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**HOMOGENEITY IN SUPPLY**

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# **HOMOGENEITY AND SUPPLY**

Jeffrey T. LaFrance and Rulon D. Pope

**Abstract:** Supply functions in the ubiquitous Gorman class are examined for their homogeneity properties. Homogeneity places surprisingly strong restrictions on functional forms. These forms facilitate testing of aggregability given homogeneity or homogeneity given aggregability or testing both.

**Keywords:** Functional form, homogeneity, supply

**Running Head:** Homogeneity and Supply

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## HOMOGENEITY AND SUPPLY

Economists' motivations for choice of functional form include: ease of implementation and interpretation, needed flexibility, and aggregability. For demand applications, one of the late Terence Gorman's great legacy is an empirically attractive generalization of previous work on representative consumers (the Gorman Polar Form), which included a readily aggregable demand system (e.g., Gorman 1981, Deaton and Muellbauer 1980). For both consumer and producer applications, this legacy is deeply embedded in agricultural economics, as evidenced by the many applications which take the Gorman class of functions (additive in functions of the aggregated variable) as the starting point for empirical analysis (e.g., Shumway 1995; Shumway and Lim 1993, Green and Alston 1990, Chambers and Pope 1991, LaFrance, Beatty, Pope and Agnew 2004 ).

Gorman class functional forms are ubiquitous because they are easily interpretable, flexible, and have the added advantage of having convenient aggregation properties. However, the bulk of agricultural economic applications, particularly to agricultural producer behavior, do not involve complete demand and supply systems and routinely involve only the supply function for a single commodity of interest (Askari and Cummings 1977). In these cases, one can employ a popular Gorman specification (without the system restrictions) or one can specify any appropriate representation of interest. In the former case, one might use a functional form that is unnecessarily inflexible because choosing a system "off of the shelf" may have imbedded restrictions that come from the economic theory of systems, e.g., symmetry and adding up of demand, expenditure or cost, or symmetry from profit maximization. In either case, it is clear that one can not consider

symmetry in these single equation cases (or adding up if applicable) but would typically impose homogeneity (and expect non-negativity and monotonicity in the own price).

One might think that the imposition of homogeneity is trivial by merely deflating by some price or index of prices. However, this can easily destroy aggregability (Lewbel 1989). Further, this does not generally allow a nested approach for testing homogeneity, a property that is notoriously rejected in empirical work (Deaton and Muellbauer 1980; or Shumway and Lim 1993). The approach we take is to isolate the homogeneity condition within the general Gorman class of supply functions.<sup>1</sup> Language and notation focus on producer supply behavior but it is readily apparent that our results apply to a variety of other settings, e.g., a factor demand equation and a normalized profit function.<sup>2</sup> Furthermore, to maximize the empirical relevance of the analysis and to reduce the notational clutter, a single supply equation is considered. However, because homogeneity is an equation-by-equation concept, the results transfer immediately to any system with the Gorman structure.

Our results are novel and to us, they are striking. We began this inquiry conjecturing that homogeneity might not substantially restrict or guide the choice of functional form. This turns out to be far from true. We find that supply functions must include explicit sums and products of power and transcendental (trigonometric) functions and must do so in rather explicit ways. However, this keenly depends on the number of independent functions of the aggregated variable, here output price, that is being utilized. That said, there is a great deal of flexibility remaining to usefully measure supply response and a great deal that is learned from considering homogeneity in isolation from other desired

properties of a supply function. We present a class or classes of functional forms that are homogeneous and can be usefully imposed directly in applied work or one can perturb them to test for implied homogeneity or aggregability.

### **Gorman Class Functions**

The rationale and approach for specifying an aggregable supply system follows Chambers and Pope 1991,1994. By focusing on supply,  $q$ , and given heterogeneity in output price presumably to temporal and spatial reasons, it is reasonable to specify supply as

$$q = \sum_{k=1}^K \alpha_k(\mathbf{w}) h_k(p) \quad (1)$$

with  $K$  smooth linearly independent functions of input prices,  $\mathbf{w}$  ( $w_1, \dots, w_n$ ), times  $K$  smooth, and linearly independent functions of output price,  $p$ . We could add fixed inputs or technical change variables. However, in order to avoid notational clutter, for the present such shifters can be subsumed in  $\alpha$  and  $h$ , but suppressed notationally. The functional form in (1) defined the Gorman-class of functions. Given monotonicity from profit maximization, we expect or impose  $\partial q / \partial p = \sum_{k=1}^K \alpha_k(\mathbf{w}) \partial h_k(p) / \partial p > 0$ .

Homogeneity ( $0^\circ$ ) of  $q$  in  $(\mathbf{w}, p)$  is most basically described by

$$q(t\mathbf{w}, tp) = q(\mathbf{w}, p), \quad \forall t > 0. \quad (2)$$

Given the smoothness assumption, homogeneity can also be written as the Euler equation form (e.g., R.G.D. Allen 1938)

$$\frac{\partial q(\mathbf{w}, p)}{\partial \mathbf{w}^\top} \mathbf{w} + \frac{\partial q(\mathbf{w}, p)}{\partial p} p = \sum_{k=1}^K \frac{\partial \alpha_k(\mathbf{w})}{\partial \mathbf{w}^\top} \mathbf{w} h_k(p) + \sum_{k=1}^K \alpha_k(\mathbf{w}) \frac{\partial h_k(p)}{\partial p} p = 0. \quad (3)$$

One could immediately impose homogeneity by dividing by  $p$  or a  $w_i$  in (1). In the first case, deflation destroys the most common sense of aggregability (at least in the conventional sense) by creating terms as functions of  $w/p$  (Lewbel 1989). In the latter case, the variables are  $p/w_i$  and  $w_j/w_i$ . This also generally destroys independence of an aggregate price index from  $w$ . In any case, if testing homogeneity is of interest, most researchers would prefer a nested test and deflation would most naturally imply a non-nested test (deflated vs. non-deflated models). Thus, the next section focuses on what restrictions on each of the functions in (1) lead to homogeneity. This requires solving the differential equation in (3).

### The Main Result

In the Appendix, the proof of the main result of the paper is presented. Not surprisingly, the results depend on the number,  $K$ , of functions included in the supply function. The explicit results for  $K \leq 3$  are highlighted because this appears to be parsimonious and sufficient for most empirical representations.<sup>3</sup>

**Proposition 1 (The Main Result):** *Let the supply function of  $q$  take the Gorman form,  $q = \sum_{k=1}^K \alpha_k(\mathbf{w})h_k(p)$ , with  $K$  smooth, linearly independent functions of input prices,  $\mathbf{w}$ , and  $K$  smooth, linearly independent functions of output price,  $p$ . If  $q$  is  $0^\circ$  homogeneous in  $(\mathbf{w}, p)$ , then each output price function is: (i)  $p^\varepsilon$ , with  $\varepsilon \in \mathbb{R}$ ; (ii)  $p^\varepsilon (\ln p)^j$ , with  $\varepsilon \in \mathbb{R}$ ,  $j \in \{1, \dots, K\}$ ; (iii)  $p^\varepsilon \sin(\tau \ln p)$ ,  $p^\varepsilon \cos(\tau \ln p)$ , with  $\varepsilon \in \mathbb{R}$ ,  $\tau \in \mathbb{R}_+$ , appearing in pairs with the same  $\{\varepsilon, \tau\}$ ; or (iv)  $p^\varepsilon (\ln p)^j \sin(\tau \ln p)$ ,  $p^\varepsilon (\ln p)^j \cos(\tau \ln p)$ , with  $\varepsilon \in \mathbb{R}$ ,  $j \in \{1, \dots, [\frac{1}{2}K]\}$ ,  $\tau \in \mathbb{R}_+$ , and  $K \geq 4$ , appearing in pairs with the same  $\{\varepsilon, j, \tau\}$ , where  $[\frac{1}{2}K]$  is the largest integer no greater than  $\frac{1}{2}K$ . If  $K \in \{1, 2, 3\}$ , then the supply of  $q$  can be written as:*

(a)  $K=1$

$$q = [p/\alpha_1(\mathbf{w})]^{\varepsilon_1};$$

(b)  $K=2$

i.  $q = [p/\alpha_1(\mathbf{w})]^{\varepsilon_1} + [p/\alpha_2(\mathbf{w})]^{\varepsilon_2};$

ii.  $q = [p/\alpha_1(\mathbf{w})]^{\varepsilon_1} \ln(p/\alpha_2(\mathbf{w}));$  or

iii.  $q = [p/\alpha_1(\mathbf{w})]^{\varepsilon_1} [\sin(\tau \ln(p/\alpha_2(\mathbf{w}))) + \cos(\tau \ln(p/\alpha_2(\mathbf{w})))];$

(c)  $K=3$

i.  $q = [p/\alpha_1(\mathbf{w})]^{\varepsilon_1} + [p/\alpha_2(\mathbf{w})]^{\varepsilon_2} + [p/\alpha_3(\mathbf{w})]^{\varepsilon_3};$

ii.  $q = [p/\alpha_1(\mathbf{w})]^{\varepsilon_1} + [p/\alpha_2(\mathbf{w})]^{\varepsilon_2} \ln(p/\alpha_3(\mathbf{w}));$

iii.  $q = [p/\alpha_1(\mathbf{w})]^{\varepsilon_1} \left\{ \alpha_2(\mathbf{w}) + [\ln(p/\alpha_3(\mathbf{w}))]^2 \right\};$  or

iv.  $q = [p/\alpha_1(\mathbf{w})]^{\varepsilon_1} + [p/\alpha_2(\mathbf{w})]^{\varepsilon_2} \left\{ \sin(\tau \ln[p/\alpha_3(\mathbf{w})]) + \cos(\tau \ln[p/\alpha_3(\mathbf{w})]) \right\}.$

In each case except (c) iii. where  $\alpha_2(\mathbf{w})$  is homogeneous of degree zero, each  $\alpha_i(\mathbf{w})$  is positively linearly homogeneous for  $i = 1, 2, 3$ .

Beginning with  $K=1$ , where there is no possibility of complex roots to (3), a rather simple power function of  $p$  emerges. Only a single linearly homogeneous index in  $\mathbf{w}$  is required. One might consider  $p/\alpha_1(\mathbf{w}) > 0$  the “real price”, in which case monotonicity of supply in the output price requires  $\varepsilon_1 > 0$ . If  $\varepsilon_1 < 1$ , then the smaller is this elasticity, the greater is the concave curvature of supply in real price, while if  $\varepsilon_1 > 1$ , then the larger is this elasticity, the greater is the convex curvature of supply in real price.

For  $K=2$ , supply curves need not go through the origin and more flexible functional forms are introduced. A linear combination of power functions in  $p$  are obtained – each



with separate exponents ((b) i.). These are generalizations of the PIGL forms developed by Muellbauer (1975) and Lewbel (1987). Other generalizations include log functions added to power functions. For the log form, the two price functions enter in a specific multiplicative way, with  $p^{\varepsilon_1}$  and  $\ln(p)p^{\varepsilon_1}$  serving as the two price functions in (1). This introduces generalizations of PIGLOG type functions developed in Muellbauer (1975) in demand analysis. Finally, for  $K=2$  (iii), the possibility of complex roots occurs and trigonometric functions are introduced with output price functions involving  $p^\varepsilon \sin(\tau \ln(p))$  and  $p^\varepsilon \cos(\tau \ln(p))$ . These provide a pair of terms similar to the Fourier series approximations developed in Gallant (1984) and applied by Chalfant (1987) and Wohlgenant (1984).

Moving on,  $K=3$  adds an additional price function. Analogous to (b) i., an additional power function term is possible as noted in (c) i. A mixing of (b) i. and (b) ii. is now possible (see (c) ii.), as well as the generalizations in case (c) iv. of the trigonometric forms discussed under  $K=2$ . Perhaps most surprising is the addition of a  $(\ln p)^2$  term as a generalization of (b) ii. These cases flow from a reduction of the Euler equations to an equivalent linear ordinary differential equation (ode; see equation (A.4) in the appendix). The solution of this ode depends in turn on whether there are unique real roots, a repeated real root and a unique real root, one real root with multiplicity 3, and a real root with a complex conjugate pair of roots.

A few comparisons of the forms in the Proposition and popular forms have been noted above but there is a large set of Gorman-class forms that could be discussed. However, the conclusion is clear even from  $K=1$  that homogeneity defines the class of func-

tions but allows a much more rich set of behaviors than were  $\varepsilon$  constrained to be 1 as in conventional systems of demand analysis. Thus, we conclude that the set of functions consistent with homogeneity is flexible and appears to be empirically meaningful. Yet, it is surprising that homogeneity alone defines the class of output price functions that can be used.

### Profit Functions

Even though a production or general equilibrium system may not be estimated, it is useful to know the profit functions corresponding to Proposition 1, so that welfare analysis can be performed (see Hausman 1981, LaFrance 1993). That is, assuming that firms solve  $\pi(p, \mathbf{w}) = \max_{q, \mathbf{x}} \{(pq - \mathbf{w}^\top \mathbf{x}) : (q, \mathbf{x}) \in \mathcal{Y}\}$  where  $\mathcal{Y}$  is the production set with input vector  $\mathbf{x}$ , assuming smoothness, and integrating the supply functions in (a)-(c) obtains the profit function,  $\pi(\mathbf{w}, p)$ .<sup>4</sup> Note that if supply contains a sum of functions, sufficiency for monotonicity is that each function is monotonically increasing in  $p$ . However, this is not necessary. Hence, it is not imposed except for  $K=1$ . We present result for  $K=1$ , 2, and  $K=3$  (c) iii. The results for the remaining  $K=3$  cases can be anticipated from the results for  $K=1$  and 2. We present the following without proof, as it only involves integration.

**Proposition 2:** *Let the supply of  $q$  take the form in Proposition 1, then homogeneity requires profit functions of the following forms:*

(a)  $K=1 \quad \varepsilon_1 > 0$

$$\pi(p, \mathbf{w}) = \frac{p}{(1 + \varepsilon_1)} \left( \frac{p}{\alpha_1(\mathbf{w})} \right)^{\varepsilon_1} - \beta(\mathbf{w});$$

(b)  $K=2$

*i.a.*  $\varepsilon_1, \varepsilon_2 \neq -1$

$$\pi(p, \mathbf{w}) = \frac{p}{(1+\varepsilon_1)} \left( \frac{p}{\alpha_1(\mathbf{w})} \right)^{\varepsilon_1} + \frac{p}{(1+\varepsilon_2)} \left( \frac{p}{\alpha_2(\mathbf{w})} \right)^{\varepsilon_2} - \beta(\mathbf{w});$$

*i.b.*  $\varepsilon_1 \neq -1, \varepsilon_2 = -1$

$$\pi(p, \mathbf{w}) = \frac{p}{(1+\varepsilon_1)} \left( \frac{p}{\alpha_1(\mathbf{w})} \right)^{\varepsilon_1} + \alpha_2(\mathbf{w}) \ln \left( \frac{p}{\beta(\mathbf{w})} \right) - \gamma(\mathbf{w});$$

*ii.a.*  $\varepsilon_1 \neq -1$

$$\pi(p, \mathbf{w}) = \frac{p}{(1+\varepsilon_1)} \left( \frac{p}{\alpha_1(\mathbf{w})} \right)^{\varepsilon_1} \left[ \ln \left( \frac{p}{\alpha_2(\mathbf{w})} \right) - \frac{1}{(1+\varepsilon_1)} \right] - \beta(\mathbf{w});$$

*ii.b.*  $\varepsilon_1 = -1$

$$\pi(p, \mathbf{w}) = \frac{1}{2} \alpha_1(\mathbf{w}) \left[ \ln \left( \frac{p}{\alpha_2(\mathbf{w})} \right) \right]^2 - \beta(\mathbf{w});$$

*iii.*

$$\begin{aligned} \pi = & \left( \frac{p}{(1+\varepsilon_1)^2 + \tau^2} \right) \left( \frac{p}{\alpha_1} \right)^{\varepsilon_1} \left[ (1+\varepsilon_1 + \tau) \sin \left( \tau \ln \left( \frac{p}{\alpha_2} \right) \right) \right. \\ & \left. + (1+\varepsilon_1 - \tau) \cos \left( \tau \ln \left( \frac{p}{\alpha_2} \right) \right) \right] - \beta; \end{aligned}$$

(c)  $K=3$

*iii.a.*  $\varepsilon_1 \neq -1$

$$\pi = \frac{p}{(1+\varepsilon_1)^3} \left( \frac{p}{\alpha_1} \right)^{\varepsilon_1} \left\{ 1 + (1+\varepsilon_1)^2 \alpha_2 + \left[ (1+\varepsilon_1) \ln \left( \frac{p}{\alpha_3} \right) - 1 \right]^2 \right\} - \beta;$$

*iii.b.*  $\varepsilon_1 = -1$

$$\pi = \alpha_1 \left[ \alpha_2 \ln \left( \frac{p}{\beta} \right) + \frac{1}{3} \left( \ln \left( \frac{p}{\alpha_3} \right) \right)^3 \right] - \gamma.$$

In each case,  $\beta(\mathbf{w})$  and  $\gamma(\mathbf{w})$  are positively linearly homogeneous functions of  $\mathbf{w}$ .

These results can be used to develop systems or partial systems of production behavior or for applied welfare analysis. For example, one could differentiate the profit functions in Proposition 2 with respect to  $p$  and  $\mathbf{w}$  and estimate a system of supply and factor demands.<sup>5</sup> Admittedly some of the forms are simpler and otherwise more attractive than others.

We have said little about regularity conditions such as monotonicity or non-negativity. In single equation problems as we consider here, these conditions are seldom imposed but merely checked after estimation. However, it is clearly possible to impose either local or in some cases global monotonicity in a single equation or a system with constrained estimation. Finally, in any application, monotonicity should be examined. To illustrate its importance, in (b) ii. for  $\alpha_1 : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_+$ , monotonicity requires that  $1 - \ln(p) > 0$ . Thus, the usefulness of this case hinges entirely on the domain of  $p$ .

### **Fixed Inputs or Other Shifters**

Fixed inputs without heterogeneity, pose no problem. They can be included with  $\mathbf{w}$  in the  $\alpha_i(\mathbf{w})$  functions,  $i=1,2,3$ . When fixed inputs are heterogeneous, then one must decide how the aggregators enter into an aggregate supply function. One can only maintain price independence of the aggregator function by adding  $\sum_{s=1}^S \psi_s(z_s) \gamma_s(\mathbf{w})$  to (1), where there are  $S$  kinds of fixed inputs. This only alters our earlier result in Proposition 1 by adding this term onto the forms (a)–(c) of the Proposition, where  $\gamma_s(\mathbf{w})$  are homogeneous of degree zero in  $\mathbf{w}$ ,  $s=1, \dots, S$ . When the aggregators of fixed inputs may depend on  $p$  or  $\mathbf{w}$  (a less common but perhaps more useful approach), a more complex analysis is required.

For example, each of the price functions  $h_k$  could be written as  $h_k(p, z)$ , but the homogeneity properties of  $h_k$  in  $p$  remain unchanged.

### **Conclusion**

Measuring producer or consumer behavior requires selection of a functional form. The Gorman class of functions is flexible and aggregable and encompasses most empirical specifications in the literature. Yet, surprisingly little is known about how homogeneity of degree zero of supply and demand functions applies. We find substantial generalizations of the functional forms found in Gorman-based systems and routinely applied in empirical demand analysis, where homogeneity, adding up, and symmetry are imposed in a complete system. The functional forms in the propositions provide a coherent approach for empirical work using any demand or supply function in the general Gorman class. Finally, although the derived functional forms are substantive generalizations of those in the complete systems literature, we are struck with the following observation: *homogeneity alone determines the admissible class of functional forms.*

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## APPENDIX

**Proposition 1:** Let the supply of  $q$  take the form  $q = \sum_{k=1}^K \alpha_k(\mathbf{w}) h_k(p)$ , with  $K$  smooth, linearly independent functions of input prices,  $\mathbf{w}$ , and  $K$  smooth, linearly independent functions of output price,  $p$ . If  $q$  is  $0^\circ$  homogeneous in  $(\mathbf{w}, p)$ , then each output price function is: (i)  $p^\varepsilon$ ,  $\varepsilon \in \mathbb{R}$ ; (ii)  $p^\varepsilon (\ln p)^j$ ,  $\varepsilon \in \mathbb{R}$ ,  $j \in \{1, \dots, K\}$ ; (iii)  $p^\varepsilon \sin(\tau \ln p)$ ,  $p^\varepsilon \cos(\tau \ln p)$ ,  $\varepsilon \in \mathbb{R}$ ,  $\tau \in \mathbb{R}_+$ , appearing in pairs with the same  $\{\varepsilon, \tau\}$ ; or (iv)  $p^\varepsilon (\ln p)^j \sin(\tau \ln p)$ ,  $p^\varepsilon (\ln p)^j \cos(\tau \ln p)$ ,  $\varepsilon \in \mathbb{R}$ ,  $j \in \{1, \dots, [\frac{1}{2}K]\}$ ,  $\tau \in \mathbb{R}_+$ ,  $K \geq 4$ , appearing in pairs with the same  $\{\varepsilon, j, \tau\}$ , where  $[\frac{1}{2}K]$  is the largest integer no greater than  $\frac{1}{2}K$ . If  $K \in \{1, 2, 3\}$ , then the supply of  $q$  can be written as:

(a)  $K=1$

$$q = [p/\alpha_1(\mathbf{w})]^{\varepsilon_1}.$$

(b)  $K=2$

i.  $q = [p/\alpha_1(\mathbf{w})]^{\varepsilon_1} + [p/\alpha_2(\mathbf{w})]^{\varepsilon_2};$

ii.  $q = [p/\alpha_1(\mathbf{w})]^{\varepsilon_1} \ln(p/\alpha_2(\mathbf{w}));$  or

iii.  $q = [p/\alpha_1(\mathbf{w})]^{\varepsilon_1} [\sin(\tau \ln(p/\alpha_2(\mathbf{w}))) + \cos(\tau \ln(p/\alpha_2(\mathbf{w})))];$

(c)  $K=3$

i.  $q = [p/\alpha_1(\mathbf{w})]^{\varepsilon_1} + [p/\alpha_2(\mathbf{w})]^{\varepsilon_2} + [p/\alpha_3(\mathbf{w})]^{\varepsilon_3};$

ii.  $q = [p/\alpha_1(\mathbf{w})]^{\varepsilon_1} + [p/\alpha_2(\mathbf{w})]^{\varepsilon_2} \ln(p/\alpha_3(\mathbf{w}));$

$$\text{iii. } q = [p/\alpha_1(\mathbf{w})]^{\varepsilon_1} \left\{ \alpha_2(\mathbf{w}) + [\ln(p/\alpha_3(\mathbf{w}))]^2 \right\};$$

$$\text{iv. } q = [p/\alpha_1(\mathbf{w})]^{\varepsilon_1} \\ + [p/\alpha_2(\mathbf{w})]^{\varepsilon_2} \left\{ \sin(\tau \ln[p/\alpha_3(\mathbf{w})]) + \cos(\tau \ln[p/\alpha_3(\mathbf{w})]) \right\}.$$

In each case except (c) iii. where  $\alpha_2(\mathbf{w})$  is homogeneous of degree zero, each  $\alpha_i(\mathbf{w})$  is positively linearly homogeneous for  $i = 1, 2, 3$ .

*Proof:* The Euler equation for  $0^\circ$  homogeneity is:

$$\sum_{k=1}^K \frac{\partial \alpha_k(\mathbf{w})}{\partial \mathbf{w}^\top} \mathbf{w} h_k(p) + \sum_{k=1}^K \alpha_k(\mathbf{w}) h'_k(p) p = 0. \quad (\text{A.1})$$

If  $K=1$  and  $h'_1(p) = 0$ , this reduces to  $\partial \alpha_1(\mathbf{w}) / \partial \mathbf{w}^\top \mathbf{w} = 0$ , so that  $h_1(p) = c$  and  $\alpha_1(\mathbf{w})$  is homogeneous of degree zero. Absorb the constant  $c$  into the price index and set  $\varepsilon=0$  to obtain a special case of (i). If either  $K=1$  and  $h'_1(p) \neq 0$  or  $K \geq 2$ , then neither sum in (A.1) can vanish without contradicting the linear independence of the  $\{\alpha_k(\mathbf{w})\}$  or the  $\{h_k(p)\}$ . Write the Euler equation as

$$\frac{\sum_{k=1}^K \alpha_k(\mathbf{w}) h'_k(p) p}{\sum_{k=1}^K \left[ \partial \alpha_k(\mathbf{w}) / \partial \mathbf{w}^\top \mathbf{w} \right] h_k(p)} = -1. \quad (\text{A.2})$$

Since the right-hand side is constant, we must be able to recombine the left-hand side to be independent of both  $\mathbf{w}$  and  $p$ . In other words, the terms in the numerator must recombine in some way so that it is proportional to the denominator, with  $-1$  as the proportionality factor. Clearly, if these two functions are proportional, the functional forms must be the same. Linear independence of  $\{h_1(p), \dots, h_K(p)\}$  then implies that each  $h'_k(p) p$  must be a linear function of  $\{h_1(p), \dots, h_K(p)\}$ , with constant coefficients:

$$h'_k(p) p = \sum_{\ell=1}^K c_{k,\ell} h_\ell(p), \quad k = 1, \dots, K. \quad (\text{A.3})$$

This is a complete system of  $K$  linear, homogeneous, ordinary differential equations (odes), of the form commonly known as Cauchy's linear differential equation. Our strategy is the following. First, we convert (A.3) through the change of variables from  $p$  to  $x = \ln p$  to a system of linear odes with constant coefficients (Cohen 1933, pp. 124-125). Second, we identify the set of solutions for the converted system of odes. Third, we return to (A.1) with these solutions in hand and identify the implied restrictions among the input price functions for each  $K=1,2,3$ .

Since  $p(x) = e^x$  and  $p'(x) = p(x)$ , defining  $\tilde{h}_k(x) \equiv h_k(p(x))$ ,  $k = 1, \dots, K$ , and applying this change of variables yields:

$$\tilde{h}'_k(x) = \sum_{\ell=1}^K c_{k,\ell} \tilde{h}_\ell(x), \quad k = 1, \dots, K. \quad (\text{A.4})$$

In matrix form, this system of linear, first-order, homogeneous odes is  $\tilde{\mathbf{h}}'(x) - \mathbf{C}\tilde{\mathbf{h}}(x) = \mathbf{0}$ , and the characteristic equation is  $|\mathbf{C} - \lambda\mathbf{I}| = 0$ . This is a  $K^{\text{th}}$  order polynomial in  $\lambda$ , for which the fundamental theorem of algebra (Gauss 1799) implies that there are exactly  $K$  roots. Some of these roots may repeat and some may be complex conjugate pairs. Let the characteristic roots be denoted by  $\lambda_k$ ,  $k = 1, \dots, K$ .

The general solution to a linear, homogeneous, ode of order  $K$  is the sum of  $K$  linearly independent particular solutions (Cohen 1933, Chapter 6; Boyce and DiPrima 1977, Chapter 5; Hirsch and Smale 1974, Chapter 5; Kamien and Schwartz 1991, Appendix B), where linear independence of the  $K$  functions,  $f_1, \dots, f_K$  of a single variable  $x$  means that there is no nonvanishing  $K$ -vector,  $(a_1, \dots, a_K)$  such that  $a_1 f_1 + \dots + a_K f_K = 0$  for all values of the variables in an open neighborhood of any point  $[x, f_1(x), \dots, f_K(x)]$ . Cohen, pp. 303-306 contains a precise statement of necessary and sufficient conditions.

Let there be  $R \geq 0$  roots that repeat and reorder the output price functions as necessary in the following way. Label the first repeating root (if one exists) as  $\lambda_1$  and let its multiplicity be denoted by  $M_1 \geq 1$ . Let the second repeating root (if one exists) be the  $M_1 + 1^{\text{st}}$  root.

Label this root as  $\lambda_2$  and its multiplicity as  $M_2 \geq 1$ . Continue in this manner until there are no more repeating roots. Let the total number of repeated roots be  $\tilde{M} = \sum_{k=1}^R M_k$ . Label the remaining  $K - \tilde{M} \geq 0$  unique roots as  $\lambda_k$  for each  $k = \tilde{M} + 1, \dots, K$ . Then (without any loss in generality) the general solution to (A.4) can be written as

$$\tilde{h}_k(x) = \sum_{r=1}^R \left[ \sum_{\ell=1}^{M_r} d_{k\ell} x^{(\ell-1)} e^{\lambda_r x} \right] + \sum_{\ell=\tilde{M}+1}^K d_{k\ell} e^{\lambda_\ell x}, \quad k = 1, \dots, K. \quad (\text{A.5})$$

Now substitute (A.5) into the supply of  $q$  to obtain:

$$\begin{aligned} q &= \sum_{k=1}^K \alpha_k(\mathbf{w}) \left[ \sum_{r=1}^R \sum_{\ell=1}^{M_r} d_{k\ell} p^{\lambda_r} (\ln p)^{(\ell-1)} + \sum_{\ell=\tilde{M}+1}^K d_{k\ell} p^{\lambda_\ell} \right] \\ &= \sum_{r=1}^R \sum_{\ell=1}^{M_r} \left[ \sum_{k=1}^K d_{k\ell} \alpha_k(\mathbf{w}) \right] p^{\lambda_r} (\ln p)^{(\ell-1)} + \sum_{\ell=\tilde{M}+1}^K \left[ \sum_{k=1}^K d_{k\ell} \alpha_k(\mathbf{w}) \right] p^{\lambda_\ell} \\ &\equiv \sum_{r=1}^R \sum_{k=1}^{M_r} \tilde{\alpha}_{kr}(\mathbf{w}) p^{\lambda_r} (\ln p)^{(k-1)} + \sum_{k=\tilde{M}+1}^K \tilde{\alpha}_k(\mathbf{w}) p^{\lambda_k}. \end{aligned} \quad (\text{A.6})$$

The terms in the first double sum give cases (i) and (ii), and case (iv) when  $K \geq 4$  and at least one pair of complex conjugate roots repeats, while the terms in the sum on the far right give cases (i) for unique real roots and (iii) for unique pairs of complex conjugate roots, completing the proof of the functional form of the output price terms.

Turn now to the representation of the supply function for  $K=1, 2$ , or  $3$ .

$$K=1: \quad q = \alpha_1(\mathbf{w}) h_1(p). \quad (\text{A.7})$$

Equation (A.3) simplifies to

$$h_1'(p) p = c_{11} h_1(p). \quad (\text{A.8})$$

Direct integration leads to  $h_1(p) = d_{11} p^{c_{11}}$ . The Euler equation then reduces to

$$\partial \alpha_1(\mathbf{w}) / \partial \mathbf{w}^\top \mathbf{w} = -c_{11} \alpha_1(\mathbf{w}). \quad (\text{A.9})$$

Hence, the input price function must be homogeneous of degree  $-c_{11}$ . Set  $\varepsilon_1 = c_{11}$ , absorb the multiplicative constant  $d_{11}$  into the homogeneous price function, and omit the

subscripts to obtain the expression found in (a) of the proposition.

$$K=2: \quad q = \alpha_1(\mathbf{w})h_1(p) + \alpha_2(\mathbf{w})h_2(p). \quad (\text{A.10})$$

The characteristic equation for (A.4) is

$$\lambda^2 - (c_{11} + c_{22})\lambda + (c_{11}c_{22} - c_{12}c_{21}) = 0. \quad (\text{A.11})$$

The characteristic roots are

$$\lambda_1, \lambda_2 = \frac{1}{2}(c_{11} + c_{22}) \pm \frac{1}{2}\sqrt{(c_{11} - c_{22})^2 + 4c_{12}c_{21}}. \quad (\text{A.12})$$

Three cases are possible:

- (a) unique real roots,  $\lambda_1 \neq \lambda_2$ ,  $\lambda_1, \lambda_2 \in \mathbb{R}$ , and  $(c_{11} - c_{22})^2 + 4c_{12}c_{21} > 0$ ;
- (b) one real root,  $\lambda_1 = \lambda_2 = \frac{1}{2}(c_{11} + c_{22}) \equiv \lambda \in \mathbb{R}$ , and  $(c_{11} - c_{22})^2 + 4c_{12}c_{21} = 0$ ; or
- (c) complex conjugate roots,  $\lambda_1 = \kappa + i\tau$ ,  $\lambda_2 = \kappa - i\tau$ ,  $\kappa = \frac{1}{2}(c_{11} + c_{22})$ ,

$$\tau = \frac{1}{2}\sqrt{|(c_{11} - c_{22})^2 + 4c_{12}c_{21}|}, \text{ and } (c_{11} - c_{22})^2 + 4c_{12}c_{21} < 0.$$

With unique roots (whether real or complex), the general solution is

$$\tilde{h}_k(x) = d_{k1}e^{\lambda_1 x} + d_{k2}e^{\lambda_2 x}, \quad k = 1, 2. \quad (\text{A.13})$$

Substituting these expressions into the supply of  $q$  yields

$$\begin{aligned} q &= [d_{11}\alpha_1(\mathbf{w}) + d_{12}\alpha_2(\mathbf{w})]p^{\lambda_1} + [d_{21}\alpha_1(\mathbf{w}) + d_{22}\alpha_2(\mathbf{w})]p^{\lambda_2} \\ &\equiv \tilde{\alpha}_1(\mathbf{w})p^{\lambda_1} + \tilde{\alpha}_2(\mathbf{w})p^{\lambda_2}. \end{aligned} \quad (\text{A.14})$$

If the roots are real, then the Euler equation is

$$\left[ \frac{\partial \tilde{\alpha}_1(\mathbf{w})}{\partial \mathbf{w}^\top \mathbf{w}} + \lambda_1 \tilde{\alpha}_1(\mathbf{w}) \right] p^{\lambda_1} + \left[ \frac{\partial \tilde{\alpha}_2(\mathbf{w})}{\partial \mathbf{w}^\top \mathbf{w}} + \lambda_2 \tilde{\alpha}_2(\mathbf{w}) \right] p^{\lambda_2} = 0. \quad (\text{A.15})$$

Linear independence of the output price functions implies that the term premultiplying each output price function vanishes. Hence,  $\tilde{\alpha}_i(\mathbf{w})$  must be homogeneous of degree  $-\lambda_i$

for  $i=1,2$ . Relabel terms so that  $\varepsilon_i = \lambda_i$  and  $\alpha_i(\mathbf{w}) = \tilde{\alpha}_i(\mathbf{w})^{-1/\varepsilon_i}$ ,  $i = 1,2$ , for case (b) i.

If the characteristic root repeats, the general solution is

$$\tilde{h}_k(x) = d_{k1}e^{\lambda x} + d_{k2}xe^{\lambda x}, \quad k = 1,2. \quad (\text{A.16})$$

Making the same substitutions as before yields:

$$q = \tilde{\alpha}_1(\mathbf{w})p^\lambda + \tilde{\alpha}_2(\mathbf{w})p^\lambda \ln p. \quad (\text{A.17})$$

The Euler equation now is

$$\left[ \left( \frac{\partial \tilde{\alpha}_1(\mathbf{w})}{\partial \mathbf{w}^\top} \mathbf{w} + \lambda \tilde{\alpha}_1(\mathbf{w}) + \tilde{\alpha}_2(\mathbf{w}) \right) + \left( \frac{\partial \tilde{\alpha}_2(\mathbf{w})}{\partial \mathbf{w}^\top} \mathbf{w} + \lambda \tilde{\alpha}_2(\mathbf{w}) \right) \ln p \right] p^\lambda = 0. \quad (\text{A.18})$$

Since  $p^\lambda > 0$  and  $\{1, \ln p\}$  is linearly independent,  $\tilde{\alpha}_2(\mathbf{w})$  must be homogeneous of degree  $-\lambda$ . Therefore, factor it and  $p^\lambda$  out on the right-hand side of (A.17),

$$q = \tilde{\alpha}_2(\mathbf{w})p^\lambda [\hat{\alpha}_1(\mathbf{w}) + \ln p], \quad (\text{A.19})$$

where  $\hat{\alpha}_1(\mathbf{w}) \equiv \tilde{\alpha}_1(\mathbf{w})/\tilde{\alpha}_2(\mathbf{w})$ . The Euler equation then simplifies to

$$\partial \hat{\alpha}_1(\mathbf{w}) / \partial \mathbf{w}^\top \mathbf{w} = -1. \quad (\text{A.20})$$

Let  $\beta(\mathbf{w}) = \exp\{-\hat{\alpha}(\mathbf{w})\}$ , note that  $\partial \beta(\mathbf{w}) / \partial \mathbf{w}^\top \mathbf{w} = -\beta(\mathbf{w}) \partial \hat{\alpha}_1(\mathbf{w}) / \partial \mathbf{w}^\top \mathbf{w} = \beta(\mathbf{w})$  if and only if  $\hat{\alpha}_1(\mathbf{w})$  satisfies (A.20). Relabel terms so that  $\varepsilon_1 = \lambda$ ,  $\alpha_1(\mathbf{w}) = \tilde{\alpha}_2(\mathbf{w})$ , and  $\alpha_2(\mathbf{w}) = \beta(\mathbf{w})$  to obtain case (b) ii.

When the roots are complex, we first require conditions on the input price functions so that  $q$  is real-valued. From (A.14), we have

$$q = p^\kappa \left[ \tilde{\alpha}_1(\mathbf{w})p^{i\tau} + \tilde{\alpha}_2(\mathbf{w})p^{-i\tau} \right], \quad (\text{A.21})$$

while deMoivre's theorem implies that

$$p^{\pm i\tau} = \cos(\tau \ln p) \pm i \sin(\tau \ln p). \quad (\text{A.22})$$

Thus, complex functions  $\tilde{\alpha}_1(\mathbf{w}) = \hat{\alpha}_0(\mathbf{w}) + i\hat{\alpha}_1(\mathbf{w})$  and  $\tilde{\alpha}_2(\mathbf{w}) = \hat{\beta}_0(\mathbf{w}) + i\hat{\beta}_1(\mathbf{w})$  are required if  $q$  is real-valued. Substituting these definitions and (A.22) into (A.21) yields:

$$q = p^\kappa \left[ (\hat{\alpha}_0 + i\hat{\alpha}_1 + \hat{\beta}_0 + i\hat{\beta}_1) \cos(\tau \ln p) + (i\hat{\alpha}_0 - \hat{\alpha}_1 - i\hat{\beta}_0 + \hat{\beta}_1) \sin(\tau \ln p) \right]. \quad (\text{A.23})$$

We must have  $\hat{\beta}_1(\mathbf{w}) = -\hat{\alpha}_1(\mathbf{w})$  for the term in front of  $\cos(\tau \ln p)$  to be real-valued and  $\hat{\beta}_0(\mathbf{w}) = \hat{\alpha}_0(\mathbf{w})$  for the term in front of  $\sin(\tau \ln p)$  to be real-valued, so that the input price functions are complex conjugates. Omitting the ^'s and the subscripts and absorbing the multiplicative constant 2 into the price functions for conciseness, we then have:

$$q = p^\kappa \left[ \alpha(\mathbf{w}) \cos(\tau \ln p) + \beta(\mathbf{w}) \sin(\tau \ln p) \right]. \quad (\text{A.24})$$

The Euler equation now has the form:

$$\begin{aligned} & \left\{ \left[ \alpha_{\mathbf{w}}(\mathbf{w})^\top \mathbf{w} - \tau\beta(\mathbf{w}) + \kappa\alpha(\mathbf{w}) \right] \cos(\tau \ln p) \right. \\ & \left. + \left[ \beta_{\mathbf{w}}(\mathbf{w})^\top \mathbf{w} + \tau\alpha(\mathbf{w}) + \kappa\alpha(\mathbf{w}) \right] \sin(\tau \ln p) \right\} p^\kappa = 0. \end{aligned} \quad (\text{A.25})$$

Define the smooth and invertible transformation

$$\begin{aligned} \alpha(\mathbf{w}) &= \tilde{\alpha}(\mathbf{w}) \left[ \cos(\tau \ln \tilde{\beta}(\mathbf{w})) - \sin(\tau \ln \tilde{\beta}(\mathbf{w})) \right], \\ \beta(\mathbf{w}) &= \tilde{\alpha}(\mathbf{w}) \left[ \sin(\tau \ln \tilde{\beta}(\mathbf{w})) + \cos(\tau \ln \tilde{\beta}(\mathbf{w})) \right], \end{aligned} \quad (\text{A.26})$$

for  $\tilde{\alpha}(\mathbf{w}) \neq 0$  any smooth, homogeneous of degree  $-\kappa$  function and  $\tilde{\beta}(\mathbf{w}) > 0$  any positive linearly homogeneous function. A direct calculation yields

$$\begin{aligned} \alpha_{\mathbf{w}}(\mathbf{w})^\top \mathbf{w} &= -\kappa\alpha(\mathbf{w}) + \tau\beta(\mathbf{w}), \\ \beta_{\mathbf{w}}(\mathbf{w})^\top \mathbf{w} &= -\kappa\alpha(\mathbf{w}) - \tau\alpha(\mathbf{w}), \end{aligned} \quad (\text{A.27})$$

as required.

Relabeling with  $\varepsilon_1 = \kappa$ ,  $\alpha_1(\mathbf{w}) = \tilde{\alpha}(\mathbf{w})^{-1/\kappa}$ , and  $\alpha_2(\mathbf{w}) = \tilde{\beta}(\mathbf{w})$  yields:



$$q = [p/\alpha_1(\mathbf{w})]^{\epsilon_1} \left\{ \left[ \cos(\tau \ln \alpha_2(\mathbf{w})) - \sin(\tau \ln \alpha_2(\mathbf{w})) \right] \cos(\tau \ln p) \right. \\ \left. + \left[ \sin(\tau \ln \alpha_2(\mathbf{w})) + \cos(\tau \ln \alpha_2(\mathbf{w})) \right] \sin(\tau \ln p) \right\}. \quad (\text{A.28})$$

A little tedious but straightforward algebra using the trigonometric identities:

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b);$$

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b);$$

$$\sin(-b) = -\sin(b); \text{ and}$$

$$\cos(-b) = \cos(b);$$

with  $a = \tau \ln p$  and  $b = -\tau \ln \alpha_2(\mathbf{w})$  then gives the form in (b) iii of the proposition.

$$K=3: \quad q = \alpha_1(\mathbf{w})h_1(p) + \alpha_2(\mathbf{w})h_2(p) + \alpha_3(\mathbf{w})h_3(p). \quad (\text{A.29})$$

In this case, the characteristic equation is a third-order polynomial in  $\lambda$ , and by the fundamental theorem of algebra, there are four mutually exclusive and exhaustive cases:

- (1) three unique real roots,  $\lambda_1 \neq \lambda_2 \neq \lambda_3$ ,  $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ ;
- (2) one repeated real root,  $\lambda_1 = \lambda_2 \in \mathbb{R}$  and one unique real root,  $\lambda_3 \in \mathbb{R}$ ;
- (3) one real root repeated thrice,  $\lambda_1 = \lambda_2 = \lambda_3 \equiv \lambda \in \mathbb{R}$ ; and
- (4) one real root,  $\lambda_1 \in \mathbb{R}$  and two complex conjugate roots,  $\lambda_2 = \kappa + i\tau$ ,  $\lambda_3 = \kappa - i\tau$ .

First, if (1) holds, the argument leading to the representation in (c) i is identical to that of the previous cases  $K=1$  or  $K=2$  when the roots are real and unique. Second, if (2) holds, then we have the sum of one term of the form given in (a) and a second term of the form given in (b) ii of the proposition, leading to case (c) ii. Third, if (4) holds, then we have the sum of one term of the form given in (a) and a second term of the form given in (b) iii of the proposition, leading to case (c) iv.

Therefore, consider case (3), for which the general solution to (A.4) has the form:

$$\tilde{h}_k(x) = d_{k1}e^{\lambda x} + d_{k2}xe^{\lambda x} + d_{k3}x^2e^{\lambda x}, \quad k = 1, 2, 3. \quad (\text{A.30})$$

Rewriting this in terms of  $p$  and the  $\{h_k(p)\}$ , substituting the result into (A.29), and re-grouping terms as before yields:

$$q = p^\lambda \left[ \tilde{\alpha}_1(\mathbf{w}) + \tilde{\alpha}_2(\mathbf{w}) \ln p + \tilde{\alpha}_3(\mathbf{w})(\ln p)^2 \right]. \quad (\text{A.31})$$

The Euler equation is:

$$\begin{aligned} & p^\lambda \left\{ \left[ \frac{\partial \tilde{\alpha}_1(\mathbf{w})}{\partial \mathbf{w}^\top \mathbf{w}} + \lambda \tilde{\alpha}_1(\mathbf{w}) + \tilde{\alpha}_2(\mathbf{w}) \right] \right. \\ & + \left[ \frac{\partial \tilde{\alpha}_2(\mathbf{w})}{\partial \mathbf{w}^\top \mathbf{w}} + \lambda \tilde{\alpha}_2(\mathbf{w}) + 2\tilde{\alpha}_3(\mathbf{w}) \right] \ln p \\ & \left. + \left[ \frac{\partial \tilde{\alpha}_3(\mathbf{w})}{\partial \mathbf{w}^\top \mathbf{w}} + \lambda \tilde{\alpha}_3(\mathbf{w}) \right] (\ln p)^2 \right\} = 0. \end{aligned} \quad (\text{A.32})$$

As before,  $p^\lambda > 0$  and the linear independence of  $\{1, \ln p, (\ln p)^2\}$  requires each sum in square brackets to vanish. In particular,  $\tilde{\alpha}_3(\mathbf{w})$  must be homogeneous of degree  $-\lambda$ , and we can factor it out of the term in square brackets in (A.31), yielding:

$$q = \tilde{\alpha}_3(\mathbf{w}) p^\lambda \left[ \hat{\alpha}_1(\mathbf{w}) + \hat{\alpha}_2(\mathbf{w}) \ln p + (\ln p)^2 \right], \quad (\text{A.33})$$

with  $\hat{\alpha}_1(\mathbf{w}) = \tilde{\alpha}_1(\mathbf{w})/\tilde{\alpha}_3(\mathbf{w})$  and  $\hat{\alpha}_2(\mathbf{w}) = \tilde{\alpha}_2(\mathbf{w})/\tilde{\alpha}_3(\mathbf{w})$ .

Now the term in brackets on the right-hand side must be homogeneous of degree zero, which implies:

$$\begin{aligned} \frac{\partial \hat{\alpha}_1(\mathbf{w})}{\partial \mathbf{w}^\top \mathbf{w}} &= -\hat{\alpha}_2(\mathbf{w}); \\ \frac{\partial \hat{\alpha}_2(\mathbf{w})}{\partial \mathbf{w}^\top \mathbf{w}} &= -2. \end{aligned} \quad (\text{A.34})$$

Therefore, define the smooth and invertible transformation

$$\begin{aligned} \hat{\alpha}_1(\mathbf{w}) &= \bar{\alpha}_1(\mathbf{w}) + [\ln \bar{\alpha}_2(\mathbf{w})]^2, \\ \hat{\alpha}_2(\mathbf{w}) &= -2 \ln \bar{\alpha}_2(\mathbf{w}), \end{aligned} \quad (\text{A.35})$$

where  $\bar{\alpha}_1(\mathbf{w})$  is an arbitrary homogeneous of degree zero function and  $\bar{\alpha}_2(\mathbf{w}) > 0$  is an

arbitrary positive linearly homogeneous function. A direct calculation shows that  $\hat{\alpha}_1(\mathbf{w})$  and  $\hat{\alpha}_2(\mathbf{w})$  satisfy (A.34) if and only if they are related to the two homogeneous functions  $\hat{\alpha}_1(\mathbf{w})$  and  $\hat{\alpha}_2(\mathbf{w})$  by (A.35). Substituting (A.35) into (A.33), grouping terms, and relabeling with  $\varepsilon = \lambda$ ,  $\alpha_1(\mathbf{w}) = \hat{\alpha}_3(\mathbf{w})$ ,  $\alpha_2(\mathbf{w}) = \hat{\alpha}_1(\mathbf{w})$ , and  $\alpha_3(\mathbf{w}) = \hat{\alpha}_2(\mathbf{w})$  yields the representation in (c) iii of the proposition. ■

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<sup>1</sup> One of the amazing, and at times frustrating, aspects of Gorman's (1981) paper is that he simultaneously imposes symmetry, adding up, and homogeneity so that the role of each is unclear (could reference our paper).

<sup>2</sup> All of the results apply – with appropriate changes in notation – to a single consumer demand equation, as well as a supply or input demand equation with profit or cost replacing output price. However, we feel that the standard supply case is most useful and illuminating. Furthermore, when applying our results to a system, the flexibility in the main proposition of this paper is restricted in a complete system of consumer demand equations due to the added properties of symmetry and adding up.

<sup>3</sup> Non-negativity and monotonicity of supply in  $p$  are usually not imposed in single equation empirical implementation but clearly are expected by the researcher.

<sup>4</sup> In some cases, the usefulness of the results appear to be limited because a closed form solution does not exist. However, numerical integration is always possible in empirical work.

<sup>5</sup> Profit could be added but adding up as well if one takes note of the statistical properties of the errors from creating profit from supply and factor demand.