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A NONLINEAR PLANCHEREL THEOREM WITH APPLICATIONS TO GLOBAL WELL-POSEDNESS FOR THE DEFOCUSING DAVEY-STEWARTSON EQUATION AND TO THE INVERSE BOUNDARY VALUE PROBLEM OF CALDERÓN

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ABSTRACT. We prove a Plancherel theorem for a nonlinear Fourier transform in two dimensions arising in the Inverse Scattering method for the defocusing Davey-Stewartson II equation. We then use it to prove global well-posedness and scattering in L^2 for defocusing DSII. This Plancherel theorem also implies global uniqueness in the inverse boundary value problem of Calderón in dimension 2, for conductivities $\sigma > 0$ with $\log \sigma \in \dot{H}^1$. The proof of the nonlinear Plancherel theorem includes new estimates on classical fractional integrals, as well as a new result on L^2-boundedness of pseudo-differential operators with non-smooth symbols, valid in all dimensions.

CONTENTS

1. INTRODUCTION

The Davey-Stewartson equations are a family of nonlinear Schrödinger (NLS) type equations in $2+1$ dimensions, which model the evolution of weakly nonlinear surface water waves traveling principally in one direction [19]. A rigorous derivation from the water wave problem in the modulational scaling regime is provided in [18]. Depending on two sign choices, one classifies the Davey-Stewartson systems as elliptic-elliptic, hyperbolic-elliptic, elliptic-hyperbolic and hyperbolic-hyperbolic. Within each class there are two nontrivial choices of parameters to be made.

In this paper we are interested in the Cauchy problem for the specific case known as the defocusing DSII problem. This model belongs to the hyperbolic-elliptic family, with a special choice for the parameters. The equations have the form

$$\begin{cases} i\partial_t q + 2(\bar\partial^2 + \partial^2)q + q(r+\overline{r}) = 0 \\ \bar\partial r + \partial(|q|^2) = 0 \\ q(0,z) = q_0(z). \end{cases} \tag{1.1}$$

Here and throughout the paper we use the notation

$$z = x_1 + ix_2$$

for points in the plane, and

$$\bar\partial = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\Big), \qquad \partial = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\Big). \tag{1.2}$$

This system (as well as all other DS systems) is mass critical, i.e. the L^2_z norm of the solution (the mass) is invariant with respect to the natural scaling associated to it,

$$q(t,z) \to \lambda q(\lambda^2 t, \lambda z).$$

Local well posedness in L^2 and global existence for small initial data have been established for the general family of Davey-Stewartson equations in [22], [33], [26] using dispersive methods. However, the large data problem has yet to be understood in general.

The defocusing DSII model considered here has the feature that it is completely integrable, as found in [4]. In this paper we use the Inverse Scattering method to investigate the Cauchy problem in L^2 for large initial data.

Precisely, our main goal here will be to prove a Plancherel theorem for a two-dimensional nonlinear Fourier transform (known as the Scattering Transform) associated to this system. We then use this result to show global well-posedness and scattering for (1.1) for any initial data in $L^2(\mathbb{R}^2)$, i.e. in the mass-critical case. Furthermore, the method yields a precise description of the large-time behaviour of the solutions for any initial data q_0 in $L^2(\mathbb{R}^2)$ in terms of its Scattering Transform. The Plancherel theorem implies completeness of the wave operators (in the sense of nonlinear scattering theory).

In a different application, we show how this nonlinear Plancherel theorem also implies global uniqueness for the inverse boundary value problem of Calderón in dimension 2, for conductivities $\sigma > 0$ with $\log\sigma \in \dot H^1$. We will briefly recall some of the background for these problems below.

1.1. **The Scattering Transform.**

We start with a quick formal definition of the scattering transform. Given a function $q(z)$ on $\mathbb{R}^2 \simeq \mathbb{C}$ and $k \in \mathbb{C}$, solve the two equations

$$\frac{\partial}{\partial \bar z} m_\pm = \pm e_{-k} q \overline{m_\pm} \tag{1.3}$$

with $m_\pm(z,k) \to 1$ as $|z| \to \infty$. (We use the notation $e_k(z) = e^{i(zk+\overline{zk})}$ first introduced in [38]). The scattering transform of q is then defined as

$$\mathcal{S}q(k) = \frac{1}{2\pi i}\int_{\mathbb{R}^2} e_k(z)\overline{q(z)}\Big(m_+(z,k) + m_-(z,k)\Big)dz, \tag{1.4}$$

where $dz = dx_1 dx_2$ denotes the Lebesgue measure on $\mathbb{R}^2$. Note that different authors have slightly different conventions.

As seen in (1.3), a key step in the analysis is to be able to invert d-bar operators L_q of the form

$$L_q u = \bar{\partial} u + q\overline{u}$$

under only the assumption that $q \in L^2$. Using the Sobolev embedding $\dot{H}^{\frac{1}{2}} \subset L^4$, it is easy to see that L_q has the following mapping property:

$$L_q : \dot{H}^{\frac{1}{2}} \to \dot{H}^{-\frac{1}{2}},$$

where $\dot{H}^s$ is the homogeneous fractional Sobolev space of order $s \in \mathbb{R}$, defined as

$$\{f \in \mathscr{S}' \,\Big|\, \hat{f} \text{ is a measurable function with } \|f\|_{\dot{H}^s} := \| |\cdot|^s \hat{f}(\cdot)\|_{L^2} < \infty\}.$$

Then it is natural to consider the solvability question for the corresponding inhomogeneous problem

$$L_q u = f \tag{1.5}$$

in the $\dot{H}^{\frac{1}{2}} \to \dot{H}^{-\frac{1}{2}}$ setting. Our main result on this problem is as follows:

Theorem 1.1. *Let $q \in L^2$. Then for each $f \in \dot{H}^{-\frac{1}{2}}$ there exists a unique solution $u \in \dot{H}^{\frac{1}{2}}$ of the inhomogeneous problem* (1.5) *with*

$$\|u\|_{\dot{H}^{\frac{1}{2}}} \leq C(\|q\|_{L^2})\|f\|_{\dot{H}^{-\frac{1}{2}}}. \tag{1.6}$$

A key point here is that the constant depends only on the L^2 norm of q. This will later allow us to show that the solutions of (1.3) are bounded by constants that depend only on the L^2 norm of q.

The above theorem is proved in Section 3. To show (1.6) we will borrow techniques developed in the modern treatment of nonlinear PDEs in critical cases: induction on energy and profile decompositions, in order to deal with the lack of compactness ([27],[21], [8]). The novelty here is that these ideas will be used in a nonstandard fashion and on the static equations (1.3), rather than on the nonlinear flow (1.1).

In Section 4 we will show how to use the above result in order to construct $m_\pm$ and $\mathcal{S}q$ assuming only $q \in L^2(\mathbb{R}^2)$.

The Scattering Transform can be viewed as a nonlinear Fourier transform, and it shares many of the same properties. The linearization of $\mathcal{S}$ at $q = 0$ is essentially the Fourier transform:

$$\mathcal{S}q(k) = \overline{\hat{q}(k)} + \mathcal{O}(q^2) \tag{1.7}$$

where

$$\hat{q}(k) = \frac{i}{\pi}\int_{\mathbb{R}^2} e_{-k}(z)q(z)dz. \tag{1.8}$$

We will use the normalization (1.8) for the Fourier transform throughout the paper (except in Section 2).

Writing $\mathbf{s} := \mathcal{S}q$, and setting

$$n_\pm := \frac{1}{2}\Big((m_+ + m_-) \pm e_{-k}\overline{(m_+ - m_-)}\Big), \tag{1.9}$$

it turns out that the functions $n_\pm(z,k)$ solve equations in k which are the same as those solved by $m_\pm(z,k)$ in z, with $q(z)$ replaced by $\mathbf{s}(k)$:

$$\frac{\partial}{\partial \overline{k}} n_\pm = \pm e_{-k} \mathbf{s} \overline{n_\pm} \tag{1.10}$$

with $n_\pm(z,k) \to 1$ as $|k| \to \infty$. The Inverse Scattering transform of $\mathbf{s}$ is then defined as

$$\mathcal{I}\mathbf{s}(z) = \frac{1}{2\pi i} \int_{\mathbb{R}^2} e_k \overline{\mathbf{s}(k)} \Big(n_+(z,k) + n_-(z,k) \Big) dk. \tag{1.11}$$

Note that $n_+ + n_- = m_+ + m_-$ and under appropriate conditions on q, one can show that $q = \mathcal{I}(\mathbf{s})$. Thus, with the above notation conventions, the scattering transform is an involution $\mathcal{S}^2 = I$.

If we now evolve the potential q according to the DSII equation (1.1), the corresponding scattering data evolves (as was shown in [10]; see also [40]) according to:

$$\frac{\partial}{\partial t} \mathbf{s}(t,k) = 2i(k^2 + \overline{k}^2) \mathbf{s}(t,k). \tag{1.12}$$

Thus, the Cauchy problem for the nonlinear equation (1.1) may be solved in a manner analogous to the use of the Fourier transform for linear PDEs, by performing forward-scattering on the initial data q_0 then evolving the scattering data linearly in time according to (1.12) and then performing Inverse Scattering to determine q at time t, namely

$$\begin{cases} \mathbf{s}_0(k) & = \mathcal{S} q_0(k) \\ \mathbf{s}(t,k) & = e^{2i(k^2+\overline{k}^2)t} \mathbf{s}_0(k) \\ q(t,z) & = \mathcal{I}\big(\mathbf{s}(t,k)\big)(z). \end{cases} \tag{1.13}$$

This Inverse Scattering approach to the solution of the DSII equations dates back to Ablowitz and Fokas ([1], [2] and [3]) and Beals and Coifman ([9], [10] and [11]). Beals and Coifman showed that for initial data in the Schwartz class, (1.3) and (1.10) are solvable, and the corresponding scattering data is also in the Schwartz class. They also proved that for potentials in the Schwartz class, the scattering transform satisfies the nonlinear Plancherel identity

$$\int |q(z)|^2 dz = \int |\mathbf{s}(k)|^2 dk, \tag{1.14}$$

and is a symplectomorphism.

Sung ([43], [44], [45]) carried out the analysis of the scattering transform and its inverse to solve the defocusing DSII for initial data $q_0 \in L^2 \cap L^p$ for some $p \in [1,2)$ with $\hat{q}_0 \in L^1 \cap L^\infty$. Brown and Uhlmann [13] proved that for $q \in L^p_c$ where $p > 2$, the scattering data $\mathbf{s} \in L^2$. Tamasan [47] proved that for $q \in W^{\varepsilon,p}_c$, where $\varepsilon > 0$ and $p > 2$, the scattering data $\mathbf{s} \in L^r$ for each $r > 2/(\varepsilon+1)$. Brown [12] proved the Plancherel identity and Lipschitz continuity of the scattering transform for $q \in L^2$ of sufficiently small norm. Brown estimated directly the series expansion of $\mathbf{s}$ in multi-linear terms in q (see also [39] for such estimates). He stated as open questions whether one can remove the smallness assumption and whether solutions to (1.1) can be constructed when q is in L^2. We will address these questions in this paper.

There has been significant recent progress on the problem of the validity of the Plancherel identity (1.14) without a smallness assumption. Perry [40] proved that for q in the weighted Sobolev space $H^{1,1}$ the scattering data $\mathbf{s} \in H^{1,1}$. In addition, he proved local Lipschitz continuity of the map $\mathcal{S}: H^{1,1} \to H^{1,1}$. He used these results to show global well-posedness for defocusing DSII for initial data in $H^{1,1}$. Astala, Faraco and Rogers [5] sharpened part of Perry's proof to show local Lipschitz continuity of the scattering map $\mathcal{S}$ from $H^{s,s}$ to L^2 for $s \in (0,1)$ thus extending the Plancherel identity to this space. Perry, Otto and Brown [14] then showed that the scattering transform maps $q \in H^{\alpha,\beta}$ to $\mathbf{s} \in H^{\beta,\alpha}$ for $\alpha, \beta > 0$ thus establishing further precise analogy between the properties of the scattering transform and the Fourier transform.

In this paper we prove the Plancherel theorem for the Scattering Transform for general q in $L^2(\mathbb{R}^2)$. To do so, we need new bounds on $\bar{\partial}^{-1}$ (or, more generally, fractional integrals), which, in the presence of an oscillatory term (see (1.3)) allow us to capture the behaviour of the functions $m_\pm(z,k) \to 1$ as $|k| \to \infty$ without assuming any smoothness on q. As well, in order to make sense of the formula (1.4), we will need a new result on the L^2-boundedness of pseudo-differential operators with non-smooth symbols. In Section 2, we give proofs of these bounds valid in any dimension, as they may be of independent interest.

We are now ready to state precisely our Plancherel theorem.

Theorem 1.2. *The nonlinear Scattering Transform $\mathcal{S}: q \mapsto \mathbf{s}$ is a C^1 diffeomorphism $\mathcal{S}: L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, satisfying:*

(1) The Plancherel Identity:

$$\|\mathcal{S}q\|_{L^2} = \|q\|_{L^2}. \tag{1.15}$$

(2) The pointwise bound:

$$|\mathcal{S}q(k)| \le C(\|q\|_{L^2}) M\hat{q}(k) \tag{1.16}$$

for a.e. k, where M denotes the Hardy-Littlewood Maximal function.

(3) Locally uniform bi-Lipschitz continuity:

$$\frac{1}{C}\|\mathcal{S}q_1 - \mathcal{S}q_2\|_{L^2} \le \|q_1 - q_2\|_{L^2} \le C\|\mathcal{S}q_1 - \mathcal{S}q_2\|_{L^2} \tag{1.17}$$

where

$$C = C(\|q_1\|_{L^2})C(\|q_2\|_{L^2}).$$

(4) Bound on the derivative:

$$\left\|\frac{\delta \mathcal{S}}{\delta q}\right\|_{L^2 \to L^2} \le C(\|q\|_{L^2}). \tag{1.18}$$

(5) Inversion Theorem:

$$\mathcal{S}^{-1} = \mathcal{S}.$$

(6) $\mathcal{S}$ is a symplectomorphism [1]: for every $q, q_1, q_2 \in L^2(\mathbb{R}^2)$

$$\omega_2\Big(\frac{\delta \mathcal{S}}{\delta q}\Big|_q q_1, \frac{\delta \mathcal{S}}{\delta q}\Big|_q q_2\Big) = \omega_1(q_1, q_2), \tag{1.19}$$

[1] We are grateful to the anonymous referee who suggested that we should also prove this additional property for the Scattering Transform.

where ω_1, ω_2 are the symplectic forms

$$\omega_1(q_1, q_2) = -\Im \int q_1(z)\overline{q_2(z)}dz, \quad \omega_2(t_1, t_2) = -\Im \int \overline{t_1(k)}t_2(k)dk.$$

We will in fact prove an identity more general than (1.15) (see Corollary 4.7) which shows to what extent S departs from being an isometry. Furthermore, explicit formulas for the derivative $\frac{\delta \mathcal{S}}{\delta q}$ and its inverse are given in Lemma 4.8.

As a consequence of properties (1), (2) and (5) above, we note the following pointwise bound on q in terms of the Fourier transform of its scattering transform.

Corollary 1.3. *If $q \in L^2(\mathbb{R}^2)$ and $\mathbf{s} = \mathcal{S}(q)$ then for a.e. z we have:*

$$|q(z)| \leq C(\|q\|_{L^2}) M\hat{\mathbf{s}}(z).$$

1.2. **Global Well-Posedness for the Defocusing DSII Problem.** One immediate application of Theorem 1.2 will be to show global well-posedness of the Cauchy problem for the defocusing Davey-Stewartson equation for arbitrary initial data in $L^2(\mathbb{R}^2)$. In particular, the above Corollary will yield pointwise control of the solution to DSII by the maximal function of a solution of the linear flow and thus will allow us to transfer Strichartz estimates on the linearization of (1.1) to bounds on the nonlinear flow.

We recall the formulation of (1.1) as an integral equation ([22]). The Cauchy problem (1.1) has a corresponding linear flow

$$\begin{cases} i\partial_t \tilde{q} + 2(\bar{\partial}^2 + \partial^2)\tilde{q} = 0 \\ \tilde{q}(0, z) = q_0(z). \end{cases} \tag{1.20}$$

Let $U(t)$ be the solution operator to the linear problem (1.20)

$$U(t)q_0 := \tilde{q}(t, \cdot) = e^{2it(\partial^2 + \bar{\partial}^2)}q_0.$$

Using Duhamel's principle, (1.1) can be written as the following nonlinear integral equation for $q(t) =: q(t, \cdot)$

$$q(t) = U(t)q_0 + \Lambda(q)(t) \tag{1.21}$$

with

$$\Lambda(q)(t) = i\int_0^t U(t - s)\Big(q(s)(\bar{\partial}\partial^{-1} + \partial\bar{\partial}^{-1})|q(s)|^2\Big)ds. \tag{1.22}$$

Ghidaglia and Saut [22] proved that for any $q_0 \in L^2(\mathbb{C})$ the problem (1.21) has a unique solution in the Strichartz type space

$$X_T := C([0, T], L^2_z(\mathbb{C})) \cap L^4_{t,z}([0, T] \times \mathbb{C})$$

for some T which depends on q_0; they also showed that for q_0 with sufficiently small L^2 norm this holds for all T. Using the Inverse Scattering method, Perry proved global well posedness for general initial data $q_0 \in H^{1,1}$. Our Plancherel Theorem yields the following:

Theorem 1.4. *(Global well-posedness for defocusing DSII on L^2) Given $q_0 \in L^2$, there exists a unique solution to the Cauchy problem (1.1) in the sense of equation (1.21) such that:*

(1) *Regularity:*

$$q(t,z) \in C(\mathbb{R}, L^2_z(\mathbb{C})) \cap L^4_{t,z}(\mathbb{R}\times\mathbb{C}).$$

(2) *Uniform bounds: conservation of mass* $\|q(t,\cdot)\|_{L^2} = \|q_0\|_{L^2}$ *for all* $t\in\mathbb{R}$ *and*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} |q(t,z)|^4 dz dt \le C(\|q_0\|_{L^2}).$$

(3) *Pointwise bound:*

$$|q(t,z)| \le C(\|q_0\|_{L^2}) M q^{lin}(t,z)$$

where

$$q^{lin}(t,\cdot) = U(t)\widehat{\overline{\mathcal{S}q_0}}.$$

(4) *Stability: if* $q_1(t,\cdot)$ *and* $q_2(t,\cdot)$ *are two solutions corresponding to initial data* $q_1(0,\cdot)$ *and* $q_2(0,\cdot)$ *with* $\|q_j(0,\cdot)\|_{L^2}\le R$ *then*

$$\|q_1(t,\cdot)-q_2(t,\cdot)\|_{L^2} \le C(R)\|q_1(0,\cdot)-q_2(0,\cdot)\|_{L^2} \qquad \text{for all } t\in\mathbb{R}.$$

We remark that much of the conclusion of this theorem closely resembles the conclusion of Dodson's result [20] for the two dimensional cubic defocusing NLS problem

$$iu_t + \Delta u = u|u|^2,$$

(see also the prior work [28]). Written in a similar format, the DSII problem has the form

$$iq_t + (\partial_1^2 - \partial_2^2)q = qL(|q|^2), \qquad L(D) = \frac{D_1^2 - D_2^2}{D_1^2 + D_2^2}.$$

Whereas the small data theory for the two problems is completely similar from a dispersive stand-point (i.e. perturbative, based on Strichartz estimates), the large-data approach in the present, completely integrable case and in Dodson's work are completely different. The large data problem for the other, non-integrable cases in the same DS family remains open at present. This includes for instance the problem

$$iq_t + (\partial_1^2 - \partial_2^2)q = q|q|^2.$$

The next theorem provides one more convincing motivation for the study of the Scattering Transform, if one seeks to understand the large-time behaviour of the solutions to the DSII equation. We first recall the definition of the wave operators, in the sense of nonlinear scattering theory.

Definition 1.5. *Let* $q_0 \in L^2(\mathbb{R}^2)$ *and let* $q(t,z)$ *be the solution to the Cauchy problem (1.1). Define* $W_+q_0 = q_+$ *if there exists a unique* $q_+\in L^2(\mathbb{R}^2)$ *such that*

$$\lim_{t\to\infty} \|q(t,\cdot) - U(t)q_+\|_{L^2(\mathbb{R}^2)} = 0.$$

Similarly $W_-q_0 = q_-$ *if*

$$\lim_{t\to-\infty} \|q(t,\cdot) - U(t)q_-\|_{L^2(\mathbb{R}^2)} = 0.$$

We can now state the following further consequence of the Plancherel theorem:

Theorem 1.6. *(Wave operators and asymptotic completeness for defocusing DSII)*
a) The Wave operators $W_\pm$ for the defocusing DSII equation are well defined on every $q_0 \in L^2(\mathbb{R}^2)$ and

$$W_\pm q_0 = \widehat{\overline{\mathcal{S}q_0}}.$$

b) The Wave operators $W_\pm$ are surjective, in fact norm-preserving diffeomorphisms of L^2.

Perry [40] established the same large-time asymptotic behaviour in the L^∞ norm, for initial data in $H^{1,1} \cap L^1$. Kiselev ([29], [30]) had similar results under more restrictive assumptions.

An interesting consequence of Theorem 1.6 is that the temporal scattering operator $W_+(W_-)^{-1}$ for the defocusing DSII equation (i.e. the operator which sends q_- to q_+) is equal to the identity.

1.3. **Application to Inverse Boundary Value Problems.** We next discuss the application of our Plancherel theorem to the Inverse Boundary Value Problem of Calderón. Let Ω be a simply connected domain in $\mathbb{R}^2 \simeq \mathbb{C}$ with $C^{1,1}$ boundary $\partial\Omega$. We denote by ν the outer unit normal to $\partial\Omega$ and τ the unit tangent in the counter-clockwise direction. Consider the Dirichlet problem

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 \text{ in } \Omega \\ u\Big|_{\partial\Omega} = g. \end{cases} \tag{1.23}$$

The Dirichlet-to-Neumann map is defined as

$$\Lambda_\sigma g := \sigma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}, \tag{1.24}$$

with u the solution to (1.23). The function σ models the inhomogeneous conductivity of Ω, and Λ_σ represents the information observable by voltage and current measurements at the boundary. Calderón posed the problem of establishing whether σ is uniquely determined by Λ_σ and, if so, of finding a way to calculate σ from knowledge of Λ_σ.

There is by now an extensive literature on this and related problems. See for instance [6] for a recent review. We only briefly recall some of the pertinent results. The first global uniqueness theorem was proved by Sylvester-Uhlmann [46] for smooth conductivities in dimensions 3 or higher. A reconstruction method was given in [36]. In three dimensions or higher, uniqueness has been shown for Lipschitz conductivities close to the identity in [25]; the smallness condition was removed in [15]. In dimensions $n = 3, 4$ Haberman [24] has proved uniqueness for conductivities in $W^{1,n}(\Omega)$.

In two dimensions, the first global uniqueness and reconstruction result was obtained in [38] for conductivities in $W^{2,p}(\Omega)$ with $p > 1$, by connecting Λ_σ to a scattering transform for a Schrödinger equation. This was refined to $W^{1,p}(\Omega)$ with $p > 2$ in [13] using the scattering transform studied in this paper. In [7], Astala and Päivärinta succeeded in proving uniqueness for general L^∞ conductivities bounded below. In [6], uniqueness is extended to a larger class of conductivities that allow some σ which need not be bounded from above or below. In a recent paper, Cârstea and Wang [16] have shown uniqueness for conductivities σ in $W^{1,2}(\Omega)$ which are bounded from below assuming $\|\nabla \log \sigma\|_{L^2}$ sufficiently small.

Here we use Theorem 1.2 to prove global uniqueness for conductivities $\sigma > 0$ a.e. with the property that

$$\log \sigma \in \dot{H}^1(\Omega), \qquad \sigma = 1 \text{ on } \partial\Omega. \tag{1.25}$$

This is in line with the sharpest results known in higher dimensions, mentioned above ([24]). Notably we do not assume any L^∞ type bounds on σ from above or below [2]. As examples, consider the conductivities $\sigma_\alpha(x) = (-\log|x|)^\alpha$, for any $\alpha \in \mathbb{R}$, on the domain $\Omega = \{x : |x| < e^{-1}\}$. We have $|\nabla \log \sigma_\alpha(x)| = -|\alpha|/|x|\log|x| \in L^2(\Omega)$, hence $\log \sigma_\alpha \in \dot{H}^1(\Omega)$. For $\alpha < 0$ these conductivities degenerate at the origin, while for $\alpha > 0$ they are unbounded; as well, for large α they are not covered by the uniqueness results in [6] (Theorem 1.9) or [16]. We first need to make sure that the Dirichlet problem (1.23) is solvable.

Theorem 1.7. *Assume that σ is as in* (1.25)*. Then for every $g \in H^1(\partial\Omega)$ there exists a unique solution u to the Dirichlet problem* (1.23) *with $\sigma^{\frac{1}{2}}\nabla u \in H^{\frac{1}{2}}(\Omega)$. Furthermore, $\partial u/\partial\nu \in L^2(\partial\Omega)$.*

In particular this insures that Λ_σ is a well-defined operator

$$\Lambda_\sigma : H^1(\partial\Omega) \to L^2(\partial\Omega).$$

Now we can state our main result on the Calderón problem:

Theorem 1.8. *Assume the conductivity $\sigma > 0$ is such that $\log\sigma \in \dot{H}^1$. We also assume, for simplicity, that $\sigma = 1$ on $\partial\Omega$. Then we can reconstruct σ from knowledge of Λ_σ.*

We will obtain Theorems 1.7 and 1.8 as consequences of corresponding results for pseudo-analytic functions, which are also of interest.

More precisely, a standard computation shows that if u is a real-valued solution for (1.23) then the function $v = \sigma^{\frac{1}{2}}\partial u$ solves the equation

$$\bar{\partial} v - q\bar{v} = 0 \text{ in } \Omega \tag{1.26}$$

with

$$q = -\frac{1}{2}\partial \log \sigma. \tag{1.27}$$

Moreover, on the boundary $\partial\Omega$ we have (using (1.2) and the assumption $\sigma = 1$ on $\partial\Omega$):

$$\frac{\partial u}{\partial \nu} = 2\Re(\nu\partial u) = 2\Re(\nu v) \tag{1.28}$$

and

$$\frac{\partial u}{\partial \tau} = -2\Im(\nu\partial u) = -2\Im(\nu v), \tag{1.29}$$

where we interpret the outer normal also as a complex-valued function $\nu = \nu_1 + i\nu_2$ on $\partial\Omega$. In particular, given $g = u|_{\partial\Omega}$ we can determine $\Im(\nu v) = -\frac{1}{2}\frac{\partial g}{\partial \tau}$ on $\partial\Omega$. We are thus led to study the following boundary value problem of pseudo-analytic function v:

[2] However, for any $1 \le p < \infty$, we do have σ and σ^{-1} in L^p, as follows from the Poincaré inequality applied to $\log\sigma$ and Theorem 7.21 in [23].

$$
(1.30)\qquad \begin{cases} \bar\partial v - q\bar v = 0 & \text{in } \Omega \\ \Im(\nu v) = g_0 & \text{on } \partial\Omega, \end{cases}
$$

for $g_0 \in L^2(\partial\Omega)$ with integral zero, and to define an associated Hilbert Transform type operator on $\partial\Omega$ as:

$$
(1.31)\qquad \mathcal{H}_q g_0 := \Re(\nu v).
$$

For then, in view of (1.28) and (1.29), we will have the following relation:

$$
(1.32)\qquad \Lambda_\sigma = -\mathcal{H}_q \frac{\partial}{\partial \tau},
$$

so that the Dirichlet-to-Neumann map Λ_σ for (1.23) will determine the boundary operator $\mathcal{H}_q$ for (1.30). In turn, $\mathcal{H}_q$ will be shown to determine the Scattering Transform of q (extended to be zero outside Ω) thus allowing the use of Theorem 1.2 (5) to complete the solution of the inverse problem. We will first prove the result on the solvability of the forward problem (1.30).

Theorem 1.9. *Assume that $q \in L^2$ is given by* (1.27) *with σ as in* (1.25)*. Then for each real-valued $g_0 \in L^2(\partial\Omega)$ with integral zero the problem* (1.30) *admits a unique solution $v \in H^{\frac{1}{2}}(\Omega)$. Furthermore, $v \in L^2(\partial\Omega)$ and*

$$
(1.33)\qquad \|v\|_{H^{\frac{1}{2}}(\Omega)} + \|v\|_{L^2(\partial\Omega)} \leq C(q)\|g_0\|_{L^2(\partial\Omega)}.
$$

Thus $\mathcal{H}_q$ is well-defined as a bounded operator on $L^2(\partial\Omega)$:

$$
(1.34)\qquad L^2 \ni \Im(\nu v) = g_0 \to \mathcal{H}_q g_0 := \Re(\nu v) \in L^2.
$$

Our main reconstruction theorem for (1.30) states that one can recover q from this boundary operator. One may consider it as analogous to the result in [7] where the Hilbert transform for a Beltrami equation is shown to determine the corresponding Beltrami coefficient.

Theorem 1.10. *Assume that $q \in L^2$ is given by* (1.27) *with σ as in* (1.25)*. Then we can reconstruct q from knowledge of $\mathcal{H}_q$.*

We will in effect consider these last two theorems as the main ones, with the results for the Calderón problem as straightforward consequences.

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2. Estimates on Fractional Integrals and Pseudo-differential Operators

This section is devoted to the proofs of new boundedness theorems on fractional integrals, pointwise multipliers in negative Besov spaces and pseudo-differential operators with non-smooth symbols. These results will be crucial in the rest of the paper. The proofs in this section are valid in all dimensions.

Recall that the Hardy-Littlewood Maximal function is defined for locally integrable functions $f : \mathbb{R}^n \to \mathbb{C}$ as:

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy$$

and yields a bounded operator on L^p for $1 < p \le \infty$ (see, for instance [42]). Also recall the mixed L^p norm:

$$\|f\|_{L^q_y L^p_x} = \Big(\int \Big(\int |f(x,y)|^p dx \Big)^{\frac{q}{p}} dy \Big)^{\frac{1}{q}}.$$

We have the following pointwise bound on fractional integrals:

Theorem 2.1. *For $0 < \alpha < n$, $f \in L^p(\mathbb{R}^n)$, $1 \le p \le 2$*

$$a)\quad \big|(-\Delta)^{-\frac{\alpha}{2}} f(x)\big| \le c_{n,\alpha} \Big(\lambda^{n-\alpha} M\hat{f}(0) + \lambda^{-\alpha} Mf(x) \Big) \qquad \text{for any } \lambda > 0$$

$$b)\quad \big|(-\Delta)^{-\frac{\alpha}{2}} f(x)\big| \le c_{n,\alpha} \Big(M\hat{f}(0) \Big)^{\frac{\alpha}{n}} \Big(Mf(x) \Big)^{1-\frac{\alpha}{n}}.$$

Proof. First we note that the restriction on f along with the Hausdorff-Young inequality assures that f and $\hat{f}$ are locally integrable and so Mf and $M\hat{f}$ are well defined. To simplify notation, we will use $\lesssim$ in place of $\le c_{n,\alpha}$. Using the Littlewood-Paley decomposition, we write

$$(-\Delta)^{-\frac{\alpha}{2}} f(x) = \frac{1}{(2\pi)^n} \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^n} \psi_j(\xi) \frac{e^{ix\cdot\xi}}{|\xi|^\alpha} \hat{f}(\xi) d\xi \tag{2.1}$$

with $\psi_j(\xi) = \psi(\xi/2^j)$ supported in $2^{j-1} < |\xi| < 2^{j+1}$. Fix j_0, for now. We estimate the terms in (2.1) with $j \le j_0$ using

$$\int_{|\xi|<r} |\hat{f}(\xi)| d\xi \le c_n r^n M\hat{f}(0) :$$

$$\begin{aligned} \sum_{j=-\infty}^{j_0} \int_{\mathbb{R}^n} \frac{\psi_j(\xi)}{|\xi|^\alpha} |\hat{f}(\xi)| d\xi &\lesssim \sum_{j=-\infty}^{j_0} 2^{-j\alpha} M\hat{f}(0) 2^{jn} \\ &\lesssim 2^{j_0(n-\alpha)} M\hat{f}(0), \end{aligned} \tag{2.2}$$

since $\alpha < n$. We bound the terms in (2.1) with $j \ge j_0$ by

$$\sum_{j=j_0}^{\infty} \int_{\mathbb{R}^n} |K_j(y)||f(x-y)|dy,$$

with

$$K_j(y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \psi_j(\xi)\frac{e^{iy\cdot\xi}}{|\xi|^\alpha}d\xi.$$

The integral kernel K_j can be estimated by

$$|K_j(y)| \lesssim |y|^{-N}2^{j(n-\alpha-N)} \tag{2.3}$$

for any integer $N \geq 0$. This estimate is obtained, as usual, by writing

$$K_j(y) = \frac{1}{(i|y|^2)^N}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{\psi(\xi/2^j)}{|\xi|^\alpha}(y\cdot\nabla_\xi)^N e^{iy\cdot\xi}d\xi.$$

and integrating by parts N times. We write

$$\begin{aligned}
\int_{|y|\geq 2^{-j}} |K_j(y)||f(x-y)|dy &= \sum_{l=-j}^{\infty}\int_{2^l\leq|y|\leq 2^{l+1}} |K_j(y)||f(x-y)|dy \\
&\lesssim \sum_{l=-j}^{\infty} 2^{j(n-\alpha-N)}2^{-lN}\int_{2^l\leq|y|\leq 2^{l+1}} |f(x-y)|dy \\
&\text{(using (2.3) with } N>n) \\
&\lesssim \sum_{l=-j}^{\infty} 2^{j(n-\alpha-N)}2^{-lN}2^{(l+1)n}Mf(x) \\
&\lesssim 2^{j(n-\alpha-N)}2^n 2^{-j(n-N)}Mf(x) \\
&\lesssim 2^{-j\alpha}Mf(x). \qquad (2.4)
\end{aligned}$$

For $|y| < 2^{-j}$ we use (2.3) with $N = 0$:

$$\begin{aligned}
\int_{|y|\leq 2^{-j}} |K_j(y)||f(x-y)|dy &\lesssim 2^{j(n-\alpha)}\int_{|y|\leq 2^{-j}} |f(x-y)|dy \\
&\lesssim 2^{j(n-\alpha)}2^{-jn}Mf(x) \\
&= 2^{-j\alpha}Mf(x). \qquad (2.5)
\end{aligned}$$

The inequalities (2.4) and (2.5) yield

$$\sum_{j=j_0}^{\infty}\int_{\mathbb{R}^n} |K_j(y)||f(x-y)|dy \lesssim Mf(x)\sum_{j=j_0}^{\infty} 2^{-j\alpha} \lesssim 2^{-j_0\alpha}Mf(x).$$

Returning to (2.1) and also using (2.2), we obtain:

$$\left|(-\Delta)^{-\frac{\alpha}{2}}f(x)\right| \lesssim 2^{j_0(n-\alpha)}M\hat{f}(0) + 2^{-j_0\alpha}Mf(x)$$

for any j_0. This proves inequality a). Inequality b) then follows by optimizing over λ. □

We state explicitly the special case of the above in the form which will be used in subsequent sections. These estimates will allow us to obtain precise control of $m(\cdot,k)$ and $\mathbf{s}(k)$ for large k without any smoothness assumptions on q.

Corollary 2.2. *For $q \in L^2(\mathbb{C})$*

$$a)\ |\bar\partial^{-1}(e_{-k}q)(x)| \lesssim \Big(M\hat q(k)\Big)^{\frac12}\Big(Mq(x)\Big)^{\frac12}$$

$$b)\ \|\bar\partial^{-1}(e_{-k}q)\|_{L^4} \lesssim \|q\|_{L^2}^{\frac12}\Big(M\hat q(k)\Big)^{\frac12}.$$

Proof. Assertion a) follows directly from Theorem 2.1 b) with $\alpha = 1$, $n = 2$ and $p = 2$. Assertion b) follows from assertion a) and the boundedness of M on L^2. □

We next use Theorem 2.1 to prove L^2 boundedness for a class of pseudo-differential operators with non-smooth symbols (See the monograph [17] for an extensive investigation of such problems). The result we need here does not appear to be available in the literature. It will allow us to show that the scattering transform is well defined and in L^2 as a function of k.

Theorem 2.3. *Let $0 \le \alpha < n$. Suppose $a(x,\xi)$ satisfies*[3]

$$i)\ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big|a(x,\xi)\big|^{\frac{2n}{n-\alpha}} d\xi dx < \infty \qquad and$$

$$ii)\ \|(-\Delta_\xi)^{\frac{\alpha}{2}} a(x,\xi)\|_{L_\xi^{\frac{2n}{n+\alpha}}} \in L_x^{\frac{2n}{n-\alpha}}.$$

Then the pseudo-differential operator

$$a(x,D)f(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat f(\xi)d\xi \tag{2.6}$$

is bounded on L^2 with

$$\|a(x,D)f\|_{L^2} \le c_{\alpha,n}\|f\|_{L^2}\|(-\Delta_\xi)^{\frac{\alpha}{2}}a(x,\xi)\|_{L_x^{\frac{2n}{n-\alpha}} L_\xi^{\frac{2n}{n+\alpha}}}. \tag{2.7}$$

Moreover, we have the pointwise bound

$$|a(x,D)f(x)| \le c_{\alpha,n}(Mf(x))^{\alpha/n}\|(-\Delta_\xi)^{\frac{\alpha}{2}}a(x,\cdot)\|_{L^{\frac{2n}{n+\alpha}}}\|f\|_{L^2}^{1-\frac{\alpha}{n}} \tag{2.8}$$

for a.e. x.

Proof. The case $\alpha = 0$ follows by Cauchy-Schwartz. To investigate the case $0 < \alpha < n$, suppose first that f is in Schwartz class. Let $b(x,\xi) = (-\Delta_\xi)^{\frac{\alpha}{2}}a(x,\xi)$. Since $a \in L^{\frac{2n}{n-\alpha}}$, then $a(x,\xi) = (-\Delta_\xi)^{-\frac{\alpha}{2}}b(x,\xi)$ and we have

$$|a(x,D)f(x)| \le \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\Big|(-\Delta_\xi)^{-\frac{\alpha}{2}}\Big(e^{ix\cdot\xi}\hat f(\xi)\Big)\Big|\ \Big|b(x,\xi)\Big|d\xi.$$

By Theorem 2.1

$$\Big|(-\Delta_\xi)^{-\frac{\alpha}{2}}\Big(e^{ix\cdot\xi}\hat f(\xi)\Big)\Big| \lesssim (Mf(x))^{\frac{\alpha}{n}}(M\hat f(\xi))^{1-\frac{\alpha}{n}}.$$

[3] Assuming that a decays in ξ at infinity, the condition (i) follows from (ii) by Sobolev embeddings, but is written separately for reference purposes.

Hence,

$$\begin{aligned}|a(x,D)f(x)| &\lesssim (Mf(x))^{\frac{\alpha}{n}} \int_{\mathbb{R}^n} (M\hat{f}(\xi))^{1-\frac{\alpha}{n}} \Big| b(x,\xi) \Big| d\xi \\ &\lesssim (Mf(x))^{\frac{\alpha}{n}} \big\| b(x,\cdot) \big\|_{L^{\frac{2n}{n+\alpha}}} \big\| (M\hat{f})^{\frac{n-\alpha}{n}} \big\|_{L^{\frac{2n}{n-\alpha}}} \\ &\lesssim (Mf(x))^{\frac{\alpha}{n}} \big\| b(x,\cdot) \big\|_{L^{\frac{2n}{n+\alpha}}} \big\| f \big\|_{L^2}^{1-\frac{\alpha}{n}}\end{aligned}$$

for a.e. x. This proves (2.8) for f in Schwartz class. We may then extend by continuity to $f \in L^2$. Therefore, we have

$$\begin{aligned}\|a(x,D)f(x)\|_{L^2} &\lesssim \|Mf\|_{L^2}^{\frac{\alpha}{n}} \|b\|_{L_x^{\frac{2n}{n-\alpha}} L_\xi^{\frac{2n}{n+\alpha}}} \|f\|_{L^2}^{1-\frac{\alpha}{n}} \\ &\lesssim \|Mf\|_{L^2}^{\frac{\alpha}{n}} \|(-\Delta_\xi)^{\frac{\alpha}{2}} a\|_{L_x^{\frac{2n}{n-\alpha}} L_\xi^{\frac{2n}{n+\alpha}}} \|f\|_{L^2}^{1-\frac{\alpha}{n}} \\ &\lesssim \|(-\Delta_\xi)^{\frac{\alpha}{2}} a\|_{L_x^{\frac{2n}{n-\alpha}} L_\xi^{\frac{2n}{n+\alpha}}} \|f\|_{L^2}.\end{aligned}$$

□

We conclude this section with an estimate on pointwise multipliers which will be needed in the proof of Theorem 1.1 in Section 3. First note that if $0 \le r < \frac{n}{2}$ and $q \in L^{\frac{n}{2r}}$ then multiplication by q yields a bounded operator from the homogeneous Sobolev space $\dot{H}^r(\mathbb{R}^n)$ to its dual $\dot{H}^{-r}(\mathbb{R}^n)$. (This follows easily from the boundedness of the Sobolev embedding of $\dot{H}^r(\mathbb{R}^n)$ in $L^{\frac{2n}{n-2r}}$). For the concentration compactness arguments in Section 3 we will need an extension of this result to a larger space of potentials q, with negative regularity index. Classes of pointwise multipliers between Sobolev spaces have been extensively studied (see for example [34], [32] and further references given there).

We show that multiplication by any q in the union of the homogeneous Besov spaces $\dot{B}_\infty^{\frac{n}{p}-2r,p}$ with $2 \le p < n/r$ yields a bounded operator from $\dot{H}^r(\mathbb{R}^n)$ to $\dot{H}^{-r}(\mathbb{R}^n)$. We use the following notation for the norm of the homogeneous Besov space:

$$\|f\|_{\dot{B}_q^{s,p}} = \Big(\sum_{k\in\mathbb{Z}} (2^{ks} \|P_k f\|_{L^p})^q \Big)^{1/q}$$

where P_k are the Littlewood-Paley projections, $1 \le q, p \le \infty$ and $s \in \mathbb{R}$.

Theorem 2.4. *Let $0 < r < n/2$ and $p \in [2, n/r)$. Then the following bilinear estimate holds:*

$$\|qu\|_{\dot{H}^{-r}(\mathbb{R}^n)} \lesssim \|q\|_{\dot{B}_\infty^{\frac{n}{p}-2r,p}(\mathbb{R}^n)} \|u\|_{\dot{H}^r(\mathbb{R}^n)}. \tag{2.9}$$

Proof. We use a dyadic Littlewood-Paley decomposition

$$1 = \sum_{k\in\mathbb{Z}} P_k$$

where P_k are the standard dyadic Littlewood-Paley operators, which are localized in the frequency regions $A_k = \{\xi : 2^{k-1} < |\xi| < 2^{k+1}\}$. We will show that the dyadic components $P_k(qu)$ of qu satisfy the correct bound with off-diagonal decay,

$$\|P_k(qu)\|_{\dot{H}^{-r}} \lesssim \|q\|_{\dot{B}_\infty^{\frac{n}{p}-2r,p}} \sum_{k''} 2^{-c|k-k''|} \|P_{k''} u\|_{\dot{H}^r}, \qquad c > 0 \tag{2.10}$$

This in turn easily implies (2.9).

To prove (2.10) we write

$$P_k(qu) = \sum_{(k',k'')\,\in\,\mathcal{A}_k} P_k\big(P_{k'} q P_{k''} u\big),$$

where the sum is taken over the set

$$\mathcal{A}_k = \{(k',k'') \in \mathbb{Z}^2 : A_k \cap (A_{k'} + A_{k''}) \neq \emptyset\}.$$

Then we have

$$\|P_k(qu)\|_{\dot H^{-r}} \lesssim \sum_{(k',k'')\,\in\,\mathcal{A}_k} 2^{-rk} \|P_k(P_{k'}qP_{k''}u)\|_{L^2}. \tag{2.11}$$

To estimate the terms in the above sum we use Bernstein inequality applied to the Littlewood-Paley projections:

$$\|P_k f\|_{L^t} \leq 2^{kn(1/s-1/t)} \|P_k f\|_{L^s} \tag{2.12}$$

for $1 \leq s \leq t \leq \infty$. We consider the three cases in the Littlewood-Paley trichotomy:

(i) Low-high interactions, $|k''-k| \leq 2$, $k' \leq k+2$. Here we estimate

$$\|P_{k'}q\|_{L^\infty} \lesssim 2^{2rk'} \|q\|_{\dot B_\infty^{\frac{n}{p}-2r,p}}$$

and thus

$$\|P_k(P_{k'}qP_{k''}u)\|_{\dot H^{-r}} \lesssim 2^{2r(k'-k'')} \|q\|_{\dot B_\infty^{\frac{n}{p}-2r,p}} \|P_{k''}u\|_{\dot H^r}$$

where the k' summation is trivial.

(ii) High-low interactions, $|k'-k| \leq 2$, $k'' \leq k+2$. Here we set

$$\tilde p = \frac{2p}{p-2}, \qquad 2 < \frac{2n}{n-r} < \tilde p \leq \infty$$

and use Bernstein to place $P_{k''}u$ in L^p, estimating

$$\begin{aligned}
\|P_k(P_{k'}qP_{k''}u)\|_{\dot H^{-r}} &\lesssim 2^{-kr} \|P_{k'}qP_{k''}u\|_{L^2} \\
&\lesssim 2^{-kr} \|P_{k'}q\|_{L^p} \|P_{k''}u\|_{L^{\tilde p}} \\
&\lesssim 2^{-kr} 2^{-(\frac{n}{p}-2r)k'} \|q\|_{\dot B_\infty^{\frac{n}{p}-2r,p}(\mathbb{R}^n)} 2^{(\frac{n}{2}-\frac{n}{\tilde p})k''} 2^{-k''r} \|P_{k''}u\|_{\dot H^r} \\
&\lesssim 2^{(r-\frac{n}{p})(k-k'')} \|q\|_{\dot B_\infty^{\frac{n}{p}-2r,p}(\mathbb{R}^n)} \|P_{k''}u\|_{\dot H^r}
\end{aligned}$$

as needed since $r < \frac{n}{p}$.

(iii) High-high $\to$ low interactions, $|k'-k''| \leq 2$, $k \leq k'+2$. Here it is more efficient to use Bernstein for the product,

$$\begin{aligned}
\|P_k(P_{k'}qP_{k''}u)\|_{\dot H^{-r}} &\lesssim 2^{-kr} \|P_k(P_{k'}qP_{k''}u)\|_{L^2} \\
&\lesssim 2^{(\frac{n}{p}-r)k} \|P_{k'}qP_{k''}u\|_{L^{\tilde p'}} \\
&\lesssim 2^{(\frac{n}{p}-r)k} \|P_{k'}q\|_{L^p} \|P_{k''}u\|_{L^2} \\
&\lesssim 2^{(\frac{n}{p}-r)(k-k'')} \|q\|_{\dot B_\infty^{\frac{n}{p}-2r,p}(\mathbb{R}^n)} \|P_{k''}u\|_{\dot H^r}
\end{aligned}$$

which again suffices. This concludes the proof of (2.10) and thus the proof of the theorem.

□

3. Concentration Compactness and a d-bar Problem

In this section we prove Theorem 1.1. To recall the set-up, we seek to show that the d-bar operator L_q defined by

$$L_q u = \bar{\partial} u + q\overline{u}, \qquad L_q : \dot{H}^{\frac{1}{2}} \to \dot{H}^{-\frac{1}{2}}$$

is invertible for all $q \in L^2$, and further that its inverse satisfies the locally uniform bound

$$\|L_q^{-1}\|_{\dot{H}^{-\frac{1}{2}} \to \dot{H}^{\frac{1}{2}}} \leq C(\|q\|_{L^2}). \tag{3.1}$$

We remark that the main novelty here, and the difficult part, is the fact that we can bound the norm of L_q^{-1} uniformly for q in a bounded set in L^2. In turn, the key ingredient in the proof is a non-standard use of the method of profile decompositions, as introduced in [21].

We begin with several preliminaries. We first recall some basic properties of the solid Cauchy transform $\bar{\partial}^{-1}$. For a proof, see for instance [38] [Lemma 1.4].

Lemma 3.1. *a) If $h \in L^p$, $1 < p < 2$ and $1/p^* = 1/p - 1/2$ then*

$$\|\bar{\partial}^{-1} h\|_{L^{p^*}} \leq c_p \|h\|_{L^p}, \tag{3.2}$$

b) If $f \in L^{p_1} \cap L^{p_2}$, with $1 < p_1 < 2 < p_2$, then the function $u = \bar{\partial}^{-1} f$ satisfies

$$\|u\|_{L^\infty} \leq c_{p_1,p_2}(\|f\|_{L^{p_1}} + \|f\|_{L^{p_2}}),$$

$$|u(z_1) - u(z_2)| \leq c_{p_2} |z_1 - z_2|^{1-\frac{2}{p_2}} \|f\|_{L^{p_2}}$$

and $\lim_{|z|\to\infty} u(z) = 0$.

Next we prove a qualitative result, which asserts that L_q^{-1} is a well defined operator from $L^{\frac{4}{3}}$ to L^4:

Lemma 3.2. *Let $q \in L^2$. Then for any $f \in L^{\frac{4}{3}}$, the equation*

$$L_q u = f \tag{3.3}$$

has a unique solution $u \in L^4$.

Proof. Write $q = q_n + q_s$, where $q_n \in L^{p_1} \cap L^{p_2}$ with $1 < p_1 < 2 < p_2$ and $\|q_s\|_{L^2}$ small (see (3.7) below). Given a solution $u \in L^4$ of (3.3), we define

$$\nu := \begin{cases} e^{\bar{\partial}^{-1}\left(q_n \frac{\overline{u}}{u}\right)} & \text{if } u \neq 0 \\ 1 & \text{if } u = 0. \end{cases} \tag{3.4}$$

Then ν and $1/\nu$ are in L^∞, in view of Lemma 3.1. So $u\nu \in L^4$. Further, since $\nu \in W^{1,p}$ for $p \in [p_1, p_2]$ then we may apply the Leibnitz rule:

$$\bar{\partial}(u\nu) = (\bar{\partial} u + q_n \overline{u})\nu = (-q_s \overline{u} + f)\nu. \tag{3.5}$$

Thus, using (3.2) we have

$$\|u\nu\|_{L^4} \leq c\|q_s\|_{L^2} \|u\nu\|_{L^4} + c\|f\|_{L^{\frac{4}{3}}} \|\nu\|_{L^\infty}. \tag{3.6}$$

To prove uniqueness for (3.3), let $f = 0$ and choose q_s with

$$\|q_s\|_{L^2} \leq 1/2c, \tag{3.7}$$

Then (3.6) yields

$$\|u\nu\|_{L^4} \le \frac{1}{2}\|u\nu\|_{L^4},$$

so $u = 0$.

To show existence, we write (3.3) as

$$\mathcal{B}u = \bar{\partial}^{-1} f, \qquad \mathcal{B} = I + \bar{\partial}^{-1}(q\,\bar{\cdot}). \tag{3.8}$$

The operator $\bar{\partial}^{-1}(q\,\bar{\cdot})$ is compact $L^4 \to L^4$ (see Lemma 7.1). It follows by the Fredholm alternative that $\mathcal{B}$ is invertible in the $L^4 \to L^4$ topology, and we can solve for $u = \mathcal{B}^{-1}\bar{\partial}^{-1} f$ in L^4. □

We continue with an easy extension of the previous Lemma.

Lemma 3.3. *For each $q \in L^2$, the operator $L_q : \dot{H}^{\frac{1}{2}} \to \dot{H}^{-\frac{1}{2}}$ is invertible and*

$$\|L_q^{-1} f\|_{\dot{H}^{\frac{1}{2}}} \le C(q)\|f\|_{\dot{H}^{-\frac{1}{2}}}. \tag{3.9}$$

Proof. Multiplication by q maps $\dot{H}^{\frac{1}{2}}$ to $\dot{H}^{-\frac{1}{2}}$, and we may rewrite (1.5) as

$$\mathcal{B}u = \bar{\partial}^{-1} f \tag{3.10}$$

where $\mathcal{B} = (I + \bar{\partial}^{-1}(q\bar{\cdot})) : \dot{H}^{\frac{1}{2}} \to \dot{H}^{\frac{1}{2}}$. Since, $\dot{H}^{\frac{1}{2}} \subset L^4$, injectivity follows from Lemma 3.2.

It remains to prove surjectivity. Let $f \in \dot{H}^{-\frac{1}{2}}$. Then $\bar{\partial}^{-1} f \in \dot{H}^{\frac{1}{2}} \subset L^4$, so the proof of Lemma 3.2 yields a solution $u \in L^4$ of $\mathcal{B}u = \bar{\partial}^{-1} f$, i.e.

$$u = -\bar{\partial}^{-1}(q\overline{u}) + \bar{\partial}^{-1} f.$$

Since $q\overline{u} \in L^{\frac{4}{3}} \subset \dot{H}^{-\frac{1}{2}}$, we have $\bar{\partial}^{-1}(q\overline{u}) \in \dot{H}^{\frac{1}{2}}$ hence also $u \in \dot{H}^{\frac{1}{2}}$ and $L_q u = f$. □

By the last lemma, the best constant $C(q)$ in (1.6) is well-defined and finite for each $q \in L^2$. The next step is to study the dependence of L_q^{-1} and of $C(q)$ on q:

Lemma 3.4. *The operator L_q^{-1} depends smoothly on $q \in L^2$, and the best constant $C(q)$ in* (1.6) *has a local Lipschitz dependence on q. More precisely, given $q_0 \in L^2$ there exists $\epsilon > 0$, depending only on $C(q_0)$, so that within the ball $B(q_0, \epsilon)$ the map*

$$q \to L_q^{-1}$$

is analytic, with a uniform Lipschitz bound

$$\|L_{q_1}^{-1} - L_{q_2}^{-1}\|_{\dot{H}^{-\frac{1}{2}} \to \dot{H}^{\frac{1}{2}}} \lesssim C(q_0)^2 \|q_1 - q_2\|_{L^2} \tag{3.11}$$

as well as

$$|C(q_1) - C(q_2)| \lesssim C(q_0)^2 \|q_1 - q_2\|_{L^2} \tag{3.12}$$

Proof. For $q \in B(q_0, \epsilon)$ we rewrite the equation

$$L_q u = f$$

as

$$L_{q_0} u = (q_0 - q)\bar{u} + f$$

and further as

$$u = L_{q_0}^{-1} f + L_{q_0}^{-1}((q_0 - q)\bar{u}).$$

If $\|q-q_0\|_{L^2} \ll C(q_0)^{-1}$ then the above equation can be solved by a Neumann series. In particular we obtain the analytic dependence of u on q, as well as the bounds

$$\|u\|_{\dot H^{\frac12}} \lesssim C(q_0)\|f\|_{\dot H^{-\frac12}}$$

and

$$\|u - L_{q_0}^{-1} f\|_{\dot H^{\frac12}} \lesssim C(q_0)^2 \|q_0 - q\|_{L^2}\|f\|_{\dot H^{-\frac12}}.$$

The latter leads to the desired Lipschitz bound for L_q^{-1}, by repeating the same argument with q, q_0 replaced by q_1, q_2 in the same ball. □

It remains to prove that the $C(q)$ bound is uniform for q in a bounded set in L^2. We denote by

$$C(R) = \sup\{C(q);\ \|q\|_{L^2} \leq R\}, \qquad C : \mathbb{R}^+ \to [0,\infty],$$

We need to prove that $C(R)$ is finite for all $R > 0$. This is the case for R small, as can be seen from the proof of the previous lemma by taking $q_0 = 0$. We also have:

Lemma 3.5. *The function $C(R)$ is nondecreasing and continuous.*

Proof. The monotonicity is obvious. The continuity is due to the uniformity in the previous lemma. Precisely, if $C(R-0) = \lim_{r\nearrow R} C(r)$ is finite then for $\|q_0\|_{L^2} < R$, the ball size ϵ in the previous lemma depends only on $C(R-0)$. This yields a uniform Lipschitz constant for $C(q)$ in $B(0, R+\epsilon)$, and the desired continuity (indeed local Lipschitz continuity) follows. □

To prove that $C(R)$ is finite for all R we argue by contradiction. Choose $R_0 > 0$ minimal so that

$$C(R_0) = \infty.$$

Then for $R < R_0$ we have $C(R) < \infty$, and, by the continuity property,

$$\lim_{R\to R_0} C(R) = \infty.$$

Thus there exists sequence q_n so that

$$R_0 > \|q_n\|_{L^2} \to R_0$$

and

$$\|L_{q_n}^{-1}\|_{\dot H^{-\frac12}\to \dot H^{\frac12}} \to \infty.$$

If we knew that q_n converged (say on a subsequence) to some $q \in L^2$ then we would have

$$\|L_{q_n}^{-1}\|_{\dot H^{-\frac12}\to \dot H^{\frac12}} \to \|L_q^{-1}\|_{\dot H^{-\frac12}\to \dot H^{\frac12}} \neq \infty$$

which would contradict the minimality of R_0.

However, there are two obvious obstructions to compactness arising from the symmetries of the problem, namely translation and scaling. Any such symmetry can be described using a positive scale factor λ and a translation distance y. We introduce the notation

$$S(\lambda, y)q = \lambda q(\lambda(x-y)).$$

Then

$$C(q) = C(S(\lambda, y)q).$$

In view of this fact, one might try to show that we have compactness up to symmetries, i.e. that (on a subsequence) there exist λ_n, y_n so that

$$S(\lambda_n, y_n)q_n \to q \qquad \text{in } L^2.$$

Since the constant $C(q)$ is easily seen to be invariant with respect to symmetries, this would again lead to contradiction.

This seems to be still too much to ask. We will prove instead a weaker compactness statement, which will nevertheless be sufficient to establish the finiteness of $C(R)$. As an intermediate step in establishing a compactness property, we first note that, in view of Theorem 2.4, we can extend the perturbative theory to a larger space, namely

$$q \in \dot{B}_{\infty}^{-\frac{1}{3},3}.$$

The exact exponents for the Besov space are not important, just the fact that this space has negative Sobolev regularity and the same scaling as L^2, and in particular we have the Sobolev embedding

$$L^2 \subset \dot{B}_{\infty}^{-\frac{1}{3},3}.$$

We note below a special case of Theorem 2.4:

Lemma 3.6. *The following bilinear estimate holds:*

$$\|qu\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|q\|_{\dot{B}_{\infty}^{-\frac{1}{3},3}} \|u\|_{\dot{H}^{\frac{1}{2}}}. \tag{3.13}$$

Proof. See Theorem 2.4. □

Using this we obtain the following extension of Lemma 3.4:

Lemma 3.7. *Given $q_0 \in L^2$ there exists $\epsilon > 0$, depending only on $C(q_0)$, so that within the ball*

$$\|q - q_0\|_{\dot{B}_{\infty}^{-\frac{1}{3},3}} \leq \epsilon,$$

the map

$$q \to L_q^{-1}$$

is analytic, with a uniform Lipschitz bound

$$\|L_{q_1}^{-1} - L_{q_2}^{-1}\|_{\dot{H}^{-\frac{1}{2}} \to \dot{H}^{\frac{1}{2}}} \lesssim C(q_0)^2 \|q_1 - q_2\|_{\dot{B}_{\infty}^{-\frac{1}{3},3}}, \tag{3.14}$$

as well as

$$|C(q_1) - C(q_2)| \lesssim C(q_0)^2 \|q_1 - q_2\|_{\dot{B}_{\infty}^{-\frac{1}{3},3}}. \tag{3.15}$$

The proof is identical to the proof of Lemma 3.4, and is omitted. This property shows that it would suffice to establish the weaker convergence property

$$S(\lambda_n, y_n) q_n \to q \qquad \text{in } \dot{B}_{\infty}^{-\frac{1}{3},3}.$$

Now we return to our compactness question. The discussion above suggests that we should look at compactness modulo symmetries. The last lemma tells us that we only need convergence in the weaker $\dot{B}_{\infty}^{-\frac{1}{3},3}$ topology. Still, for an arbitrary sequence q_n which is bounded in L^2 even this is too much to hope for, as the q_n's may be split into pieces which are driven by different symmetries. The situation is very accurately described using a profile decomposition, see [21] and also [41]:

Proposition 3.8. *Let q_n be a bounded sequence in L^2. Then up to the extraction of a subsequence, it can be decomposed in the following way:*

$$\forall l \in \mathbb{N}, q_n = \sum_{k=1}^{l} S(\lambda_n^k, y_n^k) q^k + q_n^l \tag{3.16}$$

where the functions q^j are in L^2 for all $j \in \mathbb{N}$, and the remainders q_n^l are uniformly bounded in L^2 and satisfy

$$\lim_{l\to\infty} \limsup_{n\to\infty} \|q_n^l\|_{\dot{B}_\infty^{-\frac{1}{3},3}} = 0, \tag{3.17}$$

and where for any $k \in \mathbb{N}$, (λ_n^k, y_n^k) is a sequence in $\mathbb{R}^+ \times \mathbb{R}^2$ with the property that for every $j \neq k$ we have either

$$\lim_{n\to\infty} \frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} = \infty \tag{3.18}$$

or

$$\lambda_n^j = \lambda_n^k, \qquad \lim_{n\to\infty} |y_n^j - y_n^k|\lambda_n^j = \infty. \tag{3.19}$$

Furthermore, for each l we have

$$\|q_n\|_{L^2}^2 = \sum_{k=1}^{l} \|q^k\|_{L^2}^2 + \|q_n^l\|_{L^2}^2 + o(1) \tag{3.20}$$

as $n \to \infty$.

We remark here that this is the elliptic version of the profile decomposition, as opposed to the wave equation version [8] or the Schrödinger version [35].

We also remark that the original elliptic profile decomposition of Gérard [21] is for $\dot{H}^s$ functions with $0 < s < 1$. The transition to the statement above is straightforward, simply by choosing $s = \frac{1}{3}$ and applying a $|D|^{\frac{1}{3}}$ operator.

We apply this decomposition to our sequence q_n, and will distinguish two scenarios:

- Exactly one profile. Then up to symmetries we have

  $$q_n \to q^1 \qquad \text{in } \dot{B}_\infty^{-\frac{1}{3},3}$$

  and, according to the prior discussion, the proof of Theorem 1.1 is concluded.
- more than one profile. Then in view of (3.20) we must have

  $$\sup_k \|q^k\|_{L^2} = R < R_0. \tag{3.21}$$

  Hence in this case we control the operator norms $\|L_{q^k}^{-1}\|$ associated to each profile uniformly, and we will use this to control $\|L_{q_n}^{-1}\|$.

To eliminate the case of multiple profiles we will use the solutions to the L_{q^k} equations to construct a solution to L_{q_n}. Precisely, in order to complete the proof of Theorem 1.1 it suffices to prove the following:

Proposition 3.9. *Let q_n be a bounded sequence of L^2 functions with a profile decomposition as above, so that* (3.21) *holds. Then we have*

$$\limsup_{n\to\infty} C(q_n) \lesssim C(R). \tag{3.22}$$

Proof. For $f \in \dot H^{-\frac{1}{2}}$ we seek to solve

$$L_{q_n} u = f.$$

By Lemma 3.7, the tails q_n^l play a perturbative role in this analysis. Precisely, by choosing l large enough and n large enough we can insure that

$$\|q_n^l\|_{\dot B_\infty^{-\frac{1}{3},3}} \ll_{C(R)} 1$$

and thus neglect them. Thus, for the rest of the proof we simply fix l and assume that

$$q_n = \sum_{k=1}^{l} S(\lambda_n^k, y_n^k) q^k.$$

Here its components $S(\lambda_n^k, y_n^k) q^k$ are localized around y_n^k at frequency scale λ_n^k, and are separating as $n \to \infty$. To take advantage of this we also split f in a linear fashion as

$$f = \sum f_n^k + f_n^{out}$$

so that f_n^k will primarily interact only with $S(\lambda_n^k, y_n^k) q^k$, and f_n^{out} does not interact with any of the $S(\lambda_n^k, y_n^k) q^k$. Then we seek an approximate solution of the form

$$u_n^{app} = \sum u_n^k + u_n^{out}, \qquad u_n^k = L^{-1}_{S(\lambda_n^k, y_n^k) q^k} f_n^k, \qquad u_n^{out} = L_0^{-1} f_n^{out}.$$

Thus we have

$$L_{q_n} u_n^{app} = \sum_{k\neq j} S(\lambda_n^k, y_n^k) q^k \bar u_n^j + \sum_k S(\lambda_n^k, y_n^k) q^k \bar u_n^{out} + f.$$

To succeed, we need to insure that we have the following properties:

(P1) Almost orthogonal decomposition for f,

$$\|f_n^{out}\|^2_{\dot H^{-\frac{1}{2}}} + \sum_k \|f_n^k\|^2_{\dot H^{-\frac{1}{2}}} \lesssim \|f\|^2_{\dot H^{-\frac{1}{2}}} + o_n(1) \|f\|^2_{\dot H^{-\frac{1}{2}}}. \tag{3.23}$$

(P2) Almost orthogonal decomposition for u_n^{app},

$$\|u_n^{app}\|^2_{\dot H^{\frac{1}{2}}} \lesssim \|u_n^{out}\|^2_{\dot H^{\frac{1}{2}}} + \sum_k \|u_n^k\|^2_{\dot H^{\frac{1}{2}}} + o_n(1) \|f\|^2_{\dot H^{-\frac{1}{2}}}. \tag{3.24}$$

(P3) Negligible off-diagonal interactions,

$$\begin{aligned} \|S(\lambda_n^k, y_n^k) q^k u_n^j\|_{\dot H^{-\frac{1}{2}}} &= o_n(1) \|f\|_{\dot H^{-\frac{1}{2}}} \qquad k \neq j, \\ \|S(\lambda_n^k, y_n^k) q^k u_n^{out}\|_{\dot H^{-\frac{1}{2}}} &= o_n(1) \|f\|_{\dot H^{-\frac{1}{2}}}. \end{aligned} \tag{3.25}$$

Here we remark that all implicit constants should be universal. However, all expressions $o_n(1)$, which decay to zero as $n \to \infty$, may have a decay rate that depends on all parameters in our problem, namely q^k, λ_n^k and y_n^k (but not on f). It is for this reason that the $o_n(1)$ term is not included in the first term on the right in (P1).

We first verify that these three properties (P1), (P2) and (P3) suffice in order to prove Proposition 3.9. To see that, we observe that in view of (3.21), the approximate solution $u_n^{app} = u_n^{app}(f)$ satisfies

$$\|u_n^{app}\|_{\dot H^{\frac12}} \lesssim (C(R) + o_n(1))\|f\|_{\dot H^{-\frac12}}, \qquad \|L_{q_n} u_n^{app} - f\|_{\dot H^{-\frac12}} \lesssim o_n(1)\|f\|_{\dot H^{-\frac12}}.$$

If n is large enough then the error can be made arbitrarily small, therefore a simple reiteration scheme would allow us to pass from an approximate solution to an exact solution. Thus Proposition 3.9 is proved.

It remains to construct a decomposition with the above properties. In order to construct the decomposition functions f_n we introduce a family of truncation operators $T^\mu(\lambda, y)$, where (λ, y) are associated to our symmetry group and $\mu \geq 1$ is an additional dimensionless scale parameter. Precisely, we set

$$T^\mu(\lambda, y) = \chi\left(\mu^{-2}\lambda(x-y)\right) P_{[\lambda/\mu, \lambda\mu]}$$

where χ is a Schwartz function with compactly supported Fourier transform, and so that near zero we have

$$1 - \chi(x) = O(|x|^N).$$

The role of the support assumption is to insure that our operators $T^\mu(\lambda, y)$ are frequency localized in the region

$$\lambda/\mu \leq |\xi| \leq \lambda\mu.$$

With this notation, the components f_n^k of f are defined by

$$f_n^k = T^{\mu_n}(\lambda_n^k, y_n^k) f$$

using a slowly increasing sequence $\mu_n \to \infty$. Here the meaning of slowly is taken relative to the growth rates in (3.18), (3.19). The reason we let $\mu_n \to \infty$ is so that in the limit this localization captures the effect of all of q^k.

In order to prove that the functions f_n^k and q_n^k have the desired properties we first consider some simple properties of the operators $T^\mu(\lambda, y)$. The first set of properties involve a single scale:

Lemma 3.10. *The operators $T^\mu(\lambda, y)$ have the following properties uniformly in $(\lambda, y) \in \mathbb{R}^+ \times \mathbb{R}^2$:*

(i) They are bounded in $\dot H^s$ for $|s| \leq \frac12$, uniformly in $\mu \geq 1$.

(ii) We have the decay property

$$\|(1 - T^{\mu^2}(\lambda, y)) T^\mu(\lambda, y)\|_{\dot H^s \to \dot H^s} \lesssim \mu^{-N}. \tag{3.26}$$

(iii) For $q \in L^2$ we have the bound

$$\lim_{\mu \to \infty} \|S(\lambda, y) q (1 - T^\mu(\lambda, y))\|_{\dot H^{\frac12} \to \dot H^{-\frac12}} = 0. \tag{3.27}$$

(iv) For $q \in L^2$ we have the commutator bound

$$\lim_{\mu \to \infty} \|[L_{S(\lambda,y)q}, T^\mu(\lambda, y)]\|_{\dot H^{\frac12} \to \dot H^{-\frac12}} = 0. \tag{3.28}$$

Proof of Lemma 3.10. We first note that all of the properties in the lemma are scale and translation invariant, therefore we can simply set $\lambda = 1$ and $y = 0$ and drop them from the notation.

Next, we note that the projector part of T^μ selects the frequencies $[\mu^{-1}, \mu]$, whereas the multiplication part is localized at frequency μ^{-2} which is much smaller. This implies that T^μ maps every dyadic frequency shell into a slight enlargement of itself. Because of this it suffices to prove the bounds in part (i) and (ii) for functions in a fixed dyadic shell, and the output will also be localized in a double dyadic shell. But on a fixed dyadic frequency shell the $\dot H^s$ weights $|\xi|^s$ have a fixed size. Consequently the $\dot H^s$ bounds in (i) and (ii) are all equivalent uniformly in $\mu \gg 1$, and we can simply set $s = 0$.

The L^2 boundedness of T^μ is trivial, uniformly in μ. Given the frequency localization of the multiplicative part we have

$$(1 - T^{\mu^2})T^\mu = (1 - \chi(\mu^{-4}x)P_{[\mu^{-2},\mu^2]})\chi(\mu^{-2}x)P_{[\mu^{-1},\mu]} = (1 - \chi(\mu^{-4}x))\chi(\mu^{-2}x)P_{[\mu^{-1},\mu]}$$

so (ii) also follows.

For (iii) we write

$$q(1 - T^\mu) = (1 - P_{[\mu^{-\frac12},\mu^{\frac12}]})q(1 - P_{[\mu^{-1},\mu]}) + P_{[\mu^{-\frac12},\mu^{\frac12}]}q(1 - P_{[\mu^{-1},\mu]}) + q(1 - \chi(\mu^{-2}x))P_{[\mu^{-1},\mu]}$$

and use the multiplicative property $L^2 \cdot \dot H^{\frac12} \subset \dot H^{-\frac12}$. The first and the last term decay since $(1 - P_{[\mu^{-\frac12},\mu^{\frac12}]})q$ and $q(1 - \chi(\mu^{-2}x))$ decay to zero in L^2. The middle term decays due to the increasing frequency separation between the two factors. Precisely, by a careful use of Bernstein's inequality we have the dyadic bound

$$\|P_{\lambda_1} q P_{\lambda_2} u\|_{\dot H^{-\frac12}} \lesssim \min\left\{\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_1}\right\}^{\frac12} \|P_{\lambda_1} q\|_{L^2} \|P_{\lambda_2} u\|_{\dot H^{\frac12}}$$

which after dyadic summation yields

$$\|P_{[\mu^{-\frac12},\mu^{\frac12}]} q(1 - P_{[\mu^{-1},\mu]})u\|_{\dot H^{-\frac12}} \lesssim \mu^{-\frac14} \|q\|_{L^2} \|u\|_{\dot H^{\frac12}}$$

For (iv) we treat separately the $\bar\partial$ and the q part of L_q. For the q part we disregard the commutator structure and write

$$[q, T^\mu] = q(T^\mu - 1) - (T^\mu - 1)q$$

where the first part decays due to (iii) and the second term is quite similar. For the $\bar\partial$ part we write

$$[\bar\partial, T^\mu] = \mu^{-2}(\bar\partial\chi)(\mu^{-2}x)P_{[\mu^{-1},\mu]}$$

which acts separately on each dyadic frequency. The $\dot H^{\frac12} \to \dot H^{-\frac12}$ norm of $P_{[\mu^{-1},\mu]}$ is μ, which is more than compensated for by the μ^{-2} factor. This completes the proof of Lemma 3.10 $\square$

The next lemma is related to the scale separation properties:

Lemma 3.11. *In the setting of Proposition 3.8, assume that $\mu_n \to \infty$ slowly enough. Then we have:*

$$\lim_{n\to\infty} \||D|^{s_1} T^{\mu_n}(\lambda_n^j, y_n^j)|D|^{s_2} T^{\mu_n}(\lambda_n^k, y_n^k)|D|^{s_3}\|_{L^2\to L^2} = 0, \qquad j \neq k, \quad s_1+s_2+s_3 = 0 \tag{3.29}$$

and the similar result with either T^{μ_n} replaced by $(T^{\mu_n})^*$.

Proof of Lemma 3.11. Denote

$$Q^{jk} = |D|^{s_1} T^{\mu_n}(\lambda_n^j, y_n^j)|D|^{s_2} T^{\mu_n}(\lambda_n^k, y_n^k)|D|^{s_3}.$$

We consider the two scenarios in (3.18) and (3.19). In the first case, for large enough n the operators $T^{\mu_n}(\lambda_n^j, y_n^j)$ and $T^{\mu_n}(\lambda_n^k, y_n^k)$ have disjoint frequency localizations so $Q^{jk} = 0$.

In the second case we have $\lambda_n^k = \lambda_n^j := \lambda_n$. Further, all operators in Q^{jk} act separately on different dyadic shells, so by orthogonality we can fix the input frequency and insert dyadic frequency localizations in all multipliers. Thus it suffices to consider operators of the form

$$\tilde{Q}^{jk} = R_\lambda^1 \chi(\mu_n^{-2}\lambda_n(x - y_n^k)) R_\lambda^2 \chi(\mu_n^{-2}\lambda_n(x - y_n^j)) R_\lambda^3$$

where R_λ^i are smooth bounded multipliers localized at a frequency $\lambda \in [\mu_n^{-1}\lambda_n, \mu_n\lambda_n]$ (which also depends on n). The condition $s_1 + s_2 + s_3 = 0$ guarantees that the Sobolev weights $|\xi_1|^{s_1}|\xi_2|^{s_2}|\xi_3|^{s_3}$ cancel out if $|\xi_j| \approx \lambda$. This applies equally whether we work with the operators $T^{\mu_n}(\lambda_n^j, y_n^j)$ or with their adjoints. Set

$$\tilde{\mu}_n^{-1}\lambda = \mu_n^2\lambda_n, \qquad \tilde{\mu}_n \in [\mu_n, \mu_n^3].$$

We can rescale to set $\lambda = 1$, with y_n^k rescaled accordingly. Then $\tilde{Q}^{jk}$ become

$$\tilde{Q}^{jk} = R^1 \chi(\tilde{\mu}_n^{-1}(x - y_n^k)) R^2 \chi(\tilde{\mu}_n^{-1}(x - y_n^j)) R^3$$

where $\tilde{\mu}_n \in [\mu_n, \mu_n^3]$ goes to infinity slowly enough, so that also

$$\tilde{\mu}_n^{-1}|y_n^k - y_n^j| \to \infty.$$

Then the operators R^i have uniformly bounded Schwartz kernels, so the desired conclusion follows from the spatial separation of the two bump functions. This completes the proof of Lemma 3.11. □

It remains to use the two lemmas above in order to prove the three properties (P1), (P2) and (P3).

For (P1) it is easily seen that the operators $\sum_k T^{\mu_n}(\lambda_n^k, y_n^k)$ are bounded in $\dot{H}^{-\frac{1}{2}}$, uniformly for large n, therefore it remains to show that in the limit f_n^k are almost orthogonal,

$$\lim_{n\to\infty} \langle f_n^k, f_n^j \rangle_{\dot{H}^{-\frac{1}{2}}} = 0, \qquad k \neq j$$

To see this we write

$$\langle f_n^k, f_n^j \rangle_{\dot{H}^{-\frac{1}{2}}} = \langle |D|^{-\frac{1}{2}} Q^{kj} |D|^{-\frac{1}{2}} f, f \rangle$$

where

$$Q^{kj} = |D|^{\frac{1}{2}} (T^{\mu_n}(\lambda_n^k, y_n^k))^* |D|^{-1} T^{\mu_n}(\lambda_n^j, y_n^j)|D|^{\frac{1}{2}}$$

Then it suffices to show that

$$\lim_{n\to\infty} \|Q^{kj}\|_{L^2 \to L^2} = 0$$

which follows from (3.29).

Now we consider the property (P2), for which it suffices to show that u_n^k are almost orthogonal in the limit,

$$\lim_{n\to\infty} \langle u_n^k, u_n^j \rangle_{\dot{H}^{\frac{1}{2}}} = 0, \qquad k \neq j.$$

Unfortunately u_n^k no longer share the sharp localization of f_n^k. However, the bulk of u_n^k does. Precisely, we split

$$u_n^k = T^{\mu_n^2}(\lambda_n^k, y_n^k) u_n^k + (1 - T^{\mu_n^2}(\lambda_n^k, y_n^k)) u_n^k. \tag{3.30}$$

For the first term the same argument as the one used above for f_n^k applies, except that we need to use the operators

$$\tilde{\tilde{Q}}^{kj} = |D|^{-\frac{1}{2}} T^{\mu_n^2}(\lambda_n^k, y_n^k)^* |D| T^{\mu_n^2}(\lambda_n^j, y_n^j) |D|^{-\frac{1}{2}}$$

and show that

$$\lim_{n\to\infty} \|\tilde{\tilde{Q}}^{kj}\|_{L^2\to L^2} = 0.$$

This again is a consequence of (3.29).

The second term, on the other hand, converges to 0. To see that we compute

$$L_{S(\lambda_n^k, y_n^k)q^k}(1 - T^{\mu_n^2}(\lambda_n^k, y_n^k))u_n^k = (1 - T^{\mu_n^2}(\lambda_n^k, y_n^k))T^{\mu_n}(\lambda_n^k, y_n^k)f - [L_{S(\lambda_n^k, y_n^k)q^k}, T^{\mu_n^2}(\lambda_n^k, y_n^k)]u_n^k$$

where we want to show that both terms decay to zero in $\dot{H}^{-\frac{1}{2}}$. But this follows from (3.26) for the first term, respectively (3.28) for the second.

Finally we consider the last property (P3). First we use the same decomposition (3.30) as above for u_n^k to write

$$\begin{aligned} S(\lambda_n^j, y_n^j)q^j u_n^k = {} & S(\lambda_n^j, y_n^j)q^j(1 - T^{\mu_n^2}(\lambda_n^k, y_n^k))u_n^k + S(\lambda_n^j, y_n^j)q^j T^{\mu_n^2}(\lambda_n^j, y_n^j)T^{\mu_n^2}(\lambda_n^k, y_n^k)u_n^k \\ & + S(\lambda_n^j, y_n^j)q^j(1 - T^{\mu_n^2}(\lambda_n^j, y_n^j))T^{\mu_n^2}(\lambda_n^k, y_n^k)u_n^k. \end{aligned}$$

The first term decays to zero since its second factor $(1 - T^{\mu_n^2}(\lambda_n^k, y_n^k))u_n^k$ decays to zero in $\dot{H}^{\frac{1}{2}}$, as established above. For the second we need to show that

$$\lim_{n\to\infty} \||D|^{\frac{1}{2}} T^{\mu_n^2}(\lambda_n^j, y_n^j) T^{\mu_n^2}(\lambda_n^k, y_n^k)|D|^{-\frac{1}{2}}\|_{L^2\to L^2} = 0, \qquad j \neq k$$

which is a consequence of (3.29). For the third we need

$$\lim_{n\to\infty} \||D|^{-\frac{1}{2}} S(\lambda_n^j, y_n^j)q^j(1 - T^{\mu_n}(\lambda_n^j, y_n^j))|D|^{-\frac{1}{2}}\|_{L^2\to L^2} = 0. \tag{3.31}$$

which follows from (3.27).

Finally for the outer part we write

$$u_n^{out} = \bar{\partial}^{-1}(1 - T^{\mu_n}(\lambda_n^k, y_n^k))f - \sum_{j\neq k} \bar{\partial}^{-1} T^{\mu_n}(\lambda_n^j, y_n^j)f.$$

The summand in the second term is nothing but u_n^j evaluated in the special case when $q_j = 0$. Hence, it is covered by the prior analysis. For the first term we write

$$\bar{\partial}^{-1}(1 - T^{\mu_n}(\lambda_n^k, y_n^k))f = (1 - T^{\mu_n}(\lambda_n^k, y_n^k))\bar{\partial}^{-1}f + \bar{\partial}^{-1}[L_0, T^{\mu_n}(\lambda_n^k, y_n^k)]\bar{\partial}^{-1}f.$$

For the first term we get decay when matched against q^k, by (3.27). For the second we disregard q^k and use instead the commutator bound (3.28). This completes the proof of Proposition 3.9. □

4. The Scattering Transform

The equations (1.1) arise as the compatibility condition of the Lax pair

$$\begin{cases} \bar{\partial} m^1 & = q m^2 \\ (\partial + ik)m^2 & = \bar{q} m^1 \end{cases} \tag{4.1}$$

and

$$\begin{cases} \quad i\partial_t m^1 + \partial^2 m^1 + 2ik\partial m^1 - q\bar{\partial} m^2 + \bar{\partial} q m^2 + 4g m^1 = 0 \\ - \quad i\partial_t m^2 + \bar{\partial}^2 m^2 - ik\partial m^2 - \overline{q}\partial m^1 + \partial \overline{q} m^1 + 4\overline{g} m^2 = 0. \end{cases} \tag{4.2}$$

The construction of the Scattering Transform only involves solutions of the Dirac system (4.1). The equations (4.2) are used afterwards, in order to establish its time evolution (1.12).

Assuming q is a Schwartz function, Beals and Coifman [10] studied Jost-type solutions to (4.1) with boundary conditions

$$\begin{cases} m^1 \to 1 \text{ as } |z| \to \infty \\ m^2 \to 0 \text{ as } |z| \to \infty. \end{cases} \tag{4.3}$$

With the substitutions

$$m_\pm = m^1 \pm e_{-k}\overline{m^2} \tag{4.4}$$

they obtained the decoupled pseudo-analytic equations (1.3) which we introduced in Section 1. They also established the dual set of equations

$$\begin{cases} \dfrac{\partial}{\partial \bar{k}} m^1 = e_{-k}\mathbf{s}\overline{m^2} \\ \dfrac{\partial}{\partial \bar{k}} m^2 = e_{-k}\mathbf{s}\overline{m^1} \end{cases} \tag{4.5}$$

which is equivalent to (1.10).

Throughout this section we will use both m^1 and m^2 as well as the functions $m_\pm$ defined in Section 1. We have

$$\begin{aligned} m^1 &= \frac{1}{2}(m_+ + m_-) = \frac{1}{2}(n_+ + n_-) \\ m^2 &= \frac{1}{2}e_{-k}(\overline{m_+ - m_-}) = \frac{1}{2}(n_+ - n_-). \end{aligned} \tag{4.6}$$

For the Scattering Transform (1.4) we will also use the expression

$$\mathcal{S}q(k) = -\frac{i}{\pi}\int_{\mathbb{R}^2} e_k(z)\overline{q(z)}m^1(z,k)dz. \tag{4.7}$$

Our goal is to solve (1.3) for $q \in L^2$ and show that the corresponding scattering data $\mathbf{s}$ is in L^2. To get started we rewrite the equations (1.3) in terms of the functions $m_\pm - 1$, which have the virtue that they decay at infinity:

$$\frac{\partial}{\partial \bar{z}}(m_\pm - 1) = \pm e_{-k} q(\overline{m_\pm} - 1) \pm e_{-k} q.$$

The L^4 solvability for equations of this type is considered in the next lemma.

Lemma 4.1. *Suppose $q \in L^2$. Then for any $f \in L^2$ and any $k \in \mathbb{C}$ such that $M\hat{f}(k) < \infty$, there is a unique solution $u(\cdot,k) \in L^4$ of*

$$\bar{\partial} u + e_{-k} q \overline{u} = e_{-k} f. \tag{4.8}$$

Moreover

$$\|u(\cdot,k)\|_{L^4} \le C(\|q\|_{L^2})\|\bar{\partial}^{-1}(e_k\overline{f})\|_{L^4} \le C(\|q\|_{L^2})\|f\|_{L^2}^{\frac{1}{2}}\big(M\hat{f}(k)\big)^{\frac{1}{2}}. \tag{4.9}$$

Proof. The uniqueness follows from Lemma 3.2. To prove existence, we first recall that in view of Corollary 2.2b), $\bar{\partial}^{-1}(e_{-k}f) \in L^4$ for a.e. k. Write

$$u = v + \bar{\partial}^{-1}(e_{-k}f). \tag{4.10}$$

Then u is a solution of (4.8) if and only if v solves

$$\bar{\partial}v + e_{-k}q\,\overline{v} = -e_{-k}q\partial^{-1}(e_k\overline{f}). \tag{4.11}$$

The term on the right is in $L^{\frac{4}{3}}$ for a.e. k. More precisely, by Corollary 2.2 we have

$$\|e_{-k}q\partial^{-1}(e_k\overline{f})\|_{L^{\frac{4}{3}}} \le c\|q\|_{L^2}\|f\|_{L^2}^{\frac{1}{2}}\big(M\hat{f}(k)\big)^{\frac{1}{2}}.$$

Thus, by Theorem 1.1 there is a unique solution $v \in \dot{H}^{\frac{1}{2}} \subset L^4$ for (4.11). Hence the function u defined by (4.10) solves (4.8) and satisfies (4.9). ☐

We are now ready to construct the Jost solutions $m_\pm$ for (1.3):

Lemma 4.2. *(Jost Solutions) Suppose that $q \in L^2$, then:*

a) For almost every k there exist unique solutions $m_\pm(z,k)$ of (1.3) with $m_\pm(\cdot,k) - 1 \in L^4$ and moreover,

$$\|m_\pm(\cdot,k) - 1\|_{L^4} + \|m^1(\cdot,k) - 1\|_{L^4} + \|m^2(\cdot,k)\|_{L^4} \le C(\|q\|_{L^2})\big(M\hat{q}(k)\big)^{\frac{1}{2}}. \tag{4.12}$$

In addition we have

$$\|m_\pm - 1\|_{L^4_kL^4_z} + \|m^1 - 1\|_{L^4_kL^4_z} + \|m^2\|_{L^4_kL^4_z} \le C(\|q\|_{L^2}), \tag{4.13}$$

as well as

$$\|\bar{\partial}m^1(\cdot,k)\|_{L^{\frac{4}{3}}} \le C(\|q\|_{L^2})\big(M\hat{q}(k)\big)^{\frac{1}{2}}. \tag{4.14}$$

b) The maps $q \to m_\pm$, $q \to m^1$ and $q \to m^2$ are locally Lipschitz from L^2 into the topologies in (4.13), (4.14). *Precisely, given q_1 and q_2 in L^2 we have the difference bounds*

$$\|\delta m_\pm\|_{L^4_kL^4_z} + \|\delta m^1\|_{L^4_kL^4_z} + \|\delta m^2\|_{L^4_kL^4_z} \le C(\|q_1\|_{L^2})C(\|q_2\|_{L^2})\|\delta q\|_{L^2} \tag{4.15}$$

as well as

$$\|\bar{\partial}\delta m^1\|_{L^4_kL^{\frac{4}{3}}_z} \le C(\|q_1\|_{L^2})C(\|q_2\|_{L^2})\|\delta q\|_{L^2}. \tag{4.16}$$

Remark 4.3. *If we use the first part of* (4.9) *in the proof below then we obtain the more refined bound*

$$\|m^1 - 1\|_{L^4_kL^4_z} + \|m^2\|_{L^4_kL^4_z} + \|\bar{\partial}m^1\|_{L^4_kL^{\frac{4}{3}}_z} \le C(\|q\|_{L^2})\|\bar{\partial}^{-1}(e_k\overline{q})\|_{L^4_kL^4_z}. \tag{4.17}$$

Proof. a) We define

$$r_\pm(\cdot,k) = m_\pm(\cdot,k) - 1. \tag{4.18}$$

Then $m_\pm$ solve (1.3) if and only if $r_\pm$ solve

$$\bar{\partial}r_\pm = \pm e_{-k}q\overline{r_\pm} \pm e_{-k}q \tag{4.19}$$

and so by Lemma 4.1 there exist unique solutions to (4.19) with

$$\|r_\pm(\cdot,k)\|_{L^4} \le C(\|q\|_{L^2})\big(M\hat{q}(k)\big)^{\frac{1}{2}}. \tag{4.20}$$

Now the bound (4.12) for $m_\pm$, m^1 and m^2 follows from (4.20) and (4.18).

The inequality (4.13) then follows by integrating (4.12) in k and using the mapping property $M : L^2 \to L^2$. Finally, for (4.14) we use the first equation in (4.1) combined with the m^2 bound in (4.13). This completes the proof of a).

Part (b) is easily obtained by repeating the same arguments in part a) for differences of Jost functions. The details are left for the reader. □

Next we turn our attention to the scattering transform $\mathbf{s}$ of q, which is defined by (4.7).

Lemma 4.4. *The scattering transform* $\mathbf{s}(k)$ *is well defined for a.e.* k *in* $\mathbb{C}$ *and satisfies*

$$\|\mathbf{s}\|_{L^2} \leq C(\|q\|_{L^2}) \tag{4.21}$$

as well as the pointwise bound

$$|\mathbf{s}(k)| \leq C(\|q\|_{L^2}) M\hat{q}(k). \tag{4.22}$$

Remark 4.5. *Using the slightly stronger bound* (4.17) *in the proof below yields the following slight improvement over* (4.22)*:*

$$\|\mathbf{s} - \hat{q}(k)\|_{L^2} \leq C(\|q\|_{L^2}) \|\bar{\partial}^{-1}(e_k \overline{q})\|_{L^4} \tag{4.23}$$

This will be useful later on in order to provide a self-contained proof of the characterization of the wave operators for the DSII problem.

Proof. We write $\mathbf{s}(k)$ in the form

$$i\mathbf{s}(k) = \frac{1}{\pi} \int e_k \overline{q} \, dz + \frac{1}{\pi} \int e_k \overline{q}(m^1 - 1) \, dz. \tag{4.24}$$

The first term is simply the Fourier transform of $\overline{q} \in L^2$ which obeys (4.21) and (4.22). For the second term, we apply Theorem 2.3 with $n = 2$ and $\alpha = 1$ for the symbol $m^1(z,k) - 1$, and $f = \overline{\hat{q}}$ (so that $\hat{f} = \overline{q}$), k playing the role of x and z playing the role of ξ. Hypothesis i) of Theorem 2.3 is satisfied by (4.13). To see that hypothesis ii) is justified, recall from (4.14) that

$$\|\bar{\partial} m^1(\cdot, k)\|_{L^{\frac{4}{3}}} \leq C(\|q\|_{L^2}) \big(M\hat{q}(k)\big)^{\frac{1}{2}}.$$

Hence,

$$\|\bar{\partial}(m^1 - 1)\|_{L^4_k L^{\frac{4}{3}}_z} \leq C(\|q\|_{L^2}) \|M\hat{q}(k)\|^{\frac{1}{2}}_{L^2} \leq C(\|q\|_{L^2}).$$

Thus, hypothesis ii) holds by the boundedness of the Beurling transform $\bar{\partial}\partial^{-1}$ on L^p for $1 < p < \infty$.

It follows that $\mathbf{s}$ is well defined and is in L^2. In addition, from (2.8),

$$|\mathbf{s}(k)| \leq C(\|q\|_{L^2}) (M\hat{q}(k))^{\frac{1}{2}} \|\bar{\partial} m^1(\cdot, k)\|_{L^{\frac{4}{3}}} \|q\|^{\frac{1}{2}}_{L^2} \leq C(\|q\|_{L^2}) M\hat{q}(k). \tag{4.25}$$

□

So far we have constructed the Scattering Transform $\mathcal{S}q$ for a fixed $q \in L^2$. Our next goal is to establish that $\mathcal{S}$ is a locally Lipschitz map. One can already view this as a consequence of the locally Lipschitz property for the Jost functions in Lemma 4.2, but the next lemma provides elegant difference formulas from which we will also obtain additional properties of the Scattering Transform and its derivative. These formulas (more precisely their consequences stated as Lemma 4.8 a) and c)) are generalizations of facts proved in [10] on tangent maps for potentials in Schwartz space and extended in [40] and [44].

We'll denote by $\langle \cdot, \cdot \rangle$ the usual inner product on L^2:

$$\langle f, g\rangle = \int \overline{f} g.$$

Lemma 4.6. *(Difference Formulas) a) Given any two potentials* q_1 *and* q_2 *in* $L^2(\mathbb{R}^2)$ *with scattering transforms* $\mathbf{s}_1$*, respectively* $\mathbf{s}_2$*, we have:*

$$\mathbf{s}_1 - \mathbf{s}_2 = T_{q_1,q_2}(q_1 - q_2) \tag{4.26}$$

where the linear operator T_{q_1,q_2} *is given by*

$$T_{q_1,q_2} f(k) = -\frac{i}{\pi}\Big(\int e_k(z)\overline{f(z)}a(z,k)dz - \int e_k(z)f(z)b(z,k)dz\Big) \tag{4.27}$$

with

$$a(z,k) = \overline{m^1_{\overline{q_2}}(z,-k)}m^1_{q_1}(z,k) \tag{4.28}$$

$$b(z,k) = \overline{m^2_{\overline{q_2}}(z,-k)}m^2_{q_1}(z,k). \tag{4.29}$$

The integrals are well defined for $f \in L^2$ *and*

$$\|T_{q_1,q_2}\|_{L^2\to L^2} \le C(\|q_1\|_{L^2})C(\|q_2\|_{L^2}). \tag{4.30}$$

b) With the same functions $a(z,k)$ *and* $b(z,k)$ *(defined in terms of* q_1 *and* q_2*) as above we also have*

$$q_1 - q_2 = W_{q_1,q_2}(\mathbf{s}_1 - \mathbf{s}_2) \tag{4.31}$$

where the linear operator W_{q_1,q_2} *is defined as:*

$$W_{q_1,q_2} g(z) = -\frac{i}{\pi}\Big(\int e_k(z)\overline{g(k)}a(z,k)dk - \int e_{-k}(z)g(k)\overline{b(z,k)}dk\Big). \tag{4.32}$$

The integrals are well defined for $f \in L^2$ *and*

$$\|W_{q_1,q_2}\|_{L^2\to L^2} \le C(\|q_1\|_{L^2})C(\|q_2\|_{L^2}). \tag{4.33}$$

c) For the operators T_{q_1,q_2} *and* W_{q_1,q_2} *we have the identity:*

$$\langle g, T_{q_1,q_2} f\rangle - \langle f, W_{q_1,q_2} g\rangle = \Im\langle g, \tilde{T}_{q_1,q_2} f\rangle, \tag{4.34}$$

valid for any $f, g, q_1, q_2 \in L^2$*, where*

$$\tilde{T}_{q_1,q_2} f(k) = -\frac{2}{\pi}\int e_k(z)f(z)b(z,k)dz. \tag{4.35}$$

Proof. a) First we will prove (4.26) formally. We will then show that the integral exists in L^2 and prove the bound (4.30).

From the definition (4.7), we have

$$\mathbf{s}_1 - \mathbf{s}_2 = -\frac{i}{\pi}\Big(\int e_k(\overline{q_1} - \overline{q_2})m^1_{q_1}dz + \int e_k\overline{q_2}(m^1_{q_1} - m^1_{q_2})dz\Big)$$

where $m^1_{q_i}$ solve (4.1) with boundary conditions (4.3), or in integral form

$$m^1_{q_i}(\cdot,k) - 1 = (I - \mathcal{A}_{q_i,k})^{-1}\mathcal{A}_{q_i,k}(1)$$

where

$$\mathcal{A}_{q,k}(\cdot) = \bar\partial^{-1}(e_{-k}q\partial^{-1}(e_k\overline{q}\,\cdot)).$$

For the second term, we have by the resolvent identity

$$\begin{aligned}
m^1_{q_1} - m^1_{q_2} &= [(I-\mathcal{A}_{q_1,k})^{-1} - (I-\mathcal{A}_{q_2,k})^{-1}]\mathcal{A}_{q_1,k}1 + (I-\mathcal{A}_{q_2,k})^{-1}(\mathcal{A}_{q_1,k}1 - \mathcal{A}_{q_2,k}1)\\
&= (I-\mathcal{A}_{q_2,k})^{-1}\{[(I-\mathcal{A}_{q_2,k}) - (I-\mathcal{A}_{q_1,k})](I-\mathcal{A}_{q_1,k})^{-1}\mathcal{A}_{q_1,k}1 + \mathcal{A}_{q_1,k}1 - \mathcal{A}_{q_2,k}1\}\\
&= (I-\mathcal{A}_{q_2,k})^{-1}(\mathcal{A}_{q_1,k} - \mathcal{A}_{q_2,k})m^1_{q_1}\\
&= (I-\mathcal{A}_{q_2,k})^{-1}(\mathcal{D}_1 + \mathcal{D}_2)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{D}_1 &= \bar\partial^{-1}(e_{-k}q_2\partial^{-1}(e_k(\overline{q_1} - \overline{q_2})m^1_{q_1}))\\
\mathcal{D}_2 &= \bar\partial^{-1}(e_{-k}(q_1 - q_2)\partial^{-1}(e_k\overline{q_1}m^1_{q_1})).
\end{aligned}$$

Then

$$\int e_k\overline{q_2}(m^1_{q_1} - m^1_{q_2})dz = \langle (I-\mathcal{A}^*_{q_2,k})^{-1}e_{-k}q_2, \mathcal{D}_1 + \mathcal{D}_2\rangle = \langle e_{-k}q_2 m^1_{\overline{q_2}}(\cdot,-k), \mathcal{D}_1 + \mathcal{D}_2\rangle.$$

Now,

$$\begin{aligned}
\langle e_{-k}q_2 m^1_{\overline{q_2}}(\cdot,-k), \mathcal{D}_1\rangle &= \langle \mathcal{A}_{\overline{q_2},-k}m^1_{\overline{q_2}}(\cdot,-k), e_k(\overline{q_1} - \overline{q_2})m^1_{q_1}\rangle\\
&= \int e_k(z)\overline{m^1_{\overline{q_2}}(z,-k)}\,\overline{(q_1(z) - q_2(z))}m^1_{q_1}(z,k)dz\\
&\quad - \int e_k(z)\overline{(q_1(z) - q_2(z))}m^1_{q_1}(z,k)dz.
\end{aligned}$$

In addition,

$$\begin{aligned}
\langle e_{-k}q_2 m^1_{\overline{q_2}}(\cdot,-k), \mathcal{D}_2\rangle &= -\langle \partial^{-1}(e_{-k}q_2 m^1_{\overline{q_2}}(\cdot,-k)), e_{-k}(q_1 - q_2)\partial^{-1}(e_k\overline{q_1}m^1_{q_1})\rangle\\
&= -\int e_k(z)\overline{m^2_{\overline{q_2}}(z,-k)}(q_1(z) - q_2(z))m^2_{q_1}(z,k)dz.
\end{aligned}$$

Combining the terms, we obtain (4.26). To prove (4.30), we write

$$T_{q_1,q_2}f = -\frac{i}{\pi}\Big(P_1f(k) + P_2f(k) + P_3f(k) + P_4f(k) + P_5f(k)\Big)$$

where

$$P_1 f(k) = \int e_z \overline{f} (\overline{m^1_{\overline{q_2}}} - 1)(m^1_{q_1} - 1) dz$$
$$P_2 f(k) = \int e_z \overline{f} (\overline{m^1_{\overline{q_2}}} - 1) dz$$
$$P_3 f(k) = \int e_z \overline{f} (m^1_{q_1} - 1) dz$$
$$P_4 f(k) = \int e_z \overline{f} dz$$
$$P_5 f(k) = -\int e_{-z} f \overline{m^2_{\overline{q_2}}} m^2_{q_1} dz.$$

For the term P_5, we have by Lemma 4.2

$$\begin{aligned} \|m^2_{\overline{q_2}}(\cdot, -k) m^2_{q_1}(\cdot, k)\|_{L^2} &\leq \|m^2_{\overline{q_2}}(\cdot, -k)\|_{L^4} \|m^2_{q_1}(\cdot, k)\|_{L^4} \\ &\leq C(\|q_2\|_{L^2}) \big(M\widehat{\overline{q_2}}(-k)\big)^{\frac{1}{2}} C(\|q_1\|_{L^2}) \big(M\widehat{q_1}(k)\big)^{\frac{1}{2}} \end{aligned} \tag{4.36}$$

Hence,

$$\begin{aligned} \|P_5 f\|_{L^2} &\leq \Big(\int \|m^2_{\overline{q_2}}(\cdot, -k) m^2_{q_1}(\cdot, k)\|^2_{L^2} dk\Big)^{\frac{1}{2}} \|f\|_{L^2} \\ &\leq C(\|q_1\|_{L^2}) C(\|q_2\|_{L^2}) \|M\widehat{\overline{q_2}}(-\,\cdot)\|^{\frac{1}{2}}_{L^2} \|M\widehat{q_1}(\cdot)\|^{\frac{1}{2}}_{L^2} \|f\|_{L^2} \\ &\leq C(\|q_1\|_{L^2}) C(\|q_2\|_{L^2}) \|f\|_{L^2} \end{aligned}$$

where the last inequality follows from the mapping property of the maximal function $M : L^2 \to L^2$. In similar fashion, we obtain

$$\|P_1 f\|_{L^2} \leq C(\|q_1\|_{L^2}) C(\|q_2\|_{L^2}) \|f\|_{L^2}.$$

To investigate the term $P_3 f$, we will apply Theorem 2.3 with $n = 2$ and $\alpha = 1$ for the symbol $m^1_{q_1}(z, k) - 1$, and $\overline{f}$ replacing $\hat{f}$ in the Theorem, k playing the role of x and z playing the role of ξ. The hypotheses of the theorem are satisfied in view of the bounds (4.13) and (4.14) in Lemma 4.2. It thus follows from (2.8), that

$$\begin{aligned} |P_3 f(k)| &\leq C(\|q_1\|_{L^2}) (M\widehat{f}(k))^{\frac{1}{2}} \|\bar{\partial}(m^1_{q_1} - 1)\|_{L_z^{\frac{4}{3}}} \|f\|^{\frac{1}{2}}_{L^2} \\ &\leq C(\|q_1\|_{L^2}) (M\widehat{f}(k))^{\frac{1}{2}} (M\hat{q}_1(k))^{\frac{1}{2}} \|f\|^{\frac{1}{2}}_{L^2}. \end{aligned}$$

Hence

$$\begin{aligned} \|P_3 f\|_{L^2} &\leq C(\|q_1\|_{L^2}) \Big(\int \big|M\widehat{f}(k) M\widehat{q_1}(k)\big| dk\Big)^{\frac{1}{2}} \|f\|^{\frac{1}{2}}_{L^2} \\ &\leq C(\|q_1\|_{L^2}) \|M\widehat{f}\|^{\frac{1}{2}}_{L^2} \|M\widehat{q_1}\|^{\frac{1}{2}}_{L^2} \|f\|^{\frac{1}{2}}_{L^2} \\ &\leq C(\|q_1\|_{L^2}) \|f\|_{L^2}. \end{aligned}$$

Likewise, for P_2 we have

$$\|P_2 f\|_{L^2} \leq C(\|q_2\|_{L^2}) \|f\|_{L^2}.$$

Combining the four terms, we obtain (4.30).

b) Because of the symmetry between the forward and inverse scattering, (4.26) also yields:

$$q_1(z) - q_2(z) = -\frac{i}{\pi}\Big(\int e_k(z)\overline{m^1_{\overline{\mathbf{s}_2}}(-z,k)}\,\overline{(\mathbf{s}_1(k) - \mathbf{s}_2(k))} m^1_{\mathbf{s}_1}(z,k)dk - \int e_k(z)\overline{m^2_{\overline{\mathbf{s}_2}}(-z,k)}(\mathbf{s}_1(k) - \mathbf{s}_2(k)) m^2_{\mathbf{s}_1}(z,k)dk\Big). \tag{4.37}$$

where, by (4.6)

$$\begin{aligned} m^1_{\mathbf{s}} &= \frac{1}{2}(n_{+,\mathbf{s}} + n_{-,\mathbf{s}}) = m^1_q \\ m^2_{\mathbf{s}} &= \frac{1}{2}e_{-k}(\overline{n_{+,\mathbf{s}} - n_{-,\mathbf{s}}}) = e_{-k}\overline{m^2_q} \end{aligned} \tag{4.38}$$

Now, consider the scattering problem for $\widetilde{q_2} =: \overline{q_2}$. For the corresponding scattering transform we have (cf. [10]):

$$\widetilde{\mathbf{s}_2}(k) = -\overline{\mathbf{s}_2(-k)}$$

Hence,

$$\frac{\partial}{\partial \bar{k}} n_{\pm,\widetilde{\mathbf{s}_2}}(z,-k) = \pm e_k(z)\overline{\mathbf{s}_2(k)}\,\overline{n_{\pm,\widetilde{\mathbf{s}_2}}(z,-k)}.$$

Comparing with the corresponding equations for $\overline{\mathbf{s}_2}$ we conclude from the uniqueness in Lemma 4.2 that

$$n_{\pm,\overline{\mathbf{s}_2}}(-z,k) = n_{\pm,\widetilde{\mathbf{s}_2}}(z,-k).$$

It follows (using (4.6)) that

$$\begin{aligned} m^1_{\overline{\mathbf{s}_2}}(-z,k) &= m^1_{\widetilde{\mathbf{s}_2}}(z,-k) = m^1_{\overline{q_2}}(z,-k) \\ \overline{m^2_{\overline{\mathbf{s}_2}}(-z,k)} &= \overline{m^2_{\widetilde{\mathbf{s}_2}}(z,-k)} = e_{-k}(z) m^2_{\overline{q_2}}(z,-k) \end{aligned}$$

Substituting in (4.37) we obtain:

$$q_1(z) - q_2(z) = -\frac{i}{\pi}\Big(\int e_k(z)\overline{m^1_{\overline{q_2}}(z,-k)}\,\overline{(\mathbf{s}_1(k) - \mathbf{s}_2(k))} m^1_{q_1}(z,k)dk - \int e_{-k}(z) m^2_{\overline{q_2}}(z,-k)(\mathbf{s}_1(k) - \mathbf{s}_2(k))\overline{m^2_{q_1}(z,k)}dk\Big)$$

which proves (4.31). The bound (4.33) is obtained in similar fashion to (4.30).

c) Since T_{q_1,q_2}, W_{q_1,q_2} and $\tilde{T}_{q_1,q_2}$ are bounded oerators on L^2, it suffices to prove (4.34) for f, g of compact support. In view of (4.27 and (4.32) we have

$$\begin{aligned} \langle g, T_{q_1,q_2} f\rangle - \langle f, W_{q_1,q_2} g\rangle &= -\frac{i}{\pi}\int\int \overline{g(k)}\Big(e_k(z)\overline{f(z)}a(z,k) - e_k(z)f(z)b(z,k)\Big)dzdk \\ &\quad + \frac{i}{\pi}\int\int \overline{f(k)}\Big(e_k(z)\overline{g(z)}a(z,k) - e_{-k}(z)g(z)\overline{b(z,k)}\Big)dkdz \\ &= -\frac{i}{2}\Big(\langle g, \tilde{T}_{q_1,q_2} f\rangle - \overline{\langle g, \tilde{T}_{q_1,q_2} f\rangle}\Big), \end{aligned}$$

which proves (4.34). □

So far, we have established that S is a Lipschitz map from $L^2(\mathbb{R}^2)$ to $L^2(R^2)$. The next step is to show that the properties (1) and (5) in Theorem 1.2 also hold. In part a) of the following Corollary, we first use (4.34) to obtain a generalization of the Plancherel identity (1.15):

Corollary 4.7. *a) For any $q_1, q_2 \in L^2(\mathbb{R}^2)$ we have*

$$\|\mathcal{S}q_1 - \mathcal{S}q_1\|_{L^2}^2 = \|q_1 - q_2\|_{L^2}^2 + \Im\langle \mathcal{S}q_1 - \mathcal{S}q_2, \tilde{T}_{q_1,q_2}(q_1 - q_2)\rangle, \tag{4.39}$$

with $\tilde{T}_{q_1,q_2}$ as defined in (4.35.)

b) The Scattering transform $\mathcal{S}$ satisfies the Plancherel identity $\|\mathcal{S}q\|_{L^2} = \|q\|_{L^2}$ for all $q \in L^2(\mathbb{R}^2)$, as well as the identity $\mathcal{S}^2 = I$.

Proof. a) Apply (4.34) with $f = q_1 - q_2$ and $g = \mathcal{S}q_1 - \mathcal{S}q_2$ and use (4.26 and (4.31).

b) The Plancherel Identity is a special case of (4.39) when $q_2 = 0$, as then $b(z, k) = 0$ so $\tilde{T}_{q_1,0} = 0$. It also follows by an extension by continuity argument from the Plancherel identity for potentials in Schwartz class in [10], together with the (locally) uniformly Lipschitz continuity of $\mathcal{S}$ which is based only on part a) of Lemma 4.6. Likewise, the identity $\mathcal{S}^2 = I$ was show in [10] for Schwartz potentials and now extends to all of L^2 in view of part a) of Lemma 4.6. □

We have shown that $\mathcal{S}$ is Lipschitz and also that $\mathcal{S}^{-1} = \mathcal{S}$, therefore $\mathcal{S}$ is a bi-Lipschitz homeomorphism of L^2.

In order to complete the proof of Theorem 1.2 it remains to show that $\mathcal{S}$ is continuously differentiable and is a symplectomorphism.

Lemma 4.8. *a) The map $q \to \mathcal{S}(q)$ is a C^1-diffeomorphism from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$, and its differential is given by*

$$\Big(\frac{\delta \mathcal{S}}{\delta q}\Big|_q \tilde{q}\Big)(k) := T_{q,q}\tilde{q}$$

with $T_{q,q}$ as defined in (4.27).

b)

$$\Big(\frac{\delta \mathcal{S}}{\delta q}\Big|_q\Big)^{-1} = T_{q,q}^{-1} = W_{q,q},$$

with $W_{q,q}$ as defined in (4.32).

c) For any $q, q_1, q_2 \in L^2(\mathbb{R}^2)$ we have

$$\Im\langle \frac{\delta \mathcal{S}}{\delta q}\Big|_q q_1, \frac{\delta \mathcal{S}}{\delta q}\Big|_q q_2\rangle = \Im\overline{\langle q_1, q_2\rangle} \tag{4.40}$$

Proof. a) Given two potentials q_1 and q_2, we need to estimate the difference

$$\delta^{(2)}\mathbf{s} = \mathbf{s}_2 - \mathbf{s}_1 - \Big(\frac{\delta \mathcal{S}}{\delta q}\Big|_{q_1} \delta q\Big)$$

in terms of $\delta q = q_2 - q_1$. It suffices to show that

$$\|\delta^{(2)}\mathbf{s}\|_{L^2} \lesssim C(\|q_1\|_{L^2})C(\|q_2\|_{L^2})\|\delta q\|_{L^2}^2. \tag{4.41}$$

Using the formula (4.26) we write

$$\delta^{(2)}\mathbf{s} = T_{q_1,\delta q}(q_2 - q_1) \tag{4.42}$$

where

$$T_{q_1,\delta q} f = \frac{i}{\pi}\Big(\int e_k(z)\overline{f}\delta a(z,k)dz - \int e_k(z) f \delta b(z,k)dz \Big)$$

with

$$\delta a(z,k) = \overline{\delta m^1_{q}(z,-k)} m^1_{q_1}(z,k) + \overline{m^1_{q_1}(z,-k)}\delta m^1_q(z,k)$$
$$\delta b(z,k) = \overline{\delta m^2_{q}(z,-k)} m^2_{q_1}(z,k) + \overline{m^2_{q_1}(z,-k)}\delta m^2_q(z,k).$$

Now the bound (4.41) is a consequence of

$$\|T_{q_1,\delta q}\|_{L^2\to L^2} \leq C(\|q_1\|_{L^2})C(\|q_2\|_{L^2})\|\delta q\|_{L^2}.$$

This in turn is proved in the same manner as (4.30), but using the difference bounds in part (b) of Lemma 4.2.

b) This follows from the arguments above, using (4.31) with $q_1 = q + \varepsilon\tilde{q}$ and $q_2 = q$.

c) This symplectomorphism property is another application of our identity (4.34). Since the right side of (4.34) is real-valued, we have

$$\Im\langle g, T_{\tilde{q}_1,\tilde{q}_2} f\rangle = \Im\langle f, W_{\tilde{q}_1,\tilde{q}_2} g\rangle, \tag{4.43}$$

for any $f, g, \tilde{q}_1, \tilde{q}_2 \in L^2(\mathbb{R}^2)$. We use this with $\tilde{q}_1 = \tilde{q}_2 = q$, $f = q_2$ and $g = \frac{\delta\mathcal{S}}{\delta q}\Big|_q q_1$. Then, in view of part b), $W_{q,q} g = q_1$ and (4.40) follows. □

5. Application to Defocusing DSII

In this section we use the properties of the nonlinear scattering transform $\mathcal{S}$ in Theorem 1.2 in order to prove the results on the defocusing DSII problem in Theorem 1.4 as well as Theorem 1.6. We first review the Inverse Scattering based construction of solutions to the DSII system (1.1). The steps are as follows, see (1.13):

(i) We define the initial data for the scattering transform,

$$\mathbf{s}_0 = \mathcal{S} q_0$$

(ii) We compute the linear evolution on the scattering transform side

$$\mathbf{s}(t,k) = e^{2i(k^2+\bar{k}^2)t}\mathbf{s}_0(k)$$

(iii) We return to the physical space via the inverse transform $\mathcal{S}^{-1} = \mathcal{S}$,

$$q(t) = \mathcal{S}\mathbf{s}(t).$$

Our starting point is the classical work of Beals and Coifman [9], [10] and [11], who show that if $q_0 \in \mathscr{S}$, then $\mathbf{s}_0 \in \mathscr{S}$ and further that $q(t) \in \mathscr{S}$ is the unique classical solution to (1.1). Our goal, on the other hand, is to show that the above algorithm is equally valid for all L^2 initial data. We begin by examining the presumptive data-to-solution map

$$q_0 \to q(t,\cdot) \tag{5.1}$$

Lemma 5.1. *The data-to-solution map* (5.1) *has the following properties:*

(i) Conserved mass:

$$\|q(t,\cdot)\|_{L^2} = \|q_0\|_{L^2}.$$

(ii) Continuity in time:

$$L^2 \ni q_0 \to q(t,\cdot) \in C(\mathbb{R}, L^2).$$

(iii) Lipschitz property: for two L^2 solutions q_1 and q_2 we have

$$\|q_1(t,\cdot) - q_2(t,\cdot)\|_{L^2} \le C(\|q_{01}\|_{L^2})C(\|q_{02}\|_{L^2})\|q_{01} - q_{02}\|_{L^2}$$

for all t.

(iv) Pointwise bound:

$$|q(t,z)| \le C(\|q_0\|_{L^2}) M q^{lin}(t,z)$$

where

$$q^{lin}(t,\cdot) = U(t)\overline{\widehat{\mathcal{S}q_0}}.$$

(v) L^4 bound:

$$\|q\|_{L^4_{t,z}} \le C(\|q_0\|_{L^2}).$$

Proof. (i) This is immediate from the Plancherel identity (1.15).

(ii) This is a consequence of the Lipschitz bound (1.17) combined with the L^2 time continuity of $e^{2i(k^2+\bar{k}^2)t}\mathbf{s}_0$.

(iii) This is also a consequence of the Lipschitz bound (1.17).

(iv) This follows from Corollary 1.3, noting that the (inverse) Fourier transform of (1.13) is:

$$\overline{\hat{\mathbf{s}}(t,\cdot)} = U(t)\overline{\hat{\mathbf{s}}_0} = q^{lin}(t,\cdot).$$

(v) From the Strichartz estimate for the linear flow we have

$$\|q^{lin}\|_{L^4_{t,z}} \le C\|\hat{\mathbf{s}}_0\|_{L^2} = C\|\mathbf{s}_0\|_{L^2} = C\|q_0\|_{L^2},$$

using the Plancherel identity (1.15). The bound (v) now follows from (iv) above and the L^4 boundedness of the Maximal function. $\square$

This lemma shows that the L^2 presumptive solutions can be viewed as the unique uniform limits of Schwartz solutions. However, it does not yet prove that these are actual solutions to (1.1). Our next step is stated separately as it no longer relies on the scattering transform, but rather on perturbative dispersive analysis:

Lemma 5.2. *The data-to-solution map* (5.1) *satisfies the Lipschitz bound*

$$\|q_1 - q_2\|_{L^4_{t,z}} \le C(\|q_{01}\|_{L^2})C(\|q_{02}\|_{L^2})\|q_{01} - q_{02}\|_{L^2}.$$

Proof. By the property (iii) in the previous lemma and a density argument for the embedding $\mathscr{S} \subset L^2$, it suffices to prove this for Schwarz data q_{01}, q_{02}. The advantage then is that we know in addition that q_1 and q_2 are classical solutions for (1.1), which we rewrite as

$$iq_t + 2(\bar{\partial}^2 + \partial^2)q = N(q) := qL|q|^2$$

where L is a zero order multiplier, which is bounded in all L^p spaces for $1 < p < \infty$. Then we can apply Strichartz estimates on any time interval $I = [0,T]$ for the difference of the two solutions to obtain

$$\begin{aligned}\|q_1 - q_2\|_{L^4_{t,z}\cap L^\infty_t L^2_z[I]} &\lesssim \|q_{01} - q_{02}\|_{L^2} + \|N(q_1) - N(q_2)\|_{L^{\frac{4}{3}}[I]} \\ &\lesssim \|q_{01} - q_{02}\|_{L^2} + \|q_1 - q_2\|_{L^4_{t,z}[I]}(\|q_1\|^2_{L^4_{t,z}[I]} + \|q_1\|^2_{L^4_{t,z}[I]})\end{aligned}$$

If we have the additional property

$$\|q_1\|_{L^4_{t,z}[I]}, \|q_1\|_{L^4_{t,z}[I]} \ll 1 \tag{5.2}$$

then we can absorb the second term on the right into the left hand side to obtain

$$\|q_1 - q_2\|_{L^4_{t,z} \cap L^\infty_t L^2_z[I]} \lesssim \|q_{01} - q_{02}\|_{L^2}$$

To use this property we take advantage of the L^4 bound in part (v) of the previous lemma in order to divide the real line into subintervals $\mathbb{R} = \cup_{j \in \mathcal{J}} I_j$ so that the property (5.2) holds for all intervals I_j. The number of such intervals is at most

$$|\mathcal{J}| \lesssim C(\|q_1\|_{L^4_{t,z}}) C(\|q_2\|_{L^4_{t,z}}) \lesssim C(\|q_{01}\|_{L^2}) C(\|q_{02}\|_{L^2})$$

Then we apply the above argument successively on all these intervals in order to obtain the conclusion of the Lemma. □

The L^4 Lipschitz bound can now be used in order to show that the Inverse Scattering construction yields solutions to (1.1).

Lemma 5.3. *For each $q_0 \in L^2$ the function $q(t)$ is a solution to* (1.1) *in the sense that* (1.21) *holds.*

The proof is straightforward, based on the Strichartz estimates for the linear flow.

This concludes the proof of Theorem 1.4. We now turn our attention to Theorem 1.6. We begin with a slight improvement of Lemma 5.2:

Lemma 5.4. *The map*

$$L^2 \ni q_0 \to q \in L^4_{t,z}$$

is smooth.

Proof. This is a standard perturbative argument which we only outline. Given a solution q_1 to DSII with initial data $q_{01} \in L^2$, we seek to solve the DSII with initial data q_{02} sufficiently close to q_{01}. Since $q_1 \in L^4_{t,z}$, we can divide the real line as in the proof of Lemma 5.2 into finitely many subintervals I_j so that $\|q_1\|_{L^4_{t,z}[I_j]}$ is small.

Then we construct the solution q_2 successively in each subinterval by reiterating the Duhamel formula (1.22). This converges due to the Strichartz estimates. □

The next lemma establishes the existence and regularity of the wave operators $W_\pm$:

Lemma 5.5. *The wave operators $W_\pm$ are well defined and locally Lipschitz in L^2.*

Proof. We begin by using the Duhamel formula to compute

$$U(-t)q(t) = q(0) + \int_0^t U(-s)N(q(s))ds$$

Since $q \in L^4_{t,z}$, it follows that $N(q) \in L^{\frac{4}{3}}$. Then by Strichartz estimates the above expression converges in L^2 as $t \to \pm\infty$, and we have

$$q_\pm = \lim_{t \to \pm\infty} U(-t)q(t) = q(0) + \int_0^{\pm\infty} U(-s)N(q(s))ds.$$

The map $q_0 \to q_\pm$ is smooth in view of the previous Lemma and Strichartz estimates. □

To see that $W_\pm q_0 = \widehat{\mathcal{S}q_0}$, and thus complete the proof of Theorem 1.6, we can now argue by density. It suffices to know that this is true for $q_0 \in \mathscr{S}$. This was already proved in [10], but for the sake of completeness we provide a self-contained argument below.

If q_0 is Schwartz then $\mathbf{s}_0$ is also Schwartz (see[10]), and $\mathbf{s}(t) = e^{2it(k^2+\bar{k}^2)}\mathbf{s}_0$. In view of the bound (4.23), applied with the roles of q and $\mathbf{s}$ reversed, it suffices to show that

$$\lim_{t\to\infty} \|\overline{\partial}_k^{-1}(e_z e^{2it(k^2+\bar{k}^2)}\mathbf{s}_0)\|_{L^4} \to 0. \tag{5.3}$$

Indeed, a direct computation shows that

$$|\overline{\partial}_k^{-1}(e_z e^{2it(k^2+\bar{k}^2)}\mathbf{s}_0)| \lesssim t^{-\frac{1}{2}}(1+|k|)^{-N}(1+t^{-\frac{1}{2}}|z-2tk|)^{-1}$$

which has an L^4 norm of size $t^{-\frac{1}{4}}$. This completes the proof of (5.3).

6. Application to Two Inverse Boundary Value Problems

In this section we prove Theorems 1.7, 1.8, 1.9, 1.10. We begin with the results for the boundary value problem (1.30), which we recall here:

$$\begin{cases} \bar{\partial}v - q\bar{v} = 0 & \text{in } \Omega \\ \Im(\nu v) = g_0 & \text{in } \partial\Omega \end{cases}$$

The motivation for the study of this problem was given in the Introduction. We start with the solvability result for this problem.

Proof of Theorem 1.9. We consider several increasingly difficult cases:

Case 1: $q = 0$**,** $\int g_0 = 0$. Then v is holomorphic in Ω so we can express it in the form

$$v = \partial u$$

with u real valued in Ω. Then u must solve the Laplace equation

$$\begin{cases} \triangle u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \tau} = -2g_0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Here $\tau = (-\nu_2, \nu_1)$ denotes the unit tangent vector to $\partial\Omega$ in the counterclockwise direction. This Dirichlet problem is uniquely solvable modulo constants if and only if $\int g_0 = 0$, and yields a solution $u \in H^{\frac{3}{2}}$ with $\frac{\partial u}{\partial \nu} \in L^2$.

Case 2: The inhomogeneous problem. Here we consider the inhomogeneous problem

$$\begin{cases} \bar{\partial}v = f_0 & \text{in } \Omega \\ \Im(\nu v) = g_0 & \text{on } \partial\Omega \end{cases} \tag{6.2}$$

with $f_0 \in L^{\frac{4}{3}}(\Omega)$ and claim that we can solve it if and only if

$$\int_\Omega \Im f_0 dz = \frac{1}{2}\int_{\partial\Omega} g_0 ds. \tag{6.3}$$

Indeed, extending f_0 by zero outside Ω we can solve $\triangle u = 4f_0$ in all of $\mathbb{R}^2$, obtaining a solution $v_0 = \partial u \in W^{1,\frac{4}{3}}_{loc}$, which is easily seen to have an L^2 trace on the boundary. Now we are left with the homogeneous problem, which is solvable provided that g_0 is in a codimension one affine subspace. This constraint is easily seen to be (6.3) by integrating the equation (6.2) over Ω and using the divergence theorem. We can restate the result as follows:

Lemma 6.1. *For each $f_0 \in L^{\frac{4}{3}}(\Omega)$ and real-valued $g_0 \in L^2(\partial\Omega)$ the problem*

$$\begin{cases} \bar\partial v = f_0 & \text{in } \Omega \\ \Im(\nu v) = g_0 + c & \text{on } \partial\Omega \end{cases} \tag{6.4}$$

admits a unique solution $(v, c) \in H^{\frac{1}{2}} \times \mathbb{R}$. Moreover, we have:

$$\|v\|_{H^{\frac{1}{2}}(\Omega)} + \|v\|_{L^2(\partial\Omega)} \leq C(\|f_0\|_{L^{\frac{4}{3}}(\Omega)} + \|g_0\|_{L^2(\partial\Omega)}). \tag{6.5}$$

We will write

$$v = Tf_0 + Bg_0$$

where

$$T : L^{\frac{4}{3}}(\Omega) \to H^{\frac{1}{2}}(\Omega), \qquad B : L^2(\partial\Omega) \to H^{\frac{1}{2}}(\Omega).$$

Note that c can be explicitly determined from f_0 and g_0.

Case 3: q small. We first solve the counterpart of (6.4), namely

$$\begin{cases} \bar\partial v - q\bar v = f_0 & \text{in } \Omega \\ \Im(\nu v) = g_0 + c & \text{on } \partial\Omega \end{cases} \tag{6.6}$$

This we can rewrite as

$$v = T(q\bar v) + Tf_0 + Bg_0$$

which is solved by a Neumann series in $H^{\frac{1}{2}}$.

Now we set $f_0 = 0$ and assume that $\int g_0 = 0$. The solution we obtain above does not *a priori* have $c = 0$, which is why we need to prove that *a posteriori*. Precisely, integrating by parts against $\sigma^{-\frac{1}{2}}$ and using $q = -\frac{1}{2}\partial \log \sigma$ we obtain

$$0 = \Im \int_\Omega (\bar\partial v - q\bar v)\sigma^{-\frac{1}{2}} dz = -\Im \int_\Omega v\bar\partial\sigma^{-\frac{1}{2}} + \sigma^{-\frac{1}{2}} q\bar v dz + \frac{1}{2}\int_{\partial\Omega} \Im(\nu v) ds = \frac{1}{2}cL$$

where L is the length of $\partial\Omega$. Therefore we conclude that $c = 0$.

Case 4: q large. We solve again (6.6). The problem is written as

$$v = T(q\bar v) + Tf_0 + Bg_0$$

The operator $v \to T(q\bar v)$ is compact in $H^{\frac{1}{2}}$, as it is bounded, linear in q and compact for smooth q, so by the Fredholm alternative it remains to show that the homogeneous problem

$$v = T(q\bar v)$$

admits no nontrivial solution.

Such a solution would solve

$$\begin{cases} \bar\partial v - q\bar v = 0 & \text{in } \Omega \\ \Im(\nu v) = c & \text{on } \partial\Omega \end{cases} \tag{6.7}$$

The constant c must be equal to zero, as in the previous case.

From here we proceed as in the global $\bar\partial$ problem. We split q into $q = q_{smooth} + q_{small}$. We seek to eliminate q_{smooth} by a gauge transformation ϕ which solves

$$\bar\partial\phi = r := -\frac{\bar v}{v}\, q_{smooth}$$

Here we need to insure that ϕ is real on the boundary. So we need to solve

$$\begin{cases} \bar{\partial}\phi = r & \text{in } \Omega \\ \Im\phi = 0 & \text{on } \partial\Omega \end{cases} \tag{6.8}$$

Solving the inhomogeneous problem we are left with

$$\begin{cases} \bar{\partial}\phi = 0 & \text{in } \Omega \\ \Im\phi = f & \text{on } \partial\Omega \end{cases} \tag{6.9}$$

where we solve first for $\Im\phi$ and then $\Re\phi$ is uniquely determined modulo constants.

Now we set

$$u = e^{\phi}v$$

which solves

$$\begin{cases} \bar{\partial}u = q_{small}\overline{u} & \text{in } \Omega \\ \Im(\nu u) = 0 & \text{on } \partial\Omega \end{cases} \tag{6.10}$$

We are now in the small q case so $u = 0$ follows. The proof of Theorem 1.9 is concluded. □

As discussed in the introduction, the above proof allows us to define a Hilbert transform operator associated to the $\bar{\partial}$ problem in Ω as

$$L^2 \ni \Im(\nu v) \to \mathcal{H}_q v := \Re(\nu v) \in L^2.$$

Next we show that the boundary data $\mathcal{H}_q$ uniquely determines q.

Proof of Theorem 1.10. The proof below is in the spirit of [36] and [38], also inspired by some arguments in [31] and [47]. Let $q \in L^2(\mathbb{R}^2)$ be defined by zero extension outside Ω,

$$q = \begin{cases} -\frac{1}{2}\partial \log \sigma & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^2 \setminus \Omega \end{cases}$$

We will show that $\mathbf{s} = \mathcal{S}q$ can be constructively determined from knowledge of $\mathcal{H}_q$. The potential q can then be recovered from $\mathbf{s}$ using the inversion Theorem 1.2 (5).

For k such that $M\hat{q}(k) < \infty$, let $m_\pm(\cdot, k)$ be the Jost solutions of (1.3) constructed in Lemma 4.2. We have

$$\begin{aligned} \mathbf{s}(k) &= \frac{1}{2\pi i}\int_{\mathbb{R}^2} e_k(z)\overline{q(z)}\Big(m_+(\cdot,k) + m_-(\cdot,k)\Big) \\ &= \frac{1}{2\pi i}\int_{\Omega} \partial\Big(\overline{m_+}(\cdot,k) - \overline{m_-}(\cdot,k)\Big) \\ &= \frac{1}{4\pi i}\int_{\partial\Omega} \overline{\nu}\Big(\overline{m_+}(\cdot,k) - \overline{m_-}(\cdot,k)\Big) \end{aligned}$$

Thus, it will suffice to show that one can compute the traces of $m_\pm(\cdot, k)$ from knowledge of $\mathcal{H}_q$ on $\partial\Omega$. Let

$$\psi_\pm(z,k) = e^{izk}m_\pm(z,k). \tag{6.11}$$

The following lemma shows that we can obtain the trace $\psi_+(\cdot,k)\Big|_{\partial\Omega}$ from $\mathcal{H}_q$.

Lemma 6.2. *Let Ω, σ, q be as in Theorem 1.8 and k as above. Then the function $\psi_+(z,k)$ restricted to $z \in \mathbb{C}\backslash\overline{\Omega}$ is the unique solution of the exterior problem*

$$\begin{cases} (i) & \bar\partial\psi_+ = 0 \text{ in } \mathbb{C}\backslash\overline{\Omega} \\ (ii) & \psi_+(z,k)e^{-izk} - 1 \in L^4(\mathbb{C}\backslash\overline{\Omega}) \cap W^{1,\frac{4}{3}}_{loc} \\ (iii) & \Re(\nu\psi_+\Big|_{\partial\Omega}) = \mathcal{H}_q(\Im(\nu\psi_+\Big|_{\partial\Omega})). \end{cases} \tag{6.12}$$

Proof. The main issue is to prove the uniqueness. Suppose uniqueness does not hold. Then there exists a function h with $e^{-izk}h \in L^4(\mathbb{C}\setminus\Omega)$ so that $\bar\partial h = 0$ and

$$\Re(\nu h) = \mathcal{H}_q(\Im(\nu h)) \qquad \text{in } \partial\Omega.$$

Using the solvability result in Theorem 1.9 we solve the problem

$$\begin{cases} \bar\partial v = qv & \text{in } \Omega \\ \Im(\nu v) = \Im(\nu h) & \text{on } \partial\Omega. \end{cases} \tag{6.13}$$

Then in view of the definition of $\mathcal{H}_q$ we must have also

$$\Re(\nu v) = \Re(\nu h) \qquad \text{on } \partial\Omega.$$

Now let

$$\phi = \begin{cases} v \text{ in } \overline{\Omega} \\ h \text{ in } \mathbb{C}\backslash\overline{\Omega}. \end{cases} \tag{6.14}$$

Then $m = \phi e^{-izk} \in L^4(\mathbb{R}^2)$. We have shown that ϕ (hence m) is continuous across $\partial\Omega$. In view of (6.12)(i) and (6.13), m is a weak solution of

$$\bar\partial m - e_{-k}q\overline{m} = 0 \tag{6.15}$$

in all of $\mathbb{R}^2$. Lemma 3.2 now shows that $m = 0$. This proves uniqueness. It is also clear that $\psi_+(z,k)$ restricted to $\mathbb{C}\backslash\overline{\Omega}$ is a solution of (6.12). This completes the proof the the Lemma. ☐

For computational purposes one can use layer potentials to reduce (6.12) to a problem on $\partial\Omega$. We will not pursue this here.

☐

Finally we return to the original Calderón problem with $\log\sigma \in \dot{H}^1$. We begin with the solvability question.

Proof of Theorem 1.7. Without loss of generality, we may assume g is real-valued. In complex notation, the equation (1.23) takes the form

$$\overline{\partial}(\sigma\partial u) + \partial(\sigma\overline{\partial}u) = 0.$$

For real valued u, the standard substitution $v = \sigma^{\frac{1}{2}}\partial u$ then yields a solution of (1.26) with q defined by (1.27). Thus, in view of (1.29), v solves the boundary value problem (1.30) with $g_0 = -\frac{1}{2}\frac{\partial g}{\partial\tau}$. But this problem is uniquely solvable by Theorem 1.9. Consequently $\sigma^{\frac{1}{2}}\partial u$ is uniquely determined. This immediately yields u. We also have (see (1.28)):

$$\frac{\partial u}{\partial\nu} = 2\Re(\nu v) \in L^2(\partial\Omega),$$

and (1.32) holds on $H^1(\partial\Omega)$. ☐

Finally, Theorem 1.8 on the Calderón problem is now an easy consequence of the previous results.

Proof of Theorem 1.8. Given Λ_σ, we use (1.32) to determine $\mathcal{H}_q$. From Theorem 1.10 we have a method to reconstruct $q = -\frac{1}{2}\partial \log \sigma$. Since $\log \sigma$ is assumed known on $\partial\Omega$, and we have determined its gradient, we can recover this function inside Ω. □

7. Appendix

The following is a restatement of Lemma 4.2 in [37]. It is reproduced here with the proof, for completeness:

Lemma 7.1. *If $a \in L^2(\mathbb{C})$ then the operator $f \mapsto \bar{\partial}^{-1}(af)$ is compact on $L^r(\mathbb{C})$, $2 < r < \infty$.*

Proof. By duality, it suffices to show that $a\partial^{-1}$ is compact on L^p, $1 < p < 2$. We have

$$\|a\partial^{-1}f\|_{L^p} \le \|a\|_{L^2}\|\partial^{-1}f\|_{L^{p^*}} \le c\|a\|_{L^2}\|f\|_{L^p}. \tag{7.1}$$

First suppose a is a C^1 function with compact support in, say, a disk D. Then by the boundedness of the Beurling transform on L^p:

$$\|\nabla(a\partial^{-1}f)\|_{L^p} \le \|\nabla a\|_{L^2}\|\partial^{-1}f\|_{L^{p^*}} + \|a\|_{L^\infty}\|\nabla\partial^{-1}f\|_{L^p} \le c\|f\|_{L^p}.$$

Thus, the image under $a\partial^{-1}$ of the unit ball in L^p lies in $\{u \in L^p(D) : \|u\|_{L^p} \le c, \|\nabla u\|_{L^p} \le c\}$, which is compact. Now let a be arbitrary in $L^2(\mathbb{C})$ and let $\{a_j\}$ be a sequence of C^1 functions of compact support converging to a in L^2. The corresponding operators $a_j\partial^{-1}$ are compact and norm convergent by (7.1), hence their limit, too, is compact. □

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