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SANTA CRUZ

**ANALYSIS, DESIGN OF CONTROL ALGORITHMS AND
APPLICATIONS FOR FORWARD INVARIANCE OF HYBRID
SYSTEMS**

A dissertation submitted in partial satisfaction of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

in

COMPUTER ENGINEERING

by

Jun Chai

December 2018

The Dissertation of Jun Chai
is approved:

Professor Ricardo G. Sanfelice, Chair

Professor Gabriel Hugh Elkaim

Professor Qi Gong

Lori Kletzer
Vice Provost and Dean of Graduate Studies

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List of Symbols

\mathbb{B}	The closed unit ball, of appropriate dimension, in the Euclidean norm
\mathbb{N}	The set of all positive integers including zero, i.e., $\{0, 1, 2, \dots\}$
\mathbb{R}	The set of all real numbers
$\mathbb{R}_{\geq 0}$	The set of all non-negative real numbers
$\text{int}K$	The set of interior points of K
$\overline{\text{con}}(K)$	The convex hull of a set K
\overline{K}	The closure of a set $K \subset \mathbb{R}^n$ defined by the intersection of all closed sets containing K
∂K	The set of boundary points of a set K
$T_K(x)$	The tangent cone of the closed set K at a point $x \in K$
x^\top	The transpose of x
(ν, w)	Given vectors $\nu \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $[\nu^\top \ w^\top]^\top$ is equivalent to (ν, w)
$\langle u, v \rangle$	The inner product of vectors $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, namely, $u^\top v$ for $u, v \in \mathbb{R}^n$
$ x $	The Euclidean vector norm $ x := \sqrt{x^\top x}$
$ x _K$	The distance from point $x \in \mathbb{R}^n$ to a closed set $K \subset \mathbb{R}^n$, i.e., $ x _K = \inf_{\xi \in K} x - \xi $
$L_V(\mu)$	The μ -sublevel set of V , i.e., $L_V(\mu) = \{x \in \text{dom } V : V(x) \leq \mu\}$
$V^{-1}(\mu)$	The μ -level set of V , i.e., $V^{-1}(\mu) = \{x \in \text{dom } V : V(x) = \mu\}$
$\text{dom } M$	The domain of $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, i.e., $\text{dom } M = \{x \in \mathbb{R}^m : M(x) \neq \emptyset\}$
$\text{gph } M$	The graph of $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, i.e., $\text{gph } M = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n : y \in M(x)\}$
$\text{rge } M$	The range of $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, i.e., $\text{rge } M = \{y \in \mathbb{R}^n : y \in M(x), x \in \mathbb{R}^m\}$

Abstract

Analysis, Design of Control Algorithms and Applications for
Forward Invariance of Hybrid Systems

by
Jun Chai

This dissertation focuses on developing tools to study the robust forward invariance of sets for systems with unknown disturbances and hybrid dynamics. In particular, the notions of robust forward invariance properties are proposed for hybrid dynamical systems modeled by differential and difference inclusions with state-depending conditions enabling flows and jumps. A set is said to enjoy robust forward invariance for a system when its solutions start within the set always stay in the set regardless of disturbances. These proposed notions allow for a diverse type of solutions (with and without disturbances), including solutions that have persistent flow and jumps, that are Zeno, and that stop to exist after finite amount of (hybrid) time. Moreover, sufficient conditions for sets to enjoy such properties are presented. The proposed conditions involve the system data and the set to be rendered robust forward invariant.

Furthermore, such conditions are exploited to derive conditions guaranteeing that sublevel sets of Lyapunov-like functions are robust forward invariant and, in turn, inspired a constructive way to design invariance-based control algorithms for a class of hybrid systems with control inputs and disturbances. More precisely, when a hybrid system have a Lyapunov-like function V satisfying a set of specific conditions, existence of feedback laws that render sublevel sets of V robustly forward invariant for the closed-loop hybrid systems are presented. In addition, two selection theorems are proposed to design invariance-based controllers for the class of hybrid systems considered.

Applications and academic examples are given to illustrate the results. In particular, the presented forward invariance analysis and design tools are applied to the design and validate of hybrid controllers for power conversion systems, specifically, a single-phase DC/AC inverter and a DC/DC boost converter. Moreover, results are applied to the estimation of weakly forward invariant sets, which is an invariance property of interest when employing invariance principles to study convergence properties of solutions. Finally, the developed algorithms are tested on the control of a constrained bouncing ball system.

Dedicated to my parents:

Guirong Yuan and Zhidong Chai

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Chapter 1

Introduction

The recent advancements in automation technology in everyday lives call for reliable algorithms to guarantee safe and efficient operation of autonomous systems; such as path planing in autonomous driving, energy generation and allocation in smart grids, cooperative control in air traffic management and motion planning in human-robot collaboration. Techniques to verify the safety properties are vital in the design of autonomous systems, which are even more valuable under the presence of disturbances. Such control specifications can be recast as the problem of rendering a set forward invariant.

Formally, a set K is said to be forward invariant for a dynamical system if every solution to the system from K stays in K . This property is also referred in the literature as *flow-invariance* [1] or *positively invariance* [2]. In [3], Aubin studies viability and invariance (or weak forward invariance and forward invariance) for continuous dynamical systems given as differential inclusions. His explanations of these concepts by quoting Jacques Monod’s book “Chance and Necessity” are very informative: the term “chance” describes the indeterministic factor that comes from the set-valuedness of system dynamics; while the phrase “necessity” captures the behavior that solutions ought to evolve within desired regions. In the presence of disturbances, one is typically interested in invariance properties that hold for all possible allowed disturbances, which has been referred in the literature as *robust*

forward invariance; see, e.g., [4]. In fact, robust forward invariance are used to characterize solution behaviors to systems with uncertain disturbances. This motivates the development of forward invariance tools for hybrid systems modeled by differential and difference inclusions with state constraints. In this dissertation, we verify the “necessity” of solution pairs evolving within the set that they are initialized from; while considering a wider range of “chance” in system dynamics compared to [3]: in addition to set-valuedness, it comes from the possible mixture of continuous and discrete behaviors and uncertain disturbances.

1.1 Background

Forward invariant sets are regions of the state space from which solutions start and stay for all future time. In addition to safety guarantees, such properties emerge in many engineering problems that seek for set reachability, control optimization and system asymptotic stability. For an obstacle avoidance problem in vehicular networks, [5] achieves safety in autonomous path planning via invariant set and reachable set analysis for a linear continuous-time system. In [6], forward invariant sets are used to determine the constraints of feasibility to model predictive control of nonlinear discrete-time systems. In [7], stability of controlled invariant sets is derived for piecewise-affine systems given by a finite collection of affine linear differential equations on polytopic subsets of the state space. Treated as reachability and invariance analysis of hybrid systems, air traffic management of multiple aircrafts with safety concerns is studied in [8].

Moreover, analysis and feedback-control designs with invariance rendering goals under the influence of disturbances have appeared in the literature featuring various engineering applications. For an adaptive cruise control problem, [9] studies the robust invariance properties of sublevel sets of control barrier functions for nonlinear continuous-time systems. For predicting threat assessment in semi-autonomous cars, [10] identifies the safe driving regions via reachability and

robust controlled invariance analysis for a discrete-time system with perturbations. To achieve stability of a network system with delays, which is modeled as a discrete-time system, a model predictive control using set invariance properties is implemented in [11]. Safety control of urban traffic network is considered as a robust controlled invariance problem in [12] for discrete-time and hybrid systems.

1.1.1 Forward Invariance for Continuous-time and Discrete-time Systems

Tools to verify invariance of a set for continuous-time and discrete-time systems have been thoroughly investigated in the literature. In the seminal article [13], the so-called *Nagumo Theorem* is established to determine forward invariance (and weak forward invariance¹) of sets for continuous-time systems with unique solutions. Given a locally compact set K that is to be rendered forward invariant and a continuous-time system with a continuous vector field, the Nagumo Theorem requires that, at each point in the boundary of K , the vector field belongs to the tangent cone to K ; see also [3, Theorem 1.2.1]. This result has been revisited and extended in several directions. In [14], conditions for weak invariance as well as invariance for closed sets are provided – a result guaranteeing finite-time weak invariance is also presented. In particular, one result shows that a closed set K is forward invariant for a continuous-time system with unique solutions if and only if the vector field and its negative version are subtangential to K at each point in it. A similar result is known as the Bony-Brezis theorem, which, instead of involving a condition on the tangent vectors, requires the vector field to have a nonpositive inner product with any (exterior) normal vector to the set K [15,16]. Taking advantage of convexity and linearity of the objects considered, [17] provides necessary and sufficient conditions for forward invariance of convex polyhedral sets for linear time-invariant discrete-time systems. Essentially, conditions in [17]

¹When solutions are nonunique and invariance only holds for some solutions from each point in K , then K is said to be weakly forward invariant – in [3] this property is called viability.

require that the new value of the state after every iteration belongs to the set that is to be rendered forward invariant. This condition can be interpreted as the discrete-time counterpart of the condition in the Nagumo Theorem. For the case of time-varying continuous-time systems, [1] provides conditions guaranteeing forward invariance properties of K given by a sublevel set of a Lyapunov-like function; see also [18–20]. The analysis of forward invariance of a set for systems under the effect of perturbations has also been studied in the literature; see [21] for the case when K is a cone, [22, 23] when K is a polyhedral, to just list a few more, [5, 10].

For systems with an input, forward invariance can be employed as a tool for control design. In particular, such techniques synthesize forward invariance specifications, that is, the question of whether, under the selected inputs, the states of the system remain contained in a desired region. Referred to as *invariance-based control*, invariance of sets are exploited for nonlinear continuous-time constrained systems [24], cascade nonlinear systems [25] and system stabilization of a wide class of systems. For example, [26] investigates the relationship between forward invariance and stability for uncertain constrained pure discrete-time and pure continuous-time systems. Conditions for existence of invariance-based controllers for linear discrete-time systems are given in [27]. In [28], using stability analysis, the authors derive equivalent conditions to the existence of forward invariant sets based on comparison principle for constrained discrete-time nonlinear systems. Under the presence of disturbances, predicting and governing the solution behaviors of the system more intricate than for nominal systems. For model-predictive controls of discrete-time systems, [29] establishes sufficient conditions on feasible controls that induce stability via invariance analysis. Minimal invariant sets are determined for discrete-time control systems in [30]. In [7], stability of controlled invariant sets is achieved for piecewise-affine systems. In recent years, more attention is drawn to control applications that seek for set invariance and safety under disturbances outside of stability theory. In [31], as a case study for ma-

nipulating genetic regulatory networks, robust invariant set is derived to keep the cellular states of a boolean network within desired set. For continuous-time monotone systems, [4] achieves energy efficiency in temperature control of ventilation in buildings via invariance analysis. For nonlinear continuous-time systems with control barrier functions, [9] studies invariance applications in adaptive cruise control. Among above application driven research, forward invariance analysis and control design tools often are different from the ones for set stability. Therefore, in this dissertation, we develop systematic tools to verify forward invariance properties of sets without stability in consideration. Moreover, we study the robust forward invariant sets for a general class of hybrid dynamical systems modeled as *hybrid inclusions* with disturbances.

Theoretical and computational results on forward invariance of sets under the presence of disturbances are available under the terms invariance-based control or controlled invariance. For example, article [2] surveys results for the design of controllers that induce forward invariance via Lyapunov approach. The authors of [32] study the invariance control of saturated linear continuous-time systems (the singular case is treated in [33]). Regarding the computation of invariant sets, algorithms for the computation of the maximum controlled invariant set for discrete-time systems are given in [6, 34, 35]. Among these, control designs for systems with inputs using control Lyapunov functions are implemented to generate state-feedback laws to assure invariance. For instance, [1] guarantees forward invariant properties of single-valued continuous-time systems with nonunique solutions via analysis of a family of Lyapunov-like functions. Utilizing a local control Lyapunov function, feedback control of linear systems under the presence of perturbations appeared in [36], which exploits a nesting property of sets to guarantee invariance of sets; see also the optimization approach in [37]. Similarly, barrier functions (and control barrier functions) have been shown to be effective for the study of safety in continuous-time systems; see [38, 39].

1.1.2 Forward Invariance for Hybrid Systems

The interests in forward invariance of sets for hybrid systems is also driven by applications. The use of forward invariance for analysis and design for hybrid systems include periodic motion analysis with impacts [40], reachability [41], and hybrid control design [42]. Safety verification in hybrid systems is studied as invariance-based control problem for hybrid automata with nonlinear continuous dynamics, disturbances, and control inputs. For example, [43, 44], a differential game approach is proposed to compute reachable sets for the verification of safety. In [45], an algorithm is proposed to approximate invariant sets of hybrid systems that have continuous dynamics with polynomial right-hand-side and that can be written as hybrid programs. In [46, 47] control barrier functions are used for verification of safety in hybrid automata with disturbances that affect the continuous dynamics.

However, the study of forward invariance in systems that combine continuous and discrete dynamics is not as mature as the continuous-time and discrete-time settings. This is because the dynamics of hybrid systems are typically governed by set-valued, nonlinear maps, which lead to nonunique solutions. Certainly, when the continuous dynamics are discretized, the methods for purely discrete-time systems mentioned above are applicable or can be extended without significant effort for certain classes of hybrid models in discrete time; see, in particular, the results for a class of piecewise affine discrete-time systems in [48]. Establishing forward invariance (both nominal and robust) is much more involved when the continuous dynamics are not discretized. Forward invariance of sets for impulsive differential inclusions, which are a class of hybrid systems without disturbances, are established in [49]. In particular, [49] proposes conditions to guarantee (weak – or viability – and strong) forward invariance of closed sets and a numerical algorithm to generate invariant kernels. Other recent contributions to the understanding of forward invariance for hybrid systems without inputs and disturbances include those for hybrid automata [50, 51] for hybrid inclusions [52, 53]. In [54],

discrete-time invariance inducing controllers are designed for continuous-time non-linear systems. The particular case of invariance-based control design for switched systems modeled by discrete-time systems without perturbations was treated in [55]. The problem of computing the controlled invariant sets for switched systems was studied in [56,57]. Game theoretic approach to reachability via forward invariance control design for a class of hybrid systems with disturbances were proposed in [58] and [59]. For the case of perturbed systems, analysis results to study robust forward invariance of sets for linear continuous-time systems with multiple operation modes are established and applied to controller design in [60]. When both control input and disturbances are considered in hybrid systems, predicting solutions behaviors is much more complicated compared to those for pure continuous-time or pure discrete-time systems. In this dissertation, we focus on the systematic analysis and control designs of a class of hybrid systems modeled as hybrid inclusions with control input and disturbances.

1.2 Contribution

Motivated by the lack of results for the study of robust and nominal forward invariance in hybrid systems, we propose tools for analyzing forward invariance properties of sets. In particular, formal notions of invariance and solution-independent conditions that guarantee desired invariance properties of sets are established for hybrid dynamical systems modeled as

$$\mathcal{H}_w \begin{cases} (x, w_c) \in C_w & \dot{x} \in F_w(x, w_c) \\ (x, w_d) \in D_w & x^+ \in G_w(x, w_d) \end{cases} \quad (1.1)$$

which we refer to as hybrid inclusions [61] and where x is the state and $w = (w_c, w_d)$ is the disturbance.²

Building from the established robust forward invariance properties for hybrid dynamical systems, we introduce the notions of robust controlled forward invari-

²See Chapter 2 for a precise definition of system notations appeared in this section.

ance for hybrid systems given as in [62]. Differential and difference inclusions with state, input and disturbance constraints are used to model the continuous-time and discrete-time dynamics of systems, respectively. More precisely, we consider hybrid systems with both disturbances $w = (w_c, w_d) \in \mathcal{W}_c \times \mathcal{W}_d$ and control inputs $u = (u_c, u_d) \in \mathcal{U}_c \times \mathcal{U}_d$ that are given by³

$$\mathcal{H}_{u,w} \begin{cases} (x, u_c, w_c) \in C_{u,w} & \dot{x} \in F_{u,w}(x, u_c, w_c) \\ (x, u_d, w_d) \in D_{u,w} & x^+ \in G_{u,w}(x, u_d, w_d), \end{cases} \quad (1.2)$$

where $x \in \mathbb{R}^n$ is the state. We focus on the controller design and synthesis for the purpose of rendering a set *robustly controlled forward invariant* for the closed-loop system that is resulted by replacing u with an admissible state-feedback pair. The main challenges in asserting these forward invariance properties and their designs include the following:

- (1) *Combined continuous and discrete dynamics*: given a disturbance signal and an initial state value, a solution to (1.1) may evolve continuously for some time, while at certain instances, jump. As a consequence, the set K must have the property that solutions stay in it when either the continuous or the discrete dynamics are active.
- (2) *Potential nonuniqueness and noncompleteness of solutions*: the fact that the dynamics of (1.1) are set valued and the existence of states from where flows and jumps are both allowed (namely, the state components of C_w and D_w may have a nonempty overlap with points from where flows are possible) lead to nonunique solutions. In particular, at points in K where both flows and jumps are allowed, conditions for invariance during flows and at jumps need to be enforced simultaneously. Furthermore, solutions may cease to exist after finite (hybrid) time due to the state reaching a point from where neither flow nor jump is possible. – these include points in the boundary of C_w that are not in D_w , from where the elements in F_w point outward the set C_w , and

³The space for disturbances and control inputs are $\mathcal{W}_c \subset \mathbb{R}^{d_c}$, $\mathcal{W}_d \subset \mathbb{R}^{d_d}$ and $\mathcal{U}_c \subset \mathbb{R}^{m_c}$, $\mathcal{U}_d \subset \mathbb{R}^{m_d}$, respectively.

points from where the jumps map the state outside where flow and jumps are possible.

- (3) *Presence of disturbances for systems with state constraints:* for it to be interesting, forward invariance of a set K for a hybrid system with disturbances is an invariance property that has to hold for all possible disturbances. In technical terms, for every x such that (x, w_c) belongs to C_w , the vectors in the set $F_w(x, w_c)$ need to be in directions that flow outside of K is impossible for all values of w_c . Similarly, for each x such that (x, w_d) belongs to D_w , the set $G_w(x, w_d)$ should be contained in K regardless of the values of w_d .
- (4) *Forward invariance analysis of intersection of sets:* when provided a Lyapunov-like function, V , for the given system, conditions to guarantee forward invariance properties will need to take advantage of the nonincreasing property of V . In such a case, the state component of the sets C_w and D_w will be intersected by sublevel sets of the given Lyapunov-like function, which require less restrictive conditions than for general sets.
- (5) *Robust controlled forward invariance for $\mathcal{H}_{u,w}$ via (κ_c, κ_d) :* we provide notions of robust controlled forward invariance of a generic set $K \subset \mathbb{R}^n$ for $\mathcal{H}_{u,w}$ via given feedback laws. When a $\mathcal{H}_{u,w}$ -admissible state-feedback pair (κ_c, κ_d) ⁴ renders a set robustly controlled forward invariant for the closed-loop system, the existence of nontrivial solution pair from every possible initial condition is guaranteed. Moreover, every maximal solution pair that starts from the set is complete and stays within the set for all future time. Such concept concerns the solution pair behavior of the closed-loop hybrid system of $\mathcal{H}_{u,w}$ under the effect of a state-feedback pair (κ_c, κ_d) , which is given by

$$\mathcal{H}_w \begin{cases} (x, w_c) \in C_w & \dot{x} \in F_w(x, w_c) \\ (x, w_d) \in D_w & x^+ \in G_w(x, w_d), \end{cases} \quad (1.3)$$

⁴A state-feedback pair (κ_c, κ_d) , where $\kappa_c : \mathbb{R}^n \rightarrow \mathbb{R}^{m_c}$ and $\kappa_d : \mathbb{R}^n \rightarrow \mathbb{R}^{m_d}$, is said to be $\mathcal{H}_{u,w}$ -admissible if the pair satisfies the dynamics of $\mathcal{H}_{u,w}$.

where the set-valued maps $F_w(x, w_c) := F_{u,w}(x, \kappa_c(x), w_c)$ and $G_w(x, w_d) := G_{u,w}(x, \kappa_d(x), w_d)$ govern the continuous-time and discrete-time evolutions of the system on the sets $C_w := \{(x, w_c) \in \mathbb{R}^n \times \mathcal{W}_c : (x, \kappa_c(x), w_c) \in C_{u,w}\}$ and $D_w := \{(x, w_d) \in \mathbb{R}^n \times \mathcal{W}_d : (x, \kappa_d(x), w_d) \in D_{u,w}\}$, respectively. Note that \mathcal{H}_w share same structure as hybrid system in (1.1). When provided different feedback pairs to $\mathcal{H}_{u,w}$, the resulting closed-loop systems have different dynamics.

- (6) *Robust forward invariance of sublevel sets of Lyapunov-like functions*: conditions to guarantee robust forward invariance properties that take advantage of the nonincreasing property of a Lyapunov-like function, V , are proposed. The sublevel sets of the given V is intersected by the state component of the sets C_w and D_w . Intricate derivations are needed to guarantee the existence of nontrivial solution pair from every point, furthermore, to guarantee the completeness of solution pairs. Note that these Lyapunov-like functions ought to satisfy inequalities over carefully constructed regions that allow increase in V in the interior of their sublevel sets. Moreover, to further relax the regularity on set C_w (compared to [63, Theorem 5.1]) via a constructive proof by investigating properties of vectors in the tangent cone of the intersected sets.
- (7) *Existence of continuous state-feedback laws using robust control Lyapunov function for forward invariance (RCLF-FI)*: we extend the concept of robust control Lyapunov function from [62] to the purpose of rendering robust controlled invariance. The proposed notion is derived from the conditions to guarantee robust forward invariance of sublevel sets of Lyapunov-like functions. Such a novel concept is exploited to determine sufficient conditions that lead to the existence of continuous state-feedback laws for robust controlled invariance, which involve the data of the system and properly constructed regulation maps. In particular, by assuring the existence of continuous selections from set-valued maps that collect all possible control inputs, forward invariance of sublevel sets is guaranteed. The invariance-based control design

approach using RCLF-FI for hybrid systems is unique and original to our best knowledge. Finally, utilizing the regulation maps built, we employ a pointwise minimum norm selection scheme to construct state-feedback laws. Such feedback pairs lead to suboptimality with respect to some meaningful cost function while ensuring robust controlled forward invariance for $\mathcal{H}_{u,w}$.

In this dissertation, we provide results that help tackle these key issues systematically. For starters, we introduce the notions for forward invariance in hybrid dynamical systems, both with and without disturbances. Then, we present the sufficient conditions to verify each proposed notions, for which, in the case with disturbances, we establish a result to guarantee existence of nontrivial solutions to the system modeled as in (1.1) and provides insight for solution behavior based on completeness. The proposed notions of robust forward invariance are uniform over all possible disturbances, and allow for solutions to be nonunique and to cease to exist in finite (hybrid) time (namely, not complete). For each notion, we propose sufficient conditions that the data of the hybrid inclusions and the set K should satisfy to render K robustly forward invariant. Results for hybrid systems without disturbances are derived as special cases of the robust ones. Compared to [49], which studies the nominal systems exclusively, we focus on a more general family of hybrid systems, where Marchaud and Lipschitz conditions are not necessarily always imposed in the flow map. As an application of the results for generic sets K , we present a novel approach to verify forward invariance of a sublevel set of a given Lyapunov-like function intersected with the sets where continuous or discrete dynamics are allowed. Such a result lays the groundwork for the design results in this thesis. Because of the nonincreasing properties of the given Lyapunov-like function along solutions, the developed conditions are less restrictive and more constructive when compared to the ones for a generic set K . Moreover, our results are also insightful for systems with purely continuous-time or discrete-time dynamics. In fact, because of the generality of the hybrid inclusions framework, the results in this paper are applicable to broader classes

of systems, such as those studied in [2, 3, 17, 19, 20].

Then, we exploit the analysis tools for verifying robust forward invariance properties for the autonomous hybrid dynamical system \mathcal{H}_w , to develop invariance-based control design using proposed RCLF-FI. We provide systematic approach for constructing invariance inducing feedbacks under the presence of a generic class of perturbations, from which the invariance-based control designs for hybrid systems without disturbances are derived as special cases.⁵ Moreover, our results are also insightful for systems with purely continuous-time or discrete-time dynamics. In fact, because of the generality of the hybrid inclusions framework, the results in this paper are applicable to broader classes of systems, such as those studied in [4, 26, 30, 32].

To illustrate analysis and design tools proposed in this dissertation, we investigated several engineering applications. In particular, we present two technological invariant-based control for power conversion systems, a single-phase DC/AC inverter and a DC/DC boost converter. Both systems with switching dynamics are modeled as hybrid systems with a logical input signal to be designed. To accomplish the design goals, we proposed control laws that result in desired output signals as close as possible to the reference signal by controlling the switch(es) in circuits. The proposed controllers trigger switch(es) based on the value of the current and voltage of the electronic filter. Results on forward invariance of sets for general hybrid systems are used to analyze the effect of the proposed controllers. Then, we study a bouncing ball system with constrain, which is modeled by hybrid inclusions in form of (1.2). Following the provided approach, the designed control input manage to accomplish the objective of maintaining the peaks of height after each bounce within desired range for the closed-loop system. This controller features robust controlled forward invariance of a desired set on the state space that corresponds to the desired height range under the uncertain coefficient of restitution during impact. Finally, for the purpose of estimating largest weakly forward

⁵The nominal version of the results in this paper appeared without proof in the conference article [64].

invariant set, which is essential in the set stabilization via La Salle’s invariance principle, we explore the proposed sufficient conditions that guarantee forward invariance property of the sublevel sets of Lyapunov-like functions. Numerical simulations are presented for applications to validate the claims. The proposed results are also demonstrated with illustrative academic examples.

1.3 Organization

The contents of this dissertation are organized into following chapters. For ease of presentation, we introduce results in each chapter for the nominal hybrid systems prior to the ones for the perturbed systems.

Chapter 2: Preliminaries In this chapter, the hybrid inclusions framework, which models hybrid systems as differential and difference inclusions with constraints, and its basic properties are presented. These will be used throughout this dissertation.

Chapter 3: Notions of Forward Invariance for Hybrid Systems In this chapter, we formally define the notions of nominal forward invariance properties of a set for hybrid system without disturbance signals. The proposed notions are thoroughly discussed concerning the existence of nontrivial solutions and their completeness. Then, extending these to the case with disturbances, we propose the notions of robust forward invariance for hybrid systems \mathcal{H}_w in (1.1). We provide several academic examples to illustrate the notions, which we revisit in later chapters.

Chapter 4: Sufficient Conditions to Verify Forward Invariance for Hybrid Systems In this chapter, we present sufficient conditions to guarantee each presented forward invariance notion in Chapter 3 for a generic set K . For the

system with disturbances, we provide conditions to check for existence of nontrivial solutions and their behaviors. Each set of conditions include those to ensure solutions to stay within K during flows and jumps. In addition, a particular condition to guarantee no finite escape time of solutions during flow is explored for possible solution-independent alternatives. We present several academic examples to illustrate the main results.

Chapter 5: Forward Invariance of Sublevel Sets of Lyapunov-like Functions In this chapter, we propose invariance inducing conditions for systems with a given Lyapunov-like function V . These conditions are less restrictive on system data, when compared to the ones in Chapter 4. Such advantage comes from the fact that the set of interest is the intersection of the sublevel sets of V and the regions where flows and jumps are enabled. These results are crucial to the upcoming analysis and control design efforts. We provide several academic examples to illustrate the results, which we revisit in later chapters.

Chapter 6: Controlled Forward Invariance using Control Lyapunov Functions In this chapter, we start with providing the definitions of (robust) controlled forward invariance for hybrid systems via feedback laws. Then, revisiting the relaxed sufficient conditions using Lyapunov methods, we give the formal definition of control Lyapunov function for forward invariance. Then, results for the existence and design of continuous state-feedback laws using such functions to render its sublevel sets (robustly) controlled forward invariant are presented. Control synthesis is also proposed using a minimal selection scheme. Academic examples for major concepts are included.

Chapter 7: Applications of Invariance-based Controls In this chapter, we investigate several engineering applications to illustrate analysis and design tools proposed in this dissertation. Two power conversion systems, a single-phase

DC/AC inverter and a DC/DC boost converter, are presented. A bouncing ball system with constraints, which is modeled by hybrid inclusions in form of (1.2), is controlled and analyzed to achieve the control goal of maintaining the peaks of height after each bounce within desired range. Finally, for the purpose of estimating largest weakly forward invariant set, we explore the proposed sufficient conditions that guarantee forward invariance property of the sublevel sets of Lyapunov-like functions. Numerical simulation results are presented for most applications to validate the claims.

Chapter 8: Conclusion and Plan of Future Work In this chapter, the results in this dissertation are summarized and several potential future directions are briefly discussed.

Chapter 2

Preliminaries on Hybrid Systems

In this dissertation, we consider hybrid system modeled as hybrid inclusions, where system dynamics are captured by differential and difference inclusions with constraints. More precisely, for hybrid system \mathcal{H}_w given as in (1.1), which has disturbance input $w = (w_c, w_d)$ and state x , we are interested in forward invariance properties of a set that are uniform in the allowed disturbances w ; while for hybrid systems \mathcal{H} studied in [61], which is considered as a special case of \mathcal{H}_w with constant zero disturbance, i.e., $w \equiv 0$. We further explore the relaxed conditions to guarantee nominal forward invariance of sublevel sets of Lyapunov-like functions. In the rest of this section, we present basic definitions and properties of \mathcal{H}_w that are important for deriving the forthcoming results.

We study the system $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$ in (1.1). The data of \mathcal{H}_w in (1.1) is defined by the flow set $C_w \subset \mathbb{R}^n \times \mathcal{W}_c$, the flow map $F_w : \mathbb{R}^n \times \mathbb{R}^{d_c} \rightrightarrows \mathbb{R}^n$, the jump set $D_w \subset \mathbb{R}^n \times \mathcal{W}_d$, and the jump map $G_w : \mathbb{R}^n \times \mathbb{R}^{d_d} \rightrightarrows \mathbb{R}^n$. The space for the state x is \mathbb{R}^n and the space for the disturbance $w = (w_c, w_d)$ is $\mathcal{W} = \mathcal{W}_c \times \mathcal{W}_d \subset \mathbb{R}^{d_c} \times \mathbb{R}^{d_d}$. The sets C_w and D_w define conditions that x and w should satisfy for flows or jumps to occur. In this paper, we assume $\text{dom } F_w \supset C_w$ and $\text{dom } G_w \supset D_w$. The considered class of disturbances are formally given as follows.

Definition 2.0.1 (*hybrid disturbance*) *A hybrid disturbance w is a function on a*

hybrid time domain that, for each $j \in \mathbb{N}$, $t \mapsto w(t, j)$ is Lebesgue measurable and locally essentially bounded on the interval $\{t : (t, j) \in \text{dom } w\}$. \square

When $w(t, j) = 0$ for every $(t, j) \in \text{dom } w$ (which means that there is no disturbance), the system \mathcal{H}_w reduces to the nominal hybrid system introduced in [61], which is given by

$$\mathcal{H} \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D, \end{cases} \quad (2.1)$$

where the system data is given by

- *flow map*, a set-value map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ describing the continuous dynamics of \mathcal{H} ;
- *flow set*, a set $C \subset \text{dom } F$ specifying the points where dynamics of F applies;
- *jump map*, a set-value map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ describing the discrete dynamics of \mathcal{H} ;
- *jump set*, a set $D \subset \text{dom } G$ specifying the points where dynamics of G applies.

Following [61], a solution to the hybrid system \mathcal{H}_w is parameterized by the concept of hybrid arcs and hybrid time domains.

Definition 2.0.2 (*hybrid time domain, [61, Definition 2.3]*) A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \dots \leq t_J$. A subset $S \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for all $(T, J) \in S$, $S \cap ([0, T], \{0, 1, \dots, J\})$ is a compact hybrid time domain. \square

The operations \sup_t and \sup_j on a hybrid time domain E return the supremum of the t and j coordinates, respectively, of points in E . A hybrid arc ϕ , satisfying system dynamics, can be defined as a set-valued mapping $\phi : \mathbb{R}^2 \rightrightarrows \mathbb{R}^n$ that is single-valued on its domain $\text{dom } \phi$.

Definition 2.0.3 (*hybrid arc, [61, Definition 2.4]*) A function $x : \text{dom } x \rightarrow \mathbb{R}^n$ is a hybrid arc if $\text{dom } x$ is a hybrid time domain and if for each $j \in \mathbb{N}$, the function $t \mapsto x(t, j)$ is locally absolutely continuous. \square

Then, solutions to (2.1) are defined in [61, Definition 2.6] in terms of hybrid time domains and hybrid arcs.

Definition 2.0.4 (*solution to \mathcal{H}*) A hybrid arc ϕ is a solution to the hybrid system (C, F, D, G) if $\phi(0, 0) \in \overline{C} \cup D$, and

(S1) for all $j \in \mathbb{N}$ such that $I^j := \{t : (t, j) \in \text{dom } \phi\}$ has nonempty interior

$$\begin{aligned} \phi(t, j) &\in C && \text{for all } t \in \text{int } I^j, \\ \dot{\phi}(t, j) &\in F(\phi(t, j)) && \text{for almost all } t \in I^j; \end{aligned}$$

(S2) for all $(t, j) \in \text{dom } \phi$ such that $(t, j + 1) \in \text{dom } \phi$,

$$\begin{aligned} \phi(t, j) &\in D, \\ \phi(t, j + 1) &\in G(\phi(t, j)). \end{aligned}$$

A solution ϕ to the hybrid system $\mathcal{H} = (C, F, D, G)$ is

- nontrivial if $\text{dom } \phi$ contains at least two points;
- complete if $\text{dom } \phi$ is unbounded;
- Zeno if it is complete and $\sup_t \text{dom } \phi < \infty$;
- maximal if there does not exist another solution ψ to \mathcal{H} such that $\text{dom } \phi$ is a proper subset of $\text{dom } \psi$ and $\phi(t, j) = \psi(t, j)$ for all $(t, j) \in \text{dom } \phi$;

- *eventually discrete if $T = \sup_t \text{dom } \phi < \infty$ and $\text{dom } \phi \cap (\{T\} \times \mathbb{N})$ contains at least two points;*
- *discrete if nontrivial and $\text{dom } \phi \subset \{0\} \times \mathbb{N}$.* □

Given $K \subset \mathbb{R}^n$, we use $\mathcal{S}_{\mathcal{H}}$ to represent the set of all maximal solutions to the hybrid system \mathcal{H} and $\mathcal{S}_{\mathcal{H}}(K)$ to denote a set that includes all maximal solutions ϕ to the hybrid system \mathcal{H} with initial condition $\phi(0, 0)$ in K .

As an extension to Definition 2.0.4, solution pairs to a hybrid system \mathcal{H}_w as in (1.1) are defined as follows.

Definition 2.0.5 (*solution pairs to \mathcal{H}_w*) *A pair (ϕ, w) consisting of a hybrid arc ϕ and a hybrid disturbance $w = (w_c, w_d)$, with $\text{dom } \phi = \text{dom } w (= \text{dom}(\phi, w))$, is a solution pair to the hybrid system \mathcal{H}_w in (1.1) if $(\phi(0, 0), w_c(0, 0)) \in \overline{C_w}$ or $(\phi(0, 0), w_d(0, 0)) \in D_w$, and*

(S1_w) for all $j \in \mathbb{N}$ such that I^j has nonempty interior

$$\begin{aligned} (\phi(t, j), w_c(t, j)) &\in C_w && \text{for all } t \in \text{int } I^j, \\ \frac{d\phi}{dt}(t, j) &\in F_w(\phi(t, j), w_c(t, j)) && \text{for almost all } t \in I^j, \end{aligned}$$

(S2_w) for all $(t, j) \in \text{dom } \phi$ such that $(t, j + 1) \in \text{dom } \phi$,

$$\begin{aligned} (\phi(t, j), w_d(t, j)) &\in D_w \\ \phi(t, j + 1) &\in G_w(\phi(t, j), w_d(t, j)). \end{aligned}$$

In addition, a solution pair (ϕ, w) to \mathcal{H}_w is said to be

- *nontrivial if $\text{dom}(\phi, w)$ contains at least two points;*
- *complete if $\text{dom}(\phi, w)$ is unbounded;*
- *maximal if there does not exist another $(\phi, w)'$ such that (ϕ, w) is a truncation of $(\phi, w)'$ to some proper subset of $\text{dom}(\phi, w)'$.* □

Similar as for \mathcal{H} , we use $\mathcal{S}_{\mathcal{H}_w}(K)$ to denote a set that includes all maximal solution pairs (ϕ, w) to \mathcal{H}_w with initial condition $\phi(0, 0)$ in K .

To formulate our results, we will need the following result from [61].

Proposition 2.0.6 ([53, Proposition 2.2]) *Consider the hybrid system $\mathcal{H} = (C, F, D, G)$. Let $\xi \in \overline{C} \cup D$. If $\xi \in D$ or*

(VC) there exist $\varepsilon > 0$ and an absolutely continuous function $z : [0, \varepsilon] \rightarrow \mathbb{R}^n$ such that $z(0) = \xi$, $\dot{z}(t) \in F(z(t))$ for almost all $t \in [0, \varepsilon]$ and $z(t) \in C$ for all $t \in (0, \varepsilon]$,

then there exists a nontrivial solution ϕ to \mathcal{H} with $\phi(0, 0) = \xi$. If (VC) holds for every $\xi \in \overline{C} \setminus D$, then there exists a nontrivial solution to \mathcal{H} from every point of $\overline{C} \cup D$, and every $\phi \in \mathcal{S}_{\mathcal{H}}$ satisfies exactly one of the following:

- a) ϕ is complete;*
- b) ϕ is not complete and “ends with flow”, with $(T, J) = \text{sup dom } \phi$, the interval I^J has nonempty interior; and either*
 - b.1) I^J is closed, in which case $\phi(T, J) \in \overline{C} \setminus (C \cup D)$; or*
 - b.2) I^J is open to the right, in which case $(T, J) \notin \text{dom } \phi$, and there does not exist an absolutely continuous function $z : \overline{I^J} \rightarrow \mathbb{R}^n$ satisfying $\dot{z}(t) \in F(z(t))$ for almost all $t \in I^J$, $z(t) \in C$ for all $t \in \text{int } I^J$, and such that $z(t) = \phi(t, J)$ for all $t \in I^J$;*
- c) ϕ is not complete and “ends with jump”: for $(T, J) = \text{sup dom } \phi$, one has $\phi(T, J) \notin \overline{C} \cup D$.*

Furthermore, if $G(D) \subset \overline{C} \cup D$, then c) above does not occur.

Our control design effort considers hybrid systems with both disturbances $w = (w_c, w_d) \in \mathcal{W}_c \times \mathcal{W}_d$ and inputs $u = (u_c, u_d) \in \mathcal{U}_c \times \mathcal{U}_d$ given by (1.2). The space for disturbances and inputs are $\mathcal{W}_c \subset \mathbb{R}^{d_c}$, $\mathcal{W}_d \subset \mathbb{R}^{d_d}$ and $\mathcal{U}_c \subset \mathbb{R}^{m_c}$, $\mathcal{U}_d \subset \mathbb{R}^{m_d}$,

respectively. We focus on the controller design and synthesis for the purpose of rendering a set *robustly controlled forward invariant* for the closed-loop systems that is resulted by replacing u with a $\mathcal{H}_{u,w}$ -admissible state-feedback law (κ_c, κ_d) .¹ In (1.2), the maps $F_{u,w}(x, u_c, w_c)$ and $G_{u,w}(x, u_d, w_d)$ are nonempty for every (x, u_c, w_c) and (x, u_d, w_d) , respectively, and capture the system dynamics when in sets $C_{u,w}$ and $D_{u,w}$, which define conditions that x, u , and w should satisfy for flows or jumps to occur, respectively. We assume that $C_{u,w}$ and $D_{u,w}$ define conditions on u that only depend on x and conditions on w that only depend on x , where $w = 0$, meaning there is no disturbance in system, always qualifies as a value for disturbance. For convenience, we define the projection of $S \subset \mathbb{R}^n \times \mathcal{W}_c$ onto \mathbb{R}^n as

$$\Pi_c^w(S) := \{x \in \mathbb{R}^n : \exists w_c \in \mathcal{W}_c \text{ s.t. } (x, w_c) \in S\},$$

and the projection of $S \subset \mathbb{R}^n \times \mathcal{W}_d$ onto \mathbb{R}^n as

$$\Pi_d^w(S) := \{x \in \mathbb{R}^n : \exists w_d \in \mathcal{W}_d \text{ s.t. } (x, w_d) \in S\}.$$

Moreover, the projection of $S \subset \mathbb{R}^n \times \mathcal{U}_c \times \mathcal{W}_c$ onto \mathbb{R}^n as

$$\Pi_c(S) := \{x \in \mathbb{R}^n : \exists u_c \in \mathcal{U}_c, w_c \in \mathcal{W}_c \text{ s.t. } (x, u_c, w_c) \in S\},$$

and the projection of $S \subset \mathbb{R}^n \times \mathcal{U}_d \times \mathcal{W}_d$ onto \mathbb{R}^n as

$$\Pi_d(S) := \{x \in \mathbb{R}^n : \exists u_d \in \mathcal{U}_d, w_d \in \mathcal{W}_d \text{ s.t. } (x, u_d, w_d) \in S\},$$

Given sets $C_{u,w}$ and $D_{u,w}$, the set-valued maps $\Psi_c^w : \mathbb{R}^n \rightrightarrows \mathcal{W}_c$ and $\Psi_d^w : \mathbb{R}^n \rightrightarrows \mathcal{W}_d$ are defined for each $x \in \mathbb{R}^n$ as

$$\begin{aligned} \Psi_c^w(x) &:= \{w_c \in \mathbb{R}^{d_c} : (x, u_c, w_c) \in C_{u,w}\}, \\ \Psi_d^w(x) &:= \{w_d \in \mathbb{R}^{d_d} : (x, u_d, w_d) \in D_{u,w}\}, \end{aligned} \tag{2.2}$$

respectively, and the set-valued maps $\Psi_c^u : \mathbb{R}^n \rightrightarrows \mathcal{U}_c$ and $\Psi_d^u : \mathbb{R}^n \rightrightarrows \mathcal{U}_d$ are defined for each $x \in \mathbb{R}^n$ as

$$\Psi_c^u(x) := \{u_c \in \mathbb{R}^{m_c} : (x, u_c, w_c) \in C_{u,w}\},$$

¹A state-feedback pair (κ_c, κ_d) , where $\kappa_c : \mathbb{R}^n \rightarrow \mathbb{R}^{m_c}$ and $\kappa_d : \mathbb{R}^n \rightarrow \mathbb{R}^{m_d}$ is said to be $\mathcal{H}_{u,w}$ -admissible if the pair satisfies the dynamics of $\mathcal{H}_{u,w}$.

$$\Psi_d^u(x) := \{u_d \in \mathbb{R}^{m_d} : (x, u_d, w_d) \in D_{u,w}\},$$

respectively.

The following regularity conditions on the system data of a hybrid system \mathcal{H}_w as in (6.2) are considered in some forthcoming results. These conditions guarantee robustness of stability of compact sets with respect to perturbations.

Definition 2.0.7 (*hybrid basic conditions*) *A hybrid system \mathcal{H}_w is said to satisfy the hybrid basic conditions if its data satisfies*

- (A1_w) C_w and D_w are closed subsets of $\mathbb{R}^n \times \mathcal{W}_c$ and $\mathbb{R}^n \times \mathcal{W}_d$ respectively;
- (A2_w) $F_w : \mathbb{R}^n \times \mathbb{R}^{d_c} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous² relative to C_w and locally bounded, and for all $(x, w_c) \in C_w$, $F_w(x, w_c)$ is convex;
- (A3_w) $G_w : \mathbb{R}^n \times \mathbb{R}^{d_d} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous relative to D_w and locally bounded.

□

Similarly, for system \mathcal{H} in (2.1), we define the following regularity conditions on its system data; see [61, Assumption 6.5] for details.

Definition 2.0.8 (*hybrid basic conditions*) *A hybrid system \mathcal{H} is said to satisfy the hybrid basic conditions if its data satisfies*

- (A1) C and D are closed subsets of \mathbb{R}^n ;
- (A2) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous relative to C and locally bounded, and for every $x \in C$, $F(x)$ is convex;
- (A3) $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous relative to D and locally bounded.

□

²See Definition A.0.2 in Appendix.

For general hybrid systems $\mathcal{H} = (C, F, D, G)$, [61, Definition 3.6] introduces the following stability notion.

Definition 2.0.9 (*Stability*) *A compact set $\mathcal{A} \subset \mathbb{R}^n$ is said to be*

- *stable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that each solution χ with $|\chi(0, 0)|_{\mathcal{A}} \leq \delta$ satisfies $|\chi(t, j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t, j) \in \text{dom } \chi$;*
- *attractive if there exists $\mu > 0$ such that every maximal solution χ with $|\chi(0, 0)|_{\mathcal{A}} \leq \mu$ is complete and satisfies $\lim_{(t, j) \in \text{dom } \chi, t+j \rightarrow \infty} |\chi(t, j)|_{\mathcal{A}} = 0$;*
- *asymptotically stable if \mathcal{A} is stable and attractive;*
- *globally asymptotically stable if the attractivity property holds for every point in $C \cup D$.*

□

Chapter 3

Notions of Forward Invariance for Hybrid Systems

In this chapter, we formally introduce the notions of forward invariance for hybrid systems. First, for the nominal system \mathcal{H} in (2.1), four notions are presented to capture the properties where solutions stay within set $K \subset \mathbb{R}^n$ when they are initialized within it. The four notions are different considering the existence of nontrivial solutions and completeness of maximal solutions. Then, for the hybrid system \mathcal{H}_w in (1.1), the *robust* notions are presented for cases where the forward invariance properties are uniformly in disturbances. These definitions provide the fundamentals for the forth coming effort in the invariance-based controls for hybrid systems with inputs in Chapter 6.

3.1 The Nominal Forward Invariant Sets

In this section, we define several forward invariance notions that, in particular, apply in situations where not every maximal solution is complete and unique, which is very common in hybrid systems. We start by defining weak forward pre-invariance of a set.

Definition 3.1.1 (*weak forward pre-invariance*) *The set $K \subset \mathbb{R}^n$ is said to be weakly forward pre-invariant for \mathcal{H} if for every $x \in K$ there exists $\phi \in \mathcal{S}_{\mathcal{H}}(x)$ with $\text{rge } \phi \subset K$. \square*

Note that the prefix “pre” captures the fact that the solution staying in K may not be complete. The weak forward pre-invariance notion requires that at least one solution exists from every point in K . Such a solution can be trivial (dom ϕ with only one point), nontrivial (dom ϕ with more than one point), but at least one maximal solution from each point in the set has to stay in the set for all future hybrid time. In the case that a trivial solution ϕ with $\phi(0, 0) \in K$ is maximal, the property of $\text{rge } \phi \subset K$ holds for free by definition of solutions. Furthermore, if K has points that are not in $\overline{C} \cup D$, the weak forward pre-invariance notion hold for these points as they have one trivial solution as the maximal solution, which is rather a trivial statement, suggesting that one may want to start from a set K that is a subset of $\overline{C} \cup D$ in the first place.

Next, we define a weak forward invariant set, which is equivalent to the one in [61, Definition 6.19].

Definition 3.1.2 (*weak forward invariance*) *The set $K \subset \mathbb{R}^n$ is said to be weakly forward invariant for \mathcal{H} if for every $x \in K$ there exists one complete solution $\phi \in \mathcal{S}_{\mathcal{H}}(x)$ with $\text{rge } \phi \subset K$. \square*

Note that this notion requires the existence of nontrivial solutions from every $x \in K$. We use the next example to illustrate such concept.

Example 3.1.3 (*weak forward invariant set*) *Consider the hybrid system $\mathcal{H} = (C, F, D, G)$ on \mathbb{R}^2 in (2.1) with system data given by*

$$F(x) := \begin{cases} (1, 1) & \text{if } x_2 > 1 - x_1 \\ \overline{\text{con}} \{(1, 1), (-1, -1)\} & \text{if } x_2 = 1 - x_1 \\ (-1, -1) & \text{if } x_2 < 1 - x_1 \end{cases}$$

for every $x \in C := \{x \in \mathbb{R}^2 : x_1 \in [0, 1], x_2 \in [0, 1]\}$, and

$$G(x) := \begin{cases} (\frac{1}{2} + \frac{1}{4}\mathbb{B}, \frac{1}{2}) & \text{if } x_2 \in \{0, 1\}, x_1 \in (0, 1) \\ \{(\frac{1}{2} + \frac{1}{4}\mathbb{B}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2} + \frac{1}{4}\mathbb{B})\} & \text{if } x \in \{(0, 0), (0, 1), (1, 1), (1, 0)\} \\ (\frac{1}{2}, \frac{1}{2} + \frac{1}{4}\mathbb{B}) & \text{if } x_1 \in \{0, 1\}, x_2 \in (0, 1) \end{cases}$$

for every $x \in D := \partial C = (\{0, 1\} \times [0, 1]) \cup ([0, 1] \times \{0, 1\})$.

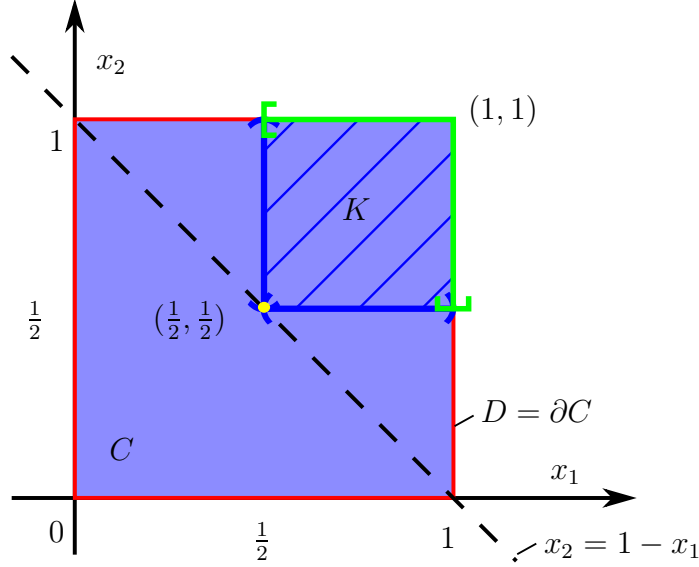


Figure 3.1: Sets pertaining to the system in Example 3.1.3.

Consider the set $K = [\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$. According to the first piece of the definition of F , every solution that starts from the set $((\frac{1}{2}, 1) \times (\frac{1}{2}, 1)) \cup (\{\frac{1}{2}\} \times (\frac{1}{2}, 1)) \cup ((\frac{1}{2}, 1) \times \{\frac{1}{2}\})$ which is represented by darker blue points in Figure 3.1, initially flows within K with vector field $[1 \ 1]^\top$. According to the definition of G , points in set $(\{1\} \times [\frac{1}{2}, 1]) \cup ([\frac{1}{2}, 1] \times \{1\})$, i.e., the points in green, are mapped via G to either outside of K (to a point in $\{x \in \mathbb{R}^2 : x_1 \in [\frac{1}{4}, \frac{1}{2}), x_2 = \frac{1}{2}\} \cup \{x \in \mathbb{R}^2 : x_2 \in [\frac{1}{4}, \frac{1}{2}), x_1 = \frac{1}{2}\}$) or mapped inside K (to a point in $\{x \in \mathbb{R}^2 : x_1 \in [\frac{1}{2}, \frac{3}{4}], x_2 = \frac{1}{2}\} \cup \{x \in \mathbb{R}^2 : x_2 \in [\frac{1}{2}, \frac{3}{4}], x_1 = \frac{1}{2}\}$). Finally, a solution that starts from $(\frac{1}{2}, \frac{1}{2})$, the yellow dot in Figure 3.1, can flow either inside or outside of K due to the second piece in the definition of F . In summary, using the solution-based approach, from

every point in K , there exists at least one complete solution that stays in K . \triangle

Similar to pure continuous-time and pure discrete-time systems, the invariance principle introduced by [65], presented originally by LaSalle [66] for differential and difference equations, is important to study convergence and stability for hybrid dynamical systems. The LaSalle's invariance principle states that bounded solutions converge to the largest invariant subset of the set, where the derivative or the difference of a suitable energy function is zero. Among the two properties that induces such invariance, i.e., backward and forward invariance [61, Definition 6.19], tools to identify the largest weak forward invariant sets are meaningful to derive stability of hybrid system. We dedicate one result in Chapter 5 to such task for estimating the largest forward invariant set for a given \mathcal{H} .

When every maximal solution starting from K stays in K , we say the set is forward pre-invariant for \mathcal{H} . This notion was introduced in [61, Definition 6.25] as “strong forward pre-invariance” in the context of invariance principles.

Definition 3.1.4 (*forward pre-invariance*) *The set $K \subset \mathbb{R}^n$ is said to be forward pre-invariant for \mathcal{H} if every $\phi \in \mathcal{S}_{\mathcal{H}}(K)$ has $\text{rge } \phi \subset K$. \square*

Finally, the strongest version of forward invariance property that requires not only that every maximal solution starting from K stays in K , but also requires completeness of all maximal solutions.

Definition 3.1.5 (forward invariance) *The set $K \subset \mathbb{R}^n$ is said to be forward invariant for \mathcal{H} if every $\phi \in \mathcal{S}_{\mathcal{H}}(K)$ is complete and satisfies $\text{rge } \phi \subset K$. \square*

The following example illustrates the concept of a forward invariant set.

Example 3.1.6 (*forward invariant set*) *Consider the hybrid system given by*

$$\mathcal{H} \begin{cases} x \in C & \dot{x} = F(x) := \begin{bmatrix} -|x_1|x_2 \\ 0 \end{bmatrix} \\ x \in D & x^+ = G(x) := x, \end{cases} \quad (3.1)$$

where the flow set is $C = \{x \in \mathbb{R}^2 : |x| \leq 1, x_1 x_2 \geq 0\}$ and the jump set is $D = \{x \in \mathbb{R}^2 : |x| \leq 1, x_1 x_2 \leq 0\}$. We observe that during flow, solutions evolve continuously within the unit circle centered at the origin; while at jumps, solutions remain at the current location. In fact, the set $K_1 = C_1 \cup D_1$ is forward invariant for \mathcal{H} , where $C_1 = \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, |x| \leq 1\}$ and $D_1 = \{x \in \mathbb{R}^2 : x_1 \leq 0, x_2 \geq 0, |x| \leq 1\}$. We show this property holds true applying results in Chapter 4 in Example 4.1.9. \triangle

The relationship among the four notions is summarized in the diagram in Figure 3.2.

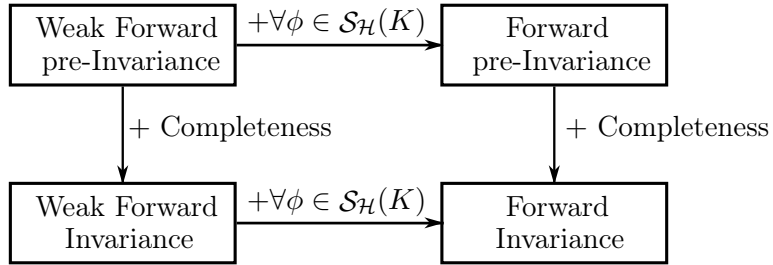


Figure 3.2: Relationships of the notions of forward invariance for a set K .

In many control applications, the properties of “every solution” start from K stays in set K is essential. For instance, safety control methodologies, such as the use of control barrier functions, call for such strict constraints. It is intuitive that when some solutions escape the “safe set,” safety constraints are violated. Hence, in the forth coming Chapter 6, we dedicate control design and synthesis to render a set forward pre-invariant and forward invariant.

Remark 3.1.7 In [49], viable and invariant sets concepts are introduced for autonomous hybrid systems that are modeled in term of impulsive differential inclusions. The viability property in [49] is equivalent to the weak forward invariance in Definition 3.1.2, while the invariant property in [49] is equivalent to the definition of forward pre-invariance in Definition 3.1.4.

3.2 The Robust Forward Invariant Sets

For hybrid systems with disturbances, i.e., \mathcal{H}_w given as in (1.1), we formally define the notions of robust forward invariant sets. In particular, a set K enjoys robust forward invariance when the state evolution begins from K and stays within K regardless of the value of the disturbance w . Similar to the nominal case, we present four classes of sets depends on solution pair behavior of \mathcal{H}_w . Such properties are uniformly in the allowed disturbances w . First, we introduce the weak versions where not every maximal solution pair to \mathcal{H}_w is necessarily that starts from K stays within K .

Definition 3.2.1 (*robust weak forward pre-invariance*) *The set $K \subset \mathbb{R}^n$ is said to be robustly weakly forward pre-invariant for \mathcal{H}_w if for every $x \in K$ there exists one solution pair $(\phi, w) \in \mathcal{S}_{\mathcal{H}_w}(x)$ such that $\text{rge } \phi \subset K$. \square*

When completeness of solution pairs are required, it leads to the notion of robust weak forward invariant set

Definition 3.2.2 (*robust weak forward invariance*) *The set $K \subset \mathbb{R}^n$ is said to be robustly weakly forward invariant for \mathcal{H}_w if for every $x \in K$ there exists a complete $(\phi, w) \in \mathcal{S}_{\mathcal{H}_w}(x)$ such that $\text{rge } \phi \subset K$. \square*

The following notions are considered stronger than the ones in Definition 3.2.1 because all maximal solution pairs that start from the set K are required to stay within K .

Definition 3.2.3 (*robust forward (pre-)invariance of a set*) *The set $K \subset \mathbb{R}^n$ is said to be robustly forward pre-invariant for \mathcal{H}_w if every $(\phi, w) \in \mathcal{S}_{\mathcal{H}_w}(K)$ is such that $\text{rge } \phi \subset K$. \square*

Then, with existence of nontrivial solution pairs from every $x \in K$ and completeness of every maximal solution pair, we derive the strongest notion of forward invariance as follows.

Definition 3.2.4 (*robust forward invariance of a set*) The set $K \subset \mathbb{R}^n$ is said to be robustly forward invariant for \mathcal{H}_w if for every $x \in K$ there exists a solution pair to \mathcal{H}_w and every $(\phi, w) \in \mathcal{S}_{\mathcal{H}_w}(K)$ is complete and such that $\text{rge } \phi \subset K$. \square

The following example illustrates the concept of weakly forward invariant

Example 3.2.5 (*robustly weakly forward invariant set*) Consider a variation of hybrid system \mathcal{H} in Example 3.1.6 with disturbances given by

$$\mathcal{H}_w \begin{cases} (x, w_c) \in C_w & \dot{x} = F_w(x, w_c) := |x_1| \begin{bmatrix} -x_2 \\ w_c x_1 \end{bmatrix} \\ (x, w_d) \in D_w & x^+ \in G_w(x, w_d) := \{R(\theta)x : \theta \in [w_d, -w_d]\}, \end{cases}$$

where $R(\theta) := \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ is a rotation matrix, $C_w := \{(x, w_c) \in \mathbb{R}^2 \times \mathbb{R} : 0 \leq w_c \leq |x| \leq 1, x_1 x_2 \geq 0\}$, and $D_w := \{(x, w_d) \in \mathbb{R}^2 \times \mathbb{R} : x_1 x_2 \leq 0, |x| \leq 1, -\frac{\pi}{4} \leq w_d \leq 0\}$. As shown in Figure 3.3, the projections of C_w and D_w onto \mathbb{R}^2 are given by $\Pi_c^w(C_w) = C_1 \cup C_2$ on \mathbb{R}^2 with $C_2 = \{x \in \mathbb{R}^2 : x_1 \leq 0, x_2 \leq 0, |x| \leq 1\}$ and by $\Pi_d^w(D_w) = D_1 \cup D_2$ with $D_2 = \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \leq 0, |x| \leq 1\}$, respectively.¹ Based on provided dynamics, solutions travel counter-clockwise during flows, while they either rotate clockwise or counter-clockwise during jumps. As a result, solutions can evolve in any of the four quadrants in \mathbb{R}^2 , either by flow or jump.

Note in Example 3.1.6, the set K_1 is forward invariant for \mathcal{H} . When small disturbances are introduced, solution pairs may escape the set of interest as shown in this example, namely, the set K_1 is only weakly forward invariant uniformly in the given disturbances w . We verify this property using the results in Chapter 4 in Example 4.2.6. \triangle

In the upcoming Chapter 6, Definition 3.2.1 - Definition 3.2.4 are presented in the context of *robust controlled forward invariance* properties of sets for $\mathcal{H}_{u,w}$ in

¹We use the same definitions for K_1 , C_1 , and D_1 as in Example 3.1.6.

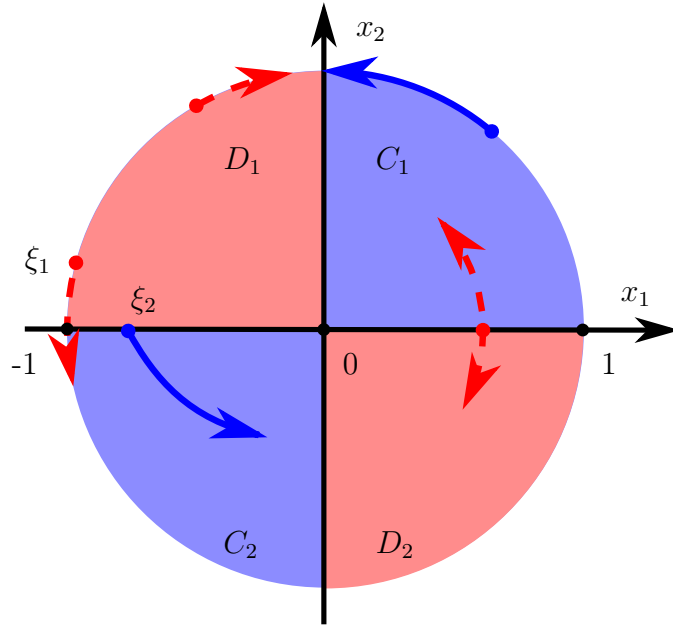


Figure 3.3: Projection onto the state space of flow and jump sets of the system in Example 3.2.5. The blue solid arrows indicate possible hybrid arcs during flow, while the red dashed arrows indicate possible hybrid arcs during jumps.

(1.2) under the effect of a given state-feedback pair (κ_c, κ_d) .

Chapter 4

Sufficient Conditions to Verify Forward Invariance for Hybrid Systems

In general, it is very difficult to directly check forward invariance of a set from the definitions, as that would require checking solutions explicitly. The solution based approach is even more exhausting for hybrid inclusions as the solutions are not unique and not necessarily bounded or complete. Therefore, in this chapter, when possible, solution independent conditions to check if a set enjoys some forward invariance properties are provided.

The presented conditions are sufficient ones to guarantee forward invariance and robust forward invariance for \mathcal{H} in (2.1) and \mathcal{H}_w in (1.1), respectively. Among these conditions, only a few is necessary since we are interested in studying a very generic class of systems without regularities such as bounded growth or linear growth during flows. We end this chapter with discussions regarding to the presented conditions that are also necessary.

4.1 Sufficient Conditions for Nominal Forward Invariant Sets

We present the sufficient conditions for forward invariance of a given set K for \mathcal{H} that involve the data (C, F, D, G) . For the discrete dynamics, namely, the jumps, such conditions involve the understanding of where G maps the state to. Inspired by the well-known Nagumo Theorem [13], for the continuous dynamics, namely, the flows, our conditions use the concept of tangent cone to the closed set K . The tangent cone at a point $x \in \mathbb{R}^n$ of a closed set $K \subset \mathbb{R}^n$ is defined using the Dini derivative of the distance to the set, and is given by¹

$$T_K(x) = \left\{ \omega \in \mathbb{R}^n : \liminf_{\tau \searrow 0} \frac{|x + \tau\omega|_K}{\tau} = 0 \right\}. \quad (4.1)$$

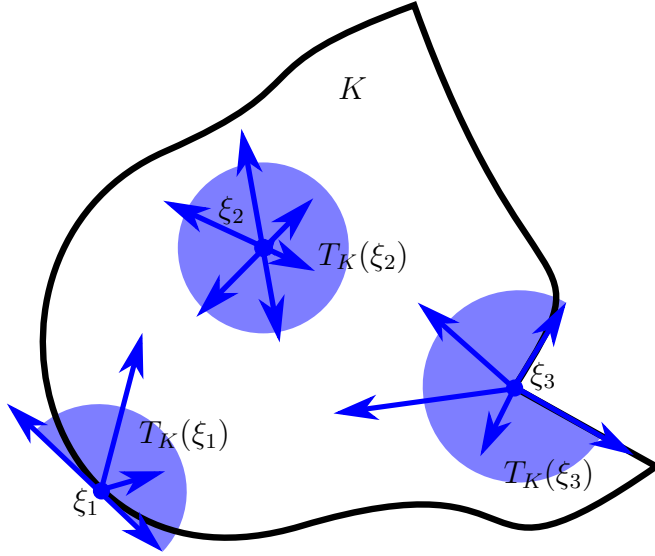


Figure 4.1: Tangent cone of a closed set K at $\xi_1, \xi_2, \xi_3 \in K$.

As shown in Figure 4.1, the set $K \in \mathbb{R}^2$ is closed. At points $\xi_1, \xi_2, \xi_3 \in K$, the tangent cone directions are represented by the light blue cones and the blue

¹In other words, ω belongs to $T_K(x)$ if and only if there exist sequences $\tau_i \searrow 0$ and $\omega_i \rightarrow \omega$ such that $x + \tau_i \omega_i \in K$ for all $i \in \mathbb{N}$; see also [3, Definition 1.1.3]. The latter property is further equivalent to the existence of sequences $x_i \in K$ and $\tau_i > 0$ with $x_i \rightarrow x, \tau_i \searrow 0$ such that $\omega = \lim_{i \rightarrow \infty} (x_i - x) / \tau_i$.

vectors initiated from each points are examples of vectors that are included in the tangent cones. Note that for $\xi_2 \in \text{int}K$, $T_K(\xi_2) = \mathbb{R} \times \mathbb{R}$.

In the literature (see, e.g., [67, Definition 4.6] and [49]), this tangent cone is also known as the sequential Bouligand tangent cone or contingent cone. In contrast to the Clarke tangent cone introduced in [67, Remark 4.7], which is always a closed convex cone for every $x \in K$, the tangent cone (possibly nonconvex) we consider in this work includes all vectors that point inward to the set K or that are tangent to the boundary of K .²

Our sufficient conditions for forward invariance require part of the data of \mathcal{H} and the set K to satisfy the following mild assumption.

Assumption 4.1.1 *The sets K, C , and D are such that $K \subset \overline{C} \cup D$ and that $K \cap C$ is closed. The map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous, locally bounded relative to $K \cap C$, and $F(x)$ is convex for every $x \in K \cap C$.*

The following result is a consequence of the forthcoming Theorem 4.2.4. Sufficient conditions for a given set K to be weakly forward pre-invariant and weakly forward invariant are presented.

Theorem 4.1.2 *(nominal weak forward pre-invariance and weak forward invariance) Given $\mathcal{H} = (C, F, D, G)$ as in (2.1) and a set K , suppose K, C, D , and F satisfy Assumption 4.1.1. The set K is weakly forward pre-invariant for \mathcal{H} if the following conditions hold:*

4.1.2.1) *For every $x \in K \cap D$, $G(x) \cap K \neq \emptyset$;*

4.1.2.2) *For every $x \in \widehat{C} \setminus D$, $F(x) \cap T_{K \cap C}(x) \neq \emptyset$;*

where $\widehat{C} := \partial(K \cap C) \setminus L$ and $L := \{x \in \partial C : F(x) \cap T_{\overline{C}}(x) = \emptyset\}$. Moreover, K is weakly forward invariant for \mathcal{H} if, in addition, $K \cap L \subset D$ and, with $K^\star = K \setminus D$,

²Note that, for a convex set, the Bouligand tangent cone coincides with the Clarke tangent cone.

$N\star$) for every $\phi \in \mathcal{S}_{\mathcal{H}}(K^*)$ with $\text{rge } \phi \subset K$, case b.2) in Proposition 2.0.6 does not hold.

This result follows from an application of Theorem 4.2.4 for a hybrid system with zero disturbance. An proof that is independent of Theorem 4.2.4 for Theorem 4.1.2 is provided as follows.

Proof Given K, C, D, F satisfying Assumption 4.1.1, we define the restriction of \mathcal{H} by K , i.e., $\tilde{\mathcal{H}} = (\tilde{C}, F, \tilde{D}, G)$, where $\tilde{C} := K \cap C$ and $\tilde{D} := K \cap D$. Since $K \subset \overline{C} \cup D$, by Definition 2.0.4, there exists a solution to $\tilde{\mathcal{H}}$ from every $\xi \in K$. Let $K_1 = \tilde{D}$ and $K_2 = K \setminus (D \cup L)$ and $K_3 = K \setminus (K_1 \cup K_2)$. By definition, every $\xi \in K_3$ is such that $\xi \in L \setminus D$ and $F(x) \cap T_{\overline{C}}(x) = \emptyset$. Then, item (a) in [61, Lemma 5.26] and Definition 2.0.4 imply there is only trivial solution from ξ to $\tilde{\mathcal{H}}$, in which case we have $\text{rge } \phi \subset K$. Otherwise, in the case where $\phi(0, 0) \in K_1 \cup K_2$, we show there exists $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}$ that is nontrivial and it has $\text{rge } \phi \subset K$ when 4.1.2.1) and 4.1.2.2) hold true. To this end, we construct a nontrivial solution from every $\xi \in K_1 \cup K_2$. Since $K_1 \cap K_2 = \emptyset$, we have the following two cases:

- i) If $\xi \in K_1$, then $\xi \in D$; hence, a jump is possible from every ξ in K_1 . Let $\tilde{\phi}(0, 0) = \xi$. By condition 4.1.2.1), there exists $\tilde{\phi}(0, 1) \in G(\xi)$ such that $\tilde{\phi}(0, 1) \in K$.
- ii) when $\xi \in K_2$: since $\xi \in \overline{C} \setminus D$, nontrivial solutions can only evolve by flowing. Conditions enforced by Assumption 4.1.1 imply that \tilde{C} is closed, F is outer semicontinuous, locally bounded and convex valued on \tilde{C} . Since $T_{\tilde{C}}(x) = \mathbb{R}^n$ for every $x \in (\text{int } \tilde{C}) \setminus (D \cup L)$, item 4.1.2.2) implies that $F(x) \cap T_{\tilde{C}}(x) \neq \emptyset$ for every $x \in K_2$. Then, by an application of (VC) in Proposition 2.0.6, there exists a nontrivial solution ϕ_b to $\tilde{\mathcal{H}}$ from every $\xi \in K_2$. By item (S1) in Definition 2.0.4, such a nontrivial solution ϕ_b is absolutely continuous on $[0, \varepsilon]$, for some $\varepsilon > 0$, with $\phi_b(0) = \xi$, $\dot{\phi}_b(t) \in F(\phi_b(t))$ for almost all $t \in [0, \varepsilon]$ and $\phi_b(t) \in \tilde{C}$ for all $t \in (0, \varepsilon]$. By closedness of \tilde{C} , we have $\phi_b(t, 0) \in K$ for every $t \in [0, \varepsilon]$.

The above shows that there exists a nontrivial solution from every point in $K_1 \cup K_2$. It also shows that from every point in K_1 , solutions can be extended to continue jumping in K_1 using the construction in case i), while from points in K_2 , solutions can be extended using the construction in case ii). Moreover, since $K \cap C$ is closed, by definition of solutions, such extensions can be defined so as they do not leave K_2 by flowing (since they can always be extended using the argument in ii)). As a consequence, from every $\xi \in K$, there exists at least one maximal solution $\tilde{\phi}$ to $\tilde{\mathcal{H}}$ that stays in K .

Next, we prove that each such $\tilde{\phi}$ is also a maximal solution to \mathcal{H} . If $\tilde{\phi}$ is complete, then $\tilde{\phi}$ is already maximal. Consider the case that $\tilde{\phi}$ is not complete. Proceeding by contradiction, suppose $\tilde{\phi}$ is not maximal for \mathcal{H} , meaning that there exists ϕ such that $\phi(t, j) = \tilde{\phi}(t, j)$ for every $(t, j) \in \text{dom } \tilde{\phi}$ and $\text{dom } \phi \setminus \text{dom } \tilde{\phi} \neq \emptyset$. Let $(T, J) = \sup \text{dom } \tilde{\phi}$. If $(T, J) \in \text{dom } \tilde{\phi}$, then, $\tilde{\phi}(T, J) \in K$ and we have the two following cases:

- $\tilde{\phi}(T, J) \in K_1 \cup K_2$, 4.1.2.1) and closeness of \tilde{C} imply that, using the arguments in i) and ii) above, it is possible for ϕ to satisfy $\phi(t, j) \in K$ for some $(t, j) \in \text{dom } \phi \setminus \text{dom } \tilde{\phi}$. By definition of solution, this contradicts with maximality of $(\tilde{\phi}, \tilde{w})$ for $\tilde{\mathcal{H}}$.
- $\tilde{\phi}(T, J) \in K_3$, by definition of L , $F(\tilde{\phi}(T, J)) \cap T_{\tilde{C}}(\tilde{\phi}(T, J)) = \emptyset$. Hence, $\sup \text{dom } \phi = (T, J)$, which contradicts with the assumption $\text{dom } \phi \setminus \text{dom } \tilde{\phi} \neq \emptyset$.

If $(T, J) \notin \text{dom } \tilde{\phi}$, according to Proposition 2.0.6, only b.2) holds.³ In such a case, there is no function $z : \overline{I^J} \rightarrow \mathbb{R}^n$ satisfying the conditions in b.2) of Proposition 2.0.6, which are needed to have a ϕ such that $\text{dom } \phi \setminus \text{dom } \tilde{\phi} \neq \emptyset$. Thus, K is weakly forward pre-invariant for \mathcal{H} .

Finally, we prove that K is weakly forward invariant for \mathcal{H} when, in addition, \mathcal{H} and K satisfy condition N \star) and $K \cap L \subset D$. We proceed by showing that from

³Case a) does not hold due to $\tilde{\phi}$ not being complete, while b.1) and c) do not hold because $(T, J) \notin \text{dom } \tilde{\phi}$.

every $\xi \in K$ there exists at least one complete solution ϕ for the K restricted system $\tilde{\mathcal{H}}$. This is because such complete solutions are also maximal for the original system \mathcal{H} and have $\text{rge } \phi \subset K$. First, since $K \cap L \subset D$, there exists nontrivial solution to $\tilde{\mathcal{H}}$ from every $x \in (K \cap L)$ via argument i) by jump. Hence, the existence of nontrivial solutions to $\tilde{\mathcal{H}}$ from every $\xi \in K$ is guaranteed by the fact that K is weakly forward pre-invariant for \mathcal{H} . By the closedness of $K \cap C$ and continuity of the function z in (VC) of Proposition 2.0.6, for every $\xi \in K \setminus D$ case b.1) is excluded for some $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(\xi)$. Moreover, case b.2) does not hold for all maximal solutions as required by condition N \star). Due to condition 4.1.2.1), from every $\xi \in K \cap D$, at least one solution can be extended by jumping to points in $G(x) \cap K$, from where the solution can be extended either by jumping or flowing afterward. Thus, case c) in Proposition 2.0.6 is not possible for such solutions. Hence, according to Proposition 2.0.6, for each $\xi \in K$, there exists $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(\xi)$ such that only a) holds. \square

To illustrate Theorem 4.1.2, we present the following example on \mathbb{R}^2 .

Example 4.1.3 (*weak forward invariant set*) Consider the hybrid system $\mathcal{H} = (C, F, D, G)$ in \mathbb{R}^2 in Example 3.1.3. We apply Theorem 4.1.2 to verify the observation that set K is weakly forward invariant for \mathcal{H} . First, \mathcal{H} and K satisfy Assumption 4.1.1. Then, according to above analysis, condition 4.1.2.1) in Theorem 4.1.2 holds, since for every $x \in K \cap D$, which is $x \in (\{1\} \times [\frac{1}{2}, 1]) \cup ([\frac{1}{2}, 1] \times \{1\})$, $G(x) \cap K \neq \emptyset$. Moreover, noting that $L = \emptyset$, we verify that condition 4.1.2.2) in Theorem 4.1.2 holds: for every point $x \in K \setminus D$, we have

$$T_{K \cap C}(x) = \begin{cases} \mathbb{R}_{\geq 0} \times \mathbb{R} & \text{if } x \in \{\frac{1}{2}\} \times (\frac{1}{2}, 1) \\ \mathbb{R} \times \mathbb{R}_{\geq 0} & \text{if } x \in (\frac{1}{2}, 1) \times \{\frac{1}{2}\} \\ \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} & \text{if } x = (\frac{1}{2}, \frac{1}{2}). \end{cases}$$

As a result, for every $x \in K \setminus D$, $F(x) \cap T_{K \cap C}(x) \neq \emptyset$. Then, since F is linear everywhere on C , condition N \star) in Theorem 4.1.2 holds. Therefore, since $L = \emptyset$, according to Theorem 4.1.2, K is weakly forward invariant for \mathcal{H} . \triangle

The next result, which is a consequence of Theorem 4.2.8 provides sufficient conditions for a set K to be forward pre-invariant and forward invariant for \mathcal{H} .

Theorem 4.1.4 (*nominal forward pre-invariance and forward invariance*) *Given $\mathcal{H} = (C, F, D, G)$ as in (2.1) and a set $K \subset \mathbb{R}^n$, suppose K, C, D , and F satisfy Assumption 4.1.1 and that F is locally Lipschitz on $(\partial K + \delta \mathbb{B}) \cap C$ for some $\delta > 0$. Let \widehat{C} and L be given as in Theorem 4.1.2. The set K is forward pre-invariant for \mathcal{H} if the following conditions hold:*

4.1.4.1) $G(K \cap D) \subset K$;

4.1.4.2) For every $x \in \widehat{C}$, $F(x) \subset T_{K \cap C}(x)$.

Moreover, K is forward invariant for \mathcal{H} if, in addition, $K \cap L \subset D$ and, with $K^* = K \cap C$, item N \star) in Theorem 4.1.2 holds.

This result follows from an application of Theorem 4.2.8 for a hybrid system with zero disturbance. An proof that is independent of Theorem 4.2.8 for Theorem 4.1.4 is provided as follows.

Proof Since conditions 4.1.4.1) and 4.1.4.2) imply conditions 4.1.2.1) and 4.1.2.2) in Theorem 4.1.2, respectively, under Assumption 4.1.1, there exist one nontrivial solution to \mathcal{H} from every point in K . Next, proceeding by contradiction, we show that K is forward pre-invariant for \mathcal{H} when conditions 4.1.4.1) and 4.1.4.2) hold. Suppose there exists a solution $\phi \in \mathcal{S}_{\mathcal{H}}(K)$ such that $\text{rge } \phi \setminus K \neq \emptyset$. Then, there exists $(t^*, j^*) \in \text{dom } \phi$ such that $\phi(t^*, j^*) \notin K$, i.e., ϕ eventually leaves K in finite hybrid time.⁴ We have the following cases:

- i) The solution ϕ “leaves K by jumping:” namely, $\phi(t, j) \in K$ for all $(t, j) \in \text{dom } \phi$ with $t + j < t^* + j^*$, and $\phi(t^*, j^* - 1) \in K \cap D$. Since $\phi(t^*, j^* - 1) \in K \cap D$, item 4.1.4.1) implies $\phi(t^*, j^*) \in K$, which is a contradiction. Then, it must

⁴Note that when $\text{rge } \phi \subset K$ and $\lim_{t+j \rightarrow \sup_t \text{dom } \phi + \sup_j \text{dom } \phi} \phi(t, j) = \infty$ (that is, ϕ stays in K but escapes to infinity, potentially in finite hybrid time) corresponds to a solution that satisfies the definition of forward invariance for K .

be the case that ϕ left K by flowing. We consider this possibility in the next item.

- ii) The solution ϕ “leaves K by flowing:” by definition of solution, ϕ leaves $K \cap C$ and enters $\overline{C} \setminus K$. Then, there exists a hybrid time instant $(\tau^*, j^*) \in \text{dom } \phi$ such that $\phi(t, j^*) \in \overline{C} \setminus K$ for all $t \in (\tau^*, t^*]$ and $\tau^* < t^*$ is arbitrarily small and positive. Moreover, by closedness of $K \cap C$, $\phi(\tau^*, j^*) \in \partial(K \cap C)$. Let $t \mapsto \chi(t) \in K \cap C$ be such that for every $t \in [\tau^*, t^*]$,

$$|z(t)|_{K \cap C} = |z(t) - \chi(t)|, \quad (4.2)$$

where $z(t) = \phi(t, j^*)$ for all $t \in [\tau^*, t^*]$. Such points exist because of the closedness of $K \cap C$. By definition of solution to \mathcal{H} , the function $t \mapsto |z(t)|_{K \cap C}$ is absolutely continuous. Thus, for almost every $t \in [\tau^*, t^*]$, $\frac{d}{dt}|z(t)|_{K \cap C}$ exists and equals to the Dini derivative of $|z(t)|_{K \cap C}$. Let t be such that both $\frac{d}{dt}|z(t)|_{K \cap C}$ and $\dot{z}(t)$ exist. We have

$$\begin{aligned} & \frac{d}{dt}|z(t)|_{K \cap C} \\ &= \liminf_{h \searrow 0} \frac{|z(t) + h\dot{z}(t)|_{K \cap C} - |z(t)|_{K \cap C}}{h}, \end{aligned}$$

which, by definition of $\chi(t)$ and (4.7), satisfies

$$\begin{aligned} & \frac{|z(t) + h\dot{z}(t)|_{K \cap C} - |z(t)|_{K \cap C}}{h} \\ & \leq \frac{|z(t) - \chi(t)| + |\chi(t) + h\dot{z}(t)|_{K \cap C} - |z(t)|_{K \cap C}}{h} = \frac{|\chi(t) + h\dot{z}(t)|_{K \cap C}}{h} \\ & \leq \frac{|\chi(t) + h\omega|_{K \cap C}}{h} + |\dot{z}(t) - \omega|, \end{aligned}$$

for every $\omega \in T_{K \cap C}(\chi(t))$. Moreover, for every such ω ,

$$\liminf_{h \searrow 0} \frac{|\chi(t) + h\omega|_{K \cap C}}{h} = 0$$

by definition of the tangent cone in (4.1). Hence, we have

$$\frac{d}{dt}|z(t)|_{K \cap C} \leq \liminf_{h \searrow 0} \frac{|\chi(t) + h\omega|_{K \cap C}}{h} + |\dot{z}(t) - \omega| = |\dot{z}(t) - \omega|.$$

Thus, for almost every $t \in [\tau^*, t^*]$,

$$\frac{d}{dt}|z(t)|_{K \cap C} \leq |\dot{z}(t)|_{T_{K \cap C}(\chi(t))}.$$

Since $K \cap C$ is closed, by definition, $\chi(t) \in K \cap C$ for every $t \in [\tau^*, t^*]$. Condition 4.1.4.2) implies that for almost all $t \in [\tau^*, t^*]$, we have

$$\begin{aligned} \frac{d}{dt}|z(t)|_{K \cap C} &\leq |\dot{z}(t)|_{T_{K \cap C}(\chi(t))} \\ &\leq |\dot{z}(t)|_{F(\chi(t))}. \end{aligned} \tag{4.3}$$

Then, because of the mapping F is locally Lipschitz on $\partial K + \delta \mathbb{B} \cap C$ for some $\delta > 0$, we can construct a neighborhood U' of $z(t)$ such that $U' \subset \chi(t) + \delta \mathbb{B}$ and $\chi(t) \in U'$ for every $t \in [\tau^*, t^*]$ and for which there exists a constant $\lambda > 0$ satisfying

$$F(z(t)) \subset F(\chi(t)) + \lambda|z(t) - \chi(t)|\mathbb{B}$$

for every $t \in [\tau^*, t^*]$. Hence, for every $t \in [\tau^*, t^*]$ and every $\eta \in F(z(t))$,

$$|\eta|_{F(\chi(t))} \leq \lambda|z(t) - \chi(t)|.$$

Moreover, since $\dot{z}(t) \in F(z(t))$ together with (4.8) and (4.7), we have that

$$\begin{aligned} \frac{d}{dt}|z(t)|_{K \cap C} &\leq |\dot{z}(t)|_{F(\chi(t))} \\ &\leq \lambda|z(t) - \chi(t)| = \lambda|z(t)|_{K \cap C}. \end{aligned}$$

Then, by the Gronwall Lemma (see [68, Lemma A.1]), for every $t \in [\tau^*, t^*]$,

$$|z(t)|_{K \cap C} = 0.$$

Since $K \cap C$ is closed, $\phi(t^*, j^*) \in K \cap C$, which contradicts the definition of t^* . Thus, there does not exist maximal solution $\phi \in \mathcal{S}_{\mathcal{H}}(K)$ that eventually leaves $K \cap C$ by flowing.

Therefore, every $\phi \in \mathcal{S}_{\mathcal{H}}(K)$ is such that $\text{rge} \subset K$.

Following the proof of Theorem 4.1.2, when $K \cap L \subset D$, with 4.1.2.1) and 4.1.2.2) satisfied, there exists a nontrivial solution ϕ with $\phi(0, 0) = \xi$ to \mathcal{H} from every $\xi \in K$. To complete the proof, we show that when \mathcal{H} and K also satisfy condition N \star), the set K is forward invariant for \mathcal{H} . Since there exists a nontrivial

solution from every point in K , we show that only case a) in Proposition 2.0.6 holds for every $\phi \in \mathcal{S}_{\mathcal{H}}(K)$. Proceeding by contradiction, suppose that there exists $\phi^* \in \mathcal{S}_{\mathcal{H}}(K)$ that is not complete. Let $(T, J) = \sup \text{dom } \phi^*$ and $T + J < \infty$. By the closedness of $K \cap C$, ϕ^* does not end as described in case b.1). Since K is forward pre-invariant for \mathcal{H} , every $\phi \in \mathcal{S}_{\mathcal{H}}(K \cap C)$ is such that $\text{rge } \phi \subset K$. Using arguments similar to those in the proof of Theorem 4.1.2, case b.2) does not hold for ϕ^* by virtue of condition $N\star$). Then, according to Proposition 2.0.6, ϕ^* satisfies c) therein. But 4.1.4.1) leads to a contradiction of the maximality of ϕ^* . More precisely, item 4.1.4.1) implies that $G(\phi^*(T, J-1)) \subset K$, so $\phi^*(T, J) \in K \subset \overline{C} \cup D$, and, hence, the solution ϕ^* can be extended either by flow or jump using the arguments in i) and ii) of proof for Theorem 4.1.2. Thus, by an application of Proposition 2.0.6, all maximal solutions to \mathcal{H} that start from K are complete and have $\text{rge } \phi \subset K$. \square

Remark 4.1.5 *Some of the conditions in Theorem 4.1.2 and Theorem 4.1.4 are weaker than those required by results in [53]. The construction of the set L in items 4.1.2.2) and 4.1.4.2) is inspired by the viability domain in [3, Definition 1.1.5]. Note that when $N\star$) holds, completeness of maximal solutions is guaranteed by ensuring that $K \cap L \subset D$, which guarantees that solutions can continue to evolve from L via a jump.*

The following example is used to illustrate Theorem 4.1.2 and Theorem 4.1.4.

Example 4.1.6 (solutions with finite escape time) *Consider the hybrid system $\mathcal{H} = (C, F, D, G)$ in \mathbb{R}^2 with system data given by*

$$F(x) := \begin{bmatrix} 1 + x_1^2 \\ 0 \end{bmatrix} \quad \forall x \in C := \{x \in \mathbb{R}^2 : x_1 \in [0, \infty), x_2 \in [-1, 1]\},$$

$$G(x) := \begin{bmatrix} x_1 + \mathbb{B} \\ x_2 \end{bmatrix} \quad \forall x \in D := \{x \in \mathbb{R}^2 : x_1 \in [0, \infty), x_2 = 0\}.$$

Let $K = C$ and note the following properties of maximal solutions to \mathcal{H} :

- For some $x \in K$, there exists $\phi = (\phi_1, \phi_2) \in \mathcal{S}_{\mathcal{H}}(x)$ with $\text{rge } \phi \subset K$, but is not complete due to $\lim_{t \searrow t^*} \phi_1(t, 0) = \infty$ with $t^* < \infty$; for instance, from $x = (0, 1)$, the solution given by $\phi(t, 0) = (\tan(t), 1)$ for every $(t, 0) \in \text{dom } \phi$ has its ϕ_1 component escape to infinite as t approaches $t^* = \pi/2$;
- From points in $D \subset K$, there exist maximal solutions that leave K and are not complete: such solutions end after a jump because their x_1 component is mapped outside of K .

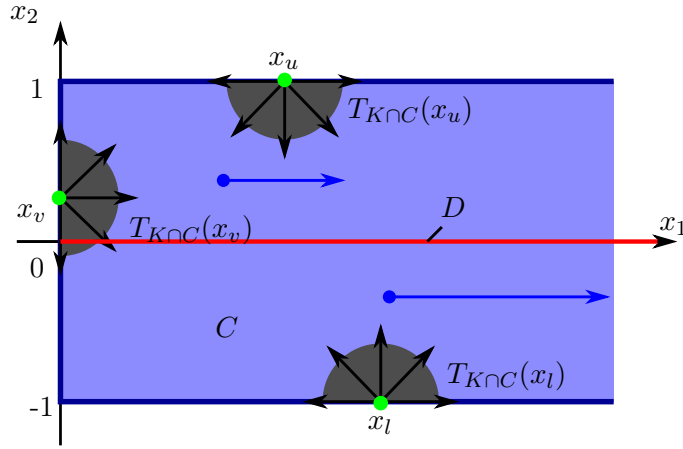


Figure 4.2: Flow and jump sets of the system in Example 4.1.6.

Thus, we verify weak forward pre-invariance of K by applying Theorem 4.1.2. The sets K, C, D and the map F satisfy Assumption 4.1.1 by construction and condition 4.1.2.1) holds for \mathcal{H} by definition of G, D and K . Since $L = \emptyset$, condition 4.1.2.2) holds because for every $x \in \widehat{C}$, $F(x)$ points horizontally, and

$$T_{K \cap C}(x) = \begin{cases} \mathbb{R} \times \mathbb{R}_{\leq 0} & \text{if } x \in \{x \in \mathbb{R}^2 : x_1 \in (0, \infty), x_2 = 1\} \\ \mathbb{R} \times \mathbb{R}_{\geq 0} & \text{if } x \in \{x \in \mathbb{R}^2 : x_1 \in (0, \infty), x_2 = -1\} \\ \mathbb{R}_{\geq 0} \times \mathbb{R}_{\leq 0} & \text{if } x = (0, 1) \\ \mathbb{R}_{\geq 0} \times \mathbb{R} & \text{if } x \in \{x \in \mathbb{R}^2 : x_1 = 0, x_2 \in (-1, 1)\} \\ \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} & \text{if } x = (0, -1). \end{cases}$$

Tangent cones of $K \cap C$ at points x_u, x_v , and x_l of K are shown in Figure 4.2.

Now, consider the same data but with G replaced by $G'(x) = G(x) \cap (\mathbb{R}_{\geq 0} \times \mathbb{R})$ for each $x \in D$. The set $K = C$ is forward pre-invariant for this system. This is because maximal solutions are not able to jump out of K as G' only maps x_1 components of solutions to $[0, +\infty)$. More precisely, the conditions in Theorem 4.1.4 hold: we have $G'(D \cap K) \subset K$, and Assumption 4.1.1 and condition 4.1.4.2) hold as discussed above. \triangle

4.1.1 Sufficient Conditions for N \star)

In Theorem 4.1.2 and Theorem 4.1.4, item N \star) excludes case b.2) in Proposition 2.0.6, where solutions escape to infinity in finite time during flows. In fact, when every solution ϕ to $\dot{x} \in F(x)$ with $\phi(0, 0) \in K^\star$ does not have a finite escape time, namely, there does not exist $t^* < \infty$ such that $\lim_{t \searrow t^*} |\phi(t)| = \infty$, item N \star) holds for \mathcal{H} and K^\star as defined in Theorem 4.1.2 and Theorem 4.1.4, respectively. Although, in principle, such a condition is solution dependent, it can be guaranteed without solving for solutions when F is single valued and globally Lipschitz. Moreover, we provide several other alternatives in the next result.

Lemma 4.1.7 (sufficient conditions for completeness) *Given $\mathcal{H} = (C, F, D, G)$ and a set $K \subset \mathbb{R}^n$, suppose K, C, D , and F satisfy Assumption 4.1.1 Condition N \star) holds if*

4.1.7.1) K^\star is compact; or

4.1.7.2) F has linear growth on K^\star .

Proof Let $\phi \in \mathcal{S}_{\mathcal{H}}(K^\star)$ with $\text{rge } \phi \subset K$ be as described by b.2) in Proposition 2.0.6; namely, $t \mapsto \phi(t, J)$ defined on I^J , where $(T, J) = \sup \text{dom } \phi, T+J < \infty$ and, for some $t^J, I^J = [t^J, T)$. Since $t \mapsto \phi(t, J)$ is locally absolutely continuous on I^J , $\lim_{t \rightarrow T} \phi(t, J)$ is finite or infinity. If it is finite, then $t \mapsto \phi(t, j)$ can be extended to $\overline{I^J}$, which contradicts with b.2). Then, it has to be that $\lim_{t \rightarrow T} \phi(t, J)$ is

infinity. When 4.1.7.1) holds, $\lim_{t \rightarrow T} \phi(t, J)$ being infinity is a contradiction since K^* is compact.

When 4.1.7.2) holds, there exists $M > 0$ such that, for each $x \in K^*$, $\eta'(x) := \sup\{|\eta| : \eta \in F(x)\} \leq M(|x| + 1)$. Because of linearity, solutions to $\dot{\xi} = M(|\xi| + 1)$ are bounded for every $\xi \in \mathbb{R}$ and $t < \infty$. Then, applying the comparison principle, for every $x \in K^*$, solutions to $\dot{\xi} = \eta'(\xi)$ are also bounded for every $t < \infty$. This contradicts with $\lim_{t \rightarrow T} \phi(t, J)$ being infinity. \square

The next example illustrates Theorem 4.1.2, Theorem 4.1.4 and Lemma 4.1.7.

Example 4.1.8 (*weakly forward invariant set*) Consider the hybrid system $\mathcal{H} = (C, F, D, G)$ in \mathbb{R}^2 given by

$$\begin{aligned} F(x) &:= (x_2, -x_1) & \forall x \in C; \\ G(x) &:= (-0.9x_1, x_2) & \forall x \in D, \end{aligned}$$

where $C := \{x \in \mathbb{R}^2 : |x| \leq 1, x_2 \geq 0\}$ and $D := \{x \in \mathbb{R}^2 : x_1 \geq -1, x_2 = 0\}$.

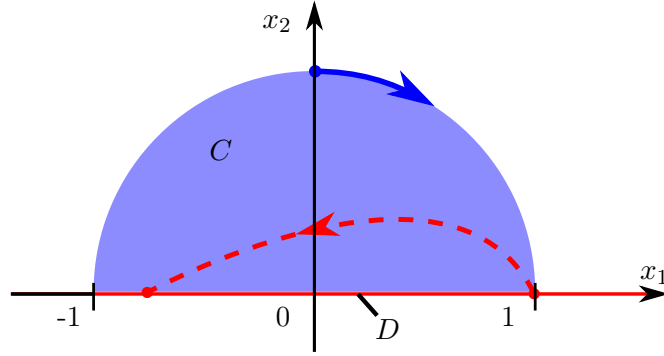


Figure 4.3: Sets and directions of flows/jumps in Example 4.1.8.

The set $K_1 = \partial C$ is weakly forward invariant for \mathcal{H} by Theorem 4.1.2. More precisely, for every $x \in K_1 \cap D$, $G(x) \in K_1$; and for every $x \in \partial(K_1 \cap C) \setminus (D \cup L) = \{x \in \mathbb{R}^2 : |x| = 1, x_2 > 0\}$, since $\langle \nabla(x_1^2 + x_2^2 - 1), F(x) \rangle = 0$, applying item 1) in Lemma A.0.15, we have $F(x) \in T_{K_1 \cap C}(x)$. Notice that the set $L = (0, 1] \times \{0\} \subset D$ for \mathcal{H} by observation. In addition, $K_1 \cap C = \partial C$ is compact, which implies

condition $N\star$) holds by Lemma 4.1.7. Thus, for every $x \in K_1$, there exists one complete solution that stays in K_1 . For example, for every $x \in [-1, 1] \times \{0\}$, there exists one complete solution that is discrete and stays in K_1 (from the origin there is also a complete continuous solution that remains at the origin), but also there exist maximal solutions that flow inside $\{x \in \mathbb{R}^2 : |x| < 1\}$ and leave K_1 .

Now consider $K_2 = C$. It is forward invariant for \mathcal{H} by applying Theorem 4.1.4. In fact, using the observations above, item 4.1.4.2) can be verified via Lemma A.0.15 since $\langle \nabla(x_1^2 + x_2^2 - 1), F(x) \rangle = 0$ for every $x \in \partial(K_2 \cap C) \setminus L = \{x \in \mathbb{R}^2 : |x| = 1, x_2 > 0\} \cup ([-1, 0] \times \{0\})$. \triangle

Condition 4.1.7.2) is typically assumed in the study of viability and invariance of differential inclusions; see, e.g., [3, 49, 69]. Condition 4.1.7.1) does not require F to be Marchaud, but impose boundedness of F and extra properties on $K \cap C$. Note that F is not necessarily required to be Marchaud in the results in this paper since linear growth is not assumed. In Lemma 4.1.7, we require F to be outer semicontinuous, locally bounded, and with nonempty, convex image, which imply that F enjoys all properties of being Marchaud except for the linear growth.

Note that one can replace condition 4.1.4.2) in Theorem 4.1.4 by

4.1.4.2') For every $x \in \partial(K \cap C)$,

$$F(x) \subset T_{K \cap C}(x) \quad \text{if } x \notin \partial C \cap D \quad (4.4)$$

$$F(x) \cap (T_C(x) \setminus T_{K \cap C}(x)) = \emptyset \quad \text{if } x \in \partial C \cap D. \quad (4.5)$$

Note that assumption (4.5) is important as in some cases, having item (4.4) only leads to solutions that escape the set K by flowing as shown in the following example. Consider the hybrid system \mathcal{H} on \mathbb{R}^2 with

$$\begin{aligned} F(x) &= (x_2, -\gamma) & \forall x \in C &:= \{x \in \mathbb{R}^2 : x_1 x_2 \geq 0\} \\ G(x) &= x & \forall x \in D &:= \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 = 0\}, \end{aligned}$$

where $\gamma > 0$. The set $K = \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$ is weakly forward invariant, and the sets K, C, D and the map F satisfies (4.4). However, at the origin, we have $F(0) = (0, -\gamma)$ and

$$\begin{aligned} T_C(0) &= (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \cup (\mathbb{R}_{\leq 0} \times \mathbb{R}_{\leq 0}), \\ T_{K \cap C}(0) &= \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}. \end{aligned}$$

Hence, at the origin, one solution can flow into $C \setminus K$ (the third quadrant) because $F(0) \in T_C(0) \setminus T_{K \cap C}(0)$.

The following example is an application of Theorem 4.1.4 and Lemma 4.1.7.

Example 4.1.9 (*forward invariant set*) Consider the hybrid system given by (3.1) as in Example 3.1.6. Applying Theorem 4.1.4, we show that the set $K_1 = C_1 \cup D_1$ is forward invariant for \mathcal{H} , where $C_1 = \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, |x| \leq 1\}$ and $D_1 = \{x \in \mathbb{R}^2 : x_1 \leq 0, x_2 \geq 0, |x| \leq 1\}$. By construction, K_1, C, D and F satisfy Assumption 4.1.1. Condition 4.1.4.1) holds since G maps the state to its current value. Condition 4.1.4.2) holds since

- for every $x \in \{x \in \partial C_1 : |x| = 1\}$, since $x_1 x_2 \geq 0$,

$$\langle \nabla(x_1^2 + x_2^2), F(x) \rangle = -2|x_1|x_1 x_2 \leq 0;$$

- for every $x \in \{x \in \partial C_1 : |x| \neq 1\}$, $F(x) = (0, 0)$, which leads to $F(x) \in T_{K_1 \cap C}(x)$.

Finally, applying Lemma 4.1.7, N_\star) holds since $K_1 \cap C$ is compact. \triangle

4.1.2 Necessary Conditions

The hybrid inclusions framework allows for an overlap between the flow set C and the jump set D . As a result, the proposed conditions are not necessary to induce forward invariance properties of sets for \mathcal{H} . When existence of nontrivial solutions and completeness are not required for every point in K , as in the “pre” notions, some of these conditions are necessary. In fact, suppose K, C, D , and F satisfy Assumption 4.1.1:

- If K is weakly forward pre-invariant for \mathcal{H} , then for every $x \in (K \cap D) \setminus C$, $G(x) \cap K \neq \emptyset$.
- If K is forward pre-invariant or forward invariant for \mathcal{H} , then condition 4.1.4.1) in Theorem 4.1.4 holds.
- If K is weakly forward invariant or forward invariant for \mathcal{H} , then for every $x \in K \setminus D$, $F(x) \cap T_{K \cap C}(x) \neq \emptyset$.⁵

Moreover, unlike [49, Theorem 3], when the flow map F is Marchaud⁶ and Lipschitz as defined in Definition A.0.3, condition $F(x) \subset T_{K \cap C}(x)$ for every $x \in K \setminus D$ is not necessary as the following example shows.

Example 4.1.10 Consider \mathcal{H} in (2.1) with data $F(x) = \begin{cases} 1 & \text{if } x > -1 \\ [-1, 1] & \text{if } x = -1 \end{cases}$ for each $x \in C := [-1, 1]$, $G(x) := \{-1, 0\}$ for each $x \in D := \{1\}$. By inspection, the set $K = C$ is forward invariant for \mathcal{H} and F is Marchaud and Lipschitz. However, at $x = -1 \in K \setminus D$, $F(-1) \supset -1$ but $-1 \notin T_{K \cap C}(-1)$. \triangle

4.2 Sufficient Conditions for Robust Forward Invariance Properties for \mathcal{H}_w

As an extension to the nominal notions, the robust forward invariance notions for \mathcal{H}_w in Definition 3.2.2 - 3.2.4 capture four types of forward invariance properties, some of which are uniform over disturbances w for \mathcal{H}_w . In this section, Theorem 4.2.4 and Theorem 4.2.8 extend Theorem 4.1.2 and Theorem 4.1.4 to hybrid systems \mathcal{H}_w given in (1.1). These results will be exploited in forward invariance-based control design for hybrid systems (with and without disturbances) in Chapter 6.

⁵A similar claim is presented in [3, Proposition 3.4.1] for continuous-time system.

⁶A map F is Marchaud on $K \cap C$ when Assumption 4.1.1 holds and F has linear growth on $K \cap C$; see [3, Definition 2.2.4].

Results in this section replay on conditions to check for existence of nontrivial solutions to \mathcal{H}_w . Hence, inspired by the conditions guaranteeing existence of solutions to \mathcal{H} (see Proposition 2.0.6), we provide the following result for guaranteeing existence of nontrivial solution pairs to \mathcal{H}_w and characterizing their possible behavior.

Proposition 4.2.1 (*basic existence under disturbances*) *Consider a hybrid system $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$ as in (1.1). Let $\xi \in \overline{\Pi_c^w(C_w)} \cup \Pi_d^w(D_w)$. If $\xi \in \Pi_d^w(D_w)$, or*

(VC_w) there exist $\varepsilon > 0$, an absolutely continuous function $\tilde{z} : [0, \varepsilon] \rightarrow \mathbb{R}^n$ with $\tilde{z}(0) = \xi$, and a Lebesgue measurable and locally essentially bounded function $\tilde{w}_c : [0, \varepsilon] \rightarrow \mathcal{W}_c$ such that $(\tilde{z}(t), \tilde{w}_c(t)) \in C_w$ for all $t \in (0, \varepsilon)$ and $\dot{\tilde{z}}(t) \in F_w(\tilde{z}(t), \tilde{w}_c(t))$ for almost all $t \in [0, \varepsilon]$, where $\tilde{w}_c(t) \in \Psi_c^w(\tilde{z}(t))$ for every $t \in [0, \varepsilon]$,

then, there exists a nontrivial solution pair (ϕ, w) from the initial state $\phi(0, 0) = \xi$. If $\xi \in \Pi_d^w(D_w)$ and (VC_w) holds for every $\xi \in \overline{\Pi_c^w(C_w)} \setminus \Pi_d^w(D_w)$, then there exists a nontrivial solution pair to \mathcal{H}_w from every initial state $\xi \in \overline{\Pi_c^w(C_w)} \cup \Pi_d^w(D_w)$, and every solution pair $(\phi, w) \in \mathcal{S}_{\mathcal{H}_w}(\overline{\Pi_c^w(C_w)} \cup \Pi_d^w(D_w))$ from such points satisfies exactly one of the following:

- a) *the solution pair (ϕ, w) is complete;*
- b) *(ϕ, w) is not complete and “ends with flow”: with $(T, J) = \text{sup dom}(\phi, w)$, the interval I^J has nonempty interior, and either*
 - b.1) *I^J is closed, in which case either*
 - b.1.1) *$\phi(T, J) \in \overline{\Pi_c^w(C_w)} \setminus (\Pi_c^w(C_w) \cup \Pi_d^w(D_w))$, or*
 - b.1.2) *from $\phi(T, J)$ flow within $\Pi_c^w(C_w)$ is not possible, meaning that there is no $\varepsilon > 0$, absolutely continuous function $\tilde{z} : [0, \varepsilon] \rightarrow \mathbb{R}^n$ and a Lebesgue measurable and locally essentially bounded function $\tilde{w}_c :$*

$[0, \varepsilon] \rightarrow \mathcal{W}_c$ such that $\tilde{z}(0) = \phi(T, J)$, $(\tilde{z}(t), \tilde{w}_c(t)) \in C_w$ for all $t \in (0, \varepsilon)$, and $\dot{\tilde{z}}(t) \in F_w(\tilde{z}(t), \tilde{w}_c(t))$ for almost all $t \in [0, \varepsilon]$, where $\tilde{w}_c(t) \in \Psi_c^w(\tilde{z}(t))$ for every $t \in [0, \varepsilon]$, or

b.2) I^J is open to the right, in which case $(T, J) \notin \text{dom}(\phi, w)$ due to the lack of existence of an absolutely continuous function $\tilde{z} : \overline{I^J} \rightarrow \mathbb{R}^n$ and a Lebesgue measurable and locally essentially bounded function $\tilde{w}_c : [0, \varepsilon] \rightarrow \mathcal{W}_c$ satisfying $(\tilde{z}(t), \tilde{w}_c(t)) \in C_w$ for all $t \in \text{int} I^J$, $\dot{\tilde{z}}(t) \in F_w(\tilde{z}(t), \tilde{w}_c(t))$ for almost all $t \in I^J$, and such that $\tilde{z}(t) = \phi(t, J)$ for all $t \in I^J$, where $\tilde{w}_c(t) \in \Psi_c^w(\tilde{z}(t))$ for every $t \in [0, \varepsilon]$;

c) (ϕ, w) is not complete and “ends with jump”: with $(T, J) = \sup \text{dom}(\phi, w) \in \text{dom}(\phi, w)$, $(T, J - 1) \in \text{dom}(\phi, w)$, and either

c.1) $\phi(T, J) \notin \overline{\Pi_c^w(C_w)} \cup \Pi_d^w(D_w)$, or

c.2) $\phi(T, J) \in \overline{\Pi_c^w(C_w)} \setminus \Pi_d^w(D_w)$,⁷ and from $\phi(T, J)$ flow within $\Pi_c^w(C_w)$ as defined in b.1.2) is not possible.

Proof To prove the existence of a nontrivial solution pair from ξ , we show that under the given assumptions, a solution pair (ϕ, w) satisfying the conditions in Definition 2.0.5 can be constructed such that $\text{dom}(\phi, w)$ contains at least two points. We have the following cases:

- i) If $\xi \in \Pi_d^w(D_w)$, then there exist w_d^* such that $(\xi, w_d^*) \in D_w$ by definition of $\Pi_d^w(D_w)$. Let the hybrid disturbance $w_1 = (w_c, w_d)$ be defined on $\text{dom } w_1 := \{(0, 0)\} \cup \{(0, 1)\}$ as $w_d(0, 0) = w_d^*$ and $w_d(0, 1) = a$, where $a \in \mathcal{W}_d$ and w_c can be arbitrary. By definition of the jump map G_w , there exists $b \in G_w(\xi, w_d^*)$. Let ϕ_1 be a hybrid arc with $\text{dom } \phi_1 = \text{dom } w_1$ defined as $\phi_1(0, 0) = \xi$ and $\phi_1(0, 1) = b$. Then, (ϕ_1, w_1) is a nontrivial solution pair to \mathcal{H}_w ;

⁷As a consequence of (ϕ, w) ending with a jump, i.e., $\phi(T, J) \notin \Pi_d^w(D_w)$, $\phi(T, J) \in \overline{\Pi_c^w(C_w)} \setminus \Pi_d^w(D_w)$ is under the condition in case c.2).

ii) If $\xi \in \overline{\Pi_c^w(C_w)} \setminus \Pi_d^w(D_w)$ and (VC_w) holds, there exist $\varepsilon > 0$, an absolutely continuous function $\tilde{z} : [0, \varepsilon] \rightarrow \mathbb{R}^n$ and a Lebesgue measurable and locally essentially bounded function $\tilde{w}_c : [0, \varepsilon] \rightarrow \mathcal{W}_c$ with $\tilde{z}(0) = \xi$ and $\tilde{w}_c(0) \in \Psi_c^w(\xi)$ satisfying $(S1_w)$ in Definition 2.0.5. Let the hybrid disturbance $w_2 = (w_c, w_d)$ be defined on $\text{dom } w_2 := [0, \varepsilon] \times \{0\}$ with $w_c(t, 0) = \tilde{w}_c(t)$ for every $t \in [0, \varepsilon)$ and let w_d be given arbitrarily. Let the hybrid arc ϕ_2 be defined on $\text{dom } \phi_2 = \text{dom } w_2$ as $\phi_2(t, 0) = \tilde{z}(t)$ for every $t \in [0, \varepsilon)$. Then, (ϕ_2, w_2) is a nontrivial solution pair to \mathcal{H}_w .

Item i) and ii) imply the existence of a nontrivial solution pair to \mathcal{H}_w from every $\xi \in \Pi_d^w(D_w)$ and every $\xi \in \overline{\Pi_c^w(C_w)} \setminus \Pi_d^w(D_w)$, respectively, that is, for every $\xi \in \overline{\Pi_c^w(C_w)} \cup \Pi_d^w(D_w)$.

Next, we prove that every maximal solution pair (ϕ, w) to \mathcal{H}_w satisfies exactly one of the properties in a), b), and c). Suppose the nontrivial solution pair (ϕ, w) is not complete, i.e., case a) does not hold and either b) or c) holds. We show that only one of these properties holds. Let $(T, J) = \sup \text{dom}(\phi, w)$.

If $(T, J) \in \text{dom}(\phi, w)$, then I^J is closed and case b.2) does not hold, for which we have either

iii) I^J is a singleton; or

iv) I^J has nonempty interior.

If iii) is true, the solution pair (ϕ, w) ends with a jump and either $\phi(T, J) \notin \overline{\Pi_c^w(C_w)} \cup \Pi_d^w(D_w)$, which directly leads to case c.1), or $\phi(T, J) \in \overline{\Pi_c^w(C_w)} \cup \Pi_d^w(D_w)$. The latter case leads to c.2) only since otherwise (ϕ, w) can be extended by flow via the functions \tilde{z} and \tilde{w}_c as described in b.1.2) or by a jump as described in item i) above with an arbitrary $w_d \in \Psi_d^w(x)$. If iv) is true, then, by item $(S1_w)$ in Definition 2.0.5, case b.1.1) holds, i.e., $\phi(T, J) \in \overline{\Pi_c^w(C_w)} \setminus (\Pi_c^w(C_w) \cup \Pi_d^w(D_w))$, or case b.1.2) holds, namely, the solution pair (ϕ, w) cannot be extended via flows. In summary, if $(T, J) \in \text{dom } \phi$, then only one among b.1.1), b.1.2), c.1) and c.2) may hold.

If $(T, J) \notin \text{dom}(\phi, w)$, then I^J is open to the right, and by maximality of (ϕ, w) , b.2) holds. \square

Proposition 4.2.1 presents conditions guaranteeing existence of nontrivial solution pairs to \mathcal{H}_w from every initial state $\xi \in \overline{\Pi_c^w(C_w)} \cup \Pi_d^w(D_w)$, as well as characterizes all possibilities for maximal solution pairs. In particular, maximal solution pairs that are not complete can either “end with flow” or “end with jump.” In short, the former means that I^J has a nonempty interior over which $(\phi(t, J), w_c(t, J)) \in C_w$ for all $t \in \text{int}I^J$ and $\frac{d\phi}{dt}(t, J) \in F_w(\phi(t, J), w_c(t, J))$ for almost all $t \in \text{int}I^J$, where $(T, J) = \text{sup dom}(\phi, w)$. In particular, case b.1.1) corresponds to a solution pair ending at the boundary of $\overline{C_w}$, case b.1.2) describes the case of a solution pair ending after flowing and at a point, where continuing to flow is not possible, while case b.2) covers the case of a solution pair escaping to infinity in finite time. The case “end with jump” means that $(T, J), (T, J - 1) \in \text{dom}(\phi, w)$, $(\phi(T, J - 1), w_d(T, J - 1)) \in D_w$, and the solution pair ends either with $\phi(T, J) \in \Pi_c^w(C_w) \cup \Pi_d^w(D_w)$ due to flow being not possible or with $\phi(T, J) \notin \Pi_c^w(C_w) \cup \Pi_d^w(D_w)$, where $(T, J) = \text{sup dom}(\phi, w)$.

Remark 4.2.2 *Case c.1) in Proposition 4.2.1 is not possible when $G_w(D_w) \subset \overline{\Pi_c^w(C_w)} \cup \Pi_d^w(D_w)$.⁸ Moreover, when the disturbance signal w_c is generated by an exosystem of the form⁹*

$$\dot{w}_c \in F_e(w_c) \quad w_c \in \mathcal{W}_c, \quad (4.6)$$

(VC_w) can be guaranteed if, for each (ξ, w'_c) , there exists a neighborhood U such that for every $(x, w_c) \in U \cap C_w$, $(F_w(x, w_c), F_e(w_c)) \cap T_{C_w}(x, w_c) \neq \emptyset$, provided that C_w is closed and (F_w, F_e) is outer semicontinuous and locally bounded with nonempty and convex values on C_w .

Similar to the results in Section 4.1, throughout this section, the following

⁸ $G_w(D_w) = \{x' \in \mathbb{R}^n : \exists(x, w_d) \in D_w, x' \in G_w(x, w_d)\}$

⁹The disturbance w_c generated by (4.6) are not necessarily differentiable but rather, absolutely continuous over each interval of flow. For examples of exosystems given as in (4.6) and having also jumps, see [70].

version of Assumption 4.1.1 with disturbances is assumed.

Assumption 4.2.3 *The sets K, C_w , and D_w are such that $K \subset \overline{\Pi_c^w(C_w)} \cup \Pi_d^w(D_w)$ and that $K \cap \Pi_c^w(C_w)$ is closed. The map F_w is outer semicontinuous, locally bounded on $(K \times \mathcal{W}_c) \cap C_w$, and $F_w(x, w_c)$ is convex for every $(x, w_c) \in (K \times \mathcal{W}_c) \cap C_w$. For every $x \in \Pi_c^w(C_w)$, $0 \in \Psi_c^w(x)$.*

Assumption 4.2.3 guarantees that all points in the set to render invariant, namely, K , are either in the projections to the state space of C_w and D_w , which is necessary for solutions from K to exist. The closedness of the set $K \cap \Pi_c^w(C_w)$ and the regularity properties of F_w are required to obtain conditions in terms of the tangent cone; see, also, [61, Proposition 6.10]. The assumption of $0 \in \Psi_c^w(x)$ for every $x \in \Pi_c^w(C_w)$ usually holds for free since systems with disturbances, such as \mathcal{H}_w , typically reduce to the nominal system, in our case \mathcal{H} , when the disturbances vanish. A similar property could be enforced for the disturbance w_d , but such an assumption is not needed in our results.

Next, we propose sufficient conditions to guarantee robust weak forward pre-invariance and robust weak forward invariance of a set for \mathcal{H}_w .

Theorem 4.2.4 *(sufficient conditions for robust weak forward (pre-) invariance of a set) Given $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$ as in (1.1) and a set $K \subset \mathbb{R}^n$, suppose C_w, F_w, D_w and K satisfy Assumption 4.2.3. The set K is robustly weakly forward pre-invariant for \mathcal{H}_w if the following conditions hold:*

4.2.4.1) *For every $x \in K \cap \Pi_d^w(D_w)$, $\exists w_d \in \Psi_d^w(x)$ such that $G_w(x, w_d) \cap K \neq \emptyset$;*

4.2.4.2) *For every $x \in \Pi_c^w(\widehat{C}_w) \setminus \Pi_d^w(D_w)$, $F_w(x, 0) \cap T_{K \cap \Pi_c^w(C_w)}(x) \neq \emptyset$;*

where $\widehat{C}_w := ((\partial(K \cap \Pi_c^w(C_w)) \times \mathcal{W}_c) \cap C_w) \setminus L_w$ and $L_w := \{(x, w_c) \in C_w : x \in \partial \Pi_c^w(C_w), F_w(x, w_c) \cap T_{\overline{\Pi_c^w(C_w)}}(x) = \emptyset\}$. Moreover, K is robustly weakly forward invariant for \mathcal{H}_w if, in addition, $K \cap \Pi_c^w(L_w) \subset \Pi_d^w(D_w)$ and, with $\widetilde{K}^ = ((K \setminus \Pi_d^w(D_w)) \times \mathcal{W}_c) \cap C_w$,*

★) For every $(\phi, w) \in \mathcal{S}_{\mathcal{H}_w}(\Pi_c^w(\tilde{K}^*))$ with $\text{rge } \phi \subset K$, case b.2) in Proposition 4.2.1 does not hold.

Proof Given C_w, F_w, D_w and K satisfying Assumption 4.2.3, zero disturbance is always admissible to \mathcal{H}_w during continuous evolution of solution pairs. We define a restriction of \mathcal{H}_w by K with zero disturbance during flows as follows: $\tilde{\mathcal{H}}_w = (\tilde{C}, \tilde{F}, \tilde{D}_w, G_w)$, where $\tilde{C} := K \cap \Pi_c^w(C_w)$, $\tilde{F}(x) = F_w(x, 0)$ for every $x \in \Pi_c^w(C_w)$ and $\tilde{D}_w := (K \times \mathcal{W}_d) \cap D_w$. Since $K \subset \overline{\Pi_c^w(C_w)} \cup \Pi_d^w(D_w)$, by Definition 2.0.5, there exists a solution pair to $\tilde{\mathcal{H}}_w$ from every $\xi \in K$. Let $K_1 = \Pi_d^w(\tilde{D}_w)$, $K_2 = K \setminus (\Pi_d^w(\tilde{D}_w) \cup \Pi_c^w(L_w))$ and $K_3 = K \setminus (K_1 \cup K_2)$. By definition, every $\xi \in K_3$ is such that $\xi \in \Pi_c^w(L_w) \setminus \Pi_d^w(\tilde{D}_w)$ and $\tilde{F}(\xi) \cap T_{\overline{\Pi_c^w(C_w)}}(\xi) = \emptyset$. Then, item (a) in [61, Lemma 5.26] and Definition 2.0.4 imply there is only trivial solution from ξ to $\tilde{\mathcal{H}}_w$, in which case we have $\text{rge } \phi \subset K$. Otherwise, in the case where $\phi(0, 0) \in K_1 \cup K_2$, we show there exists $(\phi, w) \in \mathcal{S}_{\tilde{\mathcal{H}}_w}$ that is nontrivial and it has $\text{rge } \phi \subset K$ when 4.2.4.1) and 4.2.4.2) hold true. To this end, we construct a nontrivial solution pair from every $\xi \in K_1 \cup K_2$. Since K_1 and K_2 are disjoint sets, we have following two cases:

- i) when $\xi \in K_1$: since $K_1 \subset \Pi_d^w(D_w)$, a jump is possible from every $\xi \in K_1$, i.e., from every $(\xi, w_d) \in \tilde{D}_w$. Let $\phi_a(0, 0) = \xi$. By condition 4.2.4.1), there exists $\tilde{w}_d \in \Psi_d^w(\xi)$, $\phi_a(0, 1) \in G_w(\xi, \tilde{w}_d)$, such that $\phi_a(0, 1) \in K$.
- ii) when $\xi \in K_2$: since $K \subset \overline{\Pi_c^w(C_w)} \cup \Pi_d^w(D_w)$, $\xi \in \overline{\Pi_c^w(C_w)} \setminus \Pi_d^w(D_w)$ and solution pairs can only evolve by flowing from ξ . Conditions enforced by Assumption 4.2.3 imply that \tilde{C} is closed, \tilde{F} is outer semicontinuous, locally bounded and convex valued on \tilde{C} . Since $T_{\tilde{C}}(x) = \mathbb{R}^n$ for every $x \in (\text{int } \tilde{C}) \setminus (\Pi_d^w(\tilde{D}_w) \cup \Pi_c^w(L_w))$, item 4.2.4.2) implies that $\tilde{F}(x) \cap T_{\tilde{C}}(x) \neq \emptyset$ for every $x \in K_2$. Then, by an application of [61, Proposition 6.10], there exists a nontrivial solution ϕ_b to $\tilde{\mathcal{H}}_w$ from every $\xi \in K_2$. By item (S1) in Definition 2.0.4, such a nontrivial solution ϕ_b is absolutely continuous on $[0, \varepsilon]$, for some $\varepsilon > 0$, with $\phi_b(0) = \xi$, $\dot{\phi}_b(t) \in \tilde{F}(\phi_b(t))$ for almost all $t \in [0, \varepsilon]$ and $\phi_b(t) \in \tilde{C}$ for all $t \in (0, \varepsilon]$. By

closedness of \tilde{C} , we have $\phi_b(t, 0) \in K$ for every $t \in [0, \varepsilon]$.

The above shows that from every point in K_1 , solution pairs to $\tilde{\mathcal{H}}_w$ can be extended via jumps with the state component staying within K using the construction in case i). While from points in K_2 , solution pairs can be extended using the construction in case ii) with the state component staying within K . As a consequence, from every point in K , there exists at least one $(\tilde{\phi}, \tilde{w}) \in \mathcal{S}_{\tilde{\mathcal{H}}_w}$ with $\text{rge } \tilde{\phi} \subset K$.

Next, we prove that each such $(\tilde{\phi}, \tilde{w})$ is also maximal to \mathcal{H}_w .¹⁰ If $(\tilde{\phi}, \tilde{w})$ is complete, then it is already maximal and a solution pair to \mathcal{H}_w . Consider the case that $(\tilde{\phi}, \tilde{w})$ is not complete. Proceeding by contradiction, suppose $(\tilde{\phi}, \tilde{w})$ is not maximal for \mathcal{H}_w , meaning that there exists (ϕ, w) such that $\phi(t, j) = \tilde{\phi}(t, j)$ and $w(t, j) = \tilde{w}(t, j)$ for every $(t, j) \in \text{dom } \tilde{\phi}$ and $\text{dom } \phi \setminus \text{dom } \tilde{\phi} \neq \emptyset$. Let $(T, J) = \text{sup dom } \tilde{\phi}$. If $(T, J) \in \text{dom } \tilde{\phi}$, then, $\tilde{\phi}(T, J) \in K$ and we have the two following cases:

- $\tilde{\phi}(T, J) \in K_1 \cup K_2$, 4.2.4.1) and closeness of \tilde{C} imply that, using the arguments in i) and ii) above, it is possible for ϕ to satisfy $\phi(t, j) \in K$ for some $(t, j) \in \text{dom } \phi \setminus \text{dom } \tilde{\phi}$. By definition of solution pairs, this contradicts with maximality of $(\tilde{\phi}, \tilde{w})$ for $\tilde{\mathcal{H}}_w$.
- $\tilde{\phi}(T, J) \in K_3$, by definition of L_w , $F_w(\tilde{\phi}(T, J), w_c) \cap T_{\overline{\Pi_c(C_w)}}(\tilde{\phi}(T, J)) = \emptyset$ for every $w_c \in \Psi_c^w(\tilde{\phi}(T, J))$. Hence, $\text{sup dom } \phi = (T, J)$, which contradicts with the assumption $\text{dom } \phi \setminus \text{dom } \tilde{\phi} \neq \emptyset$.

If $(T, J) \notin \text{dom } \tilde{\phi}$, according to Proposition 4.2.1, only b.2) holds.¹¹ In such a case, there is no function $z : \overline{I^J} \rightarrow \mathbb{R}^n$ satisfying the conditions in b.2) of Proposition 4.2.1, which are needed to have a (ϕ, w) such that $\text{dom } \phi \setminus \text{dom } \tilde{\phi} \neq \emptyset$. Thus, K is robustly weakly forward pre-invariant for \mathcal{H}_w .

The last claim requires to show that among these maximal solution pairs to \mathcal{H}_w that stay in K for all future time, there exist one complete solution pair from

¹⁰During flows, we have $(\tilde{\phi}, 0)$.

¹¹Case a) does not hold due to $(\tilde{\phi}, \tilde{w})$ not being complete, while b.1) and c) do not hold because $(T, J) \notin \text{dom } \tilde{\phi}$.

every point in K when, in addition, $(K \cap \Pi_c^w(L_w)) \subset \Pi_d^w(D_w)$ and item \star) hold. To this end, first, note that the existence of a nontrivial solution pair to \mathcal{H}_w from every $x \in K$ follows from $(K \cap \Pi_c^w(L_w)) \subset \Pi_d^w(D_w)$, which implies $K_3 = \emptyset$. Then, we apply Proposition 4.2.1 to complete the proof. Proceeding by contradiction, given any $\xi \in K$, suppose every $(\phi^*, w^*) \in \mathcal{S}_{\mathcal{H}_w}(\xi)$ is not complete, i.e., $(T, J) = \sup \text{dom } \phi^*, T + J < \infty$, and case a) in Proposition 4.2.1 does not hold. Such a solution pair (ϕ^*, w^*) is not as described in case b.1.1) in Proposition 4.2.1 due to the closeness of $K \cap \Pi_c^w(C_w)$. Case c.1) does not hold for (ϕ^*, w^*) either, since $\text{rge } \phi^* \subset K$ and $K \subset \overline{\Pi_c^w(C_w)} \cup \Pi_d^w(D_w)$. Thus, by Proposition 4.2.1, (ϕ^*, w^*) can only end as described by case b.1.2), b.2) or c.2).

- The solution pair ends because the functions described in case b.1.2) or c.2) of Proposition 4.2.1, i.e., \tilde{z} does not exist for $(\phi^*(T, J), w^*(T, J))$. However, using the same argument in item ii) above with $\tilde{w}_c \equiv 0$, for every $(x, 0) \in K_1 \times 0$ there exists \tilde{z} such that b.1.2) holds, which leads to a contradiction.
- If (ϕ^*, w^*) is as described by case b.2), $\phi^*(0, 0) \notin \Pi_c^w(\tilde{K}^*)$ by assumption \star). More precisely, $\phi^*(0, 0) \in K_1$, hence, the solution pair can be extended following the same construction in i) above, which contradicts with the maximality of (ϕ^*, w^*) .

□

Condition 4.2.4.1) in Theorem 4.2.4 guarantees that for every $x \in K \cap \Pi_d^w(D_w)$ such that there exists $w_d \in \Psi_d^w(x)$, the jump map contains an element that also belongs to K . Under the stated assumptions, condition 4.2.4.2) implies the satisfaction of (VC_w) with zero disturbance w_c , which suffices for the purpose of Theorem 4.2.4 as it is about weak forward invariance notions. While involving the tangent cone of $K \cap \Pi_c^w(C_w)$ in condition 4.2.4.2) is natural, such solution property is more than needed for robust weak forward pre-invariance of K as defined in Definition 3.2.2. Similarly to Lemma 4.1.7, solution-independent conditions that imply \star) are derived for the disturbance case.

Lemma 4.2.5 (*sufficient conditions for completeness*) Given $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$ and a set $K \subset \mathbb{R}^n$, suppose K, C_w, D_w , and F_w satisfy Assumption 4.2.3. Condition \star) in Theorem 4.2.4 holds if

4.2.5.1) \tilde{K}^\star is compact; or

4.2.5.2) F_w has linear growth on \tilde{K}^\star .

The following example illustrates Theorem 4.2.4.

Example 4.2.6 (*robustly weakly forward invariant set*) Consider the hybrid system \mathcal{H}_w in Example 3.2.5 with disturbances. we apply Theorem 4.2.4 to conclude robust weak forward invariance of the set $K_1 = C_1 \cup D_1$ for \mathcal{H}_w . Assumption 4.2.3 holds for K_1, C_w, D_w and F_w by construction. Since the set L_w is empty, condition 4.2.4.1) holds since for every $(x, w_d) \in (K_1 \times \mathcal{W}_d) \cap D_w$, the selection $x^+ = R(0)x$ always results in $x^+ \in K_1$. Condition 4.2.4.2) holds since, applying item 1) in Lemma A.0.15, for every $x \in \partial C_1 \setminus \Pi_d^w(D_w)$, since $x_1 x_2 \leq 0$, we have

$$\begin{aligned} \langle \nabla(x_1^2 + x_2^2 - 1), F_w(x, 0) \rangle &= 2x_1(-x_2|x_1|) + 2x_2(w_c x_1 |x_1|) \\ &= -2x_1 x_2 |x_1| \leq 0. \end{aligned}$$

Then, the robust weak forward invariance of K_1 follows from 4.2.5.2) in Lemma 4.2.5 and Theorem 4.2.4. Note that the property is weak due to the following observations:

- Because of the set-valuedness of the map G_w , there exists a solution pair from a point $\xi_1 \in D_1$ that jumps to a point in C_2 that is not in K_1 , as depicted in Figure 3.3. On the other hand, from the same point ξ_1 , there exists a solution pair that keeps jumping from and to ξ_1 , and stays within $D_1 \subset K_1$;
- Because of the overlap between $\Pi_c^w(C_w)$ and $\Pi_d^w(D_w)$, there exists a solution pair that starts from a point $\xi_2 \in D_1$ and flows to a point in C_2 that is not in K_1 , as depicted in Figure 3.3. On the other hand, the solution pair that jumps from and to ξ_2 from ξ_2 stays within $D_1 \subset K_1$.

△

To derive a set of sufficient conditions guaranteeing the stronger robust forward invariance property of K , i.e., every solution pair to \mathcal{H}_w is such that its state component stays within the set K , when starting from K , we require the disturbances w and the set K to satisfy the following assumption.

Assumption 4.2.7 *For every $\xi \in (\partial K) \cap \Pi_c^w(C_w)$, there exists a neighborhood U of ξ such that $\Psi_c^w(x) \subset \Psi_c^w(\xi)$ for every $x \in U \cap \Pi_c^w(C_w)$.*

The next result provides conditions implying robust forward pre-invariance and robust forward invariance of a set for \mathcal{H}_w .

Theorem 4.2.8 *(sufficient conditions for robust forward (pre-) invariance of a set) Given $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$ as in (1.1) and a set $K \subset \mathbb{R}^n$, suppose C_w, F_w, D_w and K satisfy Assumption 4.2.3. Furthermore, suppose the mapping $x \mapsto F_w(x, w_c)$ is locally Lipschitz uniformly in w_c on $((\partial K + \delta\mathbb{B}) \times \mathcal{W}_c) \cap C_w$ for some $\delta > 0$. The set K is robustly forward pre-invariant for \mathcal{H}_w if the following conditions hold:*

4.2.8.1) *For every $(x, w_d) \in (K \times \mathcal{W}_d) \cap D_w$, $G_w(x, w_d) \subset K$;*

4.2.8.2) *For every $(x, w_c) \in \widehat{C}_w$, $F_w(x, w_c) \subset T_{K \cap \Pi_c^w(C_w)}(x)$.*

where \widehat{C}_w and L_w be given as in Theorem 4.2.4. Moreover, K is robustly forward invariant for \mathcal{H}_w if, in addition, $K \cap \Pi_c^w(L_w) \subset \Pi_d^w(D_w)$ and, with $\widetilde{K}^* = (K \cap \Pi_c^w(C_w)) \times \mathcal{W}_c \cap C_w$, condition \star) in Theorem 4.2.4 holds.

Proof Since condition 4.2.8.1) and 4.2.8.2) imply condition 4.2.4.1) and 4.2.4.2), respectively, under given conditions, which include the fact that C_w, F_w, D_w and K satisfy Assumption 4.2.3, the set K is robustly weakly forward pre-invariant for \mathcal{H}_w by Theorem 4.2.4.

Now we show that every $(\phi, w) \in \mathcal{S}_{\mathcal{H}_w}(K)$ has $\text{rge } \phi \subset K$. Proceeding by contradiction, suppose there exists a solution pair $(\phi, w) \in \mathcal{S}_{\mathcal{H}_w}(K)$ such that $\text{rge } \phi \setminus K \neq \emptyset$. Then, there exists $(t^*, j^*) \in \text{dom } \phi$ such that $\phi(t^*, j^*) \notin K$, i.e., ϕ eventually leaves K in finite hybrid time.¹² Then, we have the two following cases:

- i) In the case that ϕ "left K by jumping," namely, $\phi(t, j) \in K$ for all $(t, j) \in \text{dom } \phi$ with $t + j < t^* + j^*$, $(\phi(t^*, j^* - 1), w_d) \in D_w$ with $\phi(t^*, j^*) \notin K$ for some $w_d \in \Psi_d^w(\phi(t^*, j^* - 1))$. This contradicts item 4.2.8.1). More precisely, since $\phi(t^*, j^* - 1) \in K \cap \Pi_d^w(D_w)$, item 4.2.8.1) implies that $\phi(t^*, j^*) \in G_w(\phi(t^*, j^* - 1), w_d(t^*, j^* - 1)) \subset K$ for every $w_d \in \Psi_d^w(\phi(t^*, j^* - 1))$. Thus, ϕ did not leave K by jumping. Then, it must be the case that ϕ left K by flowing, which is treated in the next item.
- ii) In the case that ϕ "left K by flowing," namely, there exists a hybrid time instant $(\tau^*, j^*) \in \text{dom } \phi$ such that $\phi(t, j^*) \in \overline{\Pi_c^w(C_w)} \setminus K$ for all $t \in (\tau^*, t^*]$ and $t^* - \tau^*$ is arbitrarily small and positive. Moreover, by closedness of $K \cap \Pi_c^w(C_w)$, we suppose that $\phi(\tau^*, j^*) \in (\partial K) \cap \Pi_c^w(C_w)$.¹³ Let $t \mapsto \chi(t) \in K \cap \Pi_c^w(C_w)$ be such that for every $t \in [\tau^*, t^*]$

$$|z(t)|_{K \cap \Pi_c^w(C_w)} = |z(t) - \chi(t)|, \quad (4.7)$$

where $z(t) = \phi(t, j^*)$ for all $t \in [\tau^*, t^*]$. Such points exist because of the closedness of $K \cap \Pi_c^w(C_w)$. By definition of solution pairs to \mathcal{H}_w , the function $t \mapsto |z(t)|_{K \cap \Pi_c^w(C_w)}$ is absolutely continuous. Thus, for almost every $t \in [\tau^*, t^*]$, $\frac{d}{dt}|z(t)|_{K \cap \Pi_c^w(C_w)}$ exists and equals to the Dini derivative of $|z(t)|_{K \cap \Pi_c^w(C_w)}$. Let

¹²Note that when $\text{rge } \phi \subset K$ and $\lim_{t+j \rightarrow \sup_t \text{dom } \phi + \sup_j \text{dom } \phi} \phi(t, j) = \infty$ (that is, ϕ stays in K but escapes to infinity, potentially in finite hybrid time) corresponds to a solution that satisfies the definition of forward invariance for K .

¹³By definition of solution pair, it is the case that ϕ left $K \cap \Pi_c^w(C_w)$ and entered $\overline{\Pi_c^w(C_w)} \setminus K$ passing through $(\partial K) \cap \Pi_c^w(C_w)$.

t be such that both $\frac{d}{dt}|z(t)|_{K \cap \Pi_c^w(C_w)}$ and $\dot{z}(t)$ exist. We have

$$\begin{aligned} & \frac{d}{dt}|z(t)|_{K \cap \Pi_c^w(C_w)} \\ &= \liminf_{h \searrow 0} \frac{|z(t) + h\dot{z}(t)|_{K \cap \Pi_c^w(C_w)} - |z(t)|_{K \cap \Pi_c^w(C_w)}}{h}, \end{aligned}$$

which, by definition of $\chi(t)$ and (4.7), satisfies

$$\begin{aligned} & \frac{|z(t) + h\dot{z}(t)|_{K \cap \Pi_c^w(C_w)} - |z(t)|_{K \cap \Pi_c^w(C_w)}}{h} \\ & \leq \frac{|z(t) - \chi(t)| + |\chi(t) + h\dot{z}(t)|_{K \cap \Pi_c^w(C_w)} - |z(t)|_{K \cap \Pi_c^w(C_w)}}{h} = \frac{|\chi(t) + h\dot{z}(t)|_{K \cap \Pi_c^w(C_w)}}{h} \\ & \leq \frac{|\chi(t) + h\omega|_{K \cap \Pi_c^w(C_w)}}{h} + |\dot{z}(t) - \omega|, \end{aligned}$$

for every $\omega \in T_{K \cap \Pi_c^w(C_w)}(\chi(t))$. Moreover, for every such ω ,

$$\liminf_{h \searrow 0} \frac{|\chi(t) + h\omega|_{K \cap \Pi_c^w(C_w)}}{h} = 0$$

by definition of the tangent cone in (4.1). Hence, we have

$$\frac{d}{dt}|z(t)|_{K \cap \Pi_c^w(C_w)} \leq \liminf_{h \searrow 0} \frac{|\chi(t) + h\omega|_{K \cap \Pi_c^w(C_w)}}{h} + |\dot{z}(t) - \omega| = |\dot{z}(t) - \omega|.$$

Thus, for almost every $t \in [\tau^*, t^*]$,

$$\frac{d}{dt}|z(t)|_{K \cap \Pi_c^w(C_w)} \leq |\dot{z}(t)|_{T_{K \cap \Pi_c^w(C_w)}(\chi(t))}.$$

Since $K \cap \Pi_c^w(C_w)$ is closed, by definition, $\chi(t) \in K \cap \Pi_c^w(C_w)$ for every $t \in [\tau^*, t^*]$. Condition 4.2.8.2) implies that for almost all $t \in [\tau^*, t^*]$, and every $w \in \Psi_c^w(\chi(t))$, we have

$$\begin{aligned} \frac{d}{dt}|z(t)|_{K \cap \Pi_c^w(C_w)} & \leq |\dot{z}(t)|_{T_{K \cap \Pi_c^w(C_w)}(\chi(t))} \\ & \leq |\dot{z}(t)|_{F_w(\chi(t), w)}. \end{aligned} \tag{4.8}$$

Since $t^* - \tau^*$ is positive and can be arbitrarily small, it is always possible to construct a neighborhood of $\chi(t)$ for every $t \in [\tau^*, t^*]$, denoted U , with $z(t) \in U$, and it is such that $\Psi_c^w(z(t)) \subset \Psi_c^w(\chi(t))$ by Assumption 4.2.7. Then, because of that and the fact that the mapping $x \mapsto F_w(x, w_c)$ is locally Lipschitz uniformly in w_c on $((\partial K + \delta\mathbb{B}) \times \mathcal{W}_c) \cap C_w$ for some $\delta > 0$, we can construct a neighborhood U' of $z(t)$ such that $U' \subset \chi(t) + \delta\mathbb{B}$ and $\chi(t) \in U'$

for every $t \in [\tau^*, t^*]$ and for which there exists a constant $\lambda > 0$ satisfying

$$F_w(z(t), w_c) \subset F_w(\chi(t), w_c) + \lambda|z(t) - \chi(t)|\mathbb{B}$$

for every $t \in [\tau^*, t^*]$ and every $w_c \in \Psi_c^w(z(t))$. Hence, for every $t \in [\tau^*, t^*]$, every $w_c \in \Psi_c^w(z(t))$, and every $\eta \in F_w(z(t), w_c)$,

$$|\eta|_{F_w(\chi(t), w_c)} \leq \lambda|z(t) - \chi(t)|.$$

Moreover, since $\dot{z}(t) \in F_w(z(t), w_c)$, for every $w_c \in \Psi_c^w(z(t))$, together with (4.8) and (4.7), we have that

$$\begin{aligned} \frac{d}{dt}|z(t)|_{K \cap \Pi_c^w(C_w)} &\leq |\dot{z}(t)|_{F_w(\chi(t), w_c)} \\ &\leq \lambda|z(t) - \chi(t)| = \lambda|z(t)|_{K \cap \Pi_c^w(C_w)}. \end{aligned}$$

Then, by the Gronwall Lemma (see [68, Lemma A.1]), for every $t \in [\tau^*, t^*]$,

$$|z(t)|_{K \cap \Pi_c^w(C_w)} = 0.$$

Since $K \cap \Pi_c^w(C_w)$ is closed, $\phi(t^*, j^*) \in K \cap \Pi_c^w(C_w)$, which contradicts the definition of t^* . Thus, there does not exist maximal solution pair $(\phi, w) \in \mathcal{S}_{\mathcal{H}_w}(K)$ that eventually leaves $K \cap \Pi_c^w(C_w)$ by flowing.

Thus, the set K is robustly forward pre-invariant for \mathcal{H}_w .

Following the proof of Theorem 4.2.4, when $K \cap \Pi_c^w(L_w) \subset \Pi_d^w(D_w)$, with 4.2.4.1) and 4.2.4.2) satisfied, there exists a nontrivial solution pair (ϕ, w) with $\phi(0, 0) = \xi$ to \mathcal{H}_w from every $\xi \in K$. Then, robust forward invariance of K follows from the addition of condition \star). As shown above, every $(\phi, w) \in \mathcal{S}_{\mathcal{H}_w}(K)$ has $\text{rge } \phi \subset K$, thus, it suffices to show that every maximal solution pair to \mathcal{H}_w is complete. We proceed by contradiction. Suppose there exists a maximal solution pair $(\phi^*, w^*) \in \mathcal{S}_{\mathcal{H}_w}(K)$ that is not complete, and $(T, J) = \text{sup dom } \phi^*$. Because every $(\phi, w) \in \mathcal{S}_{\mathcal{H}_w}(K)$ has $\text{rge } \phi \subset K$, by an application of Proposition 4.2.1, (ϕ^*, w^*) only satisfies one of the cases described in item b.1.2), b.2), and c.2). In particular, condition \star) eliminates case b.2) by assumption. Then, condition 4.2.8.1) and condition 4.2.8.2) imply that (ϕ^*, w^*) can be extended within K by jumps and flows, respectively. More precisely, when $\phi^*(T, J) \in \Pi_c^w(C_w)$, condi-

tions in Assumption 4.2.3 and item 4.2.8.2) imply the function $\tilde{z} : [0, \varepsilon] \rightarrow \mathbb{R}^n$ as described in (VC_w) in Proposition 4.2.1 exists with $\tilde{w}_c(t) = 0$ for every $t \in [0, \varepsilon]$, and such (\tilde{z}, \tilde{w}_c) can be used to extend (ϕ^*, w^*) to hybrid instant $(T + \varepsilon, J)$, which contradicts the maximality of (ϕ^*, w^*) .¹⁴ When $\phi^*(T, J) \in \Pi_d^w(D_w)$, jumps are always possible by virtue of condition 4.2.8.1). Therefore, the set K is robustly forward invariant for \mathcal{H}_w . \square

Remark 4.2.9 *In comparison to Theorem 4.2.4, Lipschitzness of the set-valued map F_w (uniformly in w) is assumed. Together with Assumption 4.2.7, they are crucial to ensure that every solution pair stays in the designated set during flows. Note that Assumption 4.2.7 guarantees such property uniformly in w_c (see the proof of Theorem 4.2.8 for details). We refer readers to the example provided below Theorem 3.1 in [2], which shows solutions leave a set due to the absence of locally Lipschitzness of the right-hand side of a continuous-time system.*

The following example shows an application of Theorem 4.2.8.

Example 4.2.10 (Example 3.2.5 revisited) *Consider the hybrid system in Example 3.2.5. We apply Theorem 4.2.8 to show the set $K_2 = \Pi_c^w(C_w) \cup \Pi_d^w(D_w)$ is robustly forward invariant for \mathcal{H}_w . Similar to Example 3.2.5, $L_w = \emptyset$, Assumption 4.2.3 and condition \star) hold for K_2, F_w, C_w and D_w . Moreover, Assumption 4.2.7 holds since $w_c \leq |x|$ for every $x \in \Pi_c^w(C_w)$ and the map F_w is locally Lipschitz on C_w by construction. Then, condition 4.2.8.1) holds since for every $(x, w_d) \in (K_2 \times \mathcal{W}_d) \cap D_w$, the map G_w only “rotates” the state variable x without changing $|x|$ within the unit circle centered at the origin. Condition 4.2.8.2) holds since*

- *for every $(x, w_c) \in (\partial K_2 \times \mathcal{W}_c) \cap C_w$, because $0 \leq w_c \leq |x| \leq 1$ and $x_1 x_2 \geq 0$,*

¹⁴Note that the resulting disturbance will be Lebesgue measurable and locally essentially bounded on interval I^J .

we have

$$\begin{aligned}
& \langle \nabla(x_1^2 + x_2^2), F_w(x, w_c) \rangle \\
&= 2x_1(-x_2|x_1|) + 2x_2(w_c x_1|x_1|) \\
&= 2x_1 x_2 (w_c - 1) |x_1| \leq 0,
\end{aligned}$$

which, applying item 1) in Lemma A.0.15, implies $F_w(x, w_c) \in T_{K_2 \cap \Pi_c^w(C_w)}(x)$;

- for every $(x, w_c) \in ((\partial(\Pi_c^w(C_w)) \setminus \partial K_2) \times \mathcal{W}_c) \cap C_w$, we have

$$T_{K_2 \cap \Pi_c^w(C_w)}(x) = \begin{cases} \mathbb{R}_{\geq 0} \times \mathbb{R} & \text{if } x \in C_1, x_1 = 0, x_2 \notin \{0, 1\} \\ \mathbb{R}_{\leq 0} \times \mathbb{R} & \text{if } x \in C_2, x_1 = 0, x_2 \notin \{0, -1\} \\ \mathbb{R} \times \mathbb{R}_{\geq 0} & \text{if } x \in C_1, x_1 \notin \{0, 1\}, x_2 = 0 \\ \mathbb{R} \times \mathbb{R}_{\leq 0} & \text{if } x \in C_2, x_1 \notin \{0, -1\}, x_2 = 0 \\ \mathbb{R}_{\geq 0}^2 \cup \mathbb{R}_{\leq 0}^2 & x = 0, \end{cases}$$

which, applying item 1) in Lemma A.0.15, implies $F_w(x, w_c) \in T_{K_2 \cap \Pi_c^w(C_w)}(x)$

holds true by definition of F_w .¹⁵

Thus, the set K_2 is robustly forward invariant for \mathcal{H}_w . △

¹⁵We recall from Example 3.2.5 that $C_1 = \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, |x| \leq 1\}$ and $C_2 = \{x \in \mathbb{R}^2 : x_1 \leq 0, x_2 \leq 0, |x| \leq 1\}$.

Chapter 5

Forward Invariance of Sublevel Sets of Lyapunov-like Functions

For many control problems, Lyapunov-like functions $V : \mathbb{R}^n \rightarrow \mathbb{R}$ for \mathcal{H} and \mathcal{H}_w can be obtained via analysis or numerical methods. For such systems, we can verify the robust and nominal forward invariance of the r -sublevel sets of V by exploiting the nonincreasing property of V along solutions. In this work, for the nominal case, conditions on the system data, namely (C, F, D, G) in Theorem 4.1.2 and Theorem 4.1.4 are explored to guarantee the forward invariance of a subset of its r -sublevel set that is given by

$$\mathcal{M}_r = L_V(r) \cap (C \cup D). \quad (5.1)$$

For the more generic study of robust forward invariance properties for \mathcal{H}_w via Lyapunov methods, we employ a different set of conditions than the ones in Theorem 4.2.8 to establish robust forward (pre-)invariance of the sublevel sets of V . In particular, given a continuous differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ for \mathcal{H}_w , we derive the sufficient conditions to render robust controlled forward (pre-)invariance of subsets of its r -sublevel set, which is given by

$$\mathcal{M}_r^w = L_V(r) \cap \Pi_c^w(C_w) \cup \Pi_d^w(D_w). \quad (5.2)$$

Results in this chapter are preliminaries of the forthcoming control effort to

select feedback laws that render robust forward invariance of the Lyapunov functions' sublevel sets.

5.1 Nominal Forward Invariance for \mathcal{H} via Lyapunov Method

The next result introduces a set of constructive conditions that induce weak forward invariance and forward invariance for \mathcal{M}_r in (5.1) for \mathcal{H} . These conditions ensure that solutions stay within \mathcal{M}_r and also guarantee existence and completeness of nontrivial solutions from every point in the set \mathcal{M}_r . For convenience, given a function V and two constants $r, r^* \in \mathbb{R}$ with $r \leq r^*$, we define the set $\mathcal{I}(r, r^*) := \{x \in \mathbb{R}^n : r \leq V(x) \leq r^*\}$.

Theorem 5.1.1 (*weak forward invariance and forward invariance of \mathcal{M}_r*) *Given a hybrid system $\mathcal{H} = (C, F, D, G)$ as in (2.1), suppose the set C is closed, the map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $F(x)$ is nonempty and convex for all $x \in C$. Suppose there exist a constant $r^* \in \mathbb{R}$ and a function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ that is continuously differentiable on an open set containing C such that*

$$\langle \nabla V(x), \eta \rangle \leq 0 \quad \forall x \in \mathcal{I}(r, r^*) \cap C, \eta \in F(x), \quad (5.3)$$

$$V(\eta) \leq r \quad \forall x \in L_V(r) \cap D, \eta \in G(x), \quad (5.4)$$

for some $r \in (-\infty, r^*)$. Moreover, suppose such r satisfies

5.1.1.1) for every $x \in V^{-1}(r)$, $\nabla V(x) \neq 0$;

5.1.1.2) for every $x \in (L_V(r) \cap \partial C) \setminus D$, $F(x) \cap T_C(x) \neq \emptyset$;

5.1.1.3) for every $x \in (V^{-1}(r) \cap \partial C) \setminus D$, the set C is regular at x and $\exists \xi \in F(x) \cap T_C(x)$, $\langle \nabla V(x), \xi \rangle < 0$;

5.1.1.4) condition $N\star$) in Theorem 4.1.2 holds for $K^* = \mathcal{M}_r \cap C$ and \mathcal{H} .

Then, for each such $r \in (-\infty, r^*)$ that defines a nonempty and closed \mathcal{M}_r , we have the following:

- The set \mathcal{M}_r is weakly forward invariant for \mathcal{H} if
 - 5.1.1.5) for every $x \in \mathcal{M}_r \cap D$, $G(x) \cap (C \cup D) \neq \emptyset$;
- The set \mathcal{M}_r is forward invariant for \mathcal{H} if
 - 5.1.1.6) $G(\mathcal{M}_r \cap D) \subset C \cup D$.

Proof Fix $r < r^*$ that satisfies the conditions in Theorem 5.1.1. The sets $K = \mathcal{M}_r, C, D$ and the map F satisfy Assumption 4.1.1. In fact, since \mathcal{M}_r is defined as the intersection of an r -sublevel set of V and the union of the flow set and the jump set, \mathcal{M}_r is a subset of $C \cup D$. Closedness of $\mathcal{M}_r \cap C$ follows from the fact that C is closed and V is continuous. The properties of F directly follow from the assumptions. Now, we apply Theorem 4.1.2 to prove weak forward invariance of the set \mathcal{M}_r .

Since set L in Theorem 4.1.2 is empty in this case, we prove that for every $x \in \partial(\mathcal{M}_r \cap C) \setminus D$,

$$F(x) \cap T_{L_V(r) \cap C}(x) \neq \emptyset. \quad (5.5)$$

To this end, we need the following properties of the sets $C, L_V(r)$ and of the map F . For every $x \in L_V(r)$, the r -sublevel set $L_V(r)$ is regular¹ at x by a direct application of [71, Corollary 2 of Theorem 2.4.7 (page 56)] with $f(x) = V(x) - r$. Moreover, since (7.63) and item 5.1.1.1) hold, for each $x \in V^{-1}(r)$, $F(x) \subset T_{L_V(r)}(x)$, and the set $L_V(r)$ admits a hypertangent² at every x applying Lemma A.0.15.³ Then, we show that (5.5) holds for every $x \in \partial(\mathcal{M}_r \cap C) \setminus D$ in

¹ The set C is regular at x provided the Bouligand tangent cone at x of C coincides with the Clarke tangent cone at x of C (see [71, Definition 2.4.6]). Furthermore, every convex set is regular – see [71, Theorem 2.4.7 and (page 55) and Corollary 2 (page 56)] for other special cases of regular sets.

²See [71, Section 2.4].

³Function $h(x) = V(x) - r$ is directional Lipschitz since V is continuously differentiable and by item (i) in [71, Theorem 2.9.4].

the following cases:

1. For every $x \in (\text{int}L_V(r) \cap \partial C) \setminus D$, since $T_{L_V(r) \cap C}(x) = T_C(x)$, 5.1.1.2) implies (5.5) holds;
2. For every $x \in V^{-1}(r) \cap \text{int}C$, we have $T_{L_V(r) \cap C}(x) = T_{L_V(r)}(x)$. This implies (5.5) holds for every such x , because $F(x) \subset T_{L_V(r)}(x)$ as shown above;
3. For every $x \in (V^{-1}(r) \cap \partial C) \setminus D$, 5.1.1.3) implies

$$T_C(x) \cap \text{int}T_{L_V(r)}(x) \neq \emptyset.$$

Then, since $L_V(r)$ and C are regular at x , we can apply [71, Corollary 2 of Theorem 2.9.8 (page 105)] with $C_1 = C$ and $C_2 = L_V(r)$ since $L_V(r)$ admits a hypertangent at x : for every $x \in (V^{-1}(r) \cap \partial C) \setminus D$, we have

$$T_C(x) \cap T_{L_V(r)}(x) = T_{L_V(r) \cap C}(x),$$

i.e., (5.5) holds.

Hence, condition 4.1.2.2) in Theorem 4.1.2 holds for the sets $C, K = \mathcal{M}_r$ and the map F .

Moreover, (7.64) implies for every $x \in \mathcal{M}_r \cap D$, $G(x) \subset L_V(r)$. Together with item 5.1.1.5), (7.64) leads to condition 4.1.2.1) in Theorem 4.1.2. Then, according to Theorem 4.1.2, \mathcal{M}_r is weakly forward invariant for \mathcal{H} as condition N \star) holds by item 5.1.1.4).

For the remainder of the proof, we show that \mathcal{M}_r is forward invariant when condition 5.1.1.6) holds. First, we prove \mathcal{M}_r is forward pre-invariant for the hybrid system \mathcal{H} .

Consider the restriction to hybrid system \mathcal{H} to the set $L_V(r^*)$, denoted $\tilde{\mathcal{H}}$ and whose data is $(\tilde{C}, F, \tilde{D}, G)$, where the flow set and the jump set are given by $\tilde{C} = L_V(r^*) \cap C$ and $\tilde{D} = L_V(r^*) \cap D$, respectively. Note that (7.64) implies for every $x \in \mathcal{M}_r \cap D$, $G(x) \subset L_V(r)$. Then, every $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{M}_r)$ has $\text{rge} \phi \subset L_V(r)$ if ϕ cannot leave $L_V(r)$ by “flowing.” We show by contradiction that this is the case. Suppose ϕ left $L_V(r)$ by “flowing” during the interval $I^{j^*} := [t_{j^*}, t_{j^*+1}]$:

namely, ϕ left $L_V(r) \cap C$ and entered $(L_V(r^*) \setminus L_V(r)) \cap C$. More precisely, since $L_V(r) \subsetneq L_V(r^*)$, by closedness of \mathcal{M}_r and item (S1) in Definition 2.0.4, there exist hybrid time instants $(t^*, j^*), (\tau^*, j^*) \in \text{dom } \phi$ with $\phi(t^*, j^*) \in (L_V(r^*) \setminus L_V(r)) \cap C$, $\phi(\tau^*, j^*) \in V^{-1}(r) \cap C$, and $\phi(t, j^*) \in (L_V(r^*) \setminus L_V(r)) \cap C$ for all $t \in (\tau^*, t^*]$, where $t_{j^*} < \tau^* < t^* \leq t_{j^*+1}$. Hence, we have

$$V(\phi(\tau^*, j^*)) = r < V(\phi(t^*, j^*)) \leq r^*. \quad (5.6)$$

By item (S1) in Definition 2.0.4, for every $t \in \text{int}I^{j^*}$, $\phi(t, j^*) \in \tilde{C}$. According to (7.63), $\frac{d}{dt}V(\phi(t, j^*)) \leq 0$ for almost all $t \in I^{j^*}$. Then, integrating both sides, we have

$$V(\phi(t^*, j^*)) \leq V(\phi(\tau^*, j^*)),$$

which contradicts with (5.6). Hence, every $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{M}_r)$ stays in \mathcal{M}_r during flow. Therefore, if ϕ left \mathcal{M}_r and entered $L_V(r) \setminus \mathcal{M}_r$, which is outside of $C \cup D$ by definition of \mathcal{M}_r , it must have left $C \cup D$ via jumps. This is not possible by virtue of 5.1.1.6). Thus, we establish the forward pre-invariance of \mathcal{M}_r for $\tilde{\mathcal{H}}$ by Definition 3.1.5.

Moreover, we verify that every $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{M}_r)$ with $\text{rge } \phi \subset \mathcal{M}_r$ is also a maximal solution to \mathcal{H} by contradiction. Suppose there exists $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{M}_r)$ with $\text{rge } \phi \subset \mathcal{M}_r$ that can be extended outside of \mathcal{M}_r for \mathcal{H} . More precisely, there exists $\psi \in \mathcal{S}_{\mathcal{H}}(\mathcal{M}_r)$, such that $\text{dom } \psi \setminus \text{dom } \phi \neq \emptyset$, for every $(t, j) \in \text{dom } \phi$, $\psi(t, j) = \phi(t, j)$ and for every $(t, j) \in \text{dom } \psi \setminus \text{dom } \phi$, $\psi(t, j) \notin \mathcal{M}_r$. Let $(T, J) = \sup \text{dom } \phi$. We have two cases:

4. ψ extends ϕ via flowing: namely, $\psi(T, J) = \phi(T, J) \in \mathcal{M}_r \cap C$, $t \mapsto \psi(t, J)$ is absolute continuous on I^J . By item (S1) in Definition 2.0.4, $\psi(t, J) \in C$ for all $t \in \text{int}I^J$. Thus, it must be the case that $\psi(t, J) \in C \setminus L_V(r)$ for some $t \in I^J$. Since $L_V(r) \subsetneq L_V(r^*)$, there exists $t^* \in I^J$ such that $\psi(t^*, J) \in L_V(r^*) \cap (C \setminus L_V(r))$. This contradicts with the maximality of ϕ to $\tilde{\mathcal{H}}$.
5. ψ extends ϕ via jumping: namely, $\psi(T, J) = \phi(T, J) \in \mathcal{M}_r \cap D$ and $\psi(T, J+1) \notin \mathcal{M}_r$. By item (S2) in Definition 2.0.4, this contradicts with the maximality of

ϕ to $\tilde{\mathcal{H}}$.

To complete the proof for forward invariance of \mathcal{M}_r for \mathcal{H} , we show that every $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{M}_r)$ is also complete. Because condition 5.1.1.6) implies 5.1.1.5), we know the set \mathcal{M}_r defined by the chosen $r < r^*$ is weakly forward invariant for \mathcal{H} . Hence, there exists a nontrivial solution to \mathcal{H} from every $x \in \mathcal{M}_r$. Case b.1) Proposition 2.0.6 is excluded for every $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{M}_r)$ since $\mathcal{M}_r \cap C$ is a closed set. Case b.2) is not possible for every maximal solutions from \mathcal{M}_r by assumption 5.1.1.4). Finally, $G(\mathcal{M}_r \cap D) \subset \mathcal{M}_r$ implies case c) in Proposition 2.0.6 does not hold. Therefore, only case a) is true for every maximal solution starting from \mathcal{M}_r . \square

Condition 5.1.1.3) together with (7.63) result in a less restrictive requirement on the flow map F when compared to the usual Lyapunov conditions for stability purposes, for instance, condition (3.2b) in [61, Theorem 3.18], which often rely on finding a qualified positive definite function with strict decrease outside the set to stabilize. It is not a trivial task to relax condition 5.1.1.3) in Theorem 5.1.1. When the set $\{\xi \in F(x) : \langle \nabla V(x), \xi \rangle < 0\}$ is empty for some $x \in V^{-1}(r) \cap C$, we have that for every $\xi \in F(x)$, $\langle \nabla V(x), \xi \rangle = 0$. With item 5.1.1.1), it is either that $F(x) = 0$ or $F(x) \neq 0$. If the former holds, condition 4.1.2.2) in Theorem 4.1.2 holds trivially. However, if the latter holds, it is possible to get $F(x) \cap T_{L_V(r) \cap C} = \emptyset$ at such x , which implies that only a trivial solution exists at such x . The following example illustrates such a case.

Example 5.1.2 Consider a system on \mathbb{R}^2 given by $\dot{x} = F(x) := (x_2, -x_1)$ with $C = (-\infty, -1] \times \mathbb{R}$ and pick V as $V(x) = x^2$ with $r^* = 2$. \mathcal{M}_r is nonempty and closed for $r \in [1, r^*)$. The conditions in Theorem 5.1.1 except for 5.1.1.3), which does not hold for $r = 1$. In fact, for $r = 1$, at the point $(-1, 0)$, the vector $F((-1, 0)) = (0, 1)$ lays in $T_C((-1, 0))$ and satisfies $\langle \nabla V(x), F(x) \rangle = 0$ for each $x \in L_V(r) \cap C$, so 5.1.1.3) does not hold. As a result $F((-1, 0)) \notin T_{L_V(r) \cap C}((-1, 0))$. \triangle

When set C is a sublevel set of a function that satisfies some mild conditions, item 5.1.1.2) and item 5.1.1.3) in Theorem 5.1.1 hold as a consequence as stated in Lemma A.0.15. The following result describes such a special case.

Lemma 5.1.3 (*special construction of C*) *Given a map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $C := \{x : h(x) \leq 0\} \subset \text{dom } F$, suppose $\nabla h(x) \neq 0$, $\nabla h(x)$ is nonempty and h is continuously differentiable at every $x \in \partial C$. Moreover, suppose for every $x \in \partial C$, there exists $\xi \in F(x)$ such that $\langle \nabla h(x), \xi \rangle < \infty$. Then, condition 5.1.1.2) and 5.1.1.3) in Theorem 5.1.1 hold.*

Remark 5.1.4 *As stated in Chapter 1, invariance is also a property that is key in the study of safety in dynamical systems. The Lyapunov-like function approach in this section resembles the idea behind the safety certificates. Note that the function V in the results in this section is not sign definite and that the aim was to assume as few properties as possible, though it should be recognized that the invariance property obtained is only for its sublevel sets. Connections between results in this section and their extensions to invariance-based control design is the focus of the upcoming second part of this paper.*

5.2 Robust Forward Invariance for \mathcal{H}_w via Lyapunov Method

When a Lyapunov-like function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ for \mathcal{H}_w is provided, one can employ a set of conditions derived from the ones in Theorem 4.2.8 to establish robust forward (pre-)invariance of the sublevel sets of V . We provide conditions for robust forward (pre-)invariance of sublevel sets of V for \mathcal{H}_w , which in turn, provide insights for the invariance-based control design methods in Chapter 6. More precisely, given a continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ for \mathcal{H}_w , we derive sufficient conditions to render its r -sublevel set \mathcal{M}_r^w .

We consider Lyapunov-like functions that are tailored to forward invariance analysis. Unlike the traditional ones for stability analysis, our Lyapunov candidates are not necessarily strictly decreasing outside of \mathcal{M}_r^w , nor that it is non-increasing when inside of \mathcal{M}_r^w . Building from Section 5.1, the next result characterizes the robust forward pre-invariance of \mathcal{M}_r^w . Following the same notation in Section 5.1, given a function V and two constants $r, r^* \in \mathbb{R}^n$ with $r \leq r^*$, we define the set $\mathcal{I}(r, r^*) := \{x \in \mathbb{R}^n : r \leq V(x) \leq r^*\}$.

Proposition 5.2.1 (*robust forward pre-invariance of \mathcal{M}_r^w*) *Given a hybrid system $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$ as in (6.2), suppose there exist a constant $r^* \in \mathbb{R}$ and a function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ that is continuously differentiable on an open set containing $\Pi_c^w(C_w)$ such that*

$$\langle \nabla V(x), \eta \rangle \leq 0 \quad \forall (x, w_c) \in (\mathcal{I}(r, r^*) \times \mathcal{W}_c) \cap C_w, \eta \in F_w(x, w_c), \quad (5.7)$$

$$V(\eta) \leq r \quad \forall (x, w_d) \in (L_V(r) \times \mathcal{W}_d) \cap D_w, \eta \in G_w(x, w_d), \quad (5.8)$$

for some $r \in (-\infty, r^*)$ such that \mathcal{M}_r^w is nonempty and closed, and

$$G_w((\mathcal{M}_r^w \times \mathcal{W}_d) \cap D_w) \subset \Pi_c^w(C_w) \cup \Pi_d^w(D_w) \quad (5.9)$$

holds. Then, the set \mathcal{M}_r^w is robustly forward pre-invariant for \mathcal{H}_w .

Proof Consider the $L_V(r^*)$ restriction to the hybrid system \mathcal{H}_w , denoted $\tilde{\mathcal{H}}$ and whose data is $(\tilde{C}, F_w, \tilde{D}, G_w)$, where the flow set and the jump set are given by $\tilde{C} = (L_V(r^*) \times \mathcal{W}_c) \cap C_w$ and $\tilde{D} = (L_V(r^*) \times \mathcal{W}_d) \cap D_w$, respectively. Fix $r \in (-\infty, r^*)$ such that (5.7), (5.8), and (5.9) hold and \mathcal{M}_r^w is nonempty and closed. For any nontrivial⁴ $(\phi, w) \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{M}_r^w)$, pick any $(t, j) \in \text{dom } \phi$ and let $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_{j+1} = t$ satisfy

$$\text{dom } \phi \cap ([0, t] \times \{0, 1, \dots, j\}) = \bigcup_{k=0}^j ([t_k, t_{k+1}] \times \{k\}).$$

Next, we show that $\text{rge } \phi \subset L_V(r)$. Proceeding by contradiction, suppose there

⁴Trivial solution pairs always stay within the set of interest.

exists $(t^*, j^*) \in \text{dom } \phi$ that $\phi(t^*, j^*) \in L_V(r^*) \setminus L_V(r)$, i.e.,

$$r < V(\phi(t^*, j^*)) \leq r^*. \quad (5.10)$$

Without lose of generality, we have the following two cases:

i) ϕ leaves $L_V(r)$ by “jumping” at (t^*, j^*) :

namely, $\phi(t, j) \in \mathcal{M}_r^w$ for all $(t, j) \in \text{dom } \phi$ with $t + j < t^* + j^*$, and $(\phi(t^*, j^* - 1), w_d(t^*, j^* - 1)) \in (L_V(r) \times \mathcal{W}_d) \cap D_w$. Hence, using (5.8), it implies $V(\phi(t^*, j^*)) \leq r$, which contradicts (5.10);

ii) ϕ leaves $L_V(r)$ by “flowing” during the interval $I^{j^*} := [t_{j^*}, t_{j^*+1}]$:

due to absolute continuity of $t \mapsto \phi(t, j)$ on I^{j^*} , ϕ leaves $L_V(r) \cap \Pi_c^w(C_w)$ and enters $(L_V(r^*) \setminus L_V(r)) \cap \overline{\Pi_c^w(C_w)}$. More precisely, since $L_V(r) \subsetneq L_V(r^*)$, by closedness of $L_V(r)$, there exists a hybrid time instant $(\tau^*, j^*) \in \text{dom } \phi$ such that $(\phi(\tau^*, j^*), w_c(\tau^*, j^*)) \in (V^{-1}(r) \times \mathcal{W}_c) \cap C_w$ and $(\phi(t, j^*), w_c(t, j^*)) \in ((L_V(r^*) \setminus L_V(r)) \times \mathcal{W}_c) \cap \overline{C_w}$ for all $t \in (\tau^*, t^*]$, where $t_{j^*} < \tau^* < t^* \leq t_{j^*+1}$. Moreover, by item (S1_w) in Definition 2.0.4, for every $t \in \text{int} I^{j^*}$, $(\phi(t, j^*), w_c(t, j^*)) \in \tilde{C}$. Then, (5.7) implies that for almost all $t \in [\tau^*, t^*]$,

$$\frac{d}{dt} V(\phi(t, j^*)) \leq 0.$$

Integrating both sides, we have

$$V(\phi(t^*, j^*)) \leq V(\phi(\tau^*, j^*)),$$

which leads to $V(\phi(t^*, j^*)) \leq V(\phi(\tau^*, j^*)) = r$. This contradicts (5.10).

Next, we establish robust forward pre-invariance of \mathcal{M}_r^w for $\tilde{\mathcal{H}}$ when (5.9) holds. By item (S1_w) in Definition 2.0.4 and closedness of \mathcal{M}_r^w , every $(\phi, w) \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{M}_r^w)$ stays within \mathcal{M}_r^w during flow. Therefore, if ϕ leaves \mathcal{M}_r^w and enters $L_V(r) \setminus \mathcal{M}_r^w$, it must have jumped. Suppose there exists $(\phi, w) \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{M}_r^w)$ that has its ϕ element left \mathcal{M}_r^w eventually, while (5.9) holds. Then, for every such (ϕ, w) , there exists $(t^*, j^*) \in \text{dom } \phi$ such that $\phi(t^*, j^*) \in L_V(r) \setminus (\Pi_c^w(C_w) \cup \Pi_d^w(D_w))$ and $(\phi(t^*, j^* - 1), w_d(t^*, j^* - 1)) \in (\mathcal{M}_r^w \times \mathcal{W}_d) \cap D_w$. This leads to a contradiction with (5.9). Thus, \mathcal{M}_r^w is robustly forward pre-invariant for $\tilde{\mathcal{H}}$.

To complete the proof, we show that every $(\phi, w) \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{M}_r^w)$ with $\text{rge } \phi \subset \mathcal{M}_r^w$ is also a maximal solution to \mathcal{H}_w . Proceeding by contradiction, suppose there exists $(\phi, w) \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{M}_r^w)$ with $\text{rge } \phi \subset \mathcal{M}_r^w$ that can be extended outside of \mathcal{M}_r^w for \mathcal{H}_w . More precisely, there exists $(\psi, v) \in \mathcal{S}_{\mathcal{H}_w}(\mathcal{M}_r^w)$, such that $\text{dom } \psi \setminus \text{dom } \phi \neq \emptyset$, for every $(t, j) \in \text{dom } \phi$, $(\psi(t, j), v(t, j)) = (\phi(t, j), w(t, j))$ and for every $(t, j) \in \text{dom } \psi \setminus \text{dom } \phi$, $\psi(t, j) \notin \mathcal{M}_r^w$. Let $(T, J) = \text{sup dom } \phi$. We have two cases:

iii) (ψ, v) extends (ϕ, w) via flowing:

namely, $(\psi(T, J), v_c(T, J)) = (\phi(T, J), w_c(T, J)) \in (\mathcal{M}_r^w \times \mathcal{W}_c) \cap C_w$, $t \mapsto \psi(t, J)$ is absolute continuous on I^J . By item (S1_w) in Definition 2.0.4, $(\psi(t, J), v_c(t, J)) \in \overline{C_w}$ for all $t \in \text{int } I^J$. Thus, it must be the case that $\psi(t, J) \in \overline{\Pi_c^w(C_w)} \setminus L_V(r)$ for some $t \in I^J$. Since $L_V(r) \subsetneq L_V(r^*)$, there exists $t^* \in I^J$ such that $\psi(t^*, J) \in L_V(r^*) \cap (\overline{\Pi_c^w(C_w)} \setminus L_V(r))$, which is an extension of (ϕ, w) for $\tilde{\mathcal{H}}$. This contradicts with the maximality of (ϕ, w) to $\tilde{\mathcal{H}}$.

iv) (ψ, v) extends (ϕ, w) via jumping:

namely, $(\psi(T, J), v_d(T, J)) = (\phi(T, J), w_d(T, J)) \in (\mathcal{M}_r^w \times \mathcal{W}_d) \cap D_w$ and $\psi(T, J+1) \notin \mathcal{M}_r^w$. By item (S2_w) in Definition 2.0.4, this contradicts with the maximality of (ϕ, w) to $\tilde{\mathcal{H}}$.

□

When given a Lyapunov-like function V and a constant $r < r^*$ as in Proposition 5.2.1, one can verify the robust forward pre-invariance of \mathcal{M}_r^w for \mathcal{H}_w . In turn, conditions (5.7), (5.8) and (5.9) can be used to check whether a designed state-feedback pair (κ_c, κ_d) renders \mathcal{M}_r^w given as in (5.2) robustly controlled forward invariant for $\mathcal{H}_{u,w}$.

A typical set of Lyapunov conditions for asymptotic stability analysis can be found in [61, Theorem 3.18], where the key is to evaluate the value of V along trajectories outside of the set of interests, i.e., \mathcal{A} . These conditions ensure the decrease of V along solutions that are initialized outside of \mathcal{A} . In comparison, forward invariance characterizes the properties of system dynamics within the set

of interest, in our case, \mathcal{M}_r^w . Comparing to [61, Definition 3.16] and [61, Theorem 3.18], a function V as in Proposition 5.2.1 is a Lyapunov function candidate that is less restrictive. Such function V is neither bounded by two class- \mathcal{K}_∞ functions, nor has its change along solutions bounded by the negative of a positive definite function of the distance to the set of interest. In particular, for the nominal case, item (3.2b) in [61, Theorem 3.18] asks $\langle \nabla V(x), \eta \rangle \leq 0$ for all $x \in L_V(r^*) \cap C$ and $\eta \in F(x)$; while (5.7) allows $\langle \nabla V(x), \eta \rangle$ to be positive for $x \in \text{int}L_V(r) \cap C$. Similarly, during jumps, item (3.2c) in [61, Theorem 3.18] demands the change $V(\eta) - V(x)$ to be nonpositive for every $x \in L_V(r) \cap D$; while (5.8) allows such changes to be positive for $x \in \text{int}L_V(r) \cap D$ as long as it is such that $V(\eta) \leq r$. Such properties of function V ensure solutions stay within $L_V(r)$ for any qualifying $r < r^*$.⁵ Note that (5.7) and (5.8) do not imply that maximal solutions are complete, neither to \mathcal{H}_w nor to the restriction of \mathcal{H}_w to $L_V(r^*)$. Other alternative conditions may involve a locally Lipschitz flow map F_w similar to Corollary 6.1.5.

Remark 5.2.2 *It is worth noting that due to being inequalities, the conditions in Proposition 5.2.1 cover the special cases where V remains constant in the continuous or discrete region. In such a case, (5.7) and (5.8) in Proposition 5.2.1 are given by*

$$\langle \nabla V(x), \eta \rangle = 0 \quad \forall (x, w_c) \in (L_V(r^*) \times \mathcal{W}_c) \cap C_w, \eta \in F_w(x, w_c), \quad (5.11)$$

$$V(\eta) - V(x) = 0 \quad \forall (x, w_d) \in (L_V(r) \times \mathcal{W}_d) \cap D_w, \eta \in G_w(x, w_d), \quad (5.12)$$

respectively. Intuitively, when V does not change on $L_V(r^)$, solution pairs to \mathcal{H}_w stay within the r -sublevel set during flows and jumps. Namely, we can employ (5.11) and (5.8), or (5.7) and (5.12), to verify robust forward pre-invariance of \mathcal{M}_r^w .*

Observations in Remark 5.2.2 are also practical for controlled systems where control inputs affect only the flow or jump map and the jump map, or, respectively,

⁵Note that solution pairs may escape $L_V(r)$ when $r = r^*$. This is because $\langle \nabla V(x), \eta \rangle$ is allowed to be zero in (5.7).

flow map ensure V does not change along flows or jumps, respectively. One such example is presented in Section 7.4, in which, a ball travels vertically and is controlled by impacts with a surface at zero height. The total energy of the ball is used to construct the V function for invariance analysis. During flows, no energy loss is considered. Hence, the total energy level of the system remains the same, which implies the special case of (5.7), i.e., (5.11) holds. The controlled single-phase DC/AC inverter system is one example where (5.12) holds, which is a special case of (5.8), as presented in Section 7.2.

Next, we derive conditions rendering the set $\mathcal{M}_r^w \subset \mathbb{R}^n$ in (5.2) robustly forward invariant for \mathcal{H}_w given as in (6.2). This conditions follow from Section 5.1 and ensures that every solution pair $\phi \in \mathcal{S}_{\mathcal{H}_w}(\mathcal{M}_r^w)$ has $\text{rge } \phi \subset \mathcal{M}_r^w$. Moreover, the proposed set of conditions guarantee existence and completeness of nontrivial solution pairs to \mathcal{H}_w .

Proposition 5.2.3 (*robustly forward invariance of \mathcal{M}_r^w*) *Given a hybrid system $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$ as in (6.2), suppose the set C_w is closed, item (A2_w) in Definition 2.0.8 holds and for every $x \in \Pi_c^w(C_w)$, $0 \in \Psi_c^w(x)$. Suppose there exist a constant $r^* \in \mathbb{R}$ and a function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ that is continuously differentiable on an open set containing $\Pi_c^w(C_w)$ such that (5.7) and (5.8) in Proposition 5.2.1 hold for some $r \in (-\infty, r^*)$ such that \mathcal{M}_r^w is nonempty and closed. Moreover, suppose*

5.2.3.1) *for every $x \in V^{-1}(r)$, $\nabla V(x) \neq 0$;*

5.2.3.2) *for every $x \in (L_V(r) \cap \partial \Pi_c^w(C_w)) \setminus \Pi_d^w(D_w)$, $F_w(x, 0) \cap T_{\Pi_c^w(C_w)}(x) \neq \emptyset$;*

5.2.3.3) *for every $x \in (V^{-1}(r) \cap \partial \Pi_c^w(C_w)) \setminus \Pi_d^w(D_w)$, the set $\Xi_x := \{\xi \in F_w(x, 0) \cap T_{\Pi_c^w(C_w)}(x) : \langle \nabla V(x), \xi \rangle < 0\}$ is nonempty;*

5.2.3.4) *$(\mathcal{M}_r^w \times \mathcal{W}_c) \cap C_w$ is compact, or F_w has linear growth on $(\mathcal{M}_r^w \times \mathcal{W}_c) \cap C_w$.*

If, furthermore, (5.9) in Proposition 5.2.1 holds, then, the set \mathcal{M}_r^w is robustly forward invariant for \mathcal{H}_w .

Proof The proof is derived from the proof of Theorem 5.1.1, being the main difference, the presence of the disturbances w_c and w_d . First, we show that \mathcal{M}_r^w given as in (5.2) is robustly weakly forward invariant for \mathcal{H}_w by applying Theorem 4.2.8. In particular, the sets $K = \mathcal{M}_r^w, C_w$, and D_w , and the map F_w satisfy Assumption 4.2.3. Then, with (5.7) and item 5.2.3.1)-5.2.3.3), we show in the following lemma that item 4.2.8.2) in Theorem 4.2.8 holds with $K = \mathcal{M}_r^w, C_w$, and F_w .

Lemma 5.2.4 *Consider a closed set $C_w \subset \mathbb{R}^n \times \mathcal{W}_c$ that has $0 \in \Psi_c^w(x)$ for every $x \in \Pi_c^w(C_w)$ and a map $F_w : \mathbb{R}^n \times \mathcal{W}_c \rightrightarrows \mathbb{R}^n$ satisfying item (A2_w) in Definition 2.0.8. Suppose there exists a pair (V, r^*) , where V is continuously differentiable on an open set containing $L_V(r^*)$ and $r^* \in \mathbb{R}$ such that for some $r < r^*$, items (5.7) and 5.2.3.1) - 5.2.3.3) hold. Then, for every $x \in \partial(\mathcal{M}_r^w \cap \Pi_c^w(C_w)) \setminus \Pi_d^w(D_w)$,*

$$F_w(x, 0) \cap T_{\mathcal{M}_r^w \cap \Pi_c^w(C_w)}(x) \neq \emptyset. \quad (5.13)$$

Proof Let $r < r^*$ satisfy the properties in the statement of the claim. Let $K_1 = \text{int}(L_V(r)) \cap \partial \Pi_c^w(C_w)$, $K_2 = V^{-1}(r) \cap \text{int}(\Pi_c^w(C_w))$, and $K_3 = V^{-1}(r) \cap \partial \Pi_c^w(C_w)$. It is obvious that K_1, K_2 , and K_3 are disjoint and $\bigcup_{i=1}^3 K_i \setminus \Pi_d^w(D_w) = \partial(\mathcal{M}_r^w \cap \Pi_c^w(C_w)) \setminus \Pi_d^w(D_w)$. We have the following three cases:

- i) For every $x \in K_1 \setminus \Pi_d^w(D_w)$, since $T_{\mathcal{M}_r^w \cap \Pi_c^w(C_w)}(x) = T_{\Pi_c^w(C_w)}(x)$, item 5.1.1.2) implies (5.13).
- ii) For every $x \in K_2 \setminus \Pi_d^w(D_w)$, we have $T_{\mathcal{M}_r^w \cap \Pi_c^w(C_w)}(x) = T_{L_V(r)}(x)$. Applying item 1) of Lemma A.0.15 to every such x with $h(x) = V(x) - r$, (hence, $S = L_V(r)$) we have that (5.7) and item 5.1.1.1) imply $F_w(x, w_c) \subset T_{L_V(r)}(x)$ for every $w_c \in \Psi_c^u(x)$. Then, with the assumption that $0 \in \Psi_c^u(x)$ for every $x \in \Pi_c^w(C_w)$, (5.13) holds.
- iii) For every $x \in K_3 \setminus \Pi_d^w(D_w)$, we argue that there exists a vector $\xi \in F_w(x, 0) \cap T_{\Pi_c^w(C_w)}(x)$ that is also contained in $T_{L_V(r) \cap \Pi_c^w(C_w)}(x)$. To this end, for every

$x \in K_3 \setminus \Pi_d^w(D_w)$, consider $\xi \in \Xi_x$ as defined in 5.1.1.3). For a given $x \in K_3 \setminus \Pi_d^w(D_w)$, let

$$\tilde{C}_x := \{x + \alpha\xi : \alpha \geq 0\} \cap \Pi_c^w(C_w).$$

If $\tilde{C}_x = \{x\}$, we have $\xi = 0$ by the fact that $x \in K_3 \subset \Pi_c^w(C_w)$ and item 5.1.1.2), which contradicts with item 5.1.1.3). Hence, for every such x , \tilde{C}_x has more than one point and $\xi \neq 0$. Then, there exists $x' \neq x$ such that $x' = (\alpha'\xi + x) \in \tilde{C}_x$. By definition of \tilde{C}_x , for each $\lambda \in [0, 1]$, $x'' = \lambda x + (1 - \lambda)x'$ is also in \tilde{C}_x . Let $C_x = \text{con}\{x, x'\}$. By construction, C_x is a convex subset of \tilde{C}_x and is not a singleton. Next, for every $x \in K_3 \setminus \Pi_d^w(D_w)$, we apply Corollary A.0.5 with $C_1 = C_x$ and $C_2 = L_V(r)$. Item 5.1.1.3) implies $T_{C_x}(x) \cap \text{int}T_{L_V(r)}(x) \neq \emptyset$. Applying Lemma A.0.15 with $h(x) = V(x) - r$, the set $L_V(r)$ admits a hypertangent at every $x \in V^{-1}(r)$. Then, [71, Corollary 2 of Theorem 2.4.7 (page 56)] implies the set $L_V(r)$ is regular at every x with $f(x) = V(x) - r$. Since set C_x is regular at x by construction, Corollary A.0.5 implied that for every $x \in K_3 \setminus \Pi_d^w(D_w)$,

$$T_{C_x}(x) \cap T_{L_V(r)}(x) = T_{L_V(r) \cap C_x}(x).$$

Because of the properties of tangent cones in [69, Table 4.3, item (1)] and the fact that $C_x \cap L_V(r) \subset \Pi_c^w(C_w) \cap L_V(r)$ by construction of C_x , we also have

$$T_{L_V(r) \cap C_x}(x) \subset T_{L_V(r) \cap \Pi_c^w(C_w)}(x).$$

Then, by definition of tangent cones, $\xi \in T_{C_x}(x)$ and $\xi \in (T_{L_V(r)}(x) \cap T_{C_x}(x)) \subset T_{L_V(r) \cap \Pi_c^w(C_w)}(x)$. Therefore, by assumption, since $\xi \in F_w(x, 0) \cap T_{\Pi_c^w(C_w)}(x)$, (5.13) holds for every $x \in K_3 \setminus \Pi_d^w(D_w)$. \square

Then, (5.9) together with (5.8) imply item 4.2.8.1) in Theorem 4.2.8, holds for $\mathcal{M}_r^w, C_w, D_w$ and G_w . Hence, \mathcal{M}_r^w is robustly weakly forward invariant for \mathcal{H}_w by Theorem 4.2.8 since condition 5.2.3.4) implies item \star) according to Lemma 4.2.5. Applying Proposition 4.2.1, there exists a nontrivial solution pair to \mathcal{H}_w from every $x \in \mathcal{M}_r^w$.

Next, it follows from Proposition 5.2.1 that \mathcal{M}_r^w is also robustly forward pre-invariant for \mathcal{H}_w . Such a property implies that every maximal solution pair to \mathcal{H}_w has $\text{rge } \phi \subset \mathcal{M}_r^w$. Finally, by applying Proposition 4.2.1, every maximal solution pair to \mathcal{H}_w starting from \mathcal{M}_r^w is also complete when item 5.2.3.4) holds. Case b.1.1) in Proposition 4.2.1 is excluded for every $(\phi, w) \in \mathcal{S}_{\mathcal{H}}(\mathcal{M}_r^w)$ since $\mathcal{M}_r^w \cap \Pi_c^w(C_w)$ is closed. Case b.1.2) and c.2) are excluded since (5.13) holds for every $x \in \mathcal{M}_r^w \setminus \Pi_d^w(D_w)$. This follows from Lemma 5.2.4, and the fact that $\mathcal{M}_r^w \subset \Pi_c^w(C_w) \cup \Pi_d^w(D_w)$ and $T_{L_V(r) \cap \Pi_c^w(C_w)}(x) = \mathbb{R}^n$ for every $x \in \text{int}(L_V(r) \cap \Pi_c^w(C_w))$. Case b.2) is not possible for every maximal solution from \mathcal{M}_r^w by assumption 5.2.3.4). Finally, when (5.9) holds, namely, $G_w((\mathcal{M}_r^w \times \mathcal{W}_d) \cap D_w) \subset \mathcal{M}_r^w$, case c.1) in Proposition 4.2.1 does not hold. Therefore, only case a) is true for every maximal solution pair starting from \mathcal{M}_r^w . \square

Compared to Theorem 5.1.1, item 5.2.3.3) does not require the set $\Pi_c^w(C_w)$ to be regular as in item 5.1.1.3).

Remark 5.2.5 *Forward invariance that is uniform in the disturbances is key for certifying safety in real-world applications. As mentioned in Chapter 1, barrier certificates are shown to be useful for the study of safety [39,46,47]. The Lyapunov-like function approach in this section resembles the idea behind barrier certificates, but we do not require the set to render invariant be defined by regions of where a function is nonnegative. In addition, the function V in this work is not sign definite and that the aim is to assume as few properties as possible, though it should be recognized that the invariance property obtained is only for its sublevel sets.*

Chapter 6

Controlled Forward Invariance using Control Lyapunov Functions

In previous chapters, we formally characterize forward invariance for the nominal hybrid systems, i.e., \mathcal{H} in (2.1), and hybrid systems with disturbances, i.e., \mathcal{H}_w in (1.1). Analysis tools to verify such properties are established for a generic set $K \subset \mathbb{R}^n$ and the sublevel set of $L_V(r)$ for a given Lyapunov-like function. Building on these results, in this chapter, we present control designs to render a set forward invariant for closed-loop hybrid systems.

We focus on the control synthesis for hybrid systems in form of $\mathcal{H}_{u,w}$ given in (1.2) and in form of \mathcal{H}_u given by

$$\mathcal{H}_u \begin{cases} (x, u_c) \in C_u & \dot{x} \in F_u(x, u_c) \\ (x, u_d) \in D_u & x^+ \in G_u(x, u_d). \end{cases} \quad (6.1)$$

As mentioned in Chapter 2, these two classes of systems have input $u = (u_c, u_d)$ that can be replaced by state-feedback pair (κ_c, κ_d) , where the result in closed-loop systems in the same form of \mathcal{H} and \mathcal{H}_w . In particular, given a static state-feedback pair $(\kappa_c : \mathbb{R}^n \rightarrow \mathbb{R}^{m_c}, \kappa_d : \mathbb{R}^n \rightarrow \mathbb{R}^{m_d})$, the closed-loop systems of \mathcal{H}_u , with some

abuse of notations, are given by

$$\mathcal{H} \begin{cases} x \in C & \dot{x} \in F(x); \\ x \in D & x^+ \in G(x), \end{cases} \quad (6.2)$$

where the set-valued maps

$$F(x) := F_u(x, \kappa_c(x)) \text{ and } G(x) := G_u(x, \kappa_d(x))$$

govern the continuous and discrete evolutions of the system on the sets

$$C := \{x \in \mathbb{R}^n : (x, \kappa_c(x)) \in C_u\}, \text{ and } D := \{x \in \mathbb{R}^n : (x, \kappa_d(x)) \in D_u\},$$

respectively. Similar to the notations for $\mathcal{H}_{u,w}$, we define the projection of $S \subset \mathbb{R}^n \times \mathcal{U}_c$ onto \mathbb{R}^n as

$$\Pi_c^u(S) := \{u_c \in \mathbb{R}^{m_c} : (x, u_c) \in C_u\},$$

and the projection of $S \subset \mathbb{R}^n \times \mathcal{U}_d$ onto \mathbb{R}^n as

$$\Pi_d^u(S) := \{u_d \in \mathbb{R}^{m_d} : (x, u_d) \in D_u\},$$

Moreover, we define the set-valued maps

$$\Psi_c^{un}(x) := \{u_c \in \mathbb{R}^{m_c} : (x, u_c) \in C_u\},$$

$$\Psi_d^{un}(x) := \{u_d \in \mathbb{R}^{m_d} : (x, u_d) \in D_u\}$$

that collects all inputs u that satisfy the system dynamics for a given x .

To obtain properties (A1)-(A3) in Definition 2.0.8 for $\mathcal{H} = (C, F, D, G)$, we have the following immediate result.

Lemma 6.0.6 (*hybrid basic conditions*) *Suppose $\kappa_c : C \rightarrow \mathcal{U}_c$ and $\kappa_d : D \rightarrow \mathcal{U}_d$ are continuous and $\mathcal{H}_u = (C_u, F_u, D_u, G_u)$ is such that*

(A1') C_u and D_u are closed subsets of $\mathbb{R}^n \times \mathcal{U}_c$ and $\mathbb{R}^n \times \mathcal{U}_d$, respectively;

(A2') $F_u : \mathbb{R}^n \times \mathbb{R}^{m_c} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous relative to C_u and locally bounded, and for every $(x, u_c) \in C_u$, $F_u(x, u_c)$ is nonempty and convex;

(A3') $G_u : \mathbb{R}^n \times \mathbb{R}^{m_d} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous relative to D_u and locally bounded, and for every $(x, u_d) \in D_u$, $G_u(x, u_d)$ is nonempty.

Then, \mathcal{H}_w satisfies conditions (A1)-(A3) in Definition 2.0.8.

Similarly, the resulting closed-loop system for $\mathcal{H}_{u,w}$ when controlled by pair (κ_c, κ_d) is given by (1.3). Then, to obtain properties (A1_w)-(A3_w) in Definition 2.0.7 for $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$, we have the following immediate result.

Lemma 6.0.7 (*hybrid basic conditions*) Suppose $\kappa_c : \Pi_c(C_{u,w}) \rightarrow \mathcal{U}_c$ and $\kappa_d : \Pi_d(D_{u,w}) \rightarrow \mathcal{U}_d$ are continuous and $\mathcal{H}_{u,w} = (C_{u,w}, F_{u,w}, D_{u,w}, G_{u,w})$ is such that

(A1'_w) $C_{u,w}$ and $D_{u,w}$ are closed subsets of $\mathbb{R}^n \times \mathcal{U}_c \times \mathcal{W}_c$ and $\mathbb{R}^n \times \mathcal{U}_d \times \mathcal{W}_d$, respectively;

(A2'_w) $F_{u,w} : \mathbb{R}^n \times \mathbb{R}^{m_c} \times \mathbb{R}^{d_c} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous relative to $C_{u,w}$ and locally bounded, and for every $(x, u_c, w_c) \in C_{u,w}$, $F_{u,w}(x, u_c, w_c)$ is convex;

(A3'_w) $G_{u,w} : \mathbb{R}^n \times \mathbb{R}^{m_d} \times \mathbb{R}^{d_d} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous relative to $D_{u,w}$ and locally bounded.

Then, \mathcal{H}_w satisfies conditions (A1_w)-(A3_w) in Definition 2.0.7.

We start this chapter by formally define the concept of (robust) controlled forward invariance for hybrid system \mathcal{H}_u and $\mathcal{H}_{u,w}$. Results that guarantee these notions for a hybrid system with given static feedback pair are derived from the sufficient conditions presented in Chapter 4. Then, adapting the Lyapunov conditions in Chapter 5, we define the concept of (robust) control Lyapunov functions for forward invariance. It turns out the existence of such functions for both \mathcal{H}_u and $\mathcal{H}_{u,w}$ leads to continuous feedback laws that render their sublevel sets (robust) forward invariant for the closed-loop system \mathcal{H} and \mathcal{H}_w . Finally, we provide constructive feedback law designs featuring pointwise minimal norm selection scheme that induces (robust) controlled forward invariance.

6.1 Controlled Forward Invariance via Static State-Feedback Laws

In this section, we investigate solutions (or solution pair) behavior of the closed-loop system \mathcal{H} in (6.2) (or \mathcal{H}_w in (1.3), respectively) of \mathcal{H}_u (or $\mathcal{H}_{u,w}$, respectively) that is resulted from a given feedback pair. We formulate these properties as the *controlled forward invariant* (or *robustly controlled forward invariant*, respectively) for the hybrid system \mathcal{H}_u (or $\mathcal{H}_{u,w}$, respectively) via (κ_c, κ_d) . For the nominal case, i.e., hybrid systems \mathcal{H}_u , we define the notions of *controlled forward invariance* based on Definition 3.1.1 - Definition 3.1.5.

Definition 6.1.1 (*controlled forward invariance of \mathcal{H}_u*) *The set $K \subset \mathbb{R}^n$ is said to be*

- N1) controlled weakly forward pre-invariant for \mathcal{H}_u via (κ_c, κ_d) if K is weakly forward pre-invariant for closed-loop system \mathcal{H} as in (6.2);*
- N2) controlled weakly forward invariant for \mathcal{H}_u via (κ_c, κ_d) if K is weakly forward invariant for closed-loop system \mathcal{H} as in (6.2);*
- N3) controlled forward pre-invariant for \mathcal{H}_u via (κ_c, κ_d) if K is forward pre-invariant for closed-loop system \mathcal{H} as in (6.2);*
- N4) controlled forward invariant for \mathcal{H}_u via (κ_c, κ_d) if K is forward invariant for closed-loop system \mathcal{H} as in (6.2). \square*

Similarly, when disturbances are present, referencing Definition 3.2.1 - Definition 3.2.4, we have the following definitions for $\mathcal{H}_{u,w}$.

Definition 6.1.2 (*robust controlled forward invariance of $\mathcal{H}_{u,w}$*) *The set $K \subset \mathbb{R}^n$ is said to be*

- R1) robustly controlled weakly forward pre-invariant for $\mathcal{H}_{u,w}$ via (κ_c, κ_d) if K is robustly weakly forward pre-invariant for closed-loop system \mathcal{H}_w as in (1.3);*

- R2) robustly controlled weakly forward invariant for $\mathcal{H}_{u,w}$ via (κ_c, κ_d) if K is robustly weakly forward invariant for closed-loop system \mathcal{H}_w as in (1.3);
- R3) robustly controlled forward pre-invariant for $\mathcal{H}_{u,w}$ via (κ_c, κ_d) if K is robustly forward pre-invariant for closed-loop system \mathcal{H}_w as in (1.3);
- R4) robustly controlled forward invariant for $\mathcal{H}_{u,w}$ via (κ_c, κ_d) if K is robustly forward invariant for closed-loop system \mathcal{H}_w as in (1.3). \square

Remark 6.1.3 *Similar to our notion of robustly controlled forward invariance, to the best of our knowledge, all other existing notions in the literature, though defined for different class of systems, rely on the existence of feasible control inputs that render sets robustly forward invariant for the closed-loop system. As mentioned in Chapterchap:intro, our notions apply to a more general class of systems, in particular, continuous-time, discrete-time, and hybrid systems with set-valued dynamics. Very importantly, we do not require uniqueness of solutions as in [2, Definition 2.3], [4, Definition 8] (for continuous-time systems) or [48, Definition 1] (for discrete-time systems).*

Next, we provide conditions guaranteeing that a static state-feedback pair renders controlled (robustly) forward invariant (in the appropriate sense) of a set for the closed-loop system resulting from controlling a hybrid system. Our conditions involve the \mathcal{H}_u -admissible¹ (or $\mathcal{H}_{u,w}$ -admissible) state-feedback pair (κ_c, κ_d) , which is considered to be given, the data of the closed-loop system it leads to, which is denoted \mathcal{H} (or \mathcal{H}_w), and the set K to render (robustly) forward invariant. The following result present conditions to render *controlled (weak) forward invariance* of K for \mathcal{H}_u , by adapting results in Section 4.1.

Corollary 6.1.4 *(controlled (weak) forward invariance) Consider a hybrid system $\mathcal{H}_u = (C_u, F_u, D_u, G_u)$ as in (6.1) and a \mathcal{H}_u -admissible state-feedback pair*

¹A state-feedback pair (κ_c, κ_d) , where $\kappa_c : \mathbb{R}^n \rightarrow \mathbb{R}^{m_c}$ and $\kappa_d : \mathbb{R}^n \rightarrow \mathbb{R}^{m_d}$ is said to be \mathcal{H}_u -admissible if the pair satisfies the dynamics of \mathcal{H}_u .

(κ_c, κ_d) . Let the closed-loop system $\mathcal{H} = (C, F, D, G)$ satisfy the conditions in Definition 2.0.8. Furthermore, suppose $K \subset \mathbb{R}^n$ is a closed subset of $\overline{\Pi_c^u(C)} \cup \Pi_d^u(D)$. Then, the set K is

- controlled weakly forward invariant for \mathcal{H}_u via (κ_c, κ_d) if
 - 6.1.4.1) For every $x \in K \cap \Pi_d^u(D_u)$, $G_u(x, \kappa_d(x)) \cap K \neq \emptyset$;
 - 6.1.4.2) For every $x \in \widehat{C} \setminus \Pi_d^u(D_u)$, $F_u(x, \kappa_c(x)) \cap T_{K \cap \Pi_c^u(C_u)}(x) \neq \emptyset$, where $\widehat{C} := \partial(K \cap \Pi_c^u(C_u)) \setminus L$ and $L := \{x \in \partial \Pi_c^u(C_u) : F_u(x, \kappa_c(x)) \cap T_{\overline{\Pi_c^u(C_u)}}(x) = \emptyset\}$.
 - 6.1.4.3) $K \cap \Pi_c^u(C_u)$ is compact or $F(x)$ has linear growth on $K \cap C$.
- controlled forward invariant for \mathcal{H}_u via (κ_c, κ_d) if 6.1.4.3) holds and
 - 6.1.4.4) For every $x \in K \cap \Pi_d^u(D_u)$, $G_u(x, \kappa_d(x)) \subset K$;
 - 6.1.4.5) For every $x \in \widehat{C}$, $F_u(x, \kappa_c(x)) \subset T_{K \cap \Pi_c^u(C_u)}(x)$;
 - 6.1.4.6) F is locally Lipschitz on $(\partial K + \delta \mathbb{B}) \cap \Pi_c^u(C_u)$ for some $\delta > 0$.

As discussed in Section 4.1.1, item 6.1.4.3) provides two solution-independent sufficient conditions to exclude the case where solution ends during flow by escaping to infinity in finite time. In turn, every maximal solution initiated from K are complete. Hence, to verify the “pre” notions, namely, controlled weak forward pre-invariance and controlled forward pre-invariance, one can simply ignore item 6.1.4.3).

Then, under the presence of disturbances, we derive conditions that a pair (κ_c, κ_d) , along with the data of the hybrid system $\mathcal{H}_{u,w}$ and a given set K , should satisfy for K to be robustly controlled invariant. Though the result is not necessarily a systematic design tool, it provides checkable solution-independent conditions.

Corollary 6.1.5 (*robust controlled forward (pre-)invariance*) Consider a hybrid system $\mathcal{H}_{u,w} = (C_{u,w}, F_{u,w}, D_{u,w}, G_{u,w})$ as in (1.2) and a $\mathcal{H}_{u,w}$ -admissible state-feedback pair (κ_c, κ_d) . Let the closed-loop system $\mathcal{H}_w = (C_w, F_w, D_w, G_w)$ satisfy

the conditions in Definition 2.0.8. Furthermore, suppose $K \subset \mathbb{R}^n$ is a closed subset of $\overline{\Pi_c^w(C_w)} \cup \Pi_d^w(D_w)$ and F_w is locally Lipschitz (as in Definition A.0.3) on $((\partial K + \delta\mathbb{B}) \times \mathcal{W}_c) \cap C_w$ for some $\delta > 0$. Then, the set K is robustly controlled forward pre-invariant for $\mathcal{H}_{u,w}$ via (κ_c, κ_d) if K and (C_w, F_w, D_w, G_w) are such that

6.1.5.1) For every $\xi \in (\partial K) \cap \Pi_c^w(C_w)$, there exists a neighborhood U of ξ such that $\Psi_c^w(x) \subset \Psi_c^w(\xi)$ for every $x \in U \cap \Pi_c^w(C_w)$;

6.1.5.2) For every $(x, w_d) \in (K \times \mathcal{W}_d) \cap D_w$, $G_w(x, w_d) \subset K$;

6.1.5.3) For every $(x, w_c) \in ((\partial(K \cap \Pi_c^w(C_w))) \times \mathcal{W}_c) \cap C_w \setminus L_w$, $F_w(x, w_c) \subset T_{K \cap \Pi_c^w(C_w)}(x)$, where L_w is given as in Theorem 4.2.4.

Moreover, K is robustly controlled forward invariant for $\mathcal{H}_{u,w}$ via (κ_c, κ_d) if, in addition

6.1.5.4) $(K \times \mathcal{W}_c) \cap C_w$ is compact, or F_w has linear growth on $(K \times \mathcal{W}_c) \cap C_w$; and

6.1.5.5) $K \cap \Pi_c^w(L_w) \subset \Pi_d^w(D_w)$.

Proof The proof exploits results in Section 4.2. Namely, applying Theorem 4.2.8, we show that the assumptions and conditions 6.1.5.1)-6.1.5.3) in Corollary 6.1.5 together imply the set K is robustly pre-forward invariant for the closed-loop system \mathcal{H}_w . In particular, $K \cap \Pi_c^w(C_w)$ is closed since K and C_w are closed sets. Because of item (A2_w) and the assumption that $0 \in \Psi_c^w(x)$ for every $x \in \Pi_c(C_{u,w})$, Assumption 4.2.3 holds for C_w, D_w, F_w and K . Hence, applying Theorem 4.2.8, since 6.1.5.2) and 6.1.5.3) imply item 4.2.8.1) and 4.2.8.2), respectively, set K is robustly controlled forward pre-invariant for $\mathcal{H}_{u,w}$ via (κ_c, κ_d) by Definition 6.1.2.

With the addition of item 6.1.5.4), Lemma 4.2.5 implies solution pairs are bounded in finite time. Then, item 6.1.5.5) guarantees existence of nontrivial solution pairs from every $x \in K$ by guaranteeing jump is possible from every

$x \in (K \cap \Pi_c^w(L_w))$. Therefore, K is robustly forward invariant for \mathcal{H}_w and robustly controlled forward invariant for $\mathcal{H}_{u,w}$ via (κ_c, κ_d) . \square

Remark 6.1.6 *The locally Lipschitzness of the set-valued map F_w is crucial to make sure that every solution pair stays in the set K during flows as shown in proof of Theorem 4.2.8. In addition, we refer readers to the example provided below [2, Theorem 3.1], which shows that, even though $f(x) \in T_K$, a continuous-time system has solutions that leave a set due to the absence of locally Lipschitzness of the right-hand side of a continuous-time system. In addition, condition 6.1.5.1) guarantees such property uniformly in w_c (see the proof of Theorem 4.2.8 for details).*

We use the next example illustrates Corollary 6.1.5.

Example 6.1.7 *(nonlinear planar system with jumps) Consider a hybrid system $\mathcal{H}_{u,w}$ with flow map*

$$F_{u,w}(x, u_c, w_c) := \left\{ \begin{bmatrix} x_1^2 - \gamma \\ x_1 x_2 \end{bmatrix} u_c w_c : \gamma \in [3, 4] \right\}$$

defined for every $(x, u_c, w_c) \in C_{u,w}$, where the flow map is given by

$$C_{u,w} := \{(x, u_c, w_c) \in \mathbb{R}^2 \times \mathbb{R} \times [0, 1] : \\ |x| \geq 1, |x_1| \geq |u_c|, (|x|^2 - 2)x_1^2 \leq u_c x_1 \leq (|x|^2 - 1)x_1^2\}$$

and jump map²

$$G_{u,w}(x, u_d, w_d) := \{-R(u_d w_d)x, R(u_d w_d)x\},$$

defined for every $(x, u_d, w_d) \in D_{u,w}$, where the jump map is given by

$$D_{u,w} : \left\{ (x, u_d, w_d) \in \mathbb{R}^2 \times \mathbb{R} \times [-1.1, 1.1] : x_1 = 0, |x| \geq 1, u_d \in \left[\frac{\pi}{4}, \frac{\pi}{2} \right] \right\}.$$

² $R(s) = \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix}$ represents a rotation matrix.

Consider the set $K = \{x \in \mathbb{R}^2 : 1 \leq |x| \leq \sqrt{2}\}$, and a continuous state-feedback pair (κ_c, κ_d) defined for every $x \in \mathbb{R}^2$ given by

$$\kappa_c(x) = \left(|x|^2 - \frac{3}{2}\right) x_1, \quad \kappa_d(x) = \frac{\pi}{3}.$$

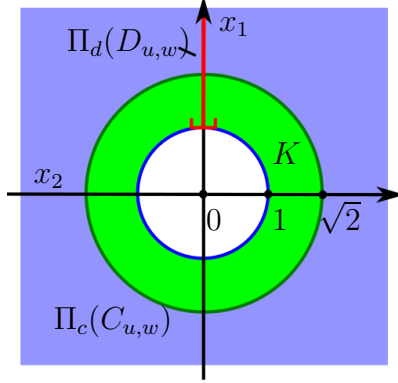


Figure 6.1: Set configuration for Example 6.1.7.

By definition of $F_{u,w}$ and κ_c , we have

$$F_w(x, w_c) := \left\{ \begin{bmatrix} x_1^2 - \gamma \\ x_1 x_2 \end{bmatrix} \left(|x|^2 - \frac{3}{2} \right) x_1 w_c : \gamma \in [3, 4] \right\},$$

which is Lipschitz on the set $\Pi_c(C_{u,w}) \cap K = K$. The assumptions and conditions 6.1.5.1) and 6.1.5.4) in Corollary 6.1.5 hold by construction of $\mathcal{H}_{u,w}$, (κ_c, κ_d) , and K . Consider a continuously differentiable function $V(x) := x_1^2 + x_2^2$ for every $x \in \mathbb{R}^2$. Since $\gamma \in [3, 4]$ and $w_c \in [0, 1]$, we have that for every x such that $|x| = 1$ and every $\xi \in F_w(x)$,

$$\langle \nabla V(x), \xi \rangle = 2x_1\xi_1 + 2x_2\xi_2 = (\gamma - 1)x_1^2 w_c \geq 0,$$

and for every x such that $|x| = 2$ and every $\xi \in F_w(x)$,

$$\langle \nabla V(x), \xi \rangle = 2x_1\xi_1 + 2x_2\xi_2 = (2 - \gamma)x_1^2 w_c \leq 0.$$

Hence, item 6.1.5.3) holds and $L_w = \emptyset$ by application of item 2) in Lemma A.0.15. Condition 6.1.5.2) holds because the rotation matrix R only changes the direction of the vector x , while its magnitude remains the same after each jump. Item 6.1.5.5) holds trivially as $L_w = \emptyset$. Therefore, by an application of Corollary 6.1.5,

the set K is robustly controlled forward invariant for system $\mathcal{H}_{u,w}$ via the given state-feedback pair (κ_c, κ_d) . \triangle

6.2 Invariance-based Control for Hybrid Systems via Control Lyapunov Functions

Utilizing Theorem 5.1.1, feedback pair (κ_c, κ_d) can be designed for \mathcal{H}_u to render a sublevel set of Lyapunov-like function $V : \mathbb{R}^n \rightarrow \mathbb{R}$, namely, \mathcal{M}_r as in (5.1), forward invariant for the closed-loop system \mathcal{H} given as in (6.2). Consequently, set \mathcal{M}_r is controlled forward invariant for \mathcal{H}_u via such pair (κ_c, κ_d) . In this section, we start with formally defining the concept of control Lyapunov functions for forward invariance (CLF-FI). When provided an CLF-FI for \mathcal{H}_u , regulation maps can be constructed for selecting state-feedbacks that induces controlled forward invariance using Theorem 5.1.1. Then, the existence of such state-feedback selections in Section 6.2.1. In addition, to show case an alternative approach to get state feedback pair that induces forward invariance featuring locally Lipschitz flow map, one result is provided in Section 6.2.2.

Motivated by achieving the regularities in Definition 2.0.8 for the resulting closed-loop system \mathcal{H} , we design state-feedback control laws by making continuous selections from sets that include all inputs ensuring required properties for forward invariance for \mathcal{H} . To this end, feedbacks are selected from the sets of inputs that keep “all solutions” within $\Pi_c^u(C_u) \cup \Pi_d^u(D_u)$. More precisely, for every $x \in \Pi_c^u(C_u)$, we consider $\Psi_c^{un}(x)$ and for every $x \in \Pi_d^u(D_u)$, we define

$$\Theta_d^n(x) := \{u_d \in \Psi_d^{un}(x) : G_u(x, u_d) \subset \Pi_c^u(C_u) \cap \Pi_d^u(D_u)\}. \quad (6.3)$$

Note that when constructing feedbacks using the methodology introduced in this section, it suffices to find subsets of $\Theta_d^n(x)$ for each $x \in D$ that satisfy the conditions in the forthcoming results. Then, we define the control Lyapunov functions for forward invariance as follows.

Definition 6.2.1 (CLF-FI for \mathcal{H}_u) Consider sets $\mathcal{U}_c \subset \mathbb{R}^{m_c}$, $\mathcal{U}_d \subset \mathbb{R}^{m_d}$, a hybrid system $\mathcal{H}_u = (C_u, F_u, D_u, G_u)$ as in (6.1), a constant $r^* \in \mathbb{R}$, and a continuous function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ that is also continuously differentiable on an open set containing $\Pi_c^u(C_u)$. Suppose there exist continuous functions $\rho_c : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\rho_d : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ such that for some $r < r^*$, we have

$$\begin{aligned} \rho_c(x) &> 0 \quad \forall x \in \mathcal{I}(r, r^*), \\ \rho_d(x) &> 0 \quad \forall x \in L_V(r). \end{aligned} \tag{6.4}$$

Then, the pair (V, r^*) defines a control Lyapunov function for forward invariance (CLF-FI) of the sublevel sets of V for \mathcal{H}_u if

$$\inf_{u_c \in \Psi_c^u(x)} \sup_{\xi \in F_u(x, u_c)} \langle \nabla V(x), \xi \rangle + \rho_c(x) \leq 0 \quad \forall x \in \mathcal{I}(r, r^*) \cap \Pi_c^u(C_u), \tag{6.5}$$

$$\inf_{u_d \in \Theta_d^n(x)} \sup_{\xi \in G_u(x, u_d)} V(\xi) + \rho_d(x) \leq r \quad \forall x \in L_V(r) \cap \Pi_d^u(D_u). \tag{6.6}$$

□

Compared to the typical control Lyapunov functions (see, e.g., [72, Definition 2.1]), the CLF-FI in Definition 6.2.1 are not constrained to be bounded by class \mathcal{K}_∞ functions. In addition, the constant r^* defining an r^* -sublevel set in Definition 6.3.1 is allowed to be negative. Note that it is possible to have $L_V(r^*) = \emptyset$, in which case we consider the pair (V, r^*) to be trivial.

6.2.1 Existence of State-Feedback Pair for Controlled Forward Invariance

Given a pair (V, r^*) defined as in Definition 6.2.1 for \mathcal{H}_u and $r < r^*$ satisfying the conditions therein, our approach consists of selecting a state-feedback law pair (κ_c, κ_d) from these inequalities. In fact, we are interested in synthesizing a pair

(κ_c, κ_d) that, in particular, satisfies

$$\begin{aligned} \sup_{\xi \in F_u(x, \kappa_c(x))} \langle \nabla V(x), \xi \rangle + \rho_c(x) &\leq 0 & \forall x \in \mathcal{I}(r, r^*) \cap \Pi_c^u(C_u), \\ \sup_{\xi \in G_{u,w}(x, \kappa_d(x))} V(\xi) + \rho_d(x) &\leq r & \forall x \in L_V(r) \cap \Pi_d^u(D_u), \end{aligned}$$

which, under certain mild conditions, with abuse of notation, renders the set

$$\mathcal{M}_r := L_V(r) \cap (\Pi_c^u(C_u) \cup \Pi_d^u(D_u)) \quad (6.7)$$

controlled forward invariant for \mathcal{H}_u . Interestingly, with a constant parameter $\sigma \in (0, 1)$, the selection of such a feedback pair can be performed by defining sets that nicely depend on the functions

$$\Gamma_c^n(x) := \begin{cases} \sup_{\xi \in F_u(x, u_c)} \langle \nabla V(x), \xi \rangle + \sigma \rho_c(x) & \text{if } x \in M_c \cap \Pi_c^u(C_u), \\ -\infty & \text{otherwise} \end{cases}$$

for each $(x, u_c) \in \mathbb{R}^n \times \mathcal{U}_c$, and

$$\Gamma_d^n(x) := \begin{cases} \sup_{\xi \in G_u(x, u_d)} V(\xi) + \sigma \rho_d(x) - r & \text{if } x \in M_d \cap \Pi_d^u(D_u), \\ -\infty & \text{otherwise,} \end{cases}$$

for each $(x, u_d) \in \mathbb{R}^n \times \mathcal{U}_d$. In fact, with these functions defined, by introducing the set-valued maps

$$\{u_c \in \Psi_c^{un}(x) : \Gamma_c^n(x) < 0\}, \quad \{u_d \in \Theta_d^n(x) : \Gamma_d^n(x) < 0\}$$

which are the so-called *regulation maps* [73], our approach is to determine a state-feedback pair (κ_c, κ_d) that is selected from these maps. In other words, the selected state-feedback pair (κ_c, κ_d) is such that

$$\kappa_c(x) \in \{u_c \in \Psi_c^{un}(x) : \Gamma_c^n(x) < 0\}, \quad \kappa_d(x) \in \{u_d \in \Theta_d^n(x) : \Gamma_d^n(x) < 0\}$$

at the appropriate values of the state x .

The goal of the remainder of this section is to formalize the approach outlined above. First, we provide key results on forward invariance of sublevel sets of CLF-like functions, which are used in our CLF approach. It turns out that when an CLF-FI for \mathcal{H}_u is provided, regulation maps as outlined above can be constructed

for selecting a state-feedback satisfying the conditions in Theorem 5.1.1; hence, the results in Section 5.1 enable us to show the desired invariance property under feedback. Since according to Lemma 6.0.6, the closed-loop system \mathcal{H} satisfies conditions (A1)-(A3) in Definition 2.0.8 when the applied state-feedback pair is continuous, we seek the design of a state-feedback pair (κ_c, κ_d) with κ_c and κ_d being continuous functions of the state. For this purpose, we first reveal conditions assuring the existence of continuous selections from the regulation maps. For ease of representation in the forthcoming sections, we define

$$M_c^n := \mathcal{I}(r, r^*) \cap \Pi_c^u(C_u), \quad M_d^n := L_V(r) \cap \Pi_d^u(D_u), \quad (6.8)$$

Then, building from Theorem 5.1.1, we establish conditions to guarantee existence of a continuous state-feedback pair (κ_c, κ_d) to render the set \mathcal{M}_r in (6.7) controlled forward pre-invariant for \mathcal{H}_u .

Theorem 6.2.2 (*existence of state-feedback pair for controlled forward pre-invariance using CLF-FI*) Consider a hybrid system $\mathcal{H}_u = (C_u, F_u, D_u, G_u)$ as in (6.1) satisfying conditions (A1')-(A3') in Lemma 6.0.6. Suppose there exists a pair (V, r^*) that defines a control Lyapunov function for forward invariance for \mathcal{H}_u as in Definition 6.2.1. Let $r < r^*$ satisfy (6.4)-(6.6) and Θ_d^n be given as in (6.3). If the following conditions hold:

6.2.2.1) The set-valued maps Ψ_c^{un} and Θ_d^n are lower semicontinuous, and Ψ_c^{un} and Θ_d^n have nonempty, closed, and convex values on the sets M_c^n and M_d^n as in (6.8), respectively;

6.2.2.2) For each $x \in M_c^n$, the function $u_c \mapsto \Gamma_c^n(x)$ is convex on $\Psi_c^u(x)$ and, for each $x \in M_d^n$, the function $u_d \mapsto \Gamma_d^n(x)$ is convex on $\Theta_d^n(x)$;

then, the set \mathcal{M}_r in (6.7) is controlled forward pre-invariant for \mathcal{H}_u via a state-feedback pair (κ_c, κ_d) with κ_c continuous on M_c^n and κ_d continuous on M_d^n .

We omit the proof for Theorem 6.2.2 as is it directly derived from Theorem 6.3.3 in the forthcoming section on invariance-based control for hybrid system

$\mathcal{H}_{u,w}$. Please see a detailed proof in Section 6.3.

The remainder of this section is dedicated to show, when provided a CLF-FI defined in Definition 6.2.1, there exists controlled forward invariance inducing feedback pair for a class of \mathcal{H}_u by enforcing additional requirements on the selections. These additional conditions guarantee existence of nontrivial solutions and completeness of maximal solutions. To this end, for every $x \in \Pi_c^u(C_u)$, we define the map

$$\Theta_c^n(x) := \begin{cases} \{u_c \in \Psi_c^{un}(x) : F_u(x, u_d) \cap T_{\Pi_c^u(C_u)} \neq \emptyset\} & \forall x \in \partial \Pi_c^u(C_u) \setminus \Pi_d^u(D_u) \\ \Psi_c^{un}(x) & \text{otherwise.} \end{cases} \quad (6.9)$$

The next result provides conditions on system data of \mathcal{H}_u for existence of continuous state-feedback pair that renders controlled forward invariance using CLF-FI in Definition 6.2.1.

Theorem 6.2.3 (*existence of state-feedback pair for controlled forward invariance using CLF-FI*) Consider a hybrid system $\mathcal{H}_u = (C_u, F_u, D_u, G_u)$ as in (6.1) satisfying conditions (A1')-(A3') in Lemma 6.0.6. Suppose there exists a pair (V, r^*) that defines a control Lyapunov function for forward invariance of the sub-level sets of V for \mathcal{H}_u as in Definition 6.2.1 with $\Psi_c^{un}(x)$ in (6.5) replaced by $\Theta_c^n(x)$ as in (6.9). Let $r < r^*$ satisfy (6.4)-(6.6) and Θ_d^n be given as in (6.3). If the following conditions hold:

6.2.3.1) The set-valued maps Θ_c^n and Θ_d^n are lower semicontinuous and Θ_c^n and Θ_d^n have nonempty, closed, and convex values on the set M_c^n and the set M_d^n , respectively;

6.2.3.2) For each $x \in M_c^n$, the map $u_c \mapsto \Gamma_c^n(x)$ is convex on $\Theta_c^n(x)$ and, for each $x \in M_d^n$, the map $u_d \mapsto \Gamma_d^n(x)$ is convex on $\Theta_d^n(x)$;

then, the set \mathcal{M}_r in (6.7) is controlled forward pre-invariant for \mathcal{H}_u via a state-feedback pair (κ_c, κ_d) with κ_c continuous on M_c^n and κ_d continuous on M_d^n . Furthermore, if item 5.1.1.4) in Theorem 5.1.1 holds for the closed-loop system \mathcal{H} as in (6.2), the pair (κ_c, κ_d) renders set \mathcal{M}_r controlled forward invariant for \mathcal{H}_u .

We omit the proof here as it resembles the one for Theorem 6.3.5. Because Θ_c^n in (6.9) is used rather than the generic map Ψ_c^{un} in selecting the inputs, the resulting regulation maps inherit the properties from Θ_c^n , namely, every selected input $\kappa_c(x)$ at x is such that $F_u(x, u_d) \cap T_{\Pi_c^u(C_u)} \neq \emptyset$. This additional constraint on feedback pair design includes condition 5.1.1.1), 5.1.1.3) and 5.1.1.4) in Theorem 5.1.1. In turn, at every $x \in \mathcal{M}_r$, the feedback law $\kappa_c(x)$ guarantees existence of nontrivial solution, moreover, the completeness of every maximal solution to the closed-loop system.

6.2.2 Existence of Lipschitz State-Feedback Pair for Controlled Forward Invariance

Results and control Lyapunov functions for forward invariance presented in Section 6.2.1 take a different approach compared to the ones in [64], which are derived from the sufficient conditions for generic sets that are presented in Section 4.1. In particular, for the “pre” notion, the continuous-time feedback law κ_c is selected from $\Psi_c^{un}(x)$ for every $x \in \mathcal{M}_r \cap \Pi_c^u(C_u)$ in Theorem 6.2.3; while result in [64] selects feedbacks that guarantee the flow condition specified in Theorem 4.1.4.

In this section, we provide an alternative approach to Theorem 6.2.3 to induce controlled forward invariance in Definition 6.1.1. The results herein resemble the ones presented in [64]. To get controlled forward invariance of the sublevel sets of a given CLF-FI, we need the flow map for the closed-loop system to be locally Lipschitz, for which we assume that the map F_u is locally Lipschitz. The next result shows that under the effect of a locally Lipschitz state-feedback κ_c , the closed-loop flow map F derived from F_u is also locally Lipschitz.

Lemma 6.2.4 (*Locally Lipschitzness of $F_u(x, u_c)$*) *Suppose $F_u : S_1 \times S_2 \rightrightarrows S_1$ is locally Lipschitz (as a set-valued map) and $\kappa_c : S_1 \rightarrow S_2$ is locally Lipschitz (as a function). Then, $F := F_u(x, \kappa_c(x))$ is locally Lipschitz on S_1 (as a set-valued map).*

Proof Since F_u is locally Lipschitz, for every $(x, u) \in S_1 \times S_2$, there exists a neighborhood $U_F = U_{F_1} \times U_{F_2}$ of (x, u) and a constant $L_F \geq 0$, where $x \in S_1, u \in S_2$, such that for every $(\xi, \delta) \in U_F \cap (S_1 \times S_2)$,

$$F_u(x, u) \subset F_u(\xi, \delta) + L_F |(x, u) - (\xi, \delta)| \mathbb{B},$$

which is equivalent to, for every $(\xi, \delta) \in (U_{F_1} \cap S_1) \times (U_{F_2} \cap S_2)$,

$$F_u(x, u) \subset F_u(\xi, \delta) + L_F \sqrt{(x - \xi)^2 + (u - \delta)^2} \mathbb{B}. \quad (6.10)$$

Then, because κ_c is a locally Lipschitz single-valued map, for every $x \in S_1$, there exist a neighborhood U_{κ_c} of x and a constant $L_{\kappa_c} \geq 0$, such that for every $\xi \in S_2$,

$$|\kappa_c(x) - \kappa_c(\xi)| \leq L_{\kappa_c} |x - \xi|,$$

i.e.,

$$(\kappa_c(x) - \kappa_c(\xi))^2 \leq L_{\kappa_c}^2 (x - \xi)^2. \quad (6.11)$$

Now, we consider the set $U_{cl} := \min\{U_{F_1}, U_{\kappa_c}\}$, it follows that, for every $x \in S_1$, (6.10) and (6.11) hold for every $(\xi, \delta) \in U_{cl} \times \{u \in S_2 : (x, u) \in S_1 \times S_2 \text{ s.t. } x \in U_{cl}\}$ and $\xi \in U_{cl}$, respectively. Then, let $u = \kappa_c(x), \delta = \kappa_c(\xi)$, for every $x \in S_1$, we have that, for every $\xi \in U_{cl}$,

$$\begin{aligned} F(x, \kappa_c(x)) &= F_u(x, u) \subset F_u(\xi, \delta) + L_F \sqrt{(x - \xi)^2 + (u - \delta)^2} \mathbb{B} \\ &\subset F_u(\xi, \delta) + L_F \sqrt{(x - \xi)^2 + L_{\kappa_c}^2 (x - \xi)^2} \mathbb{B} \\ &\subset F_u(\xi, \kappa_c(x)) + (L_F \sqrt{1 + L_{\kappa_c}^2}) |x - \xi| \mathbb{B}, \end{aligned}$$

which is equivalent to, for every $\xi \in U_{cl}$,

$$F(x) \subset F(\xi) + \lambda |x - \xi| \mathbb{B},$$

where $\lambda = L_F \sqrt{1 + L_{\kappa_c}^2}$, and U_{cl} is a neighborhood of x . Thus, F is locally Lipschitz on S_1 . \square

Since forward invariance requires that every solution to \mathcal{H} stays in \mathcal{M}_r , we define the following two set-valued maps. With $\Pi_c^u(\cdot)C_u$ closed, for each $x \in$

$\Pi_c^u(())C_u$), we define

$$\tilde{\Theta}_c(x) := \{u_c \in \Psi_c^{un}(x) : F_u(x, u_c) \subset T_{\Pi_c^u(())C_u}(x)\}, \quad (6.12)$$

and for each $x \in \Pi(D_u)$,

$$\tilde{\Theta}_d(x) := \{u_d \in \Psi_d^{un}(x) : G_u(x, u_d) \subset (\Pi_c^u(())C_u \cup \Pi_d^u(())D_u)\}. \quad (6.13)$$

Then, the following proposition establishes conditions that guarantee the existence of a continuous state-feedback pair $((\kappa_c, \kappa_d))$ for \mathcal{H}_u to render the set \mathcal{M}_r forward invariant.

Theorem 6.2.5 (*existence of state-feedback pair for forward invariance*) *Consider a hybrid system $\mathcal{H}_u = (C_u, F_u, D_u, G_u)$ as in (6.1) satisfying conditions (A1')-(A3') in Lemma 6.0.6. Suppose the flow map F_u is locally Lipschitz on C_u and there exists a pair (V, r^*) that defines a control Lyapunov function for forward invariance of the sublevel sets of V for \mathcal{H}_u as in Definition 6.2.1 with $\Psi_c^{un}(x)$ in (6.5) replaced by $\tilde{\Theta}_c(x)$ as in (6.12) and $\Theta_d^n(x)$ in (6.6) replaced by $\tilde{\Theta}_d(x)$ as in (6.13). Let $r < r^*$. If the following conditions hold:*

6.2.5.1) *The set-valued map $\tilde{\Theta}_c$ is locally Lipschitz on M_c^n and $\tilde{\Theta}_c$ has nonempty, compact and convex values; the set-valued map $\tilde{\Theta}_d$ is lower semicontinuous, $\text{gph } \tilde{\Theta}_d$ is open relative to $\text{gph } \Psi_d^{un}$, and $\tilde{\Theta}_d$ has nonempty and convex values;*

6.2.5.2) *For each $x \in M_c^n$, the map $u_c \mapsto \Gamma_c^n(x)$ is convex on $\tilde{\Theta}_c(x)$ and, for each $x \in M_d^n$, the map $u_d \mapsto \Gamma_d^n(x)$ is convex on $\tilde{\Theta}_d(x)$,*

then, the set \mathcal{M}_r in (6.7) is controlled forward invariant for \mathcal{H}_u via a state-feedback pair (κ_c, κ_d) with κ_c continuous on M_c^n and κ_d continuous on M_d^n .

Condition 6.2.5.1) in Theorem 6.2.5 is restrictive in the sense that it requires a locally Lipschitz property of $\tilde{\Theta}_c$ rather than of the general input projection Ψ_c^{un} . This is due to the fact that, in general, intersections of locally Lipschitz maps are not locally Lipschitz. However, as the following lemma suggests, it is possible to relax that condition for special cases.

Lemma 6.2.6 *In Theorem 6.2.5, when either*

- 1) *for each $x \in \Pi_c^u(\cdot)C_u$, $F_u(x, u_c) \subset T_{\Pi_c^u(\cdot)C_u}(x)$ for each $u_c \in \Psi_c^{un}(x)$; or*
- 2) *there exist Lipschitz functions $\gamma : \Pi_c^u(\cdot)C_u \rightarrow \mathbb{R}_{>0}$ and $\varepsilon : \Pi_c^u(\cdot)C_u \rightarrow (0, 1)$ such that $\Psi_c^{un}(x) \cap \varepsilon(x)r(x)\mathbb{B} \neq \emptyset$, and for every $x \in \Pi_c^u(\cdot)C_u$, $\tilde{\Theta}_c(x) = \Psi_c^{un}(x) \cap r(x)\mathbb{B}$, condition 6.2.5.1) in Theorem 6.2.5 can be replaced by*

6.2.5.1)* *The set-valued map Ψ_d^{un} is lower semicontinuous, $\text{gph } \tilde{\Theta}_d$ is open relative to $\text{gph } \Psi_d^{un}$, and the set-valued map Ψ_c^{un} is locally Lipschitz.*

Proof Item 1) implies that $\tilde{\Theta}_c(x) = \Psi_c^{un}(x)$ for every $x \in \Pi_c^u(\cdot)C_u$. It follows directly from the definitions of $\tilde{\Theta}_c$ and $uxcn$.

Item 2) follows from the application of Proposition A.0.12, where $S(x) = \tilde{\Theta}_c(x)$, $W(x) = \Psi_c^{un}(x)$ for every $x \in \Pi_c^u(\cdot)C_u$. \square

6.3 Invariance-based Control for Hybrid Systems via Robust Control Lyapunov Functions

For systematic invariance-based feedback design for $\mathcal{H}_{u,w}$ given as in (1.2), we propose control Lyapunov functions that are tailored to forward invariance properties. We refer to these functions as *robust control Lyapunov functions for forward invariance*. Under appropriate conditions, these functions can be used to systematically design state-feedback laws that render a particular sublevel set robustly forward invariant. In simple words, a robust control Lyapunov function for forward invariance, denoted as V , allows to select the inputs of $\mathcal{H}_{u,w}$ as a function of the state x so that a set \mathcal{M}_r^w given as in (5.2), which is a subset of the r -sublevel set of V , has the robust controlled forward invariance property introduced in Definition 6.1.2. As expected, and as formally stated next, the function V needs to satisfy certain CLF-like properties involving the constant r

defining the level of the sublevel set $L_V(r)$ and the data of $\mathcal{H}_{u,w}$. In its definition, we employ the set-valued map

$$\Theta_d(x) := \{u_d \in \Psi_d^u(x) : G_{u,w}(x, u_d, \Psi_d^w(x)) \subset \Pi_c(C_{u,w}) \cup \Pi_d(D_{u,w})\}, \quad (6.14)$$

for every $x \in \Pi_d(D_{u,w})$, which, at each such x , collects all inputs u_d such that, regardless of the value of the disturbance, the state x after jumps is in $\Pi_c(C_{u,w}) \cup \Pi_d(D_{u,w})$.

Definition 6.3.1 (RCLF-FI for $\mathcal{H}_{u,w}$) Consider sets $\mathcal{U}_c \subset \mathbb{R}^{m_c}$, $\mathcal{U}_d \subset \mathbb{R}^{m_d}$, $\mathcal{W}_c \subset \mathbb{R}^{d_c}$, $\mathcal{W}_d \subset \mathbb{R}^{d_d}$, a hybrid system $\mathcal{H}_{u,w} = (C_{u,w}, F_{u,w}, D_{u,w}, G_{u,w})$ as in (1.2), a constant $r^* \in \mathbb{R}$, and a continuous function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ that is also continuously differentiable on an open set containing $\Pi_c(C_{u,w})$. Suppose there exist continuous functions $\rho_c : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\rho_d : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ such that for some $r < r^*$, we have

$$\rho_c(x) > 0 \quad \forall x \in \mathcal{I}(r, r^*), \quad (6.15)$$

$$\rho_d(x) > 0 \quad \forall x \in L_V(r). \quad (6.16)$$

Then, the pair (V, r^*) defines a robust control Lyapunov function for forward invariance (RCLF-FI) of the sublevel sets of V for $\mathcal{H}_{u,w}$ if

$$\inf_{u_c \in \Psi_c^u(x)} \sup_{w_c \in \Psi_c^w(x)} \sup_{\xi \in F_{u,w}(x, u_c, w_c)} \langle \nabla V(x), \xi \rangle + \rho_c(x) \leq 0 \quad \forall x \in \mathcal{I}(r, r^*) \cap \Pi_c(C_{u,w}), \quad (6.17)$$

$$\inf_{u_d \in \Theta_d(x)} \sup_{w_d \in \Theta_d(x)} \sup_{\xi \in G_{u,w}(x, u_d, w_d)} V(\xi) + \rho_d(x) \leq r \quad \forall x \in L_V(r) \cap \Pi_d(D_{u,w}). \quad (6.18)$$

□

Remark 6.3.2 Compared to a typical control Lyapunov function (see, e.g., [72, Definition 2.1]), the RCLF-FI in Definition 6.3.1 is not constrained to be bounded by class \mathcal{K}_∞ functions. In addition, (6.17) does not impose condition in the interior of $L_V(r)$; while, for almost every $x \in L_V(r) \cap \Pi_d(D_{u,w})$, (6.18) allows the change of V to be positive during jumps. Note that the strict positivity requirements in (6.15) and (6.16) are essential to make continuous selections in the forthcoming result.

Given a pair (V, r^*) defined as in Definition 6.3.1 for $\mathcal{H}_{u,w}$ and $r < r^*$ satisfying the conditions therein, our approach consists of selecting a state-feedback law pair (κ_c, κ_d) from these inequalities. In fact, we are interested in synthesizing a pair (κ_c, κ_d) that, in particular, satisfies

$$\begin{aligned} \sup_{w_c \in \Psi_c^w(x)} \sup_{\xi \in F_{u,w}(x, \kappa_c(x), w_c)} \langle \nabla V(x), \xi \rangle + \rho_c(x) &\leq 0 \quad \forall x \in \mathcal{I}(r, r^*) \cap \Pi_c(C_{u,w}), \\ \sup_{w_d \in \Theta_d(x)} \sup_{\xi \in G_{u,w}(x, \kappa_d(x), w_d)} V(\xi) + \rho_d(x) &\leq r \quad \forall x \in L_V(r) \cap \Pi_d(D_{u,w}), \end{aligned}$$

which, under certain mild conditions, renders the set \mathcal{M}_r^w in (7.16) robustly controlled forward invariant for $\mathcal{H}_{u,w}$. Interestingly, with a constant parameter $\sigma \in (0, 1)$, the selection of such a feedback pair can be performed by defining sets that nicely depend on the functions

$$\Gamma_c(x, u_c) := \begin{cases} \sup_{w_c \in \Psi_c^w(x)} \sup_{\xi \in F_{u,w}(x, u_c, w_c)} \langle \nabla V(x), \xi \rangle + \sigma \rho_c(x) & \text{if } (x, u_c) \in \Delta_c, \\ -\infty & \text{otherwise} \end{cases} \quad (6.19)$$

for each $(x, u_c, w_c) \in \mathbb{R}^n \times \mathcal{U}_c \times \mathcal{W}_c$, and

$$\Gamma_d(x, u_d) := \begin{cases} \sup_{w_d \in \Theta_d(x)} \sup_{\xi \in G_{u,w}(x, u_d, w_d)} V(\xi) + \sigma \rho_d(x) - r & \text{if } (x, u_d) \in \Delta_d, \\ -\infty & \text{otherwise,} \end{cases} \quad (6.20)$$

for each $(x, u_d, w_d) \in \mathbb{R}^n \times \mathcal{U}_d \times \mathcal{W}_d$, where $\Delta_c := \{(x, u_c) : (x, u_c, w_c) \in (M_c \times \mathcal{U}_c \times \mathcal{W}_c) \cap C_{u,w}\}$ and $\Delta_d := \{(x, u_d) : (x, u_d, w_d) \in (M_d \times \mathcal{U}_d \times \mathcal{W}_d) \cap D_{u,w}\}$. In fact, with these functions defined, by introducing the set-valued maps

$$\{u_c \in \Psi_c^u(x) : \Gamma_c(x, u_c) < 0\}, \quad \{u_d \in \Theta_d(x) : \Gamma_d(x, u_d) < 0\}$$

which are the so-called *regulation maps* [73], our approach is to determine a state-feedback pair (κ_c, κ_d) that is selected from these maps. In other words, the selected state-feedback pair (κ_c, κ_d) is such that

$$\kappa_c(x) \in \{u_c \in \Psi_c^u(x) : \Gamma_c(x, u_c) < 0\}, \quad \kappa_d(x) \in \{u_d \in \Theta_d(x) : \Gamma_d(x, u_d) < 0\}$$

at the appropriate values of the state x .

The goal of the remainder of this section is to formalize the approach outlined above. First, we provide key results on robust forward invariance of sublevel sets of CLF-like functions, which are used in our CLF approach. It turns out that when an RCLF-FI for $\mathcal{H}_{u,w}$ is provided, regulation maps as outlined above can be constructed for selecting a state-feedback satisfying the conditions in the forthcoming Proposition 5.2.1 and Proposition 5.2.3; hence, the results in Section 5.2 enable us to show the desired invariance property under feedback. Since according to Lemma 6.0.6, the closed-loop system \mathcal{H}_w satisfies conditions (A1_w)-(A3_w) in Definition 2.0.8 when the applied state-feedback pair is continuous, we seek the design of a state-feedback pair (κ_c, κ_d) with κ_c and κ_d being continuous functions of the state. For this purpose, we first reveal conditions assuring the existence of continuous selections from the regulation maps. Our main design results are in Section 6.3.2, where we provide an explicit construction of (κ_c, κ_d) with pointwise minimum norm.

For ease of representation in the forthcoming sections, we define

$$M_c := \mathcal{I}(r, r^*) \cap \Pi_c(C_{u,w}), \quad M_d := L_V(r) \cap \Pi_d(D_{u,w}), \quad (6.21)$$

6.3.1 Existence of State-Feedback Pair for Robust Controlled Forward Invariance

Next, building from Proposition 5.2.1, we establish conditions to guarantee existence of a continuous state-feedback pair (κ_c, κ_d) to render the set \mathcal{M}_r^w robustly controlled forward pre-invariant for $\mathcal{H}_{u,w}$.

Theorem 6.3.3 (*existence of state-feedback pair for robust controlled forward pre-invariance using RCLF-FI*) Consider a hybrid system $\mathcal{H}_{u,w} = (C_{u,w}, F_{u,w}, D_{u,w}, G_{u,w})$ as in (1.2) satisfying conditions (A1'_w)-(A3'_w) in Lemma 6.0.6. Suppose there exists a pair (V, r^*) that defines a robust control Lyapunov function for forward invariance for $\mathcal{H}_{u,w}$ as in Definition 6.3.1. Let $r < r^*$ satisfy (6.15)-(6.18) and Θ_d be given as in (6.14). If the following conditions hold:

6.3.3.1) The set-valued maps Ψ_c^u and Θ_d are lower semicontinuous, and Ψ_c^u and Θ_d have nonempty, closed, and convex values on the sets M_c and M_d as in (6.21), respectively;

6.3.3.2) For each $x \in M_c$, the function $u_c \mapsto \Gamma_c(x, u_c)$ is convex on $\Psi_c^u(x)$ and, for each $x \in M_d$, the function $u_d \mapsto \Gamma_d(x, u_d)$ is convex on $\Theta_d(x)$;

then, the set \mathcal{M}_r^w in (5.2) is robustly controlled forward pre-invariant for $\mathcal{H}_{u,w}$ via a state-feedback pair (κ_c, κ_d) with κ_c continuous on M_c and κ_d continuous on M_d .

Proof To establish the result, we first show the existence of continuous control laws for a restricted version of the original hybrid system $\mathcal{H}_{u,w}$ that is given by

$$\tilde{\mathcal{H}}_{u,w} \begin{cases} \dot{x} \in F_{u,w}(x, u_c, w_c) & (x, u_c, w_c) \in \tilde{C}_{u,w} \\ x^+ \in G_{u,w}(x, u_d, w_d) & (x, u_d, w_d) \in \tilde{D}_{u,w}, \end{cases}$$

where $\tilde{C}_{u,w} := (M_c \times \mathcal{U}_c \times \mathcal{W}_c) \cap C_{u,w}$ and $\tilde{D}_{u,w} := (M_d \times \mathcal{U}_d \times \mathcal{W}_d) \cap D_{u,w}$. To this end, using Γ_c and Γ_d given as in (6.19) and (6.20), for each $x \in \mathbb{R}^n$, we define the set-valued maps

$$\begin{aligned} \tilde{S}_c(x) &:= \{u_c \in \Psi_c^u(x) : \Gamma_c(x, u_c) < 0\}, \\ \tilde{S}_d(x) &:= \{u_d \in \Theta_d(x) : \Gamma_d(x, u_d) < 0\}. \end{aligned}$$

By definition of Θ_d and condition 6.3.3.1), the maps Ψ_c^u and Θ_d are lower semicontinuous and for every $x \in M_d$, $\Theta_d(x)$ is a nonempty, convex subset of $\Psi_d^u(x)$. Then, we show the maps \tilde{S}_c and \tilde{S}_d are lower semicontinuous by applying Corollary A.0.7. First, we establish that functions Γ_c and Γ_d are upper semicontinuous by observing the properties of maps $\Psi_c^w, \Psi_d^w, F_{u,w}$ and $G_{u,w}$.

- i) The maps Ψ_c^w and Ψ_d^w are upper semicontinuous by direct application of [61, Lemma 5.10 and Lemma 5.15]: item (A1'_w) of Lemma 6.0.6 imply maps Ψ_c^w and Ψ_d^w defined in (2.2) have closed graphs and by system assumption, Ψ_c^w and Ψ_d^w are locally bounded;
- ii) The maps Ψ_c^w and Ψ_d^w have compact values, which follows from the fact that Ψ_c^w and Ψ_d^w are locally bounded and $C_{u,w}$ and $D_{u,w}$ are closed;

- iii) The maps $F_{u,w}$ and $G_{u,w}$ are upper semicontinuous by direct application of [61, Lemma 5.10 and Lemma 5.15] while noting that item (A2' $_w$) and (A3' $_w$) of Lemma 6.0.6 hold;
- iv) The maps $F_{u,w}$ and $G_{u,w}$ have compact values, which follows from the fact that $F_{u,w}$ and $G_{u,w}$ are locally bounded and $C_{u,w}$ and $D_{u,w}$ are closed;

Since for every $x \in M_d$, $\Theta_d(x)$ is a nonempty, convex subset of $\Psi_d^u(x)$, Θ_d is also upper semicontinuous and has compact values from properties i) and ii). Then, applying [73, Proposition 2.9], Γ_c and Γ_d are upper semicontinuous on the closed sets Δ_c and Δ_d , respectively, since V is continuously differentiable on an open set containing $\Pi_c(C_{u,w})$, and ρ_c and ρ_d are continuous on M_c and M_d , respectively. Then, Γ_c and Γ_d are upper semicontinuous because for every $(x, u_c, w_c) \in (\mathbb{R}^n \times \mathcal{U}_c \times \mathcal{W}_c) \setminus \tilde{C}_{u,w}$, $\Gamma_c(x, u_c) = -\infty$ and for every $(x, u_d, w_d) \in (\mathbb{R}^n \times \mathcal{U}_d \times \mathcal{W}_d) \setminus \tilde{D}_{u,w}$, $\Gamma_d(x, u_d) = -\infty$. Then, applying Corollary A.0.7, with $z = x, z' = u_c$ (or $z' = u_d$), $W = \Psi_c^u$ (or $W = \Theta_d$) and $w = \Gamma_c$ (or $w = \Gamma_d$, respectively) \tilde{S}_c (or \tilde{S}_d , respectively) is lower semicontinuous. The maps \tilde{S}_c and \tilde{S}_d have nonempty values on M_c and M_d , respectively. This is because, first, Ψ_c^u and Θ_d have nonempty values on M_c and M_d , respectively. In addition, since the inequalities in (6.17) and (6.18) hold, for each $(x, u_c) \in \Delta_c$,

$$\Gamma_c(x, u_c) + \sigma \rho_c(x) \leq 0,$$

and for each $(x, u_d) \in \Delta_d$,

$$\Gamma_d(x, u_d) + \sigma \rho_d(x) \leq 0.$$

Then, since the functions ρ_c and ρ_d have positive values on $\mathcal{I}(r, r^*)$ and $L_V(r)$, respectively, and constant parameter $\sigma \in (0, 1)$, for every $x \in M_c$ (every $x \in M_d$), there exists $u_c \in \Psi_c^u(x)$ (exists $u_d \in \Theta_d(x)$) such that $\Gamma_c(x, u_c) < 0$ (respectively, $\Gamma_d(x, u_d) < 0$). Then, by the convexity of functions Γ_c and Γ_d in condition 6.3.3.2) and of values of the set-valued maps Ψ_c^u and Θ_d in 6.3.3.1), we have that the maps \tilde{S}_c and \tilde{S}_d have convex values on M_c and M_d , respectively.

Then, to use [72, Lemma 4.2] for deriving regulation maps that are also lower semicontinuous, for each $x \in \mathbb{R}^n$, we define the set-valued maps

$$\begin{aligned} S_c(x) &:= \begin{cases} \tilde{S}_c(x) & \text{if } x \in M_c, \\ \mathbb{R}^{m_c} & \text{otherwise,} \end{cases} \\ S_d(x) &:= \begin{cases} \tilde{S}_d(x) & \text{if } x \in M_d, \\ \mathbb{R}^{m_d} & \text{otherwise,} \end{cases} \end{aligned} \tag{6.22}$$

In addition, S_c and S_d also have nonempty and convex values due to the nonemptiness and convex-valued properties of \tilde{S}_c and \tilde{S}_d .

Now, according to Michael's Selection Theorem, namely, Theorem A.0.8, there exist continuous functions $\tilde{\kappa}_c : \mathbb{R}^n \rightarrow \mathbb{R}^{m_c}$ and $\tilde{\kappa}_d : \mathbb{R}^n \rightarrow \mathbb{R}^{m_d}$ such that, for all $x \in \mathbb{R}^n$,

$$\tilde{\kappa}_c(x) \in \overline{S_c(x)}, \quad \tilde{\kappa}_d(x) \in \overline{S_d(x)}.$$

Now, we define functions $\kappa_c : \mathbb{R}^n \rightarrow \mathbb{R}^{m_c}$ and $\kappa_d : \mathbb{R}^n \rightarrow \mathbb{R}^{m_d}$ such that

$$\begin{aligned} \kappa_c(x) &= \tilde{\kappa}_c(x) \in \mathcal{U}_c & \forall x \in M_c, \\ \kappa_d(x) &= \tilde{\kappa}_d(x) \in \mathcal{U}_d & \forall x \in M_d, \end{aligned} \tag{6.23}$$

where the functions κ_c and κ_d inherit the continuity of $\tilde{\kappa}_c$ and $\tilde{\kappa}_d$ on M_c and M_d , respectively. Applying Lemma 6.0.6, the closed-loop system resulting from controlling $\tilde{\mathcal{H}}_{u,w}$ by κ_c and κ_d in (6.23) satisfies the hybrid basic conditions in Definition 2.0.8. More precisely, this is because $\tilde{\mathcal{H}}_{u,w}$ satisfies conditions (A1')-(A3') in Lemma 6.0.6, and the state-feedback pair (κ_c, κ_d) is continuous on $\Pi_c(\tilde{\mathcal{C}}_{u,w}) \cup \Pi_d(\tilde{\mathcal{D}}_{u,w})$. With these properties and the fact that ∇V is continuous, it follows that

$$\begin{aligned} \kappa_c(x) \in \Psi_c^u(x), \quad \Gamma_c(x, \kappa_c(x)) &\leq 0 & \forall x \in M_c, \\ \kappa_d(x) \in \Theta_d(x), \quad \Gamma_d(x, \kappa_d(x)) &\leq 0 & \forall x \in M_d, \end{aligned}$$

which lead to

$$\sup_{\xi \in F_{u,w}(x, \kappa_c(x), w_c)} \langle \nabla V(x), \xi \rangle + \rho_c(x) \leq 0 \quad \forall (x, \kappa_c(x), w_c) \in \tilde{C}_{u,w}, \quad (6.24)$$

$$\sup_{\xi \in G_{u,w}(x, \kappa_d(x), w_d)} V(\xi) + \rho_d(x) - r \leq 0 \quad \forall (x, \kappa_d(x), w_d) \in \tilde{D}_{u,w}. \quad (6.25)$$

The state feedback laws κ_c and κ_d can be extended – not necessarily continuously – to every point in $\Pi_c(C_{u,w})$ and $\Pi_d(D_{u,w})$, respectively, by selecting values from the nonempty sets $\Psi_c^u(x)$ for every $x \in \Pi_c(C_{u,w})$ and $\Theta_d(x)$ for every $x \in \Pi_d(D_{u,w})$.

To complete the proof, we establish the robust controlled forward pre-invariance of \mathcal{M}_r^w . For this purpose, we apply Proposition 5.2.1 to the closed-loop system of $\mathcal{H}_{u,w}$ controlled via the extended state-feedback pair (κ_c, κ_d) that is defined on $\Pi_c(C_{u,w}) \cup \Pi_d(D_{u,w})$. Relationship (6.24) and (6.25) imply

$$\begin{aligned} \langle \nabla V(x), \xi \rangle &\leq 0 & \forall (x, w_c) &\in (\mathcal{I}(r, r^*) \times \mathcal{W}_c) \cap C_w, \xi \in F_w(x, w_c) \\ V(\xi) &\leq r & \forall (x, w_d) &\in (L_V(r) \times \mathcal{W}_d) \cap D_w, \xi \in G_w(x, w_d), \end{aligned}$$

respectively. Thus, it is the case that (5.7) and (5.8) hold for the resulting closed-loop system. Moreover, since $\kappa_d(x) \in \Theta_d(x)$ for every $x \in M_d$, (6.14) implies (5.9) for \mathcal{H}_w . Hence, according to Definition 6.1.2, the extended state-feedback pair (κ_c, κ_d) renders the set \mathcal{M}_r^w as in (5.2) robustly controlled forward pre-invariant for $\mathcal{H}_{u,w}$. \square

Remark 6.3.4 *Item 6.3.3.1) in Theorem 6.3.3 imposes lower semicontinuity of the mappings from state space to the input spaces during flows and jumps. For systems that does not have convex-valued Ψ_c^u and Θ_d on M_c and M_d , respectively, Theorem 6.3.3 can still be applied, if there exist nonempty, closed and convex subsets of $\Psi_c^u(x)$ and $\Theta_d(x)$ for every $x \in M_c$ and $x \in M_d$, respectively, such that item 6.3.3.2) holds for these subsets. Similar comments apply to the forthcoming results.*

To show existence of state feedback pair (κ_c, κ_d) that renders \mathcal{M}_r^w as in (5.2) robustly forward invariant, we need further conditions on the regulation maps to

ensure existence of a solution pair from every $\Pi_c(C_{u,w})$. Hence, we dedicate the remainder of this section to show, with a variation of RCLF-FI in Definition 6.3.1, there exists a feedback pair for a class of $\mathcal{H}_{u,w}$ that induces robust controlled forward invariance of \mathcal{M}_r^w by applying Proposition 5.2.3. In particular, the next result resembles Theorem 6.3.3, but employs different regulation maps to guarantee existence of nontrivial solution pairs and their completeness. To this end, for every $x \in \Pi_c(C_{u,w})$, we define the map

$$\Theta_c(x) := \begin{cases} \{u_c \in \Psi_c^u(x) : F_{u,w}(x, u_d, 0) \cap T_{\Pi_c(C_{u,w})} \neq \emptyset\} & \forall x \in \partial\Pi_c(C_{u,w}) \setminus \Pi_d(D_{u,w}) \\ \Psi_c^u(x) & \text{otherwise.} \end{cases} \quad (6.26)$$

Theorem 6.3.5 (*existence of state-feedback pair for robust controlled forward invariance using RCLF-FI*) Consider a hybrid system $\mathcal{H}_{u,w} = (C_{u,w}, F_{u,w}, D_{u,w}, G_{u,w})$ as in (1.2) satisfying conditions (A1'_w)-(A3'_w) in Lemma 6.0.6. Suppose there exists a pair (V, r^*) that defines a robust control Lyapunov function for forward invariance of the sublevel sets of V for $\mathcal{H}_{u,w}$ as in Definition 6.3.1 with $\Psi_c^u(x)$ in (6.17) replaced by $\Theta_c(x)$ as in (6.26). Let $r < r^*$ satisfy (6.15)-(6.18) and Θ_d be given as in (6.14). If the following conditions hold:

6.3.5.1) The set-valued maps Θ_c and Θ_d are lower semicontinuous and Θ_c and Θ_d have nonempty, closed, and convex values on the set M_c and the set M_d , respectively;

6.3.5.2) For each $x \in M_c$, the map $u_c \mapsto \Gamma_c(x, u_c)$ is convex on $\Theta_c(x)$ and, for each $x \in M_d$, the map $u_d \mapsto \Gamma_d(x, u_d)$ is convex on $\Theta_d(x)$;

then, the set \mathcal{M}_r^w in (5.2) is robustly controlled forward pre-invariant for $\mathcal{H}_{u,w}$ via a state-feedback pair (κ_c, κ_d) with κ_c continuous on M_c and κ_d continuous on M_d . Furthermore, if item 5.1.1.4) in Proposition 5.2.3 holds for the closed-loop system \mathcal{H}_w as in (1.3), the pair (κ_c, κ_d) renders set \mathcal{M}_r^w robustly controlled forward invariant for $\mathcal{H}_{u,w}$.

Proof The robust forward pre-invariance of \mathcal{M}_r^w for $\mathcal{H}_{u,w}$ follows from a direct application of Theorem 6.3.3. More precisely, when conditions in Theorem 6.3.5 hold, every condition in Theorem 6.3.3 holds for a hybrid system $\tilde{\mathcal{H}}$ that has flow map, jump map, and jump set given as $F_{u,w}$, $G_{u,w}$, and $D_{u,w}$, respectively, and flow set given by

$$\tilde{C}_{u,w} = (\mathbb{R}^n \times \{u_c \in \Theta_c(x) : x \in \Pi_c(C_{u,w})\} \times \mathcal{W}_c) \cap C_{u,w}.$$

Note that $\tilde{C}_{u,w}$ is closed since $C_{u,w}$ is close, Θ_c has closed values on M_c and $\partial\Pi_c(C_{u,w}) \setminus \Pi_d(D_{u,w}) \subset M_c$. Hence, there exists a state-feedback pair (κ_c, κ_d) that renders \mathcal{M}_r^w robustly controlled forward pre-invariant for $\tilde{\mathcal{H}}$ with κ_c and κ_d being continuous on M_c and M_d , respectively. Since for every $x \in \Pi_c(C_{u,w})$, such $\kappa_c(x) \in \Theta_c(x) \subset \Psi_c^u(x)$, this implies such pair (κ_c, κ_d) is also $\mathcal{H}_{u,w}$ -admissible. Moreover, the closed-loop system resulting from $\tilde{\mathcal{H}}$ controlled by (κ_c, κ_d) is also the closed-loop system of $\mathcal{H}_{u,w}$ controlled by the same pair (κ_c, κ_d) . Hence, by Definition 6.1.2, such (κ_c, κ_d) renders \mathcal{M}_r^w robustly controlled forward pre-invariant for $\mathcal{H}_{u,w}$.

According to Theorem A.0.9, since the set M_c is closed, there exists a continuous extension of κ_c from $\mathcal{I}(r, r^*) \cap \Pi_c(C_{u,w})$ to \mathbb{R}^n with $\kappa_c(x) \in \mathbb{R}^m$ for every $x \in \text{int}L_V(r) \cap \Pi_c(C_{u,w})$.³ Then, applying such pair (κ_c, κ_d) , with κ_c and κ_d being continuous on $L_V(r) \cap \Pi_c(C_{u,w})$ and M_d , respectively, Lemma 6.0.6 implies the closed-loop system is such that F_w is outer semicontinuous, locally bounded and has nonempty and convex values on $(\mathcal{M}_r^w \times \mathcal{W}_c) \cap C_w$. Hence, item (A2_w) in Definition 2.0.8 holds for closed-loop system $\tilde{\mathcal{H}}$. Then, applying Proposition 5.2.3, we show that the pair (κ_c, κ_d) renders set \mathcal{M}_r^w robustly controlled forward invariant for $\tilde{\mathcal{H}}$. For every $x \in \Pi_c^w(C_w)$, $0 \in \Psi_c^w(x)$ by assumption. Next, inequalities (5.7) and (5.8) follow from (6.24) and (6.25) for the given pair (V, r^*) . In addition, (6.24) implies condition 5.1.1.1), and condition 5.1.1.2) follows from the definition of Θ_c . Then, (6.24) and definition of Θ_c together implies item 5.1.1.3). Item 5.1.1.4) holds by assumption. The definition of Θ_d imply (5.9) holds. Hence, the

³Note that the selected κ_c in proof of Theorem 6.3.3 is not necessarily continuous on $\Pi_c(C_{u,w})$.

set \mathcal{M}_r^w is robustly controlled forward invariant for $\tilde{\mathcal{H}}$ via the selected (κ_c, κ_d) . Furthermore, as showed above, the pair (κ_c, κ_d) is $\mathcal{H}_{u,w}$ -admissible and renders the set \mathcal{M}_r^w robustly controlled forward invariant for $\mathcal{H}_{u,w}$ by Definition 6.1.2. \square

Theorem 6.3.5 uses an alternative RCLF-FI that is defined based on Θ_c as in (6.26) instead of Ψ_c^u as in Definition 6.3.1, which leads to the existence of state-feedbacks rendering robust controlled forward invariance for $\mathcal{H}_{u,w}$. By selecting κ_c from the map Θ_c in (6.26) rather than the generic Ψ_c^u , we guarantee existence of nontrivial solution pair from every $x \in \mathcal{M}_r^w \setminus \Pi_d(D_{u,w})$. This follows from an application of Lemma 5.2.4 and the fact that items 5.1.1.1), 5.1.1.3) and 5.1.1.4) in Proposition 5.2.3 hold. Moreover, item 5.1.1.4) ensures completeness of every $(\phi, w) \in \mathcal{S}_{\mathcal{H}_w}(\mathcal{M}_r^w)$.

Remark 6.3.6 *Selection results for nominal hybrid system without perturbations are published in [64], which are developed from a different set conditions and control Lyapunov functions for forward invariance; see details in [64, Definition 4.1]. Results in [64] are derived from the forward invariance inducing sufficient conditions for generic sets⁴ and are not tailored to sublevel sets of function V . In particular, to guarantee the state component of every solution pair stays within \mathcal{M}_r^w , the feedback law κ_c needs to be locally Lipschitz, see [64, Theorem 4.7, R4)]. To get such a property, condition [64, Theorem 4.7, R1')]* asks the regulation map $\tilde{\Theta}_c$ to be locally Lipschitz, and κ_c is a Lipschitz selection, which is more intricate than a continuous selection. By virtue of results in Section 5.2, Theorem 6.3.5 only require κ_c to be a continuous selection.

Remark 6.3.7 *In the case where control inputs are only applied during jumps, conditions in Theorem 6.3.3 leads to robustly controlled forward invariance of $\mathcal{H}_{u,w}$, provided (5.7) holds during flows. Similarly, when control inputs are only applied over flow dynamics, the $F_{u,w}$ and $C_{u,w}$ related conditions in Theorem 6.3.3*

⁴The equivalent results of Corollary 6.1.5 in Section 6.1.

together with (5.8) lead to robust controlled forward invariance of \mathcal{M}_r^w . In addition, results in this section can be applied to pure continuous-time or pure discrete-time systems by defining RCLF-FI only based on (6.17) or (6.18), respectively.

6.3.2 Systematic Design of Pair (κ_c, κ_d) for Robust Controlled Forward Invariance

Inspired by the pointwise minimum norm results in [73] and [62, Theorem 5.1], we construct state-feedback pairs rendering robust controlled forward invariance of sets in form of \mathcal{M}_r^w as in (5.2). To this end, for given pair (V, r^*) defines a RCLF-FI as in Definition 6.3.1 of hybrid system $\mathcal{H}_{u,w}$, we employ Theorem 6.3.3. As a result, state-feedbacks constructed via minimal selections ought to render set \mathcal{M}_r^w robustly controlled forward pre-invariant for hybrid system $\mathcal{H}_{u,w}$. Such a claim relies on constructing appropriate functions Γ_c, Γ_d and regulation maps S_c, S_d in Section 6.3.1.

Consider the maps S_c and S_d defined in (6.22). When 6.3.3.2) in Theorem 6.3.3 holds, $u_c \mapsto \Gamma_c(x, u_c)$ and $u_d \mapsto \Gamma_d(x, u_d)$ is convex on $\Psi_c^u(x)$ for every $x \in M_c$ and on $\Theta_d(x)$ for every $x \in M_d$, respectively. Hence, the maps S_c and S_d have nonempty and convex values on \mathbb{R}^n . According to [74, Theorem 4.10], for every $x \in L_V(r^*) \cap \Pi_c(C_{u,w})$ and $x \in L_V(r^*) \cap \Pi_d(D_{u,w})$, respectively, the closure of $S_c(x)$ and $S_d(x)$, i.e., $\overline{S_c(x)}$ and $\overline{S_d(x)}$, have unique element of minimum norm. Thus, we construct the state-feedback laws $\kappa_c^m : L_V(r^*) \cap \Pi_c(C_{u,w}) \rightarrow \mathcal{U}_c$ and $\kappa_d^m : L_V(r^*) \cap \Pi_d(D_{u,w}) \rightarrow \mathcal{U}_d$ that are given by

$$\begin{aligned} \kappa_c^m(x) &:= \arg \min_{u_c \in \overline{S_c(x)}} |u_c| & \forall x \in L_V(r^*) \cap \Pi_c(C_{u,w}) & (6.27) \\ \kappa_d^m(x) &:= \arg \min_{u_d \in \overline{S_d(x)}} |u_d| & \forall x \in L_V(r^*) \cap \Pi_d(D_{u,w}). \end{aligned}$$

Moreover, such state-feedback pair enjoy continuity when map Ψ_c^u and Θ_d satisfies 6.3.3.1). We capture above claims in the following result, which is a direct application of Theorem 6.3.3.

Theorem 6.3.8 (*pointwise minimum norm state-feedback laws for robust controlled forward pre-invariance*) Consider a hybrid system $\mathcal{H}_{u,w}$ as in (1.2) satisfying conditions (A1'_w)-(A3'_w) in Lemma 6.0.6. Suppose there exists a pair (V, r^*) that defines a robust control Lyapunov function for forward invariance for $\mathcal{H}_{u,w}$ as in Definition 6.3.1. Let $r < r^*$ satisfy (6.15)-(6.18) and Θ_d be given as in (6.14). Furthermore, suppose conditions 6.3.3.1) and 6.3.3.2) in Theorem 6.3.3 hold. Then, the state-feedback pair (κ_c^m, κ_d^m) given as in (6.27) renders the set \mathcal{M}_r^w in (5.2) robustly controlled forward pre-invariant for $\mathcal{H}_{u,w}$. Moreover, κ_c^m and κ_d^m are continuous on set M_c and M_d as in (6.21), respectively.

Proof The first claim follows from similar proof steps in Theorem 6.3.3. In particular, since κ_c^m and κ_d^m are selected from the closure of S_c and S_d , i.e.,

$$\kappa_c^m(x) \in \overline{S_c(x)}, \quad \text{and} \quad \kappa_d^m(x) \in \overline{S_d(x)},$$

it follows that

$$\begin{aligned} \kappa_c^m(x) \in \Psi_c^u(x), \Gamma_c(x, \kappa_c^m(x)) &\leq 0 & \forall x \in M_c, \\ \kappa_d^m(x) \in \Theta_d(x), \Gamma_d(x, \kappa_d^m(x)) &\leq 0 & \forall x \in M_d, \end{aligned}$$

which lead to

$$\begin{aligned} \sup_{\xi \in F_{u,w}(x, \kappa_c^m(x), w_c)} \langle \nabla V(x), \xi \rangle + \rho_c(x) &\leq 0 & \forall (x, \kappa_c^m(x), w_c) \in \tilde{C}_{u,w}, \\ \sup_{\xi \in G_{u,w}(x, \kappa_d^m(x), w_d)} V(\xi) + \rho_d(x) - r &\leq 0 & \forall (x, \kappa_d^m(x), w_d) \in \tilde{D}_{u,w}. \end{aligned}$$

The feedback pair (κ_c^m, κ_d^m) can be extended to every point in $\Pi_c(C_{u,w})$ and $\Pi_d(D_{u,w})$, respectively, by selecting values from the nonempty sets $\Psi_c^u(x)$ for every $x \in \Pi_c(C_{u,w})$ and $\Theta_d(x)$ for every $x \in \Pi_d(D_{u,w})$. Then, applying Proposition 5.2.1, we establish the robust controlled forward pre-invariance of \mathcal{M}_r^w for $\mathcal{H}_{u,w}$ via (κ_c^m, κ_d^m) .

Finally, the continuity of κ_c^m and κ_d^m follow directly from Proposition A.0.10. In particular, maps $\overline{S_c}$ and $\overline{S_d}$ are lower semicontinuous with nonempty closed convex values as shown in proof of Theorem 6.3.3. \square

A similar result to Theorem 6.3.8 can be derived using Theorem 6.3.5 to render \mathcal{M}_r^w robustly controlled forward invariant for $\mathcal{H}_{u,w}$ via (κ_c^m, κ_d^m) . In such a case, the feedback law κ_c^m is selected from the closure of a map S_c that is defined using Θ_c given as in (6.26) instead of Ψ_c^u . More precisely, we consider the state feedback laws κ_c^m defined as in (6.27) with S_c given by

$$S_c(x) := \begin{cases} \{u_c \in \Theta_c(x) : \Gamma_c(x, u_c) < 0\} & \text{if } x \in M_c, \\ \mathbb{R}^{m_c} & \text{otherwise.} \end{cases} \quad (6.28)$$

In addition to conditions 6.3.5.1) and 6.3.5.2) in Theorem 6.3.5, robustly controlled forward invariance of \mathcal{M}_r^w requires item 5.1.1.4) in Proposition 5.2.3 to hold for the closed-loop system \mathcal{H}_w . We formally present such a result as follows.

Theorem 6.3.9 (*pointwise minimum norm state-feedback laws for robust controlled forward invariance*) Consider a hybrid system $\mathcal{H}_{u,w}$ as in (1.2) satisfying conditions $(A1'_w)$ - $(A3'_w)$ in Lemma 6.0.6. Suppose there exists a pair (V, r^*) that defines a robust control Lyapunov function for forward invariance for $\mathcal{H}_{u,w}$ as in Definition 6.3.1. Let $r < r^*$ satisfy (6.15)-(6.18), Θ_c and Θ_d be given as in (6.26) and (6.14), respectively. Furthermore, suppose conditions 6.3.5.1) and 6.3.5.2) in Theorem 6.3.5 hold. Then, the state-feedback pair (κ_c^m, κ_d^m) given as in (6.27) defined using S_c as in (6.28) renders the set \mathcal{M}_r^w in (5.2) robustly controlled forward invariant for $\mathcal{H}_{u,w}$ if condition 5.1.1.4) in Proposition 5.2.3 holds for the closed-loop system \mathcal{H}_w . Moreover, κ_c^m and κ_d^m are continuous on set M_c and M_d as in (6.21), respectively.

We omit the proof here as it resembles the one for Theorem 6.3.8. Note that the design results (Theorem 6.3.8 and Theorem 6.3.9) in this section naturally apply to hybrid systems without disturbances.

Chapter 7

Applications of Invariance-based Control

In this chapter, we illustrate the analysis and control design tools developed in previous chapters to several engineering applications. Relying on these tools, we complete tasks that are recast as forward invariance inducing problems. We motivate and explain the hybrid nature for each system. Moreover, their implementations focus on the importance and applicability of forward invariance tools developed in this thesis. Among these applications, we pay special attention to two power conversion problems; see Section 7.2 and Section 7.3.

7.1 Applying Forward Invariance Tools to Power Conversions in Smart Grids

Under the name “smart grid,” future power generation and distribution systems ought to provide efficient, reliable and environmental-friendly power generation, conversion and transmission to customers. In particular, advanced power conversion methods from renewable energy sources are required. The newly developed hybrid system tools have the potential to address such challenges; see examples

in [75–80]. Tools for hybrid systems in [61] can be applied to the modeling, controller design, and analysis of these power conversion systems. In particular, controllers are designed and mathematically analyzed for the two design problems using forward invariance properties. We show that the application of hybrid system theories to power conversion not only provides implementable controllers, but also are useful in highlighting the robustness introduced by such (hybrid) feedback.

In the following sections, two feedback control design problems for power systems, one pertaining to DC/DC conversion and the other to DC/AC inversion, are presented for the use of tools for hybrid systems in power conversion. The single-phase DC/AC inverter circuit, a switching system, is capable of transforming a DC input voltage into an AC output voltage. As shown in Figure 7.1a (see Chapter 7.2 for notations), by controlling the positions of the four switches of the inverter, the sign of the input DC voltage to the RLC filter changes, and when appropriately controlled, the voltage across the capacitor and the current through the inductor can evolve almost sinusoidally. Similarly, the DC/DC boost converter circuit in Figure 7.1b (see Chapter 7.3 for notations), also a switching system, is able to convert a lower DC voltage input to a higher DC voltage output by switching the switch on and off based on control logics.

Typically, both circuits are controlled using Pulse Width Modulation (PWM) techniques [81, 82]. The PWM-based controllers change switch configuration of the circuits based on the sign changes in the difference between a carrier signal, usually a triangular wave, and a reference signal. The performance of PWM-based controllers has been thoroughly studied in literatures [83–85]. Critical issues in power conversion have led to the development of new control algorithms relying on recent advances in the theory of switching and hybrid systems [77, 86, 87]. For example, in the control of inverter, one of the shortcomings of the PWM-based controllers is that the control of the output voltage magnitude is not robust to changes of the input DC voltage. Without a DC voltage regulator at its input, the

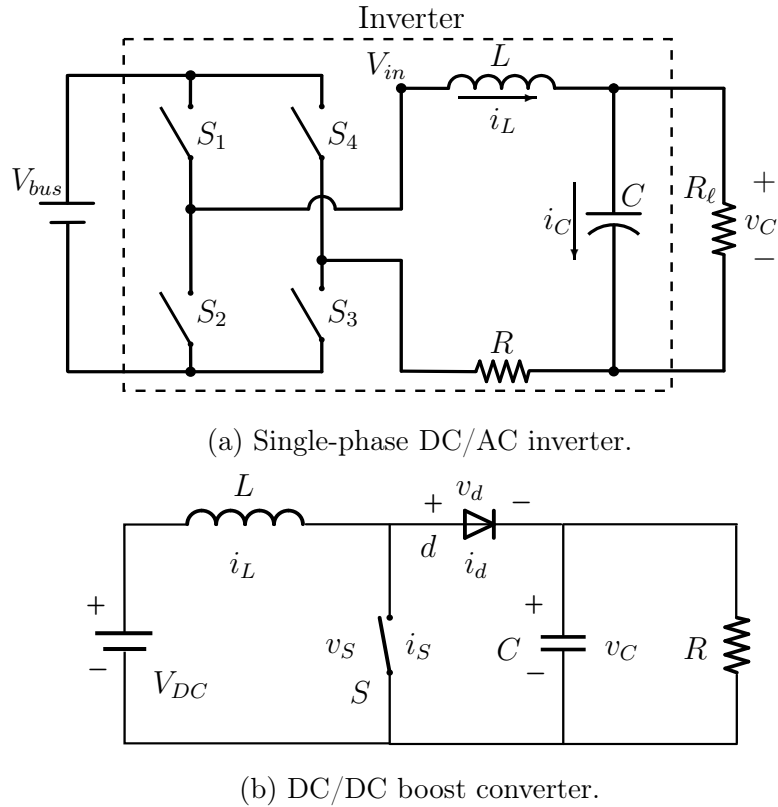


Figure 7.1: Circuit diagrams for power conversion control design problems.

“sinusoidal” output would be significantly affected, while the proposed controller using forward invariance tools for hybrid systems manages to maintain a consistent sinusoidal-like output.

The systems involved in these problems have the following challenging features, which make tools for hybrid systems very fitting (if not mandatory) for their analysis and design:

- *Systems involve nonsmooth dynamics under constraints due to the presence of switches and/or diodes.* Most power conversion circuits include some sort of switching mechanism as well as passive components for filtering. The switching mechanism typically introduces changes in the dynamics, which define different modes of operation and associated discrete dynamics. The

passive components for filtering and other analog tasks introduce continuous dynamics into the system. In this way, depending on the configuration of the switches and/or diodes in the circuit, the system operates at different modes and switches between them. By controlling the configuration of the switches with an appropriate algorithm, the closed-loop system generates desired output signals. Popular control algorithms for such purpose are designed using pulse width modulation (PWM), which is a technique that changes the configuration of the switches by comparing a carrier signal (e.g., a triangular signal for DC/AC inverter) and a reference signal (e.g., a sinusoidal signal for DC/AC inverter).

Due to the switching nature of these power converters/inverters and the associated continuous dynamics, systems have nonsmooth dynamics, which can be modeled as differential equations/inclusions with constraints as in [61]. For example, the single phase DC/AC inverter with H-bridge in Fig. 7.1a has three operation modes, and each mode corresponds to a vector field that is described by one set of differential inclusions; see Section 7.2.

- *Stabilization goals require recurrent switching.* Unfortunately, the desired output of these systems cannot be generated by choosing a single mode of operation for all time. In fact, for each fixed configuration of the switches, the resulting system has an equilibrium point that does not represent the desired output. More precisely, for example, for the circuit in Fig 7.1a, in which i_L and v_C represent the current through inductor and voltage across the capacitor, when $S_1 = S_3 = \text{ON}$ and $S_2 = S_4 = \text{OFF}$, the resulting equilibrium is for $i_L = 0$ and $v_C = V_{DC}$, while, when $S_1 = S_3 = \text{OFF}$ and $S_2 = S_4 = \text{ON}$, the equilibrium condition is $i_L = 0$ and $v_C = -V_{DC}$ (other equilibrium points can be computed similarly). Due to this, a control algorithm that changes the configuration of the switches recurrently is required to achieve the desired AC output for the inverter. Similarly, algorithms with the same feature are required for the DC/DC boost converter in Fig. 7.1b,

where the control algorithm needs to switch between two operation modes to generate an approximate DC output signal.

- *Systems have state perturbations and unmodeled dynamics.* In addition to stability properties, the hybrid analysis tools from [61] allow us to conclude robustness properties of the power conversion systems. In particular, having the closed-loop systems with designed controller to satisfy conditions in [61, Assumption 6.5] directly leads to robustness to small state perturbations and unmodeled dynamics. Moreover, these tools also benefits the modeling and analysis of hybrid systems that require periodic-like solutions, which can be studied using the forward invariance for sets.

Both power conversion systems are modeled as hybrid systems, i.e., \mathcal{H} in (2.1). Control algorithms that leads to desired output signals, which use hybrid feedback control scheme, are presented in the following sections with analysis featuring forward invariance properties of sets. Simulations for both problems confirm the usefulness of hybrid systems methods in power conversion. The detailed work in Section 7.3 and Section 7.2 are presented in [88] and [89], respectively.

In addition, to allow simulation-based quantifiable performance comparison between our control algorithms and others, we propose benchmark tests that focus on switching properties of these power systems. In particular, the proposed benchmark tests are relevant when assessing durability of the switching devices used in hardware/software implementations. For the DC/DC boost converter controller with spatial regulation, we study the number of switches per second during its “steady state”; see Section 7.3. For the DC/AC inverter controller, the benchmark tests consists of determining the number of switches per period of the generated sinusoidal-like signal; see Section 7.2. Furthermore, we indicate that both control algorithms have the flexibility of changing how often the switches happen by adjusting a corresponding controller parameter.

7.2 A Single Phase DC/AC Inverter

In this work, we consider a single-phase DC/AC inverter circuit that consists of a full H-bridge connecting to a series RLC filter, as shown in Figure 7.1a.

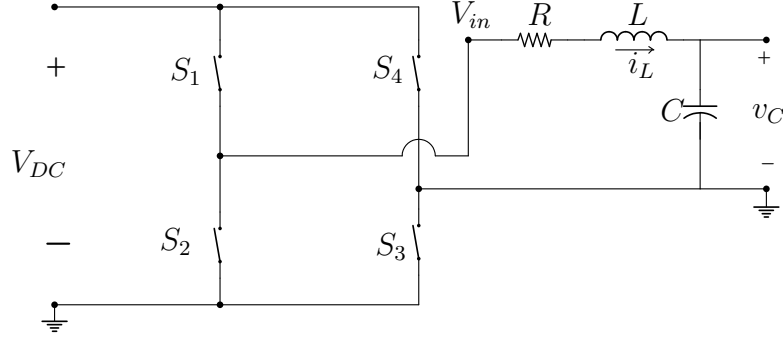


Figure 7.2: Single-phase DC/AC inverter circuit diagram.

In particular, we design a controller for the plant with an external time varying (positive) input signal V_{bus} , which is a perturbed DC signal that is given by

$$t \mapsto V_{bus} = V_{DC} + d_v(t), \quad (7.1)$$

where V_{DC} is the constant nominal voltage of signal V_{bus} and the term $d_v(t) \in \mathbb{R}$ is 0 for all $t \geq \mathbb{R}_{\geq 0}$ when no disturbance is present and a function of t otherwise. Moreover, the inverter plant has the voltage across the capacitor C , denoted by v_C , and the current through the inductor L , denoted by i_L , as its output signals. Specifically, in this paper, we design and validate a controller that selects the operation configurations of the full H-bridge such that the generated output v_C approximates a sinusoidal signal with desired frequency and amplitude by appropriately toggling the switches. The presence of the full H-bridge, i.e., the switches $S_1 - S_4$, in the circuit introduces non-smooth dynamics. By controlling the position of the switches at every t , to either “ON” or “OFF” position, we consider the operation modes where the input voltage to the RLC filter, i.e., V_{in} , is either V_{bus} , $-V_{bus}$, or 0. More precisely, we study the system with dynamics

given by

$$\dot{i}_L = \begin{cases} -\frac{R}{L}i_L - \frac{1}{L}v_C + \frac{V_{bus}}{L} & \text{when } S_1 = S_3 = \text{ON and } S_2 = S_4 = \text{OFF}; \\ -\frac{R}{L}i_L - \frac{1}{L}v_C - \frac{V_{bus}}{L} & \text{when } S_1 = S_3 = \text{OFF and } S_2 = S_4 = \text{ON}; \\ -\frac{R}{L}i_L - \frac{1}{L}v_C & \text{when } S_1 = S_4 = \text{OFF and } S_2 = S_3 = \text{ON} \end{cases} \quad (7.2)$$

$$\dot{v}_C = \frac{1}{C}i_L,$$

where R, L, C are parameters of the circuit. Let $z := (i_L, v_C) \in \mathbb{R}^2$ and q be a logic variable that describes the operation configurations of the full H-bridge. Then, $q \in Q := \{-1, 0, 1\}$ leads to the compact form of (7.2) given as

$$\dot{z} = f_q(z, t) := \begin{bmatrix} -\frac{R}{L}i_L - \frac{1}{L}v_C + \frac{V_{bus}}{L}q \\ \frac{1}{C}i_L \end{bmatrix}. \quad (7.3)$$

Remark 7.2.1 *Note that the full H-bridge may present other operation configurations in between switches among the ones that correspond to $q \in Q$. We address these “transient” operation modes as unmodeled system dynamics in robustness analysis for the closed-loop system in Section 7.2.3.1. In addition, we explore robustness to stability under the presence of measurement noise of state $z \in \mathbb{R}^2$.*

7.2.1 State-dependent Control Law

Our control objective is to design a control law for the inverter system in Figure 7.1a such that the output v_C approximates a sinusoidal signal with desired frequency and amplitude. In this paper, we consider the reference signal given by

$$t \mapsto V_r(t) = b \sin(\omega t + \theta) \quad (7.4)$$

where b is the targeted magnitudes, $\omega > 0$ is the targeted angular frequency and θ is the initial phase.¹ By Kirchoff’s law, v_C and i_L in the RLC filter always satisfy relationship

$$\dot{v}_C(t) = \frac{1}{C}i_L(t),$$

¹Note tracking phase of reference signal is out of our project scope.

hence, when v_C approximates signal given by (7.4), i_L approximates the signal

$$t \mapsto I_r(t) = C\omega b \cos(\omega t + \theta). \quad (7.5)$$

Following the exosystem approach, the signals given in (7.4) and (7.22) can be generated by

$$\dot{z}_r = \begin{bmatrix} 0 & -C\omega^2 \\ \frac{1}{C} & 0 \end{bmatrix} z_r, \quad z_r(0) = \begin{bmatrix} C\omega b \cos(\theta) \\ b \sin(\theta) \end{bmatrix}. \quad (7.6)$$

Let $a = C\omega b$, on the (i_L, v_C) plane, the z_r trajectory describes an ellipse with semi-major axis a and semi-minor axis b , and aspect ratio $\frac{a}{b} = C\omega$. Referred to as the *reference trajectory* on the (i_L, v_C) plane, such an ellipse is give by

$$S_r := \left\{ z_r : \left(\frac{z_{r1}}{a} \right)^2 + \left(\frac{z_{r2}}{b} \right)^2 = 1 \right\}. \quad (7.7)$$

For every $z \in \mathbb{R}^2$, we define

$$V(z) := \left(\frac{i_L}{a} \right)^2 + \left(\frac{v_C}{b} \right)^2. \quad (7.8)$$

Then, taking advantage of the function $V(z)$, we provide an alternative control strategy with arbitrary precision to the traditional PWM techniques. More precisely, rather than using a PWM-based controller to generate switching signal V to the RLC filter, our control law properly assign $q \in Q$ to the H-bridge based on spatial feedback logics that uses a tunable neighborhood, which is referred to as the *invariant band*, around the S_r . Given the design parameters $c_i \in (0, 1)$ and $c_o > 1$, the invariant band is given by

$$K_z := \{z \in \mathbb{R}^2 : c_i \leq V(z) \leq c_o\}. \quad (7.9)$$

The proposed control law guarantees solutions to the resulting closed-loop system converge to the set K_z in finite time, and after that, stays within it for all future time. For this purpose, Section 7.2.1.1 introduces a switching logic that guarantees forward invariance of K_z , and after that, a global convergence controller is provided in Section 7.2.1.2. The supervisor controller presented in Section 7.2.1.3 provides the logic to determine the appropriate controller in the loop for the full closed-loop system.

A sample solution to the full closed-loop system with the proposed controller is shown in Figure 7.3. On the (i_L, v_C) plane, the invariant band K_z has outer boundary $S_o^* = \{z \in \mathbb{R}^2 : V(z) = c_o\}$ (the outer green dashed line) and inner boundary $S_i^* = \{z \in \mathbb{R}^2 : V(z) = c_i\}$, (the inner green dash line). The reference trajectory (the blue solid line) is enclosed by the invariant band K_z . As Figure 7.3 depicts, the solution trajectory initialized within K_z , represented by the red solid line on the (i_L, v_C) plane, remains in the invariant band K_z for all time and “approximates” the reference trajectory; while the output signals i_L and v_C are “periodic-like.”

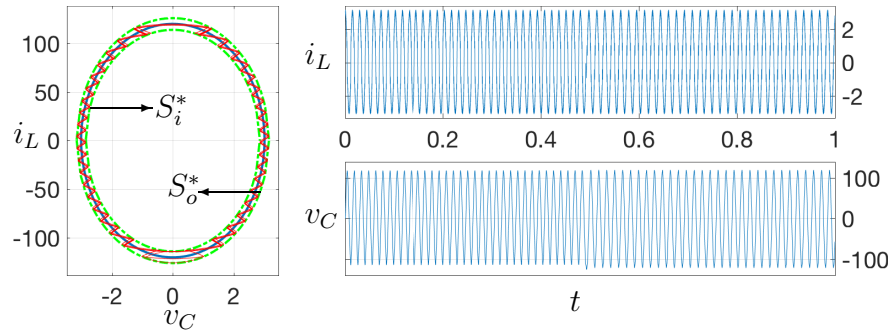


Figure 7.3: A sample trajectory resulting from using the proposed control law with circuit parameters as $R = 1\Omega$, $L = 0.106\text{H}$, $C = 66.3\mu\text{F}$, $V_{bus} \equiv 220\text{V}$, $b = 120$, $\omega = 120\pi$, $c_i = 0.9$ and $c_o = 1.1$.

To ensure existence and completeness of solutions under the presence of small noises, we “inflate” the boundary sets S_i and S_o outside of the invariant band K_z . More precisely, consider a small inflation factor $\delta \geq 0$, the sets S_i^* and S_o^* are redefined as

$$S_i := \{z \in \mathbb{R}^2 : c_i - \delta \leq V(z) \leq c_i\} \quad (7.10)$$

and

$$S_o := \{z \in \mathbb{R}^2 : c_o \leq V(z) \leq c_o + \delta\}. \quad (7.11)$$

Note that when $\delta = 0$, S_i and S_o becomes the sets S_i^* and S_o^* , respectively.

7.2.1.1 Control Logics within the Invariant Band

In this section, we present the control logic induces forward invariance of the invariant band K_z for the closed-loop system by switching $q \in Q$ appropriately. More precisely, such control logics ensures every maximal solution initiated within K_z stays in it for all future time. To this end, given the design parameters c_o and a positive design parameter ε , we define special “buffering” regions on the outer boundary of K_z , which are given by

$$\begin{aligned} M_1 &= \{z \in S_o : 0 \leq i_L \leq k\varepsilon, v_C \geq 0\}; \\ M_2 &= \{z \in S_o : (k-1)\varepsilon \leq i_L \leq 0, v_C \geq 0\}; \\ M_3 &= \{z \in S_o : -k\varepsilon \leq i_L \leq 0, v_C \leq 0\}; \\ M_4 &= \{z \in S_o : 0 \leq i_L \leq (1-k)\varepsilon, v_C \leq 0\}, \end{aligned}$$

where, with $\alpha = LC\omega^2 - 1$,

$$k = \begin{cases} 1 & \text{if } \alpha < 0, \\ 0 & \text{otherwise} \end{cases} \quad (7.12)$$

Then, the proposed control algorithm for forward invariance switches q according to the following rules:

1. if $z \in S_i \cap (\mathbb{R}_{\geq 0} \times \mathbb{R})$ and $q \neq 1$, switch q to 1;
2. if $z \in S_i \cap (\mathbb{R}_{\leq 0} \times \mathbb{R})$ and $q \neq -1$, switch q to -1 ;
3. if $z \in (S_o \setminus (M_2 \cup M_3)) \cap (\mathbb{R}_{\leq 0} \times \mathbb{R})$ and $q \neq 1$, switch q to 1;
4. if $z \in (S_o \setminus (M_1 \cup M_4)) \cap (\mathbb{R}_{\geq 0} \times \mathbb{R})$ and $q \neq -1$, switch q to -1 ;
5. if $z \in M_1 \cup M_4$ and $q = 1$, switch q to 0;
6. if $z \in M_2 \cup M_3$ and $q = -1$, switch q to 0,

The values of k decides whether M_1, M_3 or M_2, M_4 are in active for switching rules iii) - vi). In particular, when system parameters are such that $\alpha < 0$, the sets M_1 and M_3 are used for “buffering”; while sets M_2 and M_4 are in use when $\alpha \geq 0$.

Such switching dynamics are designed to ensure solution trajectories always stay within the invariant band K_z given in (7.9) on the (i_L, v_C) plane.

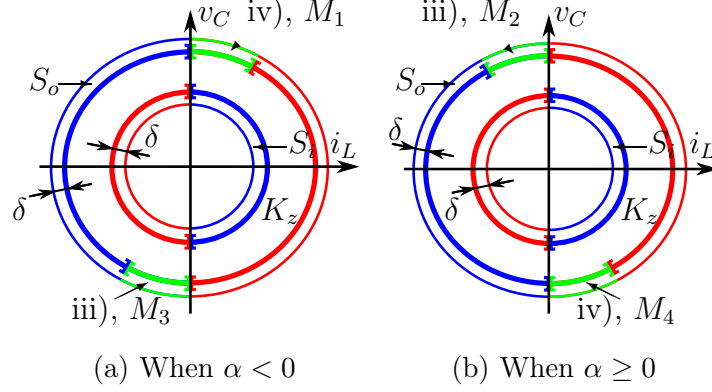


Figure 7.4: Switching regions of proposed control law.

Figure 7.4 illustrates the switching regions on the (i_L, v_C) plane following rules i)-vi). In particular, to ensure trajectories remain in K_z , the blue colored region calls for $q = 1$; while the red colored region calls for $q = -1$. Then, depending on the sign of α , different green colored regions are in active for rule iii) and iv), which call for $q = 0$. Let $V_{min} := \min_t \{V_{bus}\}$, we define the set

$$\Gamma = \{z \in \mathbb{R}^2 : -V_{min} \leq -Ri_L + \alpha v_C \leq V_{min}\}.$$

The forward invariance of K_z induced control logics i) - vi) builds on the following properties of the plant vector fields.

Lemma 7.2.2 (*inner product properties*) *Given positive system constants $R, L, C, \omega, V_{min}, b$, the followings hold:*

1. For every $(q, z) \in (\{1\} \times (\Gamma \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}))) \cup (\{-1\} \times (\Gamma \cap (\mathbb{R}_{\geq 0} \times \mathbb{R})))$, $\langle \nabla V(z), f_q(z) \rangle \leq 0$
2. For every $(q, z) \in (\{-1\} \times (\Gamma \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}))) \cup (\{1\} \times (\Gamma \cap (\mathbb{R}_{\geq 0} \times \mathbb{R})))$, $\langle \nabla V(z), f_q(z) \rangle \geq 0$;
3. When $\alpha \leq 0$, for every $z \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}) \cup (\mathbb{R}_{\leq 0} \times \mathbb{R}_{\leq 0})$, $\langle \nabla V(z), f_0(z) \rangle \leq 0$;

4. When $\alpha \geq 0$, for every $z \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\leq 0}) \cup (\mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0})$, $\langle \nabla V(z), f_0(z) \rangle \leq 0$;

Proof The inner product between the vector field f_q in (7.2) and the function V in (7.25) is given by

$$\langle \nabla V(z), f_q(z) \rangle = \frac{2i_L}{a^2} \left(\frac{-Ri_L - v_C + V_{min}q}{L} \right) + \frac{2v_C}{b^2} \left(\frac{i_L}{C} \right) = \sigma_q(z)i_L,$$

where, for every $(q, z) \in Q \times \mathbb{R}^2$,

$$\sigma_q(z) = \frac{2}{a^2 L} (-Ri_L + \alpha v_C + V_{min}q).$$

Since a and L are all positive constants, for every $z \in \Gamma$, we have

$$\sigma_1(z) \geq 0 \text{ and } \sigma_{-1}(z) \leq 0.$$

Hence,

$$1) \langle \nabla V(z), f_q(z) \rangle = \begin{cases} \sigma_1(z)i_L \leq 0 & \forall z \in \Gamma \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}), \text{ when } q = 1, \\ \sigma_{-1}(z)i_L \leq 0 & \forall z \in \Gamma \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}), \text{ when } q = -1; \end{cases}$$

$$2) \langle \nabla V(z), f_q(z) \rangle = \begin{cases} \sigma_1(z)i_L \geq 0 & \forall z \in \Gamma \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}), \text{ when } q = 1, \\ \sigma_{-1}(z)i_L \geq 0 & \forall z \in \Gamma \cap (\mathbb{R}_{\leq 0} \times \mathbb{R}), \text{ when } q = -1; \end{cases}$$

Item 1) leads to claim a), while item 2) leads to claim b).

Next, when $\alpha \leq 0$, we have the followings:

3) When $i_L \geq 0, v_C \geq 0, \sigma_0(z) \leq 0$, and $\langle \nabla V(z), f_0(z) \rangle = \sigma_0(z)i_L \leq 0$;

4) When $i_L \leq 0, v_C \leq 0, \sigma_0(z) \geq 0$, and $\langle \nabla V(z), f_0(z) \rangle = \sigma_0(z)i_L \leq 0$.

Therefore, item c) holds. Similarly, when $\alpha \geq 0$, item d) holds. \square

The sign properties listed in Lemma 7.2.2 indicate the proposed control law leads to forward invariance of the set K_z for the closed-loop system, which we will formally characterize in Section 7.2.2. The design parameters c_i and c_o allow tunable ‘‘tracking’’ precisions of the resulting signal compared to the references (7.4) and (7.22). Note that, according to Lemma 7.2.2, the design of c_o and δ need to be such that $S_o \subset \Gamma$. It is obvious when $c_o - c_i \rightarrow 0$, solution trajectories

are “periodic-like” and “imitate” S_r in (7.24) on (i_L, v_C) plane, while the H-bridge switches arbitrarily fast.

The regions $M_1 - M_4$ are designed to avoid the fast switchings at the intersection points of S_o and i_L -axis, from where, when $q \in \{-1, 1\}$, \dot{z} points “horizontally” (to the left or to the right) on the (i_L, v_C) plane. Moreover, with careful designs, appropriate value of ε avoids solution trajectory stuck near these intersection points; see details at Section 7.2.4.1. For every explicitly given set of system parameters, i.e., $b, c_i, c_o, V_{min}, \omega, R, L$, and C , the exact lower bound for ε can be determined by the critical point from intersecting the boundary S_o^* and a critical solution of the system controlled by rules i)-iv).² The explicit expression of such a bound is complicated, though, one can use the conservative choice of $\varepsilon = a\sqrt{c_o - c_i}$. In addition, irregular designs of S_i^* and S_o^* , i.e., having varying c_i and c_o rather than keeping them constants, may reduce the possibility of solution trajectories stuck near $(0, \pm b)$.

The presented control law is formulated in the hybrid inclusions framework introduced in [61]. With state $q \in Q$ and input $z \in \mathbb{R}^2$, the forward invariance controller, denoted by \mathcal{H}_{fw} , has data $(\mathcal{C}_{fw}, f_{fw}, \mathcal{D}_{fw}, G_{fw})$, which is given by

$$\mathcal{H}_{fw} \begin{cases} \dot{q} = f_{fw}(q, z) := 0 & (q, z) \in \mathcal{C}_{fw} \\ q^+ \in G_{fw}(q, z) & (q, z) \in \mathcal{D}_{fw}, \end{cases}$$

where the flow set \mathcal{C}_{fw} is defined as³

$$\begin{aligned} \mathcal{C}_{fw} := & (Q \times K_z) \cup (\{0\} \times (M_1 \cup M_2 \cup M_3 \cup M_4)) \cup \\ & \{(q, z) \in Q \times S_o : i_L q \leq 0, q \neq 0\} \cup \{(q, z) \in Q \times S_i : i_L q \geq 0, q \neq 0\}, \end{aligned}$$

²In particular, if $\alpha < 0$, the critical points near $(0, b)$ or $(0, -b)$ present on S_o^* and are in the first or the third, respectively, quadrant of the (i_L, v_C) plane. When z is initialized at such points, the resulting solutions pass through $(0, b\sqrt{c_i})$ or $(0, -b\sqrt{c_i})$, respectively, after exactly one jump. Similarly, if $\alpha \geq 0$, the critical points near $(0, b)$ or $(0, -b)$ present on S_o^* and are in the second or the fourth, respectively, quadrant of the (i_L, v_C) plane. Solutions pass through these points when z is initialized at $(0, b\sqrt{c_i})$ or $(0, -b\sqrt{c_i})$, respectively.

³When $\delta = 0$, we have $\mathcal{C}_{fw} = Q \times K_z$.

the jump map G_{fw} is defined as

$$G_{fw}(q, z) := \begin{cases} -1 & \text{if } q \neq -1, ((z \in S_o \setminus (M_1 \cup M_4), i_L \geq 0) \\ & \text{or } (z \in S_i, i_L < 0)); \\ 0 & \text{if } |i_L| \neq \varepsilon, ((q = 1, z \in (M_1 \cup M_4)) \\ & \text{or } (q = -1, z \in (M_2 \cup M_3))); \\ 1 & \text{if } q \neq 1, ((z \in S_o \setminus (M_2 \cup M_3), i_L \leq 0) \\ & \text{or } (z \in S_i, i_L > 0)); \\ \{0, 1\} & \text{if } q = -1, z \in S_o, i_L = -\varepsilon, v_C \geq 0; \\ \{-1, 0\} & \text{if } q = 1, z \in S_o, i_L = \varepsilon, v_C \leq 0; \\ \{-1, 1\} & \text{if } q = 0, z \in S_i, i_L = 0; \end{cases}$$

where k is given as in (7.12), and the jump set \mathcal{D}_{fw} is defined as

$$\mathcal{D}_{fw} := \{(q, z) \in Q \times S_o : i_L q \geq 0, q \neq 0\} \cup \{(q, z) \in Q \times S_i : i_L q \leq 0, q \neq 0\} \\ \cup (\{0\} \times S_i).$$

For the ease of forthcoming analysis in Section 7.2.2, we introduce the closed-loop system \mathcal{H}_{fw}^{cl} with controller \mathcal{H}_{fw} in the loop. The output of \mathcal{H}_{fw} , i.e., q , is the input to the plant given in (7.3), while the output z from the plant becomes the input to \mathcal{H}_{fw} . Hence, \mathcal{H}_{fw}^{cl} is autonomous with state variable $\xi := (q, z) \in Q \times (S_i \cup K_z \cup S_o)$, and it is given by

$$\mathcal{H}_{fw}^{cl} \begin{cases} \dot{\xi} = f_{fw}^{cl}(\xi) := (0, f_q(z)) & \xi \in \mathcal{C}_{fw} \\ \xi^+ \in G_{fw}^{cl}(\xi) := (G_{fw}(\xi), z) & \xi \in \mathcal{D}_{fw}. \end{cases} \quad (7.13)$$

7.2.1.2 Control Logics for Global Convergence

When solutions are initialized outside of the invariant band K_z , the controller introduced in this section guarantees global convergence to K_z in finite time. The following control logics are considered:

- vii) if $z \in \{z \in \mathbb{R}^2 : V(z) \geq c_o\}$ and $q \neq 0$, switch to $q = 0$;

viii) if $z \in \{z \in \mathbb{R}^2 : V(z) \leq c_i\}$ and $q = 0$, switch to either $q = 1$ or $q = -1$.

More precisely, the vector field $f_0(z)$ of (7.3) steers solutions to S_o from set $\{z \in \mathbb{R}^2 : V(z) \geq c_o\}$; while $f_{-1}(z)$ and $f_1(z)$ steer solution to S_i from $\{z \in \mathbb{R}^2 : V(z) \leq c_i\}$. We dedicate the following result to capture such properties.

Lemma 7.2.3 (*convergence properties*) *Given positive system constants $R, L, C, \omega, V_{min}, b$ and $c_i < c_o$ such that $V_{min} > b\sqrt{c_o}$, the followings hold:*

- e) *Every solution to $\dot{z} = f_0(z)$ initialized in $\{z \in \mathbb{R}^2 : V(z) \geq c_o\}$ converges to S_o^* in finite time;*
- f) *Every solution to $\dot{z} = f_1(z)$ and every solution to $\dot{z} = f_{-1}(z)$ initialized in $\{z \in \mathbb{R}^2 : V(z) \leq c_i\}$ converge to S_i^* in finite time.*

Proof We rewrite (7.3) in a compact form as

$$\dot{z} = f_q(z) = Az + Bq,$$

where $A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}$, and $B = (\frac{V_{bus}}{L}, 0)$. Then, solutions to $\dot{z} = f_q(z)$ is given by

$$\phi(t) = \exp(At)\phi(0) + \int_0^t \exp(A(t-\tau))Bd\tau. \quad (7.14)$$

The matrix A has eigenvalues given by

$$\lambda_{1,2} = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{CL}}}{2}.$$

Since R, L, C are all positive constants, $\lambda_{1,2}$ have negative real part.

When $q = 0$, the origin is asymptotic stable for $\dot{z} = f_0(z)$, the solution has explicit expression $\phi(t) = \exp(At)\phi(0)$ from (7.14). Let $\phi \in \mathcal{S}_{\dot{z}=f_0(z)}(\{z \in \mathbb{R}^2 : V(z) \geq c_o\})$ and $T = \sup \text{dom } \phi$. We show $T < \infty$ for item e). By definition of S_o , for every $t \in [0, T)$,

$$|\phi(t)| \geq |\phi(t)|_{S_o} + \min\{a, b\}\sqrt{c_o} \geq \min\{a, b\}\sqrt{c_o}.$$

Then, pick $\max_{i \in \{1,2\}} \operatorname{Re}(\lambda_i) < \beta < 0$, there exists $\gamma > 0$ such that, for every $t \in [0, T)$,

$$|\phi(t)| = |\exp(At)\phi(0)| \leq |\exp(At)| |\phi(0)| \leq \gamma \exp(\beta t) |\phi(0)|.$$

Hence, for every $t \in [0, T)$,

$$\min\{a, b\} \sqrt{c_o} \leq |\phi(t)| \leq \gamma \exp(\beta t) |\phi(0)|.$$

Since $\beta < 0, \gamma > 0$ and $|\phi(0)| > 0$, we have, for every $t \in [0, T)$,

$$0 < \frac{\min\{a, b\} \sqrt{c_o}}{\gamma |\phi(0)|} \leq \exp(\beta t) \leq 1,$$

thus,

$$0 \leq T \leq \frac{1}{\beta} \ln \left(\frac{\min\{a, b\} \sqrt{c_o}}{\gamma |\phi(0)|} \right) < \infty.$$

When $q = 1$ (or $q = -1$), the varying equilibrium $(0, V_{bus})$ (or $(0, -V_{bus})$) is asymptotic stable for $\dot{z} = f_1(z)$ (or $\dot{z} = f_{-1}(z)$, respectively). Since $V_{min} > b\sqrt{c_o}$, $(0, V_{bus}), (0, -V_{bus}) \notin \{z \in \mathbb{R}^2 : V(z) \leq c_i\}$. Hence, following similar arguments above, every $\phi \in \mathcal{S}_{\dot{z}=f_1(z)}(\{z \in \mathbb{R}^2 : V(z) \leq c_i\})$ (or $\phi \in \mathcal{S}_{\dot{z}=f_{-1}(z)}(\{z \in \mathbb{R}^2 : V(z) \leq c_i\})$) with $T = \sup \operatorname{dom} \phi$ has $T < \infty$. Therefore, item f) holds. \square

We model the controller as a hybrid system with state $q \in Q$ and input $z \in \mathbb{R}^2$, denoted by \mathcal{H}_g , which has flow map $f_g(q, z) := 0$ defined for every point on the flow set

$$\mathcal{C}_g := (\{0\} \times \{z \in \mathbb{R}^2 : V(z) \geq c_o\}) \cup (\{-1, 1\} \times \{z \in \mathbb{R}^2 : V(z) \leq c_i\}),$$

and jump map

$$G_g(q, z) = \begin{cases} 0 & \text{if } V(z) \geq c_o, \\ \{-1, 1\} & \text{if } V(z) \leq c_i, \end{cases}$$

defined on the jump set

$$\mathcal{D}_g := (\{-1, 1\} \times \{z \in \mathbb{R}^2 : V(z) \geq c_o\}) \cup (\{0\} \times \{z \in \mathbb{R}^2 : V(z) \leq c_i\}).$$

Remark 7.2.4 *In fact, Lemma 7.2.3 implies the set K_z given in (7.9) is uniformly globally pre-attractive when controller \mathcal{H}_g is active according to [61, Definition 3.6]. Such a property is important to establish the global asymptotic stability*

in Section 7.2.2.

7.2.1.3 Supervisor Controller

With appropriately chosen parameters for controllers \mathcal{H}_{fw} and \mathcal{H}_g , we can globally “track” any reference trajectory $(I_r(t), V_r(t))$ described by (7.24). For this purpose, we introduce a hybrid supervisor controller denoted \mathcal{H}_s that uses information of the location of z and switches between controller \mathcal{H}_{fw} and \mathcal{H}_g to guarantee global convergence and forward invariance of the invariant band K_z . Figure 7.5 shows the feedback control architecture.

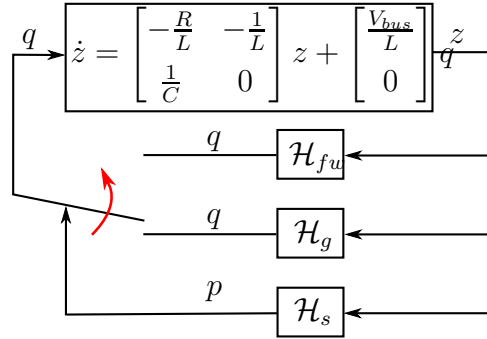


Figure 7.5: Full closed-loop system with \mathcal{H}_s , \mathcal{H}_g , and \mathcal{H}_{fw} .

The supervisor $\mathcal{H}_s = (\mathcal{C}_s, f_s, \mathcal{D}_s, g_s)$ has state p and input z . The state variable p takes values from $P := \{1, 2\}$, which denotes the following:

$$p = \begin{cases} 1 & \text{indicates } \mathcal{H}_{fw} \text{ is in the loop,} \\ 2 & \text{indicates } \mathcal{H}_g \text{ is in the loop.} \end{cases}$$

The dynamics of controller \mathcal{H}_s is given by

$$\mathcal{H}_s \begin{cases} \dot{p} = f_s(p, z) := 0 & (p, z) \in \mathcal{C}_s \\ p^+ = g_s(p, z) := 1 & (p, z) \in \mathcal{D}_s := \{2\} \times K_z \end{cases}$$

with flow set defined as

$$\mathcal{C}_s := (\{1\} \times (S_i \cup K_z \cup S_o)) \cup (\{2\} \times (\{z \in \mathbb{R}^2 : V(z) \geq c_o\} \cup \{z \in \mathbb{R}^2 : V(z) \leq c_i\})).$$

Remark 7.2.5 *Note that we constraint the definitions of \mathcal{C}_s and \mathcal{D}_s such that jumps from $p = 1$ to $p = 2$ are not allowed. This is because once \mathcal{H}_{fw} is active, the preferable action is to keep \mathcal{H}_{fw} in the loop for all future time. We formally characterize these properties in Section 7.2.2. Such a design also avoids Zeno behavior on the boundaries of the invariant band K_z . However, under the presence of disturbances on z , nontrivial solution may fail to exist due to the fact that p cannot be switched from 1 to 2. The controller design with $\delta > 0$, which induces some robustness properties, ensures existence and completeness of solutions from the set $(Q \times K_z) + \delta\mathbb{B}$. We formally characterize the properties in Section 7.2.3.1. When $\delta = 0$, a temporal regularization of the controller using appropriately designed timers can be implemented to ensure the jumps do not happen consecutively, for which we can allow p jumps from 1 to 2.*

7.2.2 Properties of the Full Closed-Loop System

In this section, we analyze the properties of the full closed-loop system \mathcal{H} that combines the dynamics of the plant in (7.3) controlled by the proposed controllers \mathcal{H}_{fw} , \mathcal{H}_g and \mathcal{H}_s . The closed-loop system is autonomous and has state variable $x := (p, q, z) \in P \times Q \times \mathbb{R}^2$. Its hybrid model is given by

$$\mathcal{H} \begin{cases} \dot{x} = f(x) & x \in \mathcal{C} \\ x^+ \in G(x) & x \in \mathcal{D} \end{cases} \quad (7.15)$$

where the flow map is given as

$$f(x) = (0, 0, f_q(z)),$$

the flow set \mathcal{C} is given as

$$\mathcal{C} = \{1\} \times \mathcal{C}_{fw} \cup \{2\} \times \mathcal{C}_g,$$

the jump map is given as

$$G(x) = \begin{cases} (1, G_{fw}(q, z), z) & \text{if } (q, z) \in \mathcal{D}_{fw} \\ (2, G_g(q, z), z) & \text{if } (q, z) \in \mathcal{D}_g, p = 2, \end{cases}$$

and the jump set is given as

$$\mathcal{D} = \{x \in P \times Q \times \mathbb{R}^2 : (p, z) \in \mathcal{D}_s\} \cup \{1\} \times \mathcal{D}_{fw} \cup \{2\} \times \mathcal{D}_g.$$

In fact, the closed-loop system \mathcal{H} satisfies the hybrid basic conditions introduced in [61, Assumption 6.5].

Lemma 7.2.6 (*hybrid basic conditions*) *The hybrid system \mathcal{H} satisfies the basic hybrid conditions, i.e., its data $(\mathcal{C}, f, \mathcal{D}, G)$ is such that*

(A1) \mathcal{C} and \mathcal{D} are closed subsets of $P \times Q \times \mathbb{R}^2$;

(A2) $f : P \times Q \times \mathbb{R}^2 \rightarrow P \times Q \times \mathbb{R}^2$ is continuous;

(A3) $G : P \times Q \times \mathbb{R}^2 \rightrightarrows P \times Q \times \mathbb{R}^2$ is outer semicontinuous, nonempty-valued and locally bounded relative to \mathcal{D} .

Proof The item (A1) and (A2) are obvious by design. The map G_{fw} is outer semicontinuous since $\mathcal{D}_{fw} = \text{dom } G_{fw}$ is closed and $Q \times K_z$ is closed. Hence, item (A3) holds. \square

Then, according to [61, Section 6.1], the full closed-loop system $\mathcal{H} = (\mathcal{C}, f, \mathcal{D}, G)$ is a well-posed system, which implies the forthcoming stability properties and their robustness with respect to small perturbations.

Next, we present the main properties of \mathcal{H} that validate the proposed controller. To this end, we define the set⁴

$$K := P \times Q \times K_z. \tag{7.16}$$

⁴The projection of K on the (i_L, v_C) plane is the invariant band K_z .

In particular, we establish the forward invariance and asymptotic stability of K for \mathcal{H} .

Proposition 7.2.7 (*forward invariance of K for \mathcal{H}*) *Given positive system constants $R, L, C, \omega, V_{min}, \varepsilon, b$ and $c_i < c_o$ such that $K_z \subset \Gamma$, K is forward invariant for \mathcal{H} given in (7.15).*

Proof The set $K \subset \mathcal{C} \cup \mathcal{D}$, $K \cap \mathcal{C} = K$ is compact and the flow map f is locally Lipschitz continuous by construction. According to Lemma 7.2.6, \mathcal{H} satisfies the hybrid basic conditions. Hence, we make the proof applying Theorem 4.1.4.

The jump map G maps p to 1 and its q component to set Q by design, while the i_L and v_C components remain the same before and after jumps. Thus, condition 4.1.4.1) in Theorem 4.1.4 holds. According to Lemma 7.2.2, by definition of tangent cones the set $K \cap \mathcal{M} = P \times (\mathcal{C}_{fw} \cap \mathcal{D}_{fw})$. Hence, condition 2.3) in Theorem 4.1.4 holds. Then, we show that for every $x \in K \setminus \mathcal{M}$, i.e., for every $x \in P \times (\mathcal{C}_{fw} \setminus \mathcal{D}_{fw})$, $f(x) \in T_{K \cap \mathcal{C}}(x)$, which is equivalent to $f(x) \in T_K(x)$ since $K \cap \mathcal{C} = K$. In particular, we have the following properties

1. For every $x \in K$ such that $z \in \text{int}K_z$, $T_K(x) = 0 \times 0 \times \mathbb{R}^2$, hence, $f(x) \in T_K(x)$.
2. By definition of tangent cone, item a), c) and d) in Lemma 7.2.2 imply that for every $x \in P \times ((Q \times S_o^*) \setminus \mathcal{D}_{fw})$, $f(x) \in T_K(x)$.
3. Similarly, for every $x \in P \times (Q \times S_i^*) \setminus \mathcal{D}_{fw}$, according to item b) of Lemma 7.2.2, $f(x) \in T_K(x)$.

□

Then, together with the properties e) and f) from Lemma 7.2.3, we present the global asymptotic stability of K for \mathcal{H} .

Theorem 7.2.8 (*global pre-asymptotic stability of K for \mathcal{H}*) *Given positive system constants $R, L, C, \omega, V_{min}, \varepsilon, b$ and $c_i < c_o$ such that $K_z \subset \Gamma$ and $V_{min} > b\sqrt{c_o}$, the (compact) set K is globally pre-asymptotically stable for \mathcal{H} given in (7.15).*

Proof Firstly, Proposition 7.2.7 established forward invariance of K . Then, because \mathcal{H}_g is designed such that e) and f) in Lemma 7.2.3 hold for every solution initiated outside of K , it is the case that K is globally uniformly pre-attractive for \mathcal{H} . By an application of [61, Proposition 7.5], since \mathcal{H} satisfies the hybrid basic conditions according to Lemma 7.2.6, the set K is globally pre-asymptotically stable for \mathcal{H} . \square

As discussed in Remark 7.2.5, due to the design of the supervisor controller, i.e., $p = 1$ jump to $p = 2$ is not allowed, some initial conditions to the full closed-loop systems \mathcal{H} only have trivial solution. However, every nontrivial maximal solution to \mathcal{H} is complete. In particular, complete solutions are guaranteed to exist for initial conditions within $\{2\} \times Q \times \mathbb{R}^2$ and $\{1\} \times Q \times (S_i \cup K_z \cup S_o)$, for which we introduce the next result.

Corollary 7.2.9 (*completeness of solutions from \mathcal{H} for certain initial conditions*)
Suppose positive system constants $R, L, C, \omega, b, V_{min}, \varepsilon$ and $c_i < c_o$ such that $K_z \subset \Gamma$ and $V_{min} > b\sqrt{c_o}$, every $\phi \in \mathcal{S}_{\mathcal{H}}$ satisfies exactly one of the following conditions:

- g) ϕ is trivial, when $\phi(0, 0) \in \mathcal{O} := \{1\} \times Q \times (\mathbb{R}^2 \setminus (S_i \cup K_z \cup S_o))$;*
- h) ϕ is complete, when $\phi(0, 0) \notin \mathcal{O}$.*

Proof Item g) directly follows from the dynamics of \mathcal{H}_s . According to Proposition 7.2.7, every $\phi \in \mathcal{S}_{\mathcal{H}}(K)$ is complete, then, we show h) holds true by showing every maximal solution is complete when $\phi(0, 0) \notin (\mathcal{O} \cup K)$. In fact, for every $\phi \in \mathcal{S}_{\mathcal{H}}((P \times Q \times \mathbb{R}^2) \setminus (\mathcal{O} \cup K))$, it is one of the following cases:

1. if $\phi(0, 0) \in \{2\} \times \mathcal{C}_g$, item e) and f) in Lemma 7.2.3 shows solutions converges to K by design of \mathcal{C}_g , hence, every maximal ϕ enters K and is complete;
2. if $\phi(0, 0) \in \{2\} \times \mathcal{D}_g$, map G_g maps the q component to Q , while p and z remain the same, hence, solutions can be extended further by flow and converge to K in finite time according to Lemma 7.2.3;

3. if $\phi(0, 0) \in P \times Q \times ((S_i \cup S_o) \setminus K_z)$, items a) - d) in Lemma 7.2.2 implies that solutions either jump in component q or flow towards set K , which also leads to complete solutions in K eventually.

□

In fact, properties g) and h) in Corollary 7.2.9 imply that hybrid system \mathcal{H} given in (7.15) is *pre-forward complete* by [61, Definition 6.12], i.e., every maximal solution to \mathcal{H} is either bounded or complete. Furthermore, together with Theorem 7.2.8, item h) in Corollary 7.2.9 indicate the set K is globally asymptotically stable for \mathcal{H} when solutions are initialized outside of \mathcal{O} because of the design of \mathcal{H}_s .

7.2.3 Analytic Validation of Proposed Algorithm

The supervisor controller \mathcal{H}_s is designed such that, \mathcal{H}_{fw} is in the loop eventually for every initial condition that leads to a complete solution. When \mathcal{H}_{fw} is in the loop, the closed-loop system \mathcal{H}_{fw}^{cl} given as (7.13) generates sinusoidal-like solutions. Hence, we validate the resulting solutions to the closed-loop system \mathcal{H} given in (7.15) always “approximate” the reference trajectory given in (7.24) by studying \mathcal{H}_{fw}^{cl} given in (7.13) with $\delta = 0$. As $(c_i, c_o) \rightarrow (1, 1)$, the outer and inner boundaries of K_z , i.e., S_o^* and S_i^* , get closer to each other, and become the set S_r when $(c_i, c_o) = (1, 1)$. Since S_o^* and S_i^* are the switching boundaries on the (i_L, v_C) plane, the amount of flow time in between jumps goes to zero as $(c_i, c_o) \rightarrow (1, 1)$. When $(c_i, c_o) = (1, 1)$ and $\delta = 0$, the sets \mathcal{C}_{fw} and \mathcal{D}_{fw} overlaps and become $Q \times S_r$, while the switching of q is arbitrarily fast and the solutions are pure discrete. In the case of limit $(c_i, c_o) \rightarrow (1, 1)$, \mathcal{H}_{fw}^{cl} behaves as a switching system with sliding mode control, where, instead of three individual switched modes correspond to $q \in Q$, the flow map is generalized as the convex hull of all available vector directions at every $z \in S_r$. We denote such a system at limit of

$(c_i, c_o) \rightarrow (1, 1)$ as \mathcal{H}^* , which is given by

$$\mathcal{H}^* \begin{cases} \dot{\xi} \in F^*(\xi) & \xi \in \mathcal{C}^* := Q \times S_r \\ \xi^+ \in G_{fw}^{cl}(\xi) & \xi \in \mathcal{D}^* := Q \times S_r, \end{cases} \quad (7.17)$$

where the flow map is defined for every $\xi \in \mathcal{C}^*$, and is given by

$$F^*(\xi) := \begin{cases} (0, \overline{\text{co}}\{f_1(z), f_0(z)\}) & \text{if } z \in M_2 \cup M_3, \\ (0, \overline{\text{co}}\{f_0(z), f_{-1}(z)\}) & \text{if } z \in M_1 \cup M_4, \\ (0, \overline{\text{co}}\{f_1(z), f_{-1}(z)\}) & \text{otherwise.} \end{cases}$$

In the next result, we establish important solution properties of the limiting system \mathcal{H}^* .

Proposition 7.2.10 *The hybrid system \mathcal{H}^* given in (7.17) satisfies the hybrid basic conditions. Suppose \mathcal{H}_{fw}^{cl} is such that $S_o \subset \Gamma$ and $V_{min} > b\sqrt{c_o}$. For every $\xi \in Q \times S_r$, every $\phi \in \mathcal{S}_{\mathcal{H}^*}(\xi)$ is complete. Moreover, from every $\xi \in Q \times S_r$,*

- i) there exists a Zeno solution; and*
- j) there exists a pure continuous solution and is such that $\phi(t + k\Delta) = \phi(t)$ for every $t \in \text{dom } \phi$ and $k \in \mathbb{N}$, where $\Delta = \frac{2\pi}{\omega}$.*

Proof Following the same steps in Lemma 7.2.6, it is obvious that \mathcal{H}^* given in (7.17) satisfies the hybrid basic conditions. Then, we apply [61, Proposition 6.10] to show every solution to \mathcal{H}^* is complete. Since $\mathcal{C}^* = \mathcal{D}^*$ by construction, (VC) holds trivially for every $\xi \in \mathcal{C}^* \setminus \mathcal{D}^*$, and there exists a nontrivial solution to \mathcal{H}^* from every $\xi \in \mathcal{C}^* \cup \mathcal{D}^* = Q \times S_r$. Item (b) in [61, Proposition 6.10] does not hold since the set \mathcal{C}^* is compact; while item (c) does not hold by construction of jump map G_{fw}^{cl} and set \mathcal{D}^* . Hence, for every $\xi \in Q \times S_r$, every $\phi \in \mathcal{S}_{\mathcal{H}^*}(\xi)$ is complete.

Item i) directly follows from the fact that $G_{fw}^{cl}(\mathcal{D}^*) = \mathcal{D}^*$. To establish item j), we show that when selecting the vector field $f_r(\xi) \in F^*(\xi)$ for every $\xi \in Q \times S_r$ following appropriate rules, there exists a complete pure continuous solution to

$\dot{\xi} = f_r(\xi)$ with period $\Delta = \frac{2\pi}{\omega}$. More precisely, for every $\xi \in Q \times S_r$, we show that

$$f_r(\xi) := (0, -C\omega^2 v_C, \frac{1}{C}i_L) \quad (7.18)$$

is such that $f_r(\xi) \in F^*(\xi)$. The first and third component of f_r are 0 and $\frac{1}{C}i_L$, respectively, which are the same as F^* . Hence, by definition of convex hull of two vectors, we show that for every $z = (i_L, v_C) \in S_r$, the term $-C\omega^2 v_C$ can be expressed as a convex combination of the first component of $f_1(z)$ and $f_0(z)$, or of $f_0(z)$ and $f_{-1}(z)$, or of $f_1(z)$ and $f_{-1}(z)$. Such a claim is equivalent to finding a constant $\lambda \in [0, 1]$ such that

$$-C\omega^2 v_C = \lambda f_q(z) + (1 - \lambda) f_{q'}(z), \quad (7.19)$$

with $q, q' \in Q$ and $q \neq q'$. Then, (7.19) leads to

$$-C\omega^2 v_C = \frac{-Ri_L - v_C}{L} + (1 - \lambda(q' - q)) \frac{V_{min}}{L},$$

i.e.,

$$-Ri_L + (LC\omega^2 - 1)v_C = (1 - \lambda(q' - q))V_{min},$$

Since $z \in S_r \subset \Gamma$, by definition of Γ , we have

$$-V_{min} \leq -Ri_L + (LC\omega^2 - 1)v_C \leq V_{min}.$$

With $V_{min} > 0$, we have $0 \leq \lambda(q' - q) \leq 2$. Hence, $\lambda \in [0, 1]$ for each possible combination of $q, q' \in Q$ and $q \neq q'$. We rewrite the map f_r given in (7.18) in compact form as

$$\dot{\xi} = f_r(\xi) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -C\omega^2 \\ 0 & \frac{1}{C} & 0 \end{bmatrix} \xi, \quad (7.20)$$

which has the z component matching the linear oscillator system in (7.6). Such a system has eigenvalues 0 and $\pm\omega i$. Hence, all solutions to (7.20) are continuous, complete and is periodic with $\Delta = \frac{2\pi}{\omega}$. \square

Without losing generality, we assume $c_i + c_o = 2$ and define $\delta^* := 1 - c_i$. Then, building on Proposition 7.2.10, the next result characterize the property of every

solution to \mathcal{H}_{fw}^{cl} “tracks” one solution to \mathcal{H}^* by showing the closedness of solutions to \mathcal{H}^* and solutions to its δ^* -perturbed system.

Proposition 7.2.11 *(consequence of well-posedness of \mathcal{H}^*) Given $\delta^* > 0$, hybrid system \mathcal{H}_{fw}^{cl} given in (7.13) and \mathcal{H}^* given in (7.17), there exist $\varepsilon^* > 0$ and $\tau^* \geq 0$ with the following property: for every $\psi \in \mathcal{S}_{\mathcal{H}_{fw}^{cl}}(Q \times K_z)$, there exists a $\phi \in \mathcal{S}_{\mathcal{H}^*}$ such that ψ and ϕ are (τ^*, ε^*) -close.⁵*

Proof Since $\delta^* = 1 - c_i$ and $c_i + c_o = 2$, we have that $K_z = S_r + \delta^*\mathbb{B}$. Then, to show the relationship between $\psi \in \mathcal{S}_{\mathcal{H}_{fw}^{cl}}(Q \times K_z)$ and a solution to \mathcal{H}_{fw}^{cl} , we construct the δ^* -perturbed system of \mathcal{H}^* as follows. With the same flow map F^* and jump map G_{fw}^{cl} , the hybrid system $\mathcal{H}_{\delta^*}^*$ has its flow set and jump set given by $\mathcal{C}_{\delta^*}^* := Q \times K_z$ and $\mathcal{D}_{\delta^*}^* := Q \times K_z$, respectively.

By Proposition 7.2.10, \mathcal{H}^* is nominally well-posed and pre-forward complete. Such properties are inherited by $\mathcal{H}_{\delta^*}^*$, i.e., $\mathcal{H}_{\delta^*}^*$ is nominally well-posed and pre-forward complete from the set $Q \times S_r$. Then, by an application of [61, Proposition 6.14], given $\delta^* > 0$, there exist a pair of $\varepsilon^* > 0$ and $\tau^* \geq 0$ such that for every $\phi_{\delta^*} \in \mathcal{S}_{\mathcal{H}_{\delta^*}^*}(Q \times K_z)$, there exists a solution $\phi \in \mathcal{S}_{\mathcal{H}_{\delta^*}^*}(Q \times S_r) \equiv \mathcal{S}_{\mathcal{H}^*}$ such that ϕ_{δ^*} and ϕ are (τ^*, ε^*) -close. This establish the result since, by definition of \mathcal{H}_{fw}^{cl} and $\mathcal{H}_{\delta^*}^*$, $\mathcal{S}_{\mathcal{H}_{fw}^{cl}}(Q \times K_z) \subset \mathcal{S}_{\mathcal{H}_{\delta^*}^*}(Q \times S_r)$. \square

The (τ^*, ε^*) -closeness property presented in Proposition 7.2.11 indicate that, given a $\delta^* > 0$, the every solution generated by \mathcal{H}_{fw}^{cl} stay close to one of the maximal solutions to the limiting system \mathcal{H}^* graphically. In particular, every $\psi \in \mathcal{S}_{\mathcal{H}_{fw}^{cl}}(Q \times K_z)$ tracks one $\phi \in \mathcal{S}_{\mathcal{H}^*}$ when projected on the (i_L, v_C) plane.

7.2.3.1 Robustness of \mathcal{H}

The proposed controllers guarantee robustness with respect to perturbations of the full closed-loop system. We formally characterize these properties in this

⁵See formal definition of (τ, ε) -closeness in [61, Definition 5.23].

section. In particular, we consider the following types of disturbances in \mathcal{H} given by (7.15):

1. A varying V_{bus} input signal given as in (7.1) rather than a constant DC voltage input;
2. Measurement noise of the output $z \in \mathbb{R}^2$;
3. Unmodeled dynamics of the closed-loop system introduced by transition modes during switches of q ;
4. State variable noise on z caused by non-ideal electronic components.

Firstly, Theorem 7.2.8 implies the globally pre-asymptotically stability of K is robust to variations in the input voltage V_{bus} described in item N1) as long as $V_{min} > b\sqrt{c_o}$. Then, according to Lemma 7.2.6, the closed-loop system \mathcal{H} satisfies the hybrid basic conditions, the global pre-asymptotic stability of K asserted by Theorem 7.2.8 is robust to small state perturbations. Such perturbations may include measurement noise $d_1 \in \mathbb{R}^2$ as described in item N2) and unmodeled dynamics $d_2 \in \mathbb{R}^2$ as described in item N3). Hence, we consider the plant in (7.3) with perturbations modeled as

$$\dot{z} = f_q(z + d_1) + d_2.$$

Let $\tilde{d}_i = (0, 0, d_i)$ for $i \in \{1, 2, 3\}$ with $d_3 \in \mathbb{R}^2$ as described in item N4). The perturbed system \mathcal{H} , denoted by $\tilde{\mathcal{H}}$, with state $x := (p, q, z)$, has dynamics

$$\begin{aligned} \dot{x} &= f(x + \tilde{d}_1) + \tilde{d}_2 & x \in \tilde{\mathcal{C}} \\ x^+ &\in G(x) & x \in \tilde{\mathcal{D}}, \end{aligned}$$

where $\tilde{\mathcal{C}} = \{x \in P \times Q \times \mathbb{R}^2 : x + \tilde{d}_3 \in \mathcal{C}\}$ and $\tilde{\mathcal{D}} = \{x \in P \times Q \times \mathbb{R}^2 : x + \tilde{d}_3 \in \mathcal{D}\}$.

The next result establishes the robustness property of stability of K for \mathcal{H} .

Theorem 7.2.12 *Suppose \mathcal{H} given as in (7.15) satisfies assumptions of Theorem 7.2.8 and the positive constant δ is such that $(S_i \cup K_z \cup S_o) \subset \Gamma$ and $V_{min} > b\sqrt{c_o} + \delta$. Then, the followings hold:*

k) there exists $\tilde{\beta} \in \mathcal{KL}$ such that, for each $\tilde{\varepsilon} \geq 0$, there exists $\tilde{\delta} > 0$ such that for any three measurable functions $\tilde{d}_1, \tilde{d}_2, \tilde{d}_3 : \mathbb{R}_{\geq 0} \rightarrow \tilde{\delta}\mathbb{B}$, every solution $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}$ is such that its $z := (i_L, v_C)$ component satisfies

$$|z(t, j)|_{K_z} \leq \tilde{\beta}(|z(0, 0)|_{K_z}, t + j) + \tilde{\varepsilon} \quad \forall (t, j) \in \text{dom } \phi,$$

where $\tilde{\varepsilon} = 0$ if $\phi(t, j) \in 1 \times Q \times K_z$;

l) given $\delta > 0$, and any three measurable functions $\tilde{d}_1, \tilde{d}_2, \tilde{d}_3 : \mathbb{R}_{\geq 0} \rightarrow \delta\mathbb{B}$, the set $P \times Q \times (S_i \cup K_z \cup S_o)$ is forward invariant for $\tilde{\mathcal{H}}$.

Proof Since for every $t \geq 0$, $\tilde{d}_1(t), \tilde{d}_2(t), \tilde{d}_3(t) \in \tilde{\delta}\mathbb{B}$, the perturbed hybrid system $\tilde{\mathcal{H}}$ can be rewritten as

$$\begin{cases} \dot{x} = f_{\tilde{\delta}}(x) & x \in \mathcal{C}_{\tilde{\delta}} \\ x^+ \in G_{\tilde{\delta}}(x) & x \in \mathcal{D}_{\tilde{\delta}} \end{cases}$$

where

$$\begin{aligned} f_{\tilde{\delta}}(x) &:= \overline{\text{co}}F((x + \tilde{\delta}\mathbb{B}) \cap \tilde{\mathcal{C}}) + \tilde{\delta}\mathbb{B} \\ G_{\tilde{\delta}}(x) &:= \left\{ \eta \in g + \tilde{\delta}\mathbb{B} : g \in G((z + \tilde{\delta}\mathbb{B}) \cap \tilde{\mathcal{D}}) \right\} \\ \mathcal{C}_{\tilde{\delta}} &:= \left\{ x : (x + \tilde{\delta}\mathbb{B}) \cap \tilde{\mathcal{C}} \neq \emptyset \right\} \\ \mathcal{D}_{\tilde{\delta}} &:= \left\{ x : (x + \tilde{\delta}\mathbb{B}) \cap \tilde{\mathcal{D}} \neq \emptyset \right\}. \end{aligned}$$

Then, by application of [61, Theorem 7.12], the globally pre-asymptotic stability and compactness of K imply the set \mathcal{B}_K^p is open and that K is \mathcal{KL} pre-asymptotically stable on \mathcal{B}_K^p . Then, applying [61, Lemma 7.20], K is semiglobally practically robustly \mathcal{KL} pre-asymptotically stable on \mathcal{B}_K^p . More precisely, according to [61, Definition 7.18], item (b), with $\omega(x) := |x|_K$ defined for every $x \in P \times Q \times \mathbb{R}^2$, for every compact subset of \mathcal{B}_K^p , there exists $\tilde{\beta} \in \mathcal{KL}$ such that, for each $\tilde{\varepsilon} > 0$, there exists $\tilde{\delta} > 0$ such that every $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}$ satisfies

$$|\phi(t, j)|_K \leq \tilde{\beta}(|\phi(0, 0)|_K, t + j) + \tilde{\varepsilon} \quad \forall (t, j) \in \text{dom } \phi.$$

Then, item k) holds because $K = P \times Q \times K_z$ and for every $(t, j) \in \text{dom } \phi$, $|\phi(t, j)|_K = |z(t, j)|_{K_z}$ and $|z(t, j)|_{K_z} = 0$ when $\phi(t, j) \in P \times Q \times K_z$.

Then item l) directly follows from the convergence properties presented in Lemma 7.2.3, the forward invariance proved in Proposition 7.2.7 and item h) in Corollary 7.2.9. \square

In addition to the nominal robustness induced by the well-posedness of \mathcal{H} presented in k) of Theorem 7.2.12, item l) asserts the robustness with respect to noise type N2), N3) and type N4) for forward invariance of K . More precisely, the design parameter δ introduced in (7.10) and (7.11) allows maximal solutions to the perturbed system $\tilde{\mathcal{H}}$ to be complete when $|d_i(t)| \leq \delta$ for all t and every $i \in \{1, 2, 3\}$. In fact, with $\delta = 0$, the completeness of solutions would fail for the perturbed system for the similar reasonings mentioned in Remark 7.2.5. The “ δ -inflation design” of S_i^* and S_o^* to S_i and S_o , respectively, preserve the forward invariance property of K for the perturbed system by guaranteeing existence of solutions from the set $P \times Q \times ((S_i \cup S_o) \setminus K_z)$.

7.2.4 Simulation Validations and Discussions

In this section, we present numerical simulations of the closed-loop system with proposed controllers, which is implemented via MATLAB Hybrid Equations Toolbox (HyEQ); see details at [90].⁶ Note that unless specified otherwise, all simulations in this section uses the following parameters: $R = 1\Omega$, $L = 0.1H$, $C = 66.6\mu F$, $V_{bus} \equiv 220V$, $b = 120V$, $\omega = 120\pi$, $c_i = 0.9$, $c_o = 1.1$ and $\delta = 0.05$. Simulations with above provided set of system parameters have $\alpha < 0$, for sample results with $\alpha \geq 0$, please see simulations in preliminary work [89,91].

7.2.4.1 Properties of \mathcal{M}_r for \mathcal{H}

The next few simulations of the closed-loop system \mathcal{H} validates the important properties of the set \mathcal{M}_r numerically. In particular, the first set of simulations have z initialized inside the interior of the invariant band K_z ; while the second

⁶Code available at <https://github.com/HybridSystemsLab/SinglePhaseInverter>.

set of simulations have $z_0 \notin K_z$. Both sets of results indicate that, when p is initialized appropriately, solutions to the closed-loop system are complete and eventually only evolve within K_z on the (i_L, v_C) plane. Then, a simulation to show the significance of design parameter ε is presented.

1. Figure 7.6 shows the solutions to the closed-loop system \mathcal{H} given in (7.15) with $z_0 = (bC\omega, 0) = (3.013, 0)$ and q_0 as either $-1, 0$ or 1 , where all three trajectories stay within the projection of \mathcal{M}_r onto the (i_L, v_C) plane, i.e., K_z .

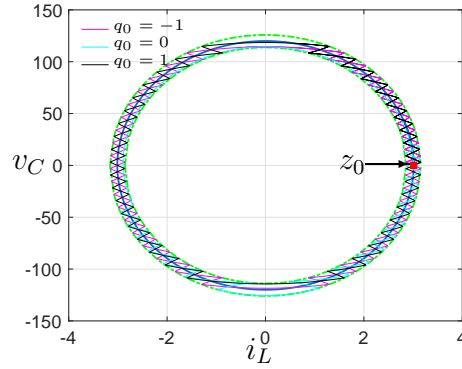


Figure 7.6: Simulations of \mathcal{H} with initial $z = (3.012, 0)$, and different initial values of q .

2. Figure 7.7 shows solutions to the closed-loop system \mathcal{H} given in (7.15) with $x_0 = (2, 1, 0, 0)$ and $x_0 = (2, 0, 0, 150)$, where both trajectories converge to and stay within K_z on the (i_L, v_C) plane.

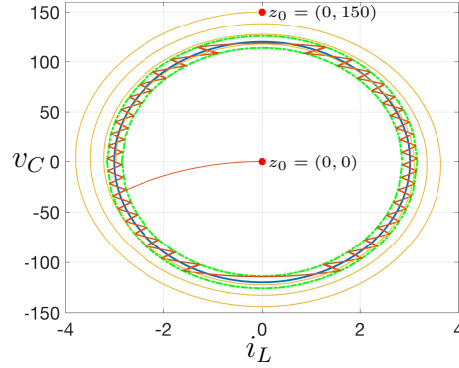


Figure 7.7: Simulations of \mathcal{H} with $z_0 = (0, 0)$ and $z_0 = (0, 150)$.

3. The next simulation uses $\varepsilon = 0.871$, which is not large enough to prevent solution trajectory to stuck near $(0, -b)$ as shown in Figure 7.8. In particular, starting around $t = 0.2s$, the trajectory evolves at the bottom of K_z (see the zoom in view at bottom right of Figure 7.8), while stay within K_z , until it reaches M_3 defined by a not large enough ε , where q is switched to $q = 0$.

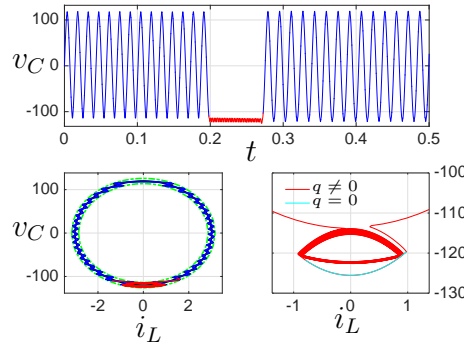


Figure 7.8: Simulations of \mathcal{H} with $\varepsilon = 0.871$.

7.2.4.2 \mathcal{H} with Perturbations

The next few simulations validate results from Section 7.2.3.1, where each of the four considered type of noises, i.e., N1)-N4), generated within range, is implemented for \mathcal{H} . For the input noise type N1), we consider the input voltage

V_{bus} given as in (7.1) has $V_{DC} = 220V$ and $d_v(t) = d_c(t) + d_s(t)$, where $t \mapsto d_c(t)$ is a sinusoidal signal given by $d_c(t) = 2 \sin(400\pi t)$, and $t \mapsto d_s(t)$ is a signal with multiple step changes that is given by

$$d_s(t) = \begin{cases} 0 & \text{if } t \in [0, 0.1) \\ -18 & \text{if } t \in [0.1, 0.2) \\ 30 & \text{if } t \in [0.2, 0.3) \\ 0 & \text{if } t \in [0.3, \infty) \end{cases}.$$

Note that this input signal has $V_{min} = 200V$, and with given system parameters, this simulation confirms that the proposed controller is robust to input noise type N1), which is a key robustness property of our controller.

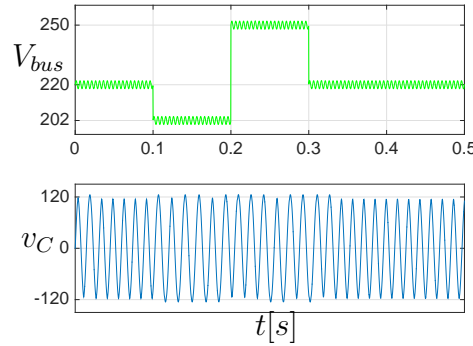


Figure 7.9: Simulations of \mathcal{H} with given V_{bus} .

As shown in Figure 7.9, the generated v_C signal is sinusoidal-like without significant transient behavior at 0.1, 0.2 and 0.3 seconds, where step changes are present in the input voltage V_{bus} . The amplitude of v_C remain within the expected 10% tolerance of b . The FFT analysis for this simulation suggest the fundamental frequency of the resulting output signals, i.e., i_L and v_C , are 60Hz with THD of % and %, respectively.

7.2.4.3 Benchmark tests

Above simulations show that the proposed controller is able to generate AC output with desired frequency and amplitude that is robust to the variations in DC input voltage. In addition, decreasing the width of the “tracking band” forces the output trajectory to be closer to the idea trajectory on the \mathbb{R}^2 plane, but results in higher number of switches within a unit of time. Thus, we propose a benchmark test for control algorithms of a single phase DC/AC inverter that focuses on the switching properties of the designed controllers. More precisely, we are interested in the average number of switches during one “period” of the output sinusoidal-like signal, v_C , from the closed-loop system. In this benchmark test, for different sets of c_i and c_o values, we record the average number of switches during a time period of $\frac{2\pi}{\omega}$ after the “transient” state⁷ of solutions from five different initial conditions. In addition, we also compute the average number of switches and its standard deviation (Std) for three different widths $c_o - c_i$ of the tracking band. The numbers of switches per period are rounded.

⁷By “transient” we mean the time after trajectories enter the set \mathcal{T} .

Table 7.1: Benchmark test for single phase DC/AC inverter

c_i & c_o	z_0	Average number of switches per period	Average & Std
$c_i = 0.9$ $c_o = 1.1$ $c_o - c_i = 0.2$	(0.1,0.009)	31	Average = 30 Std = 0.313
	(0.15,0)	30	
	(0.005,0.0119)	30	
	(0.08,0.01)	30	
	(0.14,0.004)	30	
$c_i = 0.95$ $c_o = 1.05$ $c_o - c_i = 0.1$	(0.1,0.009)	60	Average = 60 Std = 0.451
	(0.15,0)	60	
	(0.005,0.0119)	60	
	(0.08,0.01)	60	
	(0.14,0.004)	60	
$c_i = 0.99$ $c_o = 1.01$ $c_o - c_i = 0.02$	(0.1,0.009)	301	Average = 300 Std = 0.972
	(0.15,0)	301	
	(0.005,0.0119)	300	
	(0.08,0.01)	299	
	(0.14,0.004)	301	

Table 7.1 shows that with smaller width of the tracking band (namely, higher precision), the switching is more frequent, which is expected. Furthermore, the number of switches varies with different initial conditions, but the average and standard deviation results reported in Table 7.1 imply that by tuning the value of c_o and c_i , it is possible to control the number of switches per “period”. The resulting data also gives a general guideline for choosing appropriate c_o and c_i values for given system parameters.

7.2.5 Hardware Implementation

A design project to develop a hardware prototype of this hybrid control algorithm is currently undergoing. An undergraduate design team, with team member Ryan Rodriguez and Benjamin Chainey and financial support from CITRIS and Dr. Ricardo G. Sanfelice, is working towards the goal of implementing the

algorithm on a self-designed circuit board with the Texas Instruments C2000 microcontroller. The resulting product ought to convert the DC power output by two photovoltaic panels into the 120Vrms power using the proposed hybrid control algorithm. A block diagram showing the complete system configuration is shown in Figure 7.10.

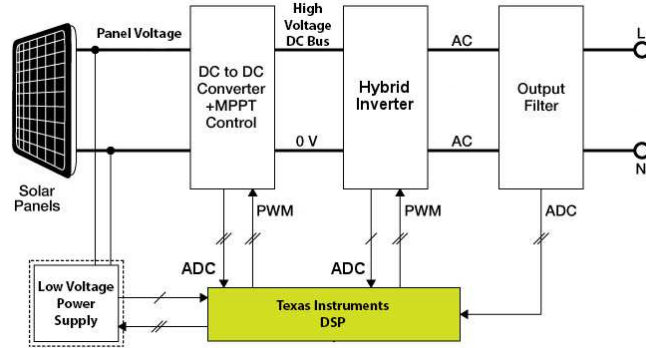


Figure 7.10: Design Block Diagram

The DC power source, two Sharp solar panels, has varying voltage and current output capabilities depending on lighting conditions. According to team research data, the maximum output power for the 170W panels occurs at $V_{panel} = 34.8V$ and $I_{panel} = 4.9A$, which has a highly nonlinear relationship between the current and voltage. Therefore, based on the design parameters and assumptions in Theorem 7.3.6, an upper limit buffer set of 200 W power from the power source is set. In addition, a DC/DC boost converter in Figure 7.11 is designed for maintain the lower limit of voltage input to the inverter circuits.

Currently, The team has accomplished the design and building of the hardware configuration shown in Figure 7.10. In addition, the PWM controller that will be used in comparison with the hybrid algorithm is implemented successfully on the Texas Instrument microcontroller. The current project progress is at the stage of debugging and program the hybrid controller into the hardware prototype.

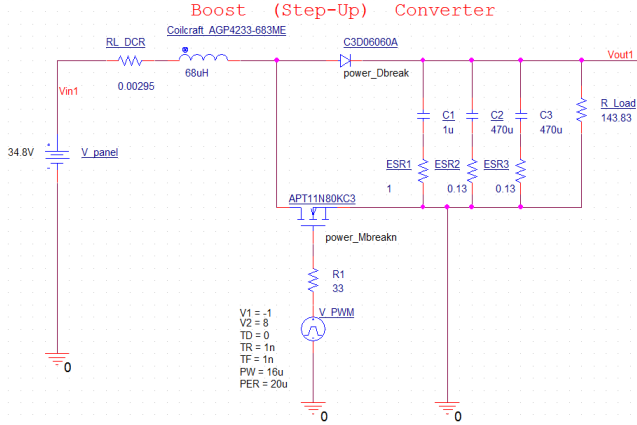


Figure 7.11: DC/DC Boost Converter Circuit Design for Hybrid Control of a Inverter

7.2.6 Control of DC/AC Inverter with Resistive Load

Unlike the control design proposed in previous sections for the inverter without load, the inverter system illustrated in Figure 7.1a has dynamics given by

$$\dot{z} = f_q(z) := \begin{bmatrix} -\left(\frac{R}{L} + \frac{1}{CR_\ell}\right) i_C - \left(\frac{1}{L} + \frac{R}{LR_\ell}\right) v_C + \frac{1}{L} V_{bus} q \\ \frac{1}{C} i_C \end{bmatrix}, \quad (7.21)$$

which has a resistive load $R_\ell < \infty$. Note in this case, we study a system with state variable (i_C, v_C) instead of (i_L, v_C) . The state variable i_C is equal to i_L in the case studied in previous sections, where its dynamics are given by (7.21) with $\frac{1}{R_\ell}$ to zero, since it is equivalent to infinity resistive load R_ℓ . The proposed control law in previous sections cannot be directly applied to the inverter plant in Figure 7.1a due to the presence of the resistive load $R_\ell < \infty$. However, we can modify the control logic to achieve the control goal of appropriately toggling the switches in the full H-bridge such that the inverter with resistive load converts the input signal V_{bus} into an sinusoidal-like output, namely, v_C .

The reference voltage signal for v_C in this design problem is given by (7.4) with b as the targeted amplitude, $\omega > 0$ as the targeted angular frequency, and θ as the initial phase. Then, since the state variables v_C and i_C in the RLC filter

always satisfy

$$\dot{v}_C(t) = \frac{1}{C}i_C(t).$$

Then, when v_C approximates $V_r(t)$ as in (7.4), i_C approximates the signal

$$t \mapsto I_r(t) = C\dot{V}_r(t) = C\omega b \cos(\omega t + \theta). \quad (7.22)$$

In fact, with $t \mapsto I_r(t)$ given as in (7.22), $V_r(t)$ in (7.4) satisfies $\ddot{V}_r(t) = -\omega^2 V_r(t)$.

Hence, a reference trajectory $(I_r(t), V_r(t))$ in coordinates (i_C, v_C) is generated by an exosystem with dynamics

$$\begin{aligned} \dot{I}_r &= -C\omega^2 V_r, \\ \dot{V}_r &= \frac{1}{C}I_r. \end{aligned}$$

The exosystem is formally defined by by the state-space model with state $z_r := (I_r, V_r) \in \mathbb{R}^2$ and dynamics

$$\dot{z}_r = \begin{bmatrix} 0 & -C\omega^2 \\ \frac{1}{C} & 0 \end{bmatrix} z_r, \quad z_r(0) = \begin{bmatrix} C\omega b \cos(\theta) \\ b \sin(\theta) \end{bmatrix}. \quad (7.23)$$

This system is a harmonic oscillator that generates solutions $t \mapsto (I_r(t), V_r(t))$ given as in (7.22) and (7.4) from initial conditions $(I_r(0), V_r(0)) = (C\omega b \cos(\theta), b \sin(\theta))$.

Let $a = C\omega b$, on the (i_C, v_C) plane, a solution to (7.23) describes an ellipse with semi-major axis a , semi-minor axis b , and aspect ratio $\frac{a}{b} = C\omega$. Referred to as the *reference trajectory* on the (i_C, v_C) plane, such an ellipse is give by the set of points

$$S_r := \left\{ z_r \in \mathbb{R}^2 : \left(\frac{z_{r1}}{a}\right)^2 + \left(\frac{z_{r2}}{b}\right)^2 = 1 \right\}. \quad (7.24)$$

For every $z \in \mathbb{R}^2$, we define

$$V(z) := \left(\frac{i_C}{a}\right)^2 + \left(\frac{v_C}{b}\right)^2. \quad (7.25)$$

Then, taking advantage of the function V , we provide a novel control strategy with arbitrary precision compare to the reference (I_r, V_r) . Given design parameters $c_i \in (0, 1)$ and $c_o > 1$, the invariant band is given by

$$K_z := \{z \in \mathbb{R}^2 : c_i \leq V(z) \leq c_o\}.$$

Note that $S_r \subset K_z$ regardless of the choice of c_o and c_i . Moreover, for ease of representation, we denote the outer boundary of K_z as $S_o^* := \{z \in \mathbb{R}^2 : V(z) = c_o\}$ and the inner boundary as $S_i^* := \{z \in \mathbb{R}^2 : V(z) = c_i\}$. To ensure existence and completeness of solutions under the presence of small state noises, we “inflate” the boundaries S_i^* and S_o^* outside of the invariant band K_z with a small inflation factor $\delta \geq 0$. More precisely, we consider sets

$$S_i := \{z \in \mathbb{R}^2 : c_i - \delta \leq V(z) \leq c_i\}$$

and

$$S_o := \{z \in \mathbb{R}^2 : c_o \leq V(z) \leq c_o + \delta\}.$$

It is obvious to see when $\delta = 0$, sets S_i and S_o become the sets S_i^* and S_o^* , respectively.

With the proposed controller, solutions to the inverter with resistive load start within the invariant band and stay within it for all future time.

7.3 A DC/DC Boost Converter

The DC/DC Boost converter is shown in Fig. 7.1b. It consists of a DC voltage source V_{DC} , a capacitor C , an ideal diode d , an inductor L , a resistor R , and an ideal switch S . The voltage across the capacitor is denoted v_C , and the current through the inductor is denoted i_L . The presence of switching elements (d and S) causes the overall system to be of a switching/hybrid nature. The purpose of the circuit is to draw power from the DC voltage source, and supply power to the load at a higher DC voltage value. This task is accomplished by first closing the switch to store energy in the inductor, and then opening the switch to transfer that energy to the capacitor, where it is available to the load. Depending on the (discrete) state of the diode and of the switch, one can distinguish four modes of

operation, see details in [76]:

$$\text{mode 1: } (S = 0, d = 1) \quad \text{mode 2: } (S = 1, d = 0)$$

$$\text{mode 3: } (S = 0, d = 0) \quad \text{mode 4: } (S = 1, d = 1)$$

Using the ideal diode model, we have

$$\text{conducting } (d = 1) : \quad i_d \geq 0, v_d = 0$$

$$\text{blocking } (d = 0) : \quad i_d = 0, v_d \leq 0$$

while using the ideal switch model, we have

$$\text{conducting } (S = 1) : \quad v_S = 0$$

$$\text{blocking } (S = 0) : \quad i_S = 0.$$

With these conditions, we derive the differential equations for each mode using state variable $x := (v_C, i_L)$, along with the specific values of S and d . After further analysis, see [88] for details, the system will take the form of a switched differential inclusion with constraints, namely

$$\dot{x} \in F_S(x) \quad x \in \widetilde{M}_S \tag{7.26}$$

where $S \in \{0, 1\}$ is the position of the switch S , and for each $S \in \{0, 1\}$, $F_S(x)$ is the Krasovskii regularization of the vector fields and \widetilde{M}_S is the corresponding regularization of the sets capturing the regions of validity for each mode. Then, the four operation modes can be combined, and expressed by two differential inclusions, as follows:

- For each $x \in \widetilde{M}_0 := \overline{M_1 \cup M_3}$, we have

$$F_0(x) := \begin{cases} \begin{bmatrix} -\frac{1}{RC}v_C + \frac{1}{C}i_L \\ -\frac{1}{L}v_C + \frac{V_{DC}}{L} \end{bmatrix} & \text{if } x \in \overline{M_1} \setminus \overline{M_3} \\ \{-\frac{1}{RC}v_C\} \times \left[-\frac{1}{L}v_C + \frac{V_{DC}}{L}, 0\right] & \text{if } x \in \overline{M_3} \end{cases}$$

where

$$M_1 = \{x \in \mathbb{R}^2 : i_L > 0\} \cup \{x \in \mathbb{R}^2 : v_C \leq V_{DC}, i_L = 0\}$$

and

$$M_3 = \{x \in \mathbb{R}^2 : v_C > V_{DC}, i_L = 0\}.$$

- For each $x \in \widetilde{M}_1 = \{x \in \mathbb{R}^2 : v_C \geq 0\}$, we have

$$F_1(x) := \left[-\frac{1}{RC}v_C \quad \frac{V_{DC}}{L} \right]^\top$$

Notice that the switching variable S can be either 0 or 1, representing different vector fields, which, at every instant, depends on the choice made by the controller. This promotes the use of hybrid system analysis and controller design tools.

The goal of the controller is to approximate a DC output with given v_C^* and i_L^* , which represent the desired voltage across capacitor and current through inductor, respectively. The equivalent design goal is to design a controller that guarantees asymptotic stability of set $\mathcal{A}_x \times \{0, 1\}$ for the closed-loop system, where $x^* = (v_C^*, i_L^*)$ and

$$\mathcal{A}_x := \{x \in \mathbb{R}^2 : x = x^*\}.$$

In addition, we conclude that $\mathcal{A} = \mathcal{A}_x \times \{0, 1\}$ is forward invariant for the closed-loop system with designed controller.

7.3.1 State-dependent Control Law

In this design problem, we present results on a CLF (control Lyapunov function)-based hybrid controller to achieve the desired DC voltage output. To this end, consider the Lyapunov-like function

$$V(x) = (x - x^*)^\top P(x - x^*), \tag{7.27}$$

where $P = \begin{bmatrix} p_{11} & 0 \\ 0 & p_{22} \end{bmatrix}$. For each $S \in \{0, 1\}$, let

$$\gamma_S(x) := \max_{\xi \in F_S(x)} \langle \nabla V(x), \xi \rangle.$$

Then, the following lemma establishes an important property of the functions $\gamma_S(x)$ that leads to the key stability result in later section. The constraints \widetilde{M}_S in (7.26) on the switching are not taken into account at this point, but are incorporated again later (see Proposition 7.3.5 below).

Lemma 7.3.1 *Let $R, V_{DC}, p_{11}, p_{22} > 0$, $\frac{p_{11}}{C} = \frac{p_{22}}{L}$, $v_C^* > V_{DC}$, and $i_L^* = \frac{v_C^{*2}}{RV_{DC}}$. Then, for each $x \in \mathbb{R}^2 \setminus \mathcal{A}_x$, there exists $S \in \{0, 1\}$ such that*

$$\gamma_S(x) < 0$$

Moreover, $\{x \in \mathbb{R}^2 : \gamma_S(x) = 0, S \in \{0, 1\}\} = \mathcal{A}_x$.

Proof Consider the functions γ_S , $S \in \{0, 1\}$, using the relationship $\frac{p_{11}}{C} = \frac{p_{22}}{L}$, we can rewrite $\gamma_S(x) = 0$ as

$$\gamma_0(x) = 2(A_0v_C^2 + B_0v_C + C_0i_L + D_0) \quad (7.28)$$

$$\gamma_1(x) = 2(A_1v_C^2 + B_1v_C + C_1i_L + D_1) \quad (7.29)$$

where the coefficients, A_0 through D_0 and A_1 through D_1 are defined as

$$\begin{aligned} A_0 &= -\frac{p_{11}}{RC} & A_1 &= -\frac{p_{11}}{RC} \\ B_0 &= \frac{p_{11}v_C^*}{RC} + \frac{p_{22}i_L^*}{L} & B_1 &= \frac{p_{11}v_C^*}{RC} \\ C_0 &= -\frac{p_{11}v_C^*}{C} + \frac{p_{22}V_{DC}}{L} & C_1 &= \frac{p_{22}V_{DC}}{L} \\ D_0 &= -\frac{p_{22}i_L^*V_{DC}}{L} & D_1 &= -\frac{p_{22}i_L^*V_{DC}}{L} \end{aligned}$$

To guarantee that for every $(v_C, i_L) \in \mathbb{R}^2 \setminus \mathcal{A}_x$ there exists an $S \in \{0, 1\}$ such that $\gamma_S(x) < 0$ and that $\{x \in \mathbb{R}^2 : \gamma_S(x) = 0, S \in \{0, 1\}\} = \mathcal{A}_x$, we consider the sets $\Gamma_S := \{x \in \mathbb{R}^2 : \gamma_S(x) < 0\}$ for $S \in \{0, 1\}$. We will also use the boundaries of the sets Γ_S given by $\Omega_S := \{x \in \mathbb{R}^2 : \gamma_S(x) = 0\}$ for $S \in \{0, 1\}$, which are parabolas. We first derive explicit expressions for Γ_S , $S \in \{0, 1\}$, next.

1. For $x \in \Gamma_0$, we have $A_0v_C^2 + B_0v_C + C_0i_L + D_0 < 0$. Substituting the coefficients A_0 through D_0 , using $\frac{p_{11}}{C} = \frac{p_{22}}{L}$ and $v_C^* > V_{DC}$ gives

$$i_L > \frac{1}{V_{DC} - v_C^*} \left(\frac{1}{R}v_C^2 - \left(\frac{v_C^*}{R} + i_L^* \right) v_C + i_L^*V_{DC} \right) \quad (7.30)$$

2. For $x \in \Gamma_1$, we have $A_1 v_C^2 + B_1 v_C + C_1 i_L + D_1 < 0$. Substituting the coefficients A_1 through D_1 , using again $\frac{p_{11}}{C} = \frac{p_{22}}{L}$ gives

$$i_L < \frac{1}{RV_{DC}} v_C^2 - \frac{v_C^*}{RV_{DC}} v_C + i_L^* \quad (7.31)$$

This gives the expressions

$$\Gamma_0 = \left\{ (v_C, i_L) \in \mathbb{R}^2 : i_L > \frac{1}{V_{DC} - v_C^*} \left(\frac{1}{R} v_C^2 - \left(\frac{v_C^*}{R} + i_L^* \right) v_C + i_L^* V_{DC} \right) \right\}$$

$$\Gamma_1 = \left\{ (v_C, i_L) \in \mathbb{R}^2 : i_L < \frac{1}{RV_{DC}} v_C^2 - \frac{v_C^*}{RV_{DC}} v_C + i_L^* \right\}$$

and similar ones for Ω_S , $S \in \{0, 1\}$. Both parabolas Ω_S , $S \in \{0, 1\}$, have their axis of symmetry parallel to the i_L -axis. Hence, we have to show now that $\Gamma_1 \cup \Gamma_2 = \mathbb{R}^2 \setminus \mathcal{A}_x$ and $\Omega_0 \cap \Omega_1 = \mathcal{A}_x$.

To show this, note that $\frac{1}{(V_{DC} - v_C^*)} < 0$ indicating that Ω_0 is a “downward” parabola (it has a maximum in i_L -direction) and Γ_0 is the region above it. Similarly, since $\frac{1}{RV_{DC}} > 0$, Ω_1 is an “upward” parabola (it has a minimum in i_L -direction) and Γ_1 is the region below it. See Figure 7.12 for an illustration. If we now can show that $\Omega_0 \cap \Omega_1 = \mathcal{A}_x$, then it follows that $\Gamma_1 \cup \Gamma_2 = \mathbb{R}^2 \setminus \mathcal{A}_x$ as in Figure 7.12, and the proof of the lemma is complete.

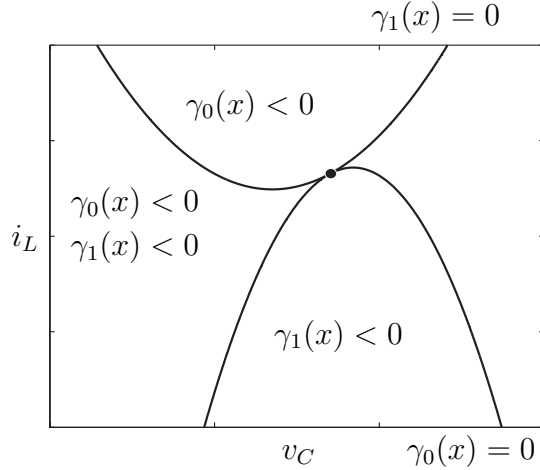


Figure 7.12: An example of a possible sign distribution for the two parabolas $\gamma_0(x) = 0$ and $\gamma_1(x) = 0$.

To show that $\Omega_0 \cap \Omega_1 = \mathcal{A}_x$, we observe that if $(v_C, i_L) \in \Omega_0 \cap \Omega_1$ we must have that the right-hand sides of (7.31) and (7.30) are equal, which leads to

$$\left(-\frac{1}{R}v_C^2 + \left(\frac{v_C^*}{R} + i_L^* \right) v_C - i_L^* V_{DC} \right) \frac{1}{V_{DC} - v_C^*} = -\frac{1}{RV_{DC}}v_C^2 + \frac{v_C^*}{RV_{DC}}v_C - i_L^*. \quad (7.32)$$

Since $i_L^* = \frac{v_C^{*2}}{RV_{DC}}$, we have

$$-\frac{v_C^*}{RV_{DC}}(v_C^2 - 2v_C^*v_C + (v_C^*)^2) = 0,$$

which has a unique solution $v_C = v_C^*$, and implies that $\Omega_0 \cap \Omega_1$ is indeed $\{(v_C^*, i_L^*)\}$.

This completes the proof. \square

The property in Lemma 7.3.1 shows that V is a CLF-like function in the sense that

$$\min_{S \in \{0,1\}} \max_{\xi \in F_S(x)} \langle \nabla V(x), \xi \rangle < 0 \quad \forall x \in \mathbb{R}^2 \setminus \mathcal{A}_x \quad (7.33)$$

This condition is used to derive the suitable stabilizing hybrid control law.

7.3.1.1 The hybrid controller

The condition obtained in (7.33) naturally leads to the following selection of the input S , which is a nonlinear system with discontinuous right-hand side (if we forget for a moment the constraints on the switching in (7.26)):

$$S = \arg \min_{S' \in \{0,1\}} \max_{\xi \in F_{S'}(x)} \langle \nabla V(x), \xi \rangle \quad (7.34)$$

However, the direct application of (7.34) as the switching law, leads to a discontinuous control law and results in chattering, which is undesirable in practice. Therefore, we will propose a modified logic-based control law (and a corresponding regularized closed-loop system), which is practically feasible. In fact, for the resulting (regularized) controller various robustness properties can be derived and proved mathematically based on the hybrid system setup particularly chosen for this purpose.

Let $q \in \{0, 1\}$ be a logic state indicating the value of the actual input S . The envisioned logic-based control law will select the input according to the current

active input q and the value of the state, namely, when certain well-designed functions $\tilde{\gamma}_q$ become zero. These functions $\tilde{\gamma}_q$ are control design parameters that are related to the functions γ_q in (7.28) and (7.29) and will be chosen as in the following lemma.

Remark 7.3.2 *The functions $\tilde{\gamma}_q$ are not chosen exactly equal to γ_q , because mode 1 would have an equilibrium $(v_C, i_L) = (V_{DC}, \frac{V_{DC}}{R})$ exactly at $\gamma_0(x) = 0$. This would prevent to achieve global asymptotic stability of the desired set point*

Lemma 7.3.3 *Let $R, V_{DC}, p_{11}, p_{22} > 0$, $\frac{p_{11}}{C} = \frac{p_{22}}{L}$, $v_C^* > V_{DC}$, and $i_L^* = \frac{v_C^{*2}}{RV_{DC}}$. For each $q \in \{0, 1\}$, let $\tilde{\gamma}_q$ be given for $x \in \mathbb{R}^2$ as*

$$\tilde{\gamma}_0(x) = \gamma_0(x) + K_0 (v_C - v_C^*)^2 \quad (7.35)$$

$$\tilde{\gamma}_1(x) = \gamma_1(x) + K_1 (v_C - v_C^*)^2 \quad (7.36)$$

and $K_0 \in (0, \frac{2p_{11}}{RC})$, $K_1 \in (0, \frac{2p_{11}}{RC})$. The following hold:

1. For $q \in \{0, 1\}$ and $x \notin \mathcal{A}_x$ we have that $\tilde{\gamma}_q(x) \geq 0$ implies $\tilde{\gamma}_{1-q}(x) < 0$;
2. For $q \in \{0, 1\}$ and $x \notin \mathcal{A}_x$ we have that $\tilde{\gamma}_q(x) \leq 0$ implies $\gamma_q(x) < 0$;
3. For $x \in \mathbb{R}^2$ it holds that

$$\begin{aligned} \frac{1}{C_0} \lim_{K_0 \rightarrow \frac{2p_{11}}{RC}} \tilde{\gamma}_0(x) &= \frac{1}{C_1} \lim_{K_1 \rightarrow \frac{2p_{11}}{RC}} \tilde{\gamma}_1(x) = -\frac{2i_L^*}{v_C^*} v_C + 2i_L, \quad \lim_{K_0 \rightarrow 0} \tilde{\gamma}_0(x) = \gamma_0(x), \\ \lim_{K_1 \rightarrow 0} \tilde{\gamma}_1(x) &= \gamma_1(x). \end{aligned}$$

Proof To show (b), note that we can rewrite (7.35) and (7.36) as

$$\gamma_0(x) = \tilde{\gamma}_0(x) - K_0 (v_C - v_C^*)^2$$

$$\gamma_1(x) = \tilde{\gamma}_1(x) - K_1 (v_C - v_C^*)^2$$

Because $K_0, K_1 > 0$, $\tilde{\gamma}_q(x) \leq 0$ implies $\gamma_q(x) < 0$ if $v_C \neq v_C^*$.

If $v_C = v_C^*$ and $i_L \neq i_L^*$ (as otherwise $x = x^*$), we have $\tilde{\gamma}_q(x) = \gamma_q(x) \leq 0$. However, we know that $\tilde{\gamma}_q(x) = \gamma_q(x) = 0$ cannot occur, as together with $v_C = v_C^*$

this would imply $i_L = i_L^*$, which would be a contradiction. Hence, also in this case $\tilde{\gamma}_q(x) = \gamma_q(x) < 0$, and the proof of property (b) is complete.

The proof of (a) follows analogously to the proof of lemma 7.3.1. First define

$$\tilde{\gamma}_0(x) = \gamma_0(x) + K_0 (v_C - v_C^*)^2 = \gamma_0(x) + K_0 \delta_0(v_C) \quad (7.37)$$

$$\tilde{\gamma}_1(x) = \gamma_1(x) + K_1 (v_C - v_C^*)^2 = \gamma_1(x) + K_1 \delta_1(v_C) \quad (7.38)$$

we consider the sets $\tilde{\Gamma}_q := \{x \in \mathbb{R}^2 : \tilde{\gamma}_q(x) < 0\}$ for $q \in \{0, 1\}$. We will also use the boundaries of the sets $\tilde{\Gamma}_q$ given by $\tilde{\Omega}_q := \{x \in \mathbb{R}^2 : \tilde{\gamma}_q(x) = 0\}$ for $q \in \{0, 1\}$, which are parabolas. Now define $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_1$ by deriving (7.37) and (7.38) in similar forms as before

$$\begin{aligned} \tilde{\Gamma}_0 = \\ \left\{ (v_C, i_L) \in \mathbb{R}^2 : i_L > \frac{1}{V_{DC} - v_C^*} \left(\frac{1}{R} v_C^2 - \left(\frac{v_C^*}{R} + i_L^* \right) v_C + i_L^* V_{DC} + K_0 \delta_0(v_C) \right) \right\} \end{aligned} \quad (7.39)$$

$$\tilde{\Gamma}_1 = \left\{ (v_C, i_L) \in \mathbb{R}^2 : i_L < \frac{1}{RV_{DC}} v_C^2 - \frac{v_C^*}{RV_{DC}} v_C + i_L^* + \frac{K_1 \delta_1(v_C)}{V_{DC}} \right\} \quad (7.40)$$

and similar ones for $\tilde{\Omega}_q$, $q \in \{0, 1\}$. Both parabolas $\tilde{\Omega}_q$, $q \in \{0, 1\}$, have their axis of symmetry parallel to the i_L -axis. Then, because $\frac{1}{(V_{DC} - v_C^*)} < 0$, $K_0 \delta_0(v_C) > 0$, and Ω_0 is a “downward” parabola, we know that $\tilde{\Omega}_0$ is also a “downward” parabola (it has a maximum in i_L -direction) and $\tilde{\Gamma}_0$ is the region above it. Similarly, since Ω_1 is an “upward” parabola and $\frac{K_1 \delta_1(v_C)}{V_{DC}} > 0$, we have $\tilde{\Omega}_1$ is also an “upward” parabola (it has a minimum in i_L -direction) and $\tilde{\Gamma}_1$ is the region below it. If we now can show that $\tilde{\Omega}_0 \cap \tilde{\Omega}_1 = \mathcal{A}_x$, then it follows that $\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2 = \mathbb{R}^2 \setminus \mathcal{A}_x$.

To show that $\tilde{\Omega}_0 \cap \tilde{\Omega}_1 = \mathcal{A}_x$, we find out the v_C, i_L value for the intersection of the two curves $\tilde{\Omega}_0$ and $\tilde{\Omega}_1$. When the right-hand-side of the inequalities in (7.39) and (7.40) equals to each other, we get a similar expression to (7.32):

$$\begin{aligned} & \frac{1}{V_{DC} - v_C^*} \left(\frac{1}{R} v_C^2 - \left(\frac{v_C^*}{R} + i_L^* \right) v_C + i_L^* V_{DC} + K_0 \delta_0(v_C) \right) \\ &= \frac{1}{RV_{DC}} v_C^2 - \frac{v_C^*}{RV_{DC}} v_C + i_L^* + \frac{K_1 \delta_1(v_C)}{V_{DC}}. \end{aligned} \quad (7.41)$$

Then, we can rewrite (7.41) in quadratic form, and find its simplified discriminant Δ to be $\left(\frac{v_C^*{}^2}{RV_{DC}}\right)^2 - i_L^*{}^2$, which equals to zero because $i_L^* = \frac{v_C^*{}^2}{RV_{DC}}$. Therefore (7.41) has a unique solution. We can find the unique solution by solving v_C from (7.41), and the result is

$$v_C = \frac{-\left(\left(\frac{1}{R} - \frac{V_{DC}-v_C^*}{RV_{DC}} + 2K_0 - \frac{2K_1(V_{DC}-v_C^*)}{V_{DC}}\right)v_C^* + i_L^*\right)}{2\left(\frac{V_{DC}-v_C^*}{RV_{DC}} - \frac{1}{R} - K_0 + \frac{K_1(V_{DC}-v_C^*)}{V_{DC}}\right)} = v_C^*$$

while $i_L = i_L^*$. This implies that $\tilde{\Omega}_0 \cap \tilde{\Omega}_1$ is the set-point $\{(v_C^*, i_L^*)\}$ and therefore completes the proof of property (a).

Property (c) can be shown by explicitly computing the limits. For finding the limit of the first two equations, we can rewrite the formulation of $\gamma_q(x)$ with $q \in \{0, 1\}$ as

$$\begin{aligned}\gamma_q(x) &= (1-q)\gamma_0(x) + q\gamma_1(x) \\ &= -\alpha(v_C - v_C^*)^2 + (\beta q + C_0) \left(-\frac{2i_L^*}{v_C^*}v_C + 2i_L\right)\end{aligned}$$

where $\alpha = \frac{2p_{11}}{RC}$, $\beta = \frac{p_{11}v_C^*}{C}$ and C_0 is given in the proof for Lemma 7.3.1. Then, we get an expression

$$\tilde{\gamma}_q(x) = (K_q - \alpha)(v_C - v_C^*)^2 + (\beta q + C_0) \left(-\frac{2i_L^*}{v_C^*}v_C + 2i_L\right) \quad (7.42)$$

We discuss the following cases when $K_q \rightarrow \alpha$,

- if $q = 0$, we have the limit of (7.42) expression as

$$\lim_{K_0 \rightarrow \alpha} \tilde{\gamma}_q(x) = C_0 \left(-\frac{2i_L^*}{v_C^*}v_C + 2i_L\right).$$

- if $q = 1$, we have the limit of (7.42) expression as

$$\begin{aligned}\lim_{K_1 \rightarrow \alpha} \tilde{\gamma}_q(x) &= (\beta + C_0) \left(-\frac{2i_L^*}{v_C^*}v_C + 2i_L\right) \\ &= \frac{p_{11}V_{DC}}{C} \left(-\frac{2i_L^*}{v_C^*}v_C + 2i_L\right) \\ &= C_1 \left(-\frac{2i_L^*}{v_C^*}v_C + 2i_L\right).\end{aligned}$$

Thus, we complete the proof for the first two limits, and the last two limits follow

naturally from the expression $\tilde{\gamma}_q(x) = \gamma_q(x) + K_q(v_C - v_C^*)^2$. \square

Based on the properties derived in the lemma above we can define an appropriate (robustly) stabilizing control law. In fact, the control law makes sure that for the current value of q and x it holds that $\tilde{\gamma}_q(x) \leq 0$, which implies by property (b) that as long as $x \notin \mathcal{A}_x$, we have that $\gamma_q(x) < 0$, which, in turn, implies that the CLF V in (7.27) is decreasing. Once $\tilde{\gamma}_q(x)$ becomes 0, a switch occurs from q to $1 - q$, and, due to property (a) in the above lemma, we have then that $\tilde{\gamma}_{1-q}(x) < 0$ if $x \notin \mathcal{A}_x$, and hence, the switching is well defined. The constants K_0 and K_1 control the shape and position of the switching boundaries, which are parabolas in the (v_C, i_L) plane. In fact, according to property (c) of Lemma 7.3.3, as K_0 and K_1 approach zero, the switching boundary approaches the zero level set of $\gamma_0(x)$ and γ_1 , respectively. Moreover, as K_0 and K_1 approach $\frac{2p_{11}}{RC}$, the switching boundaries approach the line given by the points x such that $-\frac{2i_L^*}{v_C^*}v_C + 2i_L = 0$.

Therefore, the closed-loop system with proposed controller can be expressed as in (2.1) with state variable $z = [x \ q]^\top$ and dynamics

$$\begin{aligned} \dot{z} &\in \begin{bmatrix} F_q(x) \\ 0 \end{bmatrix} =: F(x, q) & (x, q) \in \mathcal{C} \\ z^+ &= \begin{bmatrix} x \\ G_q(x) \end{bmatrix} =: G(x, q) & (x, q) \in \mathcal{D} \end{aligned} \tag{7.43}$$

where

$$\begin{aligned} \mathcal{C} &= \left\{ z : x \in \widetilde{M}_0, \tilde{\gamma}_0(x) \leq 0, q = 0 \right\} \cup \\ &\quad \left\{ z : x \in \widetilde{M}_1, \tilde{\gamma}_1(x) \leq 0, q = 1 \right\} \\ \mathcal{D} &= \left\{ z : x \in \widetilde{M}_0, \tilde{\gamma}_0(x) = 0, q = 0 \right\} \cup \\ &\quad \left\{ z : x \in \widetilde{M}_1, \tilde{\gamma}_1(x) = 0, q = 1 \right\} \end{aligned}$$

and, for each $z \in \mathbb{R}^2 \times \{0, 1\}$, the q is given by

$$G_q(x) = \begin{cases} \{1\} & \text{if } q = 0 \\ \{0\} & \text{if } q = 1 \end{cases}$$

Sample contour plots and switching boundaries $\gamma_q(x) = 0$ and $\tilde{\gamma}_q(x) = 0$ of the proposed controller for a particular set of parameters ($x^* = (7, 3.27)$, $V_{DC} = 5V$, $R = 3\Omega$, $C = 0.1F$, $L = 0.2H$, $p_{11} = \frac{C}{2}$, $p_{22} = \frac{L}{2}$, and varying K_0 and K_1) are shown in Figure 7.13.

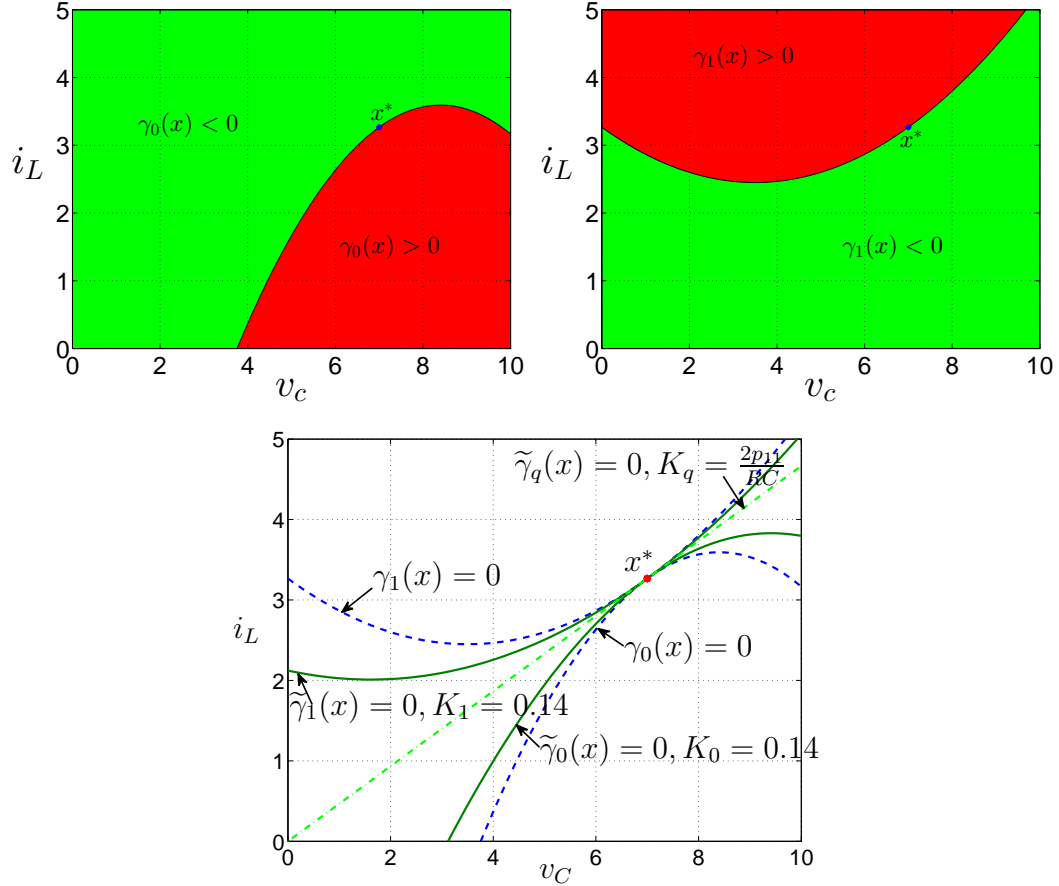


Figure 7.13: The switching boundaries $\gamma_q(x) = 0$ and $\tilde{\gamma}_q(x) = 0$ with different values for K_0 and K_1 .

By varying the constants $K_0 \in (0, \frac{2p_{11}}{RC})$ and $K_1 \in (0, \frac{2p_{11}}{RC})$, the shape and the position of the switching boundaries can be controlled. Some examples are shown in Figure 7.13. Note that the switching boundaries can also be modified by changing system parameters R and V_{DC} (because of uncertainties in supply and demand of renewable energy sources).

7.3.1.2 Properties of closed-loop system

First, note that system data of \mathcal{H} to satisfy the hybrid basic conditions.

Lemma 7.3.4 *The closed-loop system \mathcal{H} given by (7.43) satisfies the hybrid basic conditions given by (A1)-(A3) in [61, Assumption 6.5].*

Proof (A1) follows from the continuity of $\tilde{\gamma}_q$ for each $q \in \{0, 1\}$ and the closedness of \tilde{M}_0 and \tilde{M}_1 . Next, (A2) follows from the Krasovskii regularization. Lastly, (A3) follows from the fact that the jump map is continuous. \square

Then, we show that the solutions to the closed-loop system \mathcal{H} are complete by applying [61, Proposition 6.10].

Proposition 7.3.5 *For each $\xi \in \mathcal{C} \cup \mathcal{D}$, every maximal solution $\chi = (x, q)$ to the hybrid system $\mathcal{H} = (\mathcal{C}, F, \mathcal{D}, G)$ in (7.43) with $\chi(0, 0) = \xi$ is complete.*

Proof We apply [61, Proposition 6.10]. First we check the viability condition (VC), which requires verifying that for each $(x, q) \in \mathcal{C} \setminus \mathcal{D}$, there exists a neighborhood U of (x, q) such that

$$F(\bar{x}, \bar{q}) \cap T_{\mathcal{C}}(\bar{x}, \bar{q}) \neq \emptyset \quad \forall (\bar{x}, \bar{q}) \in U \cap \mathcal{C} \quad (7.44)$$

In fact, note that if $(x, q) \in \mathcal{C} \setminus \mathcal{D}$, then for any sufficiently small neighborhood U of (x, q) , it holds that $(\bar{x}, \bar{q}) \in U \cap \mathcal{C}$ implies $(\bar{x}, \bar{q}) \in \mathcal{C} \setminus \mathcal{D}$ due to continuity of $\tilde{\gamma}_S$, $S \in \{0, 1\}$. Therefore, it suffices to show that (we dropped the bars in \bar{x}, \bar{q})

$$F(x, q) \cap T_{\mathcal{C}}(x, q) \neq \emptyset \quad \forall (x, q) \in \mathcal{C} \setminus \mathcal{D}$$

To do so, we will first compute the tangent cones $T_{\mathcal{C}}(x, q)$ for the set \mathcal{C} for $(x, q) \in \mathcal{C} \setminus \mathcal{D}$:

- $q = 0, i_L > 0$: $T_{\mathcal{C}}(x, q) = \mathbb{R}^2 \times \{0\}$
- $q = 0, i_L = 0$: $T_{\mathcal{C}}(x, q) = \mathbb{R} \times \mathbb{R}_+ \times \{0\}$
- $q = 1, v_C > 0$: $T_{\mathcal{C}}(x, q) = \mathbb{R}^2 \times \{0\}$

- $q = 1, v_C = 0: T_C(x, q) = \mathbb{R}_+ \times \mathbb{R} \times \{0\}$

Using these calculations, we have the following:

1. For $(x, q) \in \mathcal{C} \setminus \mathcal{D}$ such that $q = 0, i_L > 0$, (7.44) trivially holds.
2. For $(x, q) \in \mathcal{C} \setminus \mathcal{D}$ such that $q = 0, i_L = 0$, we have to distinguish two cases based on different set-valued vector fields depending if $x \in \overline{M_1} \setminus \overline{M_3}$ (i.e. $v_C < V_{DC}$) or $x \in \overline{M_3}$ (i.e. $v_C \geq V_{DC}$).
 - (a) If $x \in \overline{M_1} \setminus \overline{M_3}$ and thus $v_C < V_{DC}$, we have the vector field

$$\begin{pmatrix} f_a(x) \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{RC}v_C + \frac{1}{C}i_L \\ -\frac{1}{L}v_C + \frac{V_{DC}}{L} \\ 0 \end{pmatrix} \in T_C(x, q)$$

because $-\frac{1}{L}v_C + \frac{V_{DC}}{L} > 0$.

- (b) When $x \in \overline{M_3}$ and thus $v_C > V_{DC}$, we have a set-valued vector field given by

$$\left\{ -\frac{1}{RC}v_C \right\} \times \left[-\frac{1}{L}v_C + \frac{V_{DC}}{L}, 0 \right] \times \{0\}$$

Since $(-\frac{1}{RC}v_C, 0, 0)$ is an element of the set above and also lies in $T_C(x, q)$, (7.44) holds.

3. For $(x, q) \in \mathcal{C} \setminus \mathcal{D}$ such that $q = 1, v_C > 0$, (7.44) trivially holds.
4. For $(x, q) \in \mathcal{C} \setminus \mathcal{D}$ such that $q = 1, v_C = 0$, the vector field is given by $F_1(x) \times \{0\}$ that only contains the element $(0, \frac{V_{DC}}{L}, 0)$, which lies in $T_C(x, q) = \mathbb{R}_+ \times \mathbb{R} \times \{0\}$. Hence, (7.44) holds.

In summary, for each $\xi \in \mathcal{C} \setminus \mathcal{D}$, there exists a neighborhood U of ξ such that (7.44) holds. Thus, according to [61, Proposition 6.10], there exists a nontrivial solution χ to \mathcal{H} for points in $\mathcal{C} \cup \mathcal{D}$.

Now, to show that every maximal solution χ is complete, we prove that cases (b) and (c) in [61, Proposition 6.10] cannot hold, and hence, only case (a) can be true.

Case (b) (finite escape time) cannot happen due to the fact that every maximal solution χ is bounded. Indeed, using lemma 7.3.1 and property (b) of lemma 7.3.3, the function V in (7.27), along a maximal solution χ , has non-positive derivative for flows and non-positive changes at jumps. Since V is quadratic it upper bounds the norm of the state (relative to the desired set point) and has compact sub-level sets. Therefore, $\lim_{t \rightarrow T} |\chi(t, j)| \leq M < \infty$ for some constant M and $T = \sup_t \text{dom} \chi$.

Case (c) (solutions jumping outside $\mathcal{C} \cup \mathcal{D}^8$) can be excluded as well, because below we will show that $G(\mathcal{D}) \subset \mathcal{C}$, and thus $G(\mathcal{D}) \subset \mathcal{C} \cup \mathcal{D}$.

In fact, to complete the proof we establish now that $G(\mathcal{D}) \subset \mathcal{C}$ and we consider two situations: I. $x \in \mathcal{D}$ and $q = 0$, and II. $x \in \mathcal{D}$ and $q = 1$.

1. Let $x \in \mathcal{D}$ and $q = 0$, and thus $\tilde{\gamma}_0(x) = 0$ and $x \in \tilde{M}_0$ (i.e. $i_L \geq 0$). We will first show that this implies that $x \in \tilde{M}_1$ (i.e. $v_C \geq 0$), i.e.

$$\left. \begin{array}{l} \tilde{\gamma}_0(x) = 0 \\ i_L \geq 0 \end{array} \right\} \Rightarrow v_C \geq 0 \quad (7.45)$$

This latter implication will follow from the fact that $\tilde{\Omega}_0 := \{x \in \mathbb{R}^2 \mid \tilde{\gamma}_0(x) = 0\}$ is a downward parabola and the fact that the minimal root $\min\{v_C \mid x \in \tilde{\Omega}_0, i_L = 0\}$ is non-negative. Indeed, since $\tilde{\Omega}_0$ is a downward parabola, these two facts would give

$$\begin{aligned} \min\{v_C \mid x \in \tilde{\Omega}_0, i_L \geq 0\} &= \min\{v_C \mid x \in \tilde{\Omega}_0, i_L = 0\} \\ &\geq 0 \end{aligned}$$

which is equivalent to (7.45).

To compute the minimal root, we can use the expression in (7.35), showing that the points $x = (v_C, i_L)$ with $\tilde{\gamma}_0(x) = 0$ ($x \in \tilde{\Omega}_0$) and $i_L = 0$ satisfy

$$i_L = -\frac{\tilde{A}_0}{C_0} v_C^2 - \frac{\tilde{B}_0}{C_0} v_C - \frac{\tilde{D}_0}{C_0} = 0$$

where $\tilde{A}_0 = A_0 + \frac{K_0}{2}$, $\tilde{B}_0 = B_0 - K_0 v_C^*$, and $\tilde{D}_0 = D_0 + \frac{K_0}{2} v_C^{*2}$, which can

⁸Note that flowing outside $\mathcal{C} \cup \mathcal{D}$ is not possible due to the closedness of \mathcal{C} and \mathcal{D} as formulated in the hybrid basic conditions.

be rewritten as

$$\left(v_C - \left(-\frac{\tilde{B}_0}{2\tilde{A}_0} \right) \right)^2 = -\frac{\tilde{D}_0}{\tilde{A}_0} + \left(-\frac{\tilde{B}_0}{2\tilde{A}_0} \right)^2$$

Then, the roots $v_{C1,2}$ of the function $\tilde{\gamma}_0(x)|_{i_L=0}$ are given by

$$\begin{aligned} v_{C1,2} &= -\frac{\tilde{B}_0}{2\tilde{A}_0} \pm \sqrt{-\frac{\tilde{D}_0}{\tilde{A}_0} + \left(-\frac{\tilde{B}_0}{2\tilde{A}_0} \right)^2} \\ &= -\left(\frac{p_{11} + \frac{p_{11}}{V_{DC}}v_C^* - RCK_0}{-2p_{11} + RCK_0} \right) v_C^* \\ &\quad \pm \sqrt{-(v_C^*)^2 + \left(-\left(\frac{p_{11} + \frac{p_{11}}{V_{DC}}v_C^* - RCK_0}{-2p_{11} + RCK_0} \right) v_C^* \right)^2} \end{aligned}$$

in terms of system constants. Because $v_C^* > V_{DC} > 0$ and $K_0 \in (0, \frac{2p_{11}}{RC})$, $\min\{v_{C1}, v_{C2}\} \geq 0$ as long as $-(v_C^*)^2 + \left(-\left(\frac{p_{11} + \frac{p_{11}}{V_{DC}}v_C^* - RCK_0}{-2p_{11} + RCK_0} \right) v_C^* \right)^2 > 0$. The left-hand side of the inequality can be rewritten as

$$\left(\frac{\frac{p_{11}}{RC} \left(1 - \frac{v_C^*}{V_{DC}} \right)}{K_0 - \frac{2p_{11}}{RC}} \right) v_C^{*2}$$

which is always positive since conditions $v_C^* > V_{DC} > 0$ and $K_0 \in (0, \frac{2p_{11}}{RC})$. This establishes (7.45). Using now property (a) of Lemma 7.3.3, and $\tilde{\gamma}_0(x) = 0$, we know that $\tilde{\gamma}_1(x) \leq 0$ and thus $G(x, 0) = (x, 1) \in \mathcal{C}$.

2. Let $x \in \mathcal{D}$ and $q = 1$, and thus $\tilde{\gamma}_1(x) = 0$ and $x \in \tilde{M}_1$ (i.e. $v_C \geq 0$). Similar to case I. we will first show that this implies that $x \in \tilde{M}_0$ (i.e. $i_L \geq 0$), i.e.

$$\left. \begin{array}{l} \tilde{\gamma}_1(x) = 0 \\ v_C \geq 0 \end{array} \right\} \Rightarrow i_L \geq 0 \quad (7.46)$$

To show this we compute $\min\{i_L \mid \tilde{\gamma}_1(x) = 0, v_C \geq 0\}$ and show it is non-negative. First we observe that $\tilde{\Omega}_1 := \{x \in \mathbb{R}^2 \mid \tilde{\gamma}_1(x) = 0\}$ is an upward parabola. Using the expression in (7.36), the points $x = (v_C, i_L)$ satisfying $\tilde{\gamma}_1(x) = 0$ are given by

$$i_L = -\frac{\tilde{A}_1}{C_1}v_C^2 - \frac{\tilde{B}_1}{C_1}v_C - \frac{\tilde{D}_1}{C_1}$$

where $\tilde{A}_1 = A_1 + \frac{K_1}{2}$, $\tilde{B}_1 = B_1 - K_1 v_C^*$, and $\tilde{D}_1 = D_1 + \frac{K_1}{2} v_C^{*2}$. The minimum value $i_{L \min} := \min\{i_L \mid \tilde{\gamma}_1(x) = 0\}$ results in two cases, namely

$$i_{L \min,1} = \frac{\tilde{B}_1^2 - 4\tilde{A}_1\tilde{D}_1}{4\tilde{A}_1 C_1}, \quad v_{C \min,1} > 0, \quad K_1 \in \left(0, \frac{p_{11}}{RC}\right) \quad (7.47)$$

$$i_{L \min,2} = -\frac{\tilde{D}_1}{C_1}, \quad v_{C \min,2} = 0, \quad K_1 \in \left[\frac{p_{11}}{RC}, \frac{2p_{11}}{RC}\right) \quad (7.48)$$

where $i_{L \min}$ is found by either the vertex of the parabola or at $v_C = 0$ due to the constraint $v_C \geq 0$, respectively. Substituting the expressions of $\tilde{A}_1, \tilde{B}_1, \tilde{D}_1$ into the right-hand side of (7.47) and (7.48), we have

$$i_{L \min,1} = \frac{\tilde{B}_1^2 - 4\tilde{A}_1\tilde{D}_1}{4\tilde{A}_1 C_1} = \frac{(6p_{11} - 4K_1 RC) v_C^{*2}}{4p_{11} R V_{DC} (2p_{11} - K_1 RC)}$$

$$i_{L \min,2} = -\frac{\tilde{D}_1}{C_1} = \frac{(2p_{11} - K_1 RC) v_C^{*2}}{2p_{11} R V_{DC}}$$

Since $R, C, V_{DC}, v_C^*, p_{11} > 0$, we obtain

$$i_{L \min} > 0$$

and thus $\min\{i_L \mid \tilde{\gamma}_1(x) = 0, v_C \geq 0\} \geq \min\{i_L \mid \tilde{\gamma}_1(x) = 0\} = i_{L \min} > 0$. This establishes (7.46). Using now property (a) of Lemma 7.3.3 and $\tilde{\gamma}_1(x) = 0$, we know that $\tilde{\gamma}_0(x) \leq 0$ and thus $G(x, 1) = (x, 0) \in C$.

This completes the proof. \square

Using these properties, we are now ready to establish the following theorem, which states global asymptotic stability of the compact set \mathcal{A} for the hybrid system \mathcal{H} .

Theorem 7.3.6 *Consider the hybrid system \mathcal{H} in (7.43) with $c, L, R, V_{DC}, K_0, K_1 > 0$. Given a desired set-point voltage and current (v_C^*, i_L^*) , where $v_C^* > V_{DC}$ and $i_L^* = \frac{v_C^{*2}}{R V_{DC}}$, then the compact set*

$$\mathcal{A} = \mathcal{A}_x \times \{0, 1\} \quad (7.49)$$

is globally asymptotically stable for \mathcal{H} .

Proof Consider the function V given in (7.27) and define $\tilde{V}(x, q) = V(x)$ for all $(x, q) \in \mathcal{C} \cup \mathcal{D}$. Note that $\tilde{V}(x, q) = 0$ when $x \in \mathcal{A}_x$ and $\tilde{V}(x, q) > 0$ for all $(x, q) \in (\mathbb{R}^2 \times \{0, 1\}) \setminus \mathcal{A}_x$. From the computation of the inner product between ∇V and the direction belonging to F_S , for each $(x, q) \in \mathcal{C}$ (see Lemma 7.3.1), we have

$$\begin{aligned} u_C(x, q) &:= \max_{\xi \in F(x, q)} \langle \nabla \tilde{V}(x, q), \xi \rangle \\ &= \begin{cases} (v_C - v_C^*)(-\frac{1}{R}v_C + i_L) + (i_L - i_L^*)(-v_C + V_{DC}) = \gamma_0(x) \leq 0 & \text{if } q = 0, \\ (v_C - v_C^*)(-\frac{1}{R}v_C) + (i_L - i_L^*)V_{DC} = \gamma_1(x) \leq 0 & \text{if } q = 1 \end{cases} \end{aligned}$$

and, for each $(x, q) \in \mathcal{D}$, we have

$$u_d(x, q) := \max_{\xi \in G(x, q)} \tilde{V}(\xi) - \tilde{V}(x, q) = 0$$

Then, by [61, Theorem 3.18], the set (7.49) is stable.

To show attractivity, we apply the invariance principle in [65, Theorem 4.7]. To this end, we compute the zero level set of u_C and u_d defined above. It follows that

$$\begin{aligned} u_C^{-1}(0) &= \{(x, q) \in \mathcal{C} : u_C(x, q) = 0\} = \mathcal{D} \\ u_d^{-1}(0) &= \{(x, q) \in \mathcal{D} : u_d(x, q) = 0\} = \mathcal{D} \end{aligned}$$

Then, each complete and bounded solution (x, q) to \mathcal{H} converges to the largest weakly invariant⁹ subset of the set

$$\left\{ (x, q) \in \mathcal{C} \cup \mathcal{D} : \tilde{V}(x, q) = r \right\} \cap (u_C^{-1}(0) \cup (u_d^{-1}(0) \cap G(u_d^{-1}(0)))) \quad (7.50)$$

for some $r \geq 0$. With the definitions above, the set of points (7.50) reduces to

$$\{(x, q) \in \mathcal{C} \cup \mathcal{D} : V(x) = r\} \cap \mathcal{D} \quad (7.51)$$

⁹For the set of hybrid trajectories \mathcal{S} , the set $\mathcal{M} \subset \mathcal{O}$ is said to be *weakly invariant* (with respect to \mathcal{S}) if it is both weakly forward invariant and weakly backward invariant; see [65, Definition 3.1], it is *weakly forward invariant* (with respect to \mathcal{S}) if for each $x^0 \in \mathcal{M}$, there exists at least one complete hybrid trajectory $x \in \mathcal{S}(x^0)$ with $x(t, j) \in \mathcal{M}$ for all $(t, j) \in \text{dom } x$. It is *weakly backward invariant* (with respect to \mathcal{S}) if for each $q \in \mathcal{M}$, $N > 0$, there exist $x^0 \in \mathcal{M}$ and at least one hybrid trajectory $x \in \mathcal{S}(x^0)$ such that some $(t^*, j^*) = q$ and $x(t, j) \in \mathcal{M}$ for all $(t, j) \leq (t^*, j^*)$, $(t, j) \in \text{dom } x$.

Note that the only invariant set for \mathcal{H} within (7.51) is \mathcal{A} since solutions cannot stay in (7.51) unless $v_C = v_C^*$ and $i_L = i_L^*$ (i.e., $r = 0$). In fact, solutions to the hybrid systems \mathcal{H} in (7.43) cannot stay in a constant level set of V since the equilibrium points of the vector field F do not belong to $\mathcal{C} \cap \mathcal{D}$ and, for points in $\mathcal{C} \setminus \mathcal{D}$, the derivative of V is negative for each $q \in \{0, 1\}$. \square

Theorem 7.3.6 implies all solutions (including solutions start from \mathcal{A}) converge to \mathcal{A} . In addition, Proposition 7.3.5 implies all solutions are complete. Thus, we can conclude solutions start from \mathcal{A} are complete and always stay in \mathcal{A} . Therefore, \mathcal{A} is forward invariant for close-loop system \mathcal{H} . In fact, the asymptotic stability of a set for a system general implies forward pre-invariance of a set due to the *stability* part of definition. However, solutions to system are required to be complete for an asymptotic stability set to be forward invariant to system.

7.3.1.3 Robustness to general perturbations

In addition, the construction of the proposed controller is such that the closed-loop system \mathcal{H} has data satisfying the hybrid basic conditions given by (A1)-(A3) in [61, Assumption 6.5]. With these properties, we have that the asymptotic stability property is robust to small perturbations; see [88, Theorem IV.6]. More precisely, we consider the following model of the plant with perturbations:

$$\dot{x} \in F_S(x + d_1) + d_2 \tag{7.52}$$

where d_1 corresponds to state noise and d_2 captures unmodeled dynamics. Then, defining $\tilde{d}_i = (d_i, 0)$, the closed-loop system \mathcal{H} results in the perturbed hybrid system, which is denoted by $\tilde{\mathcal{H}}$, with dynamics

$$\begin{aligned} \dot{z} &\in F(z + \tilde{d}_1) + \tilde{d}_2 & z + \tilde{d}_1 &\in \tilde{C} \\ z^+ &\in G(z) & z + \tilde{d}_1 &\in \tilde{D} \end{aligned}$$

The following result establishes a nominal robustness property of \mathcal{H} .

Theorem 7.3.7 *Under the assumptions of Theorem 7.3.6, there exists $\tilde{\beta} \in \mathcal{KL}$ such that, for each $\tilde{\varepsilon} > 0$ and each compact set $\mathcal{M}_r \subset \mathbb{R}^2$, there exists $\delta > 0$ such*

that for any two measurable functions $\tilde{d}_1, \tilde{d}_2 : \mathbb{R}_{\geq 0} \rightarrow \delta\mathbb{B}$, every solution $\tilde{\chi} = (\tilde{x}, \tilde{q})$ to $\tilde{\mathcal{H}}$ with $\tilde{\chi}(0, 0) \in \mathcal{M}_r \times \{0, 1\}$ is such that its \tilde{x} component, namely, (v_C, i_L) , satisfies

$$|\tilde{x}(t, j)|_{\mathcal{A}_x} \leq \tilde{\beta}(|\tilde{x}(0, 0)|_{\mathcal{A}_x}, t + j) + \tilde{\varepsilon} \quad \forall (t, j) \in \text{dom } \tilde{\chi}$$

The proof is given in [88, Section III.C].

Unlike previous results in the literature, this robustness property implies that our controller is robust to small measurement noise and unmodeled dynamics. In addition to the robustness to general perturbations shown above, the asymptotic stability of \mathcal{A} is robust to slow variations of the system parameters, such as input voltage V_{DC} and load R . Such a result follows from a direct application of [61, Corollary 7.27].

7.3.1.4 Robustness to spatial regularization

In addition to robustness to small perturbations, the fact that (A1)-(A3) are satisfied imply that the closed-loop system is robust to spatial regularization, which can be employed to alleviate possible arbitrarily fast switching. More precisely, we use the condition $\tilde{\gamma}_q(x) = \rho$ rather than $\tilde{\gamma}_q(x) = 0$ as switching boundaries at the controller level, where ρ is a small positive constant. The motivation for such a modification on the controller is to reduce the number of switches and enlarge the time between switches by allowing a neighborhood around x^* between the two switching boundaries, rather than having them intersect at the point x^* . The regularized system is denoted as

$$\mathcal{H}^\rho = (\mathcal{C}_\rho, F, \mathcal{D}_\rho, G) \tag{7.53}$$

which share the same flow and jump maps as in \mathcal{H} , while the switching boundaries

$$\tilde{\gamma}_q(x) = \rho$$

are used to define the flow and jump sets.

i.e.,

$$\begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} \in \begin{bmatrix} F_q(x) \\ 0 \end{bmatrix} \quad (x, q) \in C_\rho$$

where, now, the flow set is replaced by

$$C_\rho = \left\{ (x, q) : x \in \widetilde{M}_0, \widetilde{\gamma}_0(x) \leq \rho, q = 0 \right\} \cup \left\{ (x, q) : x \in \widetilde{M}_1, \widetilde{\gamma}_1(x) \leq \rho, q = 1 \right\}$$

Furthermore, the jump map is given by

$$\begin{aligned} x^+ &= x \\ q^+ &\in G_q(x) \end{aligned} \quad (x, q) \in D_\rho$$

where, now, the jump set is given by

$$D_\rho = \left\{ (x, q) : x \in \widetilde{M}_0, \widetilde{\gamma}_0(x) = \rho, q = 0 \right\} \cup \left\{ (x, q) : x \in \widetilde{M}_1, \widetilde{\gamma}_1(x) = \rho, q = 1 \right\}$$

and

$$G_q(x) = \begin{cases} \{1\} & \text{if } q = 0, \widetilde{\gamma}_0(x) \geq \rho \\ \{0, 1\} & \text{if } \gamma_0(x) \geq \rho, \gamma_1(x) \geq \rho \\ \{0\} & \text{if } q = 1, \widetilde{\gamma}_1(x) \geq \rho \end{cases}$$

Under the given assumptions, it can be shown that the solutions satisfy a practical \mathcal{KL} bound for any solutions, namely, for every $\epsilon > 0$, there is such that solutions to the closed-loop system converge to a neighborhood of \mathcal{A}_x after finite hybrid time (that depends on ϵ).

Theorem 7.3.8 *Under the assumptions of Theorem 7.3.6, there exists $\beta \in \mathcal{KL}$ such that, for each $\varepsilon > 0$ and each compact set $\mathcal{M}_r \subset \mathbb{R}^2$, there exists $\rho^* > 0$ guaranteeing the following property: for each $\rho \in (0, \rho^*]$ every solution $\chi = (x, q)$ to \mathcal{H}^ρ with $\chi(0, 0) \in \mathcal{M}_r \times \{0, 1\}$ is such that its x component satisfies*

$$|x(t, j)|_{\mathcal{A}_x} \leq \beta(|x(0, 0)|_{\mathcal{A}_x}, t + j) + \varepsilon \quad \forall (t, j) \in \text{dom } \chi$$

The proof follows analogously to the proof of Theorem 7.3.7. A similar result can be obtained using temporal regularization.

For the spatially regularized control algorithm, no Zeno behavior occurs and certainly no “eventually discrete” solutions (in the sense of the solution that after

some time t only jumps) exist due to the uniformly finite (nonzero) separation between the flow and jump sets—this property follows from [65, Lemma 2.7] since the closed-loop system satisfies the properties listed in Proposition 7.3.5.

7.3.2 Simulations Validations and Discussions

In this section, simulations of the closed-loop system \mathcal{H} and its variations are performed using $V_{DC} = 5V$, $R = 3\Omega$, $C = 0.1F$, $L = 0.2H$, $P = \begin{bmatrix} \frac{C}{2} & 0 \\ 0 & \frac{L}{2} \end{bmatrix}$, and $x^* = (7, 3.27)$. We used the Hybrid Equations (HyEQ) Toolbox via Simulink (see [90]) for performing the simulations.

7.3.2.1 Simulations of the closed-loop system \mathcal{H}

The simulation results for the closed-loop system \mathcal{H} are shown in Fig. 7.14. As can be seen, the solution components (v_C, i_L) converge from both initial conditions to the set \mathcal{A}_x in correspondence with the globally asymptotic stability property of the closed-loop system. Here we use $K_0 = 0.05$, $K_1 = 0.12$, and q is only drawn for the simulation using $x_0 = (0, 5)$.

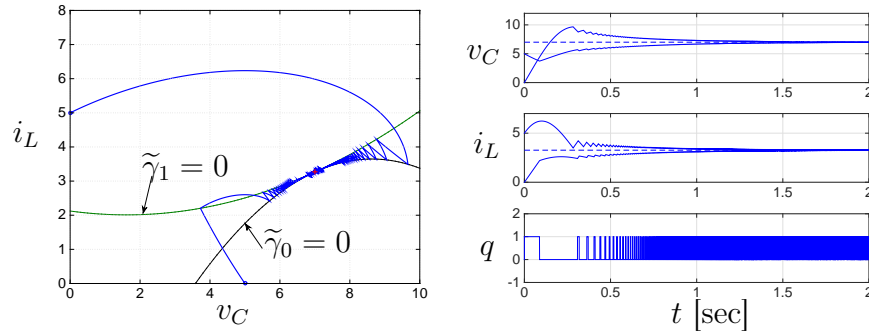


Figure 7.14: Simulation results for the closed-loop system \mathcal{H} with initial conditions $x_0 = (0, 5)$, $q_0 = 0$ and $x_0 = (5, 0)$, $q_0 = 1$.

7.3.2.2 Simulations of robustness to perturbation

The perturbed closed-loop system $\tilde{\mathcal{H}}$ with $\delta = 0.5$, $K_0 = 0.28$, and $K_1 = 0.23$ is simulated, as shown in Fig. 7.15, and a sinusoidal perturbation injected in the state x , resulting in $d_1 = 0.5 \sin(500t)$ and $d_2 = 0$ in (7.52). The Boost converter reaches a neighborhood of x^* and remains fluctuating due to the presence of the perturbation.

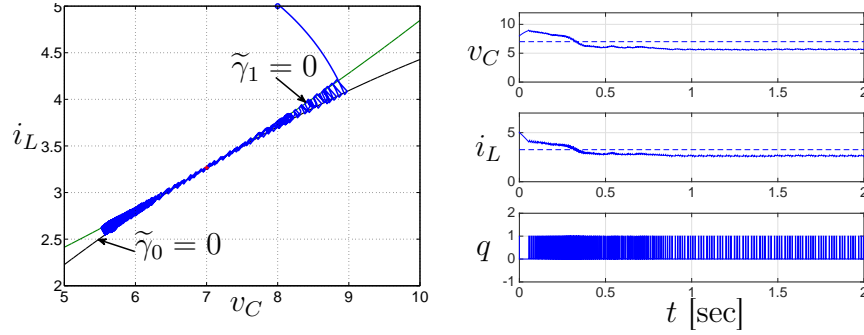


Figure 7.15: Simulation results for the perturbed closed-loop system $\tilde{\mathcal{H}}$ for initial conditions $x_0 = (8, 5)$, $q_0 = 1$.

7.3.2.3 Simulations of robustness to regularization

Simulation results are shown in Fig. 7.16 for the spatial regularized system \mathcal{H}_ρ with $\rho = 0.2$, $K_0 = 0.28$, and $K_1 = 0.12$. q is only drawn for the simulation using $x_0 = (5, 0)$. As suggested by the plot of q v.s. t , the number of switches is reduced significantly with the regularized system \mathcal{H}_ρ , which is preferred for the purpose of hardware implementations.

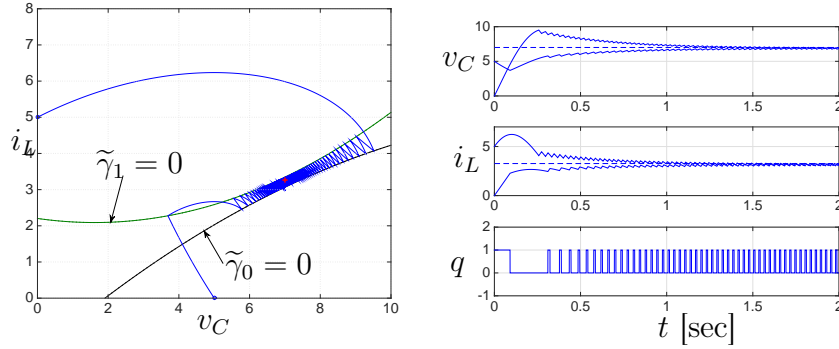


Figure 7.16: Simulation results for the spatially regularized closed-loop system \mathcal{H}^ρ for initial conditions $x_0 = (0, 5)$, $q_0 = 1$ and $x_0 = (5, 0)$, $q_0 = 0$.

7.3.2.4 Simulations of robustness to changes in supply and demand

The input voltage V_{DC} and load R are now varied to assure the robustness to such changes. In the next simulation, V_{DC} is increased from 2.5V to 5V and afterwards R is decreased from 3Ω to 2Ω . By doing this, i_L^* varies as well and also the switching boundaries change during process.

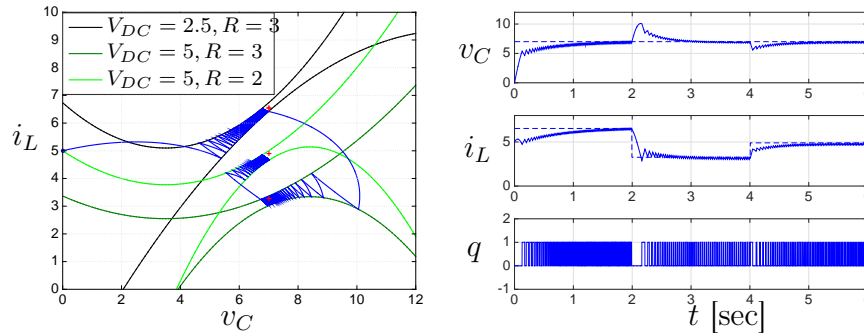


Figure 7.17: Simulation results for a varying input voltage V_{DC} and a varying load R with $\rho = 0.5$ for initial conditions $x_0 = (0, 5)$, $q_0 = 0$.

Figure 7.17 shows a simulation where V_{DC} is increased after 2 seconds and R is decreased after 4 seconds. It shows the boundaries for the three different situations. As it can be seen, a neighborhood of v_C^* is reached in three situations, which means that the controller is able to cope with variations in the supply V_{DC} and demand R .

7.3.2.5 Benchmark tests

Using the regularized system in (7.53), we propose the following benchmark test for the DC/DC boost converter. Given a constant ϵ representing the maximum deviation of the output range v_C from v_C^* , we determine the average number of switches per second for five different initial conditions after the system reaches its steady state region, i.e., when solutions reach the set

$$\{z \in \mathbb{R}^2 \times \{0, 1\} : |v_C - v_C^*| \leq \epsilon\}$$

and remain in it. Moreover, we also compute the average number of switches and its standard deviation (Std) for three different values of ϵ . We use the same system parameters and the relationship $\epsilon \approx 1.3\rho$ from [88, Table II] for this benchmark test. In addition, for each value of ϵ , we present the average dwell time for switching. The number of switches per second reported are rounded.

Table 7.2: Benchmark test for DC/DC boost converter

ϵ	x_0	Average number of switches per second	Average & Std	Average dwell-time between switches
0.01	(0,5)	1467	Average = 1587 Std = 61.21	$S = \text{ON} : 9 \times 10^{-4}s$ $S = \text{OFF} : 3.6 \times 10^{-4}s$
	(4,3)	1625		
	(6,2)	1625		
	(3,8)	1592		
	(5,0)	1625		
0.05	(0,5)	348	Average = 356 Std = 4.37	$S = \text{ON} : 4 \times 10^{-3}s$ $S = \text{OFF} : 1.6 \times 10^{-3}s$
	(4,3)	360		
	(6,2)	360		
	(3,8)	355		
	(5,0)	359		
0.1	(0,5)	179	Average = 179 Std = 0.075	$S = \text{ON} : 8 \times 10^{-3}s$ $S = \text{OFF} : 3.1 \times 10^{-3}s$
	(4,3)	179		
	(6,2)	179		
	(3,8)	179		
	(5,0)	179		

The numerical results in Table 7.2 indicate that, for smaller ϵ values, switching

happens more frequently. While the number of switches varies with the initial condition, the standard deviations suggest that the dispersion around the average is small. In addition, the average dwell time results indicate that the switch stays at the “ON” position longer than at the “OFF” position, which is expected, but more importantly, indicate that the time between consecutive switching times has a reasonable lower bound. The smaller time duration between the two, when switch is “ON” and when it is “OFF”, can be used as a factor to determine how fast the switch happens during the “steady state”.

7.4 A Constrained Bouncing Ball System

We demonstrate our main results in a constrained mechanical system with control inputs; namely, a bouncing ball moving vertically that is controlled by impacts on a controlled surface at zero height. In addition, we attach one end of a nonelastic string with length h_{\max} to zero height and the other end to the ball; see Figure 7.18 for system configuration. Compared to a typical bouncing ball

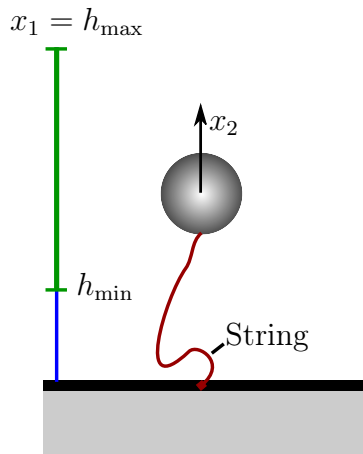


Figure 7.18: Bouncing ball system configuration.

system [61, Example 1.1], the model considered here has an additional “pulling phase” when the ball reaches the height h_{\max} with possibly nonzero velocity. The

possible pulls from the string at height h_{\max} and the impacts between the ball and the controlled surface both lead to jumps of the state. In addition to assuming unitary mass of the ball and negligible weight of the string, forces, and friction, we consider the following:

- C1) at impacts with the controlled surface, the uncertain coefficient of restitution is within the range $[e_1, e_2]$, where $0 < e_1 < e_2 < 1$;
- C2) the string breaks when the ball pulls with velocity larger than v_{\max} ;
- C3) at pulls of the string, the restitution coefficient is $e_p \in (0, 1]$.

With $x = (x_1, x_2) \in \mathbb{R}^2$, x_1 and x_2 model the height (position) and velocity of the ball, respectively. Then, with gravity constant $\gamma > 0$, the flow map is defined on $\mathbb{R}_{\geq 0} \times \mathbb{R}$ and is given by¹⁰

$$f(x) := (x_2, -\gamma).$$

To formulate the flow and jump set, we define a function $E : \mathbb{R}^2 \rightarrow \mathbb{R}$ that describes the total energy of the system

$$E(x) = \frac{x_2^2}{2} + \gamma x_1.$$

According to C2), the string remains attached to the ball when $x_1 \in [0, h_{\max}]$ and $x_2 \leq v_{\max}$, i.e., $E(x) \leq E_{\max}$ with $E_{\max} := E(h_{\max}, v_{\max})$. After impacts with the controlled surface, the ball position x_1 remains unchanged, while the velocity x_2 is updated to a function of the disturbance $w_d \in \mathcal{W}_d := [e_1, e_2]$, which represents the uncertain coefficient of restitution, and input $u_d \in \mathcal{U}_d := [0, u_{\max}]$ with $u_{\max} = \sqrt{2E_{\max}}$, which represents the velocity change caused by the controlled surface. Hence, we model impacts between the ball and the controlled surface as $x_1 = 0$ as

$$G_1(x, u_d, w_d) := (x_1, u_d - w_d x_2)$$

when $x_2 \leq 0$. Before every impact x_2 is nonpositive, after each impact the ball velocity is updated according to G_1 . Then, with a small positive constant $\delta_p <$

¹⁰Note that since there are no disturbances and inputs for flow, we omit the subscripts for f and C in this model.

v_{\max} , the map

$$G_2(x) := (x_1, \min\{-e_p x_2, -\delta_p\})$$

models the pulls between the ball and the string when $x_1 = h_{\max}$ and $x_2 \in [0, v_{\max}]$. Since before every pull x_2 is nonnegative, after each pull the ball velocity reverses its sign and updates according to G_2 . Note that since closed jump sets are preferred as suggested in (A1_w) of Definition 2.0.8, our model only allows the x_2 component to jump to a strictly negative value that is lower bounded (and controllable) by $-\delta_p$.

Then, the hybrid system $\mathcal{H}_{u,w} = (C_{u,w}, F_{u,w}, D_{u,w}, G_{u,w})$ with control input u_d and disturbance w_d has state x and dynamics on space $\mathcal{X} = \mathbb{R}^2 \times \mathcal{U}_d \times \mathcal{W}_d$ that is given by

$$\begin{aligned} \dot{x} &= f(x) & x &\in C, & (7.54) \\ x^+ &= G_{u,w}(x, u_d, w_d) & (x, u_d, w_d) &\in D_{u,w}, \end{aligned}$$

where the flow set C is given by

$$C := \{x \in \mathbb{R}^2 : 0 \leq x_1 \leq h_{\max}, E(x) \leq E_{\max}\},$$

and the jump set $D_{u,w}$ is given by $D_{u,w} := D_1 \cup D_2$ with

$$\begin{aligned} D_1 &:= \{(x, u_d, w_d) \in \mathcal{X} : x_1 = 0, x_2 \in [-\sqrt{2E_{\max}}, 0]\}, \\ D_2 &:= \{(x, u_d, w_d) \in \mathcal{X} : x_1 = h_{\max}, x_2 \in [0, v_{\max}]\}, \end{aligned}$$

where the jump map $G_{u,w}$ is given by

$$G_{u,w}(x, u_d, w_d) := \begin{cases} G_1(x, u_d, w_d) & \text{if } (x, u_d, w_d) \in D_1 \\ G_2(x) & \text{if } (x, u_d, w_d) \in D_2. \end{cases}$$

We have the following control design goal: under the presence of disturbances w_d , control the ball at impacts so that when the ball starts from $x(0,0) = (x_1(0,0), x_2(0,0))$ such that $x_1(0,0) \in [h_{\min}, h_{\max}]$ and $E(x(0,0)) \in [0, E_{\max}]$, the string remains attached to the ball, and the peak height of the ball after each

bounce is at least h_{\min} . This objective is achieved by rendering the set

$$\mathcal{M}_r^w = L_V(-\gamma h_{\min}) \cap (C \cup \Pi_d(D_{u,w})) \quad (7.55)$$

robustly controlled forward pre-invariant for $\mathcal{H}_{u,w}$, where $V(x) := -E(x)$ for every $x \in C \cup \Pi_d(D_{u,w})$.

Given system parameters e_1, e_2, e_p, v_{\max} and h_{\max} , we choose system parameters h_{\min} such that $\sqrt{\gamma(h_{\min} + \frac{\varepsilon}{2})} < e_1\sqrt{E_{\max}}$ and with $\varepsilon > 0$,

$$\gamma(h_{\min} + \varepsilon) \leq \frac{(1 + e_1 - e_2)^2}{2} E_{\max} \quad (7.56)$$

Since the control input appears in the map G_1 only, for every $x \in \Pi_d(D_1)$, according to (6.14), the set Θ_d in (6.14) is given by

$$\Theta_d(x) = [0, \sqrt{2E_{\max}} + e_2x_2].$$

In fact, Θ_d collects all control input values u_d such that $G_1(x, u_d, w_d) \in C \cup \Pi_d(D_{u,w})$ for all $w_d \in [e_1, e_2]$; i.e., every such u_d is such that $E(0, G_1(x, u_d, e_2)) \leq E_{\max}$. Then, since $M_d = \{0\} \times [-\sqrt{2E_{\max}}, -\sqrt{\gamma h_{\min}}]$ and $\Psi_d^u(x) = \mathcal{U}_d$, 6.3.3.1) in Theorem 6.3.3 holds for $\mathcal{H}_{u,w}$. Now, consider the constant $r^* = -\gamma(h_{\min} - \varepsilon)$ and the function ρ_d defined as $\rho_d(x) = \gamma\varepsilon$ for every $x \in L_V(r)$. We show that the pair (V, r^*) defines a RCLF-FI as in Definition 6.3.1. First, (5.11) and (5.7) hold on C since, for every $x \in C$,

$$\langle \nabla V(x), f(x) \rangle = -x_2(-\gamma) - \gamma x_2 = 0. \quad (7.57)$$

Then, we show the pair (V, r^*) is such that (6.18) holds for constant $r = -\gamma h_{\min} < r^*$. Moreover, for every $x \in L_V(r) \cap \Pi_d(D_1)$, we have

$$\begin{aligned} & \min_{u_d \in \Theta_d(x)} \max_{w_d \in [e_1, e_2]} V(G_1(x, u_d, w_d)) \\ &= \min_{u_d \in \Theta_d(x)} \max_{w_d \in [e_1, e_2]} \left\{ -\frac{(u_d - w_d x_2)^2}{2} \right\} \\ &= -\frac{(\sqrt{2E_{\max}} + e_2 x_2 - e_1 x_2)^2}{2} \end{aligned}$$

Since $x_2 \in [-\sqrt{2E_{\max}}, -\sqrt{2\gamma h_{\min}}]$ and condition (7.56), we have

$$\begin{aligned} & \min_{u_d \in \Theta_d(x)} \max_{w_d \in [e_1, e_2]} V(G_1(x, u_d, w_d)) + \rho_d(x) \\ & \leq -\frac{(\sqrt{2E_{\max}} + (e_2 - e_1)(-\sqrt{2E_{\max}}))^2}{2} + \rho_d(x) \\ & = -\frac{(1 + e_1 - e_2)^2}{2} E_{\max} + \gamma\varepsilon \leq -\gamma h_{\min} = r \end{aligned}$$

For every $x \in L_V(r) \cap \Pi_d(D_2)$, we have $x_2 \in [0, v_{\max}]$ and

$$\min_{u_d \in \Theta_d(x)} \max_{w_d \in [e_1, e_2]} V(G_2(x)) = -\frac{(\min\{-e_p x_2, -\delta_p\})^2}{2} - \gamma h_{\max} < r. \quad (7.58)$$

Hence, the pair (V, r^*) defines a robust control Lyapunov function for forward invariance for $\mathcal{H}_{u,w}$ according to Remark 6.3.7 and Definition 6.3.1.

Next, following the steps in Section 6.2.1, we construct the regulation map Γ_d . Since there is no control input during flows, we omit defining Γ_c . Moreover, the input u_d is only active when $(x, u_d, w_d) \in D_1$, we define the selection map Γ_d based on G_1 only. Then, for $r = -\gamma h_{\min}$ and for every $(x, u_d) \in \{(x, u_d) \in \mathbb{R}^2 \times \mathcal{U}_d : (x, u_d, w_d) \in (L_V(r) \times \mathcal{U}_d \times \mathcal{W}_d) \cap D_1\}$, with $\sigma = \frac{1}{2}$, Γ_d is given by

$$\begin{aligned} \Gamma_d(x, u_d) &= \max_{w_d \in [e_1, e_2]} V(G_1(x, u_d, w_d)) + \frac{\rho_d(x)}{2} - r \\ &= -\frac{(u_d - e_1 x_2)^2}{2} + \gamma \left(\frac{\varepsilon}{2} + h_{\min} \right). \end{aligned}$$

Item 6.3.3.2) in Theorem 6.3.3 holds since the function $u_d \mapsto \Gamma_d(x, u_d)$ is convex on $\Theta_d(x)$ for each $x \in M_d$. For each $x \in \mathbb{R}^2$, the map S_d in (6.22) is given by

$$S_d(x) := \begin{cases} \{u_d \in \Theta_d(x) : \gamma(\frac{\varepsilon}{2} + h_{\min}) - \frac{(u_d - e_1 x_2)^2}{2} < 0\} & \text{if } x \in L_V(r) \cap \Pi_d(D_1), \\ \mathbb{R}^{m_d} & \text{otherwise.} \end{cases} \quad (7.59)$$

In addition, $\mathcal{H}_{u,w}$ given in (7.54) satisfies conditions (A1'_w) - (A3'_w) in Lemma 6.0.6. According to Theorem 6.3.3, there exists a state feedback $\kappa_d : \mathbb{R}^2 \rightarrow \mathbb{R}$ that is continuous on M_d . In particular, such a feedback is selected from the closure of

map S_d given in (7.59), which reduces to an interval:

$$\overline{S}_d(x) := \left[\max \left\{ \sqrt{2\gamma \left(\frac{\varepsilon}{2} + h_{\min} \right)} + e_1 x_2, 0 \right\}, \sqrt{2E_{\max}} + e_2 x_2 \right]. \quad (7.60)$$

One such continuous selection is

$$\kappa_d(x) := \sqrt{\frac{\gamma \left(\frac{\varepsilon}{2} + h_{\min} \right)}{E_{\max}}} x_2 + \sqrt{2\gamma \left(\frac{\varepsilon}{2} + h_{\min} \right)}. \quad (7.61)$$

Since Corollary 6.1.5 provides conditions guaranteeing robust controlled forward invariance for hybrid systems without a Lyapunov function, we verify that our design of κ_d in (7.61) indeed renders \mathcal{M}_r^w robustly controlled forward invariant for $\mathcal{H}_{u,w}$. To this end, first, \mathcal{M}_r^w is a subset of $C \cup \Pi_d(D_{u,w})$, F is Lipschitz and $F(x)$ is convex on C by construction and 6.1.5.4) holds since $\mathcal{M}_r^w \cap C$ is compact. Then, item 6.1.5.1) and 6.1.5.5) hold true trivially; while item 6.1.5.3) holds since (7.57) and item 1) of Lemma A.0.15. Finally, for the closed-loop system with u_d replaced by κ_d in (7.61), we check the extreme cases for every $x \in \mathcal{M}_r^w \cap \Pi_d(D_1)$ and every $x \in \mathcal{M}_r^w \cap \Pi_d(D_2)$. More precisely, the worst case for impact with zero height is $x_2 = -\sqrt{2\gamma h_{\min}}$ before the impact and x_2 is updated by map $G_1(x, \kappa_d(x), e_1)$, i.e.,

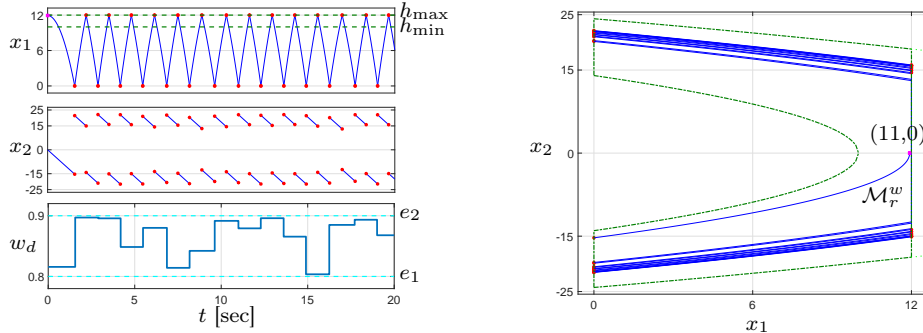
$$\begin{aligned} G_1(x, \kappa_d(x), e_1) &= \sqrt{\frac{\gamma \left(\frac{\varepsilon}{2} + h_{\min} \right)}{E_{\max}}} x_2 + \sqrt{2\gamma \left(\frac{\varepsilon}{2} + h_{\min} \right)} - e_1 x_2 \\ &= \sqrt{2\gamma \left(\frac{\varepsilon}{2} + h_{\min} \right)} + \left(\sqrt{\frac{\gamma \left(\frac{\varepsilon}{2} + h_{\min} \right)}{E_{\max}}} - e_1 \right) (-\sqrt{2\gamma h_{\min}}), \end{aligned}$$

which is greater than $\sqrt{2\gamma \left(\frac{\varepsilon}{2} + h_{\min} \right)}$ since $\sqrt{\gamma \left(\frac{\varepsilon}{2} + h_{\min} \right)} < e_1 \sqrt{E_{\max}}$. Then, 6.1.5.2) holds for every $x \in \mathcal{M}_r^w \cap \Pi_d(D_2)$ since (7.58).

Simulations are generated to show solutions to $\mathcal{H}_{u,w}$ controlled by κ_d in (7.61) with system parameters $\gamma = 9.81, h_{\min} = 10, h_{\max} = 12, v_{\max} = 6\sqrt{\gamma}, e_1 = 0.8, e_2 = 0.9, e_p = 0.95, \varepsilon = 0.1$, and $\delta_p = 0.01$.¹¹ Over the simulation horizon, the disturbance w_d is randomly generated within interval $[e_1, e_2]$ after each impact. One solution that starts from the initial condition for x given by (11, 0)

¹¹All simulations in this section are generated via the Hybrid Equations (HyEQ) Toolbox for MATLAB; see [90]. Code available at <https://github.com/HybridSystemsLab/InvariantBouncingBall>.

is shown in Figure 7.19. Figure 7.19(a) presents the randomly generated w_d disturbance for $\mathcal{H}_{u,w}$. Moreover, even under uncertain disturbances, the peaks of the resulting height reach values larger than h_{\min} and smaller than h_{\max} as Figure 7.19(a) shows. Figure 7.19(b) shows, on the (x_1, x_2) plane, that solution stays within the set \mathcal{M}_r^w , which is the region bounded by dark green dashed line.



(a) Ball position and velocity and w_d . (b) Solution on the (x_1, x_2) plane.

Figure 7.19: Simulation of $\mathcal{H}_{u,w}$ controlled by κ_d in (7.61).

Next, using the results in Section 6.3.2, a control law with minimum point-wise norm rendering the set \mathcal{M}_r^w in (7.55) robustly controlled forward invariant for $\mathcal{H}_{u,w}$ is provided. Such a feedback is given by

$$\kappa_d^m(x) = \arg \min_{u_d \in \overline{S_d(x)}} |u_d|,$$

where $\overline{S_d(x)}$ is as in (7.60) and it leads to the continuous feedback law

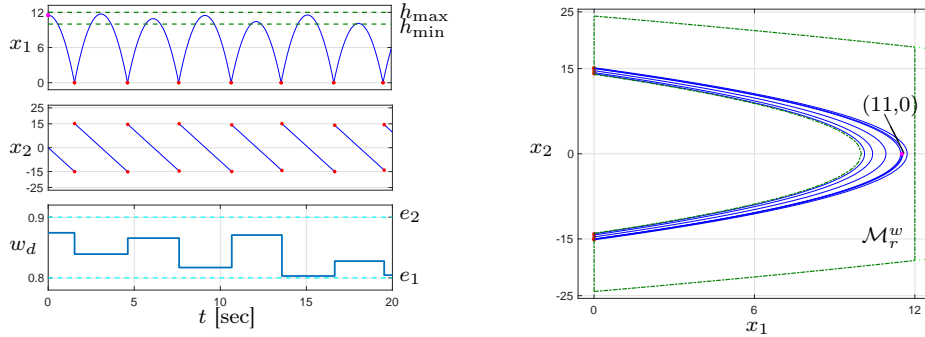
$$\kappa_d^m(x) = \max \left\{ \sqrt{2\gamma \left(\frac{\varepsilon}{2} + h_{\min} \right)} + e_1 x_2, 0 \right\}, \quad (7.62)$$

for every $x \in \mathcal{M}_r^w \cap \Pi_d(D_1)$. Following same steps as above, it can be shown that \mathcal{M}_r^w in (7.55) is robustly controlled forward invariant for $\mathcal{H}_{u,w}$ via κ_d^m applying Corollary 6.1.5.

Simulations are generated for $\mathcal{H}_{u,w}$ controlled by κ_d^m given as in (7.62) with the same system settings as above. One solution that starts from the same initial condition $x = (11, 0)$ is shown in Figure 7.20. As shown in Figure 7.20(a), the peaks of the height in between impacts are between $h_{\min} = 10$ and $h_{\max} = 12$,

while on the (x_1, x_2) plane, the trajectory stays within the set \mathcal{M}_r^w , which is the region bounded by dark green dashed lines.

As expected, compared to Figure 7.19(a), we observe that there are only 7 impacts with the controlled surface within the time span of 0 to 20 seconds in Figure 7.20(a); while there are 14 impacts in Figure 7.19(a) and every impact is followed with a pull. This indicates that less energy is used to bounce the ball at the controlled surface to maintain peak position within range $[h_{\min}, h_{\max}]$.



(a) Ball position and velocity and w_d . (b) Solution on the (x_1, x_2) plane.

Figure 7.20: Simulation of $\mathcal{H}_{u,w}$ controlled by κ_d^m in (7.62).

7.5 An Estimate of Weakly Forward Invariant Sets using Lyapunov-like Functions

We dedicate results in this section to applying

In [49, Section IV], the authors introduce the concept of viability kernel (respectively, invariance kernel) for a given set that is not viable (respectively, invariant) to the given hybrid system in the impulse differential inclusions framework. An iterative algorithm to find such set(s) is presented by using a specifically defined set operation. For the purpose of invariant-based hybrid controller design, we adapt the concept of viability kernel and invariance kernel for estimating the weakly forward invariant set to a given hybrid system in (2.1). To this end, first,

we characterize the forward invariance properties of sets that are sublevel sets of Lyapunov-like functions.

Proposition 7.5.1 (*Forward pre-Invariance of Sublevel Sets*) Consider the hybrid system $\mathcal{H} = (C, F, D, G)$ in (2.1). Let $c \geq 0$ and $W : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuously differentiable on an open set containing¹² $\overline{C} \cap L_W(c)$ and such that it satisfies

$$\langle \nabla W(x), \eta \rangle \leq 0 \quad \forall x \in C \cap L_W(c), \eta \in F(x), \quad (7.63)$$

$$W(\eta) - W(x) \leq 0 \quad \forall x \in D \cap L_W(c), \eta \in G(x), \quad (7.64)$$

$$G(x) \subset \mathcal{M}_r \quad \forall x \in D \cap L_W(c), \quad (7.65)$$

where $\mathcal{M}_r = L_W(c) \cap (C \cup D)$. In addition, let the set \mathcal{M}_r and $\tilde{\mathcal{H}} = (\mathcal{M}_r \cap C, F, D \cap \mathcal{M}_r, G)$ satisfy Assumption 4.1.1. Then, the set \mathcal{M}_r is forward pre-invariant for $\tilde{\mathcal{H}}$.

Proof Let $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{M}_r)$. Pick any $(t, j) \in \text{dom } \phi$ and let $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_{j+1} = t$ satisfy

$$\text{dom } \phi \cap ([0, t] \times \{0, 1, \dots, j\}) = \bigcup_{k=0}^j ([t_k, t_{k+1}] \times \{k\}).$$

For every $k \in \{0, \dots, j\}$ and almost all $s \in [t_k, t_{k+1}]$, $\phi(s, k) \in C \cap L_W(c)$. Then, (7.63) implies that, for each $k \in \{0, 1, \dots, j\}$ and for almost all $s \in [t_k, t_{k+1}]$,

$$\frac{d}{ds} W(\phi(s, k)) \leq 0.$$

Integrating both sides, we have that for each $k \in \{0, 1, \dots, j\}$

$$W(\phi(t_{k+1}, k)) \leq W(\phi(0, k)). \quad (7.66)$$

Similarly, for each $k \in \{1, 2, \dots, j\}$, we have $\phi(t_k, k-1) \in D \cap L_W(c)$. Thus, we obtain from (7.64) that for each $k \in \{0, 1, \dots, j\}$

$$W(\phi(t_k, j)) \leq W(\phi(t_k, 0)). \quad (7.67)$$

With $W(\phi(0, 0)) \leq c$, (7.66) and (7.67) yield

$$W(\phi(t, j)) \leq W(\phi(0, 0)) \leq c$$

¹²The c -sublevel set of the function $W : \mathbb{R}^n \rightarrow \mathbb{R}$ is denoted by $L_W(c) = \{x : W(x) \leq c\}$.

for all $(t, j) \in \text{dom } \phi$. Consequently, every solution $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{M}_r)$ stays in $L_W(c)$ for all $(t, j) \in \text{dom } \phi$. In addition, solution ϕ is not allowed to jump outside of D because of the assumption on G in (7.65). Also, by (S1) of Definition 2.0.4, ϕ does not escape \bar{C} by flow. Thus, ϕ stays in $\mathcal{M}_r = L_W(c) \cap (\bar{C} \cup D)$ for all $(t, j) \in \text{dom } \phi$. In other words, \mathcal{M}_r is forward pre-invariant for the system $\tilde{\mathcal{H}} = (\mathcal{M}_r \cap C, F, \mathcal{M}_r \cap D, G)$. \square

Proposition 7.5.1 establishes a forward pre-invariance property of sublevel sets of Lyapunov-like functions for a modified version of a hybrid system \mathcal{H} , namely $\tilde{\mathcal{H}}$. In particular, $\tilde{\mathcal{H}}$ has the same flow and jump map as the original system \mathcal{H} , but its flow set and jump set are intersected by the sublevel set $L_W(c)$. We provide sufficient conditions for the set \mathcal{M}_r to be forward invariant for $\tilde{\mathcal{H}}$ by applying Theorem 4.1.4.

Proposition 7.5.2 (*Forward Invariance of a Sublevel Set for $\tilde{\mathcal{H}}$*) *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$, $c \geq 0$, and W (as well as \mathcal{M}_r) be such that the conditions in Proposition 7.5.1 hold. Then, the set \mathcal{M}_r is forward invariant for $\tilde{\mathcal{H}} = (C \cap \mathcal{M}_r, F, D \cap \mathcal{M}_r, G)$ if at least one of the following condition holds:*

- *For every $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{M}_r)$, case (b.2) in Proposition 2.0.6 does not hold;*
- *Either $\mathcal{M}_r \cap C$ is compact or F is bounded on $\mathcal{M}_r \cap C$.*

Proof Proposition 7.5.1 implies that the set \mathcal{M}_r is forward pre-invariant for the hybrid system $\tilde{\mathcal{H}}$, i.e, for each $x \in \mathcal{M}_r$, every maximal solution $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{M}_r)$ stays in \mathcal{M}_r for all hybrid time. Thus, by proving that all maximal solutions are complete, we complete the proof.

Since $G(D \cap L_W(c)) \subset \mathcal{M}_r$, case (c) in Proposition 2.0.6 does not hold. Then, the tangent cone condition guarantees solutions will not end on the boundary of $\mathcal{M}_r \cap C$ by flow, thus, case (b.1) in Proposition 2.0.6 does not hold.

In addition, with given assumptions on $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{M}_r)$, i.e, case (b.2) in Proposition 2.0.6 does not hold, we conclude that all solutions $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{M}_r)$ are complete.

Therefore, \mathcal{M}_r is forward invariant for $\tilde{\mathcal{H}}$. When $\mathcal{M}_r \cap C$ is compact or F is bounded on $\mathcal{M}_r \cap C$, solutions of $\tilde{\mathcal{H}}$ do not escape to infinity in finite time for all $x \in \mathcal{M}_r \cap C$, which excludes the case (b.2) in Proposition 2.0.6. Therefore, it also leads to completeness of all solutions $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{M}_r)$ from every point in \mathcal{M}_r to $\tilde{\mathcal{H}}$. \square

In addition to the results from Proposition 7.5.1, Proposition 7.5.2 states that if every solution $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{M})$ is complete, the set \mathcal{M} is forward invariant for the modified hybrid system $\tilde{\mathcal{H}}$. With these results, we provide a result that can be used to estimate weakly forward invariant sets of the original hybrid system \mathcal{H} .

Theorem 7.5.3 (*Weak forward invariance of a set for \mathcal{H}*) Consider the hybrid system \mathcal{H} in (2.1). For each $i \in \{1, 2, \dots, N\}$, let c_i and \mathcal{M}_i satisfy the conditions in Proposition 7.5.2 some function W_i . Then, the set

$$\mathcal{W} = \bigcup_{i \in \{1, 2, \dots, N\}} \mathcal{M}_i$$

is weakly forward invariant for \mathcal{H} .

Proof For each i , the set \mathcal{M}_i is forward invariant for $\tilde{\mathcal{H}}_i$, which implies that, for every $x \in \mathcal{M}_i$, every solution $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}_i}(x)$ is such that $\text{rge } \phi \subset \mathcal{M}_i$. Because this property holds for every $\mathcal{M}_i, i \in \{1, 2, \dots, N\}$, we know that there exists at least one solution $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{W})$ that is complete and $\text{rge } \phi \subset \mathcal{W}$, where $\tilde{\mathcal{H}} = (\mathcal{W} \cap C, F, \mathcal{W} \cap D, G)$. This property extends to the original hybrid system \mathcal{H} . In fact, the data of \mathcal{H} has $\tilde{C} = \mathcal{W} \cap C \subset C$ and $\tilde{D} = \mathcal{W} \cap D \subset D$, and share the same flow map F and jump map G with $\tilde{\mathcal{H}}$. As a result, the existing (complete) solutions $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathcal{W})$ are also solutions to the original system \mathcal{H} . Therefore, there exists at least one complete solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{W})$ that satisfies $\text{rge } \phi \subset \mathcal{W}$ for every $x \in \mathcal{W}$, i.e., \mathcal{W} is weakly forward invariant for \mathcal{H} . \square

Remark 7.5.4 Note that $\mathcal{S}_{\mathcal{H}}(\mathcal{W})$ may include more solutions than $\bigcup_{i \in \{1, 2, \dots, N\}} \mathcal{S}_{\tilde{\mathcal{H}}_i}(\mathcal{M}_i)$, due to $C_i = \mathcal{M}_i \cap C$ and $D_i = \mathcal{M}_i \cap D$ for each i , where $\tilde{\mathcal{H}}_i = (\mathcal{M}_i \cap C, F, \mathcal{M}_i \cap$

D, G). These extra solutions may be allowed to flow or jump outside of \mathcal{W} , therefore, we cannot guarantee forward invariance of the set \mathcal{W} for \mathcal{H} . On the other hand, if every $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{W})$ is unique, $\mathcal{S}_{\mathcal{H}}(\mathcal{W})$ is equal to $\bigcup_{i \in \{1, 2, \dots, N\}} \mathcal{S}_{\tilde{\mathcal{H}}_i}(\mathcal{M}_i)$ and we can conclude that \mathcal{W} is forward invariant for \mathcal{H} .

The following example illustrates Proposition 7.5.1.

Example 7.5.5 (Forward pre-Invariance of \mathcal{M}) Consider the hybrid system $\mathcal{H} = (C, f, D, g)$ in \mathbb{R}^2 given by

$$f(x) := Ax := \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} x \quad \forall x \in C := \mathbb{B},$$

$$g(x) := \begin{cases} 2x & \text{if } x \in D_1 := \{x \in \mathbb{R}^2 : x \notin \mathbb{B}\} \\ -x & \text{if } x \in D_2 := \{x \in \mathbb{R}^2 : x_2 = 0, |x| \leq 1\}, \end{cases} \quad \forall x \in D := D_1 \cup D_2.$$

First, we note that the matrix A is Hurwitz, so the origin is a stable focus, i.e., solutions to $\dot{x} = f(x)$ spiral toward the origin.

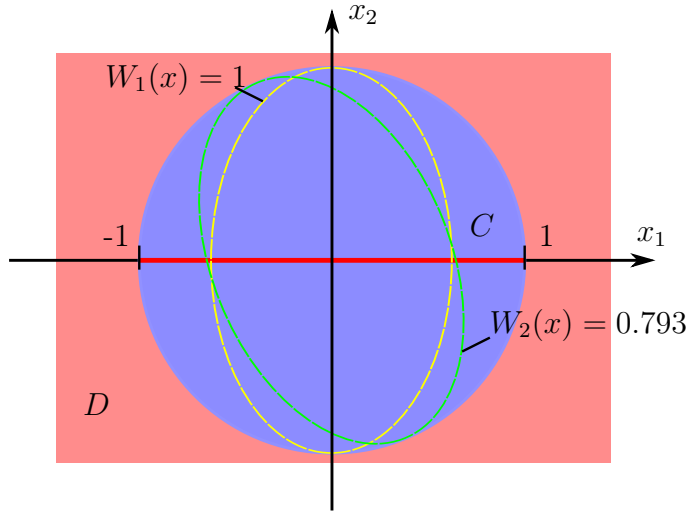


Figure 7.21: Sublevel sets of proposed Lyapunov functions Example 7.5.5.

We consider the function $W_1(x) = x^\top P_1 x$, where $P_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Then, we have,

for each $x \in C$,

$$\begin{aligned}
\langle \nabla W_1(x), f(x) \rangle &= 4x_1(-2x_1 + x_2) + 2x_2(x_1 - 2x_2) \\
&= -8x_1^2 + 6x_1x_2 - 4x_2^2 \\
&= -\frac{23}{4}x_1^2 - 4\left(\frac{3}{4}x_1 - x_2\right)^2,
\end{aligned} \tag{7.68}$$

which is guaranteed to be less than or equal to zero for every $x \in \mathbb{R}^2$. We consider the largest sublevel set of W_1 within $C = \mathbb{B}$, which is $L_{W_1}(c_1)$ with $c_1 = 1$ and is shown as yellow dashed line in Figure 7.21. In addition, $g(x) = -x$ gives $W_1(g(x)) - W_1(x) = 0$ for every $x \in L_{W_1}(c_1) \cap D$. Thus, according to Proposition 7.5.1, $\mathcal{M}_1 = L_{W_1}(c_1)$ is forward pre-invariant for $\tilde{\mathcal{H}}_1 = (\mathcal{M}_1 \cap C, f, \mathcal{M}_1 \cap D, g)$.

Now we apply Theorem 4.1.4 to check the forward pre-invariance property. By observing system data, we know $\mathcal{M}_1 \cap D = \{x : x_1 \in [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}], x_2 = 0\}$, and for every $x \in \mathcal{M}_1 \cap D$, $G(x) = -x$, therefore, $g(\mathcal{M}_1 \cap D) = \mathcal{M}_1 \cap D$, i.e., item 4.1) in Theorem 4.1.4 holds. Next, for closed set $\mathcal{M}_1 \cap C = L_{W_1}(1)$, the tangent cone $T_{\mathcal{M}_1 \cap C}(x) = \mathbb{R}^2$ for every interior point in $L_{W_1}(1)$, i.e., $f(x) \in T_{\mathcal{M}_1 \cap C}(x)$; also, $T_{\mathcal{M}_1 \cap C}(x)$ includes all vectors that are tangent to or pointing inward to the level set $W_1(x) = c_1$ for every point on the boundary of $L_{W_1}(1)$, the result in (7.68) implies $f(x) \in T_{\mathcal{M}_1 \cap C}(x)$ for these boundary points, and item 4.2) in Theorem 4.1.4 holds. Thus, \mathcal{M}_1 is forward pre-invariant for $\tilde{\mathcal{H}}_1$.

Similarly, we consider the function $W_2(x) = x^\top P_2 x$, where $P_2 = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 1 \end{bmatrix}$.

Then, we have, for each $x \in C$,

$$\begin{aligned}
\langle \nabla W_2(x), f(x) \rangle &= (4x_1 + x_2)(-2x_1 + x_2) + (2x_2 + x_1)(x_1 - 2x_2) \\
&= -7x_1^2 - 3x_2^2 + 2x_1x_2 \\
&= -6x_1^2 - 2x_2^2 - (x_1 - x_2)^2,
\end{aligned}$$

which is guaranteed to be less than or equal to zero for all points on \mathbb{R}^2 . Again, we consider the largest sublevel set for W_2 within $C = \mathbb{B}$, which is $L_{W_2}(c_2)$ with

$c_2 \approx 0.793$ and is shown as green dash line in Figure 7.21. Similar to the case for W_1 , $g(x) = -x$ gives $W_2(g(x)) - W_2(x) = 0$ for every $x \in L_{W_2}(c_2) \cap D$. Thus, according to Proposition 7.5.1, $\mathcal{M}_2 = L_{W_2}(c_2)$ is forward pre-invariant for $\tilde{\mathcal{H}}_2 = (\mathcal{M}_2 \cap C, f, \mathcal{M}_2 \cap D, g)$. This property can be verified using Theorem 4.1.4.

Following the same procedures, we can find more forward pre-invariant sets \mathcal{M} to the corresponding $\tilde{\mathcal{H}}$ based on different Lyapunov-like functions that satisfy the conditions in (7.63) and (7.64). △ △

In the following example, we explain the importance of (7.65) in Proposition 7.5.1.

Example 7.5.6 (Data restrictions on system in Proposition 7.5.1) Consider the hybrid system $\mathcal{H}_1 = (C_1, f_1, D_1, g_1)$ in \mathbb{R} given by

$$f_1(x) := -x \quad \forall x \in C_1 := \{1\}, \quad g_1(x) := 0 \quad \forall x \in D_1 := \{1\}.$$

We use quadratic function $W_1(x) = x^2$. It follows that $W_1, c_1 = 1$ and given system \mathcal{H}_1 satisfies conditions in (7.63) and (7.64) since for $x = 1$, $\langle \nabla W_1(x), f_1(x) \rangle = -2x < 0$ and $W_1(g_1(x)) - W_1(x) = -1 < 0$. However, because $g_1(x) = 0$ and $\mathcal{M}_1 = \{1\}$, system \mathcal{H}_1 does not satisfy condition (7.65). As a result, the only nontrivial solution $\phi_1(0, 0) = 1, \phi_1(0, 1) = 0$ jumps outside of \mathcal{M}_1 , i.e, jumps inside $L_{W_1}(c_1)$, but outside of $C_1 \cup D_1$. Thus, the set \mathcal{M}_1 is not forward pre-invariant for \mathcal{H}_1 . Therefore, without guaranteeing (7.65), we cannot conclude forward pre-invariance of \mathcal{M}_1 for $\tilde{\mathcal{H}}_1$.

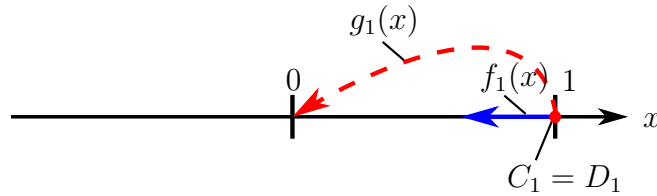


Figure 7.22: System \mathcal{H}_1 in Example 7.5.6.

Then, consider the hybrid system \mathcal{H}_2 in \mathbb{R}^2 given by

$$\begin{aligned} f_2(x) &:= [-x_1 \ -x_2]^\top & \forall x \in C_2 &:= [-2, 2] \times [-2, 2] \setminus \{x \in \mathbb{R}^2 : x^2 < 1\}; \\ g_2(x) &:= [0 \ 0]^\top & \forall x \in D_2 &:= \{x \in \mathbb{R}^2 : x^2 = 1\}. \end{aligned}$$

We choose quadratic function $W_2(x) = x^2$. It follows that $W_2, c_2 = 4$ and given system \mathcal{H}_2 satisfies conditions in (7.63) and (7.64) since for every $x \in C \cap L_{W_2}(c_2)$, $\langle \nabla W_2(x), f_2(x) \rangle = -2(x_1^2 + x_2^2) \leq -2$, and for every $x \in D \cap L_{W_2}(c_2)$, $W_2(g_2(x)) - W_2(x) = 0 - (x_1^2 + x_2^2) = -1 < 0$. However, because $g_2(x) = [0 \ 0]^\top$ and the origin $x = (0, 0) \notin \mathcal{M}_2$, system \mathcal{H}_2 does not satisfy condition (7.65). As a result, all maximal solution $\phi \in \mathcal{S}_{\mathcal{H}_2}(\mathcal{M}_2)$ jump outside of \mathcal{M}_2 , i.e, jump inside $L_{W_2}(c_2)$, but outside of $C_2 \cup D_2$. Thus, set \mathcal{M}_2 is not forward pre-invariant for \mathcal{H}_2 . Similar as the one dimension case above, the jump map G_2 mapped solution out of \mathcal{M}_2 , while (7.63) and (7.64) are satisfied. This is because the Lyapunov like function condition only stated the fact that solution will jump towards smaller sublevel sets, which will not lead to weak forward pre-invariance only, when the smaller sublevel sets are excluded from the interest set \mathcal{M}_2 .

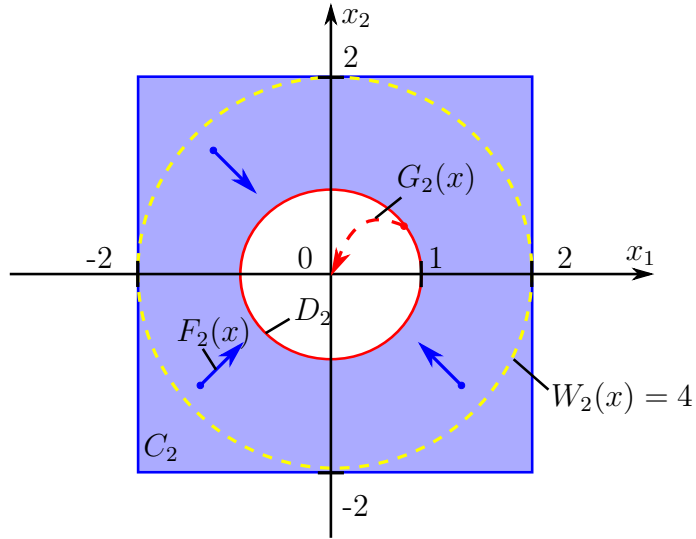


Figure 7.23: System \mathcal{H}_1 in Example 7.5.6.

△

Proposition 7.5.1 establishes a forward pre-invariance property of sublevel sets of Lyapunov-like functions for a modified version of a hybrid system \mathcal{H} , namely $\tilde{\mathcal{H}}$. In particular, $\tilde{\mathcal{H}}$ has the same flow and jump map as the original system \mathcal{H} , but its flow set and jump set are intersected by the sublevel set $L_W(c)$. We provide sufficient conditions for the set \mathcal{M} to be forward invariant for $\tilde{\mathcal{H}}$ by applying Theorem 4.1.4.

An example illustrating Theorem 7.5.3 is presented next.

Example 7.5.7 (Estimating Weakly Forward Invariant Set) *Consider the hybrid system $\mathcal{H} = (C, F, D, g)$ in \mathbb{R}^2 given by*

$$F(x) := \begin{cases} \begin{bmatrix} -x_2 \\ x_1 - 0.5 \end{bmatrix} & \text{if } x_1 > 0 \\ [0 \ -0.5]^\top & \text{if } x_1 = 0 \\ \begin{bmatrix} -x_2 \\ x_1 + 0.5 \end{bmatrix} & \text{if } x_1 < 0 \end{cases} \quad \forall x \in C := ((0, 0.5) + 0.5\mathbb{B}) \cup ((0, -0.5) + 0.5\mathbb{B})$$

$$g(x) := \begin{cases} \begin{bmatrix} 1 - x_1 \\ 0 \end{bmatrix} & \text{if } x \in D_1 \\ \begin{bmatrix} -1 - x_1 \\ 0 \end{bmatrix} & \text{if } x \in D_2 \end{cases} \quad \forall x \in D := D_1 \cup D_2,$$

where $D_1 := \{x \in \mathbb{R}^2 : x_2 = 0, x_1 \geq 0.5\}$ and $D_2 := \{x \in \mathbb{R}^2 : x_2 = 0, x_1 \leq -0.5\}$.

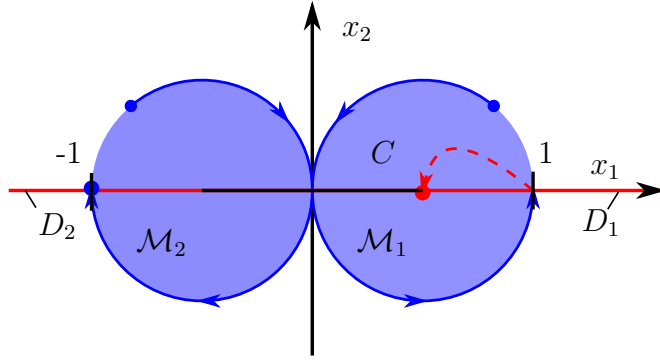


Figure 7.24: Possible solution trajectories configuration of Example 7.5.7.

It is not possible to include every point in C using a single sublevel set of a Lyapunov like function. However, it is possible to use two different functions W_1 and W_2 such that every point within C is captured in the union of two sublevel sets. We propose two candidates $W_1(x) = (x_1 - 0.5)^2 + x_2^2$ and $W_2(x) = (x_1 + 0.5)^2 + x_2^2$. For each $x \in \{x \in C : x_1 > 0\}$, the inner product between F and ∇W_1 is

$$\langle \nabla W_1(x), F(x) \rangle = (2x_1 - 1)(-x_2) + 2x_2(x_1 - 0.5) = 0;$$

for each $x \in \{x \in C : x_1 < 0\}$,

$$\langle \nabla W_2(x), F(x) \rangle = (2x_1 + 1)(-x_2) + 2x_2(x_1 + 0.5) = 0;$$

and at the origin: $\langle \nabla W_1(x), F(x) \rangle = \langle \nabla W_2(x), F(x) \rangle = 0$.

Then, we check W at jumps. For every point in D_1 we have

$$W_1(g(x)) - W_1(x) = (0.5 - x_1)^2 + x_2^2 - (x_1 - 0.5)^2 - x_2^2 = 0,$$

and for every point in D_2 , we have

$$W_2(g(x)) - W_2(x) = (-0.5 - x_1)^2 + x_2^2 - (x_1 - 0.5)^2 - x_2^2 = 0.$$

We choose $\mathcal{M}_1 = L_{W_1}(c_1)$ and $\mathcal{M}_2 = L_{W_2}(c_2)$, which are subsets of C , for \mathcal{H} with $c_1 = c_2 = 1$. Then, according to Proposition 7.5.1, \mathcal{M}_1 is forward pre-invariant for $\tilde{\mathcal{H}}_1 = (\mathcal{M}_1 \cap C, F, \mathcal{M}_1 \cap D, g)$, and \mathcal{M}_2 is forward pre-invariant for $\tilde{\mathcal{H}}_2 = (\mathcal{M}_2 \cap C, F, \mathcal{M}_2 \cap D, g)$. We verify that \mathcal{M}_1 and \mathcal{M}_2 are forward invariant for $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$, respectively, by applying Proposition 7.5.2. According to the data

of $\tilde{\mathcal{H}}_i, i \in \{1, 2\}$, solutions to $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$ can always be extended, respectively, by either flowing or jumping on \mathcal{M}_1 and \mathcal{M}_2 , respectively.

In addition, as shown in Figure 7.24, solutions starting from the origin ($x = 0$) can either flow into the left circle or right circle according to F . Therefore, we know neither \mathcal{M}_1 nor \mathcal{M}_2 is forward invariant set for the given \mathcal{H} . On the other hand, Theorem 7.5.3 implies that the sets $\mathcal{M}_1, \mathcal{M}_2$, and $\mathcal{W} = \mathcal{M}_1 \cup \mathcal{M}_2$ are weakly forward invariant for \mathcal{H} . △ △

Without completeness of each $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}_i}(\mathcal{M}_i)$ for every $i \in \{1, \dots, N\}$, when extending the solutions to the original system \mathcal{H} , $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{W})$ may be allowed to flow outside of \mathcal{W} due to the changes of $\text{dom } F$ and (or) $\text{dom } G$. An example to illustrate the idea is as follow.

Example 7.5.8 Consider hybrid system $\mathcal{H} = (C, f, D, g)$ in \mathbb{R} given by

$$f(x) := 1 \quad \forall x \in C := \mathbb{R}, \quad g(x) = 0 \quad \forall x \in D := \emptyset.$$

Consider sets $\mathcal{M}_1 = [0, 1]$ and $\mathcal{M}_2 = [1, 2]$. \mathcal{M}_1 is forward pre-invariant for $\tilde{\mathcal{H}}_1 = (\mathcal{M}_1 \cap C, f, \mathcal{M}_1 \cap D, g)$ and \mathcal{M}_2 is forward pre-invariant for $\tilde{\mathcal{H}}_2 = (\mathcal{M}_2 \cap C, f, \mathcal{M}_2 \cap D, g)$. By observing, we know solutions to $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$ are not complete, but rather stop at the right boundary of \mathcal{M}_1 and \mathcal{M}_2 . Since $f(x)$ for \mathcal{H} are allowed to flow continuously on \mathbb{R} , set $\mathcal{W} = \mathcal{M}_1 \cup \mathcal{M}_2$ is not weakly forward invariant for \mathcal{H} . △ △

Next, we propose an alternative construction for the restricted hybrid system in Proposition 7.5.1 that does not require (7.65) to hold for the jump map G and jump set D .

Given $\mathcal{H} = (C, F, D, G)$ in \mathbb{R}^n and $\mathcal{M} \subset \mathbb{R}^n$, let the hybrid system $\tilde{\mathcal{H}} = (\tilde{C}, F, \tilde{D}, \tilde{G})$ be given by

$$\tilde{\mathcal{H}} \begin{cases} x \in \tilde{C} & \dot{x} \in F(x) \\ x \in \tilde{D} & x^+ \in \tilde{G}(x), \end{cases} \quad (7.69)$$

where $\tilde{C} := C \cap \mathcal{M}$, $\tilde{D} := \text{dom } \tilde{G}$, and $\tilde{G}(x) := \begin{cases} G(x) & \text{if } G(x) \cap \mathcal{M} \neq \emptyset; \\ \emptyset & \text{otherwise} \end{cases} \quad \forall x \in \mathbb{R}^n$.

Lemma 7.5.9 *If $C \subset \text{dom } F$ and $D \subset \text{dom } G$, then $\tilde{C} \subset \text{dom } F$ and $\tilde{D} \subset \text{dom } \tilde{G}$.*

Proposition 7.5.10 *Let $W : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be continuous and $c \geq 0$. Pick $\mathcal{M} = \{x \in \mathbb{R}^n : W(x) \leq c\}$ and let the data of $\tilde{\mathcal{H}}$ in (7.69) be generated with such a choice of \mathcal{M} . Define $K := \mathcal{M} \cap (\tilde{C} \cup \tilde{D})$. If K is forward invariant for $\tilde{\mathcal{H}}$ then it is also weakly forward invariant for \mathcal{H} .*

Proof From the definition of K , we know that $K \subset \tilde{C} \cup \tilde{D}$. Since $\tilde{C} \subset C$ and $\tilde{D} \subset D$, $K \subset C \cup D$. By definition, K being forward invariant for $\tilde{\mathcal{H}}$ means the following: for every point in K , there exist at least one solution ϕ to $\tilde{\mathcal{H}}$, and all solutions to $\tilde{\mathcal{H}}$ are complete and stay in K for all hybrid time. Because both \mathcal{H} and $\tilde{\mathcal{H}}$ share the same flow map F , and jump set \tilde{G} for $\tilde{\mathcal{H}}$ is given by G on $\{x \in \mathbb{R}^n : G(x) \cap \mathcal{M} \neq \emptyset\}$, we know that $\mathcal{S}_{\tilde{\mathcal{H}}}(K) \subset \mathcal{S}_{\mathcal{H}}(K)$. Thus, for every point in K , there exist at least one complete solution $\phi \in \mathcal{S}_{\mathcal{H}}(K)$ that stays in K for all hybrid time. \square

Remark 7.5.11 *Proposition 7.5.10 only guarantees a weak forward invariant property for the set K . This is because when $\tilde{D} = \{x \in D : G(x) \cap \mathcal{M} \neq \emptyset\}$ and $K \subset \mathcal{M}$, for every $x \in D \setminus \tilde{D}$ we have $G(x) \cap K = \emptyset$. As a result, some $\phi \in \mathcal{S}_{\mathcal{H}}(K)$ will jump out of K by G . In other words, when $(\tilde{C} \setminus \tilde{D}) \cap D \neq \emptyset$, we have $\mathcal{S}_{\mathcal{H}}(K) \setminus \mathcal{S}_{\tilde{\mathcal{H}}}(K) \neq \emptyset$.*

The invariance principle introduced in [61, Theorem 8.2] requires the computation of (the largest) weakly invariant sets (inside some particular set) to characterize the set to which solutions that are bounded and complete converge. Theorem 4.1.2 can be helpful in such computation, in particular, to determine weakly forward invariant sets. The following example illustrates such an application of Theorem 4.1.2.

Example 7.5.12 (Determining Largest Invariant Sets) Consider the hybrid system $\mathcal{H} = (C, f, D, g)$ in \mathbb{R}^2 given by

$$f(x) := \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \quad \forall x \in C := \mathbb{R} \times [0, +\infty),$$

$$g(x) := \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \quad \forall x \in D := \mathbb{R} \times (-\infty, 0].$$

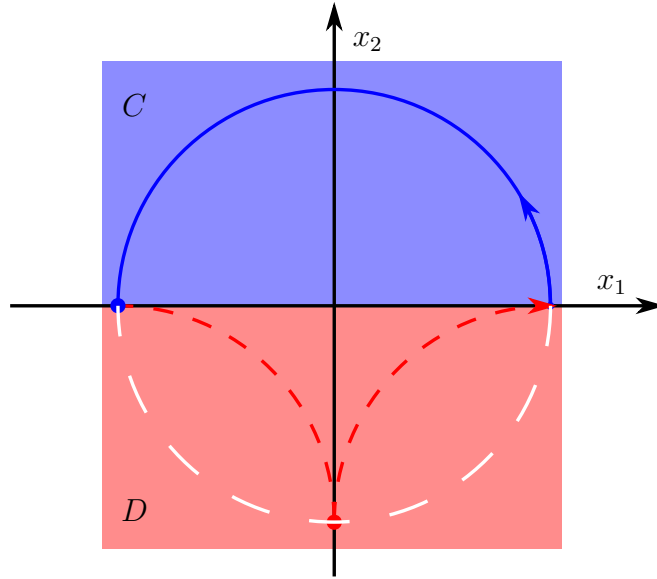


Figure 7.25: Possible solution trajectories configuration of Example 7.5.12.

To determine where solutions to \mathcal{H} converge to, using [61, Theorem 8.2], we take the Lyapunov-like function $W(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$, and define the functions $u_C(x)$ and $u_D(x)$ as

$$u_C(x) := \begin{cases} \langle \nabla W(x), f(x) \rangle = 0 & \text{if } x \in C \\ -\infty & \text{otherwise} \end{cases}$$

$$u_D(x) := \begin{cases} W(g(x)) - W(x) = 0 & \text{if } x \in D \\ -\infty & \text{otherwise} \end{cases}$$

Then, following [61, Theorem 8.2], we compute the zero level set of u_C and u_D

defined above. It follows that

$$u_C^{-1}(0) = \mathbb{R} \times [0, +\infty) \quad u_D^{-1}(0) = \mathbb{R} \times (-\infty, 0].$$

Furthermore, we have

$$g(u_D^{-1}(0)) = [0, +\infty) \times \mathbb{R}.$$

Thus, [61, Theorem 8.2] implies that every maximal solution to \mathcal{H} approaches the largest weakly invariant set given by

$$W^{-1}(r) \cap \mathbb{R}^2 \cap [(\mathbb{R} \times [0, +\infty)) \cup ((-\infty, 0] \times (-\infty, 0])].$$

Then, given an arbitrary choice of r , this set can be rewritten as

$$K = \{x \in \mathbb{R}^2 : |x| = r, x_1 \geq 0\} \cup \{x \in \mathbb{R}^2 : |x| = r, x_2 \geq 0\}.$$

The set K is weakly forward invariant according to Theorem 4.1.2. In fact, condition 3.1) holds since for every point in $K \cap D$, the jump map returns a point in K , and for every point in $K \cap (\overline{C} \setminus D)$ the linear oscillator dynamics permits flowing within the flow set. Condition 3.2) holds due to the properties of the flow map. △

Chapter 8

Conclusion and Future Work

In this dissertation, the forward invariance properties of a set are thoroughly investigated. We present a summary of major content and several potential future research directions in this chapter.

8.1 Conclusion

The hybrid inclusions is a modeling framework that models a wide range of dynamical systems, for example, pure continuous-time systems, pure discrete-time systems, and hybrid systems. This modeling framework also allows for nonlinear set-valued system dynamics, presence of disturbances, and overlaps between the flow and jump sets. Overcoming these complex characteristics, we provide a systematic approach to analyze forward invariance properties of sets and synthesis control designs that induce these important properties.

For starters, notions of nominal and robust forward invariance of sets for autonomous hybrid systems are formally provided. The notions are established based on solution (or solution pair) properties. In particular, we characterize the property of whether a solution (or a solution pair) starts from a set stays in that set for all future time. Moreover, the completeness of solutions (or solution pair) is also considered as key feature to differentiate the notions.

To prepare for the analysis effort in Chapter 4, the result [61, Proposition 2.10] is extended to hybrid systems with disturbances. It provides an in-depth understanding on existence of nontrivial solution pairs to hybrid system $\mathcal{H}_{u,w}$ and their behaviors based on their domains. For each type of forward invariance, a set of sufficient conditions is presented for verifying such property for generic sets. These conditions include a jump related condition, which ensures solutions (or solution pairs) jump back to the set of interest, a flow related condition, which ensures solutions (or solution pairs) flow in the set of interest, and a finite escape time condition, which, together with the two other conditions, leads to completeness of solutions (or solution pairs). Among these conditions, for some of the notions, locally Lipschitzness and a uniform property on disturbances during flows are enforced to guarantee every solution (pair) satisfies the desired invariance property.

When provided a Lyapunov-like function V for the system, conditions presented in Chapter 4 can be modified to derive invariance properties for the sublevel sets of V , namely, \mathcal{M}_r (or \mathcal{M}_r^w). In particular, during flow, solutions (solution pairs) cannot escape the sublevel sets because the non-increasing properties of the Lyapunov-like function V along solutions within a thin band outside of the sublevel sets. Such a property guarantees all solutions initialized within \mathcal{M}_r (or \mathcal{M}_r^w) stay in it by absolute continuity of the solutions (solution pairs) during flow. During jumps, the Lyapunov-like function V ought to remain less than the value that defines the sublevel set such that the solutions (solution pairs) do not jump to higher level sets. Then, we modify a condition from Chapter 4 to ensure solutions (or solution pairs) always jump back to the union of jump and flow sets on the state space. As a result, one achieves forward pre-invariance, which does not require completeness of solutions (or solution pairs), of the sets \mathcal{M}_r (or \mathcal{M}_r^w). In addition, to guarantee existence of nontrivial solutions from every point in the flow set intersected with \mathcal{M}_r (or \mathcal{M}_r^w), a set of conditions that involve the flow set, flow map and the Lyapunov-like function V are derived. These conditions

explore the tangent cone properties of intersected sets, in our case, the flow set and the sublevel set of V . As a result, (robust) forward invariance notions that require completeness of solutions (or solution pairs) are achieved according to the modified result of [61, Proposition 2.10].

Using a variation of proposed Lyapunov conditions for (robust) forward invariance in Chapter 5, control Lyapunov functions are defined for forward invariance purposes, namely, CLF-FI for \mathcal{H}_u and RCLF-FI for $\mathcal{H}_{u,w}$. Then, regulation maps are built to include all possible inputs that induce forward invariance of the sublevel sets of CLF-FI (or RCLF-FI). More precisely, these maps collect all \mathcal{H}_u -admissible (or $\mathcal{H}_{u,w}$ -admissible) inputs during flows and jumps such that the closed-loop systems satisfy the flow and jump conditions derived in Chapter 5. Then, to ensure the closed-loop systems satisfy the hybrid basic conditions for robustness with respect to small state perturbations, continuous control feedback laws are constructed using these regulation maps. The existence of such continuous feedback selections is established by checking a few mild conditions on system data that induce the lower semicontinuity of the regulation maps. Finally, building on the existence results, we present a constructive state-feedback design that features a minimal norm selection scheme.

We also provide several engineering applications to illustrate the analysis and control design tools developed in this dissertation for hybrid systems modeled as hybrid inclusions model. Among these applications, (robust) forward invariance properties are crucial to achieve analysis and control goals for each design problem. In particular, for a DC/AC inverter control design, forward invariance of a band around the reference trajectory on the state space for the closed-loop system implies the resulting solutions evolve near the reference trajectory for all time. Together with a global convergence property, the closed-loop system generates the desired output sinusoidal-like signal. Then, for a DC/DC converter, by rendering forward invariance of a small neighborhood around the set point on the state space, a switching control law is derived to guarantee the DC-like output with

higher voltage. Such a property is also crucial to achieve asymptotic stability of the set for the closed-loop system. Then, state-feedback control laws are derived for a constrained bouncing ball system with uncertain coefficient of restitution, so that the peak height after each bounce remains within a desired range. This is achieved by rendering robust forward invariance of the intersection of the flow set and the sublevel set of a RCLF-FI, which is defined based on the energy level of the bouncing ball system. Finally, we adapt our results to the use of estimating the largest weakly invariant set for invariance principle of hybrid system.

8.2 Future Directions

Forward invariance properties for hybrid dynamical systems have great potential in many motivational applications as presented in Chapter 7. Possible future developments on theoretical analysis and invariance-based controls with applications arises in the following topics.

- **A Differential Game using Forward Invariance of Hybrid Systems**

The pointwise minimal norm selection scheme presented in Section 6.3.2 features a invariance inducing control strategy that requires minimal control effort. In fact, the constructed feedback laws are suboptimal with respect to some meaningful cost function for a zero-sum *hybrid game*. Inspired by [73, Chapter 4.2], the cost function include a quadratic cost component that evaluates the cost during continuous evolution of the solutions for each initial conditions. In addition, a cost component that characterizes the cost during jumps can be established in similar forms.

In [3, Chapter 14], the optimality of a differential game is considered in the sense of “playability” that is derived from forward invariance properties that come from feedback control designs. One can derive the similar concept for the hybrid inclusions.

- **Safety and Control Barrier Functions for Hybrid Systems**

As mentioned repeatedly in this dissertation, forward invariance of sets are widely considered to represent “safety” specifications for control systems. For hybrid automata, Tomlin and co-authors established in-depth work to formulate “safe” sets at forward invariant sets, [8]. Moreover, these “safe” sets are computed numerically via reachability analysis [43, 58]; while controller guaranteeing the “safety” specifications for hybrid automata are designed by solving hybrid games [44].

In addition, extending [9, 38], control barrier functions defined for safety guarantees for hybrid inclusions are under development using the tools proposed in this dissertation. In particular, such functions surpass the traditional control Lyapunov functions in safety applications, since they are often defined by the “boundaries” of the safe sets [46, 47]. A popular approach to solve for control barrier function based designs is to formulate it to quadratic programming problems [39]. Our results for hybrid inclusions can be extended in these directions.

- **Forward Invariance in Model Predictive Control of Hybrid Systems**

Similar to the “playability” in [3, Chapter 14], for model predictive control (MPC), forward invariant sets that satisfy control constraints are considered to be the feasible regions where feedback laws are selected from. For instance, in [6, 11], forward invariant sets, which are characterized by control horizon, prediction horizon and terminal constraint, are used to design a MPC controller for nonlinear systems. In [10] and [12], forward invariant sets featuring “safety” specifications based on environments are used to assess the thread during a semi-autonomous driving scenario and to guarantee safety in traffic networks, respectively.

Motivated by these applications, MPC feasibility problems for a wide class of systems can be formulated as solving the controlled forward invariance sets for hybrid systems. Such an algorithm can be computational challenging due

to the use of tangent cone properties and defining the appropriate control Lyapunov functions. For starters, one can extend the MPC algorithm in [6] for computing the maximal controlled invariant set to the hybrid inclusions.

- **Event-triggered Control for Forward Invariance in Hybrid Systems**

As an efficient control implementation strategy, event-triggered control that achieves stability draw increasing attention in the research community. This is due to its nature of reducing the need to periodically update the control input by triggering the updates only when necessary, i.e., a control condition is violated. Among these, our recent effort in the modeling and analysis of event-triggered controlled systems for hybrid inclusions [92] calls for in-depth analysis of forward invariance inducing control implementations. Such result has the potential to tackle the Zeno solution behavior near the stabilization sets for event-triggered controlled systems in general.

- **Hybrid Basic Conditions for Forward Invariance in Hybrid Systems**

In [61], robustness of asymptotic stability of compact sets are guaranteed by enforcing a set of conditions call the *hybrid basic conditions*; see Definition 2.0.8. Outside of the context of asymptotically stability of compact sets, the sublevel sets of control Lyapunov functions for forward invariance in Chapter 6 has the potential to achieve stability properties in a neighborhood of itself. Such a property can be achieved by studying contractivity of invariance sets for hybrid inclusions. Moreover, by clarifying the effect of hybrid basic conditions on the forward invariance properties of sets, we can better understand the connection between robustness of stability and robust forward invariance of sets.

Other areas of science and engineering that would benefit from the theoretical and practical results in this dissertation include several problems in biology, economics and social science. For instance, according to Darwinian's natural selection, sys-

tems and organisms in nature evolve in nondeterministic ways; while the nature, through physical constraints, “selects” the species that have better “survival” potentials based on their abilities to “fit in” the environment. The concept of forward invariance for hybrid systems with disturbances and constraints can be extended to describe such behaviors. More precisely, the evolution history of the “survival” species can be seen as solutions to the natural system that start within the sets, which describe constraints like temperature, humidity, and pressure, and stay within it for all time during their existence. Similar practices appear in economics and social science, where the time span of system evolution is significantly shorter than the classical “natural selection” problem. For example, determining implicit evaluation of the volatility of portfolios and management of renewable resources [93, Chapter 7 and Chapter 15] can be solved using forward invariance tools.

We believe this dissertation contributes both to the theory and applications of hybrid control systems by introducing new analysis, and design tools that are useful for the control community and other related fields.

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Appendix A

Set-valued Analysis Tools

Definition A.0.1 (*local boundedness*) A set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is locally bounded at $x \in \mathbb{R}^m$ if there exists a neighborhood U_x of x such that $M(U_x) \subset \mathbb{R}^n$ is bounded. The mapping M is locally bounded if it is locally bounded at each $x \in \mathbb{R}^m$. Given a set $S \subset \mathbb{R}^m$, the mapping M is locally bounded relative to S if the set-valued mapping from \mathbb{R}^m to \mathbb{R}^n defined by $M(x)$ for $x \in S$ and \emptyset for $x \notin S$ is locally bounded at each $x \in S$. \square

Definition A.0.2 (*outer semicontinuity of set-valued maps*) A set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is outer semicontinuous at $x \in \mathbb{R}^n$ if for each sequence $\{x_i\}_{i=1}^{\infty}$ converging to a point $x \in \mathbb{R}^n$ and each sequence $y_i \in S(x_i)$ converging to a point y , it holds that $y \in S(x)$; see [94, Definition 5.4]. Given a set $K \subset \mathbb{R}^n$, it is outer semicontinuous relative to K if the set-valued mapping from \mathbb{R}^n to \mathbb{R}^m defined by $S(x)$ for $x \in K$ and \emptyset for $x \notin K$ is outer semicontinuous at each $x \in K$. \square

Definition A.0.3 (*Lipschitz continuity of set-valued maps*) Given a set-valued map $F : \mathbb{R}^n \times \mathcal{W}_c \rightrightarrows \mathbb{R}^n$, the mapping $x \mapsto F(x, w)$ is locally Lipschitz uniformly in w at x , if there exists a neighborhood U of x and a constant $\lambda \geq 0$ such that

for every $\xi \in U$

$$F(x, w) \subset F(\xi, w) + \lambda |x - \xi| \mathbb{B}$$

$$\forall w \in \{w \in \mathcal{W}_c : (U \times \mathcal{W}_c) \cap \text{dom } F\}.$$

Furthermore, $x \mapsto F(x, w)$ is locally Lipschitz uniformly in w on set $K \subset \text{dom } F$ when it is locally Lipschitz uniformly in w at each $x \in \Pi(K)$. \square

Definition A.0.4 (lower semicontinuous set-valued maps) A set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is lower semicontinuous if for every $x \in \mathbb{R}^n$, one has that $\liminf_{x_i \rightarrow x} S(x_i) \supset S(x)$, where

$$\liminf_{x_i \rightarrow x} S(x_i) := \{z : \forall x_i \rightarrow x, \exists z_i \rightarrow z \text{ s.t. } z_i \in S(x_i)\}$$

is the inner limit of S (see [94, Chapter 5.B]). \square

Corollary A.0.5 ([71, Corollary 2 of Theorem 2.9.8]) Let $C_1, C_2 \subset \mathbb{R}^n$ and that $x \in C_1 \cap C_2$. Suppose that

$$T_{C_1} \cap \text{int} T_{C_2}(x) \neq \emptyset,$$

and that C_2 admits at least one hypertangent vector at x . Then, if C_1 and C_2 are regular at x , one has

$$T_{C_1} \cap T_{C_2}(x) = T_{C_1 \cap C_2}(x).$$

Proposition A.0.6 ([73, Proposition 2.11]) Let set-valued maps $H, L : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be such that $\text{gph } L \subset \text{gph } H$. If $H(x)$ is lower semicontinuous and $\text{gph } L$ is open relative to $\text{gph } H$, then, L is lower semicontinuous.

Corollary A.0.7 ([73, Corollary 2.13]) Given a lower semicontinuous set-valued map W and an upper semicontinuous function w , the set-valued map defined for each z as $S(z) := \{z' \in W(z) : w(z, z') < 0\}$ is lower semicontinuous.

Theorem A.0.8 (*Michael's Selection Theorem, [73, Theorem 2.18]*) Given a lower semicontinuous set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with nonempty, convex, and closed values, there exists a continuous selection $s : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

Theorem A.0.9 (*[95, Theorem 4.1]*) Given a closed set $A \subset \mathbb{R}^n$ and a continuous map $s : A \mapsto \mathbb{R}^m$. Then there exists a continuous extension $\tilde{s} : \mathbb{R}^n \mapsto \mathbb{R}^m$ of s . Furthermore, $\tilde{s} \subset \overline{\text{co}}(s(A))$.

Proposition A.0.10 (*Minimal Selection Theorem [73, Proposition 2.19]*) Let the set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be lower semicontinuous with closed graph and nonempty closed convex values. Then the minimal selection $m : \mathbb{R}^n \rightarrow \mathbb{R}^m$, which is given by

$$m(x) := \arg \min \{|z| : z \in S(x)\},$$

is locally bounded and $\text{gph } m$ is closed and continuous.

Definition A.0.11 (*Hypertangent*) A vector $v \in \mathcal{X}$ is said to be hypertangent to the set C at the point $x \in C$ if, for some $\varepsilon > 0$,

$$y + tw \in C \text{ for all } y \in (x + \varepsilon\mathbb{B}) \cap C, w \in v + \varepsilon\mathbb{B}, t \in (0, \varepsilon).$$

□

Proposition A.0.12 (*[73, Proposition 2.16]*) Let $W : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be locally Lipschitz continuous with convex values. Let $r : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$ and $\varepsilon : \mathbb{R}^n \rightarrow (0, 1)$ be locally Lipschitz continuous, and suppose we have $W(x) \cap \varepsilon(x)r(x)\mathbb{B} \neq \emptyset$ for all $x \in \mathbb{R}^n$. Then, $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined by $S(x) := W(x) \cap \varepsilon(x)r(x)\mathbb{B}$ is locally Lipschitz continuous.

Proposition A.0.13 ([73, Proposition 2.14]) *Given a locally Lipschitz set-valued map W with nonempty compact convex values. Let w be locally Lipschitz continuous and such that the mapping $z' \mapsto w(z, z')$ is convex for each fixed z . Then the set-valued map defined by $S(z) := \{z' \in W(z) : w(z, z') < 0\}$ is locally Lipschitz continuous on $\text{dom } S$.*

Proposition A.0.14 ([73, Proposition 2.20]) *Given a locally Lipschitz set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with nonempty, closed, convex values. Then, there exists a locally Lipschitz selection such that $s : \mathbb{R}^n \rightarrow \mathbb{R}^m$.*

Lemma A.0.15 ([71, Theorem 2.9.10]) *Given a set $S := \{x : h(x) \leq 0\}$, suppose that, for every $x \in \{x : h(x) = 0\}$, h is continuously differentiable at x with $0 \notin \nabla h(x) \neq \emptyset$ and the collection of vectors $Y := \{y : \langle \nabla h(x), y \rangle < \infty\}$ is nonempty. Then, the set S admits a hypertangent at x and*

- 1) $y \in T_S(x)$ if $\langle \nabla h(x), y \rangle \leq 0$;
- 2) $\exists y \in \text{int}T_S(x) \cap \text{int}Y$ such that $\langle \nabla h(x), y \rangle < 0$.

Appendix B

List of Publications

Journal Publications

- [1] T. A. F. Theunisse, J. Chai, R. G. Sanfelice, and M. Heemels. Robust Global Stabilization of the DC-DC Boost Converter via Hybrid Control, *IEEE Transactions on Circuits and Systems*, vol. 62, pp. 1052–1061, April, 2015.
- [2] J. Chai, and R. G. Sanfelice. On Robust Controlled Forward Invariance of Sets for Hybrid Dynamical Systems (Part I), *IEEE Transactions on Automatic Control*, (accepted)
- [3] J. Chai, P. Casau, and R. G. Sanfelice. Analysis of Event-triggered Control Algorithms using Hybrid Systems Tools, *Automatica*, (in preparation)
- [4] J. Chai, and R. G. Sanfelice. On Robust Controlled Forward Invariance of Sets for Hybrid Dynamical Systems (Part II), *IEEE Transactions on Automatic Control*, (in preparation)
- [5] J. Chai, and R. G. Sanfelice. A Robust Hybrid Control Algorithm for a Single-Phase DC/AC Inverter with Variable Input Voltage, *IEEE Transactions on Circuits and Systems*, (in preparation)

Peer-Reviewed Conference Proceedings

- [1] T. A. F. Theunisse, J. Chai, R. G. Sanfelice, and W.P.M.H. Heemels. Hybrid Control of the Boost Converter: Robust Global Stabilization, In *Proceedings of*

- the 52th IEEE Conference on Decision and Control*, pp. 3635–3640, 2013.
- [2] J. Chai, and R. G. Sanfelice. A Robust Hybrid Control Algorithm for a Single-Phase DC/AC Inverter with Variable Input Voltage, In *Proceedings of the American Control Conference*, pp. 1420–1425, 2014.
- [3] J. Chai, and R. G. Sanfelice. Hybrid Feedback Control Methods for Robust and Global Power Conversion, In *Proceedings of the 5th Analysis and Design of Hybrid Systems*, pp. 298–303, October, 2015.
- [4] J. Chai, and R. G. Sanfelice. On Notions and Sufficient Conditions for Forward Invariance of Sets for Hybrid Dynamical Systems, In *Proceedings of the 54th IEEE Conference on Decision and Control*, pp. 2869–2874, December, 2015.
- [5] J. Chai, and R. G. Sanfelice. Results on Feedback Design for Forward Invariance of Sets in Hybrid Dynamical Systems, In *Proceedings of the 55th IEEE Conference on Decision and Control*, pp. 622–627, December, 2016.
- [6] J. Chai, and R. G. Sanfelice. On Robust Forward Invariance of Sets for Hybrid Dynamical Systems, In *Proceedings of the American Control Conference*, pp. 1199–1204, 2017.
- [7] J. Chai, P. Casau, and R. G. Sanfelice. Analysis of Event-triggered Control Algorithms using Hybrid Systems Tools, In *Proceedings of the 56th IEEE Conference on Decision and Control*, pp. 6057–6062, 2017.

Patent

- [1] R. G. Sanfelice and J. Chai. Robust single-phase DC/AC inverter for highly varying DC voltages, US Pat. No. 9876442, 2018.