Title

Stochastic volatility model and technical analysis of stock price

Journal

Acta Mathematica Sinica, English Series, 27(7)

ISSN

1439-8516

Authors

Liu, Wei
Zheng, Wei An

Publication Date

2011-07-01

DOI

10.1007/s10114-011-9468-1

Acta Mathematica Sinica, English Series
Jul., 2011, Vol. 27, No. 7, pp. 1283–1296
Published online: June 15, 2011
DOI: 10.1007/s10114-011-9468-1
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Stochastic Volatility Model and Technical Analysis of Stock Price

Wei LIU
Department of Statistics, Shanghai Finance University, Shanghai 201209*, P. R. China*
E-mail: *limitliuwei@yahoo.com*

Wei An ZHENG
Department of Statistics, East China Normal University, Shanghai 200062*, P. R. China*
E-mail: *wzheng@uci.edu*

Abstract In the stock market, some popular technical analysis indicators (e.g. Bollinger Bands, RSI, ROC, ...) are widely used by traders. They use the daily (hourly, weekly, ...) stock prices as samples of certain statistics and use the observed relative frequency to show the validity of those well-known indicators. However, those samples are not independent, so the classical sample survey theory does not apply. In earlier research, we discussed the law of large numbers related to those observations when one assumes Black–Scholes' stock price model. In this paper, we extend the above results to the more popular stochastic volatility model.

Keywords Stochastic volatility model, asymptotic stationary process, law of large numbers, convergence rate, technical analysis indicators

MR(2000) Subject Classification 62H10, 62P20, 65C50

1 Introduction

Liu et al. discussed in [1] the Bollinger bands for the Black–Scholes model. The following observation is interesting: Black–Scholes stock price model has Markov property — given the present, the future is independent of the past, while the formulation of the Bollinger bands largely depends on the past. So at the first glance, it looks that Bollinger bands property was unthinkable to us. However, Liu et al. proved in [1] that Black–Scholes model really possesses the Bollinger bands property of practical stock market. Under Black–Scholes model, we introduced the statistics $\{U_t^{(n)}\}$ calculated according to the formulation of the Bollinger bands, which is stationary and $\{U_{s+kn}^{(n)}\}_{k=1,2,\dots}$ are mutually independent for each fixed $s \geq 0$. Zhu [2] extended our result to another indicator RSI.

It has been noticed in [3] that "technical analysis has been a part of financial practice for many decades, but this discipline has not received the same level of academic scrutiny and acceptance as more traditional approaches such as fundamental analysis" and "a simulated

Received August 25, 2009, accepted April 2, 2010
Partially supported by National Natural Science Foundation of China (Grant No. 10971068), National Basic Research Program of China (973 Program) (Grant No. 2007CB814904) and Key Subject Construction Project of Shanghai Education Commission (Grant No. J51601)

sample is only one realization of geometric Brownian motion, so it is difficult to draw general conclusions about the relative frequencies". However, we show here that if we recognize the current popular stock price models, then we can do statistics based on relative frequency of occurrence for some technical analysis indicators.

In this paper we extend our above result to the stochastic volatility model, which is only implicitly Markovian and has no independent logarithm increment property [4]. Let us introduce the definitions of Bollinger bands, RSI and ROC, which are popular technical analysis indicators in the stock market. Denote by S_t the observed stock price.

1) Bollinger Bands Definition:

Middle Bollinger Band = 12-day weighted moving average.

Upper Bollinger Band = Middle Bollinger Band $+2 \times$ 12-day standard deviation.

Lower Bollinger Band = Middle Bollinger Band $-2 \times$ 12-day standard deviation.

If we denote

$$\hat{S}_t^{12} = \frac{1}{78}\sum_{i=0}^{11}(12-i)S_{t-i} \quad \text{(weighted mean)}, \quad \overline{S}_t^{12} = \frac{1}{12}\sum_{i=0}^{11} S_{t-i} \quad \text{(simple mean)}$$

and

$$\sigma_t = \left[\frac{1}{11}\sum_{i=0}^{11}(S_{t-i} - \overline{S}_t^{12})^2\right]^{\frac{1}{2}},$$

then the curve $\gamma_t^- = \hat{S}_t^{12} - 2\sigma_t$ is called the lower Bollinger band and $\gamma_t^+ = \hat{S}_t^{12} + 2\sigma_t$ is called the upper Bollinger band. Closer the price moves to the upper band, more overbought the market and closer to the opposite direction on behalf of oversold. So when stock price fluctuates between upper band and lower band we say the price changes within a normal scope, i.e. the process $\{\frac{S_t - \hat{S}_t^{12}}{\sigma_t}\}$ changes in a normal scope.

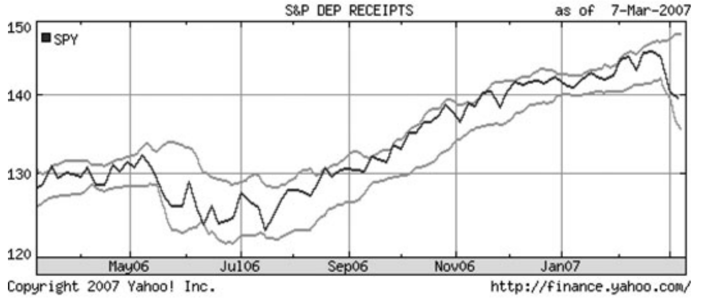

Figure 1 S&P500 annual Bollinger Bands until 7-Mar-2007

2) RSI (Relative Strength Indicator) Definition:

Mean value of rising = mean value of the stock price's rising in 14 days.

Mean value of dropping = mean value of the stock price's dropping in 14 days.

RSI = 100 $\times$ Mean value of rising / (Mean value of rising + Mean value of dropping).

If we denote

$$\Delta S_t^+ = (S_{t+1} - S_t) \vee 0, \quad \Delta S_t^- = (S_t - S_{t+1}) \vee 0,$$

14-day RSI is defined as

$$RSI_t^{(14)} = 100 \times \frac{\sum_{i=1}^{14} \Delta S_{t-i}^{+}}{\sum_{i=1}^{14} \Delta S_{t-i}^{+} + \sum_{i=1}^{14} \Delta S_{t-i}^{-}}, \quad \forall\, t > 14.$$

RSI takes its values in $[0, 100]$. In general, RSI value maintains above 50 for a strong trend market and below 50 for a weak trend one. 14-day RSI above 80 may be regarded as ultra-buy area, and below 20 may be regarded as ultra-sell area. When RSI fluctuates between [20, 80], we say RSI changes within a normal scope.

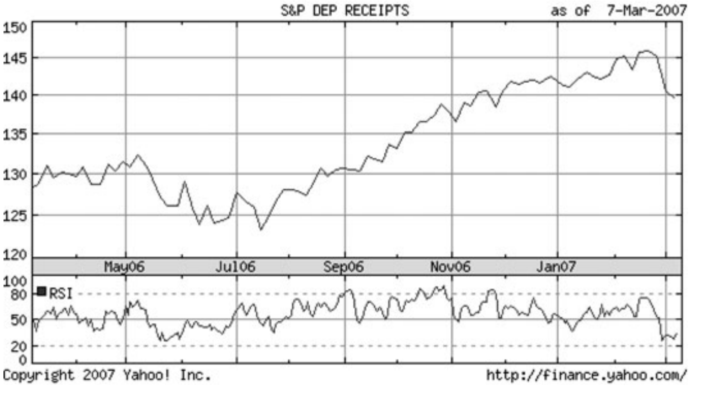

Figure 2 S&P500 annual RSI until 7-Mar-2007

3) ROC (Rate of Change Indicator) Definition:

ROC = 100 × (closing price - closing price of 12-day before) / closing price of 12-day before.

That is, 12-day ROC is defined as

$$\text{ROC}_t^{(12)} = 100 \times \frac{S_t - S_{t-12}}{S_{t-12}}, \quad \forall\, t > 12.$$

ROC has three pairs of indefinite antennas and groundings to show the ultra-buy and ultra-sell area, which are defined by the historical prices of previous year. When ROC undulates in "normal scope", it is time to sell out stock while ROC rises to the first antenna and to buy in when ROC drops to the first grounding. For example, in Figure 3, the first antenna is 5 and the first grounding is -5. When ROC fluctuates between $[-5, 5]$ we say ROC changes within a normal scope.

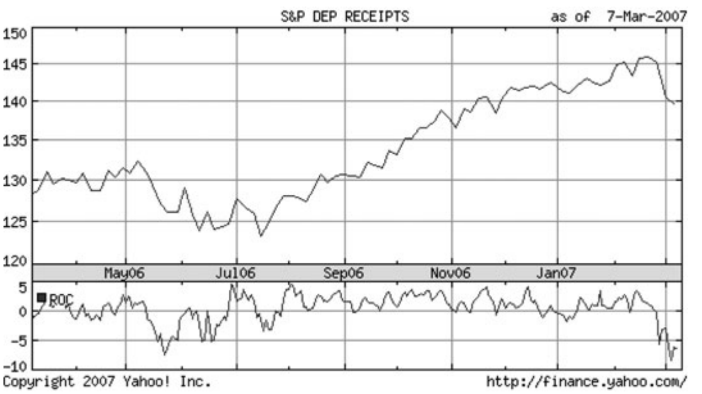

Figure 3 S&P500 annual ROC until 7-Mar-2007

We traced 13 years of the SPY daily closing prices. Our result in the following table shows that in every year more than 94% of daily closing price are between the Bollinger bands (B-B) and more than 77% of daily RSI are between $[20, 80]$. Just as introduced above, ROC has three antennas and grounds. In every year more than 74% of daily ROC are between $[-5, 5]$, more than 95% between $[-10, 10]$ and 100% between $[-20, 20]$.

Year	Price$\in$ B-B	RSI$\in [20,80]$	ROC$\in [-5,5]$	ROC$\in [-10,10]$	ROC$\in [-20,20]$
1993	95.90%	93.90%	100%	100%	100%
1994	94.21%	85.66%	100%	100%	100%
1995	97.54%	79.67%	98.35%	100%	100%
1996	96.30%	77.96%	95.45%	100%	100%
1997	98.76%	87.70%	92.53%	99.59%	100%
1998	95.85%	87.24%	80.83%	98.33%	100%
1999	97.93%	86.89%	87.97%	100%	100%
2000	97.93%	92.62%	83.40%	98.76%	100%
2001	97.90%	87.08%	85.65%	97.05	100%
2002	99.17%	87.30%	74.27%	95.44	100%
2003	97.93%	89.75%	97.10%	100%	100%
2004	97.49%	88.38%	99.16%	100%	100%
2005	97.51%	88.89%	100%	100%	100%

Table 1 Ratio of SPY 13 years historical price

In the original Black–Scholes model,

$$S_t = S_0 \exp\left\{\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)t\right\}, \tag{1.1}$$

the risk is quantified by a constant volatility parameter. It has been proposed by many authors that the volatility should be modeled by a stochastic process to obtain a more practical model. In this paper, we consider the model given by

$$\begin{aligned} dS_t &= \mu S_t dt + \mathrm{e}^{Y_t} S_t dW_t \\ dY_t &= a(b - Y_t)dt + \sigma dB_t, \\ Y_0 &= c, \end{aligned} \tag{1.2}$$

where Y_t is an O–U process beginning with an arbitrary status c, W_t and B_t are independent Brownian motions, a, b, c and σ are arbitrary constants and $a > 0$, $b \geq 0$. For more discussion about stochastic volatility model, see [5–8]. The corresponding S_t is derived as

$$S_t = S_0 \exp\left\{\int_0^t \left(\mu - \frac{1}{2}\mathrm{e}^{2Y_s}\right)ds + \int_0^t \mathrm{e}^{Y_s} dW_s\right\}, \tag{1.3}$$

where the joint process (Y_t, S_t) has Markov property: given the present (Y_t, S_t), the future is independent of the past. However, the formulations of the Bollinger bands, RSI and ROC largely depend on the past. We prove that under the stochastic volatility model some statistics drawing from these indicators have asymptotic stationary property. If we set an assumption that the process Y_t starts with the initial condition Y_0, which is a random variable independent of $(B_t, W_t)_{(t>0)}$ and having the stationary distribution, we can get strictly stationary property. However, this assumption is not very practical in application, on which we'll say more in Remark 2.2.

2 Asymptotic Stationary Property

Let S_t be the stock price generated by the stochastic volatility model (1.3). We have the following:

Theorem 2.1 *Let f, g be measurable functions*: $\mathbb{R}^{n^2} \to \mathbb{R}$, *satisfying*

(1) $f(ax) = af(x)$, $\forall\, a > 0$, $x = (x_{i,j}, 0 \le i, j < n) \in \mathbb{R}^{n^2}$;

(2) $g(ax) = ag(x)$, $\forall\, a > 0$, $x = (x_{i,j}, 0 \le i, j < n) \in \mathbb{R}^{n^2}$.

Then $\{U_t\}_{t\ge n} = \{A_t/B_t\}_{t\ge n}$ *is an asymptotic stationary process, where* $A_t = f(S_{t-i}-S_{t-j}, 0 \le i, j < n), B_t = g(S_{t-i} - S_{t-j}, 0 \le i, j < n)$.

Proof For $0 \le i, j \le n-1, t \ge n$,

$$
\begin{aligned}
f(S_{t-i} - S_{t-j}) &= f(ax_{i,j}, 0 \le i, j < n) = af(x_{i,j}, 0 \le i, j < n),\\
g(S_{t-i} - S_{t-j}) &= g(ax_{i,j}, 0 \le i, j < n) = ag(x_{i,j}, 0 \le i, j < n),
\end{aligned}
$$

where

$$
\begin{aligned}
a &= S_0 \exp\left\{ \int_0^{t-n+1} \left(\mu - \frac{1}{2}\mathrm{e}^{2Y_s}\right) ds + \int_0^{t-n+1} \mathrm{e}^{Y_s} dW_s \right\},\\
x_{i,j} &= \exp\left\{ \int_{t-n+1}^{t-i} \left(\mu - \frac{1}{2}\mathrm{e}^{2Y_s}\right) ds + \int_{t-n+1}^{t-i} \mathrm{e}^{Y_s} dW_s \right\}\\
&\quad - \exp\left\{ \int_{t-n+1}^{t-j} \left(\mu - \frac{1}{2}\mathrm{e}^{2Y_s}\right) ds + \int_{t-n+1}^{t-j} \mathrm{e}^{Y_s} dW_s \right\}.
\end{aligned}
$$

So

$$
\{U_t\}_{t\ge n} = \{A_t/B_t\}_{t\ge n} = \{f(S_{t-i} - S_{t-j})/g(S_{t-i} - S_{t-j})\}_{t\ge n}
$$

is just a function of $\{x_{i,j}, 0 \le i, j < n\}$, i.e., a function of $(Y_s, W_s)_{t-n+1\le s\le t}$. In the stochastic volatility model, W_t is independent of B_t, so it is also independent of Y_t. For $\{Y_t\}$ is O–U process, it is an asymptotic stationary process. Without loss of generality, we can assume that it starts from its stationary distribution. For simplifying our notation, we denote $X(=)Y$ if two random vectors X and Y have the same distribution. From the fact that $(Y_s, W_s)_{s\ge t-n+1}$ is a two-dimensional stationary process, we have for fixed $\alpha \ge 0$,

$$
\{U_t\}_{t\ge n}(=)\{U_{t+\alpha}\}_{t\ge n}.
$$

However, $\{Y_t\}$ is an asymptotic stationary process, so we get the conclusion that $\{U_t\}_{t\ge n}$ is asymptotic stationary. □

Remark 2.2 Assume the process $\{Y_t\}$ starts with Y_0, a random having stationary distribution, we can come to a conclusion that under the stochastic volatility model, those statistics drawn from technical indicators are strictly stationary. However, it has been noticed by the applied financial statisticians that, when we use the stochastic volatility model to describe the fluctuation of stock price, we are not sure whether the process has been stationary which is the premise the statistical analysis bases on. So it is important to decide how long it will take the process to become stationary, i.e., how many historical data one should have in order to have statistical significance. This is why we just suppose that the process $\{Y_t\}$ starts with an arbitrary stat c. We will come back to this issue in Remark 3.5.

By Theorem 2.1, we can draw a series of asymptotic stationary processes from the stochastic volatility model. This fact can be applied to the technical analysis indicators which we have introduced before. Asymptotic properties are listed in the following corollaries.

Corollary 2.3 *Let S_t be stock price generated by the stochastic volatility model* (1.3)*. Denote for each $n \geq 2$ and $t \geq n$ that*

$$\overline{S}_t^{(n)} = \frac{1}{n}\sum_{i=0}^{n-1} S_{t-i}, \quad \hat{S}_t^{(n)} = \frac{1}{\sum_{i=1}^{n} i}\sum_{i=0}^{n-1}(n-i)S_{t-i}$$

and

$$\sigma_t^{(n)} = \sqrt{\frac{1}{n-1}\sum_{i=0}^{n-1}(S_{t-i} - \overline{S}_t^{(n)})^2}.$$

The process

$$L_t^{(n)} = \frac{S_t - \hat{S}_t^{(n)}}{\sigma_t^{(n)}}, \quad \forall\, t \geq n$$

is asymptotic stationary.

Corollary 2.4 *Let S_t be the stock price generated by the stochastic volatility model* (1.3)*. For $1 \leq i \leq n, t > n$, denote*

$$\Delta S_t^{+} = (S_{t+1} - S_t) \vee 0, \quad \Delta S_t^{-} = (S_t - S_{t+1}) \vee 0.$$

Then

$$I_t^{(n)} = \frac{\sum_{i=1}^{n} \Delta S_{t-i}^{+}}{\sum_{i=1}^{n} \Delta S_{t-i}^{+} + \sum_{i=1}^{n} \Delta S_{t-i}^{-}}, \quad \forall\, t > n$$

is asymptotic stationary.

Corollary 2.5 *Let S_t be stock price generated by the stochastic volatility model* (1.3)*. Using a similar method we can get*

$$R_t^{(n)} = \frac{S_t - S_{t-n}}{S_{t-n}}, \quad \forall\, t > n$$

is asymptotic stationary.

In the above section, we use S_t, which is stock price generated by the stochastic volatility model (1.3), and get the asymptotic stationary processes $\{L_t^{(n)}\}_{t\geq n}$, $\{I_t^{(n)}\}_{t>n}$ and $\{R_t^{(n)}\}_{t>n}$. When t is sufficiently large, we can just consider the process as stationary without loss of generality. Therefore by Birkhoff's ergodic theorem (see [9, p. 296]) (it is easy to see the invariance of σ-field) we get that for any bounded measurable function f,

$$\frac{1}{N+1}\sum_{i=0}^{N} f(L_i^{(n)}) \xrightarrow{\text{a.s.}} E[f(L_\infty^{(n)})], \quad N\to\infty,$$

$$\frac{1}{N+1}\sum_{i=0}^{N} f(I_i^{(n)}) \xrightarrow{\text{a.s.}} E[f(I_\infty^{(n)})], \quad N\to\infty,$$

and

$$\frac{1}{N+1}\sum_{i=0}^{N} f(R_i^{(n)}) \xrightarrow{\text{a.s.}} E[f(R_\infty^{(n)})], \quad N\to\infty,$$

where we used $L_\infty^{(n)}$, $I_\infty^{(n)}$ and $R_\infty^{(n)}$ to denote the random variables with the corresponding stationary distributions.

3 Rate of Convergence

In general, to obtain the rate of convergence in Birkhoff's ergodic theorem may be a little complicated. But in the special case of this article, we can obtain the answer. Let S_t be a stock price given by the stochastic volatility model (1.3) and let $\{U_t\}_{t\geq n}$ be the asymptotic stationary process satisfying Theorem 2.1.

Let F_t be the natural σ-algebra generated by (S_t, Y_t). Denote for $i\geq n$, $K_{\lambda,i}^{(n)} = I_{[|U_i^{(n)}|\geq\lambda]}$. Then $E[K_{\lambda,i}^{(n)}] = P[|U_i^{(n)}|\geq\lambda]$. Let

$$V_{N,\lambda}^{(n)} = \frac{1}{N+1}\sum_{i=0}^{N} K_{\lambda,n+i}^{(n)}$$

which is the observed frequency of the events $[|U_{n+i}^{(n)}|\geq\lambda]$ $(i=0,1,\ldots,N)$.

Lemma 3.1 *The following inequlity holds*:

$$E\left|V_{N,\lambda}^{(n)} - \frac{1}{N+1}\sum_{i=0}^{N} P[|U_{n+i}^{(n)}|\geq\lambda|F_i]\right|^2 \leq \frac{n}{N+1}. \tag{3.1}$$

Proof We have

$$E\left|V_{N,\lambda}^{(n)} - \frac{1}{N+1}\sum_{i=0}^{N} P[|U_{n+i}^{(n)}|\geq\lambda|F_i]\right|^2 = E\left\{\frac{1}{N+1}\sum_{i=0}^{N}[K_{\lambda,n+i}^{(n)} - P[|U_{n+i}^{(n)}|\geq\lambda|F_i]]\right\}^2.$$

Denote for each fixed j,

$$X_j = \sum_{\{k;0\leq kn+j\leq N\}} (K_{\lambda,(k+1)n+j}^{(n)} - P[|U_{(k+1)n+j}^{(n)}|\geq\lambda|F_{kn+j}]).$$

Then,

$$\begin{aligned} E\left|V_{N,\lambda}^{(n)} - \frac{1}{N+1}\sum_{i=0}^{N} P[|U_{n+i}^{(n)}|\geq\lambda|F_i]\right|^2 &= E\left|\frac{1}{N+1}\sum_{j=0}^{n-1} X_j\right|^2 \\ &\leq n\sum_{j=0}^{n-1} E\left[\frac{1}{N+1}X_j\right]^2. \end{aligned}$$

Let

$$Z_{k,j} = K^{(n)}_{\lambda,(k+1)n+j} - P[|U^{(n)}_{(k+1)n+j}| \geq \lambda | F_{kn+j}],$$

then

$$\begin{aligned} E\Big(\frac{1}{N+1}X_j\Big)^2 &= \frac{1}{(N+1)^2}E\Big(\sum_{\{k;0\leq kn+j\leq N\}} Z_{k,j}\Big)^2 \\ &= \frac{1}{(N+1)^2}\sum_{\{k;0\leq kn+j\leq N\}} EZ^2_{k,j}\,, \end{aligned}$$

where we used the orthogonality of $\{Z_{k,j}\}$. It is easy to see that $E[Z^2_{k,j}] \leq P[U^{(n)}_{(k+1)n+j} \geq \lambda] \leq 1$ and we get the conclusion. □

On the other hand, $U^{(n)}_{n+i}$ is a function of

$$\mathrm{e}^{\int_i^{n+i-j}(\mu-\frac{1}{2}\mathrm{e}^{2Y_s})ds+\int_i^{n+i-j}\mathrm{e}^{Y_s}dW_s}, \quad 0\leq j\leq n-1,$$

thus

$$P[|U^{(n)}_{n+i}| \geq \lambda | F_i] = P[|U^{(n)}_{n+i}| \geq \lambda | Y_i]$$

where $\{Y_t\}$ is the O–U process in (1.2). Let

$$P_0[|U^{(n)}_n| \geq \lambda] = E(P[|U^{(n)}_{n+i}| \geq \lambda | Y_i \text{ is stationary}]),$$

which is just the occurrence probability of the events $[U^{(n)}_{n+i}| \geq \lambda]\,(i = 0,1,2,\dots)$ when the process $\{U^{(n)}_t\}_{t>n}$ becomes stationary. Before obtaining the conclusion, we need the following lemma firstly.

Lemma 3.2 *$\{Y_t\}$ is a stochastic process, $f(\cdot)$ is a measurable function and $|f(\cdot)| \leq 1$. For $i,j,N \in \mathbb{Z}^+$, $0 \leq i < j \leq N$ and any $0<\alpha<1$, if $|Ef(Y_i)f(Y_j)| \leq C\mathrm{e}^{-D(j-i)}$ when $j-i > N^\alpha$, where C, D are positive constants, then*

$$E\Big|\frac{1}{N+1}\sum_{i=0}^{N} f(Y_i)\Big|^2 \leq \frac{5}{N^{1-\alpha}} + 7C\mathrm{e}^{-DN^\alpha}. \tag{3.2}$$

Without losing generality, we can choose $N > N_0 = \inf\{m; \frac{5\mathrm{e}^D}{7CD} < m < \frac{5\mathrm{e}^{mD}}{7CD}, m \in \mathbb{Z}^+, \frac{5\mathrm{e}^D}{7CD} < m' < \frac{5\mathrm{e}^{m'D}}{7CD}, \forall\, m' > m, m' \in \mathbb{Z}^+\}$, then the following inequality holds:

$$E\Big|\frac{1}{N+1}\sum_{i=0}^{N} f(Y_i)\Big|^2 \leq \log\Big(\frac{7\mathrm{e}CD}{5}\Big)^{\frac{5}{D}}\frac{1}{N} + \frac{5}{D}\frac{\log N}{N}. \tag{3.3}$$

Proof For $f(\cdot)$ is a measurable function and $|f(\cdot)| \leq 1$, we have

$$\begin{aligned} E\Big|\frac{1}{N+1}\sum_{i=0}^{N} f(Y_i)\Big|^2 &= \frac{1}{(N+1)^2}\sum_{i=0}^{N} Ef^2(Y_i) + \frac{2}{(N+1)^2}\sum_{\substack{0\leq i<j\leq N,\\ 0<j-i<N^\alpha}} Ef(Y_i)f(Y_j) \\ &\quad + \frac{2}{(N+1)^2}\sum_{\substack{0\leq i<j\leq N,\\ N^\alpha\leq j-i\leq N}} Ef(Y_i)f(Y_j) \\ &\leq \frac{1}{N+1} + \frac{2N^{1+\alpha} - 2N - N^{2\alpha} + 3N^\alpha - 2}{(N+1)^2} \end{aligned}$$

$$+\frac{N^2-2N^{1+\alpha}+3N+N^{2\alpha}-3N^\alpha+2}{(N+1)^2}\sup_{\substack{0\le i<j\le N,\\ N^\alpha\le j-i\le N}} Ef(Y_i)f(Y_j)$$
$$\le \frac{5}{N^{1-\alpha}}+7\sup_{\substack{0\le i<j\le N,\\ N^\alpha\le j-i\le N}} C\mathrm{e}^{-D(j-i)}$$
$$\le \frac{5}{N^{1-\alpha}}+7C\mathrm{e}^{-DN^\alpha}.$$

For $N>N_0$, we have

$$\inf_{0<\alpha<1}\left\{\frac{5}{N^{1-\alpha}}+7C\mathrm{e}^{-DN^\alpha}\right\}=\frac{5}{N^{1-\alpha'}}+7C\mathrm{e}^{-DN^{\alpha'}},$$

where $\alpha'=\log_N\log(\frac{7NCD}{5})^{\frac{1}{D}}$ and $0<\alpha'<1$. Thus we get the conclusion. □

Lemma 3.3 *The following inequality holds*:

$$E\left|\frac{1}{N+1}\sum_{i=0}^{N}P[|U_{n+i}^{(n)}|\ge\lambda|F_i]-P_0[|U_n^{(n)}|\ge\lambda]\right|^2\le\frac{5}{N^{1-\alpha}}+7\mu\mathrm{e}^{-\nu N^\alpha},\tag{3.4}$$

where

$$\mu=\frac{2a(b+1)^2\mathrm{e}^{\frac{2a(b+1)^2\mathrm{e}^{-\frac{a}{2}}}{\sigma^2(1-\mathrm{e}^{-2a})}}}{\sigma^2(1-\mathrm{e}^{-2a})\sqrt{1-\mathrm{e}^{-2a}}}+\frac12+\frac38(1-\mathrm{e}^{-2a})^{-\frac52}+\frac{|\sigma|(\mathrm{e}^{\frac{2a}{\sigma^2}}+1-\mathrm{e}^{-a})}{\sqrt{a\pi}(1-\mathrm{e}^{-a})},$$
$$\nu=\frac a2\wedge\frac{a^2}{4\sigma^2}\tag{3.5}$$

and $\alpha\in(0,1)$ is an arbitrary constant.

Proof Denote

$$P[|U_{n+i}^{(n)}|\ge\lambda|F_i]-P_0[|U_n^{(n)}|\ge\lambda]=P[|U_{n+i}^{(n)}|\ge\lambda|Y_i]-P_0[|U_n^{(n)}|\ge\lambda]=f(Y_i),$$

where $f(\cdot)$ is a bounded deterministic function not depending on i and $|f(\cdot)|\le1$. Then

$$E\left|\frac{1}{N+1}\sum_{i=0}^{N}P[|U_{n+i}^{(n)}|\ge\lambda|F_i]-P_0[|U_n^{(n)}|\ge\lambda]\right|^2=E\left|\frac{1}{N+1}\sum_{i=0}^{N}f(Y_i)\right|^2.$$

$\{Y_t\}$ is the O–U process in (1.2), it can be derived that

$$Y_s|_{Y_0}\sim N\left(\mathrm{e}^{-as}(Y_0-b)+b,\frac{\sigma^2}{2a}(1-\mathrm{e}^{-2as})\right),$$
$$Y_t-Y_s|_{Y_s}\sim N\left(\mathrm{e}^{-a(t-s)}(Y_s-b)+b,\frac{\sigma^2}{2a}(1-\mathrm{e}^{-2a(t-s)})\right),\quad 0<s<t$$

and the stationary distribution of $\{Y_t\}$ is $N(b,\frac{\sigma^2}{2a})$. Denote by $\rho(x)$, $\rho(y|x)$, $\rho(x,y)$ and $\rho_0(x)$ the probability density of $Y_s|_{Y_0}$, $Y_t-Y_s|_{Y_s}$, (Y_t,Y_s) and $N(b,\frac{\sigma^2}{2a})$ respectively. From the markov property of $\{Y_t\}$, it is clear that $\rho(x,y)=\rho(x)\rho(y|x)$ and $\int f(x)\rho_0(x)=0$.

So

$$Ef(Y_i)f(Y_j)=\iint f(x)f(y)\rho(x)[\rho(y|x)-\rho_0(y)]dxdy.$$

For any α in $(0,1)$, we use N^α to denote the integral part of N^α for simplicity, which doesn't change the convergence rate. Let $N_0=\inf\{m;\mathrm{e}^{am^\alpha/4}\ge 2b+2|c|+2,m\in\mathbb{Z}^+\}$, and without loss of generality we can choose the observed time $N \geq N_0$. Denote $M = \mathrm{e}^{a(j-i)/4}$, $G(x,y) = f(x)f(y)\rho(x)[\rho(y|x) - \rho_0(x)]$. Then

$$\begin{aligned} Ef(Y_i)f(Y_j) = & \int_{-M}^{M}\int_{-M}^{M} G(x,y)dxdy + \int_{-M}^{M}\int_{-\infty}^{-M} G(x,y)dxdy + \int_{-M}^{M}\int_{M}^{\infty} G(x,y)dxdy \\ & + \int_{-\infty}^{-M}\int_{-\infty}^{\infty} G(x,y)dxdy + \int_{M}^{\infty}\int_{-\infty}^{\infty} G(x,y)dxdy. \end{aligned}$$

When $j - i > N^\alpha$, it can be calculated that

$$\begin{aligned} & \int_{-M}^{M}\int_{-M}^{M} G(x,y)dxdy \\ & \leq \int_{-M}^{M}\int_{-M}^{M} \rho(x)[\rho(y|x) - \rho_0(y)]dxdy \\ & \leq \int_{-M}^{M}\int_{-M}^{M} \rho(x)\rho_0(y)\Big[\frac{1}{\sqrt{1-\mathrm{e}^{-2a(j-i)}}}\mathrm{e}^{\frac{2a\mathrm{e}^{-a(j-i)}(M+b)^2}{\sigma^2(1-\mathrm{e}^{-2a(j-i)})}} - 1\Big]dxdy \\ & \leq \frac{1}{\sqrt{1-\mathrm{e}^{-2aN^\alpha}}}\mathrm{e}^{\frac{2a(\mathrm{e}^{-aN^\alpha/2}+2b\mathrm{e}^{-3aN^\alpha/4}+b^2\mathrm{e}^{-aN^\alpha})}{\sigma^2(1-\mathrm{e}^{-2a})}} - 1 \\ & \leq \frac{1}{\sqrt{1-\mathrm{e}^{-2aN^\alpha}}}\mathrm{e}^{\frac{2a(b+1)^2\mathrm{e}^{-aN^\alpha/2}}{\sigma^2(1-\mathrm{e}^{-2a})}} - 1 \\ & = \frac{1}{\sqrt{1-\mathrm{e}^{-2aN^\alpha}}}\big[\mathrm{e}^{\frac{2a(b+1)^2\mathrm{e}^{-aN^\alpha/2}}{\sigma^2(1-\mathrm{e}^{-2a})}} - 1\big] + \frac{1}{\sqrt{1-\mathrm{e}^{-2aN^\alpha}}} - 1 \\ & \leq \frac{2a(b+1)^2\mathrm{e}^{\frac{2a(b+1)^2\mathrm{e}^{-\frac{a}{2}}}{\sigma^2(1-\mathrm{e}^{-2a})}}}{\sigma^2(1-\mathrm{e}^{-2a})\sqrt{1-\mathrm{e}^{-2a}}}\mathrm{e}^{-\frac{a}{2}N^\alpha} + \frac{\mathrm{e}^{-2aN^\alpha}}{2} + \frac{3\mathrm{e}^{-4aN^\alpha}}{8(1-\mathrm{e}^{-2a})^{\frac{5}{2}}} \\ & \leq \Bigg[\frac{2a(b+1)^2\mathrm{e}^{\frac{2a(b+1)^2\mathrm{e}^{-\frac{a}{2}}}{\sigma^2(1-\mathrm{e}^{-2a})}}}{\sigma^2(1-\mathrm{e}^{-2a})\sqrt{1-\mathrm{e}^{-2a}}} + \frac{1}{2} + \frac{3}{8}(1-\mathrm{e}^{-2a})^{-\frac{5}{2}}\Bigg]\mathrm{e}^{-\frac{a}{2}N^\alpha}. \end{aligned} \tag{3.6}$$

For $-M \leq x, y \leq M$ and $M = \mathrm{e}^{\frac{a}{4}(j-i)} \geq \mathrm{e}^{\frac{a}{4}N^\alpha}$, we have the second and the third inequalities. By using the facts $\mathrm{e}^x - 1 \leq x\mathrm{e}^x$ $(x > 0)$ and $\frac{1}{\sqrt{1-x}} - 1 = \frac{x}{2}(1-x)^{-\frac{3}{2}}|_{x=0} + \frac{3x^2}{8}(1-\xi)^{-\frac{5}{2}}|_{0\leq\xi\leq x\leq \mathrm{e}^{-2a}}$, we get the fifth inequality.

Moreover,

$$\begin{aligned} & \int_{-M}^{M}\int_{-\infty}^{-M} G(x,y)dxdy \\ & \leq \int_{-M}^{M}\int_{-\infty}^{-M} \rho(x)\rho(y|x)dxdy \\ & \leq \int_{-M}^{M}\int_{-\infty}^{-M} \rho(x)\frac{1}{\sqrt{2\pi\frac{\sigma^2}{2a}(1-\mathrm{e}^{-2a(j-i)})}}\mathrm{e}^{-\frac{[y-b+\mathrm{e}^{-a(j-i)}(M+b)]^2}{2\frac{\sigma^2}{2a}(1-\mathrm{e}^{-2a(j-i)})}}dxdy \\ & \leq \frac{\sqrt{\frac{\sigma^2}{2a}(1-\mathrm{e}^{-2a(j-i)})}}{\sqrt{2\pi}[M+b-\mathrm{e}^{-a(j-i)}(M+b)]}\mathrm{e}^{-\frac{[M+b-\mathrm{e}^{-a(j-i)}(M+b)]^2}{2\frac{\sigma^2}{2a}(1-\mathrm{e}^{-2a(j-i)})}} \\ & \leq \frac{|\sigma|}{2\sqrt{a\pi}(1-\mathrm{e}^{-a})}\mathrm{e}^{-\frac{a}{\sigma^2}(\mathrm{e}^{\frac{a}{4}N^\alpha}-2)} \\ & = \frac{|\sigma|\mathrm{e}^{\frac{2a}{\sigma^2}}}{2\sqrt{a\pi}(1-\mathrm{e}^{-a})}\mathrm{e}^{-\frac{a^2}{4\sigma^2}N^\alpha}, \end{aligned} \tag{3.7}$$

where $y - b - \mathrm{e}^{-a(j-i)}(x-b) \le y - b + \mathrm{e}^{-a(j-i)}(M+b) < 0$ with $-M \le x \le M$ and $y \le -M$ in the first inequality. The third one is by using the fact that $\int_{-\infty}^{-M} \mathrm{e}^{-\frac{(x-b)^2}{2\sigma^2}} dx \le \frac{\sigma^2}{M+b} \mathrm{e}^{-\frac{(M+b)^2}{2\sigma^2}}$ when $-M < b$. Moreover, with $M = \mathrm{e}^{\frac{a}{4}(j-i)} \ge \mathrm{e}^{\frac{a}{4}N^\alpha} \ge 2b + 2|c| + 2$, we have

$$M + b - \mathrm{e}^{-a(j-i)}(M+b) > 1 - \mathrm{e}^{-a}$$

and

$$\begin{aligned}[M + b - \mathrm{e}^{-a(j-i)}(M+b)]^2 &> (M+b)^2(1 - 2\mathrm{e}^{-a(j-i)}) > M(M+2b)(1 - 2\mathrm{e}^{-a(j-i)/4}) \\ &> M(1 - 2\mathrm{e}^{-a(j-i)/4}) > \mathrm{e}^{\frac{a}{4}N^\alpha} - 2,\end{aligned}$$

by which we get the fourth inequality.

Similarly, we have

$$\int_{-M}^{M} \int_{M}^{\infty} G(x,y) dx dy \le \frac{|\sigma| \mathrm{e}^{\frac{2a}{\sigma^2}}}{2\sqrt{a\pi}(1 - \mathrm{e}^{-a})} \mathrm{e}^{-\frac{a^2}{4\sigma^2}N^\alpha}. \tag{3.8}$$

We can also calculate that

$$\begin{aligned}\int_{-\infty}^{-M} \int_{-\infty}^{\infty} G(x,y) dx dy &\le \int_{-\infty}^{-M} \frac{1}{\sqrt{2\pi \frac{\sigma^2}{2a}(1 - \mathrm{e}^{-2ai})}} \mathrm{e}^{-\frac{[x - b - \mathrm{e}^{-a(j-i)}(c-b)]^2}{2\frac{\sigma^2}{2a}(1 - \mathrm{e}^{-2ai})}} dx \\ &\le \frac{\sqrt{\frac{\sigma^2}{2a}(1 - \mathrm{e}^{-2ai})}}{\sqrt{2\pi}[M + b + \mathrm{e}^{-ai}(c-b)]} \mathrm{e}^{-\frac{[M + b + \mathrm{e}^{-ai}(c-b)]^2}{2\frac{\sigma^2}{2a}(1 - \mathrm{e}^{-2ai})}} \\ &\le \frac{|\sigma|}{2\sqrt{a\pi}} \mathrm{e}^{-\frac{a}{\sigma^2}\mathrm{e}^{aN^\alpha/4}} \\ &\le \frac{|\sigma|}{2\sqrt{a\pi}} \mathrm{e}^{-\frac{a^2}{4\sigma^2}N^\alpha}.\end{aligned} \tag{3.9}$$

In fact, when $M = \mathrm{e}^{\frac{a}{4}(j-i)} \ge \mathrm{e}^{\frac{a}{4}N^\alpha} \ge 2b + 2|c| + 2$, we have $M + b + \mathrm{e}^{-ai}(c-b) > (M+b) \wedge (M+c) > 1$ and $[M + b + \mathrm{e}^{-ai}(c-b)]^2 > (M+b)^2 \wedge (M+c)^2 > [M(M+2b)] \wedge [M(M+2c)] > M$, by which we get the third inequality.

Similarly, we have

$$\int_{M}^{\infty} \int_{-\infty}^{\infty} G(x,y) dx dy \le \frac{\sigma}{2\sqrt{a\pi}} \mathrm{e}^{-\frac{a^2}{4\sigma^2}N^\alpha}. \tag{3.10}$$

Therefore, if $j - i > N^\alpha$, by (3.6)–(3.10), we can get $Ef(Y_i)f(Y_j) \le \mu \mathrm{e}^{-\nu N^\alpha}$. Thus by Lemma 3.2 we get the conclusion. □

Then we get by Chebyshev's inequalities that

Theorem 3.4 *We have for each* $\kappa > 0$, *that*

$$\begin{aligned}P[|V_{N,\lambda}^{(n)} - P_0[|U_n^{(n)}| \ge \lambda]| > \kappa] &\le \kappa^{-2} E|V_{N,\lambda}^{(n)} - P_0[|U_n^{(n)}| \ge \lambda]|^2 \\ &\le \kappa^{-2}\Big\{ \log \Big[\mathrm{e}^n \Big(\frac{7\mathrm{e}\mu\nu}{5} \Big)^{\frac{5}{\nu}} \Big] \frac{1}{N} + \frac{5}{\nu} \frac{\log N}{N} \Big\},\end{aligned} \tag{3.11}$$

where μ, ν *are given by* (3.5).

Proof From (3.1) and (3.4), we have

$$E|V_{N,\lambda}^{(n)} - P_0[|U_n^{(n)}| \ge \lambda]|^2 \le \frac{n}{N+1} + \frac{5}{N^{1-\alpha}} + 7\mu \mathrm{e}^{-\nu N^\alpha}, \tag{3.12}$$

where for μ, ν see also (3.5), and $\alpha \in (0,1)$ is an arbitrary constant. And without loss of generality we can choose $N > N_0$ in (3.12), where

$$N_0 = \inf\Big\{m; m \geq \frac{5(2b+2|c|+2)^{4\nu/a}}{7\mu\nu},\ \frac{5\mathrm{e}^{\nu}}{7\mu\nu} < m < \frac{5\mathrm{e}^{m\nu}}{7\mu\nu},\ \frac{5\mathrm{e}^{\nu}}{7\mu\nu} < m' < \frac{5\mathrm{e}^{m'\nu}}{7\mu\nu},\ \forall\, m' > m,\ m, m' \in \mathbb{Z}^+\Big\}.$$

For any $N > N_0$, $N \geq \frac{5(2b+2|c|+2)^{4\nu/a}}{7\mu\nu}$, it can be derived that $\mathrm{e}^{\frac{a}{4}N^{\log_N \log(\frac{7N\mu\nu}{5})^{\frac{1}{\nu}}}} \geq 2b+2|c|+2$. Denote $\alpha' = \log_N \log(\frac{7N\mu\nu}{5})^{\frac{1}{\nu}}$. By using the fact $\frac{5\mathrm{e}^{\nu}}{7\mu\nu} < N < \frac{5\mathrm{e}^{N\nu}}{7\mu\nu}$, we can get $0 < \alpha' < 1$. Let α be equal to α' in (3.12), we get

$$\begin{aligned} E|V_{N,\lambda}^{(n)} - P_0[|U_n^{(n)}| \geq \lambda]|^2 &\leq \frac{n}{N+1} + \frac{5}{N^{1-\alpha'}} + 7\mu\mathrm{e}^{-\nu N^{\alpha'}} \\ &< \frac{n}{N} + \frac{5}{N^{1-\alpha'}} + 7\mu\mathrm{e}^{-\nu N^{\alpha'}}. \end{aligned}$$

By Lemma 3.2,

$$\inf_{0<\alpha<1}\Big\{\frac{5}{N^{1-\alpha}} + 7\mu\mathrm{e}^{-\nu N^{\alpha}}\Big\} = \frac{5}{N^{1-\alpha'}} + 7\mu\mathrm{e}^{-\nu N^{\alpha'}} = \log\Big(\frac{7\mathrm{e}\mu\nu}{5}\Big)^{\frac{5}{\nu}}\frac{1}{N} + \frac{5}{\nu}\frac{\log N}{N}.$$

Thus we get the conclusion. □

Remark 3.5 It was noticed that one can also apply the existing results under the "strong mixing"condition to get some similar conclusions. However, our error estimate is more explicit in a way such that there is no unknown constants, which is important in statistical applications. From the above theorem, it is reasonable to use the observed frequency $V_{N,\lambda}^{(n)}$ to calculate the asymptotic stationary distribution of $U_t^{(n)}$ and we can give the accurate error between them. We can apply this conclusion to $L_t^{(n)}$, $I_t^{(n)}$ and $R_t^{(n)}$. All of the three processes change between a normal scope just as we have introduced in the beginning. Out of this normal scope, investors may buy or sell stocks, which can make the values of $L_t^{(n)}$, $I_t^{(n)}$ and $R_t^{(n)}$ come back into the normal scope. For example, the normal scope of $L_t^{(n)}$ is $[-2,2]$. And $I_t^{(n)}$ always changes between 20 and 80. $R_t^{(n)}$ also has indefinite antennas and grounds. Those applications are illustrated in the following corollaries.

Corollary 3.6 *Denote for* $i \geq n$, $H_i^{(n)} = I_{[|L_i^{(n)}|\geq 2]}$. *Then* $E[H_i^{(n)}] = P[|L_i^{(n)}| \geq 2]$. *Let*

$$J_N^{(n)} = \frac{1}{N+1}\sum_{i=0}^{N} H_{n+i}^{(n)}, \tag{3.13}$$

which is the observed frequency of the events $[|L_{n+i}^{(n)}| \geq 2]$ $(i = 0, 1, \ldots, N)$, *i.e. the frequency of the stock falling out of the Bollinger Bands. Let* $P_0(\cdot)$ *denote the stationary distribution of the asymptotic stationary process* $\{L_t^{(n)}\}_{t\geq n}$. *Then we have the same conclusion as* (3.11) *if we substitute* $J_N^{(n)}$ *for* $V_{N,\lambda}^{(n)}$ *and replace* $U_n^{(n)}$ *with* $L_n^{(n)}$. μ, ν, N *satisfy the conditions in Theorem* 3.4.

Corollary 3.7 *Denote for* $i \geq n$, $H_i^{(n)} = I_{[I_i^{(n)}\leq 20 \ or \ I_i^{(n)}\geq 80]} = I_{[I_i^{(n)}\in\Gamma]}$, *where* $\Gamma = [0,20] \cup [80,100]$. *Then* $E[H_i^{(n)}] = P[I_i^{(n)} \in \Gamma]$. *Define* $J_N^{(n)}$ *as* (3.13) *which is the observed frequency* *of the events* $[I_i^{(n)} \leq 20$ *or* $I_i^{(n)} \geq 80]$ $(i = 0, 1, \ldots, N)$, *i.e. the frequency of the* RSI *breaking through* 20 *or* 80. *Let* $P_0(\cdot)$ *denote the stationary distribution of the asymptotic stationary process* $\{I_t^{(n)}\}_{t \geq n}$. *Then we have the same conclusion as* (3.11) *if we substitute* $J_N^{(n)}$ *for* $V_{N,\lambda}^{(n)}$ *and replace* $U_n^{(n)}$ *with* $I_n^{(n)}$. μ, ν, N *satisfy the conditions in Theorem* 3.4.

Corollary 3.8 *Denote for* $i \geq n$, $H_i^{(n)} = I_{[R_i^{(n)} \leq \epsilon \text{ or } R_i^{(n)} \geq \varepsilon]} = I_{[R_i^{(n)} \in \Lambda]}$, *where* $\Lambda = [-\infty, \ \epsilon] \cup [\varepsilon, \ \infty]$ *and* $(\epsilon, \ \varepsilon)$ *is the indefinite antenna and ground of* ROC. *Then* $E[H_i^{(n)}] = P[\text{ROC}_i^{(n)} \in \Lambda]$. *Define* $J_N^{(n)}$ *as* (3.13) *which is the observed frequency of the events* $[R_i^{(n)} \leq \epsilon$ *or* $R_i^{(n)} \geq \varepsilon]$ $(i = 0, 1, \ldots, N)$, *i.e. the frequency of the* ROC *breaking through* ϵ *or* ε. *Let* $P_0(\cdot)$ *denote the stationary distribution of the asymptotic stationary process* $\{R_t^{(n)}\}_{t \geq n}$. *Then we have the same conclusion as* (3.11) *if we substitute* $J_N^{(n)}$ *for* $V_{N,\lambda}^{(n)}$ *and replace* $U_n^{(n)}$ *with* $R_n^{(n)}$. μ, ν, N *satisfy the conditions in Theorem* 3.4.

4 Conclusion

This paper proves the asymptotic stationary and convergent properties of some most popular technical analysis indicators of the stocks (i.e. Bollinger Bands, RSI, ROC) based on stochastic volatility model. Also we apply a technique to the sample surveys in which samples are not independent.

It is interesting to work between the stock price model and the real stock market. For example, we shall use the technical indicators' VaR (Value at Risk) to get more profitable investments. As a popular technology, VaR has been widely used in the field of risk control, masses of which is calculated based on asset price model. Now we have known that the technical indicators are stationary or asymptotic stationary process based on Black–Scholes model and stochastic volatility model. If we work out the VaR of some technical indicator, we shall reduce the risk of the investment guided by technical analysis.

As mentioned earlier, we found many useful properties of the indicators, which can successfully explain the rationality of technical analysis and could be widely used in financial area. However, we may also do some research in another perspect. For example, we may use the historic data of technical indicators to check the validity of some existing stock price model. There are many stock price models and technical indicators, the related questions both in financial investment tactics and theoretical research should be worthy for further study.

References

[1] Liu, W., Huang, X., Zheng, W.: Black–Scholes model and Bollinger bands. *Phys. A*, **371**, 565–571 (2006)

[2] Zhu, W.: Statistic Analysis on Technical Indicators of Stock (in Chinese). Master thesis, Department of Statistics, ECNU, Shanghai, China, 2006

[3] Lo, A. W., Mamaysky, H., Wang, J.: Foundation of technical analysis: computational algorithms, statistical inference, and empirical implementation. *J. Finance*, **55**, 1705–1770 (2000)

[4] Genon-Catalot, V., Jeantheau, T., Larédo, C.: Stochastic volatility model as hidden Markov models and statistical applications. *Bernoulli*, **6**(6), 1051–1079 (2000)

[5] Brown, L. D., Wang, Y., Zhao, L.: Statistical equivalence at suitable frequencies of GARCH and stochastic volatility models with the corresponding diffusion model. *Statist. Sinica*, **13**, 993–1013 (2003)

[6] Fouque, J. P., Papanicolaou, G., Sircar, R.: Derivatives in Financial Markets with Stochastic Volatility, Cambridge University Press, Cambridge, 2000

[7] Fouque, J. P., Sircar, R., Solna, K.: Stochastic volatility effects on defaultable bonds. *Appl. Math. Finance*, **13**(3), 215–244 (2006)

[8] Wang, Y.: Asymptotic nonequivalence of GARCH models and diffusions. *Ann. Statist.*, **30**, 754–783 (2002)

[9] Durrett, R.: Probability: Theory and Examples, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, California, 1991