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Lecture XVIII

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INTEGRAL EQUATIONS (OTHER METHODS)

1. Introduction

The intimate connection between differential equations and problems in the calculus of variations furnishes the bridge between linear differential eigenvalue problems and their abstract formulation in terms of stationary values of functionals or, equivalently, of eigenvalues of linear bounded operators on an abstract vector space. Although this reformulation is troublesome and is to be avoided, if possible, in particular calculations, it is of inestimable conceptual advantage since it strips away the superfluous notions that enter into the original statement of the problem and leaves one only with considerations on the primitive objects that are really genuine to the problem, namely functions.

The connection which was alluded to above between differential eigenvalue problems and the calculus of variations consists, of course, in the well-known necessary condition that $x(t)$ give a stationary value of the functional,

$$J(x) = \int_a^b F(x, \mathring{x}, t)dt ,$$

namely, that $x(t)$ satisfy the Euler equation

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{x}} - \frac{\partial F}{\partial x} = 0 \text{ , where } x(a) = x_0, \quad x(b) = x_1 \quad .$$

If one considers, instead, the problem of finding a stationary value of

$$J(x) = \int_a^b F(x, \dot{x}, t)dt$$

while

$$K(x) = \int_a^b G(x, \dot{x}, t)dt = \text{constant},$$

one is led by the method of the Lagrangean multiplier λ to the Euler equation

$$\frac{d}{dt}\frac{\partial (F - \lambda G)}{\partial \dot{x}} - \frac{\partial (F - \lambda G)}{\partial x} = 0 \text{ , } \quad x(a) = x_0; \quad x(b) = x_1,$$

which represents a differential eigenvalue problem.

In order, now, to relate a given differential eigenvalue problem to one concerning functionals defined on a space of functions one must tread the same path in the reverse direction and discover the calculus of variations problem with side conditions which corresponds to the given situation.

The way of doing this with as much generality as is at present possible is described in reference (2) and is quite complicated. Here, a recipe shall merely be given in general, a few examples presented to make the recipe plausible, and the admonition be impressed on the practitioner to test the recipe in the particular case to which it is to be applied.

2. General Procedure

Suppose, then, that A and B are linear differential operators and that it is required to determine numbers λ and functions f(P) such that

$$A\, f(P) = \lambda\, B\, f(P) \qquad (1)$$

where $P \in \mathcal{D}$, a simply connected domain in n-dimensional Euclidean space and f(P) is a function possessed of sufficiently many continuous derivatives to make Eq. (1) meaningful while satisfying the linear homogeneous boundary conditions $\Lambda_i\, f = 0$ on C, the frontier of $\mathcal{D}$. The recipe to be applied for the conversion of the problem (1) into one of the calculus of variations is

$$\lambda^{-1} = \sup \int_{\mathcal{D}} f\, B\, f(P) dv \qquad \text{and} \qquad \int_{\mathcal{D}} f\, A\, f(P) dv = 1.$$

If A and B are of even order, which is the only case that has as yet been treated with any generality, then one can reduce the order of the differential operator occurring under the integral sign by partial integration (Green's formula, Gauss' theorem, etc.) and use of the boundary conditions with the result that if A, which is supposed to have the highest order, is of order 2t, only boundary conditions of order $< t$ remain explicitly.

It is at this point that the reformulation of (1) into a problem of the calculus of variation has rendered its chief service by defining properly the class of functions that must be considered. For, the remaining boundary conditions of order $< t$, define, because of their linear-homogeneous character, a class of functions that constitutes a linear manifold that can be completed into a Hilbert space.

In order to complete the above mentioned linear manifold, it is necessary to introduce a metric, that is, a distance function for elements of the linear manifold and thereupon to add sufficiently many elements, functions, to it so that one is assured that every sequence of elements that converges in the metric converge to a point of the space.

In some cases such a completion may not be necessary since the problem may have solutions in the incomplete manifold defined by the boundary conditions.

3. Examples

3.1 Example 1:

$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \lambda u \quad \text{in } \mathfrak{D} \text{ with } u = 0 \text{ on } C.$$

$$A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$B = 1$$

$$\lambda = \sup \iint_{\mathfrak{D}} u^2 \, dx\, dy$$

while

$$-\iint_{\mathfrak{D}} u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dx\, dy = \text{constant}.$$

Since

$$A = -\operatorname{div}\operatorname{grad} \quad \text{and} \quad u \operatorname{div}\operatorname{grad} u = \operatorname{div}(u \operatorname{grad} u) - (\operatorname{grad} u)^2$$

we can use Gauss' formula:

$$\iint_{\mathfrak{D}} \operatorname{div} \vec{w} \, dx\, dy = -\int_C \vec{w} \times \vec{n} \, d\sigma$$

to find that

$$\iint_{\mathcal{D}} -u\left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}\right) dx\, dy = +\int_C u \text{ grad } u \cdot \vec{n}\, d\sigma + \iint_{\mathcal{D}} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\, dy.$$

But on using the boundary condition $u = 0$ on C

$$-\iint_{\mathcal{D}} u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dx\, dy = +\iint_{\mathcal{D}} \left[\left(\frac{\partial u}{x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\, dy$$

and the ultimate formulation of the problem is

$$\lambda = \sup \iint_{\mathcal{D}} u^2\, dx\, dy$$

while

$$\iint \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial y}\right)^2\right] dx\, dy = 1 .$$

Note that in the final formulation, the differentiability conditions are relaxed while a first order boundary condition such as grad $u = 0$ is inadmissible.

<u>3.2</u> Example 2:

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial x^4} = \lambda u , \quad u = 0 , \quad \text{grad } u = 0 .$$

$$A = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} \quad \text{and} \quad B = 1 .$$

Noting that $A =$ div grad div grad and using the same tools as in example 1, we arrive at the final formulation

$$\sup \iint_{\mathcal{D}} u^2\, dx\, dy = \lambda \quad \text{while} \quad \iint_{\mathcal{D}} (\Delta u)^2\, dx\, dy = 1$$

where Δ represents the Laplacian.

In both of the above examples a situation obtains that is typical for most problems, namely that $\int u\,A\,u\,dv$ is a positive definite quadratic form. It therefore serves to define an inner product and norm on our manifold of functions. In example 1,

$$(u, v) = \iint_{\mathfrak{D}} \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dx\,dy$$

and in example 2

$$(u, v) = \iint \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial y^2} \right) dx\,dy$$

and $\|u\| = \sqrt{(u, u)}$ in both cases.

4. Continuation of Procedure

The only task remaining is to complete the function space under the norm $\|u\| = \iint u\,A\,u\,dx\,dy$, i.e., the corresponding metric $\|u - v\| = \text{dist}\,(u, v)$. Once this is done we are in firm possession of a Hilbert space. Our calculus of variations problem is translated then into the problem of finding the stationary values of a quadratic form $\iint u\,B\,u\,dx\,dy$ for functions u such that $\|u\| = 1$. By a theorem of Riesz, however, this is equivalent to find the stationary values of (Ku, u) where K is some bounded linear operator which is for suitable B and boundary conditions self-adjoint and completely continuous.

The remainder of this paper will therefore treat the following problem: Let H be a Hilbert space and K a completely continuous hermitian operator thereon. Let it be required to calculate numbers λ and to determine vectors x such that

$$Kx = \lambda x . \qquad (2)$$

The following theorems serve to lay the rigorous foundation for the Rayleigh-Ritz method of solving eigenvalue problems such as 1.

Theorem 1:

If K is a self-adjoint operator on a Hilbert space H, then the norm of K, defined as $\sup_{\|x\| = 1} \| K x \|$, is

$$\| K \| = \sup_{f \in H} \frac{|(K f, f)|}{(f, f)} .$$

Theorem 2: (Maximum Principle)

If K is self-adjoint and K is completely continuous (i.e., has only a point spectrum) then $\|K\|$ or $-\|K\|$ is an eigenvalue of K.

Theorem 3: (Monotony Principle)

If $H' \subset H$ is a subspace of H and K' is the operator K considered as an operator on H', and K is completely continuous and K is self-adjoint then

$$\lambda_k' \leq \lambda_k$$

i.e., the k'th eigenvalue of K' is less than or equal to the k'th eigenvalue of K. (The ordering is by decreasing magnitude and if H' is of dimension N, then the eigenvalue $\lambda'_{N+1} = 0$ by convention.)

The Rayleigh-Ritz Method:

In this method the given Hilbert space H is approximated by means of a finite dimensional subspace H'.

Let H' be subtended by basis vectors $w_1, w_2, \ldots, w_n$. The matrix elements of K', the restriction of K to H' are given by

$$k_{ij} = \frac{(P K w_i, w_j)}{(w_i, w_j)}$$

where the projection operator P has the effect of cutting off those components of $K\,w_i$ that do not lie in H'.

The eigenvalue problem for the matrix $\{k_{ij}\}$ can then be solved by means of the determinantal equation

$$\left| k_{ij} - \lambda s_{ij} \right| = 0$$

or else by recourse to the variational methods. To illustrate the method, the trivial example of the harmonic oscillator is given below:

$$\frac{d^2y}{dx^2} + ky = 0\,, \qquad y(0) = y(\pi) = 0\,.$$

$$k = \max \frac{-\int_0^{\pi} y^2\,dx}{\int_0^{\pi} y\,\frac{d^2y}{dx^2}\,dx} = \max \frac{\int_0^{\pi} y^2\,dx}{\int_0^{\pi} \left(\frac{dy}{dx}\right)^2 dx}$$

by the recipe presented in the introduction.

Consider the one-dimensional space subtended by the function

$$w_1 = x(x - \pi)\,.$$

Then the problem becomes that of minimizing

$$\frac{c^2 \int_0^{\pi} \left[x^2 - 2\pi x^3 + \pi^2 x^2\right] dx}{c^2 \int_0^{\pi} (2x - \pi)^2\,dx} = \frac{\left.\frac{x^5}{5} - \frac{\pi x^4}{2} + \frac{\pi^2 x^3}{3}\right|_0^{\pi}}{\left.\frac{4x^3}{3} - 2\pi x^2 + \pi^2 x\right|_0^{\pi}}$$

$$= \frac{\pi^2}{10} \leq 1\,.$$

which is a good approximation to the largest eigenvalue 1 of the inverse operator and is less as demanded by the monotony theorem.

The Rayleigh-Ritz method only furnishes lower bounds to the eigenvalues. The Weinstein method tries to remedy this defect by using the monotony principle in reverse, that is, by constructing a superspace enclosing the given Hilbert space and solving the eigenvalue problem for operators that are extensions of the given operator. Although the method is rather complicated it has found employment in the solution of eigenvalue problems for vibrating clamped plates.

5. Bibliography

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