556-2821(97)04912-6]

PACS number(s): 14.60.Pq, 26.30.+k, 95.30.Sf

I. INTRODUCTION

Two decades ago an experimental connection between quantum mechanics and gravitation was observed [1]—essentially, a gravitational analogue of the Aharanov-Bohm effect. These neutron interferometry experiments can be well described by including a gravitational potential energy in the Hamiltonian of the nonrelativistic Schrödinger equation. More recently, the effects of gravitation on another quantum-mechanical phenomenon—neutrino oscillations—have been discussed by a number of authors [2–5], some of whose results appear to conflict. In this paper we will provide a simple framework for studying neutrino oscillations in curved spacetime. We hope to clarify some issues left unclear by previous treatments [2–4], and provide a more transparent route to some of the results obtained in Ref. [5] by more formal methods than we employ here.

As an illustration of our treatment, we will do explicit calculations in Schwarzschild geometry. For radially propagating neutrinos, the oscillation formulas will be different from those that appear in flat spacetime. This is not surprising, since the gravitational redshift for radial neutrino propagation is well known [6]. The existence of a circular orbit for massless particles in the Schwarzschild geometry allows computation of the oscillation phase for purely azimuthal propagation as well. In this case, it will be seen that the oscillation formulas are identical to those obtained in flat spacetime.

We hope our calculations will clarify some issues that we feel have been left unclear by previous treatments of gravitational effects on neutrino oscillations. In Ref. [7], a semiclassical approximation—in which the action of a massive particle is taken as a quantum phase—is employed to analyze gravitationally induced fringe shifts in interference experiments. The authors of Ref. [2] apply this technique to neutrino oscillations, employing the action of a massive particle as a quantum phase for each mass eigenstate. In Ref. [4] some weaknesses of this treatment are addressed, but neither of these papers discusses matter effects or gravitational effects on the spin of the neutrinos [5] (see Sec. III). Furthermore, Refs. [2,7] separate out a ''gravitational contribution'' to the neutrino oscillation phase. Such a separation is only possible for weak fields, making the ''gravitationally induced phase'' a concept of limited utility. Related to this, some authors have used methods that appear to mix flat- and curved-spacetime thinking [2,3]. These particular treatments can leave the reader confused about the precise meaning of quantities such as the ''energy,'' and the nature of the coordinates (i.e., do they reflect proper time and distance?). Covariant calculations do not suffer from these difficulties of interpretation, and are thus preferable.

In Sec. II we formulate a standard treatment of neutrino oscillations in a more geometric framework. In Sec. III we generalize our treatment to curved spacetime, with calculations in Schwarzschild geometry in Sec. IV (vacuum oscillations) and Sec. V [Mikheyev-Smirnov-Wolfenstein (MSW) effect]. Conclusions are given in Sec. VI. We set $G=\hbar=c=1$ throughout this paper.

II. SIMPLE GEOMETRIC TREATMENT OF NEUTRINO OSCILLATIONS: FLAT SPACETIME

In this section we briefly review a simple, standard treatment of neutrino oscillations [8], and then present a geometric version.

In a standard treatment, the neutrino state is written

$$|\Psi_\alpha(x,t)\rangle=\sum_j U_{\alpha j}\exp[-i(Et-P_jx)]|\nu_j\rangle. \qquad (1)$$

Here flavor (mass) indices are in Greek (Latin) letters. The matrix elements $U_{\alpha j}$ comprise the transformation between the flavor and mass bases. The subscript α on the left-hand side indicates that the neutrino was in flavor state α at the initial position $x=0$ and time $t=0$. The mass eigenstates are taken to be energy eigenstates with a common energy E;[1] the three-momenta of the mass eigenstates are then

$$P_j=\sqrt{E^2-m_j^2}\approx E-\frac{m_j^2}{2E}, \qquad (2)$$

[1]For discussion on whether neutrino mass eigenstates should be considered momentum eigenstates, energy eigenstates, or neither, see, for example, Refs. [3,9] and references therein. These technicalities are not crucial in the present context.

where m_i is the rest mass corresponding to mass eigenstate $|\nu_i\rangle$. To compute the oscillation probability at position x, a massless neutrino trajectory is assumed, i.e., $x=t$:

$$|\Psi_\alpha(x,x)\rangle=\sum_j U_{\alpha j}\exp\left[-i\left(\frac{m_j^2}{2E}\right)x\right]|\nu_j\rangle. \tag{3}$$

This state is then used to compute the oscillation amplitude. We note that the assumption of a null trajectory is necessary for the observation of oscillations; if the mass eigenstates could be measured at different positions (or times), the interference pattern would be destroyed.[2]

While the form of Eq. (1) may not openly suggest it, the neutrino state at a given point in spacetime is frame-invariant. The quantities on the right-hand side of Eq. (1)—the transformation coefficients between flavor and mass bases, the phase, the mass eigenstates—all are frame independent quantities. Since the connection between quantities in flat and curved spacetime is most apparent when expressions are written in a manifestly covariant manner, we introduce a generalized form of Eq. (1):

$$|\Psi_\alpha(\lambda)\rangle=\sum_j U_{\alpha j}\exp\left(i\int_{\lambda_0}^{\lambda}\vec{P}\cdot\vec{p}_{\rm null}d\lambda\right)|\nu_j\rangle. \tag{4}$$

In this expression, $\vec{P}$ is the four-momentum operator that generates spacetime translation of the mass eigenstates. The quantity $\vec{p}_{\rm null}=d\vec{x}/d\lambda$ is the (null) tangent vector to the neutrino's world line $\vec{x}(\lambda)=[t(\lambda),x(\lambda),y(\lambda),z(\lambda)]$; λ is an affine parameter of the world line.

We now show that Eq. (4) is equivalent to Eq. (1), by simplifying Eq. (4) for neutrino propagation in the x direction. Let $i\Omega$ denote the argument of the exponential in Eq. (4). With $\vec{P}=(E,P^x,0,0)$, and employing the metric $\eta_{\mu\nu}=\mathrm{diag}[-1,1,1,1]$, we have

$$\Omega=-\int_{\lambda_0}^{\lambda}\left(E\frac{dt}{d\lambda}-P_x\frac{dx}{d\lambda}\right)d\lambda, \tag{5}$$

where

$$P_x=\eta_{x\mu}P^\mu\simeq E-\frac{M^2}{2E}, \tag{6}$$

and M is the mass operator. After the mass operator has done its work, we have

$$\omega_j=-\int_{\lambda_0}^{\lambda}\left[E\frac{(dt/d\lambda)}{(dx/d\lambda)}-\left(E-\frac{m_j^2}{2E}\right)\right]\frac{dx}{d\lambda}d\lambda, \tag{7}$$

where ω_j is the phase of the jth mass eigenstate. Since

$$\frac{(dt/d\lambda)}{(dx/d\lambda)}=\frac{p^t_{\rm null}}{p^x_{\rm null}}=1, \tag{8}$$

Eq. (7) reduces to

$$\omega_j=-\int_{x_0}^{x}\frac{m_j^2}{2E}dx=-\frac{m_j^2}{2E}(x-x_0). \tag{9}$$

This phase agrees with that in Eq. (3) (in which $x_0=0$), suggesting that the neutrino state as written in Eq. (4) is suitable for calculating the vacuum oscillation amplitude. Furthermore, the form of Eq. (4) suggests a straightforward generalization to curved spacetime.

We now review how contributions to effective neutrino mass arising from neutrino forward scattering off background matter can be included in the above formalism. These effects are important because they can give rise to, for example, the MSW effect. Our treatment is essentially that found in Ref. [10].

As an example, we take neutrino propagation through an electron background. In this case the Dirac equation can be cast in the form

$$[\gamma^\mu(\partial_\mu+iA_{f\mu}\mathcal{P}_L)+M_f]\psi_f=0. \tag{10}$$

(See Ref. [11] for the convention for the Dirac matrices γ^μ that we employ.) Here ψ_f is a column vector of spinors of different neutrino flavors, and M_f is the vacuum mass matrix in the flavor basis:

$$M_f^2=U\begin{pmatrix}m_1^2 & 0\\ 0 & m_2^2\end{pmatrix}U^\dagger. \tag{11}$$

with

$$U=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}. \tag{12}$$

The vector A_f^μ is the flavor-basis effective potential matrix for an interaction with the electron background:

$$A_f^\mu=\begin{pmatrix}-\sqrt{2}G_F N_e^\mu & 0\\ 0 & 0\end{pmatrix}. \tag{13}$$

In this expression, G_F is the Fermi constant, and $N_e^\mu=n_e u^\mu$ is the number current of the electron fluid; n_e is the electron density in the fluid rest frame, and u^μ is the fluid's four-velocity. $\mathcal{P}_L$ is the left-handed projection operator. [An index labeling the neutrino's helicity must now be included in the eigenkets of Eq. (4).] The form of Eq. (10) suggests that the mass shell relation read

$$(P^\mu+A_f^\mu\mathcal{P}_L)(P_\mu+A_{f\mu}\mathcal{P}_L)=-M_f^2. \tag{14}$$

This expression can be derived by iteration of the Dirac equation, with the assumption that the neutrino momentum is much larger than the inverse scale height of the background matter. Assuming that the electron background is at rest with

[2]One might argue that to account for observation of the interference pattern, an *average* (rather than null) trajectory should be employed; see Ref. [3]. In this case we would have $x=(\overline{P}/E)t$, where $\overline{P}$ is the average momentum of the mass eigenstates having a common energy E. However, this reasonable argument has no consequences for our study, since it merely introduces an overall phase common to all eigenstates. Here will we follow (perhaps unfortunate) tradition and employ null trajectories throughout this paper.

respect to the oscillation experiment, keeping only terms to first order in G_F, and writing $\vec{P}=(E,P^x,0,0)$, we find

$$P_x \simeq E - \frac{1}{2E}(M_f^2 - V_f), \tag{15}$$

with

$$V_f = \begin{pmatrix} -2\sqrt{2}G_F E n_e \mathcal{P}_L & 0 \\ 0 & 0 \end{pmatrix}. \tag{16}$$

The three-momentum operator P_x now includes effective mass contributions from background matter, and can be used in Eq. (5).

III. SIMPLE GEOMETRIC TREATMENT OF NEUTRINO OSCILLATIONS: CURVED SPACETIME

In applying Eq. (4) in curved spacetime, we see that evaluation of the argument in the exponential can become more involved. This complication results from the dependence of the metric on position. Additionally, we may worry about another gravitational effect: since gravitational fields can cause gyroscopes to precess, perhaps gravitational fields can also cause neutrino spin flips [5].

How can effects on spin be incorporated into Eq. (4)? Gravitational effects on the spin arise through the ''spin connection'' Γ_μ appearing in the Dirac equation in curved spacetime [12] (we here ignore background matter effects):

$$[\gamma^a e_a^\mu (\partial_\mu + \Gamma_\mu) + M]\psi = 0. \tag{17}$$

In this equation and in the rest of this section, Greek indices refer to general curvilinear coordinates, while the Latin indices a,b,c,d refer to locally inertial (Minkowski) coordinates. The tetrads e_a^μ connect these sets of coordinates. The explicit expression for Γ_μ is

$$\Gamma_\mu = \tfrac{1}{8}[\gamma^b,\gamma^c] e_b^\nu e_{c\nu;\mu}. \tag{18}$$

Effects on spin can be incorporated into the three-momentum operator [such as Eq. (6)] in an analogous manner to background matter effects.

We must first simplify the Dirac matrix product in the spin connection term. It can be shown that

$$\gamma^a[\gamma^b,\gamma^c] = 2\eta^{ab}\gamma^c - 2\eta^{ac}\gamma^b - 2i\epsilon^{dabc}\gamma_5\gamma_d, \tag{19}$$

where η^{ab} is the metric in flat space and ϵ^{abcd} is the (flat space) totally antisymmetric tensor, with $\epsilon^{0123}=+1$. With Eq. (19), the nonvanishing contribution from the spin connection is

$$\gamma^a e_a^\mu \Gamma_\mu = \gamma^a e_a^\mu \left\{ iA_{G\mu}\left[-(-g)^{-1/2}\frac{\gamma_5}{2}\right]\right\}, \tag{20}$$

where

$$A_G^\mu \equiv \frac{1}{4}(-g)^{1/2} e_a^\mu \epsilon^{abcd}(e_{b\nu,\sigma} - e_{b\sigma,\nu}) e_c^\nu e_d^\sigma. \tag{21}$$

In these equations, $(-g)^{1/2}=[\det(g_{\mu\nu})]^{1/2}$, where $g_{\mu\nu}$ is the metric of curved spacetime. The expression in Eq. (20) treats left- and right-handed states differently. In order to group it with terms arising from matter effects, we can without physical consequence add a term proportional to the identity to obtain

$$\gamma^a e_a^\mu \Gamma_\mu = \gamma^a e_a^\mu (iA_{G\mu}\mathcal{P}_L). \tag{22}$$

Proceeding as in the discussion of matter effects in the last section, the three-momentum operator used in neutrino oscillation calculations can be computed from the mass shell condition

$$(P_\mu + A_{G\mu}\mathcal{P}_L)(P^\mu + A_G^\mu \mathcal{P}_L) = -M^2, \tag{23}$$

where we have not included background matter effects.

An important point is that the gravitational contribution A_G^μ is proportional to the identity matrix in flavor space, and diagonal in spin space. It cannot induce spin flips on its own. Therefore, it will not have any observable effects unless there are other off-diagonal terms in spin space (e.g., from the interaction of a neutrino magnetic moment with a magnetic field) [5].

Another complication in applying Eq. (4) in curved spacetime is related to the nature of the neutrino trajectories. In flat spacetime, the neutrino trajectories are straight lines. The propagation can be taken to be in one spatial dimension, and the variable of integration becomes the spatial variable corresponding to that direction of propagation, as in Sec. II. However, the neutrino trajectories in curved spacetime are typically parametrized curves involving more than one spatial variable: $\vec{x}(\lambda)=[x^0(\lambda),x^1(\lambda),x^2(\lambda),x^3(\lambda)]$.

For general trajectories it therefore may be convenient to leave the affine parameter λ as the variable of integration, as in Eq. (4). The tangent vector to the null world line, $\vec{p}\equiv\vec{p}_{\rm null}=d\vec{x}/d\lambda$, can be found from the geodesic equation or Hamilton-Jacobi equation. The four-momentum operator $\vec{P}$ can be constructed as follows: (1) take the neutrinos to be energy eigenstates, and set $P^0=p^0$; (2) demand that the three-momenta of $\vec{P}$ and $\vec{p}$ be parallel, i.e., $P^i=p^i(1-\epsilon)$ with $i=1,2,3$; and (3) $(P_\mu+A_\mu\mathcal{P}_L)(P^\mu+A^\mu\mathcal{P}_L)=-M^2$, with A^μ now representing both matter and ''spin connection'' contributions. For relativistic neutrinos ($\epsilon\ll 1$), ignoring terms of $O(A^2)$ and $O(AM^2)$, and remembering that $\vec{p}$ is a null vector, we find

$$(g_{0i}p^0p^i + g_{ij}p^ip^j)\epsilon = \frac{M^2}{2} + \vec{p}\cdot\vec{A}\mathcal{P}_L. \tag{24}$$

(Here the indices i,j refer to the spacelike general curvilinear coordinates, not locally inertial coordinates.) From this it follows that the quantity $\vec{P}\cdot\vec{p}$ appearing in Eq. (4) is simply

$$\vec{P}\cdot\vec{p} = -\left(\frac{M^2}{2} + \vec{p}\cdot\vec{A}\mathcal{P}_L\right). \tag{25}$$

It is convenient to define a column vector of flavor amplitudes. For example, for mixing between ν_e and ν_τ,

$$\chi(\lambda) \equiv \begin{pmatrix} \langle \nu_e|\Psi(\lambda)\rangle \\ \langle \nu_\tau|\Psi(\lambda)\rangle \end{pmatrix}. \tag{26}$$

Equation (4) can be written as a differential equation for $\chi(\lambda)$,

$$i\frac{d\chi}{d\lambda}=\left(\frac{M_f^2}{2}+\vec{p}\cdot\vec{A}_f\mathcal{P}_L\right)\chi, \tag{27}$$

where the subscript f denotes ''flavor basis.'' Equation (27) can be integrated (numerically if necessary) to yield the neutrino flavor evolution. A similar equation was obtained in Ref. [5] by more formal methods.

IV. NEUTRINO OSCILLATIONS IN SCHWARZSCHILD SPACETIME: VACUUM OSCILLATIONS

While Eq. (27) may be useful for calculating the neutrino flavor evolution for general neutrino trajectories in general spacetimes, it does not yield a great deal of physical insight. In this and the following section we do example calculations in Schwarzschild geometry. This example geometry is simple enough that the oscillation formulas can be cast in a form that resembles the flat space case.

Before proceeding, we wish to emphasize that we consider vacuum neutrino oscillations in curved spacetime for pedagogical purposes only. In supernovae—a potential physical application of neutrino oscillations in curved spacetime—matter effects dominate, making vacuum oscillations irrelevant. We discuss matter effects on neutrino oscillations in curved spacetime and a possible application to supernovae in the following sections. Recognizing, however, that gravitational effects on vacuum neutrino oscillations may be of interest at some point in the future, we include in the Appendix a discussion of requirements on the neutrino wave packet necessary for the observation of vacuum oscillations in curved spacetime.

In this section we contrast radially propagating neutrinos with azimuthally propagating neutrinos in order to demonstrate how gravity affects vacuum neutrino oscillations. The geometry in a spherically symmetric, static spacetime can be globally represented by the Schwarzschild coordinate system $\{x^\mu\}\Rightarrow(t,r,\theta,\phi)$. We can take the Schwarzschild line element, which serves to define these coordinates, as

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=-e^{2\Phi(r)}dt^2+e^{2\Lambda(r)}dr^2+r^2d\theta^2 +r^2\sin^2\theta d\phi^2. \tag{28}$$

Using the tetrads

$$e_a^\mu=\mathrm{diag}\left[e^{-\Phi(r)},e^{-\Lambda(r)},\frac{1}{r},\frac{1}{r\sin\theta}\right], \tag{29}$$

direct calculation yields $A_G^\mu=0$. This is perhaps expected from spherical symmetry, and is in agreement with Ref. [5], in which terms arising from the spin connection vanish in the Schwarzschild geometry. Since the components of the metric are independent of the timelike coordinate t, there is a conserved quantity, the timelike covariant momentum component $P_t\equiv-E_*$. (The relation between covariant and contravariant components is $P_\mu=g_{\mu\nu}P^\nu$.) We take the neutrino states to be eigenstates of this quantity.

Denoting a differential physical distance at constant t by $d\ell$, we can write

$$d\lambda=d\ell\left(g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\right)^{-1/2}=d\ell\left[-g_{00}\left(\frac{dx^0}{d\lambda}\right)^2\right]^{-1/2}, \tag{30}$$

where we have used the facts that the neutrino trajectory is null and that the Schwarzschild metric does not mix time and space components. Using this expression for $d\lambda$ and Eq. (25), we obtain in vacuum

$$\Omega=\int_{\lambda_0}^{\lambda}\vec{P}\cdot\vec{p}_{\mathrm{null}}d\lambda=-\int_{\ell_0}^{\ell}\frac{M^2}{2E_l}d\ell. \tag{31}$$

In this equation, $E_l=E_*e^{-\Phi(r)}$ is the energy measured by a locally inertial observer momentarily at rest in the Schwarzschild spacetime (and presumably at rest with respect to the ''oscillation experiment''). Therefore, the integrand in Eq. (31) is formally the same as the corresponding integrand in flat space.

For radial propagation, $d\ell=e^{\Lambda(r)}dr$ is a differential element of physical distance for constant t,θ,ϕ, and so

$$\Omega=-\int_{\ell_0}^{\ell}\frac{M^2}{2E_l}d\ell=-\int_{r_0}^{r}\frac{M^2}{2E_*e^{-\Phi(r)}}e^{\Lambda(r)}dr. \tag{32}$$

We see that unlike the flat space case, the integral in terms of physical distance is not trivial, due to the gravitational redshift of the ''local energy'' E_l and the radial dependence of $d\ell$. In this manner, spacetime curvature (gravity) makes its impact on the oscillation amplitude.

Of course, in vacuum above a spherical, static source of gravitational mass $\mathcal{M}$, we have

$$e^{2\Phi(r)}=\left(1-\frac{r_s}{r}\right), \tag{33}$$

$$e^{2\Lambda(r)}=\left(1-\frac{r_s}{r}\right)^{-1}, \tag{34}$$

where the Schwarzschild radius is $r_s\equiv2\mathcal{M}$. Then Eq. (32) is trivially integrated:

$$\Omega=-\frac{M^2}{2E_*}(r-r_0). \tag{35}$$

Again, the coordinate difference $(r-r_0)$ does *not* reflect a physical distance. [The *physical* distance ℓ corresponding to the *coordinate* difference $(r-r_0)$ is $\ell=\int_{r_0}^{r}\sqrt{g_{rr}}dr=\int_{r_0}^{r}e^{\Lambda(r)}dr$.] Likewise, E_* does *not* represent the neutrino energy measured by a locally inertial observer at rest at finite radius, but rather the energy of the neutrino measured by such an observer at rest at infinity. It is generally not possible to extract a separate ''gravitational phase'' from this expression; nevertheless, it is clear that gravity has an effect on the oscillations of radially propagating neutrinos. In the weak field limit one could define a ''gravitational phase,'' however.

In the Schwarzschild spacetime there are circular orbits of radius $r=R\equiv3\mathcal{M}$ for massless particles. Consideration of neutrino oscillations in such an orbit (which we take to be in

the plane defined by $\sin\theta=1$) can provide insight into gravitational effects in the azimuthal direction.

For azimuthal propagation in this orbit, $d\ell = R d\phi$, and the local energy $E_l = E_*(1-r_s/R)^{-1/2}$ is constant along the neutrino trajectory, in contrast to the case of radial propagation. From Eq. (31) we find

$$\Omega = -\frac{M^2}{2E_l} R(\phi-\phi_0). \tag{36}$$

This expression involves local energy and physical distance, making it precisely the same as the corresponding flat space expression. Gravity has no effect here.[3]

V. NEUTRINO OSCILLATIONS IN SCHWARZSCHILD SPACETIME: MSW EFFECT

In this section we study an example of gravitational effects on MSW resonant neutrino transformations [13,14]. In particular, in Schwarzschild spacetime we find the resonance position and calculate the adiabaticity parameter for a radially propagating two-flavor neutrino system in an electron background with monotonically decreasing density profile.

It will here be convenient to write the neutrino evolution equation in terms of column vectors of flavor amplitudes:

$$\chi(r) \equiv \begin{pmatrix} \langle \nu_e | \Psi(r) \rangle \\ \langle \nu_\tau | \Psi(r) \rangle \end{pmatrix}. \tag{37}$$

For radial propagation we obtain

$$\chi(r) = \exp\left[-i \int_{r_0}^{r} \frac{1}{2E_*} [e^{\Phi(r)} M_f^2 + V_f(r)] e^{\Lambda(r)} dr \right] \chi(r_0). \tag{38}$$

In this equation, M_f^2 is the vacuum mass matrix in the flavor basis. The contribution from the background matter is

$$V_f(r) = \begin{pmatrix} v(r) & 0 \\ 0 & 0 \end{pmatrix}, \tag{39}$$

with $v(r) = 2\sqrt{2} G_F E_* n_e$ and $n_e = u_\mu N_e^\mu$ is the locally measured electron density.

Equation (38) can also be written as a Schrödinger-like equation,

$$i\frac{d\chi(r)}{dr} = \frac{\widetilde{M}_f^2}{2E_*} \chi(r), \tag{40}$$

where the effective mass matrix in the flavor basis is

$$\widetilde{M}_f^2 = e^{[\Lambda(r)+\Phi(r)]} M_f^2 + e^{\Lambda(r)} V_f(r). \tag{41}$$

The mixing angle in matter, $\widetilde{\theta}$, is defined in terms of the diagonalization of $\widetilde{M}_f^2$:

$$\widetilde{M}^2 = \widetilde{U}^\dagger \widetilde{M}_f^2 \widetilde{U} = \begin{pmatrix} \widetilde{m}_1^2 & 0 \\ 0 & \widetilde{m}_2^2 \end{pmatrix}, \tag{42}$$

$$\widetilde{U} = \begin{pmatrix} \cos\widetilde{\theta} & \sin\widetilde{\theta} \\ -\sin\widetilde{\theta} & \cos\widetilde{\theta} \end{pmatrix}. \tag{43}$$

The difference of the squares of the neutrino mass eigenvalues in matter, $\widetilde{\Delta} \equiv \widetilde{m}_2^2 - \widetilde{m}_1^2$, is given in terms of $\Delta \equiv m_2^2 - m_1^2$ as

$$\widetilde{\Delta} = e^{\Lambda} [(v - e^{\Phi}\Delta\cos 2\theta)^2 + (e^{\Phi}\Delta \sin 2\theta)^2]^{1/2}. \tag{44}$$

The mixing angle in matter is given by

$$\tan 2\widetilde{\theta} = \frac{e^{\Phi}\Delta \sin 2\theta}{-v + e^{\Phi}\Delta\cos 2\theta}. \tag{45}$$

The ''resonance'' occurs for $\sin^2 2\widetilde{\theta} = 1$, where a mass level crossing occurs and $\widetilde{\Delta}$ is a minimum. The resonance condition is $v(r) = e^{\Phi(r)}\Delta\cos 2\theta$, or

$$\sqrt{2} G_F n_e = \frac{\Delta}{2e^{-\Phi}E_*}\cos 2\theta = \frac{\Delta}{2E_l}\cos 2\theta, \tag{46}$$

where $E_l = -\vec{p}\cdot\vec{u} = E_* e^{-\Phi}$ is the redshifting ''local energy'' introduced in the last section. Unlike flat space, the resonance condition is here determined in part by an energy which may be redshifted from the energy the neutrino was born with at the production site.

After dropping a term proportional to the identity matrix that yields an overall phase, Eq. (40) can be written in the basis of instantaneous mass eigenstates as

$$i\frac{d\widetilde{\chi}(r)}{dr} = \begin{pmatrix} -\widetilde{\Delta}/4E_* & -i d\widetilde{\theta}/dr \\ i d\widetilde{\theta}/dr & \widetilde{\Delta}/4E_* \end{pmatrix} \widetilde{\chi}(r), \tag{47}$$

where

$$\widetilde{\chi}(r) = \begin{pmatrix} \langle \widetilde{\nu}_1 | \Psi(r) \rangle \\ \langle \widetilde{\nu}_2 | \Psi(r) \rangle \end{pmatrix}, \tag{48}$$

and $|\widetilde{\nu}_1\rangle$ and $|\widetilde{\nu}_2\rangle$ are the instantaneous mass eigenstates. The adiabaticity parameter $\gamma(r)$, which compares the relative magnitudes of the diagonal and off-diagonal terms in Eq. (47), is defined to be

$$\gamma(r) = \frac{\widetilde{\Delta}}{4E_* |d\widetilde{\theta}/dr|}. \tag{49}$$

For $\gamma \gg 1$, the neutrino evolution can be approximated by a constant superposition of slowly varying instantaneous mass eigenstates, except for a small probability $\exp[-(\pi/2)\gamma(r_{\rm res})]$ for one mass eigenstate to jump to the other at resonance, where $r_{\rm res}$ is the position of the resonance [14]. We find

$$\gamma(r_{\rm res}) = \frac{\Delta \sin^2 2\theta}{e^{-\Phi}E_* \cos 2\theta} \left| e^{-\Lambda} \frac{d}{dr} \ln(e^{-\Phi} n_e) \right|^{-1}$$

[3]We note that a ''gravitational phase'' for motion transverse to the radial direction is given in Ref. [2], in contrast with our result. The discrepancy arises because those authors attempt to use a gravitational potential energy instead of employing covariant methods.

$$= \frac{\Delta \sin^2 2\theta}{E_l \cos 2\theta} \left| e^{-\Lambda} \frac{d}{dr} \ln(-\vec{p}\cdot\vec{A}) \right|^{-1}, \qquad (50)$$

where $\vec{A}$ is the matter potential four-vector introduced in Sec. I. Thus the adiabaticity of the evolving neutrino (the degree to which the ''jump probability'' is unimportant) is affected by the spatial dependence of the metric, which appears for example in the determination of the local energy.

VI. CONCLUSION

We have developed a simple formalism for treating neutrino oscillations in curved spacetime. This formalism can accomodate matter effects and gravitational contributions to neutrino spin precession in the presence of a magnetic field.

We have done explicit calculations in Schwarzschild spacetime. Our simple formalism has verified the result of Ref. [5] that gravitational contributions to spin precession vanish in spherically symmetric, static Schwarzschild spacetime. However, we have found that the oscillation formulas for radially propagating neutrinos are altered by gravity. This alteration results from the metrical properties of curved spacetime. The basis of the effects we found are closely related to the gravitational redshift, in the case of both vacuum oscillations and the MSW effect. In contrast, azimuthally propagating neutrinos show no alterations to their oscillation formulas in the spherically symmetric, static case.

In applications where strong gravitational effects on neutrino oscillations are of possible interest (e.g., supernovae), matter effects will generally make vacuum oscillations (and therefore gravitational effects on the vacuum oscillation phase) unimportant. However, gravitational effects on the resonance position and adiabaticity of the MSW effect are of potential interest.

For example, the requirement for successful r-process nucleosynthesis in a neutron-rich post–core-bounce supernova environment has been used to delineate values of neutrino mass difference and mixing angle which favor and/or disfavor heavy element production [15]. These limits arise because the ν_μ and ν_τ neutrinos emitted from the supernova have a higher average energy than the emitted ν_e neutrinos. Therefore, a MSW resonant transformation of ν_μ or ν_τ neutrinos provides a population of higher energy ν_e neutrinos that tend to drive the material outside the nascent neutron star toward less neutron-rich conditions. In order to preclude r-process nucleosynthesis, the MSW transformation must be sufficiently adiabatic (conversion efficiency $\sim 30\%$), and must occur before the radius where the neutron-to-proton ratio freezes out (''weak freeze-out radius''). As we saw in Sec. V, gravitation affects both the adiabaticity parameter and the position of the resonance.

Current supernova models indicate that the general relativistic effects we consider here are probably not very important. However, the equation of state of nuclear matter is not well understood, and it may be that during the time frame of interest for nucleosynthesis some proto-neutron stars may become very relativistic [16]. If this turns out to be the case, we may hazard the following conjectures regarding gravitational effects on limits on neutrino mass difference and mixing angle from r-process considerations. The MSW transformation may become more adiabatic because of the redshifting energy appearing in the denominator of the adiabaticity parameter. This would extend the limit on neutrino mixing to exclude smaller values of the vacuum mixing angle. On the other hand, the smallest neutrino mass difference excluded by r-process considerations is, roughly, that mass difference for which the resonance position of an average energy μ (or τ) neutrino coincides with the weak freeze-out radius. The redshifting energy appearing in the resonance condition will tend to pull the resonance position closer to the neutron star.[4] However, the weak freeze-out radius is determined by the competition between the weak interaction rates and the expansion rate of the material outside the the neutron star, and the weak interaction rates are proportional to the *square* of the redshifting neutrino energy. Therefore, for a given mass difference, the redshift will reduce the separation of the resonance position and the weak freeze-out radius. This would *weaken* the mass difference boundary of the excluded region. Of course, these conjectures are preliminary, as they are based principally on redshift effects. Other effects of a supernova core of sufficiently small radius for gravitational effects to become interesting may be relevant. Such effects might include changes in the density scale height and expansion rate of the background matter in the supernova envelope, and alteration of the neutrino spectrum. In fact, if the ν_e and $\overline{\nu_e}$ are significantly redshifted, then r-process nucleosynthesis may be precluded anyway [6].

ACKNOWLEDGMENTS

We are grateful for helpful conversations with A. B. Balantekin, T. Bhattacharya, T. Goldman, S. Habib, H. J. Lipkin, E. Mottola, and Y.-Z. Qian. This work was supported by Grant Nos. NSF PHY95-03384 and NASA NAG5-3062 at UCSD.

APPENDIX

The purpose of this appendix is to point out a condition on the neutrino wave packet that must be satisfied if neutrino oscillations are to be observed. This condition, which arises from the need for quantum-mechanical interference, is derived here for the case of vacuum neutrino oscillations in curved spacetime. We do not explore similar conditions that may apply to the MSW effect, which is a level crossing phenomenon [14].

Because of the difference in their masses, the wave packets corresponding to the different mass eigenstates will separate with time. If vacuum neutrino oscillations are to be observed, the wave packets must overlap. A simple calculation shows that, in flat spacetime, interference at $x=t$ is possible only if the width Δx of the wave packets corresponding to

[4]This statement assumes a given average neutrino energy, obtained, for example, from supernova models. If one instead assumes a given fixed neutrino energy at infinity (obtained from observations), the resonance position will be pushed further from the neutron star surface by redshift effects. Either way, the qualitative result given below remains the same: redshift effects tend to reduce the separation between the resonance position and the weak freeze-out radius.

the different mass eigenstates satisfies $\Delta x \gtrsim (\Delta m^2/2E^2)x$ after traveling a distance x. For a beam of monoenergetic particles, as in the beamlines of terrestrial neutrino experiments, this condition is easily satisfied. However, if one wanted to observe vacuum neutrino oscillations in some astrophysical environment, one would have to check that this condition would be satisfied.

We now generalize the above condition on the width of the neutrino mass eigenstate wave packets to curved spacetime. We assume that there is some coordinate system in which the neutrino source and detector are at rest and in which there is no mixing of time and space components. A differential physical distance at constant time x^0 in this coordinate system is given by $d\ell = (g_{ij}dx^i dx^j)^{1/2}$. The desired condition is given by

$$\Delta\ell \gtrsim \int (g_{ij}P_2^i P_2^j)^{1/2} d\lambda - \int (g_{ij}P_1^i P_1^j)^{1/2} d\lambda, \quad \text{(A1)}$$

where (following Sec. III)

$$P_{1,2}^0 = p^0, \quad \text{(A2)}$$

$$P_{1,2}^i = p^i(1-\epsilon_{1,2}). \quad \text{(A3)}$$

In these expressions the subscripts 1,2 denote the two mass eigenstates. The quantities $\epsilon_{1,2}$ are determined from the conditions $p^2 = 0$ and $P_{1,2}^2 = -m_{1,2}^2$, and are given by

$$\epsilon_{1,2} = \frac{m_{1,2}^2}{2g_{ij}p^i p^j}. \quad \text{(A4)}$$

Putting this all together, we find

$$\Delta\ell \gtrsim \frac{\Delta m^2}{2} \int (g_{ij}p^i p^j)^{-1/2} d\lambda \quad \text{(A5)}$$

$$= \frac{\Delta m^2}{2} \int \frac{d\ell}{(g_{ij}p^i p^j)} \quad \text{(A6)}$$

$$= \frac{\Delta m^2}{2} \int \frac{d\ell}{[-g_{00}(p^0)^2]}. \quad \text{(A7)}$$

In flat spacetime, this obviously corresponds to the condition given in the previous paragraph. For radial propagation in Schwarzschild geometry,

$$\Delta\ell \gtrsim \frac{\Delta m^2}{2} \int \frac{d\ell}{E_l^2} \quad \text{(A8)}$$

$$= \frac{\Delta m^2}{2} \int \frac{e^{\Lambda(r)}dr}{e^{-2\Phi(r)}E_*^2} \quad \text{(A9)}$$

$$= \frac{\Delta m^2}{2E_*^2} \int \sqrt{1-\frac{r_s}{r}}\, dr. \quad \text{(A10)}$$

[1] R. Collela, A. W. Overhauser, and S. A. Werner, Phys. Rev. Lett. **34**, 1472 (1975).

[2] D. V. Ahluwalia and C. Burgard, Gen. Relativ. Gravit. **28**, 1161 (1996); D. V. Ahluwalia and C. Burgard, Los Alamos Report No. LA-UR-96-2031, gr-qc/9606031, 1996 (unpublished).

[3] Y. Grossman and H. J. Lipkin, Phys. Rev. D **55**, 2760 (1997).

[4] T. Bhattacharya, S. Habib, and E. Mottola, Los Alamos Report No. gr-qc/9605074, 1996 (unpublished).

[5] D. Píriz, M. Roy, and J. Wudka, Phys. Rev. D **54**, 1587 (1996).

[6] G. M. Fuller and Y.-Z. Qian, Nucl. Phys. **A606**, 167 (1996).

[7] L. Stodolsky, Gen. Relativ. Gravit. **11**, 391 (1979).

[8] See, e.g., B. Kayser, Phys. Rev. D **24**, 110 (1981).

[9] T. Goldman, Los Alamos Report No. LA-UR-96-1349, hep-ph/9604357, 1996 (unpublished).

[10] G. M. Fuller, R. W. Mayle, J. R. Wilson, and D. N. Schramm, Astrophys. J. **322**, 795 (1987).

[11] S. W. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, England, 1995), Sec. 5.4.

[12] See, e.g., S. W. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), Sec. 12.5; D. R. Brill and J. A. Wheeler, Rev. Mod. Phys. **29**, 465 (1957).

[13] L. Wolfenstein, Phys. Rev. D **17**, 2369 (1978); S. Mikheyev and A. Yu. Smirnov, Nuovo Cimento Soc. Ital. Fis. C **9**, 17 (1986).

[14] W. C. Haxton, Phys. Rev. D **35**, 2352 (1987); A. B. Balantekin, S. H. Fricke, and P. J. Hatchell, *ibid.* **38**, 935 (1988).

[15] G. M. Fuller, Phys. Rep. **227**, 149 (1993); Y.-Z. Qian, G. M. Fuller, G. J. Mathews, R. W. Mayle, J. R. Wilson, and S. E. Woosley, Phys. Rev. Lett. **71**, 1965 (1993); Y.-Z. Qian and G. M. Fuller, Phys. Rev. D **51**, 1479 (1995).

[16] G. E. Brown and H. A. Bethe, Astrophys. J. **423**, 659 (1994).