UNIVERSITY OF CALIFORNIA,
IRVINE

Conformal Geometry and Prescribed Scalar Curvature on S^2

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

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2021

Contents

Acknowledgements

I would like to thank my advisor, Professor Jeffrey Streets, for the years he has patiently worked with me on these problems.

I would also like to thank my committee members, Professors Patrick Guidotti and Jeffrey Viaclovsky.

I would like to thank all of the other UCI faculty who I have learned from over my time as a graduate student here.

Finally I would like to thank my family for the times when they encouraged me in my work.

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Abstract of the Dissertation

Conformal Geometry and Prescribed Scalar Curvature on S^2

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University of California, Irvine, 2021

Professor Jeffrey Streets, Chair

In this dissertation, we seek to understand prescribed scalar curvature through the gradient flow of conformal metrics. On S^2, we will define a modified Liouville energy and derive a geometric flow equation related to the energy functional. We will prove longtime existence for solutions of this equation with arbitrary data through the methods used by Gursky and Streets [6]. We will then show that Gauss curvature retains its regularity under evolution through the flow assuming bounds on the Gauss curvature. We will finally show that the solution is stable when converging to constant curvature if the initial curvature is close to the geometry of S^2.

Chapter 1

Prescribed Scalar Curvature Problem and Background

1.1 Curvature of a Manifold

Let M be a smooth manifold of dimension $n \geq 2$, we will introduce the notion of curvature on a manifold. For an in-depth development of smooth manifolds c.f. [10] and for in-depth development of curvature and related properties c.f. [9].

Definition 1.1.1. (Riemannian Metric) A Riemannian metric on M is a smooth symmetric covariant 2-tensor field, g, that is positive definite at each point.

We call the pair (M, g) of a smooth manifold M and a Riemannian metric g a Riemannian manifold.

Let $p, q \in M$, in order to translate a vector from tangent space T_pM to tangent space T_qM, we need an additional structure on (M, g). [9] chapter 4

Definition 1.1.2. (Connection) For vector bundle $\pi : E \to M$, let $\Gamma(E)$ denote the smooth sections of E and let TM denote the tangent bundle of M. A connection in E is map

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$

written $(X, Y) \mapsto \nabla_X Y$ such that

$$(a) \quad \nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$$

$$(b) \quad \nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$$

and

$$(c) \quad \nabla_X(fX) = f\nabla_X Y + (Xf)Y$$

The quantity $\nabla_X Y$ is often called the covariant derivative of Y in direction of X.

We need to add two restrictions to our connection. [9] chapter 5

Definition 1.1.3. (Compatible Connection) For Riemannian manifold (M, g), a connection ∇ is compatible with g if for any vector fields X, Y, Z on M

$$\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

Definition 1.1.4. (Symmetric Connection) A connection is symmetric (or torsion-free) if for vector fields X, Y on M

$$\nabla_X Y - \nabla_Y X \equiv [X, Y]$$

Here $[X, Y] \equiv XY - YX$ is the Lie bracket.

Theorem 1.1.5. *(Fundamental Lemma of Riemannian Geometry) For Riemannian mani-*

fold (M, g) there exists a unique connection on M that is compatible with g and symmetric.

This connection is called the Riemannian connection or the Levi-Civita connection. It allows us to define the curvature of a manifold. [9] chapter 7

Definition 1.1.6. (Riemann Curvature Tensor) For Riemannian manifold (M, g) with Riemannian connection ∇, we define the Riemann curvature R tensor $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{1.1.1}$$

We also write

$$Rm(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle \tag{1.1.2}$$

We will use the coordinate notation as well.

$$R_{ijkl} = \langle R(\partial_i, \partial_j)\partial_k, \partial_l \rangle \tag{1.1.3}$$

The Riemann curvature tensor contains all information about the curvature of M. Fortunately for us, we can focus on a simpler quantity.

Definition 1.1.7. (Scalar Curvature) For Riemannian manifold (M, g) with Riemannian connection ∇, scalar curvature, S, is defined by contracting the Riemann curvature tensor R with respect to g, specifically

$$S = g^{ij} g^{km} R_{kijm} \tag{1.1.4}$$

On a 2-manifold, we define the Gauss Curvature by

$$K = \frac{1}{2}S \tag{1.1.5}$$

Both K and S fully characterize the curvature of surface M. [9] page 144

1.2 Prescribed Scalar Curvature Problem

Let M be a smooth manifold of dimension $n \geq 2$. Given a smooth real-valued function f on M, does there exist a Riemannian metric g on M such that the scalar curvature of (M, g) is equal to f? This question is called the Prescribed Scalar Curvature problem and will be hereby referred to as the PSC problem.

The PSC problem is related to the Yamabe problem, which asks if for a given smooth Riemannian manifold (M, g), does there exist a conformal metric g' such that the scalar curvature of (M, g') is constant. Yamabe sought to prove the Poincaré conjecture and thought this problem would be an intermediate step. While Yamabe's proof was incomplete, the Yamabe problem has been affirmatively solved. [2] chapter 5.

1.2.1 History of Problem

The prescribed scalar curvature problem has been solved for some conditions. Although these results are not directly related to the work in this dissertation, we will briefly survey some of them.

The following three theorems are due to Kazdan and Warner. Through these theorems, they effectively solve the scalar curvature problem for manifolds with dimension 3 or greater. [7]

c.f. [2] chapter 6

Theorem 1.2.1. *Let M be a C^∞ compact manifold of dimension $n \geq 3$. If $f \in C^\infty(M)$ is negative somewhere, then there exist a Riemannian metric with scalar curvature f.*

Theorem 1.2.2. *Let M be a C^∞ compact manifold of dimension $n \geq 3$ which admits a metric with positive scalar curvature, then for any $f \in C^\infty(M)$, M admits a metric with scalar curvature f.*

Theorem 1.2.3. *Let M be a non-compact manifold of dimension $n \geq 3$ that is diffeomorphic to an open submanifold of a compact manifold. Then for every $f \in C^\infty(M)$, there exist a Riemannian metric on M with scalar curvature f.*

When the dimension of M is equal to 2 the problem is more difficult, but nonetheless follows the same classification regime suggested by the theorems above.

1.2.2 Conformal Metrics

For Riemannian manifold (M, g_0), a conformal metric is a Riemannian metric g' on M such that for some positive function h on M

$$g' = hg_0 \tag{1.2.1}$$

We call the set of metrics on M that are conformal to g_0 the conformal class of (M, g_0). It is easy to see that a conformal class is an equivalence class.
There are different conventions for representing conformal metrics and classes, but we will identify a function $u : M \to \mathbb{R}$ with a conformal metric g_u given by

$$g_u = e^{2u}g_0 \tag{1.2.2}$$

We will denote the conformal class of g_0 on M by $[g_0]$. That is

$$[g_0] = \{g_u = e^{2u} g_0 \mid u \in C^\infty(M)\} \tag{1.2.3}$$

In this dissertation we will work on a compact Riemannian surface (M, g_0) of genus 0. We also assume M is boundary-free manifold. Any such manifold is homeomorphic to S^2.

The surface (M, g_0) has Gauss curvature K_0 which we will assume is positive.

We are interested in the metrics in $[g_0]$ which preserve positive Gauss curvature so we introduce

$$\Gamma_1^+ = \{g_u = e^{2u} g_0 \in [g_0] \mid K_u = K_{g_u} > 0\} \tag{1.2.4}$$

to denote the space of these metrics following the notation laid out in [6].

Since length, angles, and curvature are defined through a Riemannian metric, we would like a means to compare these quantities on M with respect to a conformal metric g_u to the background metric g_0. It is easy to show that conformal transforms preserve angles between vectors. The curvature is not invariant, but K_u is related to K_0 by the following formula. [6] sec. 3, pg. 7 and c.f. [2] pg. 196

Proposition 1.2.4. *(Gauss Curvature Equation) For $g_u = e^{2u} g_0$, the Gauss curvature of Riemannian surface (M, g_u), K_u is related to Gauss curvature of (M, g_0), K_0, by the formula*

$$K_u = e^{-2u}(K_0 - \Delta_0 u)$$

Remark 1.2.5. Throughout this dissertation, Δ will denote the Laplacian with respect to metric g_u. We will use Δ_0 to denote Laplacian with respect to metric g_0. Notice that

$\Delta = e^{-2u}\Delta_0.$

Similarly we will use $\langle \cdot, \cdot \rangle$ to denote the inner product and $|\cdot|$ to denote the norm with respect to metric g_u, while $\langle \cdot, \cdot \rangle_0$ and $|\cdot|_0$ are with respect to the background metric g_0. Notice $\langle \cdot, \cdot \rangle = e^{2u}\langle \cdot, \cdot \rangle_0$ and $|\cdot| = e^{2u}|\cdot|_0$.

Remark 1.2.6. We will be integrating with respect to both the background metric g_0 and the metric g_u throughout this dissertation. We will denote their respective area measures as dA_0 and dA_u. For local coordinates (x^1, x^2), they are related by

$$\begin{aligned} dA_u &= \sqrt{|\det g_u|}\; dx^1 \wedge dx^2 \\ &= \sqrt{(e^{2u})^2 |\det g_0|}\; dx^1 \wedge dx^2 \\ &= e^{2u}\sqrt{|\det g_0|}\; dx^1 \wedge dx^2 \\ &= e^{2u}\; dA_0 \end{aligned} \tag{1.2.5}$$

1.3 Prescribed Scalar Curvature for Conformal Metric on Surfaces

Since scalar curvature completely characterizes the curvature tensor on a two dimensional manifold, the prescribed scalar curvature problem is a greater interest on a surface. [9] page 144

Given a Riemannian surface (M, g_0) and a real-valued function f on M, we wish to find a metric g so that the scalar curvature of M with respect to g is equal to f. Instead of considering metrics from the entire set of Riemannian metrics on M, we wish to prescribe curvature only considering metrics g conformal to g_0.

From the Gauss Curvature Equation 1.2.4 we see that for $n = 2$, this problem is equivalent to solving the PDE

$$K_0 - \Delta_0 u = f e^{2u} \tag{1.3.1}$$

We briefly survey some results pertaining to prescribed scalar curvature for conformal metrics on surfaces. The results fall into distinct categories depending on the sign of the background scalar curvature.

First we consider manifolds where the scalar curvature of the background metric is negative. To state the result, we must first introduce the Yamabe functional as in [2] chapter 5, page 150

Definition 1.3.1. (Yamabe's Functional) On Riemannian manifold (M^n, g) with scalar curvature $S(x)$.
For $n \geq 3$, let $N = \frac{2n}{n-2}$, for $2 \leq q \leq N$, Yamabe's functional is given by

$$J_q(\phi) = \Big[4\frac{n-1}{n-2}\int_M |\nabla\phi|^2 \, dV + \int_M S(x)\phi^2 \, dV\Big]||\phi||_q^{-2} \tag{1.3.2}$$

We also define

$$\mu_q = \inf\{J_q(\phi)|\phi \in H^1(M), \phi \geq 0, \phi \not\equiv 0\} \tag{1.3.3}$$

and

$$\mu = \mu_N \tag{1.3.4}$$

We have the follow results is due to Aubin. c.f. [2] chapter 6, page 197

Theorem 1.3.2. *Let (M^n, g) be C^∞-Riemannian manifold with $\mu < 0$ and $n \geq 3$ and assume scalar curvature S_g is negative. Given C^∞ function $f < 0$, there exists a unique conformal metric with scalar curvature f.*

This result can be strengthened for $n = 2$. [2] page 203

Theorem 1.3.3. *For (M^2, g_0), if f is negative somewhere and $f \leq C$ for some positive C, then there exist a conformal metric with scalar curvature f.*

In the case where we have background scalar curvature of zero. [2] page 204

Theorem 1.3.4. *Let (M^2, g) be a Riemannian surface with $S_g = 0$, then a smooth function f is the scalar curvature of a conformal metric if and only if f changes sign and $\int f \, dA < 0$.*

We will be dealing with the sphere S^2, which has positive scalar curvature under the standard metric, for the remainder of this dissertation. We will however give one result about another manifold with positive scalar curvature first.

The following theorem is due to M.S. Berger and J. Moser. [12] c.f. [2] page 209

Theorem 1.3.5. *For the real projective space, $(\mathbb{RP}^2, g)$, any smooth function f on $\mathbb{RP}^2$ with* $\sup f > 0$ *is the scalar curvature of a metric conformal to g.*

1.3.1 Nirenberg Problem

From here on we will focus our attention to a special problem related to the general prescribed scalar curvature problem for conformal metrics.

The Nirenberg problem poses the following: Given a positive smooth function h on (S^2, g_0), is h the Gauss curvature of a metric g conformal to g_0? Here g_0 is the standard metric on the sphere with Gauss curvature 1. We will also assume h is close to 1. [2] page 230

By the Gauss Curvature formula 1.2.4, solving the Nirenberg problem is equivalent to solving the partial differential equation

$$he^{2u} = 1 - \Delta_0 u \tag{1.3.5}$$

The following theorem is due to Moser. [12], c.f. [2] page 232

Theorem 1.3.6. *On (S^2, g_0), let $f \in C^\infty(S^2)$ be a function which is invariant under the antipodal map, i.e. $f(x) = f(-x)$ for all $x \in S^2$, then f is the scalar curvature of a metric conformal to g_0.*

1.4 Our Contributions

In this section, we will briefly summarize the contribution to the Nirenberg problem which we make in this dissertation.

In chapter 3, section 3.2.2, we introduce a modified version of the Liouville energy in definition

3.1.5

$$G[u] = \int_M |\nabla u|_0^2 \; dA_0 + 2\int_M K_0 u \; dA_0 - (\int_M K_0 \; dA_0) \log(\fint e^{2u} \Big| f) \tag{1.4.1}$$

where we define the symbol

$$\begin{aligned} \fint e^{2u} \Big| f &= \frac{\int_M e^{2u} \; f \; dA_0}{\int_M f \; dA_0} \\ &= \frac{\int_M f \; dA_u}{\int_M f \; dA_0} \end{aligned} \tag{1.4.2}$$

for $f \in C^\infty(M)$ with $f > 0$.

We also define

$$\begin{aligned} \tilde{K_u} &= (\frac{\int_M K_u \; dA_u}{\int_M f \; dA_u}) f \\ &= \frac{4\pi}{\int_M f \; dA_u} f \end{aligned} \tag{1.4.3}$$

The term $\tilde{K_u}$ is the mean curvature weighted by f. We will see that $\tilde{K_u}$ is the value for the Gauss curvature at which the flow equation is stationary.

From our functional G, we will derive the inverse Gauss curvature flow equation 3.2.1,

$$\frac{\partial u}{\partial t} = -\frac{K_u - \tilde{K_u}}{K_u} \tag{1.4.4}$$

This is the main geometric flow which we study. We will simply call it the flow equation.

We will show that for $K_0 > 0$, any solution to 3.2.1 exists for all time as stated in Theorem 3.2.17.

Theorem 1.4.1. *For compact Riemannian manifold (M^2, g_0) with $K_0 > 0$, the solution to the flow equation 3.2.1 exists for all time.*

In chapter 4, we will show that Gauss curvature K_u retains regularity in all of its derivatives if we assume that for some $\lambda > 0$ for some $C_1, C_2 > 0$

$$\frac{1}{\lambda} < K_u < \lambda \tag{1.4.5}$$
$$|\nabla K_u|^2 \leq C_1$$
$$|\nabla^2 K_u|^2 \leq C_2$$

We will prove Theorem 4.0.3 and Corollary 4.0.5

Theorem 1.4.2. *Let u be a solution to flow equation that satisfies 1.4.5, then for all order i there exist some constant C_i such that for large t*

$$|\nabla^i \frac{1}{K_u}|^2 \leq \frac{C_i}{t^{i-2}} \tag{1.4.6}$$

The constant C_i depends only on the lower order derivatives, C_2, C_1, and λ.

Corollary 1.4.3. *Let u be a solution to flow equation that satisfies 1.4.5, then for all order i, there exist some C_i' depending only on the lower order derivatives, C_2, C_1, and λ such that*

$$|\nabla^i K_u|^2 \leq C_i' \tag{1.4.7}$$

Remark 1.4.4. We notice that each $\nabla^i K_u$ is pointwise bounded as a consequence of Corollary 4.0.5.

In chapter 5, section 5.2, we will show that the solutions of the flow equation are stable when $f \equiv 1$. Specifically under the assumptions that for some $\delta > 0$

$$\begin{aligned} |u| &< \delta \\ |K_u - 1| &< \delta \\ |\nabla u| &< \delta \\ |\nabla K_u| &< \delta \end{aligned} \tag{1.4.8}$$

we can prove that u decays exponentially to 0 in some cases, as given in Theorem 5.2.5.

Theorem 1.4.5. *For u satisfying the flow equation such that $||u-\bar{u}||_{L^2}$ decays exponentially, there exist some $\delta > 0$ such that if the assumptions 1.4.8 are satisfied then for all integer $i \geq 1$*

$$||\nabla^i u||^2_{L^2(M,g_u)} \leq A_i e^{-B_i t} \tag{1.4.9}$$

for some $A_i, B_i > 0$.

By Corollary 5.2.6, we have exponential decay for antipodal invariant solutions to the flow equation.

Corollary 1.4.6. *Let u be an antipodal invariant solution to the flow equation 5.0.3, then there exist some $\delta > 0$ such that if the assumptions 1.4.8 are satisfied then for all integer*

$i \geq 1$

$$||\nabla^i u||^2_{L^2(M,g_u)} \leq A_i e^{-B_i t} \tag{1.4.10}$$

for some $A_i, B_i > 0$.

In future work, we would like to prove results similar to Theorem 5.2.5 without assuming $f \equiv 1$. This will allow our method to be applied to a broader class of prescribed scalar curvatures rather than only constant scalar curvature.

Chapter 2

Analytic Tools

We will outline some tools from geometry and analysis which we will use throughout this dissertation. This material will be familiar to anyone with significant experience in geometric analysis and may be skipped or referred back to at the pleasure of the reader. This is called Green's Identity, it is the analogous integration by parts on manifolds. [9] page 44

Theorem 2.0.1. *(Green's Indentity) For a compact, connected, oriented Riemannian manifold (M, g), for $u, v \in C^\infty(M)$ we have Green's Identity*

$$\int_M u\Delta v \; dV + \int_M \langle \nabla u, \nabla v \rangle \; dV = \int_{\partial M} u \; Nv \; d\tilde{V} \tag{2.0.1}$$

where $Nv = \nabla v \cdot N$ and N is the outward pointing unit normal vector of ∂M.

For this dissertation, we will be working on closed manifolds, hence $\partial M = \emptyset$ and

$$\int_M u\Delta v \; dV = -\int_M \langle \nabla u, \nabla v \rangle \; dV \tag{2.0.2}$$

We define differential operators which we will use later. [5] chapters 6, 7 and [2] ch.3, sec. 6

Definition 2.0.2. We define a (time dependent) differential operator

$$Eu = -\sum_{i,j} a^{i,j}(x,t)u_{x_i,x_j} + \sum_i b^i(x,t)u_{x_i} + c(x,t)u$$

Here each coeffecient $a^{i,j}, b^j, c$ is a function.

We call E a uniformly elliptic operator if there exist some $\theta > 0$ so that

$$\sum_{i,j} a^{i,j}(x,t)\eta_i\eta_j \geq \theta|\eta|^2$$

for all x and t in the domain of $a^{i,j}$ and all vectors η

An operator of the form

$$Pu = \frac{\partial u}{\partial t} + Eu$$

is called a parabolic operator when E is a uniformly elliptic.

Remark 2.0.3. It is immediate that the Laplacian $E = -\Delta$ is a uniformly elliptic operator. In fact, if $\alpha(x,t)$ is a positive function bounded away from zero, then $E = -\alpha\Delta$ is a uniformly elliptic operator.

We will be using elliptic and parabolic operators to justify estimates of functions, the Scalar Maximum Principle is an important tool for this and we will use it throughout this dissertation. [1] page 99 and [2] page 130

Theorem 2.0.4. *(Scalar Maximum Principle) Let L be a parabolic operator such that $Lu = \frac{\partial u}{\partial t} + Eu - F(u,t)$ where where E is a linear uniformly elliptic operator and $F(u,t)$ is* *a continuous function in t and locally Lipschitz in u.*

Suppose u is C^2 and satisfies $Lu \leq 0$ on $M \times [0,T)$, and $u(x,0) \leq c$ for all $x \in M$. Let $\phi(t)$ be the solution to the associated ODE:

$$\frac{d\phi}{dt} = F(\phi, t) \tag{2.0.3}$$
$$\phi(0) = c$$

then

$$u(x,t) \leq \phi(t) \tag{2.0.4}$$

for all $x \in M$ and all $t \in [0,T)$ in the interval of existence of ϕ.

We will use this version more often than the alternative version where all inequalities are reversed.

The Comparison Principle is closely related to the Maximum Principle and will be used frequently as well. [1] page 98

Theorem 2.0.5. *(Comparison Principle)*
Suppose u and v are C^2 and satisfy $Lv \leq Lu$ on $M \times [0,T)$ and $v(x,0) \leq u(x,0)$ for all $x \in M$, then

$$v(x,t) \leq u(x,t)$$

holds on $M \times [0,T)$.

Here L is the same parabolic operator as in Theorem 2.0.4.

We will define Hölder and Sobolev Spaces. Both are common spaces encountered when studying partial differential equations. [5] ch. 5, [2] ch. 2

Definition 2.0.6. (Hölder Norm) For open set $U \subset (M^n, g_0)$, if $u : U \to \mathbb{R}$ is bounded and continuous, we write

$$||u||_{C(\bar{U})} = \sup_{x \in U} |u(x)| \tag{2.0.5}$$

For $0 < \gamma \leq 1$, we define the γ-Hölder seminorm of u by

$$[u]_{C^{0,\gamma}(\bar{U})} = \sup_{x,y \in U} \frac{|u(x) - u(y)|}{d_0(x,y)^\gamma}$$

where d_0 is the distance function given by the metric g_0.

We define γ-Hölder norm by

$$||u||_{C^{0,\gamma}(\bar{U})} = ||u||_{C(\bar{U})} + [u]_{C^{0,\gamma}(\bar{U})} \tag{2.0.6}$$

Definition 2.0.7. (Hölder Space) The Hölder Space $C^{k,\gamma}(\bar{U})$ is the space of all $u \in C^k(\bar{U})$ such that the norm

$$||u||_{C^{k,\gamma}(\bar{U})} = \sum_{|\alpha| \leq k} ||\nabla^\alpha u||_{C(\bar{U})} + \sum_{|\alpha| = k} [\nabla^\alpha u]_{C^{0,\gamma}(\bar{U})} \tag{2.0.7}$$

is finite.

Remark 2.0.8. It is straightforward to check that $C^{k,\gamma}(\bar{U})$ is a Banach space.

Definition 2.0.9. (Sobolev Space) For open subset U of M^n and positive integers k, p, the

Sobolev space is the set of functions

$$W^{k,p}(U) = \{u : U \to \mathbb{R} |\ u \in L^1_{loc}(U),\ \nabla^\alpha u \in L^p(U)\ for\ |\alpha| \leq k\} \tag{2.0.8}$$

here α is a multi-index.

$W^{k,p}(U)$ is a normed space with Sobolev norm

$$||u||_{W^{k,p}(U)} = (\sum_{|\alpha|\leq k} \int_U |\nabla^\alpha u|^p\ dx)^{\frac{1}{p}} \tag{2.0.9}$$

When $p = 2$, $W^{k,2}$ is frequently denoted by H^k.

Remark 2.0.10. It is straightforward to show $W^{k,p}$ is a Banach space. H^k is a Hilbert space.

There are several theorems which are sometimes called the Sobolev inequality. Roughly speaking, a Sobolev inequality bounds the norm of a function in one Sobolev space by its norm in another. This allows one to embed one Sobolev space into another provided certain assumptions are met.

We will state one which we will use in this dissertation. When we use the next theorem, $p = 2$, $k = 2$, and $n = 2$. [5] page 286-287

Theorem 2.0.11. *(Sobolev Inequality) For U a bounded open subset of M^n with C^1 boundary and $u \in W^{k,p}(u)$ with $k > \frac{n}{p}$ then*

$$||u||_{C^{k-[\frac{n}{p}]-1,\gamma}(\bar{U})} \leq C||u||_{W^{k,p}(U)} \tag{2.0.10}$$

where

$$\gamma = \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p} & \frac{n}{p} \text{ is not an integer} \\ \text{any positive number } < 1 & [\frac{n}{p}] \text{ in an integer} \end{cases}$$

We will be using the case where $p = 2$, $k = 2$, and $n = 2$ and have

$$\begin{aligned} ||u||_{L^\infty(U)} &\leq ||u||_{C^{0,\gamma}(\bar{U})} \\ &\leq C||u||_{H^2(U)} \end{aligned} \tag{2.0.11}$$

This next theorem is often called the Poincaré inequality. [5] page 292

Here $\bar{u} = \frac{\int_U u \ dV}{\int_U \ dV}$ is the mean value of u on U.

Theorem 2.0.12. *(Poincaré Inequality) Let U be a bounded, connected, open subset of (M, g) with a C^1 boundary, then there exist a constant C, depending only on n, p, and U such that*

$$||u - \bar{u}||_{L^p(U)} \leq C||\nabla u||_{L^p(U)}$$

for each function $u \in W^{1,p}(U)$.

Remark 2.0.13. On a closed manifold (M, g), we have

$$||u - \bar{u}||_{L^2} \leq \lambda||\nabla u||_{L^2}$$

where λ can be chosen to be the reciprocal of the smallest nonzero eigenvalue of Δ. If u is orthogonal to the first nontrivial eigenspace then λ can be taken to be the reciprocal of the next eigenvalue. This is sometimes called the sharp Poincaré inequality.

We will use the next inequality ubiquitously. We will call it the Arithmetic-Geometric Mean Inequality. [5] pg 708

Lemma 2.0.14. *(Arithmetic-Geometric Mean Inequality)*

$$|ab| \leq \frac{1}{2}a^2 + \frac{1}{2}b^2 \tag{2.0.12}$$

In fact, in general for $\epsilon > 0$

$$|ab| \leq \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2 \tag{2.0.13}$$

We we also use a generalization of the Arithmetic-Geometric Mean Inequality, called Young's Inequality

Lemma 2.0.15. *(Young's Inequality) For* $\epsilon > 0$ *and* $\frac{1}{p} + \frac{1}{q} = 1$

$$|AB| \leq \frac{\epsilon^p}{p}|A|^p + \frac{1}{q\epsilon^q}|B|^q \tag{2.0.14}$$

The next theorem is known as the Moser-Trudinger inequality for S^2. [11]

Theorem 2.0.16. *(Moser-Trudinger Inequality) For smooth function* u *on* S^2 *satisfying conditions*

$$\int_{S^2} |\nabla u|^2 \, dA \leq 1 \tag{2.0.15}$$

$$\int_{S^2} u \, dA = 0$$

then there exists a constant C such that

$$\int_{S^2} e^{4\pi u^2}\, dA \leq C \tag{2.0.16}$$

This next theorem is due to Michael Struwe. [13] Theorem 3.2

Theorem 2.0.17. *(Struwe's Theorem) Let $g_n = e^{2u_n} g_0$ be family of smooth metrics on M with unit volume and bounded Calabi energy, i.e. for some $C > 0$, for all n*

$$Ca(g_n) = \int_M |K_{u_n} - K_0|\, dA_{u_n} \leq C$$

then either

i) sequence $\{u_n\}$ is bounded in $H^2(M, g_0)$

or

ii) There exist $x_1, ..., x_L \in M$ and subsequence $\{u_{n_k}\}$ such that for any $R > 0$ and $l = 1, ..., L$

$$\liminf \int_{B_R(x_l)} |K_{u_{n_k}}|\, dA_{u_{n_k}} \geq 2\pi$$

We will use the Evans-Krylov Theorem, proved independently by Evans and Krylov. [4] and [8], c.f. [3]

Theorem 2.0.18. *(Evans-Krylov Theorem) For u, a smooth solution of a uniformly parabolic, fully non-linear, convex equation*

$$F(D^2 u) = 0 \text{ in the unit ball } B_1 \subset \mathbb{R}^n \tag{2.0.17}$$

then

$$||u||_{C^{2,\alpha}(B_{1/2})} \leq C ||u||_{C^{1,1}(B_1)} \tag{2.0.18}$$

with C depending on ellipticity of F.

We will make use of Interior Schauder Estimates. [2] chapter 3, page 88

Theorem 2.0.19. *(Interior Schauder Estimates) For linear second order, uniformly parabolic operator L with Hölder continuous C^α coefficients then*

$$||u||_{C^{2,\alpha}(K)} \leq C(||Lu||_{C^\alpha} + ||u||_{L^\infty}) \tag{2.0.19}$$

We will use the Gauss-Bonnet. [9] page 167

Theorem 2.0.20. *(Gauss-Bonnet Theorem) For a Riemannian surface (M, g) with Gauss curvature K_g and Euler characteristic $\chi(M)$, if M is compact and oriented, then*

$$\int_M K_g \, dA_g = 2\pi\chi(M) \tag{2.0.20}$$

Chapter 3

Energy Functional and Flow Equation

We wish to consider a one-parameter family of functions $\{u_t \mid t \in [0,T)\}$ on M by which we will perform a conformal change to metric g_0 given by

$$g_{u_t} = e^{2u_t} g_0 \tag{3.0.1}$$

As it is conventional, we will think of the variable $t \in [0,T)$ as "time" and the $x \in M$ as "space". We therefore write $u(x,t)$ as a function of both space and time.

We seek a differential equation by which u_t will evolve in time. We will then seek to show that the solution to this equation exists for all time.

Remark 3.0.1. We will frequently use of notation $\dot{u} = \frac{\partial u}{\partial t}$. The notation u_t does not refer to a derivative with respect to t. We write $u_0 = u(x,0)$.

The approach in this chapter is based on the approach taken in [6], section 6.

3.1 The Energy Functional and its Properties

We introduce the Liouville energy

Definition 3.1.1. (Liouville Energy Functional) The Liouville energy functional is given by

$$J[u] = \int_M |\nabla u|_0^2 \, dA_0 + 2\int_M K_0 u \, dA_0 \tag{3.1.1}$$

Remark 3.1.2. The Liouville energy is analogous to Yamabe's functional for dimension 2.

Remark 3.1.3. We see that $J[u]$ is well-defined for a compact manifold M.

Remark 3.1.4. We use K_0 to denote Gauss curvature with respect to g_0. This is not the same as K_{u_0}, the value of curvature at time $t = 0$.

We will modify this functional to fit our problem. Specifically we wish our functional to be scale invariant. For any constant c, we see that

$$g_{u+c} = e^{2(u+c)} g_0 = e^{2c} e^{2u} g_0 = C g_u$$

For this reason, we want functions u and $u + c$ to have the same energy value.

We introduce notation for a normalized integral with respect to a smooth function f. We will see that the function f, once weighted properly, will be the value to which the curvature of M will flow towards under the flow equation which we will derive shortly.

Definition 3.1.5. For a fixed $f \in C^\infty(M)$ with $\int_M f \, dA_0 < \infty$ and $f > 0$. We define the normalized integral by

$$\fint e^{2u}\Big| f = \frac{\int_M e^{2u}\ f\ dA_0}{\int_M f\ dA_0} = \frac{\int_M f\ dA_u}{\int_M f\ dA_0} \tag{3.1.2}$$

Here we introduce our notation

Definition 3.1.6.

$$B_u = \int f\ dA_u \quad \text{and} \quad B_0 = \int f\ dA_0$$

Remark 3.1.7. We have

$$\fint e^{2u}\Big| f = \frac{B_u}{B_0}$$

We consider the normalized functional

Definition 3.1.8. We define the functional $G : W^{1,2} \to \mathbb{R}$ by

$$G[u] = \int_M |\nabla u|_0^2\ dA_0 + 2\int_M K_0 u\ dA_0 - (\int_M K_0\ dA_0)\log(\fint e^{2u}\Big| f) \tag{3.1.3}$$

Remark 3.1.9. This functional is normalized in the sense that

$$\begin{aligned} G[u+c] &= \int_M |\nabla_0 u|_0^2\ dA_0 + 2\int_M K_0 u\ dA_0 + 2c\int_M K_0\ dA_0 \\ &\quad - (\int_M K_0\ dA_0)(2c + \log(\int_M f e^{2u}\ dA_0) - \log(\int_M f\ dA_0)) \\ &= G[u] \end{aligned} \tag{3.1.4}$$

as desired.

Remark 3.1.10. We note the $G[u]$ is well-defined since

$$\int f\ e^{2u}\ dA_0 \leq (\int f^2\ dA_0)^{\frac{1}{2}}(\int e^{4u}\ dA_0)^{\frac{1}{2}} < \infty \tag{3.1.5}$$

where the first inequality is due to Cauchy-Schwarz and the second is due to Moser-Trudinger inequality, Theorem 2.0.16.

We will need the derivative of $J[u]$

Proposition 3.1.11.

$$J'_u(v) = 2\int_M vK_u\ dA_u \tag{3.1.6}$$

Proof.

$$\begin{aligned}
(J')_u(v) &= \frac{d}{ds}J[u+sv]\Big|_{s=0} \\
&= \frac{d}{ds}\Big|_{s=0}\int(|\nabla u + s\nabla v|_0^2 + 2K_0u + 2sK_0v\ dA_0) \\
&= 2\int\langle\nabla u, \nabla v\rangle_0\ dA_0 + 2\int_M K_0v\ dA_0 \\
&= -2\int v\Delta_0 u\ dA_0 + 2\int_M K_0v\ dA_0 \\
&= 2\int(K_0 - \Delta_0 u)v\ dA_0 \\
&= 2\int vK_u\ dA_u
\end{aligned} \tag{3.1.7}$$

□

The next step is to compute the derivative of $G[u]$. We introduce the term $\tilde{K_u}$ which we think of as the mean curvature normalized by f. In fact, we will see that $\tilde{K_u}$ is the value for the Gauss curvature at which the flow equation is stationary.

Definition 3.1.12.

$$\begin{aligned}\tilde{K}_u &= \Big(\frac{\int_M K_u \ dA_u}{\int_M f \ dA_u}\Big)f \\ &= 2\pi f \frac{\chi(M)}{\int_M f \ dA_u} \\ &= \frac{4\pi f}{B_u}\end{aligned} \tag{3.1.8}$$

Here $\chi(M)$ is the Euler characteristic of M. Since we will be working on a topological sphere, we have $\chi(M) = 2$.

Proposition 3.1.13.

$$G'_u[v] = 2\int_M v(K_u - \tilde{K}_u) \ dA_u \tag{3.1.9}$$

Proof. Since

$$G[u] = J[u] - \Big(\int K_0 \ dA_0\Big)\Big(\log(⨍ e^{2u}\Big| f)\Big)$$

we have

$$G'_u[v] = J'_u[v] - \Big(\int K_0 \ dA_0\Big)\frac{d}{ds}\Big|_{s=0} \log(⨍ e^{2u+2sv}\Big| f)$$

We calculate

$$\frac{d}{ds}\Big|_{s=0} \log(\fint e^{2u+2sv}\Big| f) = \frac{d}{ds}\Big|_{s=0}\Big[\log(\int e^{2sv}\ e^{2u}\ f\ dA_0) - \log(\int f\ dA_0)\Big] \tag{3.1.10}$$
$$= \frac{\int 2ve^{2sv}\ e^{2u}\ f\ dA_0}{\int e^{2sv}\ e^{2u}\ f\ dA_0}\Big|_{s=0}$$
$$= \frac{2\int v\ e^{2u}\ f\ dA_0}{\int e^{2u}\ f\ dA_0}$$

and

$$-2\Big(\int K_0\ dA_0\Big)\frac{\int v\ e^{2u}\ f\ dA_0}{\int e^{2u}\ f\ dA_0} = -2\frac{\int K_0\ dA_0}{\int f\ dA_u}\int vf\ dA_u \tag{3.1.11}$$
$$= -2\frac{\int K_u\ dA_u}{\int f\ dA_u}\int vf\ dA_u$$
$$= -2\int v\tilde{K}_u\ dA_u$$

where the second equality is the result of the Gauss-Bonnet Theorem 2.0.20

We have the derivative of $J[u]$ from Proposition 3.1.11 and this yields

$$G'_u[v] = 2\int_M v(K_u - \tilde{K}_u)\ dA_u \tag{3.1.12}$$

□

3.1.1 Formal Inner Product

Proposition 3.1.13 will motivate the definition of a formal inner product on $C^\infty(M)$, the space of real-valued, smooth functions on M.

We define an inner product on $C^\infty(M)$.

Definition 3.1.14. For $v, w \in C^\infty(M)$

$$\langle v, w \rangle = \int_M vw\ K_u\ dA_u \tag{3.1.13}$$

Proposition 3.1.13 can be rewritten as

$$\begin{aligned} G'_u[v] &= 2\int_M v(K_u - \tilde{K}_u)\ dA_u \\ &= \langle v, \frac{K_u - \tilde{K}_u}{K_u} \rangle \end{aligned} \tag{3.1.14}$$

This will motivate our focus on the flow equation in the next section.

3.2 Flow Equation

We see from the variation equation for G'_u that we can show the existence of gradient flow by proving a partial differential equation has a smooth solution.

We will call the following equation the inverse Gauss curvature flow equation or simply the flow equation. It will be central for the remainder of this dissertation.

Definition 3.2.1. (Inverse Gauss Curvature Flow Equation)

$$\frac{\partial u}{\partial t} = -\frac{K_u - \tilde{K}_u}{K_u} \tag{3.2.1}$$

Now since curvature with respect to metric $g_u = e^{2u} g_0$ is given by

$$K_u = e^{-2u}(K_0 - \Delta_0 u)$$

and we have

$$\tilde{K}_u = \frac{4\pi f}{B_u}$$

the equation 3.2.1 becomes

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\tilde{K}_u}{K_u} - 1 \\ &= \frac{4\pi}{B_u} \frac{e^{2u} f}{K_0 - \Delta_0 u} - 1 \end{aligned} \tag{3.2.2}$$

3.2.1 Evolution Equations

This section is devoted to a large handful of computational lemmas. They are not very interesting on their own but will be important later in proving key theorems in the next section.

Lemma 3.2.2. *Let u be a solution to flow equation 3.2.1, then*

$$\frac{\partial u}{\partial t} = \frac{\tilde{K_u}}{K_u^2}\Delta u - 1 + \frac{2\tilde{K_u}}{K_u} - e^{-2u}\frac{K_0\tilde{K_u}}{K_u^2}$$

Proof. From the 1.2.4, we calculate

$$\begin{aligned} -\frac{\tilde{K_u}}{K_u^2}\Delta u &= \frac{\tilde{K_u}}{K_u^2}[K_u - e^{-2u}K_0] \\ &= \frac{\tilde{K_u}}{K_u} - e^{-2u}\frac{K_0\tilde{K_u}}{K_u^2} \end{aligned} \tag{3.2.3}$$

so then using flow equation 3.2.1, we have

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\tilde{K_u}}{K_u^2}\Delta u &= -\frac{K_u - \tilde{K_u}}{K_u} + \frac{\tilde{K_u}}{K_u} - e^{-2u}\frac{K_0\tilde{K_u}}{K_u^2} \\ &= -1 + \frac{2\tilde{K_u}}{K_u} - e^{-2u}\frac{K_0\tilde{K_u}}{K_u^2} \end{aligned} \tag{3.2.4}$$

and hence

$$\frac{\partial u}{\partial t} = \frac{\tilde{K_u}}{K_u^2}\Delta u - 1 + \frac{2\tilde{K_u}}{K_u} - e^{-2u}\frac{K_0\tilde{K_u}}{K_u^2} \tag{3.2.5}$$

as desired. □

Lemma 3.2.3. *Let u be a solution to flow equation 3.2.1, then*

$$\begin{aligned} \frac{\partial}{\partial t}K_u &= -2(\tilde{K_u} - K_u) - \Delta(\frac{\tilde{K_u}}{K_u}) \\ &= -2(\tilde{K_u} - K_u) - \frac{4\pi}{B_u}\frac{\Delta f}{K_u} + \frac{8\pi}{B_u}\frac{\langle \nabla f, \nabla K\rangle}{K_u^2} - 2\tilde{K_u}\frac{|\nabla K_u|^2}{K_u^3} + \tilde{K_u}\frac{\Delta K_u}{K_u^2} \end{aligned} \tag{3.2.6}$$

Proof.

$$\begin{aligned}\frac{\partial}{\partial t}K_u &= \frac{\partial}{\partial t}e^{-2u}[-\Delta_0 u + K_0] \\ &= -2\dot{u}e^{-2u}[-\Delta_0 u + K_0] - e^{-2u}\Delta_0\dot{u} \\ &= -2\dot{u}K_u - \Delta\dot{u} \\ &= -2(\tilde{K_u} - K_u) - \Delta(\frac{\tilde{K_u}}{K_u})\end{aligned} \tag{3.2.7}$$

We now compute $\Delta(\frac{\tilde{K_u}}{K_u})$ using normal coordinates

$$\begin{aligned}\Delta(\frac{\tilde{K_u}}{K_u}) &= \nabla\ \nabla(\frac{\tilde{K_u}}{K_u}) \\ &= \nabla(\frac{K_u\nabla\tilde{K_u} - \tilde{K_u}\nabla K_u}{K_u^2}) \\ &= \nabla(\frac{\nabla\tilde{K_u}}{K_u}) - \nabla(\frac{\tilde{K_u}\nabla K_u}{K_u^2}) \\ &= \frac{K_u\Delta\tilde{K_u} - \langle\nabla\tilde{K_u}, \nabla K_u\rangle}{K_u^2} - \frac{K_u^2\nabla(\tilde{K_u}\nabla K_u) - 2\tilde{K_u}K_u\langle\nabla K_u, \nabla K_u\rangle}{K_u^4} \\ &= \frac{\Delta\tilde{K_u}}{K_u} - \frac{\langle\nabla\tilde{K_u}, \nabla K_u\rangle}{K_u^2} - \frac{1}{K_u^2}(\langle\nabla\tilde{K_u}, \nabla K_u\rangle + \tilde{K_u}\Delta K_u) + 2\tilde{K_u}\frac{|\nabla K_u|^2}{K_u^3} \\ &= \frac{\Delta\tilde{K_u}}{K_u} - 2\frac{\langle\nabla\tilde{K_u}, \nabla K_u\rangle}{K_u^2} - \frac{\tilde{K_u}\Delta K_u}{K_u^2} + 2\tilde{K_u}\frac{|\nabla K_u|^2}{K_u^3} \\ &= \frac{4\pi}{B_u}\frac{\Delta f}{K_u} - \frac{8\pi}{B_u}\frac{\langle\nabla f, \nabla K\rangle}{K_u^2} + 2\tilde{K_u}\frac{|\nabla K_u|^2}{K_u^3} - \tilde{K_u}\frac{\Delta K_u}{K_u^2}\end{aligned} \tag{3.2.8}$$

□

We rewrite the the equation as

$$\frac{\partial}{\partial t}K_u - \frac{\tilde{K_u}}{K_u^2}\Delta K_u - \frac{8\pi}{B_u}\frac{\langle \nabla f, \nabla K_u \rangle}{K_u^2} + 2\tilde{K_u}\frac{|\nabla K_u|^2}{K_u^3} = -2\tilde{K_u} - \frac{4\pi}{B_u}\frac{\Delta f}{K_u} + 2K_u \tag{3.2.9}$$

We denote the left-hand of Equation 3.2.9 side by operator H acting on K_u, i.e.

$$H(K_u) \equiv \frac{\partial}{\partial t}K_u - \frac{\tilde{K_u}}{K_u^2}\Delta K_u - \frac{8\pi}{B_u}\frac{\langle \nabla f, \nabla K_u \rangle}{K_u^2} + 2\tilde{K_u}\frac{|\nabla K_u|^2}{K_u^3} \tag{3.2.10}$$

We see H is a parabolic operator and removing the negative term from the right-hand side attain

$$\begin{aligned} H(K_u) &= -2\tilde{K_u} - \frac{4\pi}{B_u}\frac{\Delta f}{K_u} + 2K_u \\ &\leq -\frac{4\pi}{B_u}\frac{\Delta f}{K_u} + 2K_u \\ &\leq \frac{\alpha}{K_u} + 2K_u \end{aligned} \tag{3.2.11}$$

We will write $\phi(x) = \frac{\alpha}{x} + 2x$, so we have

$$H(K_u) \leq \phi(K_u) \tag{3.2.12}$$

Proposition 3.2.4. *Let u be a solution to flow equation 3.2.1, then K_u has sub-exponential growth. Specifically there exist numbers P_1, P_2 so that*

$$K_u \leq P_1 + P_2 e^{2t} \tag{3.2.13}$$

Proof. Let function $v(t)$ be the solution to ODE

$$\frac{d}{dt}v(t) = \phi(v) \tag{3.2.14}$$
$$v(0) = \sup_{M\times\{0\}} K_0$$

where $\phi(x) = \frac{\alpha}{x} + 2x$.

Let $H_0 = H - \phi$. We see that H_0 is a parabolic operator and from inequality 3.2.11, we have

$$H_0(K_u) = H(K_u) - \phi(K_u) \leq 0$$

for u be a solution to flow equation 3.2.1.

By the Maximum Principle (Theorem 2.0.4), we conclude

$$\sup K_u \leq v(t) \tag{3.2.15}$$

To show the growth of K_u is sub-exponential, we assume $v(t) \geq 1$, then

$$v'(t) \leq \alpha + 2v \tag{3.2.16}$$

We can solve the ODE

$$w'(t) = \alpha + 2w \tag{3.2.17}$$
$$w(0) = P$$

to find

$$w(t) = P_1 + P_2 e^{2t}$$

for some P_1 and P_2.

We know $v'(t) \leq w'(t)$ and assuming that $P \geq \sup_{M\times\{0\}} v(x,0)$, we have $v(t) \leq w(t)$ and

$$K_u \leq P_1 + P_2 e^{2t}$$

as desired. □

Lemma 3.2.5. *Let u be a solution to flow equation 3.2.1, then*

$$\frac{\partial}{\partial t}\left(\frac{1}{K_u}\right) = 2\frac{\tilde{K_u}}{K_u^2} - \frac{2}{K_u} + \frac{1}{K_u^2}\Delta\left(\frac{\tilde{K_u}}{K_u}\right) \tag{3.2.18}$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{1}{K_u}\right) &= -\frac{1}{K_u^2}\frac{\partial K_u}{\partial t} \\
&= -\frac{1}{K_u^2}(-2\dot{u}K - e^{-2u}\Delta_0\dot{u}) \\
&= -\frac{1}{K_u^2}(2(K_u - \tilde{K_u}) - \Delta(\frac{\tilde{K}}{K})) \\
&= 2\frac{\tilde{K_u}}{K_u^2} - \frac{2}{K_u} + \frac{1}{K_u^2}\Delta\left(\frac{\tilde{K_u}}{K_u}\right)
\end{aligned} \tag{3.2.19}$$

□

Lemma 3.2.6. *Let u be a solution to flow equation 3.2.1 and G be as in definition 3.1.8,* *then*

$$\frac{dG}{dt} = 8\pi - 2\frac{(4\pi)^2}{B_u^2}\int_M \frac{f^2}{K_u}\ dA_u \tag{3.2.20}$$

Proof. Using Proposition 3.1.13 and Equation 3.2.1, we compute

$$\begin{aligned}
\frac{dG}{dt} &= G'[\dot{u}] = 2\int_M \dot{u}(K_u - \tilde{K_u})\ dA_u \qquad (3.2.21)\\
&= -2\int_M (1-\frac{\tilde{K_u}}{K_u})^2\ K_u\ dA_u\\
&= -2\int_M (1-2\frac{\tilde{K_u}}{K_u} + \frac{\tilde{K_u}^2}{K_u^2})\ K_u\ dA_u\\
&= -2\Big(\int_M K_u\ dA_u - 2\int_M \tilde{K_u}\ dA_u + \int_M \frac{\tilde{K_u^2}}{K_u}\ dA_u\Big)\\
&= 2\Big(-4\pi + 2\frac{4\pi}{B_u}\int f\ dA_u - \frac{(4\pi)^2}{B_u^2}\int \frac{f^2}{K_u}\ dA_u\Big)\\
&= 2\Big(4\pi - \frac{(4\pi)^2}{B_u^2}\int_M \frac{f^2}{K_u}\ dA_u\Big)
\end{aligned}$$

□

Lemma 3.2.7. *Let u be a solution to flow equation 3.2.1, then*

$$B(t) = B(0)\exp\{\frac{1}{4\pi}(G[u(0)] - G[u(t)])\}$$

here $B(t) = \int_M f\ dA_{u(t)}$

Proof.

$$\begin{aligned}
\frac{d}{dt}B(t) &= 2\int_M \dot{u}e^{2u}f\ dA_0 && (3.2.22)\\
&= 2\int_M (-1+\frac{\tilde{K_u}}{K_u})f\ dA_u\\
&= B(t)\big(-2+\frac{2}{B(t)}\int_M \frac{\tilde{K_u}}{K_u}f\ dA_u\big)\\
&= B(t)\big(-2+\frac{8\pi}{B(t)^2}\int_M \frac{1}{K_u}f^2\ dA_u\big)\\
&= \frac{-1}{4\pi}B(t)\frac{dG}{dt}
\end{aligned}$$

We used Lemma 3.2.6 in the last step. □

Lemma 3.2.8. *Let u be a solution to flow equation 3.2.1, then for u at all t*

$$\int_M (|\nabla u|_0^2 + 2K_0 u)\ dA_0 = \int_M (|\nabla u_0|_0^2 + 2K_0 u_0)\ dA_0 \tag{3.2.23}$$

Proof. We know

$$\begin{aligned}
G[u] &= \int_M (|\nabla u|_0^2 + 2K_0 u)\ dA_0 - (\int_M K_0\ dA_0)\log(\fint e^{2u}\Big|f) && (3.2.24)\\
&= \int_M (|\nabla u|_0^2 + 2K_0 u)\ dA_0 - 4\pi(\log B_u - \log B_0)
\end{aligned}$$

By Lemma 3.2.7

$$\begin{aligned}\int_M (|\nabla u|_0^2 + 2K_0 u)\ dA_0 &= G[u] + 4\pi(\log B_u - \log B_0) \\ &= G[u] + 4\pi(\log B_0 + \frac{1}{4\pi}(G[u_0] - G[u]) - \log B_0) \\ &= G[u_0] = \int_M (|\nabla u_0|_0^2 + 2K_0 u_0)\ dA_0\end{aligned} \tag{3.2.25}$$

□

Lemma 3.2.9. *For solution u to the flow equation 3.2.1,*

$$\inf_{M\times\{t\}} u \geq \inf_M u_0 - t \tag{3.2.26}$$

Proof.

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\tilde{K_u} - K_u}{K_u} \\ &= \frac{\tilde{K_u}}{K_u} - 1 \geq -1\end{aligned} \tag{3.2.27}$$

The result follows from the Comparison Principle (Theorem 2.0.5). □

3.2.2 First Results

We are now ready for our first bounds concerning u of real significance.

Proposition 3.2.10. *Let u be a solution to the equation 3.2.1, then there exists constant C,* *depending only on u_0 and M, such that for all t for which u exists, we have*

$$||\nabla u||^2_{L^2(M,g_0)} \leq C(1+t) \tag{3.2.28}$$

Proof. By Lemma 3.2.8 and Lemma 3.2.9

$$\begin{aligned} ||\nabla u(t)||^2_{L^2(M,g_0)} &= \int_M (|\nabla u|^2_0 + 2K_0 u - 2K_0 u)\ dA_0 \\ &= \int_M (|\nabla u|^2_0 + 2K_0 u)\ dV_0 - 2K_0 \int_M u\ dA_0 \\ &= \int_M (|\nabla u_0|^2_0 + 2K_0 u_0)\ dV_0 - 2K_0 \int_M u\ dA_0 \\ &\leq C - C \inf_M u \\ &\leq C(1+t) \end{aligned} \tag{3.2.29}$$

□

Proposition 3.2.11. *Let u be a solution to the flow equation 3.2.1 on [0, T), then there exist constant C, depending only on u at $t = 0$, T, and M, such that for all t for which u exists, we have*

$$\sup_{M\times[0,T)} |u| \leq C \tag{3.2.30}$$

Proof. Since K_u is positive,

$$\begin{aligned}\int_{B_R(x_0)} |K_u| \ dA_u &= \int_{B_R(x_0)} K_u e^{2u} \ dA_0 \\ &\leq C \int_{B_R(x_0)} e^{2u} \ dA_0 \\ &\leq C[\int_{B_R(x_0)} e^{4u} \ dA_0]^{\frac{1}{2}} [\int_{B_R(x_0)} 1 \ dA_0]^{\frac{1}{2}} \\ &\leq CR[\int_{B_R(x_0)} e^{4u} \ dA_0]^{\frac{1}{2}}\end{aligned} \tag{3.2.31}$$

We used Hölder's inequality for the last line.

We denote

$$\bar{u}_0 = \frac{\int_M u \ dA_0}{\int_M \ dA_0}$$

that is the mean value of u with respect to the background metric g_0.

After some manipulation, we use the Moser-Trudinger inequality 2.0.16 to see

$$\begin{aligned}\int_{B_R(x_0)} e^{4u} \ dA_0 &\leq e^{4\bar{u}_0} \int_{B_R(x_0)} e^{4(u-\bar{u}_0)} \ dA_0 \\ &\leq e^{4\bar{u}_0} \int_{B_R(x_0} e^{\frac{\epsilon}{2}(u-\bar{u}_0)^2 + \frac{16}{2\epsilon}} \ dA_0 \\ &= e^{4\bar{u}_0} e^{\frac{8}{\epsilon}} \int_{B_R(x_0)} e^{\frac{\epsilon}{2}(u-\bar{u}_0)^2} \ dA_0 \\ &\leq C\end{aligned} \tag{3.2.32}$$

We choose $\epsilon > 0$ to be small enough so that may use the inequality.

Now for small R, we have

$$\int_{B_R(x_0)} |K_u| \, dA_u \leq C_1 R < 2\pi \tag{3.2.33}$$

We now use the Theorem of Struwe 2.0.17 to conclude that u is bounded in $H^2(M, g_0)$, we then use the Sobolev inquality, Theorem 2.0.11, to conclude u is uniformly bounded. □

Proposition 3.2.12. *Let u be solution to flow equation 3.2.1 on $[0, T)$, then there exists constant $C = C(T, u_0)$ such that*

$$\frac{1}{K_u} \leq C \tag{3.2.34}$$

Proof. We let the parabolic operator $H_l = \frac{\partial}{\partial t} - \frac{\tilde{K_u}}{K_u^2}\Delta$ act on $\Phi = \frac{\tilde{K_u}}{K_u} + Au$ for some $A > 0$ we will choose later.

We use Lemma 3.2.3 to evaluate

$$
\begin{aligned}
H_l(\frac{\tilde{K_u}}{K_u}) &= \frac{\partial}{\partial t}(\frac{\tilde{K_u}}{K_u}) - \frac{\tilde{K_u}}{K_u^2}\Delta(\frac{\tilde{K_u}}{K_u}) \\
&= \frac{1}{K_u}(\frac{\partial}{\partial t}\tilde{K_u}) - \frac{\tilde{K_u}}{K_u^2}(\frac{\partial}{\partial t}K_u) - \frac{\tilde{K_u}}{K_u^2}\Delta(\frac{\tilde{K_u}}{K_u}) \\
&= \frac{1}{K_u}(\frac{\partial}{\partial t}\tilde{K_u}) - \frac{\tilde{K_u}}{K_u^2}(\frac{\partial}{\partial t}K_u + \Delta(\frac{\tilde{K_u}}{K_u})) \\
&= \frac{1}{K_u}(\frac{\partial}{\partial t}\tilde{K_u}) - \frac{\tilde{K_u}}{K_u^2}(-2(\tilde{K_u} - K_u)) \\
&= \frac{1}{K_u}(\frac{\partial}{\partial t}\tilde{K_u}) + 2(\frac{\tilde{K_u}}{K_u})^2 - 2(\frac{\tilde{K_u}}{K_u}) \\
&= \frac{\tilde{K_u}}{K_u}[-(\frac{B'_u}{B_u}) + 2\frac{\tilde{K_u}}{K_u} - 2]
\end{aligned} \tag{3.2.35}
$$

For the last line, we computed

$$\frac{\partial}{\partial t}\tilde{K_u} = Cf\frac{\partial}{\partial t}(\frac{1}{B_u}) = -\tilde{K_u}(\frac{B'_u}{B_u})$$

where

$$B'_u = \frac{\partial}{\partial t}B_u = 2\int_M \frac{\partial u}{\partial t}f \; dA_u$$

We observe that $B'_u = 2\int_M \frac{\partial u}{\partial t}f \; dA_u \leq C_0 \int_M f \; dA_u = B_u$ for some C_0 and thus $\frac{B'_u}{B_u} \leq C_0$.

By Lemma 3.2.2

$$H_l(u) = -1 + 2\frac{\tilde{K_u}}{K_u} - e^{-2u}\frac{K_0\tilde{K_u}}{K_u^2}$$

Putting the parts together and estimating, we get

$$
\begin{aligned}
H_l(\Phi) &= H_l(\frac{\tilde{K_u}}{K_u}) + AH_l(u) \qquad (3.2.36)\\
&= \frac{\tilde{K_u}}{K_u}[-(\frac{B'_u}{B_u}) + 2\frac{\tilde{K_u}}{K_u} - 2] + A[-1 + 2\frac{\tilde{K_u}}{K_u} - e^{-2u}\frac{K_0\tilde{K_u}}{K_u^2}]\\
&= K_u^{-2}[\tilde{K_u}K_u(\frac{-B'_u}{B_u}) + 2\tilde{K_u}^2 - 2\tilde{K_u}^2 K_u - AK_u^2 + 2A\tilde{K_u}K_u - Ae^{-2u}K_0\tilde{K}]\\
&\leq K_u^{-2}[CK_u + 2C + ACK_u - \delta A]
\end{aligned}
$$

We choose C so that $\tilde{K_u},\ \tilde{K_u}^2, C_0 < C$ and δ so that $-e^{-2u}K_0\tilde{K_u} \leq -\delta$ and assumed $A > 1$ We want $CK_u + 2C + ACK_u - \delta A \leq 0$, we assume K_u^{-1} attains a large maximum so that $K_u^{-1} \geq \frac{2C}{\delta}$, then $\frac{2CAK_u}{\delta} \leq A$ and

$$
CK + 2C + ACK_u - \delta A \leq CK_u + 2C - A\frac{\delta}{2} < 0
$$

for large enough A.

By the Comparison Principle (Theorem 2.0.5), Φ is bounded and therefore K_u^{-1} is bounded as desired. □

These next two lemmas will be used to prove Proposition 3.2.16, which gives a uniform bound for the Hessian of u.

Lemma 3.2.13. *Let u be a solution to flow equation 3.2.1, then*

$$
\frac{\partial}{\partial t}|\nabla u|_0^2 \leq \frac{\tilde{K_u}}{K_u^2}\Delta|\nabla u|_0^2 - \frac{2\tilde{K_u}}{K_u^2}e^{-2u}|\nabla_0^2 u|_0^2 + C_1(|\nabla u|_0^2 + 1) \qquad (3.2.37)
$$

Proof. We calculate

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla u &= \nabla \frac{\partial}{\partial t} u \\
&= \nabla(\frac{\tilde{K_u}}{K_u} - 1) \\
&= \frac{K\nabla \tilde{K_u} - \tilde{K_u}\nabla K_u}{K_u^2} \\
&= -\frac{\tilde{K_u}}{K_u^2}\nabla K_u + \frac{\nabla \tilde{K_u}}{K_u}
\end{aligned} \tag{3.2.38}$$

Now

$$\begin{aligned}
\nabla K_u &= \nabla[e^{-2u}(K_0 - \Delta_0 u)] \\
&= e^{-2u}\nabla(K_0 - \Delta_0 u) + (K_0 - \Delta_0 u)(-2e^{-2u})\nabla u \\
&= -e^{-2u}\nabla(\Delta_0 u) - 2K_u\nabla u + e^{-2u}\nabla K_0
\end{aligned} \tag{3.2.39}$$

which gives

$$\frac{\partial}{\partial t}\nabla u = \frac{\tilde{K_u}}{K_u^2} e^{-2u}\nabla(\Delta_0 u) + 2\frac{\tilde{K_u}}{K_u}\nabla u - \frac{\tilde{K_u}}{K_u^2} e^{-2u}\nabla K_0 + \frac{1}{K_u}\nabla \tilde{K_u} \tag{3.2.40}$$

We then calculate

$$
\begin{aligned}
\frac{\partial}{\partial t}|\nabla u|_0^2 &= \frac{\partial}{\partial t}\langle \nabla u, \nabla u\rangle_0 \\
&= 2\langle \nabla u, \frac{\partial}{\partial t}\nabla u\rangle_0 \\
&= 2\langle \nabla u, \frac{\tilde{K_u}}{K_u^2}e^{-2u}\nabla(\Delta_0 u) + 2\frac{\tilde{K_u}}{K_u}\nabla u - \frac{\tilde{K_u}}{K_u^2}e^{-2u}\nabla K_0 + \frac{\nabla \tilde{K_u}}{K_u}\rangle_0 \\
&= 2\langle \nabla u, \frac{\tilde{K_u}}{K_u^2}e^{-2u}\nabla(\Delta_0 u) + X_1 * \nabla u + X_2\rangle_0 \\
&= 2\langle \nabla u, \frac{\tilde{K_0}}{K_0^2}e^{-2u}\nabla(\Delta_0 u)\rangle_0 + \langle \nabla u, X_1 * \nabla u\rangle + \langle \nabla u, X_2\rangle_0 \\
&= \frac{\tilde{K_u}}{K_u^2}e^{-2u}\Delta_0|\nabla u|_0^2 - \frac{2\tilde{K_u}}{K_u^2}e^{-2u}|\nabla^2 u|_0^2 + \langle \nabla u, X_1 * \nabla u\rangle_0 + \langle \nabla u, X_2\rangle_0 \\
&\leq \frac{\tilde{K_u}}{K_u^2}\Delta|\nabla u|_0^2 - \frac{2\tilde{K_u}}{K_u^2}e^{-2u}|\nabla^2 u|_0^2 + C_1(|\nabla u|_0^2 + 1)
\end{aligned}
\tag{3.2.41}
$$

In the second to last line we used the identity

$$\Delta|\nabla u|^2 = 2|\nabla^2 u|^2 + 2\langle \nabla u, \nabla(\Delta u)\rangle$$

□

Lemma 3.2.14. *Let u be a solution to flow equation 3.2.1 and Z be a vector field, then*

$$\frac{\partial}{\partial t}\nabla_Z^0\nabla_Z^0 u \leq \frac{\tilde{K_u}}{K_u^2}\Delta\nabla_Z^0\nabla_Z^0 u + \nabla_0 Z * \nabla_0^3 u + X_1 * \nabla_Z^0\nabla_Z^0 u + X_2 * (\nabla u)^{*2} + X_3 \tag{3.2.42}$$

Proof. We calculate

$$
\begin{aligned}
\frac{\partial}{\partial t}\nabla^0_Z\nabla^0_Z u &= \nabla^0_Z\nabla^0_Z[\frac{\tilde{K_u}}{K_u}-1] \qquad (3.2.43)\\
&= \nabla^0_Z[\frac{1}{K_u}\nabla^0_Z\tilde{K_u} - \frac{\tilde{K_u}}{K_u^2}\nabla^0_Z K_u]\\
&= -\frac{\tilde{K_u}}{K_u^2}\nabla^0_Z\nabla^0_Z K_u + \frac{2\tilde{K_u}}{K_u^3}\nabla^0_Z K_u\otimes\nabla^0_Z K_u - \frac{2}{K_u^2}\nabla^0_Z\tilde{K}\otimes\nabla^0_Z K_u - \frac{1}{K_u}\nabla^0_Z\nabla^0_Z\tilde{K_u}\\
&= -\frac{\tilde{K_u}}{K_u^2}[(\nabla^0_Z\nabla^0_Z e^{-2u})e^{2u}K_u + 2\nabla^0_Z e^{-2u}\otimes\nabla^0_Z(e^{2u}K_u) + e^{-2u}\nabla^0_Z\nabla^0_Z(-\Delta_0 u + K_0)]\\
&\quad + \frac{2\tilde{K_u}}{K_u^3}\nabla^0_Z K_u\otimes\nabla^0_Z K_u - \frac{2}{K_u^2}\nabla^0_Z\tilde{K_u}\otimes\nabla^0_Z K_u - \frac{1}{K_u}\nabla^0_Z\nabla^0_Z\tilde{K_u}\\
&\le \frac{\tilde{K_u}}{K_u^2}e^{-2u}\nabla^0_Z\nabla^0_Z\Delta_0 u + X_1 * \nabla^0_Z\nabla^0_Z u + X_2 * (\nabla u)^{*2} + X_3
\end{aligned}
$$

Now

$$
\begin{aligned}
\nabla^0_Z\nabla^0_Z\Delta_0 u &= \nabla^0_Z\nabla^0_Z\nabla_i\nabla_i u \qquad (3.2.44)\\
&= \nabla^0_Z(\nabla^0_i\nabla^0_Z\nabla^0_i u + K * \nabla^0_i u)\\
&= \nabla^0_Z(\nabla^0_i\nabla^0_i\nabla^0_Z u + K * \nabla^0_i u + K * u)\\
&= \nabla^0_i\nabla^0_Z\nabla^0_i\nabla^0_Z u + K * \nabla^0_i u + K * \nabla^0_i\nabla^0_Z u + \nabla^0_Z(K * \nabla^0_i u) + \nabla^0_Z(K * u)\\
&= \Delta_0(\nabla^0_Z\nabla^0_Z u(Z,Z)) + \nabla_0 Z * \nabla^3_0 u + \nabla^2_0 Z * \nabla^3_0 u\\
&\quad + \nabla^2_0 u * \nabla^2_0 Z + K * \nabla^2_0 u + \nabla_0 K * \nabla_0 u + \nabla_0 K * u + K * \nabla^0_Z u
\end{aligned}
$$

which leaves us with

$$\frac{\partial}{\partial t}\nabla^0_Z\nabla^0_Z u \leq \frac{\tilde{K_u}}{K_u^2}e^{-2u}\Delta_0\nabla^0_Z\nabla^0_Z u + \nabla_0 Z * \nabla^3_0 u + X_1 * \nabla^0_Z\nabla^0_Z u + X_2 * (\nabla u)^{*2} + X_3 \tag{3.2.45}$$

as desired. □

Remark 3.2.15. We will write $\nabla^2_0 u(Z,Z)$ in the place of $\nabla^0_Z\nabla^0_Z u$ in the next proof.

We will now show that for a solution to the flow equation, the Hessian is uniformly bounded on its interval of existence, $[0,T)$.

Proposition 3.2.16. *For a solution to flow equation 3.2.1 on $[0,T)$, there exist a C so that*

$$\sup_{M\times[0,T)} |\nabla^2_0 u| \leq C \tag{3.2.46}$$

Proof. We define

$$\beta(p) = \max_{X\in T_pM\setminus\{0\}} \frac{\nabla^2_0 u(X,X)}{|X|^2}$$

If we find an upper bound for β, then we have an upper bound for $\nabla^2_0 u$.

For fixed constant $A > 0$ we consider the function

$$\Phi(x,t) = t\beta(x) + A|\nabla u|^2_0 \;\text{ for }\; t\in[0,1]$$

We will show that if A is sufficiently large, then we will have an *a priori* bound for an interior maximum of Φ.

Let (x_0, t_0) be the point at which $\Phi(x,t)$ attains an interior spacetime maximum and let $Z_0 \in T_{x_0}M$ be a unit vector so that $\beta(x) = \nabla_0^2 u(Z_0, Z_0)$. We extend Z_0 by parallel transport along radial geodesics to a vector field Z on $B_\epsilon(x_0)$ so that

$$\begin{aligned} |Z| &= 1 \text{ on } B_\epsilon(x_0) \\ \nabla_X^0 Z(x_0) &= 0 \text{ for all vector fields } X \\ |\nabla_0^2 Z|(x_0) &\le C_0(g_0) \end{aligned} \tag{3.2.47}$$

Now we define

$$\Psi(x,t) = t\nabla_0^2 u(Z,Z) + A|\nabla u|_0^2$$

and see that Ψ also attains a spacetime maximum at (x_0, t_0) as well.
Let $H_l = \frac{\partial}{\partial t} - \frac{\tilde{K}}{K^2}\Delta$ as before.

H_l is an parabolic operator and we use Lemma 3.2.13 and Lemma 3.2.14

$$\begin{aligned} H_l\Psi &= \frac{\partial \Psi}{\partial t} - \frac{\tilde{K}}{K^2}\Delta\Psi \qquad (3.2.48)\\ &= \nabla_0^2 u(Z,Z) + t(\frac{\partial}{\partial t}\nabla_0^2 u(Z,Z)) + A\frac{\partial}{\partial t}|\nabla u|_0^2 - t\frac{\tilde{K}}{K^2}\Delta(\nabla_0^2 u(Z,Z)) - A\frac{\tilde{K}}{K^2}\Delta|\nabla u|_0^2 \\ &= \nabla_0^2 u(Z,Z) + t(\frac{\partial}{\partial t}\nabla_0^2 u(Z,Z) - \frac{\tilde{K}}{K^2}\Delta(\nabla_0^2 u(Z,Z))) + A(\frac{\partial}{\partial t}|\nabla u|_0^2 - \frac{\tilde{K}}{K^2}\Delta|\nabla u|_0^2) \\ &\le C_0 + (\nabla_0 Z * \nabla_0^3 u + X_1 * \nabla_Z^0 \nabla_Z^0 u + X_2 * (\nabla u)^{*2} + X_3) \\ &+ A(-\frac{2\tilde{K}}{K^2}e^{-2u}|\nabla_0^2 u|_0^2 + C_1(|\nabla_0 u|_0^2 + 1)) \\ &\le -\delta|\nabla_0^2 u|_0^2 + C \\ &\le C \end{aligned}$$

We achieve the penultimate inequality from the uniform bounds for u and ∇u and applying Cauchy-Schwartz inequality and Arithmetic-Geometric Mean Inequality

$$\begin{aligned} |X_1 * \nabla_0^2 u|_0 &\le C_0|\nabla_0^2 u|_0 \qquad (3.2.49)\\ &= (\delta^{-1/2}C_0)(|\nabla_0^2 u|_0 \delta^{1/2}) \\ &\le \frac{\delta}{2}|\nabla_0^2 u|_0^2 + C \end{aligned}$$

We then choose A large enough so that the $\frac{\delta}{2} - A(\frac{2\tilde{K}}{K^2}e^{-2u}) \le -\delta$.

Since $H_l\Psi$ bounded and $\Psi(x,0) = A|\nabla u|_0^2$ we can conclude Ψ is bounded and hence $|\nabla_0^2 u|$ is bounded. □

Theorem 3.2.17. *For compact Riemannian manifold (M^2, g_0) with $K_0 > 0$, the solution to the flow equation 3.2.1 exists for all time.*

Proof. The operator

$$\Phi(u) = \frac{\tilde{K_u} - K_u}{K_u}$$

is a convex operator in the leading order term. Furthermore since curvature is uniformly bounded, the equation is uniformly parabolic.

By Evans-Krylov Theorem we obtain a priori $C^{2,\alpha}$ estimate for u, that is

$$||u||_{C^{2,\alpha}} \leq C_1$$

since we have a bound for the Hessian of u from Proposition 3.2.16.

We bootstrap Schauder estimates to yield estimates for $C^{k,\alpha}$ norm of u. Since all derivatives exist and are controlled, the solution exists on $[0, \infty)$. □

Chapter 4

Regularity Estimates

Our goal for this section is to show the regularity of the curvature K_u as u evolves. To do this we must start with assumptions for the regularity of K_u and extrapolate stronger regularity from them.

We will assume K_u is bounded away from 0 and infinity, that is for some $\lambda > 0$

$$\frac{1}{\lambda} < K_u < \lambda \tag{4.0.1}$$

We also assume that its first and second derivatives are bounded, i.e. for some C_1, C_2

$$\begin{aligned} |\nabla K_u|^2 &\leq C_1 \\ |\nabla^2 K_u|^2 &\leq C_2 \end{aligned} \tag{4.0.2}$$

Remark 4.0.1. Following the convention laid out earlier, the undecorated ∇ denote covariant derivatives under the metric g_u.

We will approach the problem by looking at the quantity

$$\kappa = \frac{1}{K_u} \tag{4.0.3}$$

instead of K_u directly.

We will express equations in terms of κ.

The flow equation 3.2.1 becomes

$$\frac{\partial u}{\partial t} = \tilde{K}_u \kappa - 1 \tag{4.0.4}$$

and our regularity assumptions become

$$\begin{aligned} \frac{1}{\lambda} < \kappa < \lambda \\ |\nabla \kappa|^2 \leq C_1' \\ |\nabla^2 \kappa|^2 \leq C_2' \end{aligned} \tag{4.0.5}$$

since

$$\begin{aligned} \nabla \kappa = -\frac{1}{K_u} \nabla K_u \\ \text{and} \\ \nabla^2 \kappa = \frac{2}{K_u^2} \nabla K_u - \frac{1}{K_u} \nabla^2 K_u \end{aligned} \tag{4.0.6}$$

Remark 4.0.2. Even though $\dot{u} = \tilde{K_u}\kappa - 1$, and $\tilde{K_u}$ depends on t and x, for the next theorem, we will assume that

$$\tilde{K_u} = 1$$

so that

$$\dot{u} = \kappa - 1$$

This assumption will greatly simplify our calculations and is justified because $\tilde{K_u}$ is close to 1 when f is close to 1 and u is small.

Theorem 4.0.3. *Let u be a solution to flow equation 4.0.4 that satisfies assumptions 4.0.1 and 4.0.2, then for all order i there exist some constant C_i such that for large t*

$$|\nabla^i \kappa|^2 \leq \frac{C_i}{t^{i-2}} \tag{4.0.7}$$

The constant C_i depends only on the lower order derivatives, C_2, C_1, and λ.

Remark 4.0.4. We notice that each $\nabla^i \kappa$ is pointwise bounded as a consequence of Theorem 4.0.3.

We express the results of the theorem in terms of Gauss curvature K_u.

Corollary 4.0.5. *Let u be a solution to flow equation 4.0.4 that satisfies assumptions 4.0.1 and 4.0.2 then for all order i, there exist some C_i' depending only on the lower order deriva-*

tives, C_2*,* C_1*, and* λ *such that*

$$|\nabla^i K_u|^2 \leq C_i' \tag{4.0.8}$$

We will outline the proof now, but the final proof will depend on several lemmas which we prove in the next section.

We will prove Theorem 4.0.3 through mathematical induction. We will assume $|\nabla^i \kappa|^2$, ... ,$|\nabla \kappa|^2$ are bounded and use that fact to show $|\nabla^{i+1}\kappa|^2$ is bounded as well.

We consider the function

$$G(x,t) = t^{i-2}|\nabla^{i+1}\kappa|^2 + At^{i-2}|\nabla^i\kappa|^2 \tag{4.0.9}$$

We see that if $G(x,t)$ is bounded, then $|\nabla^{i+1}\kappa|^2$ is bounded as stated in Theorem 4.0.3; we will show G is bounded by applying the Maximum Principle to H_l acting on G.

We compute

$$\begin{aligned} H_l G &= (\frac{\partial}{\partial t} - \kappa^2\Delta)(t^{i-1}|\nabla^{i+1}\kappa|^2 + At^{i-2}|\nabla^i\kappa|^2) \qquad (4.0.10)\\ &= (i-1)t^{i-2}|\nabla^{i+1}\kappa|^2 + A(i-2)t^{i-3}|\nabla^i\kappa|^2 \\ &+ t^{i-1}(\frac{\partial}{\partial t}|\nabla^{i+1}\kappa|^2 - \kappa^2\Delta|\nabla^{i+1}\kappa|^2) + At^{i-2}(\frac{\partial}{\partial t}|\nabla^i\kappa|^2 - \kappa^2\Delta|\nabla^i\kappa|^2) \end{aligned}$$

In order to estimate $H_l G$, we must find an appropriate bound for

$$H_l(|\nabla^i \kappa|^2) = \frac{\partial}{\partial t}|\nabla^i \kappa|^2 - \kappa^2 \Delta |\nabla^i \kappa|^2$$

and

$$H_l(|\nabla^i \kappa|^2) = \frac{\partial}{\partial t}|\nabla^{i+1} \kappa|^2 - \kappa^2 \Delta |\nabla^{i+1} \kappa|^2$$

This will be given by Lemma 4.0.13, where we see, for any $\epsilon > 0$,

$$H_l(|\nabla^i \kappa|^2) \leq (-2\kappa^2 + \epsilon)|\nabla^{i+1}\kappa|^2 + |\kappa * ... * \nabla^{i-1}\kappa||\nabla^i \kappa|^2 + |\kappa^m * \nabla\kappa * ... * \nabla^{i-1}\kappa|$$

and

$$H_l(|\nabla^{i+1} \kappa|^2) \leq (-2\kappa^2 + \epsilon)|\nabla^{i+2}\kappa|^2 + |\kappa * ... * \nabla^{i}\kappa||\nabla^{i+1} \kappa|^2 + |\kappa^m * \nabla\kappa * ... * \nabla^{i}\kappa|$$

4.0.1 Proof of Theorem

We will start by proving a series of lemmas which will lead us to Lemma 4.0.13 and the final proof of Theorem 4.0.3

Lemma 4.0.6. *Let u be a solution to flow equation 4.0.4 and $\kappa = \frac{1}{K_u}$,*

$$\dot{\kappa} = 2\kappa(\tilde{K}_u \kappa - 1) + \kappa^2 \Delta(\tilde{K}_u \kappa)$$

Proof.

$$\begin{aligned}\dot{\kappa} &= \frac{d}{dt}\left(\frac{1}{K_u}\right) \\ &= \frac{d}{dt}\left(\frac{e^{2u}}{1-\Delta_0 u}\right) \\ &= \frac{2\dot{u}e^{2u}(1-\Delta_0 u) + e^{2u}\Delta_0\dot{u}}{(1-\Delta_0 u)^2} \\ &= 2\kappa\dot{u} + \kappa^2 e^{-2u}\Delta_0\dot{u} \\ &= 2\kappa(\tilde{K}_u\kappa - 1) + \kappa^2\Delta(\tilde{K}_u\kappa)\end{aligned} \tag{4.0.11}$$

□

Remark 4.0.7. In accordance with Remark 4.0.2, we will write

$$\dot{\kappa} = 2\kappa^2 - 2\kappa + \kappa^2\Delta\kappa \tag{4.0.12}$$

in order to make computations more tractable.

Remark 4.0.8. We introduce notation $A_1 * ... * A_k$ to denote a linear combination of tensor products involving some of the terms A_1, ..., A_k. These may include contractions with respect to g_u so that each expression of the form $A_1 * ... * A_k$ is a tensor of appropriate rank.

Lemma 4.0.9. *Let u be a solution to flow equation 4.0.4, then*

$$\frac{\partial}{\partial t}|\nabla^i\kappa|^2 = 2\langle\nabla^i\kappa, \nabla^i\dot{\kappa}\rangle + 2i(\kappa-1)|\nabla^i\kappa|^2 + \kappa * \nabla\kappa * ... * \nabla^i\kappa$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^i\kappa|^2 &= \frac{\partial}{\partial t}\langle\nabla^i\kappa, \nabla^i\kappa\rangle \\
&= \frac{\partial}{\partial t}(e^{2iu}\langle\nabla^i\kappa, \nabla^i\kappa\rangle_0) \\
&= 2e^{2iu}\langle\nabla^i\kappa, \frac{\partial}{\partial t}(\nabla^i\kappa)\rangle_0 + 2\dot{u}e^{2iu}|\nabla^i\kappa|_0^2 \\
&= 2\langle\nabla^i\kappa, \frac{\partial}{\partial t}(\nabla^i\kappa)\rangle + 2\dot{u}|\nabla^i\kappa|^2
\end{aligned} \tag{4.0.13}$$

Because we are taking covariant derivatives with respect to the time-dependent metric g_u, we must apply the Leibniz rule to ∇. c.f. [1], Lemma 3.4, page 55.

$$\begin{aligned}
\frac{\partial}{\partial t}(\nabla\kappa) &= \nabla\dot{\kappa} + (\frac{\partial}{\partial t}\nabla)\kappa \\
&= \nabla\dot{\kappa} + \nabla\dot{u} * \kappa \\
&= \nabla\dot{\kappa} + \nabla\kappa * \kappa
\end{aligned} \tag{4.0.14}$$

and hence

$$\frac{\partial}{\partial t}(\nabla^i\kappa) = \nabla^i\dot{\kappa} + \kappa * \nabla\kappa * ... * \nabla^i\kappa \tag{4.0.15}$$

We now have

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^i\kappa|^2 &= 2\langle\nabla^i\kappa, \nabla^i\dot{\kappa}\rangle + \kappa * \nabla\kappa * ... * \nabla^i\kappa + 2\dot{u}|\nabla^i\kappa|^2 \\
&= 2\langle\nabla^i\kappa, \nabla^i\dot{\kappa}\rangle + 2(\kappa - 1)|\nabla^i\kappa|^2 + \kappa * \nabla\kappa * ... * \nabla^i\kappa
\end{aligned} \tag{4.0.16}$$

as claimed. □

We now state a lemma which relates derivatives of the curvature tensor to κ

Lemma 4.0.10. *Let* u *be a solution to flow equation 4.0.4, for* (M, g) *with Riemann curvature tensor* R *and* $\kappa = \frac{1}{K_u}$*, then*

$$\nabla^i R = \kappa^m * \nabla\kappa * ... * \nabla^i \kappa$$

for an integer m *with* $|m| \leq i + 1$

Proof. We know (c.f. [9], Lemma 8.7, page 144), that

$$R \sim \frac{1}{\kappa} \tag{4.0.17}$$

It follows that

$$\nabla R \sim -\frac{1}{\kappa^2}\nabla\kappa \tag{4.0.18}$$

and

$$\nabla^2 R \sim \frac{1}{\kappa^2} * \nabla^2\kappa + \frac{1}{\kappa^3} * \nabla\kappa$$

We follow the patten to find

$$\nabla^i R = \kappa^m * \nabla\kappa * ... * \nabla^i \kappa \tag{4.0.19}$$

where m is integer with $|m| \leq m + 1$ □

We are are now ready to work the expression for an ith derivative

Recall Lemma 4.0.9

$$\frac{\partial}{\partial t}|\nabla^i \kappa|^2 = 2\langle \nabla^i \kappa, \nabla^i \dot{\kappa}\rangle + 2i(\kappa - 1)|\nabla^i \kappa|^2 + \kappa * \nabla\kappa * ... * \nabla^i \kappa \tag{4.0.20}$$

In order to compute this, we need an expression for $\nabla^i \dot{\kappa}$.

We know that

$$\begin{aligned} \nabla^i \dot{\kappa} &= \nabla^i (2\kappa^2 - 2\kappa + \kappa^2 \Delta\kappa) \\ &= 2\nabla^i \kappa^2 - 2\nabla^i \kappa + \nabla^i(\kappa^2 \Delta\kappa) \\ &= \kappa^2 \nabla^i(\Delta\kappa) + \kappa * \nabla\kappa * \nabla^{i-1}(\Delta\kappa) + \kappa * \nabla\kappa * ... * \nabla^i \kappa \end{aligned} \tag{4.0.21}$$

We need to commute derivatives in $\nabla^i(\Delta\kappa)$.

Lemma 4.0.11. *Let u be a solution to flow equation 4.0.4 and $\kappa = \frac{1}{K_u}$, then*

$$\nabla^i(\Delta\kappa) = \Delta(\nabla^i \kappa) + \kappa^m * \nabla\kappa * ... * \nabla^i \kappa \tag{4.0.22}$$

Proof. We know

$$\begin{aligned} \nabla(\Delta\kappa) &= \Delta(\nabla\kappa) + R * \nabla\kappa \\ &= \Delta(\nabla\kappa) + \kappa^m * \nabla\kappa \end{aligned} \tag{4.0.23}$$

by use of Lemma 4.0.10

We will use mathematical inductiuon, assume

$$\nabla^{i-1}(\Delta\kappa) = \Delta(\nabla^{i-1}\kappa) + \kappa^m * \nabla\kappa * ... * \nabla^{i-1}\kappa$$

then

$$\begin{aligned}
\nabla^i(\Delta\kappa) &= \nabla(\nabla^{i-1}(\Delta\kappa)) \\
&= \nabla(\Delta(\nabla^{i-1}\kappa) + \kappa^m * \nabla\kappa * ... * \nabla^{i-1}\kappa) \\
&= \nabla(\Delta(\nabla^{i-1}\kappa)) + \kappa^m * \nabla\kappa * ... * \nabla^i\kappa \\
&= \Delta(\nabla^i\kappa) + R * \kappa^m * \nabla\kappa * ... * \nabla^i\kappa + \kappa^m * \nabla\kappa * ... * \nabla^i\kappa \\
&= \Delta(\nabla^i\kappa) + \kappa^{m'} * \nabla\kappa * ... * \nabla^i\kappa
\end{aligned} \tag{4.0.24}$$

as desired. We used Lemma 4.0.10 and m' may take a value one integer less than m. □

Lemma 4.0.12. *Let u be a solution to flow equation 4.0.4 and $\kappa = \frac{1}{K_u}$, then*

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^i\kappa|^2 &= \kappa^2\Delta|\nabla^i\kappa|^2 - 2\kappa|\nabla^{i+1}\kappa|^2 + \kappa * \nabla\kappa * ... * \nabla^{i+1}\kappa \\
&\quad + 2i(\kappa - 1)|\nabla^i\kappa|^2 + \kappa^m * \nabla\kappa * ... * \nabla^i\kappa^2
\end{aligned} \tag{4.0.25}$$

Proof. We are now ready to compute the term $\langle\nabla^i\kappa, \nabla^i\dot{\kappa}\rangle$ from Lemma 4.0.9.

From equation 4.0.21

$$\begin{aligned}\langle \nabla^i \kappa, \nabla^i \dot{\kappa} \rangle &= \langle \nabla^i \kappa, \kappa^2 \nabla^i (\Delta \kappa) + \nabla \kappa^2 * \nabla^{i-1} (\Delta \kappa) + \kappa * \nabla \kappa * ... * \nabla^i \kappa \rangle \\ &= \kappa^2 \langle \nabla^i \kappa, \nabla^i (\Delta \kappa) \rangle + \langle \nabla^i \kappa, \kappa * \nabla \kappa * \nabla^{i-1} (\Delta \kappa) \rangle + \langle \nabla^i \kappa, \kappa * \nabla \kappa * ... * \nabla^i \kappa \rangle\end{aligned} \tag{4.0.26}$$

We need to dig into the first and second terms in the last line. We use Lemma 4.0.11

$$\begin{aligned}\kappa^2 \langle \nabla^i \kappa, \nabla^i (\Delta \kappa) \rangle &= \langle \nabla^i \kappa, \Delta (\nabla^i \kappa) + \kappa^m * \nabla \kappa * ... * \nabla^i \kappa \rangle \\ &= \kappa^2 \langle \nabla^i \kappa, \Delta (\nabla^i \kappa) \rangle + \langle \nabla^i \kappa, \kappa^m * \nabla \kappa * ... * \nabla^i \kappa \rangle \\ &= \frac{1}{2} \kappa^2 \Delta |\nabla^i \kappa|^2 - \kappa^2 |\nabla^{i+1} \kappa|^2 + \kappa^m * \nabla \kappa * ... * \nabla^i \kappa^2\end{aligned} \tag{4.0.27}$$

For the last line, we used the identity

$$\langle F, \Delta F \rangle = \frac{1}{2} \Delta |F|^2 - |\nabla F|^2 \tag{4.0.28}$$

For the second term, we see

$$\langle \nabla^i \kappa, \kappa * \nabla \kappa * \nabla^{i-1} (\Delta \kappa) \rangle = \kappa * \nabla \kappa * ... * \nabla^{i+1} \kappa \tag{4.0.29}$$

We now have the formula

$$\frac{\partial}{\partial t}|\nabla^i \kappa|^2 = \kappa^2 \Delta |\nabla^i \kappa|^2 - 2\kappa |\nabla^{i+1}\kappa|^2 + \kappa * \nabla\kappa * ... * \nabla^{i+1}\kappa \tag{4.0.30}$$
$$+ 2i(\kappa - 1)|\nabla^i \kappa|^2 + \kappa^m * \nabla\kappa * ... * \nabla^i \kappa^2$$

□

We are now ready to prove Lemma 4.0.13 which will be the key inequality in the proof of Theorem 4.0.3.

Lemma 4.0.13. *Let u be a solution to flow equation 4.0.4 and $\kappa = \frac{1}{K_u}$, then for any $\epsilon > 0$*

$$H_l(|\nabla^i \kappa|^2) \leq (-2\kappa^2 + \epsilon)|\nabla^{i+1}\kappa|^2 + |\kappa * ... * \nabla^{i-1}\kappa||\nabla^i \kappa|^2 \tag{4.0.31}$$
$$+ |\kappa^m * \nabla\kappa * ... * \nabla^{i-1}\kappa|$$

Proof. Let $\epsilon > 0$, recall $H_l = \frac{\partial}{\partial t} - \kappa^2\Delta$, Lemma 4.0.12 can be rewritten as

$$H_l|\nabla^i \kappa|^2 = -2\kappa|\nabla^{i+1}\kappa|^2 + \kappa * \nabla\kappa * ... * \nabla^{i+1}\kappa \tag{4.0.32}$$
$$+ 2i(\kappa - 1)|\nabla^i \kappa|^2 + \kappa^m * \nabla\kappa * ... * \nabla^i \kappa^2$$

We use the Arthmetic-Geometric Mean Inequality (Lemma 2.0.14) to find that

$$|\kappa * \nabla\kappa * ... * \nabla^{i+1}\kappa| \leq \frac{1}{4\epsilon}|\kappa * ... * \nabla^i \kappa|^2 + \epsilon|\nabla^{i+1}\kappa|^2$$

Hence

$$H_l|\nabla^i\kappa|^2 \leq (-2\kappa^2+\epsilon)|\nabla^{i+1}\kappa|^2 + |\kappa * ... * \nabla^i\kappa|^2 + 2i(\kappa-1)|\nabla^i\kappa|^2 + \kappa^m * \nabla\kappa * ... * \nabla^i\kappa^2 \tag{4.0.33}$$

$$\leq (-2\kappa^2+\epsilon)|\nabla^{i+1}\kappa|^2 + |\kappa * ... * \nabla^{i-1}\kappa||\nabla^i\kappa|^2 + |\kappa^m * \nabla\kappa * ... * \nabla^{i-1}\kappa|$$

as claimed. □

We are now ready for the proof of Theorem 4.0.3.

Proof. The proof will proceed by induction, by assumptions 4.0.1 and 4.0.2 we know that $\nabla\kappa$ and $\nabla^2\kappa$ are bounded. We assume that

$$\kappa,\ \nabla\kappa, ...,\ \nabla^i\kappa$$

are bounded as in Theorem 4.0.3

We will show $\nabla^{i+1}\kappa$ is bounded by showing that the function

$$G(x,t) = t^{i-1}|\nabla^{i+1}\kappa|^2 + At^{i-2}|\nabla^i\kappa|^2$$

is bounded. We will show G is bounded by applying the Maximum Principle to H_l acting on G.

We compute

$$H_l G = (\frac{\partial}{\partial t} - \kappa^2\Delta)(t^{i-1}|\nabla^{i+1}\kappa|^2 + A\ t^{i-2}|\nabla^i\kappa|^2) \tag{4.0.34}$$
$$= (i-1)t^{i-2}|\nabla^{i+1}\kappa|^2 + A(i-2)t^{i-3}|\nabla^i\kappa|^2$$
$$+ t^{i-1}(\frac{\partial}{\partial t}|\nabla^{i+1}\kappa|^2 - \kappa^2\Delta|\nabla^{i+1}\kappa|^2) + At^{i-2}(\frac{\partial}{\partial t}|\nabla^i\kappa|^2 - \kappa^2\Delta|\nabla^i\kappa|^2)$$

We use Lemma 4.0.13

$$H_l G = (i-1)t^{i-2}|\nabla^{i+1}\kappa|^2 + A(i-2)t^{i-3}|\nabla^i\kappa|^2 \tag{4.0.35}$$
$$+ t^{i-1}(\frac{\partial}{\partial t}|\nabla^{i+1}\kappa|^2 - \kappa^2\Delta|\nabla^{i+1}\kappa|^2) + At^{i-2}(\frac{\partial}{\partial t}|\nabla^i\kappa|^2 - \kappa^2\Delta|\nabla^i\kappa|^2)$$

$$\leq (i-1)t^{i-2}|\nabla^{i+1}\kappa|^2 + A(i-2)t^{i-3}|\nabla^i\kappa|^2$$
$$+ t^{i-1}\big[(-2\kappa^2+\epsilon)|\nabla^{i+2}\kappa|^2 + |\kappa * ... * \nabla^i\kappa||\nabla^{i+1}\kappa|^2 + |\kappa^m * \nabla\kappa * ... * \nabla^i\kappa|\big]$$
$$+ At^{i-2}\big[(-2\kappa^2+\epsilon)|\nabla^{i+1}\kappa|^2 + |\kappa * ... * \nabla^{i-1}\kappa||\nabla^i\kappa|^2 + |\kappa^m * \nabla\kappa * ... * \nabla^{i-1}\kappa|\big]$$

$$= t^{i-1}(-2\kappa^2+\epsilon)|\nabla^{i+2}\kappa|^2 + \big[(i-1)t^{i-2} + t|\kappa * ... * \nabla^i\kappa| + A(-2\kappa^2+\epsilon)\big]|\nabla^{i+1}\kappa|^2$$
$$+ |t * A * \kappa^m * \nabla\kappa * ... * \nabla^i\kappa|$$

We choose $\epsilon > 0$ to be small enough so that $-2\kappa^2 + \epsilon < 0$. By our inductive hypothesis $|\kappa * ... * \nabla^i\kappa|$ is bounded so we can choose $A > 0$ large enough so that

$$1 + t|\kappa * ... * \nabla^i\kappa| + A(-2\kappa^2+\epsilon) < 0$$

Furthermore by the inductive hypothesis $|t * A * \kappa^m * \nabla\kappa * ... * \nabla^i\kappa|$ is also bounded, thus there exist some C so that

$$H_l G \leq C \tag{4.0.36}$$

and hence

$$H_l(G - Ct) \leq 0 \tag{4.0.37}$$

By the induction assumption, see that for

$$G(x, 0) \leq AC_i$$

By the Maximum Principle, we can see that

$$G(x, t) \leq Ct + AC_i$$

and we conclude that for large t, for some C_{i+1}

$$|\nabla^{i+1}\kappa|^2 \leq \frac{C_{i+1}}{t^{i-1}} \tag{4.0.38}$$

□

Chapter 5

Stability and Exponential Decay

We will now turn our attention the to Nirenberg Problem. In the Nirenberg, we assume that the background metric is the standard metric of the sphere, that is $K_0 \equiv 1$, and we want the flow equation to be stationary at the standard metric so we set $f \equiv 1$.

With $f \equiv 1$, $\tilde{K_u}$ is the mean value of K_u on (M, g_u). For aesthetic reasons, we will denote

$$\Lambda = \tilde{K_u} \tag{5.0.1}$$

Furthermore we will assume the metric g_u is close to the background metric g_0. Specifically we assume that u is close to zero and K_u is close to 1. We assume both have small derivatives. That is for some small $\delta > 0$

$$
\begin{aligned}
|u| &< \delta \\
|K_u - 1| &< \delta \\
|\nabla u| &< \delta \\
|\nabla K_u| &< \delta
\end{aligned} \tag{5.0.2}
$$

We will show the for δ sufficiently small, certain functions u converge exponentially to 0 under flow equation 4.0.4 and g_u converges to background metric g_0. We can say g_0 is stable in a strong sense.

We rewrite the flow equation 3.2.1 again

Lemma 5.0.1. *The flow equation 3.2.1 can be written as*

$$
\frac{\partial u}{\partial t} = \Lambda\kappa^2 \Delta u - (\Lambda\kappa - 1)^2 + \Lambda\kappa^2(\Lambda - e^{-2u}) \tag{5.0.3}
$$

Proof. We have the formula

$$
\frac{\partial u}{\partial t} = \frac{\bar{K}}{K^2}\Delta u - 1 + \frac{2\bar{K}}{K} - e^{-2u}\frac{K_0\bar{K}}{K^2} \tag{5.0.4}
$$

from [6] sec. 6, Lemma 6.13.

Since $f \equiv 1$, we have $\tilde{K_u} = \bar{K_u}$, where $\bar{K_u}$ denotes the mean value of K_u, i.e. $\bar{K_u} = \frac{4\pi}{A_u}$, where $A_u = \int_M \, dA_u$ is the surface area under metric g_u.

Since we assume $K_0 \equiv 1$ and denote $\Lambda = \bar{K_u}$, the equation becomes

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\Lambda}{K_u^2}\Delta u - 1 + \frac{2\Lambda}{K_u} - e^{-2u}\frac{\Lambda}{K^2} \qquad (5.0.5)\\
&= \frac{\Lambda}{K_u^2}\Delta u - \frac{K_u^2 - 2\Lambda K_u + e^{-2u}\Lambda}{K_u^2}\\
&= \frac{\Lambda}{K_u^2}\Delta u - \frac{(K_u - \Lambda)^2 - \Lambda(\Lambda - e^{-2u})}{K_u^2}\\
&= \frac{\Lambda}{K_u^2}\Delta u - (\frac{K_u - \Lambda}{K})^2 + \frac{\Lambda(\Lambda - e^{-2u})}{K_u^2}\\
&= \Lambda\kappa^2\Delta u - (\Lambda\kappa - 1)^2 + \Lambda\kappa^2(\Lambda - e^{-2u})
\end{aligned}$$

□

Remark 5.0.2. From assumptions 5.0.2 and flow equation 5.0.3, we can see that $|\dot{u}| < \delta_1$ for any $\delta_1 > 0$ when δ is sufficiently small.

Remark 5.0.3. We can write equation 5.0.3 as

$$\frac{\partial u}{\partial t} - \Lambda\kappa^2\Delta u = -(\Lambda\kappa - 1)^2 + \Lambda\kappa^2(\Lambda - e^{-2u})$$

and since $\kappa \approx 1$, $u \approx 0$, and $\Lambda \approx 1$ we see

$$\frac{\partial u}{\partial t} - \kappa^2\Delta u \approx 0$$

Hence solutions to the equation

$$H_l u = 0$$

approximate solutions to the flow equation.

5.1 Useful Estimates

We will need the follow lemmas in order to prove our result in the next section. These will compare several quantities that appear in those results. We introduce notation for the mean of u with respect to metrics g_u and g_0.

Definition 5.1.1. We will denote the mean of u with respect to metrics g_0 by

$$\bar{u}_0 = \frac{\int u \; dA_0}{\int \; dA_0}$$

We will continue to denote the mean of u with respect to metrics g_u by

$$\bar{u} = \frac{\int u \; dA_u}{\int \; dA_u}$$

When u is close to 0, $\bar{u}$ is close to $\bar{u}_0$, we will make this idea precise with the following lemmas.

Lemma 5.1.2. *For u satisfying equation 5.0.3, there exist some $\delta > 0$ such that if $|u| < \delta$ then*

$$|\bar{u} - \bar{u}_0| \le C \int (u - \bar{u}_0)^2 \; dA_0$$

Proof. Given the definitions $\bar{u} = \frac{\int u \; dA_u}{\int \; dA_u}$ and $\bar{u}_0 = \frac{\int u \; dA_0}{\int \; dA_0}$,

$$\begin{aligned} \bar{u} - \bar{u}_0 &= \frac{\int u \; dA_u}{\int \; dA_u} - \frac{\int u \; dA_0}{\int \; dA_0} \\ &= \frac{A_0 \int u \; dA_u - A_u \int u \; dA_0}{A_u A_0} \end{aligned} \tag{5.1.1}$$

We estimate the numerator

$$A_0 \int u \, dA_u - A_u \int u \, dA_0 = \int A_0 e^{2u} u \, dA_0 - \int A_u u \, dA_0 \qquad (5.1.2)$$
$$= \int (A_0 e^{2u} - A_u) u \, dA_0$$

We use Taylor's Theorem to estimate $A_0 e^{2u} - A_u$ since $e^{2u} = 1 + 2u + 2u_1^2$ for small $|u|$

$$A_0 e^{2u} - A_u \approx A_0(1 + 2u + 2u_1^2) - \int (1 + 2u + 2u_1^2) \, dA_0 \qquad (5.1.3)$$
$$= A_0(1 + 2u + 2u_1^2) - A_0 - 2 \int u \, dA_0 - 2A_0 u_1^2$$
$$= 2A_0 u - 2 \int u \, dA_0$$

From this we find

$$A_0 \int u \, dA_u - A_u \int u \, dA_0 = \int (A_0 e^{2u} - A_u) u \, dA_0 \qquad (5.1.4)$$
$$\approx \int (2A_0 u - 2 \int u \, dA_0) u \, dA_0$$
$$= 2A_0 \int u^2 \, dA_0 - 2(\int u \, dA_0)^2$$
$$= 2A_0 \int (u - \bar{u}_0)^2 \, dA_0$$

For the last line, we used an identity similar to the Computational Variance Formula from probability theory.

$$\begin{aligned}
\int (u-\bar{u}_0)^2 \; dA_0 &= \int u^2 - 2\bar{u}_0 u + (\bar{u}_0)^2 \; dA_0 \qquad (5.1.5)\\
&= \int u^2 \; dA_0 - 2\bar{u}_0 \int u \; dA_0 + \bar{u}_0^2 \int \; dA_0\\
&= \int u^2 \; dA_0 - 2\bar{u}_0^2 A_0 + \bar{u}_0^2 A_0\\
&= \int u^2 \; dA_0 - \bar{u}_0^2 A_0\\
&= \int u^2 \; dA_0 - A_0 (\int u \; dA_0 / A_0)^2\\
&= \int u^2 \; dA_0 - \frac{1}{A_0} (\int u \; dA_0)^2\\
&= \frac{1}{A_0} [A_0 \int u^2 \; dA_0 - (\int u \; dA_0)^2]
\end{aligned}$$

and so

$$A_0 \int (u-\bar{u}_0)^2 \; dA_0 = A_0 \int u^2 \; dA_0 - (\int u \; dA_0)^2$$

It follows that for some C when $|u| < \delta$ for some δ

$$|\bar{u} - \bar{u}_0| \leq C \int (u - \bar{u}_0)^2 \; dA_0$$

□

Lemma 5.1.3. *For u satisfying equation 5.0.3, there exist some $\delta > 0$ such that if $|u| < \delta$ then*

$$\int (u-\bar{u}_0)^2 \; dA_0 \leq C \int (u-\bar{u})^2 \; dA_u$$

Proof. We begin

$$(u - \bar{u}_0)^2 = [(u - \bar{u}) + (\bar{u} - \bar{u}_0)]^2 \tag{5.1.6}$$
$$= (u - \bar{u})^2 + 2(u - \bar{u})(\bar{u} - \bar{u}_0) + (\bar{u} - \bar{u}_0)^2$$
$$\leq 2(u - \bar{u})^2 + 2(\bar{u} - \bar{u}_0)^2$$

Here we used the Arithmetic-Geometric Mean Inequality (Lemma 2.0.14).

So we have

$$\int (u - \bar{u}_0)^2 \; dA_0 \leq 2 \int (u - \bar{u})^2 \; dA_0 + 2 \int (\bar{u} - \bar{u}_0)^2 \; dA_0$$

We will further estimate the second term using Lemma 5.1.2

$$\int (\bar{u} - \bar{u}_0)^2 \; dA_0 = A_0 (\bar{u} - \bar{u}_0)^2 \tag{5.1.7}$$
$$\leq C(\int (u - \bar{u}_0)^2 \; dA_0)^2$$

and so we have

$$\int (u - \bar{u}_0)^2 \; dA_0 \leq 2 \int (u - \bar{u}_0)^2 \; dA_0 + C(\int (u - \bar{u}_0)^2 \; dA_0)^2 \tag{5.1.8}$$

We can assume $|u| < \delta$ for small enough δ so that $\int (u - \bar{u}_0)^2 \, dA_0 < \frac{\epsilon}{C}$ for some $\epsilon < 1$ and so

$$\int (u - \bar{u}_0)^2 \, dA_0 \leq 2 \int (u - \bar{u}_0)^2 \, dA_0 + \epsilon \int (u - \bar{u}_0)^2 \, dA_0 \tag{5.1.9}$$

and hence

$$(1 - \epsilon) \int (u - \bar{u}_0)^2 \, dA_0 \leq 2 \int (u - \bar{u})^2 \, dA_0$$

From here we can conclude

$$\int (u - \bar{u}_0)^2 \, dA_0 \leq C \int (u - \bar{u})^2 \, dA_u$$

□

Lemma 5.1.4. *For u satisfying equation 5.0.3, there exist some $\delta > 0$ such that if $|u| < \delta$ then*

$$|\Lambda - e^{-2u}| \leq C|u - \bar{u}_0|$$

Proof. We have

$$\begin{aligned} \Lambda - e^{-2u} &= \frac{4\pi}{\int e^{2u} \, dA_0} - e^{-2u} \\ &= \frac{4\pi e^{-2\bar{u}_0}}{\int e^{2(u-\bar{u}_0)} \, dA_0} - \frac{e^{-2u} \int e^{2(u-\bar{u}_0)} \, dA_0}{\int e^{2(u-\bar{u}_0)} \, dA_0} \\ &= \frac{1}{\int e^{2(u-\bar{u}_0)} \, dA_0} (4\pi e^{-2\bar{u}_0} - e^{-2u} \int e^{2(u-\bar{u}_0)} \, dA_0) \end{aligned} \tag{5.1.10}$$

We expand $e^{-2(u-\bar{u}_0)}$ through its Taylor series

$$e^{2(u-\bar{u}_0)} = 1 + 2(u - \bar{u}_0) + 2(u - \bar{u}_0)^2 + ... \tag{5.1.11}$$

Specifically we will be using

$$\begin{aligned} -e^{2(u-\bar{u}_0)} &= -1 - 2(u - \bar{u}_0) - 2(u - \bar{u}_0)^2 + ... \\ &\leq -1 - C(u - \bar{u}_0) \end{aligned} \tag{5.1.12}$$

when $|u - \bar{u}_0|$ is small.

So

$$\begin{aligned} -\int e^{2(u-\bar{u}_0)} \, dA_0 &\leq \int (-1 - C(u - \bar{u}_0)) \, dA_0 \\ &= -A_0 - C \int (u - \bar{u}_0) \, dA_0 \\ &= -A_0 \end{aligned} \tag{5.1.13}$$

where

$$A_0 = \int \, dA_0$$

We apply this to the previous calculation

$$\begin{aligned}\Lambda - e^{-2u} &= \frac{1}{\int e^{2(u-\bar{u}_0)}\ dA_0}(4\pi e^{-2\bar{u}_0} - e^{-2u}\int e^{2(u-\bar{u}_0)}\ dA_0) \\ &\leq C(4\pi e^{-2\bar{u}_0} - A_0 e^{-2u}) \\ &\leq C(4\pi(e^{-2\bar{u}_0} - e^{-2u})) \\ &= C_0(e^{-2\bar{u}_0} - e^{-2u})\end{aligned} \tag{5.1.14}$$

Now $A_0 = 4\pi$ and choose constant C so that

$$\frac{1}{\int e^{2(u-\bar{u}_0)}} < C$$

Now we look at the term $e^{-2\bar{u}_0} - e^{-2u}$

$$\begin{aligned}e^{-2\bar{u}_0} - e^{-2u} &= e^{-2\bar{u}_0}(1 - e^{-2(u-\bar{u}_0)}) \\ &\leq e^{-2\bar{u}_0}(-C(-2(u-\bar{u}_0)))\end{aligned} \tag{5.1.15}$$

Thus

$$e^{-2\bar{u}_0} - e^{-2u} \leq Ce^{-2\bar{u}_0}(u - \bar{u}_0)$$

so we have

$$\Lambda - e^{-2u} \leq Ce^{-2\bar{u}_0}(u - \bar{u}_0) \tag{5.1.16}$$

By a similar calculation, we also have

$$e^{-2u} - \Lambda \leq Ce^{-2\bar{u}_0}(u - \bar{u}_0)$$

We conclude that for some C

$$|\Lambda - e^{-2u}| \leq C|u - \bar{u}_0|$$

□

Lemma 5.1.5. *For u satisfying equation 5.0.3, there exist some $\delta > 0$ such that if $|u| < \delta$ then*

$$\int (\Lambda - e^{-2u})^2 \ dA_u \leq C \int (u - \bar{u})^2 \ dA_u$$

Proof.

$$\begin{aligned} \int (\Lambda - e^{-2u})^2 \ dA_u &\leq C \int (\Lambda - e^{-2u})^2 \ dA_0 \\ &\leq C \int (u - \bar{u}_0)^2 \ dA_0 \\ &\leq C \int (u - \bar{u})^2 \ dA_u \end{aligned} \tag{5.1.17}$$

by Lemma 5.1.4 and Lemma 5.1.3. □

5.2 Exponential Decay

Proposition 5.2.1. *For u satisfying the flow equation 4.0.4, for any $\epsilon > 0$ there exist some $\delta > 0$ such that if the assumptions of 5.0.2 are satisfied then for*

$$f(t) = ||u(\ \cdot, t)||^2_{L^2(M,g_u)} = \int (u(x,t) - \bar{u}(t))^2 \ dA_u$$

We have

$$f'(t) \leq (-1+\epsilon) \int |\nabla u|^2 \ dA_u + (2+\epsilon) \int (u-\bar{u})^2 \ dA_u \tag{5.2.1}$$

Proof. Let

$$f(t) = \int (u-\bar{u})^2 \ dA_u$$

$$\begin{aligned} f'(t) &= \frac{\partial}{\partial t} \int (u-\bar{u})^2 e^{2u} \ dA_0 \\ &= \int (u-\bar{u})^2 2e^{2u}\dot{u} + 2(u-\bar{u})(\dot{u}-\dot{\bar{u}})e^{2u} \ dA_0 \\ &= 2\int (u-\bar{u})^2 \dot{u} \ dA_u + 2\int (u-\bar{u})(\dot{u}-\dot{\bar{u}}) \ dA_u \\ &= 2\int (u-\bar{u})^2 \dot{u} \ dA_u + 2\int \dot{u}(u-\bar{u}) \ dA_u - 2\dot{\bar{u}} \int (u-\bar{u}) \ dA_u \end{aligned} \tag{5.2.2}$$

We know

$$\int (u - \bar{u})\ dA_u = \bar{u} \int\ dA_u - \int u\ dA_u = 0$$

and so

$$\begin{aligned} f'(t) &= 2\int (u - \bar{u})^2 \dot{u}\ dA_u + 2\int \dot{u}(u - \bar{u})\ dA_u \\ &\leq \epsilon_1 \int (u - \bar{u})^2\ dA_u + 2\int \dot{u}(u - \bar{u})\ dA_u \end{aligned} \tag{5.2.3}$$

We used assumption 5.0.2 to claim $|\dot{u}| < \epsilon_1$.

For the second term, we use flow equation 5.0.3 to see

$$\begin{aligned} \int \dot{u}(u - \bar{u})\ dA_u &= \int (\Lambda\kappa^2 \Delta u - (\Lambda - \kappa)^2 + \Lambda\kappa^2(\Lambda - e^{-2u}))(u - \bar{u})\ dA_u \\ &= \Lambda \int \kappa^2 (u - \bar{u})\Delta u\ dA_u - \int (\Lambda\kappa - 1)^2 (u - \bar{u})\ dA_u \\ &+ \Lambda \int \kappa^2(\Lambda - e^{-2u})(u - \bar{u})\ dA_u \end{aligned} \tag{5.2.4}$$

We need to estimate the three integrals in the last line.

For the first term

$$\begin{aligned} \int \kappa^2 \Delta u\ (u - \bar{u})\ dA_u &= \int \Delta u (u - \bar{u})\ dA_u + \int (\kappa^2 - 1)\Delta u (u - \bar{u})\ dA_u \\ &\leq - \int |\nabla u|^2\ dA_u + \epsilon_2 \int |\nabla u|^2\ dA_u \end{aligned} \tag{5.2.5}$$

We used assumption 5.0.2 so that $|\kappa^2 - 1| < \epsilon_2$, and we used Green's identity on the other integral.

For the second term we use assumption 5.0.2 to say $|\Lambda\kappa - 1| < \epsilon_3$. We see that

$$\Lambda\kappa - 1 = \frac{1}{K_u}((\Lambda - e^{-2u}) + \Delta u) \tag{5.2.6}$$

and find that

$$\begin{aligned} \Big|\int (\Lambda\kappa - 1)^2 (u - \bar{u})\ dA_u\Big| &\leq \epsilon_3 \Big|\int (\Lambda\kappa - 1)(u - \bar{u})\ dA_u\Big| \\ &\leq \epsilon_3 \Big|\int (\frac{1}{K_u}(\Lambda - e^{-2u}) + \Delta u)(u - \bar{u})\ dA_u\Big| \\ &\leq \epsilon_3 \Big[C_0 \Big|\int (\Lambda - e^{-2u})(u - \bar{u})\ dA_u\Big| + \Big|\int (\Delta u)(u - \bar{u})\ dA_u\Big|\Big] \\ &\leq C_1 \epsilon_3 \int (u - \bar{u})^2\ dA_u + C_0 \epsilon_3 \int |\nabla u|^2\ dA_u \end{aligned} \tag{5.2.7}$$

For the last line we used the Arithmetic-Geometric Mean Inequality and Lemma 5.1.5 on the first term and we used Green's identity on the second term.

For the third term

$$\begin{aligned} \int \kappa^2 (\Lambda - e^{-2u})(u - \bar{u})\ dA_u &= \int (\Lambda - e^{-2u})(u - \bar{u})\ dA_u \\ &+ \int (\kappa^2 - 1)(\Lambda - e^{-2u})(u - \bar{u})\ dA_u \end{aligned} \tag{5.2.8}$$

We use Taylor's Theorem to see that for $|u| < \epsilon_4$ there is some C_2 so that

$$\begin{aligned}
\int (\Lambda - e^{-2u})(u - \bar{u})\ dA_u &= \Lambda \int (u - \bar{u})\ dA_u - \int e^{-2u}(u - \bar{u})\ dA_u \\
&= - \int e^{-2u}(u - \bar{u})\ dA_u \\
&= - \int (1 - 2u + 2u^2 + ...)(u - \bar{u})\ dA_u \\
&\leq - \int (1 - 2u + C_2 u^2)(u - \bar{u})\ dA_u \\
&\leq 2 \int (u - \bar{u})^2\ dA_u + C_2 \epsilon_4 \int (u - \bar{u})^2\ dA_u
\end{aligned} \tag{5.2.9}$$

and we see

$$\begin{aligned}
\Big| \int (\kappa^2 - 1)(\Lambda - e^{-2u})(u - \bar{u}) \Big| &\leq \epsilon_1 \int |\Lambda - e^{-2u}||u - \bar{u}|\ dA_u \\
&\leq \epsilon_1 \big[\frac{1}{2} \int (\Lambda - e^{-2u})^2\ dA_u + \frac{1}{2} \int (u - \bar{u})^2\ dA_u \big] \\
&\leq C_3 \epsilon_1 \int (u - \bar{u})^2\ dA_u
\end{aligned} \tag{5.2.10}$$

We now have the bound for $f'(t)$. Putting the estimates for the the three integrals together, we see

$$\begin{aligned}\int \dot{u}(u-\bar{u}) \leq &-\int |\nabla u|^2 \, dA_u + \epsilon_2 \int |\nabla u|^2 \, dA_u \\ &+ C_1\epsilon_3 \int (u-\bar{u})^2 \, dA_u + C_0\epsilon_3 \int |\nabla u|^2 \, dA_u \\ &+ 2\int (u-\bar{u})^2 \, dA_u + C_2\epsilon_4 \int (u-\bar{u})^2 \, dA_u + C_3\epsilon_1 \int (u-\bar{u})^2 \, dA_u\end{aligned} \tag{5.2.11}$$

so for some $\epsilon > 0$, we have

$$f'(t) \leq (-1+\epsilon) \int |\nabla u|^2 \, dA_u + 2\int (u-\bar{u})^2 \, dA_u + \epsilon \int (u-\bar{u})^2 \, dA_u \tag{5.2.12}$$

as claimed. □

Remark 5.2.2. We would like to show that $||u-\bar{u}||_{L^2(M,g_u)}$ decays exponentially. In order to get exponential decay for

$$f(t) = \int (u-\bar{u})^2 \, dA_u$$

however, we need

$$f'(t) \leq -\epsilon \int (u-\bar{u})^2 \, dA_u \tag{5.2.13}$$

in order to use ODE comparison, but we are only able to show in Proposition 5.2.1 that

$$f'(t) \leq (-1+\epsilon) \int |\nabla u|^2 \, dA_u + (2+\epsilon) \int (u-\bar{u})^2 \, dA_u \tag{5.2.14}$$

In order to attain the stronger inequality, we need to perform a conformal diffeomorphism to M and subtract out the first nontrivial eigenspace of Δ which has eigenvalue of 2 and is L^2-orthogonal to $(u-\bar{u})^2$. This will allow us to attain inequality 5.2.13.

While we don't have the result in general, we do have the result if u is invariant under the antipodal map.

Proposition 5.2.3. *Let u be an antipodal invariant solution to flow equation 5.0.3, there exist some $\delta > 0$ such that if the assumptions of 5.0.2 are satisfied then*

$$||u-\bar{u}||_{L^2(M,g_u)} \leq A_0 e^{-B_0 t}$$

for some $A_0, B_0 > 0$. .

Proof. Let u be invariant under the antipodal map, i.e. $u(x,t) = u(-x,t)$ for all $x \in S^2$. The first nonzero eigenvalue of Δ_0 is 2 and its eigenfunctions are the standard coordinate functions of $\mathbb{R}^3$. By symmetry, u is L^2-orthogonal to the first nontrivial eigenspace of Δ_0.

It follows that

$$\begin{aligned} -\int |\nabla u|^2 \, dA_u &\leq -C_1 \int |\nabla u|_0^2 \, dA_0 \\ &\leq -6C_1 \int (u-\bar{u}_0)^2 \, dA_0 \\ &\leq -6C_1C_2 \int (u-\bar{u})^2 \, dA_u \end{aligned} \tag{5.2.15}$$

Here C_1 can be made close to 1 and C_2 can be made close to 1/2 (see proof of Lemma 5.1.3).

For middle inequality, we have a sharp Poincaré inequality (see Remark 2.0.13)

$$-\int |\nabla u|_0^2 \, dA_0 \leq -6 \int (u-\bar{u}_0)^2 \, dA_0$$

since $u - \bar{u}_0$ is orthogonal to the 0 and 2 eigenspaces of Δ_0 and the next eigenvalue is 6.

Hence

$$-\int |\nabla u|^2 \, dA_u \leq -C \int (u - \bar{u})^2 \, dA_u$$

where C is approximately 3.

It follows from Proposition 5.2.1 and Remark 5.2.2 that $||u - \bar{u}||_{L^2(M, g_u)}$ decays exponetially.

□

The next proposition shows that the $L^2(M, g_u)$ norm of ∇u decays exponentially in t when u is sufficiently small and $||u - \bar{u}||_{L^2}$ decays exponentially.

Proposition 5.2.4. *For u satisfying the flow equation 5.0.3 such that $||u - \bar{u}||_{L^2}$ decays exponentially, there exist some $\delta > 0$ such that if the assumptions of 5.0.2 are satisfied then*

$$||\nabla u||^2_{L^2(M, g_u)} \leq A_1 e^{-B_1 t} \tag{5.2.16}$$

for some $A_1, B_1 > 0$.

Proof. Let

$$f(t) = \int |\nabla u|^2 \, dA_u$$

then

$$
\begin{aligned}
f'(t) &= \frac{d}{dt}\int |\nabla u|^2 \ dA_u \\
&= \frac{d}{dt}\int e^{2u}\langle \nabla u, \nabla u\rangle_0 e^{2u} \ dA_0 \\
&= \int 4\dot{u}e^{4u}\langle \nabla u, \nabla u\rangle_0 + 2e^{4u}\langle \nabla u, \nabla \dot{u}\rangle_0 \ dA_0 \\
&= 4\int \dot{u}|\nabla u|^2 \ dA_u + 2\int \langle \nabla u, \nabla \dot{u}\rangle \ dA_u
\end{aligned} \tag{5.2.17}
$$

We will seek to bound both of these terms. For the first term, since $\dot{u} = \Lambda\kappa - 1$ is close to zero by assumption 5.0.2, we have

$$|\dot{u}| < \delta_1$$

and

$$|\int \dot{u}|\nabla u|^2 \ dA_u| \leq \delta_1 \int |\nabla u|^2 \ dA_u \tag{5.2.18}$$

For the second term, we use Green's identity and flow equation 5.0.3

$$
\begin{aligned}
\int \langle \nabla u, \nabla \dot{u}\rangle \ dA_u &= -\int \dot{u}\Delta u \ dA_u \\
&= -\Lambda \int \kappa^2 (\Delta u)^2 \ dA_u + \int (\Lambda\kappa - 1)^2 \Delta u \ dA_u \\
&\quad - \Lambda \int \kappa^2 (\Lambda - e^{-2u})\Delta u \ dA_u
\end{aligned} \tag{5.2.19}
$$

We need to bound each of these terms, for the first term for some C_1,

$$-\Lambda \int \kappa^2 (\Delta u)^2 \; dA_u \leq -C_1 \int (\Delta u)^2 \; dA_u \tag{5.2.20}$$

For the second term

$$\begin{aligned} |\int (\Lambda\kappa - 1)^2 \Delta u \; dA_u| &\leq \int (\Lambda\kappa - 1)^2 |\Delta u| \; dA_u \\ &\leq \delta_1 \int |\Lambda\kappa - 1||\Delta u| \; dA_u \end{aligned} \tag{5.2.21}$$

Again we have $|\Lambda\kappa - 1| < \delta_1$ from assumption 5.0.2

Now for some constant C_2,

$$\begin{aligned} |\Lambda\kappa - 1| &= \frac{1}{K_u} |K_u - \Lambda| \\ &= \frac{1}{K_u} |e^{-2u}(1 - \Delta_0 u) - \Lambda| \\ &= \frac{1}{K_u} |(e^{-2u} - \Lambda) - \Delta u| \\ &\leq C_2 |(e^{-2u} - \Lambda) - \Delta u| \end{aligned} \tag{5.2.22}$$

We have

$$|\int (\Lambda\kappa - 1)^2 \Delta u \; dA_u| \leq C_2\delta_1 \int |(e^{-2u} - \Lambda) - \Delta u||\Delta u| \; dA_u \tag{5.2.23}$$
$$\leq C_2\delta_1 \int |e^{-2u} - \Lambda||\Delta u| + (\Delta u)^2 \; dA_u$$
$$\leq C_2\delta_1 \int \frac{1}{2}(e^{-2u} - \Lambda)^2 + \frac{1}{2}(\Delta u)^2 + (\Delta u)^2 \; dA_u$$
$$\leq \frac{C_2}{2}\delta_1 \int (u - \bar{u})^2 \; dA_u + \frac{3C_2}{2}\delta_1 \int (\Delta u)^2 \; dA_u$$

We used the triangle inequality, the Arithmetic-Geometric Mean Inequality and Lemma 5.1.5 in the last calculation.

For the third term

$$|\Lambda \int \kappa^2(\Lambda - e^{-2u})\Delta u \; dA_u| \leq C \int |(\Lambda - e^{-2u})\Delta u| \; dA_u \tag{5.2.24}$$
$$\leq \frac{C}{2\epsilon} \int (\Lambda - e^{-2u})^2 \; dA_u + C\frac{\epsilon}{2} \int (\Delta u)^2 \; dA_u$$
$$\leq C_3 \int (u - \bar{u})^2 \; dA_u + C_4\epsilon \int (\Delta u)^2 \; dA_u$$

For the last line we used Lemma 5.1.5.

We put these together and have the inequality

$$
\begin{aligned}
f'(t) &= 4\int \dot{u}|\nabla u|^2\ dA_u + 2\int \langle \nabla u, \nabla \dot{u}\rangle\ dA_u \qquad (5.2.25)\\
&\le \delta_1 \int |\nabla u|^2\ dA_u - 2\Lambda \int \kappa^2(\Delta u)^2\ dA_u + 2\int (1-\Lambda\kappa)^2 \Delta u\ dA_u - 2\Lambda \int \kappa^2(\Lambda - e^{-2u})\Delta u\ dA_u\\
&\le \delta_1 \int |\nabla u|^2\ dA_u - C_1 \int (\Delta u)^2\ dA_u + \frac{C_2}{2}\delta_1 \int (u-\bar{u})^2\ dA_u + \frac{3C_2}{2}\delta_1 \int (\Delta u)^2\ dA_u\\
&+ C_3 \int (u-\bar{u})^2\ dA_u + C_4\epsilon \int (\Delta u)^2\ dA_u\\
&\le \delta_1 \int |\nabla u|^2\ dA_u + C_5 \int (u-\bar{u})^2\ dA_u - (C_1 - \frac{3C_2}{2}\delta_1 - C_4\epsilon) \int (\Delta u)^2\ dA_u\\
&\le \delta_1 \int |\nabla u|^2\ dA_u + C_5 \int (u-\bar{u})^2\ dA_u - C_6 \int (\Delta u)^2\ dA_u
\end{aligned}
$$

We choose δ_1 and ϵ small enough so that the coefficient of $\int (\Delta u)^2\ dA_u$ has a negative sign.

Now for any $A > 0$ we have inequality

$$
-C \int (\Delta u)^2\ dA_u \le C(A^2 \int (u-\bar{u})^2\ dA_u - 2A \int |\nabla u|^2\ dA_u) \qquad (5.2.26)
$$

For large enough A, this gives

$$
\begin{aligned}
f'(t) &\le \delta_1 \int |\nabla u|^2\ dA_u + C_5 \int (u-\bar{u})^2\ dA_u - C_6 \int (\Delta u)^2\ dA_u \qquad (5.2.27)\\
&\le \delta_1 \int |\nabla u|^2\ dA_u + C_5 \int (u-\bar{u})^2\ dA_u - C_6(A^2 \int (u-\bar{u})^2\ dA_u - 2A \int |\nabla u|^2\ dA_u)\\
&\le -C_6 \int |\nabla u|^2\ dA_u + C_7 \int (u-\bar{u})^2\ dA_u
\end{aligned}
$$

Since we have exponential decay for $\int (u-\bar{u})^2\ dA_u$ by hypothesis, we can use ODE comparison and conclude that we have exponential decay for $f(t)$. □

We now prove stability for all derivatives of u. That is if u is sufficiently small and $||u-\bar{u}||_{L^2}$ decays exponentially, then the $L^2(M, g_u)$ norm of $\nabla^i u$ decays exponentially in t.

Theorem 5.2.5. *For u satisfying the flow equation 5.0.3 such that $||u-\bar{u}||_{L^2}$ decays exponentially, there exist some $\delta > 0$ such that if the assumptions of 5.0.2 are satisfied then for all integer $i \geq 1$*

$$||\nabla^i u||^2_{L^2(M,g_u)} \leq A_i e^{-B_i t} \tag{5.2.28}$$

for some $A_i, B_i > 0$.

Proof. The proof will use induction and, similar the proof for Proposition 5.2.4, will use ODE comparison. Proposition 5.2.4 provides our base case.

We will assume that $||\nabla u||$, ..., $||\nabla^{i-1}u||$ are decaying exponentially.

Let

$$f_i(t) = \int |\nabla^i u|^2 \; dA_u$$

we find

$$\begin{aligned} f'(t) &= \frac{d}{dt}\int \langle \nabla^i u, \nabla^i u\rangle \; dA_u \\ &= \frac{d}{dt}\int e^{2iu}\langle \nabla^i u, \nabla^i u\rangle_0 \; e^{2u} dA_0 \\ &= (2i+2)\int \dot{u}|\nabla^i u|^2 \; dA_u + 2\int \langle \nabla^i u, (\frac{\partial}{\partial t}\nabla^i)u\rangle \; dA_u + 2\int \langle \nabla^i u, \nabla^i \dot{u}\rangle \; dA_u \end{aligned} \tag{5.2.29}$$

We already know from assumption 5.0.2 that for small $\delta_1 > 0$

$$\int \dot{u}|\nabla^i u|^2 \; dA_u \leq \delta_1 \int |\nabla^i u|^2 \; dA_u \tag{5.2.30}$$

For the the second integral, as in the proof of Lemma 4.0.9, we see

$$\begin{aligned}
(\frac{\partial}{\partial t}\nabla^i)u &= \nabla^i \dot{u} * u + ... + \nabla \dot{u} * \nabla^{i-1} u \\
&= \nabla^i \kappa * u + ... + \nabla \kappa * \nabla^{i-1} u \\
&= \nabla^i \kappa * u + \nabla \kappa * ... * \nabla^{i-1}\kappa * \nabla u * ... * \nabla^{i-1} u
\end{aligned} \tag{5.2.31}$$

By induction we have exponential decay for the second term in the last line, but we must be careful with the first term.

$$\begin{aligned}
\int \langle \nabla^i u, (\frac{\partial}{\partial t}\nabla^i)u \rangle \; dA_u &= \int \langle \nabla^i u, \nabla^i \kappa * u + \nabla \kappa * ... * \nabla^{i-1}\kappa * \nabla u * ... * \nabla^{i-1} u \rangle \; dA_u \\
&= \int \langle \nabla^i u, \nabla^i \kappa * u \rangle \; dA_u \\
&\quad + \int \langle \nabla^i u, \nabla \kappa * ... * \nabla^{i-1}\kappa * \nabla u * ... * \nabla^{i-1} u \rangle \; dA_u \\
&\leq C||\nabla^i u|| + C||\nabla^i u||^2 + C||\nabla u * ... * \nabla^{i-1} u||^2
\end{aligned} \tag{5.2.32}$$

We used the Cauchy-Schwarz inequality and Arithmetic-Geometric Mean Inequality for the last line and bounded $\nabla^k \kappa$ terms with Theorem 4.0.3

To handle the term $||\nabla^i u||$, we employ Green's identity, Cauchy-Schwarz inequality, and Young's inequality (Lemma 2.0.15).

$$
\begin{aligned}
||\nabla^i u|| &= \Big(\int |\nabla^i u|^2 \; dA_u\Big)^{\frac{1}{2}} \\
&= \Big(-\int \langle \nabla^{i+1} u, \nabla^{i-1} u\rangle \; dA_u\Big)^{\frac{1}{2}} \\
&\leq \Big(\int |\nabla^{i+1} u||\nabla^{i-1} u| \; dA_u\Big)^{\frac{1}{2}} \\
&\leq \Big(\int |\nabla^{i+1} u|^2 \; dA_u\Big)^{\frac{1}{4}}\Big(\int |\nabla^{i-1} u|^2 \; dA_u\Big)^{\frac{1}{4}} \\
&\leq \epsilon\frac{1}{4}\int |\nabla^{i+1} u|^2 \; dA_u + \epsilon^{-1}\frac{3}{4}\big(\int |\nabla^{i-1} u|^2 \; dA_u\big)^{\frac{1}{3}}
\end{aligned}
\tag{5.2.33}
$$

We now have the bound

$$
\begin{aligned}
\int \langle \nabla^i u, (\frac{\partial}{\partial t}\nabla^i)u\rangle \; dA_u \leq \epsilon||\nabla^{i+1} u||^2 + C||\nabla^i u||^2 + C||\nabla^{i-1} u||^{\frac{1}{3}} \\
+ C||\nabla u * ... * \nabla^{i-1} u||^2
\end{aligned}
\tag{5.2.34}
$$

We now turn to the third integral and use flow equation 5.0.3. We have

$$
\begin{aligned}
\int \langle \nabla^i u, \nabla^i \dot{u}\rangle \; dA_u &= \Lambda \int \langle \nabla^i u, \nabla^i(\kappa^2 \Delta u)\rangle \; dA_u \\
&- \int \langle \nabla^i, \nabla^i(\Lambda\kappa - 1)^2\rangle \; dA_u + \Lambda\int \langle \nabla^i u, \nabla^i(\kappa^2(\Lambda - e^{-2u}))\rangle \; dA_u
\end{aligned}
\tag{5.2.35}
$$

We will break each of these integrals apart.

$$\nabla^i(\Lambda\kappa - 1)^2 = \kappa * \nabla\kappa * ... * \nabla^i\kappa \tag{5.2.36}$$

so

$$\begin{aligned} \Big|\int \langle \nabla^i, \nabla^i(\Lambda\kappa-1)^2\rangle \; dA_u\Big| &= \Big|\int \langle \nabla^i, \kappa * \nabla\kappa * ... * \nabla^i\kappa\rangle \; dA_u\Big| \\ &\leq C||\nabla^i u|| \end{aligned} \tag{5.2.37}$$

In the last line, we used the Cauchy-Schwarz inequality and Theorem 4.0.3.

For the next integral, we first see

$$\nabla^i(\kappa^2(\Lambda - e^{-2u})) = u * \nabla^i\kappa + \kappa * ... * \nabla^i\kappa * \nabla u * ... * \nabla^{i-1}u + \kappa * u * \nabla^i u \tag{5.2.38}$$

and so using Cauchy-Schwarz inequality and Arithmetic-Geometric Mean Inequality and bounding $\nabla^k\kappa$ terms with Theorem 4.0.3, we have

$$\Big|\int \langle \nabla^i u, \nabla^i(\kappa^2(\Lambda - e^{-2u}))\rangle \; dA_u\Big| = \Big|\int \langle \nabla^i u, u * \nabla^i\kappa + \kappa * ... * \nabla^{i-1}u + \kappa * u * \nabla^i u\rangle \; dA_u\Big| \tag{5.2.39}$$

$$\begin{aligned} &\leq C||\nabla^i u|| + C||\nabla^i u||^2 + C||\nabla u * ... * \nabla^{i-1}u||^2 \\ &\leq \epsilon||\nabla^{i+1}u||^2 + C||\nabla^i u||^2 + C||\nabla^{i-1}u||^{\frac{1}{3}} \\ &+ C||\nabla u * ... * \nabla^{i-1}u||^2 \end{aligned}$$

For the last line, we used the same calculation as in 5.2.33.

We turn our attention to term $\int \langle \nabla^i u, \nabla^i(\kappa^2 \Delta u)\rangle \ dA_u$.

We compute

$$\nabla^i(\kappa^2\Delta u) = \nabla^i(\Delta u) + \kappa * \nabla\kappa * \nabla^{i+1}u + \kappa * ... * \nabla^i\kappa * \nabla^2 u * ... * \nabla^i u \tag{5.2.40}$$

and have

$$\begin{aligned}\int \langle \nabla^i u, \nabla^i(\kappa^2\Delta u)\rangle \ dA_u &= \int \langle \nabla^i u, \nabla^i(\Delta u)\rangle \ dA_u \\ &+ \int \langle \nabla^i u, \kappa * \nabla\kappa * \nabla^{i+1}u\rangle \ dA_u \\ &+ \int \langle \nabla^i u, \kappa * ... * \nabla^i\kappa * \nabla^2 u * ... * \nabla^i u\rangle \ dA_u\end{aligned} \tag{5.2.41}$$

Using Arithmetic-Geometric Mean Inequality, we have

$$\begin{aligned}\int \langle \nabla^i u, \kappa * \nabla\kappa * \nabla^{i+1}u\rangle \ dA_u &\leq C||\nabla^i u||||\nabla^{i+1}u|| \\ &\leq C(\frac{1}{2\epsilon}||\nabla^i u||^2 + \frac{\epsilon}{2}||\nabla^{i+1}u||^2)\end{aligned} \tag{5.2.42}$$

and

$$\int \langle \nabla^i u, \kappa * ... * \nabla^i \kappa * \nabla^2 u * ... * \nabla^i u \rangle \; dA_u \leq C||\nabla^i u||^2 + C||\kappa * ... * \nabla^i \kappa * \nabla^2 u * ... * \nabla^{i-1} u||^2 \tag{5.2.43}$$

For the other integral $\int \langle \nabla^i u, \nabla^i(\Delta u) \rangle \; dA_u$, we must commute derivatives similarly to Lemma 4.0.11.

$$\begin{aligned} \nabla^i(\Delta u) &= \Delta(\nabla^i u) + R * \nabla^{i+1} + \nabla R * \nabla^i u + ... + \nabla^i R * \nabla u \\ &= \Delta(\nabla^i u) + \frac{1}{\kappa} * \nabla^{i+1} u + \kappa * \nabla \kappa * ... * \nabla^i \kappa * \nabla u * ... * \nabla^i u \end{aligned} \tag{5.2.44}$$

And so

$$\begin{aligned} \int \langle \nabla^i u, \nabla^i(\Delta u) \rangle \; dA_u = \int \langle \nabla^i u, \Delta(\nabla^i u) \rangle \; dA_u + \int \langle \nabla^i u, \frac{1}{\kappa} * \nabla^{i+1} u \rangle \; dA_u \\ + \int \langle \nabla^i u, \kappa * \nabla \kappa * ... * \nabla^i \kappa * \nabla u * ... * \nabla^i u \rangle \; dA_u \end{aligned} \tag{5.2.45}$$

We bound the last two integrals like before

$$\begin{aligned} \int \langle \nabla^i u, \frac{1}{\kappa} * \nabla^{i+1} u \rangle \; dA_u &\leq C||\nabla^i u|| ||\nabla^{i+1} u|| \\ &\leq C(\frac{1}{2\epsilon}||\nabla^i u||^2 + \frac{\epsilon}{2}||\nabla^{i+1} u||^2) \end{aligned} \tag{5.2.46}$$

and

$$\int \langle \nabla^i u, \kappa * ... * \nabla^i \kappa * \nabla u * ... * \nabla^i u \rangle \, dA_u \leq C||\nabla^i u||^2 \\ + C||\kappa * ... * \nabla^i \kappa * \nabla u * ... * \nabla^{i-1} u||^2 \tag{5.2.47}$$

Finally we apply Green's identity to the first integral

$$\int \langle \nabla^i u, \Delta(\nabla^i u) \rangle \, dA_u = -\int |\nabla^{i+1} u|^2 \, dA_u \\ = -||\nabla^{i+1} u||^2 \tag{5.2.48}$$

We are now ready to put all of these pieces together, we have the inequality

$$f'(t) \leq (-1+\epsilon)||\nabla^{i+1} u||^2 + C||\nabla^i u||^2 + C||\nabla^{i-1} u||^{\frac{1}{3}} + C(\nabla u * ... * \nabla^{i-1} u) \tag{5.2.49}$$

We have such a C since the terms κ, ..., $\nabla^i \kappa$ and we can choose $\epsilon < 1$. By the inductive hypothesis, the final term in the expression is already decaying exponentially.

We now consider the function

$$F_i(t) = ||\nabla^i u||^2 + A||\nabla^{i-1}u||^2 \tag{5.2.50}$$
$$= f_i(t) + Af_{i-1}(t)$$

We see using inequality 5.2.49

$$\frac{d}{dt}F_i(t) \leq (-1+\epsilon_1)||\nabla^{i+1}u||^2 + C||\nabla^i u||^2 + C||\nabla^{i-1}u||^{\frac{1}{3}} + C(\nabla u * ... * \nabla^{i-1}u) \tag{5.2.51}$$
$$+ A\big((-1+\epsilon_2)||\nabla^i u||^2 + C||\nabla^{i-1}u||^2 + C||\nabla^{i-2}u||^{\frac{1}{3}} + C(\nabla u * ... * \nabla^{i-2}u)\big)$$
$$\leq (-1+\epsilon_1)||\nabla^{i+1}u||^2 - C||\nabla^i u||^2 + C(\nabla u * ... * \nabla^{i-1}u)$$
$$\leq C(\nabla u * ... * \nabla^{i-1}u) + C||\nabla^{i-1}u||^{\frac{1}{3}} + C||\nabla^{i-2}u||^{\frac{1}{3}}$$

By the induction hypothesis, all terms of the final line are decaying exponentially. By ODE comparison, F_i must decay exponentially and therefore

$$f_i(t) = ||\nabla^i u||^2$$

does as well. □

We have the result for an antipodal invariant solution to the flow equation.

Corollary 5.2.6. *Let u be an antipodal invariant solution to the flow equation 5.0.3, then there exist some $\delta > 0$ such that if the assumptions of 5.0.2 are satisfied then for all integer*

$i \geq 1$

$$||\nabla^i u||^2_{L^2(M,g_u)} \leq A_i e^{-B_i t} \tag{5.2.52}$$

for some $A_i, B_i > 0$.

Proof. By Proposition 5.2.3, we have exponential decay for $||u - \bar{u}||_{L^2}$. By Theorem 5.2.5, we conclude that $||\nabla^i u||_{L^2}$ decays exponentially for each $i \geq 1$. □

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