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Spectral Theory for Semiclassical Operators and Artificial Black Holes

A dissertation submitted in partial satisfaction
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by

Michael Allen Hall

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Abstract of the Dissertation

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Professor Michael Hitrik, Chair

In this thesis we study several problems related to the spectral theory of semiclassical pseudodifferential operators, as well as artificial black holes in a curved spacetime. For nonselfadjoint perturbations of selfadjoint operators in dimension 2, we show that one can recover the (quantum) Birkhoff normal form of the operator near a Lagrangian torus satisfying a Diophantine condition from an appropriate portion of the spectrum, provided the unperturbed operator is known and under analyticity assumptions. Also working in dimension 2, we use a quantum version of the method of averaging, combined with techniques inspired by secular perturbation theory, to derive microlocal normal forms for selfadjoint semiclassical operators in dimension 2 with periodic classical flow. Finally, for stationary metrics in 2 space dimensions, we exhibit artificial black holes where the ergosphere and event horizon meet at isolated points, and which display a complicated dynamical structure.

The dissertation of Michael Allen Hall is approved.

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University of California, Los Angeles

2013

To Mom & Dad,

Nancy, Ron, Carmel, Oren, and David;

and Linda too.

Table of Contents

List of Figures

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Publications

Diophantine tori and nonselfadjoint inverse spectral problems, (*to appear in Math. Research Letters*)

(with Jonas Azzam and Robert Strichartz) Conformal energy, conformal Laplacian, and energy measures on the Sierpinski gasket, *Trans. Amer. Math. Soc. 360 (2008), no. 4, 2089–2130.*

CHAPTER 1

Introduction

1.1 Overview

Here we will give brief description of the general areas of study, and further below we will summarize our results in more detail.

Semiclassical analysis concerns the study of partial differential equations with a small parameter, which we will usually denote h. In the spectral theory of semiclassical operators, we are typically interested in the relationship between the spectrum of a semiclassical pseudodifferential operator P and the classical dynamics of its principal symbol p. This thesis concerns two types of results:

Inverse spectral problems: From the knowledge of (a part of) the spectrum of an operator, we ask what data about the symbol of an operator can be recovered (e.g. "Can one hear the shape of a drum?").

In chapter 2, which includes results published in [Hal], we study a semiclassical inverse spectral problem for non-selfadjoint perturbations of selfadjoint h-pseudodifferential operators in dimension 2.

Microlocal normal forms: Under assumptions on the classical dynamics of the system, we attempt to find a microlocal normal form, which may be applied to derive spectral estimates of the operator.

In chapter 3, we carry out the reduction to a microlocal normal form near an invariant Lagrangian torus for a *selfadjoint* perturbation of a selfadjoint operator with periodic classical flow in dimension 2.

Artificial black holes are regions of a curved spacetime from which null-geodesics cannot escape. The singular spacetime metric is not necessarily governed by Einstein's equations (hence "artificial"). The study of analogue models of gravity has attracted increased attention recently, especially in physics, where it is hoped that such phenomena may be produced in the laboratory and studied as models of gravitational black holes. At the same time, the area presents a number of questions of independent interest in hyperbolic PDE, as some of the basic problems come down to simple questions about the wave equation.

In chapter 4, we study stationary spacetime metrics in 2 space dimensions with a singularity. We can exhibit a number of examples of black holes where the event horizon is tangent to the ergosphere at several points.

1.2 Statement of Results

1.2.1 Nonselfadjoint operators in dimension 2

In the chapter 2 we solve a semiclassical inverse spectral problem for nonselfadjoint perturbations of selfadjoint operators in dimension two. As mentioned above, we are able to show that from an appropriate portion of the spectrum of certain operators of the form $P + i\varepsilon Q$ and knowledge of the unperturbed operator P, one can recover the Birkhoff normal form of $P + i\varepsilon Q$ near a Diophantine torus.

The case of non-selfadjoint operators in dimension two is special because the eigenvalues may have an explicit description in terms of a Bohr-Sommerfeld quantization condition, an idea first explored in Melin-Sjöstrand [MS03] in a case where the real and imaginary parts of the symbol of the operator are nearly in involution on an energy surface.

Our inverse result is based on spectral asymptotics obtained by M. Hitrik, J. Sjöstrand, and S. Vũ Ngọc in a related setting [HSN07], where they were able to describe large portions of the spectrum of certain perturbations $P + i\varepsilon Q$ in a regime where $h^N \leq \varepsilon \leq h^\gamma$, where $0 < \gamma < 1$ and $N \geq 1$ are arbitrary but fixed. Here P and Q are h-pseudodifferential operators with analytic symbols and the unperturbed operator P is elliptic at infinity and is assumed to be formally selfadjoint. The real part of the symbol of Q satisfies a nondegeneracy condition.

Letting H_p be the Hamilton vector field of p, the principal symbol of P, it is assumed that within the compact, connected, non-critical energy surface $p^{-1}(0)$ there are finitely many invariant tori Λ_j, each carrying analytic coordinates (x_1, x_2) such that $H_p = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2}$ along Λ_j, where the vector of frequencies $\vec{a} = (a_1, a_2)$ is *Diophantine*, i.e.

$$|\vec{a} \cdot \vec{n}| \geq \frac{C}{|\vec{n}|^M}, \quad 0 \neq \vec{n} \in \mathbf{Z}^2,$$

for some constants $C, M > 0$. We will refer to such a torus as a *Diophantine torus.*

One of the main theorems in [HSN07] gives asymptotic expansions for eigenvalues of $P + i\varepsilon Q$ which belong to an h-dependent window in the complex plane of the form

$$[-h^\delta, h^\delta] + i(F + [-\varepsilon h^\delta, \varepsilon h^\delta]). \tag{1.1}$$

Here $F = \langle \text{Re } q \rangle_{\Lambda_j}$ is the common long time average with respect to the H_p flow of the real part of the principal symbol of Q over finitely many Diophantine tori Λ_j. The Diophantine assumption ensures that this average exists and is the same for all points on any such torus. For the inverse result, we only consider the case of a single Diophantine torus Λ with average F.

It is necessary to make some additional technical assumptions which ensure good separation of the value F from the value of the flow average at points not too close to Λ. One may consult [HSN07] for precise statements in the most general case, while in this chapter we use a simplified formulation where we assume the Hamilton flow H_p is completely integrable.

The eigenvalues in (2.3) form a distorted lattice with horizontal spacing $\sim h$ and vertical spacing $\sim \varepsilon h$, each such eigenvalue having an asymptotic expansion of the form

$$P^{(\infty)}\left(h\left(k - \tfrac{k_0}{4}\right) - \tfrac{S}{2\pi}, \varepsilon, h\right) + \mathcal{O}(h^\infty), \tag{1.2}$$

for some $k \in \mathbf{Z}^2$, where $P^{(\infty)}(\xi, \varepsilon, h)$ is the Birkhoff normal form of $P + i\varepsilon Q$ near Λ, which is an asymptotic series in (ξ, ε, h).[1]

To construct the Birkhoff normal form near the Diophantine torus Λ, one makes a sequence of analytic, symplectic changes of variables, obtaining local coordinates (x, ξ) so that $\Lambda = \{\xi = 0\}$, and the symbol of $P + i\varepsilon Q$ becomes independent of x to higher and higher order in ξ.

Loosely stated, our main result is:

Theorem 1.1. *Assuming the unperturbed operator P is known, the eigenvalues of $P + i\varepsilon Q$ in in (2.3) for all sufficiently small values of h, and all ε in the range $h^N \leq \varepsilon \leq h^\gamma$ determine the Birkhoff normal form of $P + i\varepsilon Q$ near Λ.*

A pleasant side effect of Theorem 1.1 is the verification that the Birkhoff normal form near the Diophantine torus Λ is in fact unique (because it can be recovered from spectral asymptotics) up to certain natural kinds of symmetries, which to our knowledge has not been proven directly. We also discuss the Birkhoff normal form construction, symmetries of the normal form, and degenerate relationships between the parameters ε and h. While it is not discussed here in the greatest possible generality, essentially the same proof shows that in degenerate situations, one can always recover the normal form insofar as it is well-defined, and also that one may obtain a partial recovery in the presence of a small amount of noise.

1.2.2 Normal forms for h-pseudodifferential operators with periodic classical flow

Chapter 3 concerns microlocal normal forms for operators $P_\varepsilon = P + \varepsilon Q$, where P and Q are selfadjoint h-pseudodifferential operators on $\mathbf{R}^2$ with smooth symbols. Here we assume P is elliptic at infinity, having periodic classical flow and vanishing subprincipal symbol, and we work in the regime where $h^{2-\delta} \ll \varepsilon = \mathcal{O}(h^\delta)$, $\delta > 0$. It is well known (see [DS99]) that

[1]To be precise, there exists a 1-1 partial function from $\mathbf{Z}^2$ to the set of eigenvalues in the window $[-h^\delta, h^\delta] + i(F + [-\varepsilon h^\delta, \varepsilon h^\delta])$ such that if $k \in \mathbf{Z}^2$ maps into the window and $\lambda_k = \lambda_k(\varepsilon, h)$ is any point with asymptotics given by the first term in (1.2), then the image of k is within $\mathcal{O}(h^\infty)$ of λ_k, uniformly in k.

the spectrum of P near 0 has a cluster structure, where the clusters are of size $\sim h^2$ and separated by a distance $\sim h$. When $\varepsilon \ll h$, a cluster structure persists for the spectrum of $P + \varepsilon Q$,

$$\mathrm{spec}(P+\varepsilon Q) \cap \mathrm{neigh}(0,\ \mathbf{R}) \subseteq \cup_k I_k(h)$$

$$I_k(h) = f(hk) + [-\mathcal{O}(\varepsilon + h^2), \mathcal{O}(\varepsilon + h^2)], \quad k \in \mathbf{Z},$$

for a function f satisfying $f(0) = 0$, $f'(0) > 0$. One would like to have precise asymptotics for individual eigenvalues within each cluster. As a first step toward such a result, we derive a microlocal normal form for P_ε near a suitable invariant Lagrangian torus.

A source of motivation for the general problem is a classic paper of Weinstein [Wei77], which used averaging on the level of operators to analyze eigenvalue clusters for the Schrödinger operator $-\Delta + V$ on a sphere, obtaining a description of the asymptotic distribution of eigenvalues within the kth cluster in the limit as $k \to \infty$. The methods we use in fact apply to semiclassical operators of the form $-h^2\Delta + \varepsilon V$ on the 2-sphere with the expanded range $h^N \ll \varepsilon \ll h$, for any $N > 1$ fixed.

Letting q be the principal symbol of Q, we set

$$\langle q \rangle = \frac{1}{T} \int_0^T q \circ \exp(tH_p)\, dt,$$

where $T > 0$ is the common period for the H_p-flow on $p^{-1}(0)$. For any $F \in \mathbf{R}$, belonging to the range of $\langle q \rangle$ along $p^{-1}(0)$, we set $\Lambda_{0,F} = \{p = 0, \langle q \rangle = F\}$, which is the union of finitely many two-dimensional Lagrangian tori assuming that $dp \wedge d\langle q \rangle \neq 0$ along $\Lambda_{0,F}$.

We combine the classical averaging method ([Wei77], [Ver96]) with further microlocal study near $\Lambda_{0,F}$, leading to a complete reduction to a translation invariant operator of the form $\widehat{P}(hD_x, \varepsilon, h^2/\varepsilon; h)$ on $\mathbf{T}^2$, where the symbol of $\widehat{P}$ is of the form

$$\widehat{P}(\xi, \varepsilon, h^2/\varepsilon; h) = p(\xi_1) + \varepsilon(r_0(\xi, \varepsilon, h^2/\varepsilon) + hr_1 + h^2 r_2 + \ldots), \quad (x, \xi) \in T^*\mathbf{T}^2$$

with $r_0 = \langle q \rangle(\xi) + \mathcal{O}(\varepsilon + h^2/\varepsilon)$ and $r_j = \mathcal{O}(1), j \geq 1$.

Such a reduction leads to formal quasi-eigenvalues $\widehat{P}(hk, h\ell, \varepsilon, h^2/\varepsilon; h)$, where $(k, \ell) \in \mathbf{Z}^2$. In a future work, we shall pursue the spectral analysis of the family P_ε further, justifying the fact that the quasi-eigenvalues asymptotically describe all of the eigenvalues of P_ε in suitable sub-clusters, corresponding to the regular value F of $\langle q \rangle$.

1.2.3 Analogue Black Holes

The setting of general relativity is a pseudo-Riemannian manifold M where the curvature and stress-energy tensors are related by Einstein's equations. Many other physical systems have a description in terms of an "effective metric", whose dynamics may not be described by Einstein's equations, but nonetheless may display features analogous to ergoregions and event horizons for relativistic black holes. Unruh [Unr81] pointed out that these and other physical features arise in models of acoustic waves in a moving medium.

For a simple intuitive example, we can imagine water swirling into a drain, with both angular and radial components of the fluid velocity growing very large near the singularity. The ergoregion is where the fluid's velocity is supersonic, and there is a black hole event horizon at the boundary of the region where the radial component is faster than the speed of sound.

The mathematical model we study is a smooth domain in $\mathbf{R}^{n+1}$ endowed with a stationary pseudo-Riemannian metric tensor, which may be studied in terms of the associated wave equation

$$\sum_{\mu,\nu=0}^{n} \frac{1}{\sqrt{(-1)^n g(x)}} \frac{\partial}{\partial x_\mu} \left(\sqrt{(-1)^n g(x)}\, g^{\mu\nu}(x) \frac{\partial u(x_0, x)}{\partial x_\nu} \right) = 0, \tag{1.3}$$

Here $(x_\nu)_{\nu=0}^{n} = (x_0, x) \in \mathbf{R}^{n+1}$, the metric tensor has signature $(+1, -1, -1, \ldots, -1)$ and does not depend on x_0, $g^{\mu\nu}(x)$ is the inverse of the metric tensor, and $g(x) = (\det[g^{\mu\nu}(x)])^{-1}$.

The ergoregion is defined as the region where $\det[g^{jk}(x)]_{j,k=1}^{n} < 0$. Working in dimension $n = 2$, we assume that the boundary of the ergoregion, called the ergosphere, is a smooth, simple, closed curve, and that the interior contains an inward or outward trapped surface. This holds for example when there is an appropriate sort of singularity in the metric.

In such a situation, Eskin [Esk10] has shown, using the Poincaré-Bendixson theorem, that if the ergosphere is either a characteristic surface for (1.3), or is nowhere characteristic, then there must exist a black hole in the interior of the ergoregion, i.e. a characteristic surface enclosing a region out of which signals may not propagate (or a white hole, which is the opposite).

In this work, we examine in detail a number of explicit examples in the case where there are characteristic points on the ergosphere, which shed light on the typical situation. We show that in general there may be a complicated dynamical picture of the ergoregion, including bifurcations of characteristic points on the ergosphere, and black hole event horizons which are C^1 curves but not smooth.

CHAPTER 2

Diophantine Tori and Nonselfadjoint Inverse Spectral Problems

2.1 Introduction

Let M denote either $\mathbf{R}^2$ or a compact, real analytic manifold of dimension 2, and let $\widetilde{M}$ denote a complexification of M, which is $\mathbf{C}^2$ in the Euclidean case, and a Grauert tube of M in the compact analytic 2-manifold case.

We study operators of the form $P_\varepsilon = P + i\varepsilon Q$, where P and Q are analytic h-pseudodifferential operators on M with principal symbols p, q, respectively, and P is selfadjoint. The principal symbol of P_ε is then $p + i\varepsilon q$, where p is real. We will also make a non-degeneracy assumption on $\operatorname{Re} q$, but we do not require Q to be selfadjoint.

Consider a Lagrangian torus Λ, contained in an energy surface $p^{-1}(0) \cap T^*M$, which is invariant with respect to the Hamilton flow of p and satisfies a Diophantine condition (see Section 2.2.2). For simplicity, we will assume:

$$\text{The Hamilton flow of } p \text{ is completely integrable in a neighborhood of } p^{-1}(0). \tag{2.1}$$

This implies that in a neighborhood of Λ the energy surface is foliated by invariant Lagrangian tori.

In action-angle coordinates (x, ξ) such that Λ is the set $\{\xi = 0\} \subseteq T^*\mathbf{T}^2_x$, where $\mathbf{T}^2 = \mathbf{R}^2/2\pi\mathbf{Z}^2$, let

$$\langle q \rangle(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbf{T}^2} q(x, \xi)\, dx \tag{2.2}$$

denote the spatial average of $\langle q\rangle$. Here we take x to be a multi-valued function whose gradient is single-valued. We assume that dp and $d\,\mathrm{Re}\langle q\rangle$ are linearly independent along Λ. Then, as is explained in Section 2.3, we may make a sequence of changes of variables which transforms the full symbol of P_ε into a Birkhoff normal form, which in this context means an asymptotic expansion in (ξ, ε, h) that is independent of x to high order. Formally we may carry out this procedure on the level of operators by conjugating by a sequence of appropriately defined Fourier integral operators, obtaining what we call a quantum Birkhoff normal form for P_ε.

Under some further technical assumptions which we will explain later on, one of the main theorems of [HSN07] establishes, for any $\delta > 0$, asymptotics for the eigenvalues of P_ε in an h-dependent window in the complex plane

$$\left\{ z \in \mathbf{C} \;\middle|\; |\,\mathrm{Re}\, z| < \frac{h^\delta}{C},\ |\,\mathrm{Im}\, z - \varepsilon\, \mathrm{Re}\, F| < \frac{\varepsilon h^\delta}{C} \right\}. \tag{2.3}$$

Here $F = \langle q\rangle|_\Lambda = \langle q\rangle(0)$, and we assume that this average is not shared by any other invariant torus. The expansions are given in terms of a Bohr-Sommerfeld type condition and the quantum Birkhoff normal form of P_ε near Λ.

Our goal in this chapter is to address the semiclassical inverse problem of determining the quantum Birkhoff normal form of P_ε from the eigenvalues in (2.3), assuming the unperturbed operator P is known.

Inverse spectral problems have been studied for many years, as surveyed for example by Zelditch [Zel04]. Recently, *semiclassical* inverse spectral problems have been investigated by several authors, such as Colin de Verdière [Ver11][Ver], Colin de Verdière-Guillemin [VG11][VG02], Guillemin-Paul-Uribe [GPU07], Guillemin-Paul [GP10], Guillemin-Uribe [GU07], Hezari [Hez09], Iantchenko-Sjöstrand-Zworski [ISZ02], Vũ Ngọc [Vu 11]. Often in inverse spectral problems one studies the wave trace, in the spirit of Guillemin [Gui96]. The non-selfadjoint case in dimension 2 is special because the eigenvalues may have an explicit description in terms of the Birkhoff normal form and Bohr-Sommerfeld type rules, an idea first explored in Melin-Sjöstrand [MS03]. In such a situation it seems most natural to recover the normal form directly from eigenvalue asymptotics. Our approach is taken very much in the spirit of Colin de Verdière [Ver].

According to [HSN07], the eigenvalues in the window (2.3) form a distorted lattice, with horizontal spacing $\sim h$ and vertical spacing $\sim \varepsilon h$. The window is of size h^δ by εh^δ, for some $0 < \delta \ll 1$, which means that the asymptotic expansions are valid for a comparatively large number of eigenvalues (on the order of $h^{2(\delta-1)}$) as $h \to 0$.

In addition, for the semiclassical inverse problem, we assume we know the eigenvalues for *each* sufficiently small value of the semiclassical parameter (or possibly for a sequence of values of h tending to 0). This provides a rich data set, from which we will recover information about the Birkhoff normal form using elementary order of magnitude arguments. In the perturbative, non-selfadjoint setting there is also the parameter ε to consider, since the results of [HSN07] apply to all values of ε such that $h^K \leq \varepsilon \leq h^\delta$. We will exploit this flexibility to assume that we can choose ε so as to rule out any sort of degenerate relationship between ε and h. For simplicity, we shall assume more strongly that we know the eigenvalues for all values of ε in such a range. See also the remarks at the end of Section 2.5.

Our main result, stated informally, is the following (see Theorem 2.10 for the precise statement)

Theorem 2.1. *The eigenvalues of P_ε in (2.3) determine the quantum Birkhoff normal form of P_ε near Λ.*

The plan of the paper is as follows. In Section 2.2, we recall the setting and technical assumptions of [HSN07] needed to apply one of the main results of that paper. In Section 2.3, we review the normal form construction in the present context, using some notation and methods due to S. Vũ Ngọc, and we also discuss symmetries and uniqueness of the normal form. In Section 2.4, we recall a spectral asymptotics result of [HSN07] in a precise form, which is the basis of the inverse result. Finally, in Section 2.5, we prove our main theorem.

2.2 Assumptions

We will state our assumptions along the lines of Section 7 of [HSN07], in particular restricting our attention to the completely integrable case (2.1) rather than the most general case treated in that work.

2.2.1 Analyticity and general assumptions

Let us assume that $P_\varepsilon = P + i\varepsilon Q$, with $\varepsilon \in \operatorname{neigh}(0, \mathbf{R})$, satisfies the same general assumptions as operators studied in [HSN07], which we recall here for convenience.

When $M = \mathbf{R}^2$, assume that $P_\varepsilon = P_\varepsilon(x, hD_x; h) = P(x, hD_x; h) + i\varepsilon Q(x, hD_x; h)$ is the Weyl quantization of a symbol which we also denote $P_\varepsilon(x, \xi, \varepsilon; h) = P(x, \xi; h) + i\varepsilon Q(x, \xi; h)$. Assume that P_ε is a holomorphic function of (x, ξ) in a complex tubular neighborhood of $\mathbf{R}^4 \subseteq \mathbf{C}^4$. Assume that

$$|P_\varepsilon(x, \xi; h)| \leq \mathcal{O}(1) g(\operatorname{Re}(x, \xi)) \tag{2.4}$$

in this neighborhood, where $g \geq 1$ is an order function in the sense that

$$g(X) \leq C \langle X - Y \rangle^M g(Y), \quad C > 0,\ M > 0,\ X, Y \in \mathbf{R}^4.$$

Assume that P and Q have asymptotic expansions

$$P(x, \xi; h) \sim \sum_{j=0}^{\infty} h^j p_j(x, \xi), \quad Q(x, \xi; h) \sim \sum_{j=0}^{\infty} h^j q_j(x, \xi) \tag{2.5}$$

valid in the space of holomorphic symbols satisfying the bound (2.4). Let us also assume that the principal symbol $p = p_0$ satisfies an ellipticity condition at infinity,

$$|p(x, \xi)| \geq \frac{1}{C} g(\operatorname{Re}(x, \xi)), \quad |(x, \xi)| \geq C.$$

When M is a compact, real analytic 2-manifold, assume that in any choice of local coordinates $P_\varepsilon = P + i\varepsilon Q$ is a differential operator of order m with analytic coefficients, which themselves have asymptotic expansions in integer powers of h. Assume also that the principal symbol p of P satisfies an ellipticity condition near infinity,

$$|p(x,\xi)| \geq \frac{1}{C}\langle \xi \rangle^m, \quad (x,\xi) \in T^*M,\ |\xi| \geq C,$$

where we implicitly assume that M has been equipped with an analytic Riemannian metric, so that the quantity $\langle \xi \rangle = \sqrt{1+|\xi|^2}$ makes sense. Assume also that the underlying Hilbert space is $L^2(M, \mu(dx))$, where μ is the Riemannian volume form on M.

In both cases, we assume that P is formally selfadjoint on L^2, which implies that p is real.

The above assumptions imply also that P_ε has a natural closed, densely defined realization on L^2, which has discrete spectrum in a fixed neighborhood of $0 \in \mathbf{C}$ for h, ε small enough. Also, we have that $\mathrm{spec}(P_\varepsilon) \cap \mathrm{neigh}(0, \mathbf{C}) \subseteq \{z \mid \operatorname{Im} z = \mathcal{O}(\varepsilon)\}$.

2.2.2 Assumptions on the classical dynamics

We assume the energy surface $p^{-1}(0) \cap T^*M$ is non-critical, i.e. $dp \neq 0$ along this set. For simplicity, we assume $p^{-1}(0) \cap T^*M$ is connected. Let

$$H_p = \sum_{j=1,2} \frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}$$

denote the Hamilton vector field of p (in any choice of canonical coordinates).

By the complete integrability assumption (2.1), there exists an analytic, real-valued function $\widetilde{p}$ such that $H_p\widetilde{p} = \{p, \widetilde{p}\} = 0$ with $d\widetilde{p}$ and dp linearly independent almost everywhere. Here, $\{\cdot,\cdot\}$ denotes the Poisson bracket. Then the energy surface $p^{-1}(0) \cap T^*M$ decomposes as a disjoint union of compact, connected H_p-invariant sets, which we assume has the structure of a graph, in which edges correspond to families of regular invariant Lagrangian tori and vertices correspond to singular invariant sets.

Near an invariant torus Λ we have real analytic action-angle coordinates (x, ξ) such that $\Lambda = \{\xi = 0\}$ and $H_p|_\Lambda = a \cdot \partial_x$ for some frequency vector $a \in \mathbf{R}^2$. We refer to Λ as a rational, irrational, or Diophantine torus if the vector a has the corresponding property. Below, we will consider a Diophantine torus, i.e. one such that the frequencies a satisfy

$$|a \cdot k| \geq \frac{1}{C_0 |k|^{N_0}}, \quad 0 \neq k \in \mathbf{Z}^n. \tag{2.6}$$

for some $C_0 > 0, N_0 > 0$. (Here, $n = 2$.)

In action-angle coordinates near any such Λ, the principal symbol p takes the form

$$p = p(\xi) = a \cdot \xi + \mathcal{O}(\xi^2). \tag{2.7}$$

In particular, it is independent of x, which means that it is in Birkhoff normal form.

Let q be the principal symbol of $Q(x, hD_x; h)$, and let $\langle q \rangle |_\Lambda$ denote the average as in (2.2) of q with respect to the natural smooth measure on Λ. We assume that the analytic function $\Lambda \mapsto \operatorname{Re} \langle q \rangle |_\Lambda$ is not identically constant on any of the aforementioned "edges", consisting of families of invariant tori.

When $T > 0$, let $\langle q \rangle_T$ denote the symmetric time T average of q along the H_p-flow:

$$\langle q \rangle_T (x, \xi) = \frac{1}{T} \int_{-T/2}^{T/2} q \circ \exp(sH_p)(x, \xi)\, ds.$$

For each invariant torus Λ, define the interval

$$Q_\infty(\Lambda) = \left[\lim_{T\to\infty} \inf_{\Lambda} \operatorname{Re}\langle q \rangle_T, \lim_{T\to\infty} \sup_{\Lambda} \operatorname{Re}\langle q \rangle_T \right]. \tag{2.8}$$

As in [Sjo00], we have that $\operatorname{spec} P_\varepsilon \cap \{|\operatorname{Re} z| \leq \delta\}$ is contained in a band

$$\frac{\operatorname{Im} z}{\varepsilon} \in \left[\inf \bigcup_\Lambda Q_\infty(\Lambda) - o(1), \sup \bigcup_\Lambda Q_\infty(\Lambda) + o(1) \right]$$

as $\varepsilon, h, \delta \to 0$.

From now on, fix a single Diophantine invariant Lagrangian torus Λ, set $F = \langle q \rangle |_\Lambda$, and assume that

$$dp \text{ and } d_\Lambda \operatorname{Re}\langle q \rangle|_\Lambda \text{ are linearly independent.} \tag{2.9}$$

With all the assumptions above, and in particular assuming complete integrability, the last global assumption needed is that

$$\operatorname{Re} F \notin Q_\infty(\Lambda'), \quad \Lambda' \neq \Lambda. \tag{2.10}$$

Without assuming complete integrability, a different assumption is needed (see [HSN07], (1.24)).

The eigenvalue asymptotics result of [HSN07] is valid in the (h, ε)-dependent rectangle (2.3) for sufficiently small h and assuming $h^K \leq \varepsilon \leq h^\delta$, where K is a fixed integer, which can be chosen arbitrarily large, and $\delta > 0$ is also fixed, and can be chosen arbitrarily small.

Remark 2.2. Note that the global condition (2.10) implies the value F is unique to Λ. The results of [HSN07] apply also to a finite collection of Diophantine tori sharing the value F, in which case the set of eigenvalues in (2.3) is simply the union of the contributions from each individual torus, modulo $\mathcal{O}(h^\infty)$. However, we will not consider the problem of separating the contributions of several tori.

2.3 Quantum Birkhoff normal form

In this section we present the quantum Birkhoff normal form construction near a Diophantine torus for a perturbed symbol, and discuss issues of uniqueness for the normal form and normalizing change of variables. Though we only need to consider dimension 2, it is natural to carry out the discussion in a general dimension n, as no changes are needed. We will work on $T^*\mathbf{T}^n$, assuming we are in a microlocal model where the Diophantine torus in question corresponds to the 0 section $\{\xi = 0\}$.

2.3.1 Normal form construction

Let us identify symbols on $T^*\mathbf{T}^n$ with their formal Weyl quantizations. The Moyal formula

$$P\#^w Q(x,\xi,\varepsilon;h) \sim \sum_{|\alpha|,|\beta|=0}^{\infty} \frac{h^{|\alpha|+|\beta|}(-1)^{|\alpha|}}{(2i)^{|\alpha|+|\beta|}\alpha!\beta!}(\partial_x^\alpha \partial_\xi^\beta P(x,\xi,\varepsilon)(\partial_\xi^\alpha \partial_x^\beta Q(x,\xi,\varepsilon))$$

defines a product operation on symbols which corresponds to composition of the corresponding operators. We denote by $[\cdot,\cdot]$ the associated bracket operation, which on the level of Weyl quantizations is simply the commutator bracket.

The normal form construction may be summarized as follows: We make a sequence of analytic, symplectic changes of variables, which transform P_ε to a symbol which is independent of x to higher and higher order in (ξ, ε, h). On the level of operators this is formally equivalent to conjugating by a sequence of Fourier integral operators. The resulting sequence of symbols is convergent in the space of formal power series in (ξ, ε, h). The quantum Birkhoff normal form (QBNF) of P_ε near the Diophantine torus Λ is an asymptotic expansion which is the formal limit of this procedure, while if we truncate the procedure after finitely many steps we get a well-defined analytic change of variables.

Later, we will often write the QBNF as a formal expansion

$$P^{(\infty)}(\xi, \varepsilon, h) \sim \sum_{j,m,n} P^{(\infty)}_{jmn}(\xi)\varepsilon^m h^n,$$

where the sum is over integers $j \geq 0$, $m \geq 0$, $n \geq 0$, and P_{jmn} is a homogeneous polynomial of degree j in ξ, and $P^{(\infty)}$ denotes the entire formal expansion.

For the moment, however, it is convenient to use slightly different notation. Consider a grading in (ξ, ε, h) which counts the power in ξ plus *twice* the power in (ε, h). Let $\mathcal{O}(N)$ denote the associated order classes. Here we do not attach any special significance to the number two, but we note the convenience of this sort of grading: because the Moyal formula has an asymptotic expansion in powers of $(\frac{h}{i}\frac{\partial}{\partial \xi}, \frac{\partial}{\partial x})$, the higher weight of h ensures that each time we lose an order in ξ we gain one in h. This implies that the main contribution in the bracket $\frac{i}{h}[\cdot,\cdot]$ comes from the Poisson bracket of the two symbols.

When $K_j = \mathcal{O}(j)$ and $K_\ell = \mathcal{O}(\ell)$, their Poisson bracket satisfies

$$\{K_j, K_\ell\} = \sum_k \frac{\partial K_j}{\partial \xi_k}\frac{\partial K_\ell}{\partial x_k} - \frac{\partial K_j}{\partial x_k}\frac{\partial K_\ell}{\partial \xi_k} = \mathcal{O}(j+\ell-1),$$

By what we have said above, we see that $\frac{i}{h}[K_j, K_\ell] = \{K_j, K_\ell\} + \mathcal{O}(j+\ell)$.

Proposition 2.3. *Suppose that $P = P_1 + \mathcal{O}(2)$ is analytic in x and ξ, where $P_1 = a \cdot \xi$, and that a satisfies the Diophantine condition (2.6). Then for all $N \geq 1$ there exist functions*

$$G^{(1)} = 0, \quad G^{(N)} = G_2 + \cdots + G_N \ (N \geq 2), \quad P^{(N)} = P_1 + P_2 + \cdots + P_N, \quad R_{N+1},$$

which are analytic in x, with G_j, P_j and R_j homogeneous of degree j with respect to the grading described above (thus polynomials in ξ), such that

$$\exp\left(\tfrac{i}{h}\operatorname{ad}_{G^{(N)}}\right) P = P^{(N)} + R_{N+1} + \mathcal{O}(N+2) \tag{2.11}$$

with each P_j independent of x.

Here, we write formally $\operatorname{ad}_G P = [G, P]$ and $\exp(\frac{i}{h}\operatorname{ad}_G)P = \exp(\frac{i}{h}G)P\exp(-\frac{i}{h}G)$, but note that these do not represent concretely defined operators. Notice also that $G^{(N)} = \mathcal{O}(2)$ for all $N \geq 1$.

Remark 2.4. Note that although it is not ruled out by the notation, it will follow from the proof that no half-powers of h or ε appear in the normal form.

Proof. We proceed by induction on the order N. By assumption the claim holds for $N = 1$, with $G_1 = 0$, and R_2 representing the homogeneous terms of degree 2. Inductively if (2.11) holds, then setting $G^{(N+1)} = G^{(N)} + G_{N+1}$, for some function G_{N+1}, homogeneous of degree $N+1$, to be determined, we claim that the only new term modulo $\mathcal{O}(N+2)$ is given by $\{G_{N+1}, P_1\}$. Indeed, by the Campbell-Hausdorff formula

$$\begin{aligned}\exp\left(\tfrac{i}{h}\operatorname{ad}_{G^{(N+1)}}\right) P &= \exp\left(\tfrac{i}{h}\operatorname{ad}_{G_{N+1}} + \tfrac{i}{h}\operatorname{ad}_{G^{(N)}}\right) P \\ &= \exp\left(\tfrac{i}{h}\operatorname{ad}_S\right) \circ \exp\left(\tfrac{i}{h}\operatorname{ad}_{G_{N+1}}\right) \circ \exp\left(\tfrac{i}{h}\operatorname{ad}_{G^{(N)}}\right) P\end{aligned}$$

where $S = \mathcal{O}(\frac{1}{2}[G_{N+1}, G^{(N)}]) = \mathcal{O}((N+1)+2-1) = \mathcal{O}(N+2)$. Here we have used that $G^{(N)} = \mathcal{O}(2)$. Meanwhile

$$\exp\left(\tfrac{i}{h}\operatorname{ad}_{G_{N+1}}\right) \circ \exp\left(\tfrac{i}{h}\operatorname{ad}_{G^{(N)}}\right) P = \exp\left(\tfrac{i}{h}\operatorname{ad}_{G_{N+1}}\right) (P^{(N)} + R_{N+1}) = \mathcal{O}(1).$$

Because of the Campbell-Hausdorff formula and the order of S, applying $\exp(\frac{i}{h}\operatorname{ad}_S)$ only affects the terms of order $\mathcal{O}(1+(N+2)-1)=\mathcal{O}(N+2)$. Therefore we have

$$\begin{aligned}
\exp\left(\tfrac{i}{h}\operatorname{ad}_{G^{(N+1)}}\right)P &= \exp\left(\tfrac{i}{h}\operatorname{ad}_{G_{N+1}}\right)(P^{(N)}+R_{N+1})+\mathcal{O}(N+2)\\
&= P^{(N)}+R_{N+1}+\tfrac{i}{h}\operatorname{ad}_{G_{N+1}}(P^{(N)}+R_{N+1})+\mathcal{O}(N+2)\\
&= P^{(N)}+R_{N+1}+\tfrac{i}{h}\operatorname{ad}_{G_{N+1}}(P_1+\mathcal{O}(2))+\mathcal{O}(N+2)\\
&= P^{(N)}+R_{N+1}+\tfrac{i}{h}\operatorname{ad}_{G_{N+1}}P_1+\mathcal{O}(N+2),
\end{aligned}$$

Because the bracket $\frac{i}{h}[\cdot,\cdot]$ reduces to the Poisson bracket when one of the arguments is at most quadratic, we have $\frac{i}{h}\operatorname{ad}_{G_{N+1}}P_1=\frac{i}{h}[G_{N+1},P_1]=\{G_{N+1},P_1\}$, so

$$\exp\left(\tfrac{i}{h}\operatorname{ad}_{G^{(N+1)}}\right)P=P^{(N)}+R_{N+1}+\{G_{N+1},P_1\}+\mathcal{O}(N+2),$$

as claimed.

To make the homogeneous order $N+1$ terms independent of x, it suffices to solve the cohomological equation,

$$\{G_{N+1},P_1\}=\langle R_{N+1}\rangle-R_{N+1}, \tag{2.12}$$

for G_{N+1}, where $\langle R_{N+1}\rangle$ is the x-average of R_{N+1} as in (2.2). Indeed, assuming that we have done so, we then set $P_{N+1}=\langle R_{N+1}\rangle$ and let R_{N+2} represent the homogenous order $N+2$ part of the $\mathcal{O}(N+2)$ error terms, which are analytic.

To solve (2.12), note that because $P_1=a\cdot\xi$, we have

$$\{G_{N+1},P_1\}=-H_{P_1}G_{N+1}=-(a\cdot\partial_x)G_{N+1}.$$

Expanding G_{N+1} and R_{N+1} in Fourier series

$$\begin{aligned}
G_{N+1}(x,\xi)&=\sum_{k\in\mathbf{Z}^n}\widehat{G}_{N+1}(k,\xi)e^{ik\cdot x}\\
R_{N+1}(x,\xi)&=\sum_{k\in\mathbf{Z}^n}\widehat{R}_{N+1}(k,\xi)e^{ik\cdot x},
\end{aligned}$$

we have

$$(a\cdot\partial_x)G_{N+1}=\sum_{k\in\mathbf{Z}^n}(ia\cdot k)\widehat{G}_{N+1}(k,\xi)e^{ik\cdot x}.$$

When $0 \neq k \in \mathbf{Z}^n$, because $a \cdot k$ does not vanish by the Diophantine condition (2.6), we may set $\widehat{G}_{N+1}(k,\xi) = -i\widehat{R}_{N+1}(k,\xi)/(a\cdot k)$, to obtain $(a\cdot\partial_x)G_{N+1} = R_{N+1} - \widehat{R}_{N+1}(0,\xi) = R_{N+1} - \langle R_{N+1}\rangle$ and thus solve the cohomological equation. Furthermore, again by (2.6), we have

$$|\widehat{G}_{N+1}(k,\xi)| = \frac{|\widehat{R}_{N+1}(k,\xi)|}{i|a\cdot k|} \leq C_0|k|^{N_0}|\widehat{R}_{N+1}(k,\xi)|,$$

and because R_{N+1} is analytic in a neighborhood of $\mathbf{T}^n \times \{0\} \subseteq (\mathbf{C}^n/2\pi\mathbf{Z}^n)\times\mathbf{C}^n$, so too is G_{N+1}.

□

Remark 2.5. As mentioned above, $\exp(\frac{i}{h}\operatorname{ad}_{G^{(N)}})$ formally represents conjugation by a microlocally defined Fourier integral operator $\exp(\frac{i}{h}G^{(N)})$, and such conjugation microlocally implements the symplectic transformation $\exp H_{G^{(N)}}$. To define such operators concretely, so that they act on microlocally defined distributions, one works on the FBI transform side in suitable weighted spaces of holomorphic functions. An additional assumption about smoothness in ε is required. See [HSN07] and the references there.

2.3.2 Symmetries and uniqueness of the normal form

In this section we discuss symmetries and uniqueness of the Birkhoff normal form.

We first remark that there is always some flexibility in choosing action-angle variables (x,ξ) near an invariant torus (here $x \in \mathbf{T}^n$ represents the angle variables, and $\xi \in \mathbf{R}^n$ the action variables). If $A \in \mathrm{GL}\,(n,\mathbf{Z})$ and ψ is any smooth function on $\mathbf{R}^n$, then

$$\kappa : (y,\eta) \mapsto (x,\xi) = (A^{-1}y + \partial\psi(\eta), A^t\eta) \tag{2.13}$$

gives a well-defined smooth, symplectic change of variables (which is analytic if ψ is analytic) on $\mathbf{T}^n \times \mathbf{R}^n \cong T^*\mathbf{T}^n$, and thus a new set of action-angle coordinates (y,η).

This transformation also preserves independence of the angle coordinate, and thus takes one asymptotic expansion which is in normal form (to order N) to another. More precisely, if

$$p(x,\xi,\varepsilon,h) = \widetilde{p}(\xi,\varepsilon,h) + \mathcal{O}(N) = a \cdot \xi + \mathcal{O}(2),$$

then in the new coordinates

$$(p \circ \kappa)(y,\eta,\varepsilon,h) = \widetilde{p}(A^t \eta,\varepsilon,h) + \mathcal{O}(N) = (Aa) \cdot \eta + \mathcal{O}(2).$$

Once we fix a choice of frequencies a, then, A must be the identity (because of the Diophantine assumption), which means we only have maps of the form

$$(y,\eta) \mapsto (y + \partial\psi, \eta), \tag{2.14}$$

which do not affect a normal form expansion because the second coordinate is unchanged.

Our aim is to show that the formula (2.14) gives all transformations which preserve independence of the angle variables in a general function or asymptotic expansion, while not affecting the frequencies a. Appendix A.1 of [HZ94] essentially contains a proof using generating functions that if a real symplectic diffeomorphism $(y,\eta) \mapsto (x,\xi)$ satisfies $\xi = b(\eta)$ with $\det(b_\eta) \neq 0$, then the mapping is of the form (2.13). The only difference is that the argument given there is local, and they end up with a more general type of transformation. In our case, it turns out one can make the formula apply globally, and then because we are on a torus, periodicity forces the simpler form (2.13), which reduces to (2.14) assuming a is unchanged. This argument will be given in Proposition 2.7 below.

Before proceeding, however, we note that because we have used complex symplectic transformations in our reduction to the normal form, it is natural to also consider symplectic biholomorphisms in a small complex neighborhood of $\mathbf{T}^n \times \{0\}$, for example allowing ψ to be complex-valued in (2.14). Here when we say a holomorphic transformation is symplectic or canonical we mean that the mapping preserves the standard symplectic form σ on $\mathbf{C}^n/2\pi\mathbf{Z}^n \times \mathbf{C}^n$, which is a form of type $(2,0)$, given in coordinates by

$$\sigma = \sum_{j=1}^{n} d\xi_j \wedge dx_j.$$

The standard fact that symplectomorphisms admit local generating functions carries over to the complex setting:

Proposition 2.6. *If $\kappa : (y, \eta) \mapsto (x, \xi) = (a(y, \eta), b(y, \eta))$ is a symplectic biholomorphism between small neighborhoods of $(y_0, \eta_0) \in \mathbf{C}^n$ and $(x_0, \xi_0) \in \mathbf{C}^n$, and $\det(b_\eta) \neq 0$, then there exists a holomorphic function $\phi(y, \xi)$ such that*

$$\kappa : (y, \partial_y \phi) \mapsto (\partial_\xi \phi, \xi).$$

The proof is exactly the same as in the real case (see appendix A.1 of [HZ94] for example), if we note that the implicit function theorem holds for holomorphic maps, and where normally we use Poincaré's lemma we instead use the Dolbeault-Grothendieck lemma.

Using this fact, we may now prove

Proposition 2.7. *Suppose $\kappa\colon (y, \eta) \mapsto (x, \xi)$ is a symplectic biholomorphism between open sets $U, V \subseteq (\mathbf{C}^n / 2\pi\mathbf{Z}^n) \times \mathbf{C}^n$, where U and V are small neighborhoods of $(\mathbf{R}^n / 2\pi\mathbf{Z}^n) \times \{0\}$. Suppose $\xi = b(\eta)$, where b is a biholomorphism defined near $0 \in \mathbf{C}^n$, such that $b(0) = 0$, $b_\eta(0) = 1$. Then in a small enough neighborhood of $(\mathbf{R}^n / 2\pi\mathbf{Z}^n) \times \{0\}$, κ satisfies*

$$\kappa : (y, \eta) \mapsto (y + \partial\psi(\eta), \eta).$$

for some analytic function ψ defined in a neighborhood of $0 \in \mathbf{C}^n$.

Proof. We may lift κ to a mapping between small neighborhoods of $\mathbf{R}^n \times \{0\} \subseteq \mathbf{C}^{2n}$. We use the same notations for the lift. By continuity, in small enough neighborhoods we have that $\det(b_\eta) \neq 0$, so by Proposition 2.6, locally we may find an analytic generating function $\phi(y, \xi)$ so that the mapping is given by

$$(y, \partial_y \phi) \mapsto (\partial_\xi \phi, \xi). \tag{2.15}$$

When two neighborhoods overlap, the local generating functions $\phi, \widetilde{\phi}$ must agree modulo constants, and in this way we may define a globally defined analytic generating function

$\phi(y, \xi)$, so that (2.15) is satisfied everywhere. Moreover the derivatives of ϕ are $2\pi\mathbf{Z}^n$-periodic, so (2.15) descends to U, V.

Still working with the lift, if $\partial_y \phi = \beta(\xi)$ is the inverse of the diffeomorphism b, then for some function $\widetilde{\psi}$, we have

$$\begin{aligned} \phi(y, \xi) &= \beta(\xi) \cdot y + \widetilde{\psi}(\xi) \\ \partial_\xi \phi &= \partial_\xi \beta \cdot y + \partial_\xi \widetilde{\psi}. \end{aligned}$$

Thus we have

$$(y, \beta(\xi)) \mapsto (\partial_\xi \beta \cdot y + \partial_\xi \widetilde{\psi}, \xi).$$

Because the mapping is $2\pi\mathbf{Z}^n$-periodic in x and y, we must have that for each $\xi_0 \in \mathbf{C}^n$, the assignment $\mathbf{Z}^n \ni n \mapsto \partial_\xi \beta(\xi_0) \cdot n$ defines an automorphism of $\mathbf{Z}^n$. By continuity, $\partial_\xi \beta \in \mathrm{GL}\,(n, \mathbf{Z})$ must be constant, and since $\beta(0) = 0$, with $\partial_\xi \beta(0) = 1$, β itself must be the identity mapping.

Therefore, setting $\psi = \widetilde{\psi} \circ b(\eta)$, the original mapping is given by

$$(y, \eta) \mapsto (y + \partial \psi(\eta), \eta),$$

with generating function $\phi(y, \xi) = \xi \cdot y + \widetilde{\psi}(\xi)$, and $\widetilde{\psi}$ is now allowed to be complex-valued.

□

Remark 2.8. Note that Proposition 2.7 only classifies transformations which preserve independence of the angular variables for a *general* symbol $\widetilde{p}(\xi, \varepsilon, h) = a \cdot \xi + \mathcal{O}(2)$ (while leaving the frequencies a unchanged). That is, if a transformation has this property with respect to *any* such $\widetilde{p}$, then it is easy to see the transformation must satisfy the hypotheses of Proposition 2.7. A priori, some specific symbol may admit more symmetries than the type described. See also Corollary 2.15.

2.4 Eigenvalue asymptotics

Under the assumptions stated in Section 2.2, Theorem 1.1 in [HSN07] implies that if $h^K \leq \varepsilon \leq h^\delta$, the eigenvalues of $P_\varepsilon = P + i\varepsilon Q$ which lie in (2.3) have an asymptotic expansion given in terms of the QBNF of P_ε.

Theorem 2.9 ([HSN07])**.** *Let P_ε satisfy all the assumptions of Section 2.2. For each $N \geq 1$, let $P^{(N)}$ be the result of applying Proposition 2.3 to P_ε, and let us write*

$$P^{(N)} = P^{(N)}(\xi, \varepsilon, h) = \sum_{j+2(m+n)\leq N} P^{(\infty)}_{jmn}(\xi)\varepsilon^m h^n, \tag{2.16}$$

where $P^{(\infty)}_{jmn}$ is a homogeneous polynomial of degree j in ξ which does not depend on N.

For any $0 < \delta < 1$, and any fixed integer K, suppose that $h^K \leq \varepsilon \leq h^\delta$. Recall the complex window (2.3), where we now take $C > 0$ to be sufficiently large. Then for each N, as $h \to 0$, the quasi-eigenvalues

$$P^{(N)}\left(h\left(k - \tfrac{k_0}{4}\right) - \tfrac{S}{2\pi}, \varepsilon, h\right), \quad k \in \mathbf{Z}^2 \tag{2.17}$$

are equal to the eigenvalues of P_ε modulo $\mathcal{O}(N+1)$ in (2.3), in the sense that for all sufficiently small h, there is a one-to-one partial function from the set of quasi-eigenvalues to spec(P_ε)*, equal to $1 + \mathcal{O}(N+1)$ uniformly, which is defined whenever a quasi-eigenvalue or the targeted true eigenvalue lies in (2.3).*

The formulation is slightly different from Theorem 1.1 in [HSN07] because of the different grading we have chosen, but the same proof applies. See also Theorems 5.1 and 5.2 of [HSN07] for somewhat more direct analogues to the above with the alternate grading.

The expression which appears in place of ξ in (2.17) is the result of a Bohr-Sommerfeld type condition. The constant vector $k_0 \in \mathbf{Z}^2$ contains the Maslov indices and $S \in \mathbf{R}^2$ the actions along a set of fundamental cycles of Λ, for example $\{x_1 = 0\}$, $\{x_2 = 0\}$, with respect to action-angle variables chosen so that Λ is represented as $\{\xi = 0\} \subseteq T^*\mathbf{T}^2$. For more details on this point, see [HSN07], as well as section 2 of [HS04].

2.5 Main Result

Our main result is a uniqueness statement asserting that if the eigenvalues corresponding to invariant torus Λ are the same for operators $P + i\varepsilon Q_1, P + i\varepsilon Q_2$, then they have the same QBNF near Λ.

Theorem 2.10. *Suppose that $P_1 = P + i\varepsilon Q_1$, $P_2 = P + i\varepsilon Q_2$ are operators satisfying the assumptions described in Section 2.2, where $P = P(x, hD_x; h)$ is a fixed, self-adjoint operator with principal symbol p, Λ is an H_p-invariant Lagrangian torus satisfying the Diophantine condition (2.6), and Q_1, Q_2 are operators with principal symbols q_1, q_2, respectively, such that $\langle q_1 \rangle |_\Lambda = \langle q_2 \rangle |_\Lambda$, and Λ satisfies the global condition (2.10) with respect to $P + i\varepsilon Q_\nu$, $\nu = 1, 2$.*

Fix $0 < \delta < 1$, and let the (ε, h)-dependent rectangle $R_\delta \subseteq \mathbf{C}$ be as described in equation (2.3),

$$R_\delta = \left\{ z \in \mathbf{C} \,\middle|\, |\operatorname{Re} z| < \frac{h^\delta}{C},\ |\operatorname{Im} z - \varepsilon \operatorname{Re} F| < \frac{\varepsilon h^\delta}{C} \right\}, \tag{2.18}$$

where $C > 0$ is large enough, with $F = \langle q_1 \rangle |_\Lambda = \langle q_2 \rangle |_\Lambda$. For $\nu = 1, 2$, let $P_\nu^{(\infty)}$, denote the QBNF of $P + i\varepsilon Q_\nu$, which is a formal asymptotic expansion of the form

$$P_\nu^{(\infty)}(\xi, \varepsilon, h) \sim \sum_{j,m,n} P_{\nu,jmn}^{(\infty)}(\xi) \varepsilon^m h^n = a \cdot \xi + i\varepsilon F + p_{01} h + \mathcal{O}((|\xi|, \varepsilon, h)^2),$$

where $P_{\nu,jmn}^{(\infty)}$, $j \geq 0$, $m \geq 0$, $n \geq 0$, is a homogeneous polynomial in ξ of degree j. Suppose that for all sufficiently small h and for all ε in the range $h^K \leq \varepsilon \leq h^\gamma$, where $K \geq 1$ and $\gamma > \frac{1}{2}$ are fixed, we have

$$\operatorname{spec}(P + i\varepsilon Q_1) \cap R_\delta = \operatorname{spec}(P + i\varepsilon Q_2) \cap R_\delta.$$

Then the QBNF's $P_1^{(\infty)}$, $P_2^{(\infty)}$ are equal, i.e., for all j, m, n we have $P_{1,jmn}^{(\infty)} = P_{2,jmn}^{(\infty)}$.

Whenever $k \in \mathbf{Z}^2$, in what follows we will use the notation $\xi_k = h(k - \frac{k_0}{4}) - \frac{S}{2\pi}$. To simplify the notation further, let us write $P_\nu^{(\infty)}(k)$ for the eigenvalue with asymptotic expansion given by $P_\nu^{(\infty)}(\xi_k, \varepsilon, h)$, according to Theorem 2.9.

We first prove a lemma which says that for an operator P_ε, we can recover the Bohr-Sommerfeld index k from the associated eigenvalue $P_\varepsilon^{(\infty)}(k)$ when, as in the hypotheses of the theorem, we have a bound $\varepsilon \lesssim h^\gamma$ for some $\gamma > \frac{1}{2}$. We remark that the exponent $\frac{1}{2}$ is not fundamental, but essentially comes out of the proof.

Lemma 2.11. *Let $P_1^{(\infty)}, P_2^{(\infty)}$ represent two QBNF's arising from operators which satisfy the hypotheses of Theorem 2.10. In particular, assume*

$$P_\nu^{(\infty)}(\xi, \varepsilon; h) = a \cdot \xi + i\varepsilon F + p_{01}h + \mathcal{O}((|\xi|, \varepsilon, h)^2), \quad \nu = 1, 2, \tag{2.19}$$

where p_{01} is a real constant and the vector a satisfies the Diophantine condition

$$|a \cdot k| \geq \frac{1}{C_0 |k|^{N_0}}, \qquad 0 \neq k \in \mathbf{Z}^2.$$

Then for any $\gamma > \frac{1}{2}$, if $\varepsilon \lesssim h^\gamma$, there exists $\beta_0 < 1$ and $h_0 > 0$ such that for all $\beta \in [\beta_0, 1)$, when $h \in [0, h_0)$, if $k, \ell \in \mathbf{Z}^2$ satisfy

$$P_1^{(\infty)}(k) = P_2^{(\infty)}(\ell) \mod \mathcal{O}(h^\infty),$$

with $P_1^{(\infty)}(k), P_2^{(\infty)}(\ell)$ lying in the window R_β, then $k = \ell$.

Remark 2.12. Note that we want $0 < \beta < 1$ in order to have many eigenvalues $P_\nu^{(\infty)}(k)$ which lie in the rectangle R_β. Indeed, the dimensions of R_β are $\sim h^\beta \times \varepsilon h^\beta$, so by the nondegeneracy assumption (2.9), $\#\,\mathrm{spec}(P_\varepsilon) \cap R_\beta \sim h^{2\beta - 2}$ and if $P_\nu^{(\infty)}(k) \in R_\beta$ then $|\xi_k| = \mathcal{O}(h^\beta)$.

Proof. By the hypotheses on $P_1^{(\infty)}, P_2^{(\infty)}$, and in particular because the the two expansions have several terms in common, as indicated in (2.19), we have

$$P_1^{(\infty)}(k) - P_2^{(\infty)}(\ell) = ha \cdot (\xi_k - \xi_\ell) + \mathcal{O}((|\xi_k|, \varepsilon, h)^2) + \mathcal{O}((|\xi_\ell|, \varepsilon, h)^2). \tag{2.20}$$

By the Diophantine condition on a, we have

$$|a \cdot (\xi_k - \xi_\ell)| = h|a \cdot (k - \ell)| \gtrsim \frac{h}{|k - \ell|^{N_0}}, \quad k \neq \ell.$$

Let us consider eigenvalues $P_\nu^{(\infty)}(k) \in R_\beta$ with $k \in \mathbf{Z}^2$, for some $\beta < 1$ to be determined. For any k, ℓ such that $|\xi_k|, |\xi_\ell| \lesssim h^\beta$, we have $h|k-\ell| = |\xi_k - \xi_\ell| \lesssim h^\beta$, so $|k-\ell| \lesssim h^{\beta-1}$. Hence,

$$h|a \cdot (k-\ell)| \gtrsim \frac{h}{|k-\ell|^{N_0}} \gtrsim \frac{h}{(h^{\beta-1})^{N_0}} = h^{1+N_0(1-\beta)}, \quad k \neq \ell. \tag{2.21}$$

The lemma will follow if we can show that by choosing β close enough to 1, the above dominates the contributions of the error terms in (2.20).

Thus, assuming $|\xi_k| \lesssim h^\beta$ and $\varepsilon \leq h^\gamma$, we estimate a typical term of one of the QBNF's:

$$|P_{\nu,jmn}^{(\infty)}(\xi)\varepsilon^m h^n \lesssim |\xi|^j \varepsilon^m h^n| \lesssim h^{j\beta+m\gamma+n}.$$

To have $P_{\nu,jmn}^{(\infty)} = o(a \cdot (k-\ell))$, in view of (2.21), it suffices to have $h^{j\beta+m\gamma+n} = o(h^{1+N_0(1-\beta)})$, or

$$1 + N_0(1-\beta) < j\beta + m\gamma + n.$$

After rearranging, this is equivalent to

$$\frac{N_0 + 1 - n - m\gamma}{N_0 + j} < \beta. \tag{2.22}$$

Examining the left hand side, we see that it is strictly less than 1, except in the following cases:

(a) $j = m = n = 0$

(b) $j = 1$ and $m = n = 0$

(c) $j = m = 0$, and $n = 1$

(d) $j = n = 0$ and $m\gamma \leq 1$.

However, the terms corresponding to these cases are exactly those written out in the right hand side of (2.19). Case (a) corresponds to the constant term 0, cases (b) and (c) correspond to the terms $a \cdot \xi$ and $p_{01}h$, respectively, and our hypothesis $\gamma > \frac{1}{2}$ is designed so that case (d) only applies when $m = 1$, which corresponds to the term $i\varepsilon F$.

It follows that the exceptional cases above occur when $j+m+n=1$. The left hand side of (2.22) must be maximized in one of the cases where $j+m+n=2$, as increasing any of j,m,n only makes the left hand side of (2.22) smaller. Therefore, if we choose β so that

$$\frac{N_0+1-n-m\gamma}{N_0+j}<\beta<1$$

in each of these finitely many cases (which we will not list), then these inequalities also hold for all j,m,n with $j+m+n\geq 2$.

Therefore, for some large M, we have that for sufficiently small h,

$$\begin{aligned}
&|P_1^{(\infty)}(k)-P_2^{(\infty)}(\ell)| \\
&\geq h|a\cdot(k-\ell)| - \sum_{2\leq j+m+n\leq M} |(P^{(\infty)}_{1,jmn}(\xi_k)-P^{(\infty}_{2,jmn}(\xi_\ell))\varepsilon^m h^n| \\
&\quad - \mathcal{O}((|\xi_k|,|\xi_\ell|,\varepsilon,h)^M) \\
&\gtrsim h^{1+N_0(1-\beta)} - \mathcal{O}(1)\sum_{2\leq j+m+n\leq M} h^{j\beta+m\gamma+n} - \mathcal{O}(h^{\frac{M}{2}}) \\
&\gtrsim h^{1+N_0(1-\beta)} - \mathcal{O}(h^{\frac{M}{2}}) \\
&\gtrsim h^{1+N_0(1-\beta)},
\end{aligned}$$

where the last estimate holds when M is chosen sufficiently large. The bound $\mathcal{O}(h^{\frac{M}{2}})$ comes from the fact that $\varepsilon\lesssim h^\gamma$ with $\gamma>\frac{1}{2}$ and $|\xi_k|,|\xi_\ell|\lesssim h^\beta$, where β is close to 1 (and so greater than $\frac{1}{2}$, we may assume). Note that the implicit constants depend only on the terms of the QBNFs and not on k,ℓ, so we get a uniform lower bound on the size of $P_1^{(\infty)}(k)-P_2^{(\infty)}(\ell)$ when $k\neq\ell$ and the eigenvalues lie in R_β.

Summing up, we have found that there exists $\beta<1$ such that if $P_1^{(\infty)}(k),P_2^{(\infty)}(\ell)\in R_\beta$, and $k\neq\ell$, then when h is sufficiently small,

$$|P_1^{(\infty)}(k)-P_2^{(\infty)}(\ell)|\gtrsim h^{1+N_0(1-\beta)}$$

where the implicit constant does not depend on k,ℓ. Therefore, when h is smaller than some small constant, which does not depend on k,ℓ, if $P_1^{(\infty)}(k)=P_2^{(\infty)}(\ell)$, we must have $k=\ell$.

□

We now proceed to the proof of the main theorem.

Proof of Theorem 2.10. Theorem 2.9 applies to P_1, P_2, meaning that their eigenvalues in a rectangle R_δ as in (2.18) have asymptotic expansions of the form (2.17), in terms of the QBNF's $P_1^{(\infty)}, P_2^{(\infty)}$. We will show one can recover the QBNF from the eigenvalues in any window R_β where $\beta_0 \leq \beta < 1$, with β_0 chosen as in Lemma 2.11. Note that $\delta \leq \beta$ implies $R_\beta \subseteq R_\delta$, so we may assume without loss of generality that $\beta = \delta$, and simply refer to both as β.

Suppose that $P^{(\infty)}_{1,jmn} \neq P^{(\infty)}_{2,jmn}$ for some index (j, m, n). Then $P^{(\infty)}_{1,jmn} - P^{(\infty)}_{2,jmn}$ is a homogeneous polynomial of degree j in ξ which does not vanish identically. By homogeneity, we can find an open subset of the unit circle $U_{jmn} \subseteq \{|\xi| = 1\} \subseteq \mathbf{R}^2$ on which this polynomial is bounded away from 0 in absolute value. For all h sufficiently small, there exists $\xi_k = h(k - \frac{k_0}{4}) - \frac{S}{2\pi}$ such that

$$|\xi_k| \sim h^\beta \quad \text{with} \quad \xi_k/|\xi_k| \in U_{jmn}. \tag{2.23}$$

For such ξ_k, by homogeneity we have

$$|P^{(\infty)}_{1,jmn}(\xi_k)\varepsilon^m h^n - P^{(\infty)}_{2,jmn}(\xi_k)\varepsilon^m h^n| \sim \varepsilon^m h^{j\beta+n}.$$

Let us also take $\varepsilon = h^\gamma$, so that $\varepsilon^m h^{j\beta+n} \sim h^{j\beta+m\gamma+n}$.

Without loss of generality we may assume that, possibly after increasing β and γ slightly, we have that $j\beta + m\gamma + n = j'\beta + m'\gamma + n'$ implies $m = m'$, $j = j'$, $n = n'$ (we just need $1, \beta, \gamma$ to be independent over the rationals). Then we have a total ordering of indices (j, m, n) according to the size of the expression $h^{j\beta+m\gamma+n}$. Also, because $\beta, \gamma > 0$, for any fixed $M > 0$, there are only finitely many indicies (j, m, n) such that $j\beta + m\gamma + n \leq M$. Therefore, of the indices (j, m, n) for which $P^{(\infty)}_{1,jmn} \neq P^{(\infty)}_{2,jmn}$, there is a unique index for which $j\beta + m\gamma + n$ is minimal. From now on, let (j_0, m_0, n_0) stand for this index.

Then with ξ_k satisfying (2.23) with $j = j_0$, $m = m_0$, $n = n_0$, and $\varepsilon = h^\gamma$, we have

$$|P_1^{(\infty)}(k) - P_2^{(\infty)}(k)| = (P^{(\infty)}_{1,j_0 m_0 n_0} - P^{(\infty)}_{2,j_0 m_0 n_0})h^{j_0\beta+m_0\gamma+n_0} + o(h^{j_0\beta+m_0\gamma+n_0}), \tag{2.24}$$

and so for h sufficiently small,

$$|P_1^{(\infty)}(k) - P_2^{(\infty)}(k)| \gtrsim h^{j_0\beta + m_0\gamma + n_0}. \tag{2.25}$$

By assumption, $P_1^{(\infty)}(k) = P_2^{(\infty)}(\ell)$ for some $\ell \in \mathbf{Z}^2$, and then by Lemma 2.11, we have $k = \ell$ for all h sufficiently small. Thus $P_1^{(\infty)}(k) = P_2^{(\infty)}(k)$, which contradicts (2.25).

Therefore, we have $P_{1,jmn}^{(\infty)} = P_{2,jmn}^{(\infty)}$ for all j, m, n, so the two QBNF's are equal.

□

Remark 2.13. We remark that the complete integrability assumptions were unnecessary in the proof of Theorem 2.10. In principle, one only needs that the asymptotic expansions given in Theorem 1.1 of [HSN07] are valid, as well as the Diophantine assumption, and so the main result may hold more generally.

Remark 2.14. We note as an addendum to the discussion in Section 2.3.2 that when an operator satisfies the hypotheses of Theorem 2.10, it implies uniqueness of the QBNF near the Diophantine torus Λ. Indeed, Theorem 2.9 describes the eigenvalues of such an operator in a window $R_\beta \subseteq \mathbf{C}$ corresponding to Λ, and Theorem 2.10 implies QBNF near Λ can be (uniquely) recovered from the eigenvalues if we know the frequencies a. This implies that the QBNF is unique up to the choice of action-angle variables. Thus we have

Corollary 2.15. *If P_ε is an operator satisfying the hypotheses on the operators in Theorem 2.10, the QBNF of P_ε near the Diophantine torus Λ is uniquely defined up to the choice of action-angle variables.*

Remark 2.16. In the proof of Theorem 2.10, we exploited the fact that Theorem 2.9 applies for all ε in a range $h^K \leq \varepsilon \leq h^\delta$ to assume that $\varepsilon = h^\gamma$, for a favorable choice of γ. It is natural to consider situations when ε is, for example, a function of h, or possibly has a more general sort of degenerate relationship with h. For example:

1. The damped wave operator on a compact manifold may be studied as a non-selfadjoint perturbation of the selfadjoint operator $-h^2\Delta$, where Δ is the Laplace-Beltrami operator, and the strength of the non-selfadjoint perturbation is $\varepsilon = h$ (see [Sjo00]). Then,

for instance, since $\varepsilon^2 = \varepsilon h = h^2$, it is meaningless to ask for the coefficients of these terms individually in some QBNF for the operator.

2. If $P^{(\infty)}$ is an asymptotic expansion satisfying (formally)

$$P^{(\infty)}(\xi, \varepsilon, h) = (\varepsilon - h)(\varepsilon^2 - h)\tilde{P}^{(\infty)}(\xi, \varepsilon, h), \tag{2.26}$$

for some asymptotic expansion $\widetilde{P}^{(\infty)}$, then when we restrict to $\varepsilon \in \{h, h^{\frac{1}{2}}\}$, $P^{(\infty)}$ represents the zero function, and a QBNF may only be uniquely-defined modulo expansions of this form.

More generally, consider situations where ε, h satisfy $r(\varepsilon, h) = 0$, where $r \in C^\infty(\text{neigh}(0,0))$ is not flat at (0,0), and $r(0,0) = 0$. Replacing r by $h^{-n_0}r$ if necessary, we may assume without loss of generality that $\partial_\varepsilon^k r(0,0) \neq 0$ for some $k \geq 1$.

If $\partial_\varepsilon r(0,0) \neq 0$, then by the implicit function theorem we have locally $\varepsilon = f(h)$ for some smooth function $f(h)$. Thus

$$r(\varepsilon, h) = c(\varepsilon, h)(\varepsilon - f(h)),$$

where $c(0,0) \neq 0$. Then $r(\varepsilon, h) = 0$ precisely when $\varepsilon = f(h)$.

In general, if we have $\partial_\varepsilon^k r(0,0) = 0$, $0 \leq k \leq m-1$, and $\partial_\varepsilon^m r(0,0) \neq 0$, then the Malgrange preparation theorem (cf. [Hor03], Section 7.5) implies a factorization

$$r(\varepsilon, h) = c(\varepsilon, h)(\varepsilon^m + a_{m-1}(h)\varepsilon^{m-1} + \ldots + a_0(h)), \quad (\varepsilon, h) \in \text{neigh}((0,0)). \tag{2.27}$$

where c and a_j, $0 \leq j \leq m-1$, are smooth functions of (ε, h) and h, respectively, with $c(0,0) \neq 0$, $a_j(0) = 0$. As $|c(\varepsilon, h)|$ is larger than some fixed, positive constant in a neighborhood of (0,0), let us assume without loss of generality that

$$r(\varepsilon, h) = \varepsilon^m + a_{m-1}(h)\varepsilon^{m-1} + \ldots + a_0(h).$$

Then by Theorem A.III.I of [Ger88], the roots of the right hand side, considered as a polynomial in ε, have formal asymptotic expansions in Puiseux series, i.e. powers of $h^{1/k}$, for some fixed $k \in \mathbf{N}$. Thus on the level of formal power series we have

$$\varepsilon^m + a_{m-1}(h)\varepsilon^{m-1} + \ldots + a_0(h) = \prod_{i=1}^{m}(\varepsilon - f^{(i)}(h^{\frac{1}{k}})),$$

where $f^{(i)}(h^{\frac{1}{k}})$ represents a formal Puiseux series,

$$f^{(i)}(h^{\frac{1}{k}}) \sim \sum_{n=0}^{\infty} c_n^{(i)} h^{\frac{n}{k}}.$$

Using a Borelian construction, we may find smooth functions $\tilde{f}^{(i)}(h)$, $1 \leq i \leq m$, defined when $h \geq 0$, with asymptotic expansion near $h = 0$ given by $f^{(i)}(h)$. Then

$$\varepsilon^m + a_{m-1}(h)\varepsilon^{m-1} + \ldots + a_0(h) = \prod_{i=1}^{m}(\varepsilon - \tilde{f}^{(i)}(h^{\frac{1}{k}}) + \mathcal{O}(h^\infty)). \tag{2.28}$$

We claim that $r(\varepsilon, h) = \mathcal{O}(h^\infty) \iff \operatorname{dist}(\varepsilon, \cup_{i=1}^m f^{(i)}(h^{\frac{1}{k}})) = \mathcal{O}(h^\infty)$.

Indeed, ($\Leftarrow$) is clear from (2.28). For ($\Rightarrow$), if $\operatorname{dist}(\varepsilon, \cup_{i=1}^m f^{(i)}(h^{\frac{1}{k}})) \neq \mathcal{O}(h^\infty)$, then, possibly after restricting to a sequence of values of h tending to zero, we have that $|\varepsilon - \tilde{f}^{(i)}(h^{\frac{1}{k}})| \geq \frac{1}{\mathcal{O}(1)} h^{N_i}$, with $N_i \in \mathbf{N}$, $1 \leq i \leq m$, hence

$$\begin{aligned} |r(\varepsilon, h)| = \prod_{i=1}^{m} |\varepsilon - \tilde{f}^{(i)}(h) + \mathcal{O}(h^\infty)| &\geq \prod_{i=1}^{m}(\frac{1}{\mathcal{O}(1)} h^{N_i} - \mathcal{O}(h^\infty)) \\ &\geq \frac{1}{\mathcal{O}(1)} h^{N_1 + \ldots + N_m} \neq \mathcal{O}(h^\infty). \end{aligned}$$

Conversely, if ε is a smooth (real-valued) function of $h^{\frac{1}{k}}$,

$$\varepsilon = f(h^{\frac{1}{k}}) \sim \sum_{n=0}^{\infty} c_n h^{\frac{n}{k}}, \quad c_n \in \mathbf{R}, \tag{2.29}$$

then by a Borelian construction, we can find smooth functions $f^{(i)}$, $0 \leq i \leq k-1$ with asymptotics given by the Puiseux conjugates of (2.29), i.e.

$$f^{(i)}(h^{\frac{1}{k}}) \sim \sum_{n=0}^{\infty} c_n (\zeta_k^i h^{\frac{1}{k}})^n,$$

where ζ_k is a primitive kth root of unity. (We can take $f^{(0)} = f$.) Then one can check that

$$r(\varepsilon, h) = \prod_{i=1}^{m}(\varepsilon - f^{(i)}(h^{\frac{1}{k}})) \tag{2.30}$$

is a smooth function near $(0,0) \in \mathbf{R}^2$, and $r(\varepsilon, h) = \mathcal{O}(h^\infty)$ when $\varepsilon = f(h)$.

If $\varepsilon \in \{f_\mu(h^{\frac{1}{k_\mu}}) \ \mid \ 1 \leq \mu \leq M\}$, where each f_μ is a smooth, real-valued function near 0, then letting $r(\varepsilon, h)$ be the product of the corresponding expressions (2.30) for each f_μ, we get a smooth function $r(\varepsilon, h)$ such that $r(\varepsilon, h) = \mathcal{O}(h^\infty)$ along the union of the curves $\varepsilon = f_\mu(h^{\frac{1}{k_\mu}})$.

To sum up the discussion, we see that there is a degenerate relationship between the parameters ε, h precisely when ε is within $\mathcal{O}(h^\infty)$ of a finite number of curves of the form $\varepsilon = \tilde{f}^{(i)}(h^{\frac{1}{k}})$, where $\tilde{f}^{(i)}$ is a smooth, real-valued function near 0.

CHAPTER 3

Normal forms for h-pseudodifferential operators with periodic classical flow

3.1 Introduction

The spectral theory of selfadjoint (pseudo)differential operators, whose associated classical flow is periodic, has a long and distinguished tradition, starting with the classical works of J. J. Duistermaat-V. Guillemin [Dui75] and A. Weinstein [Wei77], in the case of compact manifolds. Subsequently, many important contributions to the theory were given, [Ver79], [MG81], [HR84], [Doz97], [Ivr98]. In the case of semiclassical pseudodifferential operators whose Hamilton flow is periodic in some energy shell, the cluster structure of the spectrum has been established in [HR84]. That work also contains some precise results concerning the semiclassical asymptotics for the counting function of eigenvalues in the clusters, with the celebrated Bohr-Sommerfeld quantization rule obtained as a special case in dimension one, see also [DS99].

The purpose of this chapter is to show how the microlocal techniques of [HS04], [HS05], developed in the context of non-selfadjoint perturbations of selfadjoint operators with periodic classical flow in dimension two, apply to a class of *selfadjoint* operators of the form

$$P_\varepsilon = P + \varepsilon Q.$$

Here $P = P(x, hD_x; h)$ and $Q = Q(x, hD_x; h)$ are selfadjoint in $L^2(\mathbf{R}^2)$, with the classical flow of P being periodic in a band of energies around 0. The parameter $\varepsilon > 0$ measures the strength of the selfadjoint perturbation, and in order to have the clustering for the spectrum of P_ε, one should have $\varepsilon \ll h$. The general problem is then to understand the structure of the spectral clusters of the perturbed operator P_ε in the semiclassical limit $h \to 0$. In this work, we make a first and essential step in the study of this problem by constructing a microlocal (quantum Birkhoff) normal form for P_ε near a suitable Lagrangian torus, for a natural range of the parameter ε, $h^{2-\delta} \leq \varepsilon \leq \mathcal{O}(h^\delta)$, $\delta > 0$. Consequences of the normal form construction for the spectral analysis of P_ε will be explored in a future work, where we expect to be able to obtain complete semiclassical asymptotic expansions for the individual eigenvalues of P_ε in subclusters, corresponding to regular values of the averaged symbol of the perturbation along the classical flow. We also remark that contrary to [HS04], [HS05], no analyticity assumptions are needed here.

The plan of the paper is as follows. In Section 3.2, we rederive the clustering of the spectrum of P, obtained by means of a direct microlocal study near the closed orbits. In Section 3.3, we carry out an averaging reduction of P_ε microlocally near the energy surface. In Section 3.4, we microlocalize further to a suitable Lagrangian torus and construct a quantum Birkhoff normal form for P_ε near the torus.

3.2 Clustering of Eigenvalues

Let $P(x, \xi; h) \in S(1)$ be a real-valued semiclassical symbol on $T^*\mathbf{R}^n \cong \mathbf{R}^{2n}$. Here and in what follows we freely use the standard notation of semiclassical analysis, [DS99], [Zwo12]. Assume that $P(x, \xi; h) \sim \sum_{j=0}^{\infty} h^j p_j(x, \xi)$ in $S(1)$, as $h \to 0$. Assume that the subprincipal symbol p_1 vanishes identically, $p_1 \equiv 0$. We shall write $p = p_0$ for the leading symbol of P. Let us make the ellipticity assumption $p(x, \xi) \geq \frac{1}{C}$ when $|(x, \xi)| \geq C$, for some $C > 1$. Assume that the compact energy surface $p^{-1}(0)$ is connected in $T^*\mathbf{R}^n$, and that $dp \neq 0$ on this set.

We let P also denote the h-Weyl quantization $P = \mathrm{Op}_h^w(P)$. Then $P = \mathcal{O}(1)\colon L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ is a selfadjoint operator on $L^2(\mathbf{R}^n)$ ([DS99]). We have the following standard result:

Proposition 3.1. *The spectrum of P in a fixed neighborhood of* 0*,* $\mathrm{spec}(P) \cap \mathrm{neigh}(0, \mathbf{R})$*,* *is discrete, for all $h > 0$ small enough.*

Proof. There exists $0 \leq \chi \in C_0^\infty(\mathbf{R}^{2n})$ such that $p + \chi \geq \frac{1}{C}$ on $T^*\mathbf{R}^n$. Then $|p + \chi - z| \geq \frac{1}{2C}$ for $z \in \mathrm{neigh}(0, \mathbf{C})$, and therefore $P + \chi^w - z$ is invertible on L^2, for $h > 0$ sufficiently small. Here χ^w stands for the h-Weyl quantization of χ. Since χ^w is a compact operator on L^2 (see [DS99], Chapter 9 or [Zwo12], Chapter 4), it follows that $P - z$ is an analytic family of Fredholm operators for $z \in \mathrm{neigh}(0, \mathbf{C})$, invertible for $z \notin \mathbf{R}$, and thus invertible outside of a discrete set, by analytic Fredholm theory ([Zwo12], Appendix D). □

When G is any smooth real-valued function on $T^*\mathbf{R}^n$, we let H_G denote the Hamilton vector field of G, which is given by

$$H_G = \sum_{j=1}^{n} \frac{\partial G}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial G}{\partial x_j}\frac{\partial}{\partial \xi_j},$$

in standard symplectic coordinates. We make the following important assumption on the Hamiltonian dynamics of p:

Main assumption: For $E \in \mathrm{neigh}(0, \mathbf{R})$ the H_p-flow is periodic on $p^{-1}(E)$ with minimal period $T(E) > 0$, which is a smooth function of E.

Remark 3.2. If we assume each H_p–orbit γ has a (not necessarily minimal) period $T(\gamma)$ which varies smoothly, then it follows already that this period is a function of the energy alone: $T(\gamma) = T(E)$, $\gamma \subseteq p^{-1}(E)$ (see Chapter 15 of [DS99]). We are assuming in addition that $T(E)$ is the minimal period of each H_p–orbit $\gamma \subseteq p^{-1}(E)$.

As in [HR84], [Ver79], [Wei77], see also [HS05], our main assumption implies that the spectrum of P near 0 has a cluster structure, each cluster being of size $\mathcal{O}(h^2)$, and with adjacent clusters separated by a distance $\sim h$. More precisely:

Theorem 3.3. *We have for all $h > 0$ small enough,* $\mathrm{spec}(P) \cap \mathrm{neigh}(0, \mathbf{R}) \subseteq \cup_{k\in\mathbf{Z}} I_k$, *where*

$$I_k = [f(h(k - \alpha/4) - S/2\pi) - \mathcal{O}(h^2), f(h(k - \alpha/4) - S/2\pi) + \mathcal{O}(h^2)],$$

for a function $f \in C^\infty(\mathrm{neigh}(0, \mathbf{R}); \mathbf{R})$, $f(0) = 0$, $f'(0) > 0$. Here $S = \int_\gamma \xi \, dx$ is the classical action for a periodic trajectory $\gamma \subseteq p^{-1}(0)$, with α equal to the Maslov index of γ.

Remark 3.4. The function f is the inverse of the function $g \in C^\infty(\mathrm{neigh}(0, \mathbf{R}); \mathbf{R})$ defined by $g'(E) = T(E)/2\pi$, $g(0) = 0$. See [DS99], chapter 15, and [HS04].

Remark 3.5. Theorem 3.3 is still valid if our assumption that the subprincipal symbol vanishes, $p_1 \equiv 0$, is replaced by the weaker condition that for E near 0, the average of p_1 along each closed H_p trajectory,

$$\langle p_1 \rangle (x, \xi) = \frac{1}{T(E)} \int_0^{T(E)} p_1(\exp(tH_p)(x, \xi))\, dt, \quad (x, \xi) \in p^{-1}(E), \tag{3.1}$$

is a function of the energy E only,

$$\langle p_1 \rangle = \langle p_1 \rangle (E), \quad E \in \mathrm{neigh}(0, \mathbf{R}).$$

Here we shall merely indicate how this weaker condition may be used to simplify our operator P. Indeed for $G \in S(1)$ real-valued and compactly supported near $p^{-1}(0)$ (say), we have

$$e^{-iG}(P + hP_1)e^{iG} = P + e^{-iG}[P, e^{iG}] + hP_1 + \mathcal{O}(h^2).$$

Here $e^{\pm iG}$ also stand for the h-Weyl quantizations of these symbols. The leading symbol of $e^{-iG}[P, e^{iG}] + hP_1$ is $h(\frac{1}{i}e^{-iG}H_p e^{iG} + p_1) = h(H_p G + p_1)$. Then if we choose G so that $H_p G + p_1 = \langle p_1 \rangle_E$ along $p^{-1}(E)$, we see that we can conjugate our operator P to one whose subprincipal symbol is constant on each energy surface.

The proof of Theorem 3.3 proceeds in several steps.

1. We first prove a standard semiclassical elliptic estimate away from the energy surface $p^{-1}(0)$.

   Suppose $u, v \in L^2(\mathbf{R}^n)$ with $(P - z)u = v$. Here $z \in \mathrm{neigh}(0, \mathbf{C})$. Let $\chi \in C_0^\infty(T^*\mathbf{R}^n; [0, 1])$ be such that $\chi = 1$ near $p^{-1}(0)$. Then we claim

$$\|(1 - \chi^w)u\| \leq \mathcal{O}(1)\|v\| + \mathcal{O}(h^\infty)\|u\|, \tag{3.2}$$

   where $\| \cdot \| = \| \cdot \|_{L^2(\mathbf{R}^n)}$.

Proof. Let $\widetilde{\chi} \in C_0^\infty(T^*\mathbf{R}^n;[0,1])$ be such that $\chi = 1$ near the support of $\widetilde{\chi}$, and $\widetilde{\chi} = 1$ in a neighborhood of $p^{-1}(0)$. If we let $e = \frac{1-\widetilde{\chi}}{p-z} \in S(1)$, $E = \mathrm{Op}_h^w(e)$, then

$$E(P-z) = (1-\widetilde{\chi}^w) + hR,$$

where $R \in \mathrm{Op}_h^w(S(1))$. We have, for all $h > 0$ small enough,

$$\begin{aligned}(1-\chi^w)(1+hR)^{-1}E(P-z) &= (1-\chi^w)(1-(1+hR)^{-1}\widetilde{\chi}^w) \\ &= 1-\chi^w-(1-\chi^w)(1+hR)^{-1}\widetilde{\chi}^w.\end{aligned}$$

Since we chose $\widetilde{\chi}$ so that $\chi = 1$ near the support of $\widetilde{\chi}$, the Weyl calculus implies $(1-\chi^w)(1+hR)^{-1}\widetilde{\chi}^w = \mathcal{O}(h^\infty) : L^2 \to L^2$. Here we have also used that $(1+hR)^{-1} \in \mathrm{Op}_h^w(S(1))$, according to Beals' lemma, [DS99]. Thus the holomorphic family of h-pseudodifferential operators $\widetilde{E} = \widetilde{E}(z) = (1-\chi^w)(1+hR)^{-1}E = \mathcal{O}(1)\colon L^2 \to L^2$ satisfies $\widetilde{E}(P-z) = 1-\chi^w + \mathcal{O}(h^\infty)\colon L^2 \to L^2$, so

$$(1-\chi^w)u = \widetilde{E}v + \mathcal{O}(h^\infty)u,$$

which implies (3.2). □

2. We would like to microlocalize to a neighborhood of $p^{-1}(0)$. For this we first microlocalize to a neighborhood of a closed H_p–orbit $\gamma \subset p^{-1}(0)$.

Lemma 3.6. *There exists a smooth canonical transformation*

$$\kappa\colon \operatorname{neigh}(\gamma, T^*\mathbf{R}^n) \to \operatorname{neigh}(\tau = x = \xi = 0, T^*(S_t^1 \times \mathbf{R}_x^{n-1}))$$

such that $\kappa(\{\gamma\}) = \{x = \xi = \tau = 0\}$, $p \circ \kappa^{-1} = f(\tau)$.

Remark 3.7. Recall that $f^{-1} = g$, where $g \in C^\infty(\operatorname{neigh}(0,\mathbf{R});\mathbf{R})$ satisfies $g'(E) = T(E)/2\pi$, $g(0) = 0$. Then $H_{g\circ p} = g'(p)H_p$, so 2π is the minimal period of the $H_{g\circ p}$–flow.

Proof. Fix any point $\rho_0 \in \gamma$ and choose local symplectic coordinates (t,τ,x,ξ) in a neighborhood of ρ_0 and vanishing at that point, where $\tau = g \circ p$. Then we have

$$\{\xi, x\} = 1, \quad \{t, x\} = 0, \quad \{\tau, \xi\} = 0, \tag{3.3}$$

and

$$H_\tau t = 1, \quad H_\tau x = H_\tau \xi = 0. \tag{3.4}$$

Putting $\tau = g \circ p$, we may extend x, ξ according to the latter conditions to a full neighborhood of γ, obtaining smooth, single-valued functions. Similarly t extends to a multi-valued function in a neighborhood of γ, in such a way that it increases by 2π each time we make a loop in the forward direction. The conditions (3.3) extend to a full neighborhood of γ, and the lemma follows. □

3. As explained in [HS04], the canonical transformation κ can be implemented by a multi-valued microlocally unitary h-Fourier integral operator $U = \mathcal{O}(1)\colon L^2(\mathbf{R}^n) \to L^2_f(S^1 \times \mathbf{R}^{n-1})$, so that the improved Egorov property holds—see the discussion in Section 2 of [HS04]. Here $L^2_f(S^1 \times \mathbf{R}^{n-1})$ is the space of multi-valued L^2 functions on $S^1 \times \mathbf{R}^{n-1}$, which satisfy the Floquet-Bloch periodicity condition,

$$u(t - 2\pi, x) = e^{\frac{2\pi i}{h}(\frac{S}{2\pi} + h\frac{\alpha}{4})} u(t, x). \tag{3.5}$$

The multi-valuedness of U is a reflection of the fact that the domain of definition of the canonical transformation κ is not simply connected, the homotopy group being generated by the trajectory γ.

It follows that there exists an operator $\widetilde{P}$ with the leading symbol $f(\tau)$ near $\{\tau = x = \xi = 0\}$ and with the vanishing subprincipal symbol so that $\widetilde{P}U = UP$ microlocally near γ, so that

$$(\widetilde{P}U - UP)\chi_1^w(x, hD_x) = \mathcal{O}(h^\infty) : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n), \tag{3.6}$$

and

$$\chi_2^w(x, hD_x)(\widetilde{P}U - UP) = \mathcal{O}(h^\infty) : L^2_f(S^1 \times \mathbf{R}^{n-1}) \to L^2_f(S^1 \times \mathbf{R}^{n-1}),$$

for every $\chi_1 \in C_0^\infty(\mathrm{neigh}(\gamma, T^*\mathbf{R}^n))$ and for every $\chi_2 \in C_0^\infty(T^*(S^1 \times \mathbf{R}^{n-1}))$ supported near $\tau = x = \xi = 0$. The operator $\widetilde{P}$ acts on the space of L^2–functions satisfying the

Floquet-Bloch condition (3.5), defined microlocally near $\tau = x = \xi = 0$ in $T^*(S^1 \times \mathbf{R}^{n-1})$.

Let us also remark that an orthonormal basis for the space $L^2_f(S^1)$ of multi-valued L^2 functions on the circle satisfying a periodicity condition analogous to (3.5), with the x-variable suppressed, consists of the functions

$$e_k(t) = \exp\left(\frac{it}{h}\left(h\left(k - \frac{\alpha}{4}\right) - \frac{S}{2\pi}\right)\right), \quad k \in \mathbf{Z},$$

which satisfy

$$f(hD_t)e_k(t) = f\left(h\left(k - \frac{\alpha}{4}\right) - \frac{S}{2\pi}\right)e_k(t).$$

It follows that if $z \in \text{neigh}(0, \mathbf{R})$ is such that $|z - f(h(k - \frac{\alpha}{4}) - \frac{S}{2\pi})| \geq Ch^2$, $k \in \mathbf{Z}$, for $C > 1$ sufficiently large but fixed, then the operator

$$\widetilde{P} - z = f(hD_t) + h^2R - z = (f(hD_t) - z)(1 + (f(hD_t) - z)^{-1}h^2R), \quad R = \mathcal{O}(1),$$

is invertible, microlocally near $\tau = x = \xi = 0$, with the norm of the inverse being $\mathcal{O}(h^{-2})$.

4. Take finitely many closed trajectories $\gamma_1, \ldots, \gamma_N \subset p^{-1}(0)$ and small open neighborhoods $\gamma_j \subseteq \Omega_j$, with Ω_j invariant under the H_p-flow, such that $p^{-1}(0) \subseteq \cup\Omega_j$, and also cutoff functions $0 \leq \chi_j \in C_0^\infty(\Omega_j)$ such that $H_p\chi_j = 0$, $\sum \chi_j = 1$ near $p^{-1}(0)$. Let U_j denote a multi-valued h-Fourier integral operator associated to γ_j, as in the previous step.

   We investigate solving the equation $(P - z)u = v$, when u, $v \in L^2$. For each j, $1 \leq j \leq N$, we have

$$\chi_j^w(P - z)u = \chi_j^w v \implies (P - z)\chi_j^w u + [\chi_j^w, P]u = \chi_j^w v.$$

   By the microlocal normal form for P in Ω_j derived above, given in (3.6),

$$U_j(P - z)\chi_j^w = (\widetilde{P} - z)U_j\chi_j^w + \mathcal{O}(h^\infty) = (f(hD_t) + h^2R - z)U_j\chi_j^w + \mathcal{O}(h^\infty),$$

with $R = \mathcal{O}(1)\colon L^2_f(S^1 \times \mathbf{R}^{n-1}) \to L^2_f(S^1 \times \mathbf{R}^{n-1})$. When $z \in \mathrm{neigh}(0, \mathbf{R})$ avoids the intervals I_k, we just saw that the operator $f(hD_t) + h^2 R - z$ possesses a microlocal inverse $(f(hD_t) + h^2 R - z)^{-1} = \mathcal{O}(1/h^2)$. For each j, we get

$$(f(hD_t) + h^2 R - z) U_j \chi_j^w u = U_j \chi_j^w v + U_j [P, \chi_j^w] u.$$

and then applying the microlocal inverse $(f(hD_t) + h^2 R - z)^{-1}$, we get

$$\|\chi_j^w u\| \leq \mathcal{O}(1/h^2)\|v\| + \mathcal{O}(1/h^2)\|[P, \chi_j^w]u\|.$$

Since the subprincipal symbol of P vanishes, we have in the operator sense, $[P, \chi_j^w] = \mathcal{O}(h^3)$. Summing over j we get

$$\|\sum \chi_j^w u\| \leq \mathcal{O}(1/h^2)\|v\| + \mathcal{O}(h)\|u\|$$

and by the elliptic bound (3.2) we have

$$\|(1 - \sum \chi_j^w) u\| \leq \mathcal{O}(1)\|v\| + \mathcal{O}(h^\infty)\|u\|.$$

Combining these, we obtain

$$\|u\| \leq \mathcal{O}(1/h^2)\|v\| + \mathcal{O}(h)\|u\|.$$

Taking h small enough, we conclude that $(P - z)^{-1}$ exists and satisfies $(P - z)^{-1} = \mathcal{O}(1/h^2) : L^2 \to L^2$.

This completes the proof of Theorem 3.3 $\Box$.

3.3 Selfadjoint perturbations and averaging along closed orbits

Let us now consider a perturbed h-pseudodifferential operator of the form $P_\varepsilon = P + \varepsilon Q$, $\varepsilon \in \mathrm{neigh}(0, \mathbf{R})$, where P has been introduced in Section 3.2, and Q is the h-Weyl quantization of a real-valued symbol also denoted by $Q = Q(x, \xi; h) \in S(1)$, which has the leading symbol q. We shall write $p_\varepsilon = p + \varepsilon q$ for the leading symbol of P_ε. By Theorem 3.3 combined with the spectral theorem, we see that the spectrum of $P+\varepsilon Q$ near 0 is discrete, for $\varepsilon \geq 0$ small enough, and

$$\operatorname{spec}(P+\varepsilon Q)\cap \operatorname{neigh}(0,\mathbf{R}) \subseteq \cup_{k\in\mathbf{Z}} f\left(h\left(k-\tfrac{\alpha}{4}\right)-\tfrac{S}{2\pi}\right)+[-\mathcal{O}(\varepsilon+h^2),+\mathcal{O}(\varepsilon+h^2)].$$

When $\varepsilon \ll h$, the spectrum of $P+\varepsilon Q$ retains a cluster structure, but for now we do not make this assumption.

Following in a long tradition dating to [Wei77], [Ver79], and others, we would like to perform an averaging procedure, replacing q by its average along closed orbits of the H_p-flow, which we will denote

$$\langle q\rangle = \frac{1}{T(E)}\int_{-T(E)/2}^{T(E)/2} q\circ \exp(tH_p)\,dt \quad \text{on } p^{-1}(E).$$

Implementing the averaging procedure on the operator level will allow us to reduce the dimension by one unit, provided that $\varepsilon = \mathcal{O}(h^\delta)$, for some $\delta > 0$.

Let $G_0 \in C_0^\infty(T^*\mathbf{R}^n;\mathbf{R})$. Then by Taylor's formula,

$$\begin{aligned} p_\varepsilon \circ \exp(\varepsilon H_{G_0}) &= p+\varepsilon q+\varepsilon H_{G_0} p+\mathcal{O}(\varepsilon^2)\\ &= p+\varepsilon(q-H_pG_0)+\mathcal{O}(\varepsilon^2). \end{aligned}$$

Since $\langle q-\langle q\rangle\rangle = 0$, locally near $p^{-1}(0)$ we may solve $H_pG_0 = q-\langle q\rangle$, the solution being a smooth function. In fact, as in [HS04], on $p^{-1}(E)$, $E\in\operatorname{neigh}(0,\mathbf{R})$, we can use the explicit formula

$$G = \frac{1}{T(E)}\int_{-T(E)/2}^{T(E)/2}\left[\mathbf{1}_{\mathbf{R}_{<0}}(t)\left(t+\frac{1}{2}T(E)\right)+\mathbf{1}_{\mathbf{R}_{>0}}(t)\left(t-\frac{1}{2}T(E)\right)\right] q\circ\exp(tH_p)\,dt.$$

Similarly, with $G_1, G_2, \ldots$ denoting a sequence of smooth real-valued functions to be determined, and $G \sim \sum_{j=0}^\infty \varepsilon^j G_j$, if we expand $p_\varepsilon \circ \exp(\varepsilon H_G)$ asymptotically, we claim that we can iteratively solve for G_j so that

$$p_\varepsilon \circ \exp(\varepsilon H_G) = p+\varepsilon\langle q\rangle + \mathcal{O}(\varepsilon^2),$$

where the $\mathcal{O}(\varepsilon^2)$ error term Poisson commutes with p modulo $\mathcal{O}(\varepsilon^\infty)$.

Explicitly, if $G_{\leq N} = G_0 + \varepsilon G_1 + \varepsilon^2 G_2 + \ldots + \varepsilon^N G_N$ satisfies

$$p_\varepsilon \circ \exp(\varepsilon H_{G_{\leq N}}) = p + \varepsilon \langle q \rangle + a_2 \varepsilon^3 + \ldots + a_N \varepsilon^N + r_{N+1} \varepsilon^{N+1} + \mathcal{O}(\varepsilon^{N+2})$$

where $\{p, a_j\} = 0$, $2 \leq j \leq N$, then for $G_{\leq N+1} = G_{\leq N} + \varepsilon^{N+1} G_{N+1}$, with $G_{N+1} \in C^\infty$ to be determined, we have by a variation on the Baker-Campbell-Hausdorff formula [Hor67], pp. 160–161,

$$\begin{aligned}\exp(\varepsilon H_{G_{\leq N+1}}) &= \exp(\varepsilon H_{G_{\leq N}} + \varepsilon^{N+1} H_{G_{N+1}}) \\ &= \exp(\varepsilon H_{G_{\leq N}}) \exp(\varepsilon^{N+1} H_{G_{N+1}})(1 + \mathcal{O}(\varepsilon^{N+2}))\end{aligned}$$

where the $\mathcal{O}(\varepsilon^{N+2})$ bound is in the C^∞-sense. This implies that

$$\begin{aligned}&p_\varepsilon \circ \exp(\varepsilon H_{G_{\leq N+1}}) \\ &\quad = p + \varepsilon \langle q \rangle + a_2 \varepsilon^2 + a_2 \varepsilon^3 + \ldots + a_N \varepsilon^N + (r_{N+1} - H_p G_{N+1}) \varepsilon^{N+1} + \mathcal{O}(\varepsilon^{N+2}).\end{aligned}$$

As above, we may find a smooth real-valued solution of $H_p G_{N+1} = r_{N+1} - \langle r_{N+1} \rangle$, defined near $p^{-1}(0)$.

The functions G_j, $j \geq 0$, may be defined in a fixed neighborhood of $p^{-1}(0)$. We extend them to globally defined, compactly supported smooth functions on all of $T^*\mathbf{R}^n$. By Borel's lemma we may choose $G \in C_0^\infty(\mathbf{R}^n; \mathbf{R})$ which is given by $\sum_{j=0}^\infty \varepsilon^j G_j$ asymptotically in the C^∞–sense, and then we have achieved that $p_\varepsilon \circ \exp(\varepsilon H_G)$ is in involution with p modulo $\mathcal{O}(\varepsilon^\infty)$ in a fixed neighborhood of $p^{-1}(0)$, as desired.

By Egorov's theorem (e.g. [Zwo12], Theorem 11.1), we may quantize the exact canonical transformation $\exp(\varepsilon H_G)$ with an elliptic h-Fourier integral operator $U = \mathcal{O}(1) \colon L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ which is microlocally unitary near $p^{-1}(0)$. Then we have that the selfadjoint operator $\widetilde{P}_\varepsilon := U^{-1} P_\varepsilon U$ is of the form $\widetilde{P}_\varepsilon \sim \sum_{j=0}^\infty h^j p_j(x, \xi, \varepsilon)$ with $p_0(x, \xi, \varepsilon) = p + \varepsilon \langle q \rangle + \mathcal{O}(\varepsilon^2)$, with the $\mathcal{O}(\varepsilon^2)$ term in involution with p modulo $\mathcal{O}(\varepsilon^\infty)$.

Furthermore, by the results of Section 2 of [HS04], if we choose the principal symbol of the Fourier integral operator U to be real, then U enjoys the improved Egorov property, namely that on the level of symbols we have $\widetilde{P}_\varepsilon = P_\varepsilon \circ \exp(\varepsilon H_G) + \mathcal{O}(h^2)$, so that the subprincipal symbol of $\widetilde{P}_{\varepsilon=0}$ vanishes. We thus reduce ourselves to the study of $\widetilde{P}_\varepsilon$, which is an h-pseudodifferential operator on $\mathbf{R}^n$ satisfying the same general assumptions as in the previous section, with principal symbol of the form $p + \varepsilon \langle q \rangle + \mathcal{O}(\varepsilon^2)$.

Remark 3.8. There is a particularly convenient global choice of the h-Fourier integral operator U. Since G is defined globally, we may use $U = e^{-i\varepsilon G/h}$, defined by the spectral theorem, which is then globally unitary on L^2. To see that this choice of U enjoys the improved Egorov property, note that $U(t) = e^{-itG/h}$ solves the operator ODE

$$hD_t U(t) + GU(t) = 0, \quad 0 \leq t \leq \varepsilon,$$

and by [Zwo12], Theorem 11.1, $U = U(\varepsilon)$ quantizes the canonical transformation $\exp(\varepsilon H_G)$. Since the subprincipal symbol of G vanishes, the principal symbol of U solves a real transport equation [Zwo12], and by Proposition 2.1 of [HS05], U enjoys the improved Egorov property.

3.4 Microlocal study near a torus when $n = 2$

We now work in dimension $n = 2$, and from now on we will assume that $\varepsilon = \mathcal{O}(h^\delta)$ for some $\delta > 0$. In particular $\mathcal{O}(\varepsilon^\infty) = \mathcal{O}(h^\infty)$. Recall from the previous section that we have reduced ourselves to an operator $\widetilde{P}_\varepsilon$ with the leading symbol of the form

$$p + \varepsilon\langle q \rangle + \mathcal{O}(\varepsilon^2),$$

where the $\mathcal{O}(\varepsilon^2)$–term Poisson commutes with p, modulo $\mathcal{O}(h^\infty)$. The subprincipal symbol of $\widetilde{P}_\varepsilon$ is $\mathcal{O}(\varepsilon)$. In what follows, when working with the operator $\widetilde{P}_\varepsilon$, to simplify the notation, we shall drop the tilde and continue to write P_ε.

Let $F_0 \in \mathbf{R}$ be such that $\min_{p^{-1}(0)} \langle q \rangle < F_0 < \max_{p^{-1}(0)} \langle q \rangle$ and assume that F_0 is a regular value of $\langle q \rangle$ restricted to $p^{-1}(0)$. After replacing q by $q - F_0$ we may assume that $F_0 = 0$, and let us consider the H_p–flow invariant set $\Lambda = \{p = 0, \langle q \rangle = 0\}$. We know that $dp, d\langle q \rangle$ are linearly independent at each point of Λ, so that Λ is a Lagrangian manifold which is a union of finitely many 2-tori. Assume for simplicity that Λ is connected so that it is equal to a single Lagrangian torus. Because the functions $p, \langle q\rangle$ are in involution, they form a completely integrable system in a neighborhood of Λ. We have action-angle coordinates near Λ [HZ94], given by a smooth canonical transformation

$$\kappa\colon \operatorname{neigh}(\xi = 0, T^*\mathbf{T}^2) \to \operatorname{neigh}(\Lambda, T^*\mathbf{R}^2),$$

mapping the zero section in $T^*\mathbf{T}^2$ onto Λ, and such that $p \circ \kappa = p(\xi)$, $\langle q\rangle \circ \kappa = \langle q\rangle(\xi)$. Here we make the identification $T^*\mathbf{T}^2 \cong (\mathbf{R}/2\pi\mathbf{Z})^2_x \times \mathbf{R}^2_\xi$. Because p has periodic flow, we may choose κ so that in fact $p \circ \kappa = p(\xi_1)$ by letting ξ_1 be the normalized action of a closed H_p trajectory — see the discussion in Section 4 of [HS04]. The linear independence of differentials of p and $\langle q\rangle$ implies that $p'(0) \neq 0$, $\partial_{\xi_2}\langle q\rangle(0) \neq 0$.

Implementing κ by means of a multi-valued microlocally unitary h-Fourier integral operator U, which also has the improved Egorov property [HS04], we get a new operator $U^{-1}P_\varepsilon U$, which will still be denoted by P_ε,

$$P_\varepsilon\colon L^2_f(\mathbf{T}^2) \to L^2_f(\mathbf{T}^2).$$

Here the selfadjoint operator P_ε is defined microlocally near $\xi = 0$ in $T^*\mathbf{T}^2$, with

$$P_\varepsilon \sim \sum_{j=0}^{\infty} h^j p_j(x, \xi, \varepsilon), \tag{3.7}$$

the principal symbol being

$$p_0(x, \varepsilon, \xi) = p(\xi_1) + \varepsilon\langle q\rangle(\xi) + \mathcal{O}(\varepsilon^2) \tag{3.8}$$

with the $\mathcal{O}(\varepsilon^2)$ error term independent of x_1 modulo $\mathcal{O}(h^\infty)$. Furtermore, $p_1(x, \xi, \varepsilon) = \mathcal{O}(\varepsilon)$. The space here $L^2_f(\mathbf{T}^2)$ stands for the subspace of $L^2_{\mathrm{loc}}(\mathbf{R}^2)$ consisting of Floquet periodic functions $u(x)$, satifying

$$u(x - \nu) = e^{i\nu\cdot\theta}u(x), \quad \nu \in (2\pi\mathbf{Z})^2, \quad \theta = \frac{S}{2\pi h} + \frac{\alpha}{4}.$$

Here $S = (S_1, S_2)$ with S_j being the action of the generator γ_j of the homotopy group of Λ, with γ_1 being given by a closed H_p–trajectory, and $\alpha = (\alpha_1, \alpha_2)$ is the corresponding Maslov index.

Removing the x_1 dependence

Our next goal will be to eliminate the x_1-dependence in p_j, $j \geq 1$ in (3.7). Let $A = A(x,\xi,\varepsilon) \in S(1)$ be real and let us consider the conjugation of P_ε by the elliptic h-pseudodifferential operator e^{iA}. We have, identifying the symbols with the corresponding h-Weyl quantizations,

$$\begin{aligned} e^{-iA}P_\varepsilon e^{iA} &= P_\varepsilon + e^{-iA}[P_\varepsilon, e^{iA}] \\ &= p_0 + h(p_1 + \{p_0, A\}) + \mathcal{O}(h^2). \end{aligned}$$

For future reference, let us notice that we can write

$$e^{-iA}P_\varepsilon e^{iA} = e^{-i\,\mathrm{ad}\,A}P_\varepsilon, \quad (\mathrm{ad}\,A)P_\varepsilon = [A, P_\varepsilon].$$

We shall now show that A can be chosen so that $p_1 + H_{p_0}A$ becomes independent of x_1, modulo $\mathcal{O}(h^\infty)$. In doing so, we shall construct the C^∞–symbol A as a formal power series in ε. Introducing the Taylor expansions,

$$p_0 \sim \sum_{\ell=0}^{\infty} \varepsilon^\ell p_{0,\ell}(x,\xi), \quad p_1 \sim \sum_{\ell=1}^{\infty} \varepsilon^\ell p_{1,\ell}(x,\xi),$$

and writing

$$A \sim \sum_{\ell=1}^{\infty} \varepsilon^\ell a_\ell(x,\xi),$$

we compute the power series expansion of the Poisson bracket,

$$H_{p_0}A \sim \sum_{k\geq 0, \ell\geq 1} \varepsilon^{k+\ell}\{p_{0,k}, a_\ell\} = \sum_{m=1}^{\infty} \varepsilon^m f_m,$$

where

$$f_m = \sum_{k+\ell=m, k\geq 0, \ell\geq 1} \{p_{0,k}, a_\ell\}.$$

We would like to choose the coefficients a_ℓ, $\ell \geq 1$, so that $p_{1,\ell} + f_\ell$ is independent of x_1, for all ℓ. When $\ell = 1$, we have $p_{1,1} + f_1 = p_{1,1} + \partial_{\xi_1} p\, \partial_{x_1} a_1$, and since $\partial_{\xi_1} p(0) \neq 0$, we can determine a_1 by solving the transport equation,

$$p_{1,1} + \partial_{\xi_1} p\, \partial_{x_1} a_1 = \langle p_{1,1} \rangle_{x_1},$$

the right hand side standing for the average with respect to x_1. Arguing inductively, assume that the smooth real-valued $a_1, \dots a_m$ have already been determined. The term $p_{1,m+1}+f_{m+1}$ is of the form

$$p_{1,m+1} + \partial_{\xi_1} p \, \partial_{x_1} a_{m+1} + \sum_{k+\ell=m+1,\, \ell<m+1} \{p_{0,k}, a_\ell\},$$

and it is therefore clear that we can choose a_{m+1} so that this expression becomes independent of x_1. Arguing in this fashion, we obtain a sequence $a_j \in C^\infty(\mathrm{neigh}(\xi = 0, T^*\mathbf{T}^2))$, a_j real-valued, so that if $A \in C^\infty$ is such that

$$A(x, \xi, \varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j a_j,$$

then the subprincipal symbol of the conjugated operator $e^{-iA} P_\varepsilon e^{iA}$ is $\mathcal{O}(\varepsilon)$ and independent of x_1, modulo $\mathcal{O}(h^\infty)$.

Assume inductively that we have found $A_0 = A, \dots A_{N-1}$ so that the operator

$$P_\varepsilon^{(N)} := e^{-i\,\mathrm{ad}(h^{N-1}A_{N-1})} \circ \dots e^{-i\,\mathrm{ad}(hA_1)} \circ e^{-i\,\mathrm{ad}(A)} P_\varepsilon$$

is of the form $\sim \sum_{j=0}^\infty h^j p_j$, where p_j are independent of x_1 modulo $\mathcal{O}(h^\infty)$, for $j \leq N$. We then look for the operator of the form $e^{ih^N A_N}$ and we see as before that the leading symbol of $e^{-ih^N A_N}[P_\varepsilon^{(N)}, e^{ih^N A_N}]$ is $h^{N+1} H_{p_0} A_N$. Therefore, $e^{-ih^N A_N} P_\varepsilon^{(N)} e^{ih^N A_N}$ is of the form $\sim \sum_{j=0}^\infty h^j \widetilde{p}_j$, where $\widetilde{p}_j = p_j$ for $j \leq N$, and $\widetilde{p}_{N+1} = p_{N+1} + H_{p_0} A_N$. It is therefore clear that we can determine A_N, as a formal power series in ε, so that $\widetilde{p}_{N+1}$ becomes independent of x_1. Using the Baker-Campbell-Hausdorff formula and Borel's lemma, we see that there exists an h-pseudodifferential operator $\mathcal{A}$ with symbol $\sim \sum_{\nu=0}^\infty h^\nu a_\nu(x, \xi, \varepsilon) \in S(1)$, with $a_0 = \mathcal{O}(\varepsilon)$, such that

$$e^{-i\,\mathrm{ad}\,\mathcal{A}} \sim \dots e^{-ih^2\,\mathrm{ad}(A_2)} \circ e^{-ih\,\mathrm{ad}(A_1)} \circ e^{-i\,\mathrm{ad}(A_0)},$$

and we conclude that the operator $\widetilde{P}_\varepsilon = e^{-i\,\mathrm{ad}\,\mathcal{A}} P_\varepsilon$ is of the form $\sum_{j=0}^\infty h^j \widetilde{p}_j(x_2, \xi, \varepsilon)$, where $\widetilde{p}_0 = p(\xi_1) + \varepsilon\langle q\rangle(\xi) + \mathcal{O}(\varepsilon^2)$ is also independent of x_1, and $\widetilde{p}_1(x_2, \xi, 0) = 0$.

Removing the x_2 dependence

In the previous subsection, using only the fact that $\partial_{\xi_1} p(0) \neq 0$, we have eliminated the x_1-dependence in the full symbol of P_ε and have reduced ourselves to an operator of the form,

$$\widetilde{P}_\varepsilon \sim \sum_{j=0}^{\infty} h^j \widetilde{p}_j(x_2, \xi, \varepsilon) \quad \text{on } L_f^2(\mathbf{T}^2),$$

where

$$\widetilde{p}_0 = p(\xi_1) + \varepsilon \langle q \rangle (\xi) + \varepsilon^2 r_2(x_2, \xi, \varepsilon) + \mathcal{O}(\varepsilon^3)$$

with the $\mathcal{O}(\varepsilon^3)$ error term independent of x_1. Also, $\widetilde{p}_1(x_2, \xi, 0) = 0$.

Arguing in the spirit of [HS04], [HS05], we shall now look for an additional conjugation by means of Fourier integral operators which eliminates the x_2-dependence in the symbol. Following [HS04], it will be convenient to construct the conjugating operator by viewing h and h^2/ε as two independent asymptotically small parameters, provided of course, that ε is not too small.

On the level of symbols, we write, using that $\widetilde{p}_1(x_2, \xi, \varepsilon) = \varepsilon q_1(x_2, \xi, \varepsilon)$,

$$\begin{aligned} \widetilde{P}_\varepsilon &= p(\xi_1) + \varepsilon(\langle q \rangle(\xi) + \mathcal{O}(\varepsilon) + h q_1(x_2, \xi, \varepsilon) + \frac{h^2}{\varepsilon}\widetilde{p}_2 + h\frac{h^2}{\varepsilon}\widetilde{p}_3 + \ldots) \\ &= p(\xi_1) + \varepsilon\left(r_0(x_2, \xi, \varepsilon, \frac{h^2}{\varepsilon}) + h r_1(x_2, \xi, \varepsilon, \frac{h^2}{\varepsilon}) + h^2 r_2 + \ldots \right), \end{aligned} \tag{3.9}$$

with

$$\begin{aligned} r_0(x_2, \xi, \varepsilon, \frac{h^2}{\varepsilon}) &= \langle q \rangle + \mathcal{O}(\varepsilon) + \frac{h^2}{\varepsilon}\widetilde{p}_2, \\ r_1(x_2, \xi, \varepsilon, \frac{h^2}{\varepsilon}) &= q_1(x_2, \xi, \varepsilon) + \frac{h^2}{\varepsilon}\widetilde{p}_3, \\ r_j(x_2, \xi, \varepsilon, \frac{h^2}{\varepsilon}) &= \frac{h^2}{\varepsilon}\widetilde{p}_{j+2}, \quad j \geq 2. \end{aligned}$$

In this work, we shall only be concerned with the case when

$$\frac{h^2}{\varepsilon} \leq \mathcal{O}(h^{\delta_1}), \tag{3.10}$$

for some fixed $\delta_1 > 0$. When $b_0 = b_0(x_2, \xi, \varepsilon, \frac{h^2}{\varepsilon})$ is such that $b_0 = \mathcal{O}(\varepsilon + h^2/\varepsilon)$ in the C^∞–sense, we consider the conjugated operator

$$e^{\frac{i}{h}B_0}\widetilde{P}_\varepsilon e^{-\frac{i}{h}B_0}, \quad B_0 = b_0(x_2, hD_x, \varepsilon, h^2/\varepsilon). \tag{3.11}$$

Since B_0 and $p(hD_{x_1})$ commute, we see that the symbol of the conjugated operator (3.11) is of the form

$$p(\xi_1) + \varepsilon\left(\widehat{r}_0 + h\widehat{r}_1 + \ldots\right),$$

where by Egorov's theorem,

$$\widehat{r}_0 = r_0 \circ \exp(H_{b_0}) = \sum_{k=0}^{\infty} \frac{1}{k!} H_{b_0}^k r_0,$$

while $\widehat{r}_j = \mathcal{O}(1)$ for $j \geq 1$. Since the canonical transformation $\exp(H_{b_0})$ is exact, we see that the conjugated operator acts on the space $L^2_f(\mathbf{T}^2)$ of Floquet periodic functions.

It follows that

$$\widehat{r}_0 = \langle q\rangle(\xi) + \mathcal{O}\left(\varepsilon + \frac{h^2}{\varepsilon}\right) - \partial_{\xi_2}\langle q\rangle \partial_{x_2} b_0 + \mathcal{O}\left((\varepsilon, \frac{h^2}{\varepsilon})^2\right),$$

and using that $\partial_{\xi_2}\langle q\rangle \neq 0$, it becomes clear how to construct a real-valued smooth symbol $b_0 = \mathcal{O}(\varepsilon + h^2/\varepsilon)$, defined near $\xi = 0$ in $T^*\mathbf{T}^2$, as a formal Taylor series in ε, h^2/ε, so that $\widehat{r}_0 = \langle q\rangle + \mathcal{O}(\varepsilon + h^2/\varepsilon)$ is independent of x, modulo $\mathcal{O}(h^\infty)$.

In what follows, we may therefore assume, for simplicity, that the conjugation by $e^{iB_0/h}$ has already been carried out, so that we are reduced to the operator $\widetilde{P}_\varepsilon$ of the form (3.9), where $r_0 = \langle q\rangle(\xi) + \mathcal{O}(\varepsilon + h^2/\varepsilon)$ is independent of x, and $r_j = \mathcal{O}(1)$, $j \geq 1$. To eliminate the x_2–dependence in the lower order terms, we could argue as in the previous step, making the terms r_j independent of x_2 one at a time, but here we would like to describe a slightly different method, which has the merit of being more direct. Let us look for a conjugation by a pseudodifferential operator of the form $e^{iB/h}$, where

$$B(x_2, \xi, \varepsilon, \frac{h^2}{\varepsilon}; h) = \sum_{\nu=1}^{\infty} h^\nu b_\nu(x_2, \xi, \varepsilon, \frac{h^2}{\varepsilon}).$$

The conjugated operator

$$e^{\frac{i}{h}B}\widetilde{P}_\varepsilon e^{-\frac{i}{h}B} = e^{\frac{i}{h}\,\mathrm{ad}\,B}\widetilde{P}_\varepsilon$$

can be expanded as follows,

$$p(\xi_1) + \varepsilon \sum_{k=0}^{\infty} \sum_{j_1=1}^{\infty} ... \sum_{j_k=1}^{\infty} \sum_{\ell=0}^{\infty} h^{\ell+j_1+..+j_k} \frac{1}{k!} \left(\frac{i}{h} \mathrm{ad}\ b_{j_1} \right) .. \left(\frac{i}{h} \mathrm{ad}\ b_{j_k} \right) r_\ell = p(\xi_1) + \varepsilon \sum_{n=0}^{\infty} h^n \widehat{r}_n. \tag{3.12}$$

Here $\widehat{r}_n$ is equal to the sum of all the coefficients for h^n coming from the expressions

$$h^{\ell+j_1+..+j_k} \frac{1}{k!} \left(\frac{i}{h} \mathrm{ad}\ b_{j_1} \right) .. \left(\frac{i}{h} \mathrm{ad}\ b_{j_k} \right) r_\ell, \tag{3.13}$$

with $\ell + j_1 + .. + j_k \leq n$ and $j_\nu \geq 1$. Then $\widehat{r}_0 = r_0$, $\widehat{r}_1 = r_1 + H_{b_1} r_0 = r_1 - H_{r_0} b_1$,..,$\widehat{r}_n = r_n - H_{r_0} b_n + s_n$, where s_n only depends on $b_1, ..., b_{n-1}$ and is the sum of all coefficients of h^n arising in the expressions (3.13) with $\ell + j_1 + .. + j_k \leq n$, $j_1, .., j_k, \ell < n$, $j_\nu \geq 1$.

It is therefore clear how to find $b_1, b_2, \ldots$ successively with $b_j = \mathcal{O}(1)$, such that all the coefficients $\widehat{r}_j$ in (3.12) are independent of x and $= \mathcal{O}(1)$.

The discussion in this section may be summarized in the following theorem, which is the main result of this chapter.

Theorem 3.9. *Let us make all the assumptions of Section (3.2) and let $F_0 \in \mathbf{R}$ be a regular value of $\langle q \rangle$ viewed as a function on $p^{-1}(0)$. Assume that the Lagrangian manifold*

$$\Lambda_{0,F_0} : p = 0, \quad \langle q \rangle = F_0$$

is connected. When γ_1 and γ_2 are the fundamental cycles in Λ_{0,F_0} with γ_1 corresponding to a closed H_p–trajectory, we write $S = (S_1, S_2)$ and $\alpha = (\alpha_1, \alpha_2)$ for the actions and the Maslov indices of the cycles, respectively. Assume furthermore that $\varepsilon = \mathcal{O}(h^\delta)$ is such that $h^2/\varepsilon \leq \mathcal{O}(h^\delta)$, for some $\delta > 0$ fixed. There exists a Lagrangian torus $\widehat{\Lambda}_{0,F_0} \subset T^*\mathbf{R}^2$, which is an $\mathcal{O}(\varepsilon)$–perturbation of Λ_{0,F_0} in the C^∞–sense, and an h-Fourier integral operator

$$U = \mathcal{O}(1) : L^2(\mathbf{R}^2) \to L^2_f(\mathbf{T}^2),$$

which has the following properties:

1. *The operator U is microlocally invertible near $\widehat{\Lambda}_{0,F_0}$: there exists an operator $V = \mathcal{O}(1) : L^2_f(\mathbf{T}^2) \to L^2(\mathbf{R}^2)$ such that for every $\chi_1 \in C_0^\infty(\mathrm{neigh}(\widehat{\Lambda}_{0,F_0}, T^*\mathbf{R}^2))$, we have*

$$(VU - 1)\, \chi_1(x, hD_x) = \mathcal{O}(h^\infty) : L^2(\mathbf{R}^2) \to L^2(\mathbf{R}^2). \tag{3.14}$$

For every $\chi_2 \in C_0^\infty(\mathrm{neigh}(\xi = 0, T^*\mathbf{T}^2))$*, we have*

$$(UV - 1)\, \chi_2(x, hD_x) = \mathcal{O}(h^\infty) : L_f^2(\mathbf{T}^2) \to L_f^2(\mathbf{T}^2).$$

2. *We have Egorov's theorem: Acting on* $L_f^2(\mathbf{T}^2)$*, there exists* $\widehat{P}\left(hD_x, \varepsilon, \frac{h^2}{\varepsilon}; h\right)$ *with the symbol*

$$\widehat{P}\left(\xi, \varepsilon, \frac{h^2}{\varepsilon}; h\right) \sim p(\xi_1) + \varepsilon \sum_{j=0}^{\infty} h^j r_j\left(\xi, \varepsilon, \frac{h^2}{\varepsilon}\right), \quad |\xi| \le \frac{1}{\mathcal{O}(1)},$$

$$r_0 = \langle q \rangle(\xi) + \mathcal{O}\left(\varepsilon + \frac{h^2}{\varepsilon}\right),$$

and

$$r_j = \mathcal{O}(1), \quad j \ge 1,$$

such that $\widehat{P}U = UP_\varepsilon$ *microlocally near* $\widehat{\Lambda}_{0,F_0}$*, i.e.*

$$\left(\widehat{P}U - UP_\varepsilon\right) \chi_1(x, hD_x) = \mathcal{O}(h^\infty), \quad \chi_2(x, hD_x)\left(\widehat{P}U - UP_\varepsilon\right) = \mathcal{O}(h^\infty),$$

for every χ_1*,* χ_2 *as above.*

Remark. In an upcoming work, consequences of Theorem 3.9 for the spectral analysis of the family of selfadjoint operators P_ε will be explored, under the assumption that $\varepsilon \ll h$. Notice that this assumption guarantees that the spectrum of the family P_ε enjoys a cluster structure, in view of Theorem 3.

Remark. Assume that the spectrum of P clusters into intervals of size $\le \mathcal{O}(1)h^{N_0}$, for some integer $N_0 \ge 2$. Following Section 12 of [HS08], in this case we expect to be able to extend the normal form construction of Theorem 3.9 to the range $h^{N_0} \ll \varepsilon \le h^\delta$. This will also be the subject of future work, and let us presently merely mention that the examples of selfadjoint operators with periodic classical flow for which the spectral clusters are of size $\mathcal{O}(h^\infty)$ (in fact, 0) include the semiclassical Laplacian on a compact rank one symmetric space [Gui78] and the two-dimensional harmonic oscillator with rationally dependent frequencies.

CHAPTER 4

Black Holes for Stationary Metrics in 2 Space Dimensions

4.1 Introduction

Consider the wave equation associated to a stationary metric on a cylindrical domain $\mathbf{R}\times\Omega \subseteq \mathbf{R}^{1+2} \cong \mathbf{R}^1_{x_0} \times \mathbf{R}^2_{(x_1,x_2)}$, which takes the form

$$\sum_{i,j=0}^{2} \frac{1}{\sqrt{g(x)}} \frac{\partial}{\partial x_i} \left(\sqrt{g(x)} g^{ij}(x) \frac{\partial u(x_0, x)}{\partial x_j} \right) = 0, \quad \vec{x} = (x_0, x) \in \mathbf{R}^{1+2}. \tag{4.1}$$

Here $g_{ij}(x) \in C^\infty(\mathbf{R}^{1+2}; \mathbf{R})$ defines a pseudo-Riemannian metric with signature $(+1, -1, -1)$ and depends only on x, with $g_{ij}(x) = g_{ji}(x)$, and $g(x) = \det[g^{ij}(x)]^{-1}$ where $(g^{ij}(x))^2_{i,j=0}$ is the inverse of the metric tensor $(g_{ij}(x))^2_{i,j=0}$.

It is well known that equations of the form (4.1) may have black holes; i.e regions which disturbances may not propagate out of. In general we call these *artificial* or *analogue* black holes the metric may not be a solution of the Einstein equations of general relativity.

Two of the most famous examples arising from physical models are *optical black holes* and *acoustic black holes*. In optics, the propagation of light in a moving medium can be modeled by an equation of the form (4.1), while in acoustics, the propagation of sound waves in a moving medium may be described by such an equation. In this work we will primarily study the acoustic model, albeit with relaxed assumptions on the "fluid flow".

Physicists are interested in the study of physical systems which may contain artificial black holes, as they may be suitable for experimental study while retaining enough similarities to provide insight into gravitational black holes. See the surveys [BLV05],[NVV02] for example.

4.1.1 Analogy with Relativity

In the present context, we define an *event horizon* to be a simple closed curve $S \subseteq \mathbf{R}^2$ such that forward null-geodesics either can not pass from the interior to the exterior of S, or vice versa. In the former case we will say that the region enclosed by S is a *black hole*, and in the latter case call it a *white hole*. It is easy to see that such a curve may only exist in the ergoregion, and must satisfy that $\mathbf{R} \times S$ is characteristic for (4.1).

We recall that a *bicharacteristic* curve for the wave equation (4.1) is a curve $(\vec{x}(t), \vec{\xi}(t))$ satisfying

$$\dot{x}_j = \sum_{k=0}^{2} g^{jk}(x)\xi_k, \quad \dot{\xi}_j = -\sum_{k=0}^{2} g^{jk}_{x_j}(x)\xi_j\xi_k, \quad j = 0, 1, 2.$$

Such a curve is called a *null-bicharacteristic* if in addition we have

$$\sum_{j,k=0}^{2} g^{jk}(x)\xi_j\xi_k = 0. \tag{4.2}$$

Note that if (4.2) holds at one time, then it holds at all other times as well. A *null-geodesic* is a curve $\vec{x}(s)$ which is the projection onto $\vec{x}$ of a null-bicharacteristic, and satisfies $\sum_{j,k=0}^{2} g_{jk}(x)\frac{dx_j}{ds}\frac{dx_k}{ds} = 0$. A *forward null-geodesic* is a geodesic satisfying $\dot{x}_0 > 0$.

With g denoting a spactime metric satisfying the assumptions as given after equation (4.1), set $\Delta(x) = g^{11}g^{22} - (g^{12})^2$. We define the *ergoregion* to be the region where $\Delta < 0$. Assume the boundary $\Delta = 0$, called the *ergosphere*, is a smooth, simple closed curve. Throughout, the ergoregion will contain a trapped surface, i.e. a stationary non-characteristic surface which forward null-geodesics may only cross in one direction.

In Eskin [Esk10], it was shown that an ergoregion containing a trapped surface must contain a black hole or a white hole if either the ergosphere is nowhere-characteristic, or is itself characteristic. In this chapter, we will discuss metrics where instead there are several isolated characteristic points on the ergosphere. As we will show, one may still have black holes, with a more complicated description of the dynamics of null-geodesics in the ergoregion.

4.1.2 Acoustic Metrics

We now assume that g is an *acoustic metric*; that is, we take a vector field $\mathbf{v} = (v_1, v_2) \in C^\infty(\mathbf{R}^2; \mathbf{R}^2)$, and define a metric by the formula

$$(g_{ij})_{i,j=0}^2 = \begin{bmatrix} 1 - |\mathbf{v}|^2 & v_1 & v_2 \\ v_1 & -1 & 0 \\ v_2 & 0 & -1 \end{bmatrix} \tag{4.3}$$

with respect to our global coordinate system (x_0, x). When $\mathbf{v}$ is the velocity field of a barotropic, inviscid, irrotational fluid flow, then the propagation of sound waves may be modeled by the wave equation (4.1) for a metric with the form (4.3) (see [Vis98]). Here we do not require that $\mathbf{v}$ satisfy these physical hypotheses, and for simplicity we have not included physical constants, which amounts to formally assuming the fluid density and speed of sound are equal to 1.

The inverse of the metric tensor is given by

$$(g^{ij})_{i,j=0}^2 = \begin{bmatrix} 1 & v_1 & v_2 \\ v_1 & v_1^2 - 1 & v_1 v_2 \\ v_2 & v_1 v_2 & v_2^2 - 1 \end{bmatrix}.$$

Thus we see that for an acoustic metric, the ergoregion is where $1 - |\mathbf{v}|^2 < 0$, i.e. where the speed of the fluid is supersonic.

Remark 4.1. Without making further assumptions on $\mathbf{v}$ such as those mentioned above, many other metrics may take the form (4.3) in an appropriate coordinate system, including the Schwarzschild metric.

4.1.3 Acoustic Metrics in Polar Coordinates

We consider an acoustic metric defined by the velocity field

$$\mathbf{v} = \frac{A}{r}\hat{\mathbf{r}} + \frac{B}{r}\hat{\theta}.$$

The case where $A < 0$ and B are constants is a physical model of fluid swirling into a drain [BLV05]. See also [Esk10]. We will take the same form, but allow A, B to vary. In polar coordinates, the inverse of the metric tensor is given by

$$(g^{ij})_{i,j=0}^2 = \begin{bmatrix} 1 & A/r & B/r^2 \\ A/r & A^2/r^2 - 1 & AB/r^3 \\ B/r^2 & AB/r^3 & B^2/r^4 - 1/r^2 \end{bmatrix}. \tag{4.4}$$

Recall that a black hole or white hole must be a closed characteristic curve. In polar coordinates, we see from (4.4) that a characteristic curve given by $S(r,\theta) = 0$ must satisfy

$$\left(\frac{A^2}{r^2} - 1\right)\left(\frac{\partial S}{\partial r}\right)^2 + 2\frac{AB}{r^3}\frac{\partial S}{\partial r}\frac{\partial S}{\partial \theta} + \left(\frac{B}{r^4} - \frac{1}{r^2}\right)\left(\frac{\partial S}{\partial \theta}\right)^2 = 0,$$

which gives

$$\frac{\partial S}{\partial \theta} = \frac{-\frac{AB}{r^3} \pm \sqrt{\frac{(AB)^2}{r^6} - \left(\frac{A^2}{r^2} - 1\right)\left(\frac{B^2}{r^4} - \frac{1}{r^2}\right)}}{\frac{B^2}{r^4} - \frac{1}{r^2}}\frac{\partial S}{\partial r},$$

or

$$\frac{\partial S}{\partial \theta} = \frac{-\frac{AB}{r} \pm \sqrt{A^2 + B^2 - r^2}}{\frac{B^2}{r^2} - 1}\frac{\partial S}{\partial r}.$$

From the latter condition we may construct a pair of vector fields $\vec{V}^{\pm} = (V_1^{\pm}, V_2^{\pm})$ whose integral curves are characteristic curves,

$$\frac{dr^{\pm}}{ds} = V_1^{\pm} = \frac{AB}{r} \mp \sqrt{A^2 + B^2 - r^2} \tag{4.5}$$

$$\frac{d\theta^{\pm}}{ds} = V_2^{\pm} = \frac{B^2}{r^2} - 1. \tag{4.6}$$

Scaling the vector fields $\vec{V}^{\pm}$ by $\dfrac{\frac{AB}{r} \pm \sqrt{A^2+B^2-r^2}}{\frac{B^2}{r^2}-1}$ yields another vector field $\vec{W}^{\pm} = (W_1^{\pm}, W_2^{\pm})$,

$$\frac{dr^{\pm}}{ds} = W_1^{\pm} = A^2 - r^2 \tag{4.7}$$

$$\frac{d\theta^{\pm}}{ds} = W_2^{\pm} = \frac{AB}{r} \pm \sqrt{A^2+B^2-r^2}. \tag{4.8}$$

Note that the scaling factor is sometimes positive, sometimes negative, and we have $\vec{V}^{\pm} = \vec{0}$ whenever $|B| = r$ and $\operatorname{sgn} A = \pm \operatorname{sgn} B$, while $\vec{W}^{\pm} = \vec{0}$ whenever $|A| = r$ and $\operatorname{sgn} A = \pm \operatorname{sgn} B$. However, the vector fields $\vec{V}^{\pm}$ and $\vec{W}^{\pm}$ have the same integral curves up to reparameterization and concatenation. It was shown in [Esk10] that it is always possible to construct a pair of nonvanishing characteristic vector fields, so there is no ambiguity in referring to two globally defined vector fields, but we will not write down an explicit formula.

4.1.4 Change of variables near the ergosphere

We introduce the new variable ρ given by $\rho^2 = A^2 + B^2 - r^2$ for $r \leq A^2 + B^2$. Noting that

$$\frac{B^2}{r^2} - 1 = \frac{B^2 - r^2}{r^2} = \frac{\rho^2 - A^2}{r^2} = \frac{\rho^2 - A^2}{A^2 + B^2 - \rho},$$

we obtain

$$\begin{aligned} 2\rho\frac{d\rho}{d\theta} &= \frac{d}{d\theta}[A^2+B^2] - 2r\frac{dr}{d\theta} \\ &= \frac{d}{d\theta}[A^2+B^2] - 2r^2\frac{AB \mp r\rho}{\rho - A^2}. \end{aligned}$$

In the new coordinates (ρ, θ) and after another rescaling, (4.5)-(4.6) become

$$\frac{d\rho^{\pm}}{ds} = F_1^{\pm} = \frac{r^2}{\rho^2 - A^2}\left[\frac{1}{2}\frac{d}{ds}[A^2+B^2] - (AB \mp r\rho)\right] \tag{4.9}$$

$$\frac{d\theta^{\pm}}{ds} = F_2^{\pm} = \rho. \tag{4.10}$$

Remark 4.2. Due to the degeneracy of the change of variables near the ergosphere, critical points of $\vec{F}^{\pm}$ do not (in general) correspond to critical points in (r, θ), but rather to characteristic points of the ergosphere.

As functions of $\rho \geq 0$ and θ, $\vec{F}^{\pm}$ are smooth in a neighborhood of $\rho = 0$ if A is bounded below for ρ small. Depending on the formulas for A and B, they may extend smoothly to $\rho < 0$. In particular, this is the case when A, B depend only on θ. In such a situation, we may analyze the behavior near critical points of $\vec{F}^{\pm}$ as the restriction to a half-space of the behavior near a regular critical point. Standard results in dynamical systems about structural stability and normal form theory then apply to the extension.

4.2 Homogeneous Flows

Let A, B depend on θ only, $A = A(\theta)$, $B = B(\theta)$.

In (r, θ) coordinates, (4.5)-(4.6), (4.7)-(4.8) retain the same expressions, while in (ρ, θ) coordinates, we obtain

$$\frac{d\rho^{\pm}}{ds} = F_1^{\pm} = AA_\theta + BB_\theta - \frac{r^2}{\rho^2 - A^2}(AB \mp r\rho) \tag{4.11}$$

$$\frac{d\theta^{\pm}}{ds} = F_2^{\pm} = \rho. \tag{4.12}$$

Then $\vec{F}^{\pm}$ extends smoothly to ρ negative and sufficiently close to 0. A crucial role will be played by the equilibrium points of $\vec{F}^{\pm}$, which only occur when $\rho = 0$, i.e. on the ergosphere. For analyzing the linearization at critical points, we compute derivatives of $\vec{F}^{\pm}$.

$$(F_1^{\pm})_\rho = \frac{-2rr_\rho}{\rho^2 - A^2}(AB \mp r\rho) + \frac{r^2}{(\rho^2 - A^2)^2}(2\rho)(AB \mp r\rho) - \frac{r^2}{\rho^2 - A^2}(\mp 1)(r + \rho r_\rho) \tag{4.13}$$

$$\begin{aligned}(F_1^{\pm})_\theta = (AA_\theta + BB_\theta)_\theta - \frac{AA_\theta + BB_\theta}{\rho^2 - A^2}(AB \mp r\rho) - \frac{r^2}{(\rho^2 - A^2)^2}(-AA_\theta)(AB \mp r\rho) \\ - \frac{r^2}{\rho^2 - A^2}(AB_\theta + A_\theta B \mp r_\theta \rho)\end{aligned} \tag{4.14}$$

while $(F_2^{\pm})_\rho = 1$, $(F_2^{\pm})_\theta = 0$.

When $\rho = 0$, we have $r_\rho = -\frac{\rho}{r} = 0$, and (4.13)-(4.14) simplify to

$$(F_1^{\pm})_\rho = \mp \frac{r^3}{A^2} \tag{4.15}$$

$$(F_1^{\pm})_\theta = (AA_\theta + BB_\theta)_\theta + (AA_\theta + BB_\theta)\frac{B}{A} + \frac{A^2 + B^2}{A^2}(AB_\theta + 2A_\theta B). \tag{4.16}$$

Next, we consider several explicit choices for A, B, and investigate the phase portraits of the associated vector fields.

4.2.1 The case $A = A_0 < 0$, $B = B_0 \cos\theta$

Consider the case where $A = A_0 < 0$ is a constant, $B = B_0 \cos\theta$, where $B_0 > 0$.

In (r, θ), (4.5)-(4.6) become

$$\frac{dr^\pm}{ds} = V_1^\pm = \frac{A_0 B_0 \cos\theta}{r} \mp \sqrt{A_0^2 + B_0^2 \cos^2\theta - r^2} \tag{4.17}$$

$$\frac{d\theta^\pm}{ds} = V_2^\pm = \frac{B_0^2 \cos^2\theta}{r^2} - 1, \tag{4.18}$$

and (4.7)-(4.8) become

$$\frac{dr^\pm}{ds} = W_1^\pm = A_0^2 - r^2 \tag{4.19}$$

$$\frac{d\theta^\pm}{ds} = W_2^\pm = \frac{A_0 B_0 \cos\theta}{r} \pm \sqrt{A_0^2 + B_0^2 \cos^2\theta - r^2}, \tag{4.20}$$

while (4.11)-(4.12) become

$$\frac{d\rho^\pm}{ds} = F_1^\pm = -B_0^2 \sin\theta\cos\theta - \frac{r^2}{\rho^2 - A_0^2}(A_0 B_0 \cos\theta \mp r\rho) \tag{4.21}$$

$$\frac{d\theta^\pm}{ds} = F_2^\pm = \rho. \tag{4.22}$$

From (4.21)-(4.22) we get critical points when

$$\rho = 0 \quad \text{and} \quad \left[B_0 \cos\theta = 0 \quad \text{or} \quad -B_0 \sin\theta - \frac{r^2}{(-A_0^2)} A_0 = 0 \right]$$

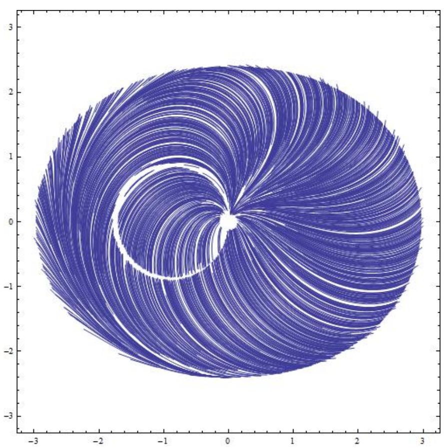
$$\iff [r = |A_0|, \theta = \pm\pi/2] \quad \text{or} \quad \left[r^2 = A_0 B_0 \sin\theta = A_0^2 + B_0^2 \cos^2\theta \right].$$

In the case where $\theta \neq \pm\pi/2$, we compute

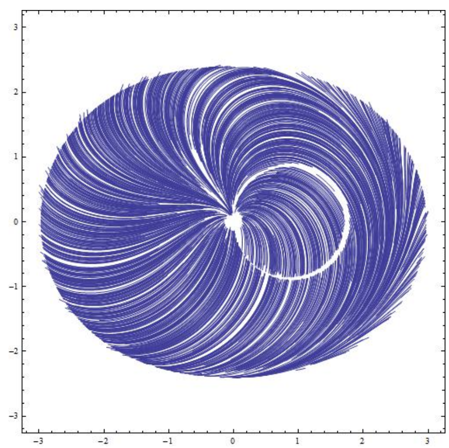
$$A_0^2 + B_0^2(1 - \sin^2\theta) = A_0 B_0 \sin\theta \tag{4.23}$$

$$\sin^2\theta + \frac{A_0}{B_0}\sin\theta - \left(\frac{A_0^2}{B_0^2} + 1 \right) = 0 \tag{4.24}$$

$$\sin\theta = \frac{1}{2}\left[-\left(\frac{A_0}{B_0}\right) \pm \sqrt{5\left(\frac{A_0}{B_0}\right)^2 + 4} \right]. \tag{4.25}$$



(a) Trajectories for r^+, θ^+



(b) Trajectories for r^-, θ^-

Figure 4.1: An assortment of numerically simulated trajectories for (4.17)-(4.18), where $A = A_0$, $B = B_0 \cos\theta$, with $A_0 = -2.4$, $B_0 = 1.75$. Numerical artifacts appear where $r = |B|, \mp B > 0$ due to the vanishing of the vector fields (4.17)-(4.18). [1]

When $|A_0| > |B_0|$, (4.25) has no solutions, so there are only the two critical points at $\theta = \pm\pi/2$. When $|A_0| < |B_0|$, (4.25) provides an additional two solutions for a total of four critical points.

- When $\rho = 0$, $\theta = \pi/2$, we have by (4.15)-(4.16)

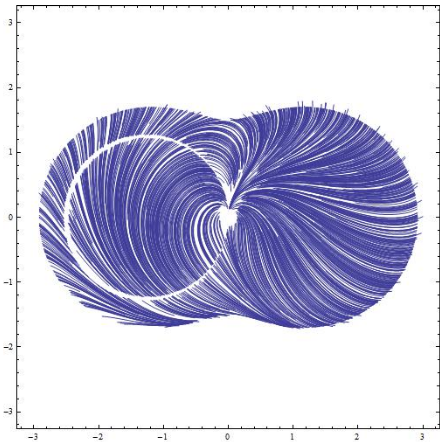
$$\nabla_{\rho,\theta}\vec{F}^{\pm} = \begin{bmatrix} \mp\frac{r^3}{A_0^2} & B_0^2 - A_0B_0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mp A_0 & B_0^2 - A_0B_0 \\ 1 & 0 \end{bmatrix},$$

  which has determinant $(A_0 - B_0)B_0 < 0$ and is thus a saddle point. The eigenvalues are $\mp B_0$, $\mp(B_0 - A_0)$ with eigenvectors $\begin{bmatrix} \mp B_0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} \mp(B_0 - A_0) \\ 1 \end{bmatrix}$. Examining the critical points:

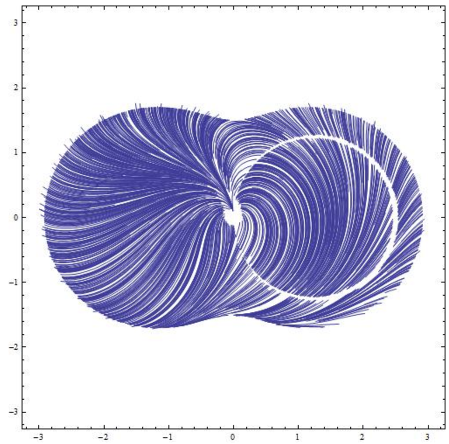
- When $\rho = 0$, $\theta = -\pi/2$, we have

$$\nabla_{\rho,\theta}\vec{F}^{\pm} = \begin{bmatrix} \mp\frac{r^3}{A_0^2} & B_0^2 + A_0B_0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mp A_0 & B_0^2 + A_0B_0 \\ 1 & 0 \end{bmatrix}$$

[1]Figures created using Mathematica [Mat10].



(a) Trajectories for r^+, θ^+



(b) Trajectories for r^-, θ^-

Figure 4.2: An assortment of numerically simulated trajectories for (4.17)-(4.18), where $A = A_0$, $B = B_0 \cos\theta$, with $A_0 = -1.5$, $B_0 = 2.5$. Numerical artifacts appear where $r = |B|, \mp B > 0$ due to the vanishing of the vector fields (4.17)-(4.18).

which has determinant $-B_0(A_0 + B_0) \begin{cases} < 0, & |A_0| < |B_0| \\ > 0, & |A_0| > |B_0| \end{cases}$, trace squared A_0^2, and discriminant $\mathcal{D} = A_0^2 + 4(B_0^2 + A_0 B_0) = (A_0 + 2B_0)^2 \geq 0$. This is a node (two real eigenvalues of the same sign) when $|A_0| > |B_0|$ and a saddle when $|A_0| < |B_0|$. The eigenvalues are $\mp B_0$, $\pm(A_0 + B_0)$ with eigenvectors $\begin{bmatrix} \mp B_0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} \pm(A_0 + B_0) \\ 1 \end{bmatrix}$, respectively.

- When $|A_0| < |B_0|$ and $\theta \neq \pm\pi/2$, $\rho = 0$, and $r^2 = A_0 B_0 \sin\theta$ ($\implies \sin\theta < 0$),

$$\nabla_{\rho,\theta} \vec{F}^{\pm} = \begin{bmatrix} \mp \frac{r^3}{A_0^2} & -B_0^2 \cos^2\theta - 2\frac{B_0^3}{A_0} \sin\theta \cos^2\theta \\ 1 & 0 \end{bmatrix},$$

which has determinant $B_0^2 \cos^2\theta + 2\frac{B_0^3}{A_0} \sin\theta \cos^2\theta \geq 0$. For $|B_0|/|A_0|$ slightly larger than 1, the two such critical points are nodes. One may verify numerically that for larger values of $|B_0|/|A_0|$ they are spirals (two imaginary eigenvalues).

In the case where $|A_0| > |B_0|$, let us sketch in detail the qualitative picture for the (+) family. By symmetry, for the (−) family the picture is reflected in the vertical axis.

(a) (ρ, θ) (b) (r, θ)

Figure 4.3: The qualitative picture near the node $r = |A_0|$, $\theta = -\pi/2$.

(a) (ρ, θ) (b) (r, θ)

Figure 4.4: The qualitative picture near the saddle point $r = |A_0|$, $\theta = \pi/2$.

By inspecting the equations (4.19)-(4.20), we see that the circle $r = |A_0|$ is characteristic. The semicircle $r = |A_0|$, $x \leq 0$ is a trajectory of the (+) family and $r = |A_0|$, $x \geq 0$ is a trajectory of the (−) family. The endpoints of these trajectories are where the circle is tangent to the ergosphere, at the two characteristic points $r = |A_0|$, $\theta = \pm\pi/2$.

Inside $r = |A_0|$, trajectories for both the (+) and (−) families converge to the singularity, as do all trajectories starting where $x > 0$ in the (+) family and all trajectories with $x < 0$ in the (−) family.

For the (+) family, there is a node in (ρ, θ) coordinates at $r = |A_0|$, $\theta = -\pi/2$, from which emerges a trajectory of (4.17)-(4.18) following the circle $r = |A_0|$ clockwise until it reaches $\theta = \pi/2$. Since $r = |A_0|$, $\cos\theta < 0$ implies $\rho = |B| = -B_0 \cos\theta = -B_0(\theta + \pi/2) + \mathcal{O}((\theta + \pi/2)^3)$, in the transformed coordinates this trajectory is always tangent to the direction corresponding to the eigenvalue $-B_0$ as calculated above (which should be $+B_0$ to have forward null-geodesics, as is apparent by comparing with (4.7)-(4.8)).

The $A_0 + B_0$ eigendirection is stronger than the $-B_0$ eigendirection when $|A_0| > 2|B_0|$, and is then pointed more toward the center of the circle. In this case, there are trajectories traveling from the node to the singularity. When $|B_0| < |A_0| < 2|B_0|$, the $-B_0$ direction is stronger and all trajectories coming out of the node stay outside $r = |A_0|$.

At $\theta = \pi/2$, in (ρ, θ) coordinates we found above that there is a saddle point. The stable trajectory for the saddle must follow the curve $r = |A_0|$, $x \leq 0$, and as in the previous paragraph it corresponds to the eigenvalue $-B_0$.

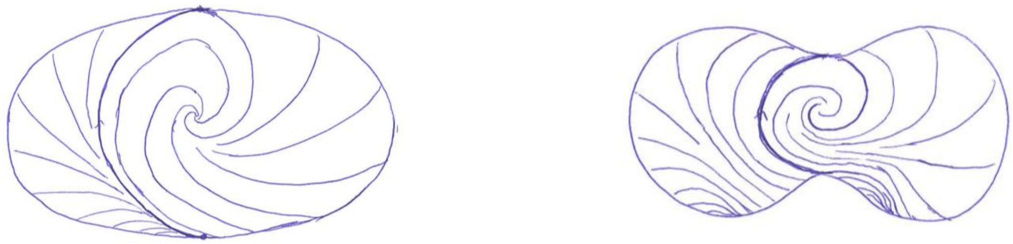
It is now clear how to take the proper sign in each vector field, and we see that the $(+)$ family points out of the ergosphere when $x \leq 0$, and into it when $x \geq 0$. Trajectories of points between $r = |A_0|$ and the ergosphere when $x \leq 0$ have no choice but to exit the ergoregion as time increases. As time decreases, they must converge to the node. Trajectories in the region where $r < |A_0|$ or $x > 0$ must converge to the singularity.

Remark 4.3. In both families, at the point $r = |A_0|$, $\theta = -\pi/2$, we have nonuniqueness of solutions for (4.17)-(4.18), with a fan of many possible trajectories originating at one point.

Remark 4.4. Consider now a small perturbation of A, for example $A = A_0(1+\varepsilon\cos\theta)$. Since the formulas for $\vec{F}^{\pm}$ change continuously, we still have two critical points, both lying on the ergosphere. By structural stability ([AP37] or [Kuz04], Theorem 2.5), the phase portraits for the $(+)$ and $(-)$ families have the same qualitative pictures, even in (ρ, θ) coordinates. Each of these families contains a trajectory which makes up one half of a black hole, whose existence may not be easy to observe by other means. We know that these two curves are tangent to the ergosphere, so the event horizon will be at least C^1, but in general it may not be C^2, so the black hole is not a C^∞ curve in the general case.

Now let us analyze the phase portrait when $|A_0| < |B_0|$. Again, by symmetry, the picture for the $(-)$ family is just the reflection of that for the $(+)$ family.

In this case, our analysis of the critical points shows that we have saddles at $\theta = \pm\pi/2$, as well as two additional critical points, which we shall assume are nodes by taking $|B_0|/|A_0|$ sufficiently close to 1. We again find that $r = |A_0|$ is characteristic, being made up of one



(a) $|A_0| > 2|B_0|$.

(b) $|A_0| < |B_0|$.

Figure 4.5: The qualitative picture for the (+) family.

trajectory of each of the (+) and (−) families, and that trajectories of points inside $r = |A_0|$ must converge to the singularity.

The (+) family, given the proper sign, alternates pointing into and out of the ergosphere as we pass the four critical points. Between $\theta = -\pi/2$ and the node clockwise around the ergosphere it points inward. As in the case when $|A_0| > |B_0|$, trajectories in the region outside $r = |A_0|$ with $x \leq 0$ have no choice but to exit the ergoregion as x_0 increases. When $x \geq 0$, we now have a family of trajectories that enter and exit the ergosphere. Indeed, if we consider the orbit which converges to $\theta = -\pi/2$ as $x_0 \to +\infty$, then as we follow it backwards in time, it has no choice but to exit the ergosphere somewhere in the segment of the ergosphere between $\theta = \pi/2$ and the critical point immediately clockwise. Then this trajectory is a separatrix for other trajectories. Trajectories to one side must converge to the singularity as $x_0 \to +\infty$, while to the other side they must exit in the segment of the ergosphere between $\theta = -\pi/2$ and the critical point immediately counterclockwise.

All together, we see that there is still a black hole. However, in this case there is a saddle-saddle connection, so the dynamics are unstable under a general perturbation, though the black hole may still be stable under special kinds of perturbations.

4.2.2 The case $A = A_0 < 0$, $B = B_0 \cos^2\theta$

Consider the case when $A = A_0 < 0, B = B_0 \cos^2\theta$. Then (4.11)-(4.12) become

$$\frac{d\rho^\pm}{ds} = F_1^\pm = -2B_0^2 \sin\theta\cos^3\theta - \frac{A_0^2 + B_0^2\cos^4\theta - \rho^2}{\rho^2 - A_0^2}(A_0B_0\cos^2\theta \mp r\rho) \tag{4.26}$$

$$\frac{d\theta^\pm}{ds} = F_2^\pm = \rho. \tag{4.27}$$

From (4.26)-(4.27) we get critical points when

$$\rho = 0 \quad \text{and} \quad \left[-2B_0^2\sin\theta\cos^3\theta + \frac{r^2}{A_0^2}A_0B_0\cos^2\theta = 0\right]$$

$$\iff [r = |A_0|, \theta = \pm\pi/2] \quad \text{or} \quad \left[r^2 = 2A_0B_0\sin\theta\cos\theta = A_0^2 + B_0^2\cos^4\theta\right].$$

When $|B_0|/|A_0|$ is small, it is clear that there are only the two critical points at $r = |A_0|$, $\theta = \pm\pi/2$, while one can verify numerically that four new critical points appear for larger values.

We have by (4.16),

$$(F_1^\pm)_\theta = \\ -2B_0^2\cos^4\theta + 6B_0^2\cos^2\theta\sin^2\theta + \frac{B_0^3}{A_0}\sin\theta\cos^5\theta + \left(\frac{A_0^2 + B_0^2\cos^4\theta}{A_0}\right)(-2B_0\sin\theta\cos\theta),$$

which means that the critical points at $\theta = \pm\pi/2$ are degenerate in (ρ, θ) coordinates.

However, it will not be necessary to analyze them in more detail.

In (r, θ), (4.5)-(4.6) become

$$\frac{dr^\pm}{ds} = V_1^\pm = \frac{A_0B_0\cos^2\theta}{r} \mp \sqrt{A_0^2 + B_0^2\cos^4\theta - r^2} \tag{4.28}$$

$$\frac{d\theta^\pm}{ds} = V_2^\pm = \frac{B_0^2\cos^4\theta}{r^2} - 1, \tag{4.29}$$

(4.7)-(4.8) become

$$\frac{dr^\pm}{ds} = W_1^\pm = A_0^2 - r^2 \tag{4.30}$$

$$\frac{d\theta^\pm}{ds} = W_2^\pm = \frac{A_0B_0\cos^2\theta}{r} \pm \sqrt{A_0^2 + B_0^2\cos^4\theta - r^2}. \tag{4.31}$$

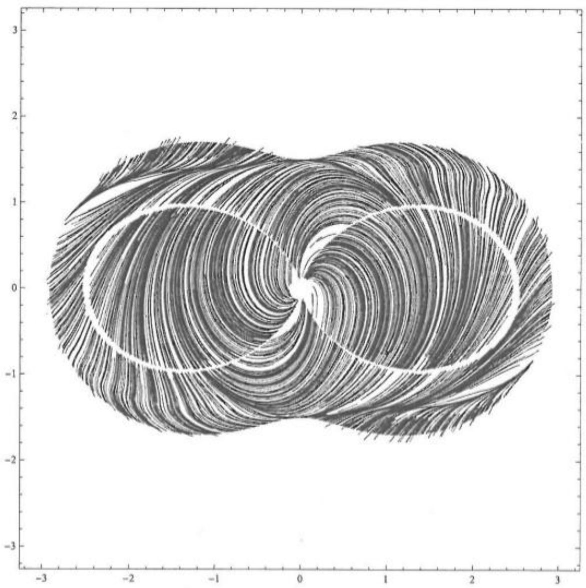


Figure 4.6: An assortment of numerically simulated trajectories for the $(-)$ family in (4.28)-(4.29), where $A = A_0$, $B = B_0 \cos^2\theta$, with $A_0 = -2.4$, $B_0 = 1.75$. Numerical artifacts appear where $r = |B|$ due to the vanishing of the vector fields (4.28)-(4.29).

When $|A_0| = r$, we have $W_2^{\pm} = -B_0 \cos^2\theta \pm |B_0 \cos^2\theta|$. For the $(+)$ family, this vanishes identically, but for the $(-)$ family, it only vanishes when $\theta = \pm\pi/2$. Thus for the $(-)$ family, the entire circle $|A_0| = r$ is a trajectory. The $(+)$ family points nontangentially into the ergosphere everywhere except at $\theta = \pm\pi/2$. Therefore, trajectories of the $(+)$ family must converge to the singularity as $x_0 \to +\infty$.

4.3 Conclusions

We have exhibited a class of examples of stationary spacetime metrics with black holes in 2 space dimensions. These examples demonstrate the possibility to have (artificial) black holes which have an interesting interplay with the velocity field and the dynamics of null geodesics, leading to a complicated structure of the ergoregion. In particular, the event horizon is tangent to the ergosphere at several characteristic points, and consists of several smooth pieces, which may not in general form a C^2 curve. In a future work, we plan to expand our range of examples, and further study the stability of such black holes under perturbations.

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