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Hirsch, MW

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# COMMON FIXED POINTS FOR TWO COMMUTING SURFACE HOMEOMORPHISMS

MORRIS W. HIRSCH

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Abstract. Let $f$ and $g$ be orientation-preserving surface homeomorphisms that commute under composition. Conditions are found ensuring that the fixed point set of $f$ contains a fixed or periodic point for $g$. Proofs are based on Brouwer's Plane Translation Theorem and the Cartwright-Littlewood Fixed Point Theorem.

## 1. Introduction

Throughout this paper we make the following assumptions:
Standing Hypothesis:

- $M$ is a connected oriented surface, with empty boundary unless the contrary is indicated
- $f$ and $g$ belong to the group $\mathcal{H}_+(M)$ of orientation-preserving homeomorphisms of $M$
- $f$ and $g$ commute under composition: $g(f(x)) = f(g(x))$ for all $x \in M$.
- The fixed point set $\mathsf{Fix}\,(f)$ is nonempty.

The main question is:

*Under what conditions do $f$ and $g$ have a common fixed point?*

More generally, we seek conditions guaranteeing that $\mathsf{Fix}\,(g^k) \cap \mathsf{Fix}\,(f) \neq \varnothing$ for some $k > 0$, with a bound on $k$. The main results are in Section 4 and 5.

To focus ideas, we hazard the following conjecture:

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**Conjecture 1.1.** *Two commuting, orientation-preserving homeomorphisms of $\mathbf{R}^2$ have a common fixed point, provided the fixed point set of one of them is nonempty and compact.*

The analogous conjecture for the unit sphere $\mathbf{S}^2 \subset \mathbf{R}^3$ is false:

**Example 1.2.** Let $f$ and $g$ be the rotations of $\mathbf{S}^2$ having the diagonal matrices with diagonal entries $(1,-1,-1)$ and $(-1,-1,1)$. Then $\mathsf{Fix}\,(f) \cap \mathsf{Fix}\,(g) = \varnothing$.

This well known counterexample refutes other plausible conjectures. It suggests a curious question: Are there four commuting, orientation-preserving homeomorphisms of $\mathbf{S}^2$, no two of which have a common fixed point?

**Structure of the paper.** The main results assume that $\mathsf{Fix}\,(f)$ has at at least one and most finitely many compact components. The general method for proving $\mathsf{Fix}\,(g^k) \cap \mathsf{Fix}\,(f) \neq \varnothing$ is as follows:

- **(1):** Find an open cell $E$ invariant under $f$ and some iterate $g^k$, $k \geq 1$, such that $E \cap \mathsf{Fix}\,(f)$ is compact and the frontier $\mathsf{Fr}(E)$ lies in $\mathsf{Fix}\,(f)$.
- **(2):** If $E \cap \mathsf{Fix}\,(f) = \varnothing$, then $E$ is disjoint from the nonwandering set $\mathsf{NW}(f)$ by a corollary of Brouwer's Plane Translation Theorem (see 2.1). Suitable assumptions imply $\mathsf{Clos}(E) \cap \mathsf{Fix}\,(g^k) \neq \varnothing$, and there is usually an explicit upper bound for $k$. The Complementary Cell Principle 3.2 is used to conclude that $\mathsf{Fr}(E)$ meets $\mathsf{Fix}\,(g^k) \cap \mathsf{Fix}\,(f)$.
- **(3):** If $E \cap \mathsf{Fix}\,(f) \neq \varnothing$, let $K \subset E$ be a compact component of $\mathsf{Fix}\,(f)$. There exists $k \in \mathbf{N}_+$ such that $g^k(K) = K$. If $K$ is acyclic, the Cartwright-Littlewood Theorem 2.2 implies $\mathsf{Fix}\,(g^k) \cap K \neq \varnothing$. If no component of $E \cap \mathsf{Fix}\,(f)$ is acyclic, then $E$ can be replaced by a different open cell permitting step (2) to be carried out. Here the Acyclic Hull Principle 2.5 is used.

After a brief discussion of earlier work, terminology and basic topological and dynamical concepts are reviewed. Basic tools are developed in Sections 2 and 3.

Section 5 shows that for general surfaces, $\mathsf{Per}(g)$ meets certain Nielsen classes in $\mathsf{Fix}\,(f)$.

Many theorems adapt readily to surfaces with nonempty boundary by assuming $\mathsf{Fix}\,(f)$ does not meet the boundary and working with $M \setminus \partial M$, or by doubling $M$ along its boundary; in a few cases this is made explicit.

**Earlier work.** On the question of existence of a common fixed point for groups of homeomorphisms, diverse sufficient conditions have been found in a variety of settings.

Shields [37] proved there is a common fixed point for any family of commuting homeomorphisms of the closed unit disk that are holomorphic in the open disk. Several generalizations have been obtained, including work by Abate [1], Behan [4], Bracci [5], Eustice [17], Kuczumow *et al.* [28], Suffridge [39] and Tauraso [40].

Lima's pioneering work [30, 31, 32] showed that commuting flows on a compact surface of nonzero Euler characteristic have a common fixed point; for a recent proof, see Turiel [42]. Lima's theorem was extended to nilpotent Lie groups by Plante [36], and to analytic actions of supersoluble Lie groups by Hirsch & Weinstein [26].

Bonatti [9] proved that two commuting real analytic flows on a compact 4-manifold of nonzero Euler characteristic must have a common fixed point.

Hirsch [23] considered a commuting family $\mathcal{F}$ of real analytic, orientation-preserving homeomorphisms of $\mathbf{R}^2$, and showed that if the fixed point set of $h \in \mathcal{F}$ is nonempty and compact, then some point of $\mathsf{Fix}\,(h)$ is periodic under every element of $\mathcal{F}$. Pairs of commuting homeomorphisms of $\mathbf{R}^2$ and $\mathbf{S}^2$ were treated briefly in Hirsch [24] (see Proposition 4.7 below).

Hu [27] investigated ergodic properties of commuting diffeomorphisms on manifolds of arbitrary dimension.

Perhaps the deepest results are due to Bonatti [8] and Handel [21]. Bonatti proved there is a neighborhood $U$ of the identity map in the group $\mathsf{Diff}_+(\mathbf{S}^2)$ of orientation-preserving $C^1$ diffeomorphisms of $\mathbf{S}^2$, such that any commuting family in $U$ has a common fixed point. Handel [21] extended Bonatti's theorem by means of an invariant $\mathcal{W}(f,g)$ (credited to J. Mather) in the fundamental group $\pi_1(\mathcal{H}_+(\mathbf{S}^2)) \cong \mathbf{Z}_2$ of the orientation-preserving homeomorphisms of $\mathbf{S}^2$, defined as follows. Given $f, g \in \mathcal{H}_+(\mathbf{S}^2)$, let $f_t$ and $g_t$ be isotopies $f_t$ and $g_t$ from $f$ and $g$ to the identity, and define $\mathcal{W}(f,g)$ to be the homotopy class of the loop $f_t g_t f_t^{-1} g_t^{-1}$. Handel proved:

**Theorem 1.3** (Handel's Theorem)**.** *Let $f, g \in \mathcal{H}_+(\mathbf{S}^2)$ commute, and suppose $\mathcal{W}(f,g) = 0$. If $f$ and $g$ are $C^1$ diffeomorphisms, or have finite fixed point sets, then* $\mathsf{Fix}\,(g) \cap \mathsf{Fix}\,(f) \neq \varnothing$.

This implies Bonatti's theorem; it has the striking consequence that $f$ and $g^2$ always have a common fixed point. Theorem 4.9 is another corollary.

It is remarkable that the theorems of Bonatti and Handel for diffeomorphisms do not need any finiteness assumptions.

**Terminology.** The sets of real numbers, integers, natural numbers, and positive natural numbers are denoted respectively by $\mathbf{R}, \mathbf{Z}$, $\mathbf{N}$ and $\mathbf{N}_+$. If $\mathsf{c}$ denotes a cardinal number, we write $\mathsf{c} = \infty$ to mean $\mathsf{c} \geq \aleph_0$. Homeomorphism is indicated by $\approx$, group isomorphism by $\cong$.

The set of connected components of a space $X$ is denoted by $\mathsf{Comp}(X)$. The closure of $S \subset X$ is denoted by $\mathsf{clos}(S)$ or $\overline{S}$ and the interior by $\mathsf{Int}(S)$. The frontier of $S$ is $\mathsf{Fr}(S) = \mathsf{clos}(S) \cap \mathsf{clos}(X \setminus S)$.

The closed unit interval is $I = [0, 1]$. Euclidean n-space is $\mathbf{R}^n$. The Euclidean norm of $x \in \mathbf{R}^n$ is written $||x||$. The closed unit disk in $\mathbf{R}^n$ is $\mathbf{D}^n$; its boundary is the unit sphere $\partial \mathbf{D}^n = \mathbf{S}^{n-1}$. A *disk, circle* or *annulus* means a homeomorph of the standard object $\mathbf{D}^n$, $\mathbf{S}^1$ or $\mathbf{S}^1 \times I$, respectively. An *open cell*, or simply a *cell*, means a homeomorph of $\mathbf{R}^2$.

Surfaces and other manifolds are assumed connected and without boundary unless the contrary is indicated. A manifold is *closed* if it is compact, connected and without boundary.

The surface $M$ has a metric $\mathbf{d}$. The *distance* from $x \in X$ to $S \subset X$ is $\mathsf{dist}(x, S) = \inf_{y \in S} \mathbf{d}(x, y)$.

Area in $M$ is defined by a Borel measure which is positive, finite and nonempty on precompact open sets.

A *continuum* is a nonempty compact, connected space.

A space is *triangulable* if it is homeomorphic to the underlying space of a simplicial complex. Surfaces and analytic varieties are triangulable.

$f$ is *piecewise linear* if there is a triangulation of $M$ having rectilinear subdivisions $\tau_0, \tau_1$ such that $f$ maps each simplex of $\tau_0$ affinely onto a simplex of $\tau_1$.

A set $S \subset M$ is *invariant* under $f$ if $S$ is nonempty and $f(S) = S$. The *orbit* of $x$ is $\gamma(x) = \{f^n(x)\}_{n \in \mathbf{Z}}$; its closure is $\overline{\gamma}(x)$. The omega and alpha *limit sets* of $x$ are respectively $\omega(x) = \bigcap_{j>0} \overline{\gamma}(f^j(x))$ and $\alpha(x) = \bigcap_{j>0} \overline{\gamma}(f^{-j}(x))$. To indicate $f$ we may write $\gamma_f(x)$, etc

$x$ is *recurrent* if $x \in \omega_f(p)$. When $f$ preserves area, Poincaré's Recurrence Theorem implies recurrent points are dense in every open invariant set having finite area.

$x$ is *nonwandering* if every neighborhood $U$ of $x$ meets $f^n(U)$ for some $n > 0$; all other points are *wandering*. The set $\mathsf{NW}(f)$ of nonwandering points is closed and invariant. The open invariant set of wandering points is denoted by $\mathsf{W}(f)$.

$x$ is *chain recurrent* if for every $\epsilon > 0$ there exists $m \in \mathbf{N}_+$ and points $x_0, \dots, x_m$ such that $x_0 = x_m = x$ and $d(x_j, f(x_{j-1})) < \epsilon,\ j = 1, \dots m$. This notion is independent of the metric when $X$ is compact. The set $\mathsf{CR}(f)$ of chain recurrent points is closed and invariant, and contains $\mathsf{NW}(f)$. For chain recurrence see Akin [2], Conley [14], Hirsch & Hurley [25].

An *attractor* for $f$ is a compact, proper invariant set $A$ possessing a neighborhood $W$ such that $\lim_{n\to\infty} \mathsf{dist}(f^n(x), A) = 0$ uniformly for $x \in N$. The union of all such sets $W$ is the *basin* of $A$. No point of the basin is chain recurrent. A *repellor* is an attractor for $f^{-1}$. When $M$ is compact, the complement of the basin of $A$ is a repellor $R$; in this case $(A, R)$ is called a *dual attractor-repellor pair.*

$\check{H}^i(X)$ denotes the $i$th Čech cohomology group of a space $X$ (Spanier [38]). Its rank is the $i$th *Čech number* $\mathsf{c}^i(X)$. Thus $\mathsf{c}^0(X)$ is the number of components of $X$. If $X$ is triangulable, $\mathsf{c}^i(X)$ equals the $i$th Betti number $\mathsf{b}_i(X)$, defined as the rank of the $i$th singular homology group $H_i(X)$.

$X$ has *finite type* if its Čech numbers are finite and have finite sum. In this case we define the *Čech characteristic* $\check{\chi}(X) = \sum(-1)^i \mathsf{c}^i(X)$. When such an $X$ is triangulable, $\check{\chi}(X)$ equals the *Euler characteristic* $\chi(X) = \sum(-1)^i \mathsf{b}_i(X)$. If $X \subsetneqq M$ then $\check{H}^i(X) = \{0\}$ for $i > 1$. The connected surface $M$ has finite type if and only if $H_1(M)$ is finitely generated.

$X$ is *acyclic* if it has the same Čech cohomology as a point; equivalently, $X$ is connected and nonempty and $\check{H}^i(X) = \{0\}$ for $i > 0$. A connected subset of $M$ is acyclic if and only if its Čech characteristic is 1. An acyclic open set in $M$ is an open cell. A compact acyclic analytic variety is a singleton (Borel & Haefliger [7]). The number of acyclic components of $X$ is denoted by $\mathsf{ac}(X)$.

The following fact is of great importance: *A continuum in* $\mathbf{R}^2$ *or* $\mathbf{S}^2$ *is acyclic if and only if its complement is connected.* This is a consequence of the Alexander duality theorem (Dold [16], Theorem VIII.8.15; Spanier [38], Theorem 6.2.16).

Suppose $X$ has finite type and $h\colon X \to X$ is continuous. The *Lefschetz number* $\mathsf{Lef}(h) \in \mathbf{Z}$ is the alternating sum of the traces of the endomorphisms induced by $f$ in the rational cohomology groups $H^i(X)\otimes\mathbf{Q}$. If $X$ is compact and triangulable and $\mathsf{Lef}(h) \neq 0$, Lefschetz's Fixed Point Theorem implies every map sufficiently close to $h$ has a fixed point.

## 2. Basic topological tools

This section presents some basic results used throughout the paper.

**Proposition 2.1** (Brouwer's Nonwandering Theorem)**.** *A homeomorphism of* $\mathbf{R}^2$ *that preserves orientation has a fixed point provided it has a nonwandering point.*

This fundamental result is a corollary of Brouwer's Plane Translation Theorem (Brouwer [10], Franks [18]).

**Proposition 2.2** (Cartwright-Littlewood Fixed Point Theorem). *An orientation-preserving homeomorphism of a surface has a fixed point in any invariant acyclic continuum.*

This was proved for $\mathbf{R}^2$ by Cartwright & Littlewood [13]. There is a two-page proof by Hamilton [20] and a one-page proof by Brown [11]. The extension to general surfaces is given in Hirsch [24], section 4.4.[1] There are generalizations to orientation-reversing homeomorphisms by Bell [6] and Kuperberg [29], and to smooth noninvertible maps by Akis [3].

**Proposition 2.3** (Brown-Kister Invariance Theorem). *Let $V$ a connected orientable manifold with empty boundary, and $h \in \mathcal{H}_+(V)$. Then every component of $V \setminus \mathsf{Fix}\,(h)$ is invariant.*

We need this useful result, due to Brown & Kister [12], only for surfaces.

**Proposition 2.4.**

- **(i):** *Let $E \subset M$ be an open cell with compact nonempty frontier $\mathsf{Fr}(E)$. Then $\mathsf{Clos}(E)$ is compact and $\mathsf{Fr}(E)$ is connected.*
- **(ii):** *Let $W$ be a complementary component of a continuum in $\mathbf{S}^2$, or a bounded complementary component of a continuum in $\mathbf{R}^2$. Then $W$ is a precompact open cell.*
- **(iii):** *Let $U, V \subset M$ be open sets with disjoint frontiers, such that $U \cap V \neq \varnothing$. Then:*
  - **(a):** *the frontier of one of the sets has a component lying in the other set*
  - **(b):** *if $U$ and $V$ are precompact cells, either $\mathsf{Clos}(U) \subset V$, or $\mathsf{Clos}(V) \subset U$, or $U \cup V = M$ and $M \approx \mathbf{S}^2$*
- **(iv):** *Assume $M \not\approx \mathbf{S}^2$ and let $T \subset \mathbf{R}^2$ be an acyclic continuum containing the boundary of an open cell $U$. Then $U \subset T$.*

Proof. A proof of (i) is given in Hirsch [24], Lemma 4.1. Part (ii) is proved in Newman [35], Theorem VI.4.1.

Under the assumptions of (iii) there exists $p \in \mathsf{Fr}\,(U \cap V)$, and $p$ lies in $\mathsf{Fr}(U)$ or $\mathsf{Fr}(V)$ but not both. We assume without loss of generality that $p \in \mathsf{Fr}(U)$. Then $p \in \overline{U} \setminus \mathsf{Fr}(V) \subset V$. Therefore the component of $p$ in $\mathsf{Fr}(U)$ must lie in $V$, proving (a).

[1] In 4.4 of [24], $g$ should be a homeomorphism.

Now assume $U$ and $V$ are precompact cells. Consider the case $U \cup V = M$. By shrinking $U$ and $V$ slightly we obtain compact surfaces $B \subset U, D \subset V$ such that $B \cup D = M$ and $\mathsf{Fr}(B) \subset \mathsf{Int}(D)$. We now have $M$ as the union of compact surfaces $B$ and $D \setminus \mathsf{Int}\, B$ that meet only in their common boundary; thus $M$ is a closed surface. If $U$ and $V$ are cells then $B$ and $D \setminus \mathsf{Int}\, B$ are disks, whence $M \approx \mathbf{S}^2$.

Suppose $U \cup V \neq M$. Then there exists $q \in \mathsf{Fr}\,(U \cup V)$, and $q$ must belong either to $\mathsf{Fr}(U) \setminus V$ or $\mathsf{Fr}(V) \setminus U$, but not both because $\mathsf{Fr}(U)$ and $\mathsf{Fr}(V)$ are disjoint. But $\mathsf{Fr}(U) \subset V$, thus $q \in \mathsf{Fr}(V) \setminus U$. Thus $\mathsf{Fr}(V)$ meets $M \setminus U$; therefore $\mathsf{Fr}(V) \subset M \setminus \mathsf{Clos}(U)$, because $\mathsf{Fr}(V)$ is connected and disjoint from $\mathsf{Fr}(U)$. This shows that the connected set $\mathsf{Clos}(U)$ lies in a complementary component of $\mathsf{Fr}(V)$. As $\mathsf{Clos}(U)$ meets $V$, we have $\mathsf{Clos}(U) \subset V$. This completes the proof of (b).

To prove (iv), let $\epsilon > 0$. Expand $T$ slightly to a polyhedral disk $B$ (Hirsch [24], Proposition 4.5), and shrink $U$ slightly to the interior of a polyhedral disk $D$ with $\mathsf{Fr}(D) \subset \mathsf{Int}(B)$, and such that $T$ and $U$ lie in the $\epsilon$-neighborhoods of $B$ and $D$, respectively. The interiors of $D$ and $B$ are overlapping cells in $M$ with disjoint boundaries. Therefore the closure of one lies in the other by (iii). This implies $D \subset B$. Since $\epsilon$ is arbitrary, it follows that $U \subset T$. □

Let $E$ be an open cell and $R \subset E$ a continuum. Among the components of $E \setminus R$ there is a unique component $W$ whose closure in $E$ is not compact; when $E = M = \mathbf{R}^2$, this is the unbounded complementary component of $R$. The *acyclic hull of $R$ in $E$* is the union $\mathsf{A}_E(R)$ of $R$ and the precompact components of $E \setminus R$; equivalently, $\mathsf{A}_E(R) = E \setminus W$. When $E = \mathbf{R}^2$ we set $\mathsf{A}(R) = \mathsf{A}_{\mathbf{R}^2}(R)$.

**Proposition 2.5** (Acyclic Hull Principle)**.** *Let $R \subset \mathbf{R}^2$ be a continuum and $W$ its unbounded complementary component.*

- **(a):** $\mathsf{A}(R)$ *is an acyclic continuum,* $\mathsf{Fr}\,(\mathsf{A}(R))$ *is a continuum, and* $\mathsf{Fr}\,(\mathsf{A}(R)) = \mathsf{Fr}\,(W)$.
- **(b):** $\mathsf{A}(R)$ *lies in every acyclic continuum containing $R$. Therefore $R$ is acyclic* $\Leftrightarrow \mathsf{A}(R) = R$*; and* $\mathsf{A}(R)$ *lies in the convex hull of $R$.*
- **(c):** *If $Q$ and $R$ are disjoint continua in the plane, then* $\mathsf{A}(Q) \cap \mathsf{A}(R) = \varnothing$*, or* $\mathsf{A}(Q) \subset \mathsf{Int}\,\mathsf{A}(R)$*, or* $\mathsf{A}(R) \subset \mathsf{Int}\,\mathsf{A}(Q)$.

Proof. **(a)** The identity $\mathsf{A}(R) = \mathbf{R}^2 \setminus W$ shows that $\mathsf{A}(R)$ and its frontier are compact and $\mathsf{Fr}(\mathsf{A}(R)) = \mathsf{Fr}(W)$; and also that $\mathsf{A}(R)$ is acyclic because $W$ is connected.

**(b)** Suppose $R$ lies in an acyclic continuum $T \subset \mathbf{R}^2$. Let $U$ be a bounded complementary component of $R$. Then $\mathsf{Fr}\,(U) \subset T$, so $U \subset T$ by Proposition 2.4(iv).

**(c)** Suppose $\mathsf{A}(Q) \cap \mathsf{A}(R) \neq \varnothing$. As $Q \cap R = \varnothing$, there exist bounded open sets $U \in \mathsf{Comp}(\mathbf{R}^1 \setminus Q)$ and $V \in \mathsf{Comp}(\mathbf{R}^1 \setminus R)$ such that $U \cap V \neq \varnothing$ and $\mathsf{Fr}(U) \cap \mathsf{Fr}(V) \neq \varnothing$. Observe that $U$ and $V$ are cells by 2.4(ii); now apply 2.4(iii). □

A cell $E \subset M$ is *complementary* to $X \subset M$ if $E$ is a component of $M \setminus X$. When $X$ is compact, every complementary cell is precompact with connected frontier (Proposition 2.4(i)).

The following result is a paraphrase of Hirsch [24], Theorem 4.6:

**Proposition 2.6.** *Assume $M$ has finite type. Let $X \subset M$ be compact and nonempty with $\mathsf{c}^0(X) < \infty$. Then $X$ admits a complementary cell in each of the following cases:*

- **(a):** *$X$ has finite type and $\check{\chi}(X) < \chi(M)$*
- **(b):** *$X$ is not of finite type*
- **(c):** *$M \setminus X$ is not of finite type* □

**Proposition 2.7.** *Let $X \subset \mathbf{R}^2$ be a nonempty closed set of which no complementary component is a cell. If $K$ is a compact component of $X$, then $\mathsf{A}(K)$ contains an acyclic component of $X$.*

Proof. Let $U$ be a bounded complementary component of $K$. Because $U$ is a cell (Proposition 2.4(ii)), the hypothesis implies $X \cap U$ is a nonempty compact set whose components are components of $X$. By Zorn 's Lemma, there is a family $\mathcal{C}$ of components of $X$ whose acyclic hulls form a nested family $\mathcal{F}$ of acyclic continua, and $\mathcal{C}$ is maximal for this property. Let $Z$ denote the intersection of the elements of $\mathcal{F}$. Then $Z$ is acyclic continuum, owing to the continuity property of Čech cohomology. It is not hard to see that $\dot{Z}$ lies in a component $J$ of $K$, and $J \in \mathcal{C}$ by maximality of $\mathcal{C}$. I claim $J$ is acyclic. Otherwise, choose any component $E$ of $\mathsf{A}(J) \setminus J$, necessarily an open cell. It is not hard to show that $E$ is a bounded complementary component of $J$, so $E$ meets $X$ by hypothesis. Therefore $E$ contains a component $Q$ of $K$; but $\mathcal{C} \cup \{Q\}$ contradicts the maximality of $\mathcal{C}$. □

The dynamical significance of cells complementary to $\mathsf{Fix}\,(f)$ is shown by the next result, in which $M$ may have nonempty boundary:

**Proposition 2.8.** *If $E$ is a precompact cell complementary to $\mathsf{Fix}\,(f)$, every point of $E$ is wandering and chain recurrent for $f$.*

PROOF. If $\partial M = \varnothing$, this is a special case of Lemma 4.2 of Hirsch [24]. The general case follows by the *doubling trick*, i.e., extending $f$ to a homeomorphism of a boundary-free surface consisting of two copies of $M$ meeting only along their common boundary $\partial M$. □

Consider the following subsets of $\mathcal{H}_+(M)$:

$$\mathcal{P}_+(M) = \{h \in \mathcal{H}_+(M)\colon \ \mathsf{Fix}\,(h) \text{ meets every precompact invariant open cell}\}$$

$$\mathcal{Q}_+(M) = \{h \in \mathcal{H}_+(M)\colon \ \mathsf{NW}(h) \text{ meets every precompact invariant open set}\}$$

These are rather broad classes of maps:

**Proposition 2.9.** *If $h \in \mathcal{H}_+(M)$, then $h \in \mathcal{Q}_+(M)$ provided one of the following conditions holds:*

- **(a):** *every point with compact orbit closure is nonwandering*
- **(b):** *area is preserved*
- **(c):** *there is a nowhere dense global attractor*
- **(d):** *there is an attractor-repellor pair $(A, R)$ such that $A \cup R$ is nowhere dense*

*Moreover $\mathcal{Q}_+(M) \subset \mathcal{P}_+(M)$.*

PROOF. Case (a) is obvious. When (b) holds, for every precompact open set $W$ we have $\mathsf{CR}(h) \cap W \neq \varnothing$ by Poincaré's Recurrence Theorem. Under (c) there is no precompact invariant open set, so the defining condition of $\mathcal{Q}_+(M)$ is vacuously satisfied by $h$; likewise for (d).

For the final conclusion, suppose $h \in \mathcal{Q}_+(M)$ and let $E$ be a precompact cell invariant by $h$. Poincaré's Recurrence Theorem implies $\mathsf{NW}(f) \cap E \neq \varnothing$, whence $\mathsf{Fix}\,(h) \cap E \neq \varnothing$ by Brouwer's Nonwandering Theorem 2.1. □

**Proposition 2.10.** *Let $E \subset M$ be an open cell. Assume $f \in \mathcal{P}_+(M)$ and $K \in \mathsf{Comp}(\mathsf{Fix}\,(f))$ is compact. If $K$ lies in an open cell $E$, then $\mathsf{A}_E(K)$ contains an acyclic component of $\mathsf{Fix}\,(f)$.*

PROOF. Note that $E \approx \mathbf{R}^2$, set $X = \mathsf{Fix}\,(f) \cap E$, and apply Proposition 2.7. □

**Proposition 2.11.** *Let $E \subset M$ be a precompact cell with triangulable closure. If a homeomorphism $E \approx E$ extends to a continuous map $h\colon \ \mathsf{Clos}(E) \to \mathsf{Clos}(E)$, then $h$ has a fixed point.*

PROOF. Note that $\mathsf{Fr}(E)$ and $\mathsf{Clos}(E)$ are invariant. Denote by $\tau', \tau$ and $\tau''$ respectively the alternating sum of the traces of the endomorphisms induced by $h$ in the singular homology groups of $\mathsf{Fr}(E)$, $\mathsf{Clos}(E)$ and $(\mathsf{Clos}(E), \mathsf{Fr}(E))$. Thus $\tau'$

and $\tau$ are the Lefschetz numbers of $h|\,\mathsf{Fr}(E)$ and $h|\,\mathsf{Clos}(E)$ respectively. Exactness of the homology sequence of the pair $(\mathsf{Clos}(E), \mathsf{Fr}(E))$ implies $\tau = \tau' + \tau''$.

We will show that $\tau'' \neq 0$. Now $H_n(\mathsf{Clos}(E), \mathsf{Fr}(E)) \cong \mathbf{Z}$ and all other homology groups of $(\mathsf{Clos}(E), \mathsf{Fr}(E))$ vanish, because the quotient space $S = \mathsf{Clos}(E)/\,\mathsf{Fr}(E)$, obtained by collapsing $\mathsf{Fr}(E)$ to a point, is a sphere. $\tau''$ is the trace of homology endomorphism in dimension $n$ of the map $S \to S$ induced by $h$. Because $h|E$ is a homeomorphism, $\tau'' \in \{\pm 1\}$.

It follows that $\tau$ or $\tau'$ is nonzero. As $\mathsf{Clos}(E)$ and $\mathsf{Fr}(E)$ are triangulable, the conclusion follows from Lefschetz's Fixed Point Theorem. □

**Example 2.12.** The conclusion of Proposition 2.11 is not valid if $\mathsf{Clos}(E)$ is not assumed triangulable. Consider a Denjoy homeomorphism $u\colon\ \mathbf{S}^1 \approx \mathbf{S}^1$, meaning that there is a invariant Cantor set $C$ which is the alpha and omega limit set of every orbit (Denjoy [15]). Suspending $u$ gives a flow $\Phi$ on a torus $V$. The complement in $V$ of the suspension of $C$ is an open 2-cell $E$ which is dense in the torus. Evidently the time-one map of $\Phi$ leaves $E$ invariant, but has no fixed point.

It is not known whether there is a nontriangulable counterexample to Proposition 2.11 in $\mathbf{R}^2$ or $\mathbf{S}^2$. A flow construction similar to Example 2.12 will not work, owing to the Poincaré-Bendixson theorem. Hence we are led to:

**Conjecture 2.13.** *Let $E \subset \mathbf{S}^2$ be an open cell. If a homeomorphism of $E$ extends to a continuous map $h\colon\ \mathsf{Clos}(E) \to \mathsf{Clos}(E)$, then $h$ has a fixed point.*

## 3. Complementary cell principles

In this section we obtain important methods for locating common fixed points. Most results are easily extended to surfaces with boundary.

**Lemma 3.1.** *Let $E$ be a precompact cell complementary to $\mathsf{Fix}\,(f)$. If $g(E)$ meets $E$, then $g(E) = E$.*

Proof. $E$ is $f$-invariant by the Brown-Kister Invariance Theorem 2.3, and $M \setminus \mathsf{Fix}\,(f)$ is $g$-invariant because $g$ is a homeomorphism that commutes with $f$. Therefore both $E$ and $g(E)$ are components of $M \setminus \mathsf{Fix}\,(f)$; as they intersect, they coincide. □

**Theorem 3.2** (Complementary Cell Principle)**.** *Let $E$ be a precompact cell that is complementary to $\mathsf{Fix}\,(f)$ and contains a nonwandering point of $g$. Then $\mathsf{Fix}\,(g) \cap E \neq \varnothing$ and $\mathsf{Fix}\,(g) \cap \mathsf{Fix}\,(f) \cap \mathsf{Fr}(E) \neq \varnothing$. For all $p \in \mathsf{Fix}\,(g) \cap E$:*

**(a):** *$\gamma_f(p)$ is an infinite discrete subset of $E \cap \mathsf{Fix}\,(g) \cap \mathsf{CR}(f) \cap \mathsf{W}(f)$*

**(b):** *$\omega_f(p)$ and $\alpha_f(p)$ are nonempty subsets of* $\mathsf{Fix}\,(g) \cap \mathsf{Fix}\,(f) \cap \mathsf{Fr}(E)$

PROOF. $E$ is $f$-invariant by the Brown-Kister Invariance Theorem 2.3, and $g$-invariant by Lemma 3.1. Brouwer's Nonwandering Theorem 2.1, applied to $g\colon E \approx E$, proves $\mathsf{Fix}\,(g) \cap E \neq \varnothing$.

The orbit $\gamma_f(p)$ is a nonempty $f$-invariant subset of $\mathsf{Fix}\,(g) \cap E$. The limit sets $\alpha_f(p)$ and $\omega_f(p)$ are nonempty because $\overline{E}$ is compact. Brouwer's Nonwandering Theorem 2.1, applied to $f\colon E \approx E$, implies $\gamma_f(p)$ has no limit points in $E$, and Proposition 2.8 shows that $E \subset \mathsf{CR}(f) \cap \mathsf{W}(f)$. This proves (a), and (b) follows. ☐

$\mathsf{Fix}\,(f)$ is triangulable provided $f$ is analytic, and in other cases as well (see Proposition 4.12). When this holds, $M$ has a triangulation in which components of $\mathsf{Fix}\,(f)$, and closures of complementary components of $\mathsf{Fix}\,(f)$, are subcomplexes. In this situation there is a conclusion similar to the Complementary Cell Principle 3.2, even without assuming that $g$ has a nonwandering point in $E$:

**Theorem 3.3.** *Let $E$ be a precompact cell complementary to* $\mathsf{Fix}\,(f)$ *and invariant under $g$. If* $\mathsf{Clos}(E)$ *is triangulable, then* $\mathsf{Fix}\,(g) \cap \mathsf{Fix}\,(f) \cap \mathsf{Fr}(E) \neq \varnothing$.

PROOF. There exists $p \in \mathsf{Fix}\,(g) \cap \mathsf{Clos}(E)$ by Proposition 2.11. If $p \in \mathsf{Fr}(E)$ the conclusion follows. If $p \in \mathsf{Int}(E)$, apply the Complementary Cell Principle. ☐

The following theorem is applicable to area-preserving $g$:

**Theorem 3.4.** *Assume* $\mathsf{Fix}\,(f)$ *is compact and $E$ is a precompact open cell complementary to* $\mathsf{Fix}\,(f)$. *Let $g \in \mathcal{Q}_+(M)$. Then* $\mathsf{Fix}\,(g) \cap E \neq \varnothing$ *is infinite and* $\mathsf{Fix}\,(g) \cap \mathsf{Fix}\,(f) \cap \mathsf{Fr}(E)$ *is nonempty.*

PROOF. Every iterate $g^n$ maps $E$ onto another complementary component of $\mathsf{Fix}\,(f)$, and the frontier of $g^n(E)$ lies in the compact set $\mathsf{Fix}\,(f)$. This implies the set $W = \bigcup_{j\in\mathbf{Z}} g^j(E)$ has compact closure. Therefore $\mathsf{NW}(g) \cap W \neq \varnothing$ because $W$ is $g$ invariant and $g \in \mathcal{Q}_+(M)$. Let $p \in \mathsf{NW}(g) \cap g^j(E)$. Then $g^{-n}(p) \in \mathsf{NW}(g) \cap E$, whence $\mathsf{Fix}\,(g) \cap E \neq \varnothing$ by Brouwer's Nonwandering Theorem 2.1. Now apply the Complementary Cell Principle 3.2. ☐

A folk theorem states that when $M$ is compact with a smooth area form, there is Baire set of area-preserving $C^1$ diffeomorphisms having discrete fixed point sets. In this sense, the condition on $h$ in the following negative result is satisfied by a generic generic set of area-preserving diffeomorph of a compact surface:

**Theorem 3.5.** *Assume* $\mathsf{Fix}\,(f)$ *is compact and some component of* $M \setminus \mathsf{Fix}\,(f)$ *is an open cell (see Proposition 2.6). If* $h \in \mathcal{Q}_+(M)$ *and* $\mathsf{Fix}\,(h)$ *is discrete, then* $h$ *does* not *commute with* $f$*.*

Proof. Otherwise Theorem 3.4, applied to $g = h$, would lead to the contradiction that the compact set $\mathsf{Clos}(E)$ contains infinitely many fixed points of $h$. ☐

## 4. Main results

Throughout this section we assume:

**Hypothesis 4.1.** $\mathsf{Fix}\,(f)$ *has exactly* $\nu \in \mathbf{N}_+$ *compact components (and perhaps some noncompact components).*

This is a severe restriction, but it holds whenever $f$ is analytic or piecewise linear and $\mathsf{Fix}\,(f)$ is compact.

**Proposition 4.2.** *Let* $L \in \mathsf{Comp}(\mathsf{Fix}\,(f))$ *be compact. There exists a smallest* $k \in \{1, \dots, \nu\}$ *such that* $g^k(L) = L$*; and* $k = 1$ *if* $g$ *is sufficiently close to* $\{f^j\}_{j \in \mathbf{N}_+}$*.*

Proof. $g$ induces a permutation of the set of compact components of $\mathsf{Fix}\,(f)$, which has cardinality $\nu$; this proves the first conclusion. Choose a neighborhood $N$ of $L$ containing no other compact component of $\mathsf{Fix}\,(f)$. If $j \in \mathbf{N}_+$ and $g$ is sufficiently close to $f^j$ in the compact open topology, then $g(L)$ is a component of $\mathsf{Fix}\,(f)$ contained in $N$, so $g(L) = L$. ☐

We denote the $k$ described in Proposition 4.2 by $\mathbf{k}(L) \in \{1, \dots, \nu\}$.

**Theorem 4.3.** *Assume* $\mathsf{Fix}\,(f)$ *is compact,* $M$ *and* $\mathsf{Fix}\,(f)$ *have finite type, and* $\chi(M) > \check{\chi}(\mathsf{Fix}\,(f))$*. If* $g \in \mathcal{Q}_+(M)$*, then* $\mathsf{Fix}\,(g)$ *is infinite and* $\mathsf{Fix}\,(g) \cap \mathsf{Fix}\,(f)$ *is nonempty.*

Proof. $\mathsf{Fix}\,(f)$ admits a precompact complementary open cell $E$ by Proposition 2.6. Now apply Theorem 3.4. ☐

**Proposition 4.4.** *Let* $K$ *be a compact component of* $\mathsf{Fix}\,(f)$ *lying in a precompact open cell* $U$*.*

- **(i):** $\mathsf{A}_U(K)$ *contains a component* $L$ *of* $\mathsf{Fix}\,(f)$ *such that* $\mathsf{Fix}\,(g^{\mathbf{k}(L)}) \cap L \neq \varnothing$*.*
- **(ii):** *If* $f \in \mathcal{P}_+(M)$*, then* $L$ *can be chosen to be acyclic.*

Proof. Finiteness of $\nu$ implies $\mathsf{Fix}\,(f)$ has a compact component $L \subset \mathsf{A}_U(K)$ such that $\mathsf{Fix}\,(f) \cap (\mathsf{A}_U(L) \setminus L) = \varnothing$. This means every precompact component of $\mathsf{A}_U(L) \setminus L$ is an open cell complementary to $\mathsf{Fix}\,(f)$.

Choose such an $L$ and set $k = \mathbf{k}(L)$. If $L$ is acyclic, $\mathsf{Fix}\,(g^k) \cap L \neq \varnothing$ by the Cartwright-Littlewood Theorem. This proves (ii), because when $f \in \mathcal{P}_+(M)$, we can choose $L$ to be acyclic by Proposition 2.10.

Suppose $L$ is not acyclic. Consider the case $g^k(\mathsf{A}_U(L)) = \mathsf{A}_U(L)$. As $\mathsf{A}_U(L)$ is compact and acyclic by the Acyclic Hull Principle 2.5, there exists $q \in \mathsf{Fix}\,(g^k \cap \mathsf{A}_U(L))$ by the Cartwright-Littlewood Theorem 2.2. If $q \in L$ the proof is over. Assume $q$ belongs to a component $E$ of $\mathsf{A}_U(L) \setminus L$, necessarily a precompact open cell. $E$ is complementary to $\mathsf{Fix}\,(f)$ by the choice of $L$. The Brown-Kister Theorem 2.3 implies $E$ is $f$-invariant, and $E$ is $g^k$-invariant because $g^k(E)$ and $E$ are components of $\mathsf{A}_U(L) \setminus L$ that contain $p$. Therefore $\mathsf{Fix}\,(g^k) \cap L \neq \varnothing$ by the Complementary Cell Principle 3.2.

Now assume $g^k(\mathsf{A}_U(L)) \neq \mathsf{A}_U(L)$. There exists a precompact component $V$ of $\mathsf{A}_U(L) \setminus L$ such that $g^k(V)$ is not a precompact component of $\mathsf{A}_U(L) \setminus L$. Now $\mathsf{Fr}(V) \subset L$, so $\mathsf{Fr}(g(V)) \subset L$. This implies $g(V)$ is a precompact cell complementary to $\mathsf{A}_U(L)$. Therefore $M$ is the disjoint union of the precompact open cell $g^k(V)$ and the acyclic continuum $\mathsf{A}_U(L)$.

It follows that $M \approx \mathbf{S}^2$, so there exists $p \in \mathsf{Fix}\,(g)$. If $p \in L$ there is nothing more to prove. If $p \notin L$ then $p$ lies in a component $E$ of $M \setminus L$. As $M \approx \mathbf{S}^2$, we see that $E$ is a precompact cell. The Complementary Cell Principle proves $\mathsf{Fix}\,(g) \cap L \neq \varnothing$. This proves (i), and Proposition 2.10 implies (ii). □

**Theorem 4.5.** *Assume $M$ is $\mathbf{R}^2$, $\mathbf{S}^2$ or $\mathbf{D}^2$. Then there is a compact component $L$ of $\mathsf{Fix}\,(f)$ such that $\mathsf{Fix}\,(g^{\mathbf{k}(L)}) \cap L \neq \varnothing$.*

Since $\mathbf{k}(L) = 1$ when $\mathsf{Fix}\,(f)$ is connected, we have:

**Corollary 4.6.** *If $\mathsf{Fix}\,(f)$ is connected, it meets $\mathsf{Fix}\,(g)$.*

Proof of Theorem 4.5. By the doubling trick we may assume $M = \mathbf{R}^2$ or $\mathbf{S}^2$. Let $K \in \mathsf{Comp}(\mathsf{Fix}\,(f))$ be compact and apply Proposition 4.4(i) to an open cell $U$ containing $K$. □

It is interesting to compare the preceding theorems to the following result from Hirsch [24]:

**Proposition 4.7.** *Take $M = \mathbf{R}^2$ or $\mathbf{S}^2$. Assume $f \in \mathcal{P}_+(M)$, with $\mathsf{Fix}\,(f)$ compact and $0 < \mathsf{c}^0(\mathsf{Fix}\,(f)) < \infty$. Then:*

- **(i):** *$\mathsf{Fix}\,(f)$ and its complement have finite type*
- **(ii):** *$\mathsf{Fix}\,(f)$ has an acyclic component, and two acyclic components when $M = \mathbf{S}^2$*
- **(iii):** *each acyclic component $L$ of $\mathsf{Fix}\,(f)$ meets $\mathsf{Fix}\,(g^{\mathbf{k}(L)})$.*

Proof. (i) proved in [24], Theorem 2.2(i) (see also Proposition 2.6). Conclusion (ii) follows from [24], Corollary 3.13 and its proof. Statement 1(iii), which does not require $f \in \mathcal{P}_+(M)$, is a consequence of the Cartwright-Littlewood Theorem. □

In 4.7, $\mathsf{Fix}\,(f)$ is assumed not to have any complementary cells. In 4.3, on the other hand, such cells are mandatory for $f$, but forbidden to $g$.

The following result extends Proposition 4.7 to surfaces of finite type:

**Theorem 4.8.** *Assume :*

- **(a):** $f \in \mathcal{P}_+(M)$
- **(b):** *$M$ and $\mathsf{Fix}\,(f)$ have finite type*
- **(c):** *there exists compact subset $X \subset \mathsf{Fix}\,(f)$ that is a nonempty union of components of $\mathsf{Fix}\,(f)$, and such that $\check{\chi}(X) < \check{\chi}(M)$*

*Then $\mathsf{Fix}\,(f)$ has an acyclic component $L$ disjoint from $X$, and $\mathsf{Fix}\,(g^{\mathbf{k}(L)}) \cap L \neq \varnothing$.*

Proof. Proposition 2.6(a) implies $X$ admits a complementary cell $U$, necessarily precompact by Proposition 2.4. But $U$ cannot be complementary to $\mathsf{Fix}\,(f)$, because $f \in \mathcal{P}_+(M)$. Therefore $U$ meets $\mathsf{Fix}\,(f)$. The fact that $\mathsf{Fr}(U) \subset X$ implies every component of $U \cap \mathsf{Fix}\,(f)$ is compact. Now apply Proposition 4.4. □

The following corollary to Handel's Theorem 1.3 has some overlap with Theorem 4.5:

**Theorem 4.9** (Handel). *$f$ has a neighborhood $\mathcal{N} \subset \mathcal{H}_+(\mathbf{S}^2)$ such that if $g \in \mathcal{N}$, then $\mathsf{Fix}\,(g) \cap \mathsf{Fix}\,(f) \neq \varnothing$ provided $f$ and $g$ are diffeomorphisms, or $f$ and $g$ have finite fixed point sets.* □

Proposition 4.7(ii)(iii) shows that when $M = \mathbf{S}^2$ and $f \in \mathcal{P}_+(M)$, at least two components of $\mathsf{Fix}\,(f)$ meet $\mathsf{Per}(g)$. The following two theorems have similar conclusions.

Let $\nu \in \mathbf{N}_+$ be as in Hypothesis 4.1.

**Theorem 4.10.** *Take $M = \mathbf{S}^2$ and assume $\nu \geq 2$. Then there are two components $K_i$, $i = 1, 2$ of $\mathsf{Fix}\,(f)$ such that $\mathsf{Fix}\,(g^{\mathbf{k}(K_i)}) \cap K_i \neq \varnothing$.*

Proof. By Theorem 4.5 there exists $K_1 \in \mathsf{Comp}(\mathsf{Fix}\,(f))$ and $p \in g^{\mathbf{k}(K_1)} \cap K_1$. As $\nu \geq 2$, the open cell $U = \mathbf{S}^2 \setminus \{p\}$ contains a component $L \neq K_1$ of $\mathsf{Fix}\,(f)$. Now apply Proposition 4.4(i) to infer that $U$ contains the required $K_2$. □

**Theorem 4.11.** *Take $M = \mathbf{S}^2$ and assume $\nu = 2$.*

- **(i):** *Each component of $\mathsf{Fix}\,(f)$ meet $\mathsf{Fix}\,(g^2)$.*

**(ii):** *If the two components are not homeomorphic, each one meets* $\mathsf{Fix}\,(g)$.

PROOF. Both conclusions follow from Theorem 4.10, because $\mathbf{k}(K) \le \nu = 2$ for each $K \in \mathsf{Comp}(\mathsf{Fix}\,(f))$, and $\mathbf{k}(K) = 1$ when the two components are not homeomorphic. □

A useful condition guaranteeing triangulability of $\mathsf{Fix}\,(f^n)$ for all $n$ is that $f$ or $f^{-1}$ is analytic. It suffices to consider $n = 1$. The graph $\Gamma(f) \subset M \times M$ of $f$ and the diagonal $\Delta$ are analytic varieties $\Gamma(f) \subset M \times M$, as is the diagonal $\Delta$. Therefore $\Gamma(f) \cap \Delta$ is also a variety, hence triangulable (Lojasiewicz [33]). Hence $\mathsf{Fix}\,(f)$ is triangulable, as it is the homeomorphic image of $\Gamma(f) \cap \Delta$ under projection on the first factor, $\Pi_1\colon\ M \times M \to M$.

A similar argument shows that $\mathsf{Fix}\,(f^n)$ is triangulable when $f$ is piecewise linear.

Call $f$ *locally subanalytic* if every point in $\Gamma(f)$ has a neighborhood in $\Gamma(f)$ that is the image of an analytic variety $V$ under an analytic map $V \to M \times M$. It is not hard to see that locally subanalytic homeomorphisms form a subgroup of $\mathcal{H}_+(M)$. The results of Hardt [22] imply:

**Proposition 4.12.** *If $f$ is locally subanalytic,* $\mathsf{Fix}\,(f^n)$ *is triangulable for all* $n$. □

**Theorem 4.13.** *Assume* $K \in \mathsf{Comp}(\mathsf{Fix}\,(f))$ *is compact and triangulable.*

**(i):** *Suppose* $\chi(K) \neq 0$. *Then* $\mathsf{Fix}\,(g^k) \cap K \neq \varnothing$ *with* $1 \le k \le \mathbf{k}(K) \cdot \max\{1, \mathsf{b}_1(K)\}$.

**(ii):** *Suppose* $\mathsf{Per}(g) \cap K = \varnothing$. *Then:*

- **(a):** *$K$ is either a circle or a compact annulus*
- **(b):** *the inclusion map $K \to M$ induces an injective homomorphism of fundamental groups*

PROOF. The only nonzero Betti numbers of $K$ are $\mathsf{b}_0(K) = 1$, and also $\mathsf{b}_0(K)$ provided $K$ is not acyclic. To prove case (i), apply to $g^{\mathbf{k}(K)}\colon\ K \approx K$ the general theorem of Fuller [19]:

> *If $h$ is a homeomorphism of a compact polyhedron $P$ having nonzero Euler characteristic, then* $\mathsf{Fix}\,(h^\nu) \neq \varnothing$, *with*
>
> $$1 \le \nu \le \max\left(\sum_i \mathsf{b}_{2i}(P), \sum_i \mathsf{b}_{2i+1}(P)\right)$$

In case (ii), note that the set of points where $K$ is not locally a manifold (possibly with boundary) is finite and invariant by $g$, hence empty because $g\colon\ K \approx K$ has no periodic points. Thus $K$ is a compact connected manifold of dimension 1 or 2, hence a circle or compact annulus because $\chi(K) = 0$ by (i). This proves (ii)(a).

We prove (ii)(b) by contradiction, assuming for this purpose either that $K$ is a circle bounding a disk $D$, or that $K$ is an annulus and each of the two boundary circles $C_i,\ i = 1, 2$ of $K$ bounds a disk $D_i$. The assumption in (ii) implies $M \not\approx \mathbf{S}^2$, by Theorem 4.5. This means that $D$ (respectively, $D_i$) is unique. Note that $D$ (respectively, $D_i$) is $f$-invariant.

Set $m = \mathsf{k}(K)$. If $K$ is a circle, $g^m(D) = D$. But then Theorem 4.5, applied to $f, g^m \colon D \approx D$, yields a contradiction.

If $K$ is an annulus, Lefschetz's Fixed Point theorem implies $g^m|K$ induces the identity automorphism of $H_1(K)$, for otherwise $\mathsf{Fix}\,(g^m) \cap K \neq \varnothing$. Because $g$ preserves orientation, it follows that the disks $D_i$ are $g$-invariant. Again Theorem 4.5 gives a contradiction. □

**Theorem 4.14.** *Assume* $\mathsf{Fix}\,(f)$ *compact and triangulable. Let $E$ be an open cell complementary to* $\mathsf{Fix}\,(f)$*, with frontier in* $L \in \mathsf{Comp}(\mathsf{Fix}\,(f))$*. Then*

$$\mathsf{Fix}\,(g^k) \cap \mathsf{Fix}\,(f) \cap \mathsf{Clos}(E) \neq \varnothing, \quad 1 \leq k \leq \mathbf{k}(L)(\mathsf{b}_1(L) + 1)$$

*and $k = 1$ provided $g$ is sufficiently near* $\{f^i\}_{i\in\mathbf{Z}}$.

Proof. $E$ is a precompact cell complementary to $L$. The number of cells complementary to $L$ is bounded by $\mathsf{b}_1(L) + 1$; this can be deduced from the Lefschetz duality isomorphism $H^1(L) \cong H_1(M, M \setminus L)$ (Spanier [38], Theorem 6.2.19). Because $g^{\mathbf{k}(L)}$ permutes these cells, there exists a smallest $n \in \{1, \ldots, \mathsf{c}^1(L) + 1\}$ such that $g^{\mathbf{k}(L)n}(\mathsf{Clos}(E)) = \overline{E}$. Set $k = \mathbf{k}(L)n$. Theorem 3.3 shows that $g^k$ has a fixed point in $L$.

$E$ is invariant under $f$ and its iterates by the Brown-Kister Theorem 2.3. It is easy to see that $g(E)$ meets $E$ provided $g$ is sufficiently near $\{f^i\}_{i\in\mathbf{Z}}$. Under this assumption $g(E) = E$ because $g$ permutes complementary components of $\mathsf{Fix}\,(f)$; hence $g(\mathsf{Fr}(E)) = \mathsf{Fr}(E)$. As $\mathsf{Fr}(E) \subset L$ and $g$ permutes components of $\mathsf{Fix}\,(f)$, we have $g(L) = L$. Hence $k = 1$. □

In Theorem 4.14, the assumption that $\mathsf{Fix}\,(f)$ admits a complementary cell holds provided $M$ has finite type and $\chi(K) < \chi(M)$. This follows from Proposition 2.6(a).

In the next result, $M$ is a closed surface whose metric is induced from a smooth Riemannian metric $\mathbf{g}$. Let $\rho = \rho(\mathbf{g}) > 0$ denote the largest real number with the following property: At every $x \in M$, the exponential map $\mathsf{exp}_x \colon M_x \to M$ is injective on the open disk $M_x(\rho)$ of radius $\rho$ about the origin in the tangent plane

$M_x$. If $\mathsf{g}$ has constant nonpositive curvature, $\rho(\mathbf{g})$ equals half the minimum of the lengths of closed geodesics. Note that $\mathsf{exp}_x(M_x(\rho))$ is an open cell which contains the open ball of radius $\rho$ about $x$.

**Theorem 4.15.** *Let $M$ be closed, with $\rho = \rho(\mathsf{g})$ as above. Assume $\mathsf{c}^0(\mathsf{Fix}\,(f)) < \infty$. If a component $K$ of $\mathsf{Fix}\,(f)$ has diameter $< \rho$, then $\mathsf{Fix}\,(g^k) \cap \mathsf{Fix}\,(f) \neq \varnothing$ with $1 \leq k \leq \mathsf{c}^0(\mathsf{Fix}\,(f))$.*

Proof. For every $x \in K$ we see that have $K$ lies in the open cell $U = \mathsf{exp}_x(M_x(\rho))$; apply Proposition 4.4(i). □

**Example 4.16.** Take $M$ to be the torus $\mathbf{S}^1 \times \mathbf{S}^1$. Give $\mathbf{S}^1$ its standard Riemannian metric, in which it has length $2\pi$. Give $M$ the corresponding product Riemannian metric $\mathbf{g}$, which is flat. Thus $M$ has diameter $\pi\sqrt{2}$, while $\rho(\mathbf{g}) = \pi$. Suppose $\mathsf{Fix}\,(f)$ has a unique component $K$, and $K$ has diameter $< \pi$. Then Corollary 4.15 implies $\mathsf{Fix}\,(g) \cap \mathsf{Fix}\,(f) \neq \varnothing$.

The upper bound $\pi$ is best possible. To see this, define $f, g \in \mathcal{H}_+(\mathbf{S}^1 \times \mathbf{S}^1)$ to be the maps covered by the maps of $\mathbf{R}^2$ that send $(x, y)$ to $(x, y + \sin x/2)$ and to $(x, y + 1)$, respectively. Then $f$ and $g$ commute, $\mathsf{Fix}\,(f)$ is the circle of diameter $\pi$ covered by $\{0\} \times \mathbf{R}$, but $\mathsf{Fix}\,(g)$ is empty. In fact $\mathsf{Per}(g)$ is empty because $\pi$ is irrational.

## 5. Nielsen classes

Here we show that certain kinds of Nielsen classes in $\mathsf{Fix}\,(f)$ meet $\mathsf{Per}(g)$.

Fix a universal covering space $\psi\colon \mathbf{R}^2 \to M$, with its group $\Gamma \subset \mathcal{H}_+(\mathbf{R}^2)$ of deck transformations. Recall that every $q \in \psi^{-1}(p)$ determines an isomorphism $\Theta_q\colon \Gamma \cong \pi_1(M, p)$, as follows: If $T \in \Gamma$ then $\Theta_q(T) \in \pi_1(M, p)$ is represented by any loop in $M$ that is covered by a path in $\hat{M}$ from $q$ to $T(q)$.

If $u$ denotes a map $M \to M$, the symbol $\hat{u}$ denotes a *lift* of $u$, i.e., a map $\hat{u} \in \mathcal{C}(\hat{M})$ such that $\psi \circ \hat{u} = u \circ \psi$. If $w \in \mathcal{C}(\hat{M})$ is another lift of $u$, there is a unique deck transformation $T$ such that $w = T \circ \hat{u}$.

For any set $X \subset M$, a *lift* of $X$ denotes a set $\hat{X} \subset \mathbf{R}^2$ mapped homeomorphically onto $X$ under $\psi$.

For $p \in \mathsf{Fix}\,(f)$ we denote by $f_{p\#}\colon \pi_1(M, p) \cong \pi_1(M, p)$ the fundamental group automorphism induced by $f$. If $q \in \psi^{-1}(p)$, the isomorphism $\Theta_q$ conjugates $f_{p\#}$ to the automorphism of $\Gamma$ given by $T \mapsto \hat{f} \circ T \circ \hat{f}^{-1}$, where $\hat{f}$ is the unique lift of $f$ that fixes $q$.

Fixed points $p, q$ of $f$ are *Nielsen equivalent* if there is a lift $\hat{f}$ having fixed points $\hat{p}$, $\hat{q}$ mapped by $\psi$ to $p$, $q$ respectively. Equivalently, there is a path $\lambda$ in $M$

joining $p$ to $q$ such that the loop obtained by following first $\lambda$ and then $f \circ \lambda$ is null homotopic. In this case $f_{q\#} = \lambda_\# \circ f_{p\#} \circ \lambda_\#^{-1}$, where $\lambda_\# \colon \pi_1(M, p) \cong \pi_1(M, q)$ is the isomorphism determined by $\lambda$.

A *Nielsen class* in $\mathsf{Fix}\,(f)$ is an equivalence class for the relation of being Nielsen equivalent. Every Nielsen class is relatively open and closed in $\mathsf{Fix}\,(f)$, hence it is a union of components of $\mathsf{Fix}\,(f)$. If its fixed point index is nonzero, the class is *essential*.

When $\mathsf{Fix}\,(f)$ is compact there are only finitely many Nielsen classes. For any lift $\hat{f}$ having a fixed point, the image of $\mathsf{Fix}\,(\hat{f}))$ under $\psi \colon \mathbf{R}^2 \to M$ is a Nielsen class for $f$. A partial converse is given in Lemma 5.2.

Each Nielsen class $N$ relatively open and closed in the fixed point set.

Suppose $h_0, h_1 \in \mathcal{H}_+(M)$ are isotopic, and let $\{h_t\}_{t \in I}$ be an isotopy from $h_0$ to $h_1$. Define $H \colon V \times I \approx V \times I$ by $H(x, t) = (h_t(x), t)$. Suppose $P_0$ is an essential Nielsen class for $h_0$. Let $Q \subset M \times I$ be the Nielsen class of $H$ that contains $P_0 \times \{0\}$. Because $P_0$ is essential, $Q \cap M \times \{1\} = P_1 \times \{1\}$ where $P_1$ is an essential Nielsen class for $h_1$. We say $P_0$ and $P_1$ are related by *continuation*.

In the rest of this section we make the following assumption:

**Hypothesis 5.1.** *$N \subset \mathsf{Fix}\,(f)$ is a compact Nielsen class such that for some (hence any) $p \in N$, the automorphism $f_{p\#}$ of $\pi_1(M, p)$ fixes only the unit element.*

Suppose this holds and $f$ is isotopic to $f_1$. If $N_1$ is a compact Nielsen class for $f_1$, and $N_1$ is related to $N$ by continuation, then it can be shown that Hypothesis 5.1 also holds for $f_1$ and $N_1$.

Suppose $M$ is closed, $f$ is pseudo-Anosov, and $p \in \mathsf{Fix}\,(f)$. Results of Nielsen (see Thurston [41]) show that $\{p\}$ is an essential Nielsen class satisfying Hypothesis 5.1.

**Lemma 5.2.** *Assume Hypothesis 5.1 and suppose $g(N) = N$. Then:*

- **(i):** *there is a lift $\hat{f}$ such that $\mathsf{Fix}\,(\hat{f})$ is a lift of $N$*
- **(ii):** *there is lift $\hat{g}$ commuting with $\hat{f}$.*

Proof. Let $p \in N$ and choose a lift $\hat{f}$ having a fixed point $\hat{p} \in \psi^{-1}(p)$. It is easy to see that $\psi \colon \mathsf{Fix}\,(\hat{f}) \to N$ is a surjective local homeomorphism, so it suffices to prove this map injective. Suppose $y, x \in \mathsf{Fix}\,(\hat{f}) \cap \psi^{-1} N$ are such that $\psi(y) = \psi(x)$. Choose a path $\gamma \colon \ I \to \mathbf{R}^2$ joining $y$ to $x$. Then $\psi \circ \gamma$ is a loop in $M$ representing a homotopy class $\alpha \in \psi_1(M, p)$ that is fixed under $f_{p\#}$. Therefore $\alpha$ is trivial, implying $y = x$.

Let $h \colon \mathbf{R}^2 \approx \mathbf{R}^2$ be a lift of $g$. For every $z \in \hat{N}$ we have $\psi \circ h(z) = g \circ \psi(z)$, which lies in $g(N) = N$. Thus $h(z) \in \psi^{-1}(N)$, so there is a unique $T_z \in \Gamma$ such that $T_z \circ h(z) \in \hat{N}$. Compactness of $\hat{N}$ and total discontinuity of $\Gamma$ implies all the $T_z$ coincide with some $T \in \Gamma$. The lift $\hat{g} = T^{-1} \circ h$ of $g$ maps $\hat{N}$ homeomorphically onto $\hat{N}$.

Because $g$ and $f$ commute, there is a deck transformation $S$ such that $\hat{g} \circ \hat{f} = S \circ \hat{f} \circ \hat{g}$; therefore

$$\hat{N} = \hat{g} \circ \hat{f}(\hat{N}) = S \circ \hat{f} \circ \hat{g}(\hat{N}) = S \circ \hat{f}(N) = S(\hat{N}).$$

Thus $\hat{p} = S(z)$ for some $z \in N$, entailing $\psi(\hat{p}) = \psi(z)$; therefore $\hat{p} = z$ because $\psi|N$ is injective. This proves $S(z) = z$, so $S$ is the identity. □

**Theorem 5.3.** *Assume Hypothesis 5.1, and let* $\mathsf{c}^0(N) = \nu \in \mathbf{N}_+$. *Then* $\mathsf{Fix}\,(g^k) \cap \mathsf{Fix}\,(f) \cap N \neq \varnothing$ *with* $k \in \{1, \dots, \nu\}$, *and* $k = 1$ *provided* $g$ *is sufficiently close to* $\{f^i\}_{n \in \mathbf{Z}}$.

Proof. By Proposition 5.2, there exist commuting lifts $\hat{f}, \hat{g}$ such that $\psi\colon \mathsf{Fix}\,(\hat{f}) \approx N$. The conclusion follows from Theorem 4.5 applied to $\hat{f}$ and $\hat{g}$. □

**Theorem 5.4.** *Let* $M$ *be closed and assume:*

- **(a):** $\mathsf{Lef}(f) \neq 0$
- **(b):** $\mathsf{c}^0(N) < \infty$ *for every essential Nielsen class* $N$
- **(c):** $f$ *is isotopic to a pseudo-Anosov homeomorphism* $h$

*Then* $\mathsf{Per}(g) \cap \mathsf{Fix}\,(f) \neq \varnothing$.

*More precisely:* $f$ *admits an essential Nielsen class. If* $N \subset \mathsf{Fix}\,(f)$ *is an essential Nielsen class and* $\mathsf{c}^0(N) = \nu \in \mathbf{N}_+$, *there exists* $l \in \mathbf{N}_+$ *such that*

$$\mathsf{Fix}\,(g^l) \cap \mathsf{Fix}\,(f) \cap N \neq \varnothing, \quad l = mk, \quad 1 \leq m \leq \mu, \quad 1 \leq k \leq \nu$$

*And* $k = 1$ *if* $g$ *is sufficiently close to* $\{f^i\}_{n \in \mathbf{Z}}$.

Proof. $h$ and $f$ have the same nonzero Lefschetz number. Therefore $h$ has a fixed point $p$, and $\{p\}$ is an essential Nielsen class for $h$ obeying Hypothesis 5.1.

There is a Nielsen class $N$ for $f$ related to $\{p\}$ by continuation; such a class is essential, and every essential Nielsen class for $f$ arises in this way. Therefore Hypothesis 5.1 holds for $f$ and $N$. The proof is completed by applying Theorem 5.3 to lifts $\hat{f}$ and $\hat{g}$. □

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University of California and University of Wisconsin

7926 N. Hill Point Rd., Cross Plains WI 53528
*E-mail address*: hirsch@math.berkeley.edu