On quasi-polynomials counting planar tight maps

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Abstract. A *tight map* is a map with some of its vertices marked, such that every vertex of degree 1 is marked. We give an explicit formula for the number $N_{0,n}(d_1,\dots,d_n)$ of planar tight maps with n labeled faces of prescribed degrees $d_1,\dots,d_n$, where a marked vertex is seen as a face of degree 0. It is a quasi-polynomial in $(d_1,\dots,d_n)$, as shown previously by Norbury. Our derivation is bijective and based on the *slice decomposition* of planar maps. In the non-bipartite case, we also rely on enumeration results for two-type forests. We discuss the connection with the enumeration of non necessarily tight maps. In particular, we provide a generalization of Tutte's classical slicings formula to all non-bipartite maps.

Keywords. Planar maps, bijective enumeration, slice decomposition

Mathematics Subject Classifications. 05A15, 05A19

1. Introduction

1.1. Tight maps

The main purpose of this paper is to study the enumeration problem for a class of maps, called *tight maps*.

Definition 1.1. A *tight map* is a connected map with some of its vertices marked, such that every vertex of degree 1 is marked. In a tight map, the faces as well as the marked vertices are called *boundaries*.

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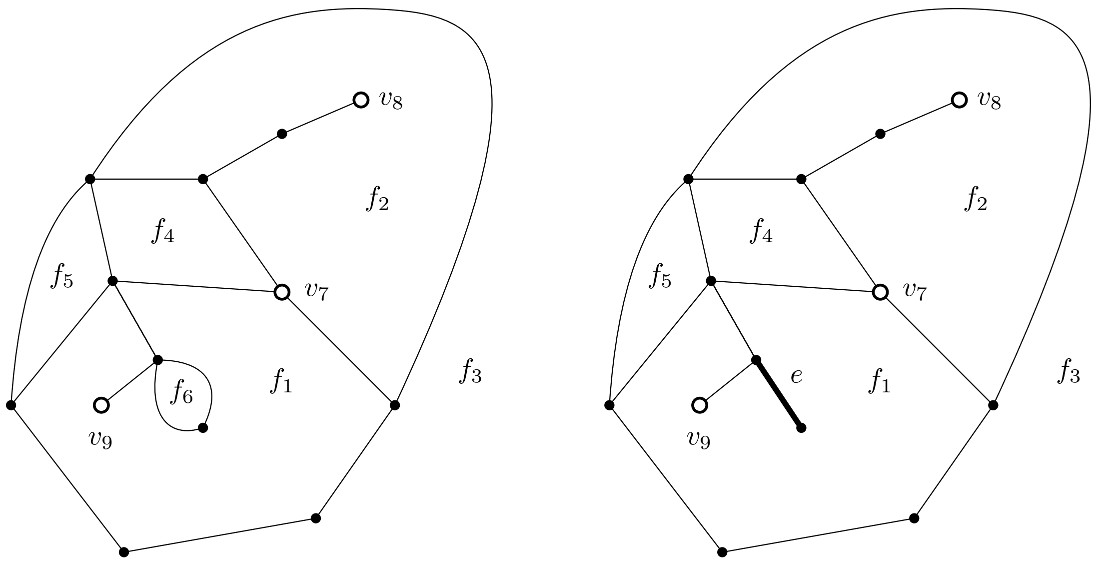

Figure 1.1: On the left, a tight map with 9 boundaries: six faces $f_1, f_2, f_3, f_4, f_5, f_6$ with respective degrees $12, 8, 5, 4, 3, 2$, and three marked vertices v_7, v_8, v_9, shown in white. On the right: a rooted tight map, whose opening operation gives the tight map on the left. As this figure demonstrates, a vertex incident to the root in a rooted tight map can be of degree 1 without being necessarily marked. These maps can be also seen as pointed (respectively pointed rooted) tight maps, for example by distinguishing the marked vertex v_9.

Even though this definition makes sense for maps on arbitrary surfaces, we will restrict in this paper to the planar case. We refer to [Sch15] and [BGM22] for the standard definitions and terminology about maps.

Usually, we will endow a tight map with some extra structure, in particular by labeling its faces and some or all of its marked vertices, or distinguishing one marked vertex. For instance, we will call *pointed tight map* a tight map with one distinguished marked vertex. We will adopt a slightly unusual notion of *rooted tight map* compared to the well-established notion of rooting of a map: in particular, the root will be an unoriented edge. If e is a distinguished edge in a map $\mathbf{m}$, there is a natural opening operation $O(\mathbf{m}, e)$ consisting in cutting along the edge, thereby creating a face of degree 2. If $\mathbf{m}$ has marked or labeled elements (vertices or faces), then $O(\mathbf{m}, e)$ naturally inherits these elements. We say that a map $\mathbf{m}$ with some of its vertices marked and a distinguished edge e is a *rooted tight map* if $O(\mathbf{m}, e)$ is a tight map. Note that $\mathbf{m}$ may not be a tight map itself, as the distinguished edge may be incident to an unmarked vertex of degree 1. Finally, a *pointed rooted tight map* is a rooted tight map with one distinguished marked vertex. See Figure 1.1 for examples of tight maps.

We define the *length* of a boundary in a tight map as being equal to its degree for a face, and to zero for a marked vertex. In other words, we interpret the marked vertices as boundaries of length 0.

The terminology of tight maps comes from [BGM22]. Let us discuss it in some detail. In a general map $\mathbf{m}$, drawn on a surface S, and with some faces and vertices marked, let us call

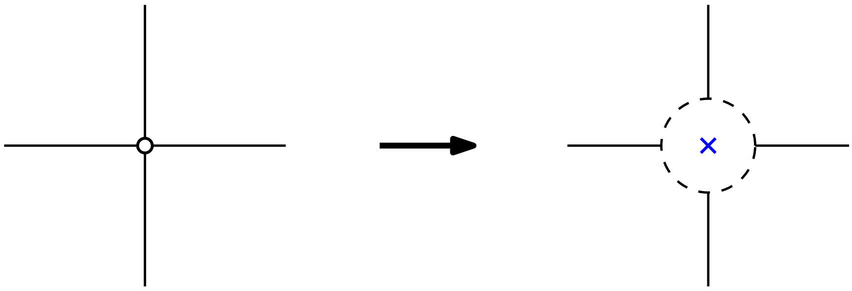

Figure 1.2: Blowing a boundary-vertex into a face of degree 0 with one puncture.

the marked elements the *boundaries*, and the unmarked elements the *internal* faces and vertices. We let S' be the space obtained from S by removing one point (i.e. creating a puncture) inside each of the boundaries of $\mathbf{m}$. A boundary of $\mathbf{m}$ is called *tight* if its contour path has minimal possible length among all paths in the map that are freely homotopic to it in S'. When the map has marked vertices, then all such vertices are automatically tight boundaries, and the previous minimality condition on the contours of the boundary-faces must be understood in a slightly modified map obtained by blowing every marked vertex of degree k into a k-cycle with edges of "length 0", hence creating a new face in the map, which we view as having degree 0, with one puncture. See Figure 1.2 for an example.

This being said, it is straightforward to see that a map is tight according to Definition 1.1, if and only if it is a map with no internal faces (i.e. all its faces are marked as boundaries), whose boundaries are all tight. Indeed, starting from a tight map, we see that all its boundaries are necessarily tight: in the modified map with every marked vertex blown into a degree-0 face, the contour of a given face is in fact the unique non-backtracking path of edges in its free homotopy class in the punctured surface S'. Conversely, a map which is not tight contains an unmarked vertex v of degree 1. The contour of the face f incident to v can be deformed into a strictly shorter path by shortcutting the edge incident to v, meaning that f is not a tight boundary.

In fact, we view the results of the present paper as a first step towards a better (in particular, bijective) understanding of the counting problem for general maps with tight boundaries and possibly with internal faces. This problem was addressed in the case of planar maps with three boundaries in [BGM22], and remains a challenge in more complex topologies.

1.2. Lattice count polynomials

For any choice of nonnegative integers $d_1, d_2, \dots, d_n$ not all equal to 0 and for any nonnegative integer g, we let $N_{g,n}(d_1, d_2, \dots, d_n)$ be the number of tight maps of genus g, with n labeled boundaries of respective lengths $d_1, d_2, \dots, d_n$, and where each map is weighted by its inverse number of automorphisms. In fact, the latter number is always 1 as soon as $n \geqslant 3$, while the only tight maps with two boundaries having a non-trivial automorphism group are (genus 0) cycles of length $p \geqslant 2$, with automorphism group $\mathbb{Z}/p\mathbb{Z}$ (such maps are thus weighted by $1/p$). For $n = 1$, the only planar ($g = 0$) example of a tight map is the (marked) vertex-map, which is excluded from the discussion since its only boundary has length 0, but in higher genera, there are many tight maps with one face, and these can have a non-trivial automorphism group.

The numbers $N_{g,n}(d_1,\ldots,d_n)$ have been extensively studied in particular by Norbury and Do [Nor10, Nor13, DN11][1], in the broader context of the study of invariants of spectral curves appearing in Eynard and Orantin's topological recursion [EO07]. Norbury proved that $N_{g,n}(d_1,\ldots,d_n)$ is a quasi-polynomial in the variables $d_1^2,\ldots,d_n^2$, depending on their parities. This means that for every $k\in\{0,1,\ldots,n\}$, there exists a polynomial $\mathsf{N}_{g,n}^{(k)}(x_1,\ldots,x_n)$ in the variables $x_1^2,\ldots,x_n^2$, symmetric under permutations of the first k variables and of the last $n-k$ variables, such that, if the numbers $d_1,\ldots,d_k$ are odd and the numbers $d_{k+1},\ldots,d_n$ are even, then

$$N_{g,n}(d_1,\ldots,d_n)=\mathsf{N}_{g,n}^{(k)}(d_1,\ldots,d_n)\,. \tag{1.1}$$

For k odd, these polynomials are equal to 0. In fact, the first two papers mentioned above assume that $d_1,\ldots,d_n$ are all non-zero, while the third paper considers the general case where some, but not all d_i may vanish. Definition 2.7 in [DN11] is indeed equivalent to our definition of tight maps, while Proposition 2.8 therein proves that the extension of the quasipolynomials to some zero values do solve the enumeration problem of tight maps with marked vertices. Norbury [Nor10] also proves that evaluating the polynomials at $(0,0,\ldots,0)$ gives interesting geometric information, although the combinatorial meaning of this evaluation is not clear. Note that the theory of enumeration of integer points in polytopes implies that $N_{g,n}(d_1,\ldots,d_n)$ is a piecewise quasi-polynomial in $d_1,\ldots,d_n$, see for instance the discussion around [Bud22b, Proposition 4] (in the case $b=0$). Therefore, it is surprising that $N_{g,n}(d_1,\ldots,d_n)$ is actually a genuine quasi-polynomial, furthermore in the squared variables.

The approach taken in [Nor10, Nor13, DN11] is to prove the wanted properties using recursions for these polynomials, called *lattice count polynomials*, that also allows one to effectively compute them. These recursions are in turn consequences of combinatorial recursion relations with a geometric flavor, similar to Tutte's equations used for instance in [Tut62], and to the topological recursion originating in Eynard and Orantin's work [EO07].

In this paper, focusing on the planar case $g=0$, our main goal in to show how one can obtain the above quasipolynomiality results by *bijective* techniques, which in passing yield new explicit formulas for the lattice count polynomials. We will use two different strategies: the first one, discussed in Section 3, is based on a *substitution* approach using as an input Tutte's classical *slicings formula* [Tut62]. This formula holds however only for planar maps which are *bipartite* or *quasi-bipartite*, namely with a number of faces of odd degree equal to 0 or 2 respectively. As a consequence, the substitution approach is limited to the enumeration of planar tight bipartite and quasi-bipartite maps, corresponding to respectively $k=0$ and $k=2$ in (1.1). The second, purely bijective, strategy is based on the so-called *slice decomposition* of maps introduced in [BG12], and its extensions developed in [BG14, Bou19]. We will first discuss it in Section 4 in the easier case of planar tight bipartite and quasi-bipartite maps, and then extend it in Section 5 to the general case of planar tight maps with an arbitrary number k of faces of odd degree. Using then the substitution approach backwards, our general expression for tight maps allows us to extend Tutte's slicings formula to non necessarily tight maps with an arbitrary number of faces of odd degree.

[1] Here, we should warn the reader that these references use the notion of fatgraphs, which is different but equivalent to the language of maps used in this paper.

The paper is organized as follows. Section 2 provides a self-contained presentation of our main results. Section 2.1 deals with the simpler case of bipartite and quasi-bipartite tight maps: after introducing and studying in Section 2.1.1 the required basic univariate and multivariable polynomials, we state our main theorems which connect these polynomials to the numbers of planar tight bipartite maps in Section 2.1.2 (Theorem 2.3) and of planar tight quasi-bipartite maps in Section 2.1.3 (Theorem 2.8). We then state our enumeration result for general tight maps in Section 2.2 (Theorem 2.12) after introducing the appropriate univariate and multivariate quasi-polynomials. Section 3 discusses the connection with the enumeration of non necessarily tight maps by the substitution approach: Section 3.1 is devoted to the derivation of Theorems 2.3 and 2.8 from Tutte's slicings formula, and Section 3.2 uses this approach backwards to obtain from Theorem 2.12 an extension of the slicings formula to maps with an arbitrary number of faces of odd degree, see Theorem 3.2. We then discuss in Sections 4 and 5 the bijective approach based on the slice decomposition of maps. Section 4 concentrates again on the simpler bipartite and quasi-bipartite cases, discussing first tight maps with a single face (Section 4.1), tight maps with two faces (Section 4.2), pointed rooted tight maps in connection with tight slices (Section 4.3) and finally tight maps which are neither pointed nor rooted (Section 4.4) using the slice decomposition of annular maps. All these bijective results are then extended to the non-bipartite or quasi-bipartite case in Section 5, which requires the preliminary enumeration of so-called petal trees (Section 5.1), petal necklaces (Section 5.2) and non-bipartite slices (Section 5.3). Our most general enumeration result for planar tight maps with arbitrary prescribed boundary lengths is given in Section 5.4 by Theorem 5.13, which presents a single formula encompassing Theorems 2.3, 2.8 and 2.12. We gather our concluding remarks in Section 6, while a few appendices detail the derivation of some technical results.

2. Main results

2.1. Polynomials counting planar tight bipartite or quasi-bipartite maps

In this section, we provide explicit expressions for the lattice count polynomials $\mathsf{N}_{0,n}^{(0)}$ and $\mathsf{N}_{0,n}^{(2)}$, which correspond to planar tight bipartite and quasi-bipartite maps, respectively.

2.1.1 Definition and properties of the polynomials

Let us start by introducing families of polynomials which appear in the explicit expression of $\mathsf{N}_{0,n}^{(0)}$. Here, we concentrate on the very definitions of these polynomials and on their resulting algebraic properties. The connection with tight map enumeration will be discussed in Section 2.1.2.

Basic univariate polynomials. Our first basic polynomials are functions of a single variable m and are defined as follows: for any integer $k \geqslant 0$, we set

$$\begin{aligned} p_k(m) &:= \frac{1}{(k!)^2} \prod_{i=1}^{k} \left(m^2 - i^2\right) = \binom{m-1}{k}\binom{m+k}{k} \\ q_k(m) &:= \frac{1}{(k!)^2} \prod_{i=0}^{k-1} \left(m^2 - i^2\right) = \binom{m}{k}\binom{m+k-1}{k} \end{aligned} \tag{2.1}$$

with the usual convention $p_0(m) = q_0(m) = 1$ for the empty product, and with

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$$

viewed as a polynomial in x.

Clearly, p_k and q_k are polynomials of degree k in the variable m^2, and $p_k(m)$ and $q_k(m)$ are integers if m is an integer. The two families of polynomials are linked by the relation

$$q_k(m) = p_k(m) + p_{k-1}(m), \qquad k \geqslant 0 \tag{2.2}$$

with the convention $p_{-1} := 0$. A combinatorial interpretation of this relation based on the enumeration of tight maps with a single face will be given in Section 2.1.2. We also record the identities

$$(k+1)p_{k+1}(m) = (m-k-1)p_k(m) + \sum_{j=1}^{m-1} (2j) p_k(j) \tag{2.3}$$

$$(k+1)q_{k+1}(m) = (m-k)q_k(m) + \sum_{j=1}^{m-1} (2j) q_k(j) \tag{2.4}$$

which are valid for m a positive integer, and which may be checked by induction.

Multivariate polynomials. The above univariate polynomials may be extended to multivariate polynomials, functions of n variables $m_1, m_2, \ldots, m_n$ as follows: for any integer $k \geqslant 0$ and any integer $n \geqslant 1$, we set

$$\begin{aligned} p_k(m_1, m_2, \ldots, m_n) &:= \sum_{\substack{k_1, k_2, \ldots, k_n \geqslant 0 \\ k_1 + k_2 + \cdots + k_n = k}} p_{k_1}(m_1) q_{k_2}(m_2) \cdots q_{k_n}(m_n), \\ q_k(m_1, m_2, \ldots, m_n) &:= \sum_{\substack{k_1, k_2, \ldots, k_n \geqslant 0 \\ k_1 + k_2 + \cdots + k_n = k}} q_{k_1}(m_1) q_{k_2}(m_2) \cdots q_{k_n}(m_n). \end{aligned} \tag{2.5}$$

(In the right-hand side of the first line, all factors except the first one are q_{k_i}'s.)

Clearly, p_k and q_k are polynomials of degree k in the variables $m_1^2, m_2^2, \ldots, m_n^2$. Note that the notation is consistent for $n = 1$ with (2.1). From the identity $p_k(1) = \delta_{k,0}$, we get the identification:

$$p_k(1, m_1, m_2, \ldots, m_n) = q_k(m_1, m_2, \ldots, m_n) \tag{2.6}$$

and, by (2.2), we find that $q_k = p_k + p_{k-1}$ for any number of variables.

For bookkeeping purposes, let us record the explicit expressions of p_k and q_k for $k = 1, 2, 3$:

$$\begin{aligned}
p_1(m_1,\ldots,m_n) &= \left(\sum_{i=1}^{n} m_i^2\right) - 1,\\
p_2(m_1,\ldots,m_n) &= \frac{1}{4}\left(\sum_{i=1}^{n} m_i^4\right) + \sum_{i<j} m_i^2 m_j^2 - \frac{5}{4}\left(\sum_{i=1}^{n} m_i^2\right) + 1,\\
p_3(m_1,\ldots,m_n) &= \frac{1}{36}\left(\sum_{i=1}^{n} m_i^6\right) + \frac{1}{4}\left(\sum_{i\neq j} m_i^4 m_j^2\right) + \sum_{i<j<h} m_i^2 m_j^2 m_h^2\\
&\qquad - \frac{7}{18}\left(\sum_{i=1}^{n} m_i^4\right) - \frac{3}{2}\left(\sum_{i<j} m_i^2 m_j^2\right) + \frac{49}{36}\left(\sum_{i=1}^{n} m_i^2\right) - 1\,,\\
q_1(m_1,\ldots,m_n) &= \sum_{i=1}^{n} m_i^2,\\
q_2(m_1,\ldots,m_n) &= \frac{1}{4}\left(\sum_{i=1}^{n} m_i^4\right) + \sum_{i<j} m_i^2 m_j^2 - \frac{1}{4}\left(\sum_{i=1}^{n} m_i^2\right),\\
q_3(m_1,\ldots,m_n) &= \frac{1}{36}\left(\sum_{i=1}^{n} m_i^6\right) + \frac{1}{4}\left(\sum_{i\neq j} m_i^4 m_j^2\right) + \sum_{i<j<h} m_i^2 m_j^2 m_h^2\\
&\qquad - \frac{5}{36}\left(\sum_{i=1}^{n} m_i^4\right) - \frac{1}{2}\left(\sum_{i<j} m_i^2 m_j^2\right) + \frac{1}{9}\left(\sum_{i=1}^{n} m_i^2\right).
\end{aligned} \tag{2.7}$$

Proposition 2.1. *For any integer $k \geqslant 0$, p_k and q_k are symmetric functions. In other words, for any integer $n \geqslant 1$, $p_k(m_1, m_2, \ldots, m_n)$ and $q_k(m_1, m_2, \ldots, m_n)$ are symmetric polynomials in $m_1, m_2, \ldots, m_n$ which satisfy the consistency relation*

$$\begin{aligned}
p_k(m_1, m_2, \ldots, m_n, 0) &= p_k(m_1, m_2, \ldots, m_n),\\
q_k(m_1, m_2, \ldots, m_n, 0) &= q_k(m_1, m_2, \ldots, m_n).
\end{aligned} \tag{2.8}$$

Proof. The symmetry of $q_k(m_1, m_2, \ldots, m_n)$ is apparent from its very definition in (2.5). As for $p_k(m_1, m_2, \ldots, m_n)$, its symmetry is also made apparent from the following alternative and manifestly symmetric expression:

$$p_k(m_1, m_2, \ldots, m_n) = \sum_{\substack{k_0,k_1,k_2,\ldots,k_n \geqslant 0\\ k_0+k_1+k_2+\cdots+k_n=k}} \binom{n-1}{k_0} p_{k_1}(m_1) p_{k_2}(m_2) \cdots p_{k_n}(m_n). \tag{2.9}$$

To get this latter expression, we use again the relation (2.2) to write, in the expression (2.5) for $p_k(m_1, m_2, \ldots, m_n)$, each q_{k_i} for $i = 2$ to n as the sum of p_{k_i} and p_{k_i-1} and distribute the two terms in the product so as to get a sum of terms of the form $p_{k'_1}(m_1)p_{k'_2}(m_2)\cdots p_{k'_n}(m_n)$ with summation variables $k'_1 = k_1$ and $k'_i = k_i$ or $k_i - 1$ for $i \geqslant 2$. The number of terms in the sum having exactly k_0 indices i for which $k_i' = k_i - 1$ is $\binom{n-1}{k_0}$ and, for such terms, the sum rule $k_1 + k_2 + \cdots + k_n = n$ in (2.5) becomes $k_0 + k_1' + k_2' + \cdots + k_n' = n$. This leads to (2.9) upon renaming the summation variable k_i' as k_i.

As for the consistency relation (2.8), it is a direct consequence of the identity $q_k(0) = \delta_{k,0}$ (here and in the following, we will always implicitly assume that k is a non-negative integer). $\square$

Finally, let us state some recursion relations obeyed by the p_k's, which we call the dilaton and string equations as we shall see later that they correspond to the recursions obtained in [Nor13] in the bipartite case.

Proposition 2.2 (Dilaton and string equations)**.** *We have the dilaton equation*

$$p_k(m_1, \ldots, m_n, 1) - p_k(m_1, \ldots, m_n, 0) = p_{k-1}(m_1, \ldots, m_n) \tag{2.10}$$

and the string equation, valid for non-negative integer $m_1, \ldots, m_n$*:*

$$(k+1)p_{k+1}(m_1, \ldots, m_n) = (m_1 + \cdots + m_n - k - 1)p_k(m_1, \ldots, m_n) + \sum_{i=1}^{n} \sum_{j=1}^{m_i - 1} (2j) p_k(m_1, \ldots, m_{i-1}, j, m_{i+1}, \ldots, m_n) \tag{2.11}$$

Proof. From (2.6) and Proposition 2.1, the dilaton equation boils down to the relation $q_k = p_k + p_{k-1}$ noted above. The string equation is nothing but the multivariate extension of (2.3), and is obtained by a linear combination of it and (2.4) (precisely, we take (2.3) at $k = k_1$ and $m = m_1$ times $q_{k_2}(m_2) \cdots q_{k_n}(m_n)$ and add, for $i = 2, \ldots, n$, the relation (2.4) at $k = k_i$ and $m = m_i$ times $p_{k_1}(m_1) \cdots q_{k_n}(m_n)$ with the factor $q_{k_i}(m_i)$ omitted). $\square$

2.1.2 Enumeration results in the bipartite case

We are now ready to state our first enumerative result:

Theorem 2.3. *For* $n \geqslant 3$ *and for non-negative integers* $m_1, m_2, \ldots, m_n$ *not all equal to zero, the number* $N_{0,n}(2m_1, 2m_2, \ldots, 2m_n)$ *of planar tight bipartite maps with* n *boundaries labeled from* 1 *to* n *with respective lengths* $2m_1, 2m_2, \ldots, 2m_n$ *is given by the polynomial*

$$\mathsf{N}_{0,n}^{(0)}(2m_1, 2m_2, \ldots, 2m_n) = (n-3)!\, p_{n-3}(m_1, m_2, \ldots, m_n). \tag{2.12}$$

For $n = 4, 5, 6$, the expression for $\mathsf{N}_{0,n}^{(0)}$ that we obtain from (2.7) is in agreement with the polynomials given in [Bud22b, Table 1] for $g = 0$ and $b = 0$, as expected. Note that the constant term of $\mathsf{N}_{0,n}^{(0)}$, obtained by setting all the m_i's to zero, is equal to $(n-3)!p_{n-3}(0, \ldots, 0) = (-1)^{n-3}(n-3)!$, and this quantity was interpreted in [Nor10] as the orbifold Euler characteristic of the moduli space $\mathcal{M}_{0,n}$.

By combining Proposition 2.2 with Theorem 2.3, we recover the string and dilaton equations found by Norbury in [Nor10] in the planar bipartite case. Note that, in this reference, the string equation corresponds to the addition of a face of degree 2, while (2.11) for $k = n - 3$ may be interpreted as the addition of a vertex. Norbury's original equation can however be recovered by combining it with the dilaton equation. Note finally that Proposition 2.2 holds more generally for any k. When $k \geqslant n-3$ we can naturally interpret it in terms of adding new marked vertices, but the combinatorial meaning of the polynomial $p_k(m_1, \ldots, m_n)$ for $k < n-3$ is more elusive.

A first derivation of Equation (2.12) will be presented in Section 3 below by showing that, up to some appropriate transformation accounting for the tight nature of the maps, it is actually equivalent to Tutte's celebrated slicings enumeration formula [Tut62]. We shall then present in Section 4 a direct bijective proof of Theorem 2.3 upon using some canonical *slice decomposition* of the maps at hand [BG12, BG14, Bou19]. As it appears, it will be convenient for that purpose to proceed gradually and first derive (2.12) for a number of specialized cases (Propositions 2.4, 2.5, 2.6 and 2.7 below) before addressing the result in all generality.

Maps with one face. Taking $n = k+3$ in (2.12) with $m_1 = m \neq 0$ and $m_2 = \cdots = m_{k+3} = 0$, we get

$$N_{0,k+3}(2m, \underbrace{0, \ldots, 0}_{k+2}) = k!\, p_k(m, \underbrace{0, \ldots, 0}_{k+2}) = k!\, p_k(m), \qquad k \geqslant 0\,, \tag{2.13}$$

which already appeared in [Nor13, Corollary 5.6][2]. Upon dividing by $k!$, which amounts to considering that all but two of the marked vertices are unlabeled, we obtain the following combinatorial interpretation of $p_k(m)$:

Proposition 2.4. *For $k \geqslant 0$ and $m \geqslant 1$, $p_k(m)$ is the number of planar tight bipartite maps with one face of degree $2m$ and $k+2$ distinct marked vertices, two of them distinguished and labeled, say as vertex 1 and vertex 2, and the remaining k unlabeled.*

Note that a planar map with a single face of degree $2m$ is nothing but a plane tree with m edges. It is tight if and only if all its leaves are marked.

Similarly, taking $n = k+3$ in (2.12) with $m_1 = m$ and $m_2 = 1$ and $m_3 = \cdots = m_{k+3} = 0$, we now get

$$N_{0,k+3}(2m, 2, \underbrace{0, \ldots, 0}_{k+1}) = k!\, p_k(m, 1, \underbrace{0, \ldots, 0}_{k+1}) = k!\, q_k(m, \underbrace{0, \ldots, 0}_{k+1}) = k!\, q_k(m), \tag{2.14}$$

where we used (2.6) (and the symmetry in exchanging the variables) to switch from p_k to q_k. Dividing by $k!$ and viewing the face of degree 2 as a split root edge, as discussed in the introduction, we obtain a combinatorial interpretation of $q_k(m)$:

Proposition 2.5. *For $k \geqslant 0$ and $m \geqslant 1$, $q_k(m)$ is the number of pointed rooted planar tight bipartite maps with one face of degree $2m$ and k additional unlabeled marked vertices (distinct from each other and from the pointed vertex).*

Proofs of Propositions 2.4 and 2.5 will be presented in Section 4.1 by a direct enumeration of the trees at hand. From the above interpretations of $p_k(m)$ and $q_k(m)$, we may now understand the identity (2.2), for integer values of m, in a combinatorial way using the map language. Indeed, for each tree enumerated by $q_k(m)$, we may transfer the marking of its root edge into a

[2]Incidentally, we note that our main results (Theorems 2.3, 2.8 and 2.12) answer the question raised in the paragraph just before this corollary about finding a general formula for $N_{0,n}$.

marking of that of its endpoints further away from the pointed vertex. Let us for clarity label the newly marked vertex as vertex 2 and the pointed vertex as vertex 1. Note that vertices 1 and 2 are necessarily distinct by construction, but that the vertex 2 may very well coincide with one of the k additional marked vertices in the map enumerated by $q_k(m)$. The marking transformation is clearly reversible, the root edge being recovered as the only edge incident to vertex 2 that belongs to the branch (that is, the unique simple path) from vertex 2 to vertex 1. We may thus interpret $q_k(m)$ as counting plane trees with m edges, and with two distinct marked vertices 1 and 2 and k other marked vertices distinct from each other and from the vertex 1. This yields a map enumerated by $p_k(m)$ when none of the k marked vertices coincide with the vertex 2—note that this may happen even if the vertex 2 is a leaf since, as an endpoint of the root edge, it needs not being marked in the map enumerated by $q_k(m)$. Otherwise, it yields a map enumerated by $p_{k-1}(m)$ by ignoring the "redundant" additional marking of vertex 2. This yields the desired relation (2.2).

Maps with two faces. Taking $n = k+3$ in (2.12) with $m_1, m_2 \geqslant 1$ and $m_3 = \cdots = m_{k+3} = 0$, we get

$$N_{0,k+3}(2m_1, 2m_2, \underbrace{0, \ldots, 0}_{k+1}) = k!\, p_k(m_1, m_2, \underbrace{0, \ldots, 0}_{k+1}) = k!\, p_k(m_1, m_2). \tag{2.15}$$

Upon dividing by $k!$ we get a combinatorial interpretation of $p_k(m_1, m_2)$:

Proposition 2.6. *For $k \geqslant 0$ and $m_1, m_2 \geqslant 1$, $p_k(m_1, m_2)$ is the number of planar tight bipartite maps with two faces of respective degrees $2m_1, 2m_2$ and $k+1$ distinct marked vertices, one of them distinguished and labeled, say as vertex 1, and the remaining k unlabeled.*

A direct bijective proof of this proposition will be presented in Section 4.2. More generally, and although we will not use it in our bijective proof in Section 4, we note the relation, valid for integers m_1, m_2, m_3 not all equal to 0

$$N_{0,k+3}(2m_1, 2m_2, 2m_3, \underbrace{0, \ldots, 0}_{k}) = k!\, p_k(m_1, m_2, m_3) \tag{2.16}$$

so that $p_k(m_1, m_2, m_3)$ counts planar tight bipartite maps with three labeled boundaries with lengths $2m_1, 2m_2, 2m_3$ and k unlabeled marked vertices. It is relatively straightforward to adapt the bijective proof of Proposition 2.6 to prove (2.16) directly, we leave it as an exercise to the reader.

Pointed rooted maps. Taking (2.12) with $n \to n+2$ and specializing to $m_{n+1} = 2$ and $m_{n+2} = 0$, we get:

Proposition 2.7. *For $n \geqslant 1$, the number of pointed rooted planar tight bipartite maps with n labeled boundaries of respective lengths $2m_1, \ldots, 2m_n$ (in addition to the marked vertex and to the root edge) is given by*

$$\mathsf{N}^{(0)}_{0,n+2}, (2m_1, 2m_2, \ldots, 2m_n, 2, 0) = (n-1)!\, q_{n-1}(m_1, m_2, \ldots, m_n). \tag{2.17}$$

Proof. This is a direct application of formula (2.12), together with the identity $p_{n-1}(m_1, m_2, \dots, m_n, 1, 0) = q_{n-1}(m_1, m_2, \dots, m_n)$ from (2.6). $\square$

The expression (2.17) will be proved bijectively in Section 4.3 by a direct decomposition of the maps into slices.

2.1.3 Enumeration results in the quasi-bipartite case

Recall that a planar quasi-bipartite map is a planar map whose all faces but two have even degree. To give the explicit expression of the corresponding lattice count polynomial $\mathsf{N}^{(2)}_{0,n}$, we need to introduce the following new family of univariate polynomials: for any integer $k \geqslant 0$, we set

$$\tilde{p}_k(m) := \frac{1}{(k!)^2} \prod_{i=1}^{k} \left(m^2 - \left(i - \frac{1}{2} \right)^2 \right) = \binom{m - \frac{1}{2}}{k} \binom{m + k - \frac{1}{2}}{k} \tag{2.18}$$

with the convention $\tilde{p}_0(m) = 1$. Note that $\tilde{p}_k$ is again a polynomial of degree k in m^2 and that $\tilde{p}_k(m)$ is an integer if m is a half-integer. It satisfies the following counterpart of (2.3)

$$(k+1)\tilde{p}_{k+1}(m) = \left(m - k - \tfrac{1}{2}\right) \tilde{p}_k(m) + \sum_{0<j<m} (2j)\tilde{p}_k(j) \tag{2.19}$$

where it is understood that m and j are now half-integers.

The multivariate extension of $\tilde{p}_k$ is then defined for any integer $k \geqslant 0$ and any integer $n \geqslant 2$ as

$$\tilde{p}_k(m_1, m_2; m_3, \dots, m_n) := \sum_{\substack{k_1, k_2, \dots, k_n \geqslant 0 \\ k_1 + k_2 + \dots + k_n = k}} \tilde{p}_{k_1}(m_1)\tilde{p}_{k_2}(m_2) q_{k_3}(m_3) \cdots q_{k_n}(m_n). \tag{2.20}$$

Again, we may append an arbitrary number of 0's to the arguments of $\tilde{p}_k$ without changing its value. Note that $\tilde{p}_k$ is in general *not symmetric* in all its variables, but only in m_1 and m_2 on the one hand, and in $m_3, \dots, m_n$ on the other hand. Note also that $\tilde{p}_k(m, 1/2) = \tilde{p}_k(m)$. The quasi-bipartite analog of Theorem 2.3 is then:

Theorem 2.8. *For $n \geqslant 3$, for $m_1, m_2 \in \mathbb{Z}_{\geqslant 0} + \frac{1}{2}$ and $m_3, \dots, m_n \in \mathbb{Z}_{\geqslant 0}$, the number $N_{0,n}(2m_1, 2m_2, 2m_3, \dots, 2m_n)$ of planar tight quasi-bipartite maps with n boundaries labeled from 1 to n with respective lengths $2m_1, 2m_2, 2m_3, \dots, 2m_n$ is given by*

$$\mathsf{N}^{(2)}_{0,n}(2m_1, 2m_2, 2m_3, \dots, 2m_n) = (n-3)!\, \tilde{p}_{n-3}(m_1, m_2; m_3, \dots, m_n). \tag{2.21}$$

In particular, for $n = k+3$ and $m_3 = \dots = m_{k+3} = 0$, we get, for m_1, m_2 half-integers:

$$N_{0,k+3}(2m_1, 2m_2, \underbrace{0, \dots, 0}_{k+1}) = k!\, \tilde{p}_k(m_1, m_2; \underbrace{0, \dots, 0}_{k+1}) = k!\, \tilde{p}_k(m_1, m_2). \tag{2.22}$$

Upon dividing by $k!$, we obtain a combinatorial interpretation of $\tilde{p}_k(m_1, m_2)$:

Proposition 2.9. *For $k \geqslant 0$ and $m_1, m_2 \in \mathbb{Z}_{\geqslant 0} + \frac{1}{2}$, $\tilde{p}_k(m_1, m_2)$ is the number of planar tight quasi-bipartite maps with two faces of odd degrees $2m_1, 2m_2$ and $k+1$ distinct marked vertices, one of them distinguished and labeled, say as vertex 1, and the remaining k unlabeled.*

Setting $m_2 = 1/2$, we obtain a combinatorial interpretation of the univariate polynomial $\tilde{p}_k(m) = \tilde{p}_k(m, 1/2)$, which will be given a direct bijective derivation in Section 4.1:

Proposition 2.10. *For $k \geqslant 0$ and $m \in \mathbb{Z}_{\geqslant 0} + \frac{1}{2}$, $\tilde{p}_k(m)$ is the number of planar tight quasi-bipartite maps with one face of odd degree $2m$, one face of degree one, and $k+1$ distinct marked vertices, one of them distinguished and labeled, say as vertex 1, and the remaining k unlabeled.*

The polynomials $\tilde{p}_k$ obey the dilaton equation

$$\tilde{p}_k(m_1, m_2; \ldots, m_n, 1) - \tilde{p}_k(m_1, m_2; \ldots, m_n, 0) = \tilde{p}_{k-1}(m_1, m_2; \ldots, m_n) \tag{2.23}$$

and the string equation (for m_i's as in Theorem 2.8)

$$\begin{aligned}(k+1)\tilde{p}_{k+1}(m_1, m_2; \ldots, m_n) = (m_1 + \cdots + m_n - k - 1)\tilde{p}_k(m_1, m_2; \ldots, m_n) + \\ \sum_{i=1}^{n} \sum_{0<j<m} (2j)\tilde{p}_k(m_1, \ldots, m_{i-1}, j, m_{i+1}, \ldots, m_n)\end{aligned} \tag{2.24}$$

where we sum over half-integer values of j for $i = 1$ and 2 and over integer values of j for $i \geqslant 3$. The proof is similar to that of Proposition 2.2 and uses now (2.19). Again, this corresponds to Norbury's dilaton and string equations in the planar quasi-bipartite case.

Remark 2.11. We have the relation $\tilde{p}_k(1/2, 1/2; m_1, \ldots, m_n) = q_k(m_1, \ldots, m_n)$ which implies that $N_{0,n+2}(2m_1, 2m_2, 2m_3, \ldots, 2m_n, 1, 1) = N_{0,n+2}(2m_1, 2m_2, 2m_3, \ldots, 2m_n, 2, 0)$ for $m_1, \ldots, m_n$ integers. The latter equality can be explained via a "slit-slide-sew" bijection in the spirit of [Bet20].

2.2. Quasi-polynomials counting planar tight maps with more odd faces

In this section, we provide explicit expressions for the lattice count polynomials $\mathsf{N}_{0,n}^{(k)}$, enumerating planar tight maps with k boundaries of odd lengths, and $n-k$ boundaries of even lengths, for an arbitrary value of $k \geqslant 3$.

To this end, similarly to the bipartite and quasi-bipartite cases discussed in the preceding section, we first need to introduce a two-parameter family of univariate polynomials which generalizes those introduced above: for k a non-negative integer and $e \in \mathbb{Z}$, we define

$$p_{k,e}(m) := \frac{1}{(k!)^2} \prod_{i=1}^{k} \left(m^2 - \left(i - \frac{e}{2} \right)^2 \right) = \binom{m + \frac{e}{2} - 1}{k} \binom{m - \frac{e}{2} + k}{k}. \tag{2.25}$$

We recover the polynomials p_k, $\tilde{p}_k$, and q_k of Section 2.1 for $e = 0, 1, 2$, respectively. We will provide combinatorial interpretations of these polynomials in Section 5.

Next, let r, s be non-negative integers and $\epsilon \in \mathbb{Z}$ be fixed. For $m \in \mathbb{Z}/2$, we let

$$\pi_{r,s}^{(\epsilon)}(m) := \begin{cases} \binom{r+s}{s} p_{r+s,s+1+\epsilon}(m) & \text{if } m - \frac{s+1+\epsilon}{2} \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.26}$$

For every choice of r, s, ϵ, this defines a quasi-polynomial in the variable $2m$. For the purposes of stating the main theorem of this section (Theorem 2.12), only the cases $\epsilon \in \{0,1\}$ will be of interest. Note that for $m=0$ we have

$$\pi^{(\epsilon)}_{r,s}(0) = \begin{cases} \delta_{r,0}\delta_{s,0} & \text{if } \epsilon = 1, \\ 0 & \text{if } \epsilon = 0. \end{cases} \tag{2.27}$$

We may now state the main theorem of this section. In (2.29) below and later, for $\epsilon \in \{0,1\}$, we will write $\bar{\epsilon} := 1-\epsilon$ to lighten the notation.

Theorem 2.12. *For $n \geqslant 3$, for $m_1, m_2, \ldots, m_n \in \mathbb{Z}_{\geqslant 0}/2$, with at least three of the m_i being half-integers, the number $N_{0,n}(2m_1, 2m_2, \ldots, 2m_n)$ of planar tight maps with n boundaries labeled from 1 to n with respective lengths $2m_1, 2m_2, \ldots, 2m_n$ is given by the symmetric quasi-polynomial*

$$N_{0,n}(2m_1, 2m_2, \ldots, 2m_n) = \sum_{\left(\substack{\epsilon_1,\ldots,\epsilon_n \\ r_1,\ldots,r_n \\ s_1,\ldots,s_n}\right) \in I_n} \left(\sum_{i=1}^n r_i\right)! \left(\sum_{i=1}^n \epsilon_i s_i\right) \left(\sum_{i=1}^n s_i - 1\right)! \prod_{i=1}^n \pi^{(\epsilon_i)}_{r_i,s_i}(m_i), \tag{2.28}$$

where I_n is the (finite) subset of $\{0,1\}^n \times \mathbb{Z}^n_{\geqslant 0} \times \mathbb{Z}^n_{\geqslant 0}$ defined by

$$I_n := \left\{ \begin{pmatrix} \epsilon_1, \ldots, \epsilon_n \\ r_1, \ldots, r_n \\ s_1, \ldots, s_n \end{pmatrix} : \begin{array}{l} \sum_{i=1}^n \epsilon_i = \sum_{i=1}^n r_i + 1 \\ \sum_{i=1}^n \bar{\epsilon}_i = \sum_{i=1}^n s_i + 2 \end{array}, \sum_{i=1}^n s_i \geqslant 1 \right\}. \tag{2.29}$$

Remark 2.13. We note that the right-hand side of (2.28) is equal to 0 when the number k of faces of odd degree is equal to 0 or 2, so that the formula does *not* hold in these cases, which have been respectively dealt with above in Theorems 2.3 and 2.8. Theorem 2.12 yields a non-trivial result only when $k \geqslant 4$ is an even number, since a map necessarily has an even number of faces of odd degrees. As a sanity check, it is not difficult to see that the right-hand side of (2.28) vanishes when k is odd. Indeed, assume without loss of generality that $2m_1, \ldots, 2m_k$ are odd numbers, and that $2m_{k+1}, \ldots, 2m_n$ are even. By (2.26) and (2.27), the product term in the sum (2.28) is non-zero only if $s_i + \epsilon_i$ is even for $1 \leqslant i \leqslant k$, and odd for $k+1 \leqslant i \leqslant n$. On the other hand, the constraints in the definition of the summation index implies that $\sum_{i=1}^n (s_i + \epsilon_i) + 2 = n$, which after reduction modulo 2, shows that $n-k$ and n have the same parity, so that k is necessarily even. A fully unified formula, encompassing Theorems 2.3, 2.8 and 2.12 is given in Theorem 5.13 below.

The proof of Theorem 2.12 will follow an architecture similar to the bijective proof of Theorems 2.3 and 2.8, building from elementary examples of maps with explicit enumeration formulas, to construct general ones. In particular, note that (2.28) reduces to $\pi^{(1)}_{r,s}(m)$ if we specialize it to $n = r+s$ with $m_1 = m$, $m_2 = \cdots = m_{s+3} = 1/2$ and $m_{s+4} = \cdots = m_{r+s} = 0$. However, it is not a priori obvious to obtain an interpretation of $\pi^{(0)}_{r,s}(m)$ by specializing formula (2.28). For

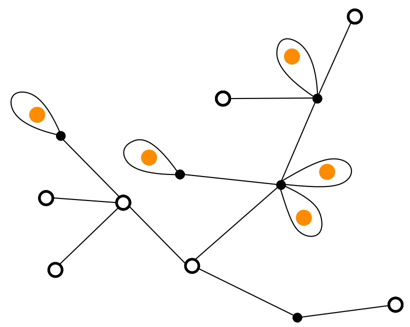

Figure 2.1: A tight petal tree with 5 petals (displayed with yellow marks) and 7 marked vertices (shown in white).

this reason, we will need to investigate in some depth these quasi-polynomials and relate them to the combinatorial notion of *petal trees*. This will be the object of Section 5.1, but let us record right away the definition of this notion so as to state one important result, Proposition 2.14, which will be used at the end of Section 3.

We call *petal* a face of degree 1. A *petal tree* is a planar map having an *exterior* face of arbitrary degree, and such that every other face is a petal. A tight petal tree is just a petal tree with marked vertices, which is tight as a map. See Figure 2.1 for an illustration.

Proposition 2.14. *For r, s nonnegative integers, $m \in \mathbb{Z}_{>0}/2$, and $\epsilon \in \{-1, 0, 1\}$, the number of tight petal trees with an exterior face of degree $2m$, $s+1+\epsilon$ petals, $1+\epsilon$ of which distinguished, and $r+1-\epsilon$ marked vertices, $1-\epsilon$ of which distinguished, is equal to $\pi^{(\epsilon)}_{r,s}(m)$.*

Remark 2.15. Proposition 2.14, which will be proved in Section 5.1, holds for $m > 0$. It can be extended to $m = 0$, provided we restrict the value of ϵ to the set $\{0, 1\}$. Indeed, in this case, (2.27) is consistent with Proposition 2.14 upon understanding the exterior face of degree 0 as a distinguished marked vertex. For $\epsilon = 0$ or 1, the only possible map with such a marked vertex, $s+1+\epsilon$ petals (and no other face) and $r+1-\epsilon$ other marked vertices is made of a single loop connecting the distinguished marked vertex and separating two petals. It has $s+1+\epsilon = 2$ and $r+1-\epsilon = 0$, hence $\epsilon = 1$ and $r = s = 0$ (note that the distinction of the two petals does not create any degeneracy).

Remark 2.16. The situation above is quite similar to the case $m = 1/2$, for which we obtain

$$\pi^{(\epsilon)}_{r,s}\left(\frac{1}{2}\right) = \begin{cases} 0 & \text{if } \epsilon = -1, \\ \delta_{r,0}\delta_{s,0} & \text{if } \epsilon = 0, \\ 0 & \text{if } \epsilon = 1. \end{cases} \tag{2.30}$$

This agrees with Proposition 2.14 since the only possible map with an exterior face of degree 1, $s+1+\epsilon$ petals and $r+1-\epsilon$ marked vertices is made of a single loop connecting a unique vertex and separating the exterior face from a unique petal. It has $s+1+\epsilon = 1$ and $r+1-\epsilon \leqslant 1$, hence $\epsilon = 0$, $s = 0$ and $r = 0$ (note that the unique vertex is therefore marked).

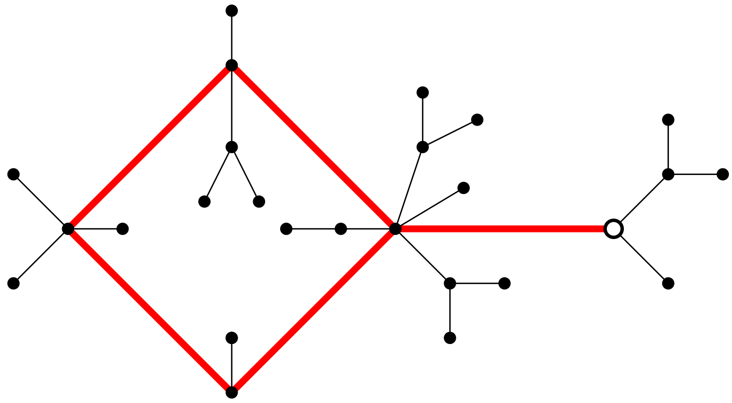

Figure 3.1: The tight core (thick red edges) of a planar bipartite map with two faces and one marked vertex (shown in white). The rest of the map consists of rooted plane trees (possibly empty) attached in the corners of the tight core.

3. Connection with the enumeration of non necessarily tight maps

3.1. Equivalence with Tutte's slicings formula in the (quasi-)bipartite case

One of the earliest results in map enumeration is Tutte's *slicings formula* [Tut62] which, in our current terminology, asserts that the number $M(\ell_1, \ldots, \ell_n)$ of planar (non necessarily tight) *bipartite* maps with $n \geqslant 3$ labeled faces of prescribed even degrees $2\ell_1, \ldots, 2\ell_n$ is given by

$$M(\ell_1, \ldots, \ell_n) = (\ell_1 + \cdots + \ell_n - 1)_{n-3} \prod_{i=1}^{n} \binom{2\ell_i - 1}{\ell_i}, \tag{3.1}$$

where $(\ell)_k := \ell(\ell-1)\cdots(\ell-k+1)$ denotes the falling factorial. Note that the faces are assumed unrooted and that the formula extends to the case where some, but not all, of the ℓ_i vanish, with the convention $\binom{-1}{0} = 1$, upon again understanding that a face of degree 0 is a marked vertex (planar maps with three or more labeled faces or vertices have no symmetries).

In this section, we explain how Tutte's slicings formula is related to Theorem 2.3 giving a formula for the number of planar *tight* bipartite maps with n boundaries of prescribed lengths. As we shall see, the two formulas can be deduced from one another.

The key observation, already made in [Bud22b, Section 4], is that an arbitrary (non necessarily tight) map, possibly with marked vertices, can be bijectively decomposed into a tight map (which we call the *tight core*) and a collection of rooted plane trees (without marked vertices) attached to the corners of the tight core. See Figure 3.1 for an illustration. Precisely, the tight core has the same number of faces and marked vertices as the arbitrary map, and a face of degree 2ℓ in the arbitrary map yields a face of degree $2m$ in the tight core, for some $m \leqslant \ell$, together with a plane forest made of $2m$ trees having $\ell - m$ edges in total. The number of such plane forests is equal to

$$A_{\ell,m} := \frac{2m}{2\ell}\binom{2\ell}{\ell - m}, \tag{3.2}$$

see for instance [FS09, ▷ I.38], with conventionally $A_{0,0} = 1$ and $A_{\ell,m} = 0$ for $m > \ell$. As a consequence, we have

$$M(\ell_1,\dots,\ell_n) = \sum_{m_1=0}^{\ell_1} \cdots \sum_{m_n=0}^{\ell_n} A_{\ell_1,m_1} \cdots A_{\ell_n,m_n} N_{0,n}(2m_1,\dots,2m_n) \tag{3.3}$$

where $N_{0,n}$ is the number of planar tight bipartite maps with n boundaries of lengths $2m_1,\dots,2m_n$, as defined in Section 1.2. Note that the matrix $(A_{\ell,m})_{\ell,m\geqslant 0}$ is unitriangular, hence the formula (3.3) can be inverted as

$$N_{0,n}(2m_1,\dots,2m_n) = \sum_{\ell_1=0}^{m_1} \cdots \sum_{\ell_n=0}^{m_n} B_{m_1,\ell_1} \cdots B_{m_n,\ell_n} M(\ell_1,\dots,\ell_n) \tag{3.4}$$

where B is the inverse of A. This inverse is given explicitly by $B_{m,\ell} = (-1)^{m-\ell}\binom{m+\ell-1}{m-\ell}$ but we will not use its expression in the following.

Now, let us substitute Tutte's slicings formula (3.1) into (3.4). By the Chu–Vandermonde identity, we may expand the falling factorial as

$$(\ell_1 + \cdots + \ell_n - 1)_{n-3} = (n-3)! \sum_{\substack{k_1,k_2,\dots,k_n \geqslant 0 \\ k_1+k_2+\cdots+k_n = n-3}} \binom{\ell_1 - 1}{k_1}\binom{\ell_2}{k_2}\cdots\binom{\ell_n}{k_n}, \tag{3.5}$$

which is nothing but an equality between polynomials in $\ell_1,\dots,\ell_n$. This yields

$$N_{0,n}(2m_1,\dots,2m_n) = (n-3)! \sum_{\substack{k_1,k_2,\dots,k_n \geqslant 0 \\ k_1+k_2+\cdots+k_n = n-3}} \hat{p}_{k_1}(m_1)\hat{q}_{k_2}(m_n)\cdots\hat{q}_{k_n}(m_n) \tag{3.6}$$

where $\hat{p}_k(m) := \sum_{\ell=0}^{m} B_{m,\ell}\binom{\ell-1}{k}\binom{2\ell-1}{\ell}$ and $\hat{q}_k(m) := \sum_{\ell=0}^{m} B_{m,\ell}\binom{\ell}{k}\binom{2\ell-1}{\ell}$. We recover Theorem 2.3, with the multivariate polynomial $p_{n-3}(m_1,m_2,\dots,m_n)$ defined via (2.5), provided that $\hat{p}_k(m)$ and $\hat{q}_k(m)$ are respectively equal to the univariate polynomials $p_k(m)$ and $q_k(m)$ defined in (2.1). This is ensured by the following:

Lemma 3.1. *The univariate polynomials $p_k(m)$ and $q_k(m)$ defined in (2.1) satisfy*

$$\sum_{m=0}^{\ell} A_{\ell,m} p_k(m) = \binom{\ell-1}{k}\binom{2\ell-1}{\ell}, \qquad \sum_{m=0}^{\ell} A_{\ell,m} q_k(m) = \binom{\ell}{k}\binom{2\ell-1}{\ell}. \tag{3.7}$$

Proof. These hypergeometric identities can be proved using algorithmic methods, see [PWZ96] and references therein.

Alternatively, a bijective proof for $\ell \geqslant 1$ follows from Propositions 2.4 and 2.5, themselves proved bijectively in Section 4.1. More precisely, the first identity is obtained by counting in two different ways plane trees with ℓ edges and $k+2$ distinct marked vertices, two of them distinguished and labeled. Namely, the left-hand side is obtained via the tight core decomposition and Proposition 2.4, while the right-hand side is obtained by a direct enumeration: $\binom{2\ell-1}{\ell}$ is the number of plane trees with ℓ edges and two distinguished labeled vertices, and $\binom{\ell-1}{k}$ is the number of ways to choose the k other marked vertices. The second identity is obtained similarly by counting in two different ways plane trees with ℓ edges, one of them marked, and $k+1$ distinct marked vertices, one of them distinguished. □

Note that, doing the above reasoning backwards, it is conversely possible to recover Tutte's slicings formula from Theorem 2.3, using (3.3). We now briefly discuss the quasi-bipartite case: let ℓ_1, ℓ_2 be positive half-integers, and $\ell_3, \dots, \ell_n$ be non-negative integers, $n \geqslant 3$. Then, by [Tut62, Section 6], the number $M(\ell_1, \dots, \ell_n)$ of planar quasi-bipartite maps with n labeled boundaries of prescribed lengths $2\ell_1, \dots, 2\ell_n$ reads

$$M(\ell_1, \dots, \ell_n) = (\ell_1 + \cdots + \ell_n - 1)_{n-3} \binom{2\ell_1 - 1}{\ell_1 - \frac{1}{2}} \binom{2\ell_2 - 1}{\ell_2 - \frac{1}{2}} \prod_{i=3}^{n} \binom{2\ell_i - 1}{\ell_i}. \tag{3.8}$$

The tight core decomposition works as before and we find that (3.3) still holds, upon understanding that the sums over m_1 and m_2 should be now taken over half-integer values, $A_{\ell,m}$ being still defined by (3.2) for ℓ, m half-integers. By a slight variant of the reasoning above, we may deduce Theorem 2.8 with the multivariate polynomial $\tilde{p}_k(m_1, m_2; m_3, \dots, m_n)$ being given by (2.20). Namely, we modify the expansion (3.5) of the falling factorial by replacing $\ell_1 - 1$ and ℓ_2 in the right-hand side by $\ell_1 - \frac{1}{2}$ and $\ell_2 - \frac{1}{2}$, respectively, and we make use of the identity

$$\sum_{m \in \{\frac{1}{2}, \frac{3}{2}, \dots, \ell\}} A_{\ell,m} \tilde{p}_k(m) = \binom{\ell - \frac{1}{2}}{k} \binom{2\ell - 1}{\ell - \frac{1}{2}} \tag{3.9}$$

valid for ℓ a positive half-integer. Again, this identity can be proved either via algorithmic methods, or via a bijective argument: it is obtained by counting in two different ways planar maps with one face of odd degree 2ℓ, one face of degree one and $k+1$ distinct marked vertices, one of them distinguished. The left-hand side is obtained by the tight-core decomposition together with Proposition 2.10 (which will be derived bijectively in Section 4.1). As for the right-hand side, note that, by collapsing the face of degree one, such maps correspond to rooted plane trees with $\ell - \frac{1}{2}$ edges: $\binom{2\ell-1}{\ell-1/2}$ is the number of such trees with one distinguished vertex, and $\binom{\ell-1/2}{k}$ is the number of ways to choose the k other marked vertices.

3.2. A non-bipartite slicings formula

By arguing similarly, we may use Theorem 2.12 to obtain a generalization of Tutte's slicings formula counting planar maps with a prescribed degree sequence. The relevant identity to use, valid for $\ell - \frac{s+1+\epsilon}{2} \in \mathbb{Z}$ and $\epsilon \in \{0, 1\}$, is

$$\sum_{m \in \mathbb{Z}_{\geqslant 0}/2} A_{\ell,m} \pi^{(\epsilon)}_{r,s}(m) = \binom{2\ell - 1}{\ell - \frac{s+1+\epsilon}{2}\,,\ \ell - r - \frac{s+1-\epsilon}{2}\,,\ r\,,\ s}, \tag{3.10}$$

where in the left-hand side, we observe by (2.26) that only the terms for which $\ell - m \in \mathbb{Z}_{\geqslant 0}$ contribute, and in the right-hand side, we use a multinomial coefficient notation. To understand this formula, recall from Proposition 2.14 that $\pi^{(\epsilon)}_{r,s}(m)$ counts tight petal trees, i.e. tight maps with an exterior face of degree $2m$, $s+1+\epsilon$ petals, $1+\epsilon$ of which are distinguished, and $r+1-\epsilon$ marked vertices, $1-\epsilon$ of which are distinguished. By applying the tight core decomposition, the left-hand side of (3.10) expresses the number of petal trees which are not necessarily tight, with an exterior face of degree 2ℓ, and with the same number of (distinguished) petals and (distinguished) marked vertices as described in the preceding sentence. Checking that this number equals the right-hand side of (3.10) is a straightforward exercise based on the methods used in Section 5.1 to prove Proposition 2.14, and is simpler due to the absence of the tightness condition.

Formula (3.3) remains unchanged if $\ell_1, \dots, \ell_n$ are allowed to take half-integer values, except that the corresponding sums should then run over half-integer m_i's as well. Substituting in (3.3) the formula of Theorem 2.12 for $N_{0,n}(2m_1, \dots, 2m_n)$, we obtain the following:

Theorem 3.2 (A census of non-bipartite slicings)**.** *The number $M(\ell_1, \dots, \ell_n)$ of planar maps with n labeled faces of degrees $2\ell_1, \dots, 2\ell_n$, at least four of which are odd, is given by*

$$M(\ell_1, \dots, \ell_n) = \sum_{\left(\substack{\epsilon_1,\dots,\epsilon_n \\ r_1,\dots,r_n \\ s_1,\dots,s_n}\right) \in I_n} \left(\sum_{i=1}^n r_i\right)! \left(\sum_{i=1}^n \epsilon_i s_i\right) \left(\sum_{i=1}^n s_i - 1\right)! \\ \times \prod_{i=1}^n \binom{2\ell_i - 1}{\ell_i - \frac{s_i+1+\epsilon_i}{2}, \ell_i - r_i - \frac{s_i+1-\epsilon_i}{2}, r_i, s_i}, \quad (3.11)$$

where I_n is as in (2.29), and where it is understood that the multinomial coefficient vanishes whenever $\ell_i - \frac{s_i+1+\epsilon_i}{2}$ is not an integer. The formula makes sense when some ℓ_i vanish, upon understanding that a face of degree 0 is in fact a vertex.

Example 3.3. For $\ell_1, \dots, \ell_4 \in \mathbb{Z}_{\geqslant 0} + \frac{1}{2}$, performing the sum in (3.11) yields a number of planar maps with four labeled faces of odd degrees $2\ell_1, \dots, 2\ell_4$ equal to

$$M(\ell_1, \dots, \ell_4) = (\ell_1 + \dots + \ell_4 - 2) \prod_{i=1}^4 \binom{2\ell_i - 1}{\ell_i - \frac{1}{2}}. \quad (3.12)$$

In particular, we find $M\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 2$ which, after rooting each face, gives a number of slicings equal to 6 consistently with [Tut62, Section 6].

To our knowledge, this extension of Tutte's slicings formula for general non-bipartite planar maps with prescribed degrees is new. Note that (3.11) does not hold when the number of faces of odd degree is zero or two, see the discussion in Remark 2.13 in the tight setting. Even though we obtain Theorem 3.2 as a consequence of Theorem 2.12, the former could be proved directly by the approach of Section 5, forgetting about the tightness constraint.

4. Bijective proofs in the bipartite and quasi-bipartite cases

4.1. Case of maps with one face

In this section, we first present a combinatorial proof of Propositions 2.4 and 2.5 by a direct enumeration of the planar tight bipartite maps with one face considered in these propositions.

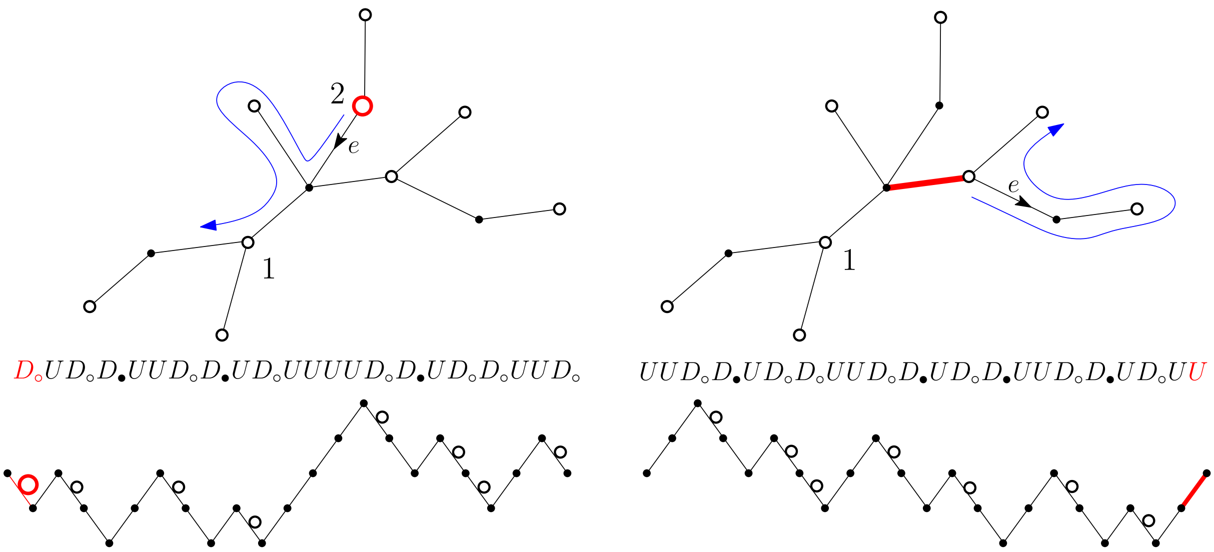

Figure 4.1: The coding of planar tight maps with one face with degree $2m$ (here $m = 11$) by dressed words with letters U, $D_\circ$ and $D_\bullet$. Left: in the context of Proposition 2.4, such a map has $k+2$ (here $k = 7$) marked vertices (represented as white, as in Figure 1.1), two of them distinguished as vertex 1 and 2 (represented by a bigger red circle). The associated word, indicated under the map, starts with a $D_\circ$. Right: in the context of Proposition 2.5, the map has $k+1$ marked vertices, one of them distinguished as vertex 1, and a root edge (represented in thick red). The associated word ends with a U. We indicated under each word its visualization as a lattice path, where the vertex markings have been transferred to descending steps (as indicated by circles). We also indicated the starting oriented edge e for the contour word of each map.

Our approach is inspired from the bijective interpretation of Narayana numbers given in [DZ80, Section 3.2].

More precisely, we wish to enumerate planar bipartite maps with one face of degree $2m$, which are nothing but *plane trees with m edges* ($m \geqslant 1$), endowed with a distinguished marked vertex labeled 1, and either a second distinguished marked vertex labeled 2 in the context of Proposition 2.4 or with a marked edge (the root edge) in the context of Proposition 2.5. These two situations will be referred to respectively as "case (p)" and "case (q)" in the following, to remind the reader that they concern the combinatorial interpretation of $p_k(m)$ and $q_k(m)$ respectively. The trees are finally decorated by the choice of k additional unlabeled marked vertices ($k \geqslant 0$), with the constraint that *all the leaves of the tree are marked vertices* (either labeled or unlabeled), except possibly the endpoints of the root edge in the case (q) if it happens that such an endpoint is a leaf.

Our proof is based on the classical coding of plane trees by their contour word, here in terms of the letters U (up) and D (down). The following discussion is illustrated on Figure 4.1. Ignoring the k unlabeled vertex markings for now, the coding that we use here is adapted to trees *with both a marked vertex* 1 *and a marked oriented edge* e whose choice in cases (p) and (q) will be discussed below. For any such tree, the contour word is obtained as follows: we start from the right of e and visit all the edge sides counterclockwise around the tree. We then record a letter U if we move away from vertex 1 on the tree, and a letter D otherwise. After one turn around the tree, we get a word of $2m$ letters containing m occurrences of U and m occurrences of D, since the two sides of any given edge give rise to exactly one U and one D. Viewing the successive U's and D's as successive up and down steps, the coding may alternatively be represented as a lattice path of length $2m$ starting and ending at the same height (a so-called bridge in the lattice path terminology) and whose nodes correspond to the successive visited corners around the tree. In this representation, the corners at vertex 1 are associated with the nodes with minimal height. Setting the minimal height to 0, the height of a node is nothing but the graph distance to vertex 1 of the vertex incident to the associated corner. The above coding by words/paths is clearly bijective.

To use this coding in the context of Propositions 2.4 and 2.5 where the trees already have a distinguished vertex 1, we need a canonical prescription for the choice of the oriented edge e at which we start the contour. In case (p) where the tree is endowed with a second distinguished vertex 2, we take for e the first edge of the branch between vertex 2 and vertex 1, oriented towards 1. Clearly the knowledge of e and that of the vertex 2 are equivalent, but we note that, by construction, the associated word necessarily *starts with a* D in case (p).

In case (q), we first orient the root edge away from vertex 1 and take for e the edge following it along the counterclockwise contour around the tree. The edge e is therefore incident to the endpoint of the root edge further away from vertex 1 and we orient it away from that vertex[3]. Clearly the knowledge of e and that of the root edge are equivalent, but we note that, by construction, the last visited edge side in the contour is that of the root edge itself, going away from vertex 1, hence the associated coding word necessarily *ends with a* U in case (q).

It remains to introduce the k additional vertex markings. The markings may be recorded in the coding word as follows: every vertex v distinct from vertex 1 may be associated bijectively with a letter D of the coding word. Indeed v is bijectively associated with the edge $e(v)$ incident to v that belongs to the branch between v and the vertex 1 and exactly one of the two sides of $e(v)$ is coded by the letter D. If v is a marked vertex, we transfer its marking to the associated letter D, which we denote by $D_\circ$ to record the marking. If v is not a marked vertex, the associated letter D will be denoted by $D_\bullet$, so that the letter D eventually appears in two flavors $D_\circ$ and $D_\bullet$, leading to dressed words made of the three letters U, $D_\circ$ and $D_\bullet$. In case (p), we also transfer the marking of vertex 2, so that the first letter (which we know is originally a D) is now a $D_\circ$. The numbers of U, $D_\circ$ and $D_\bullet$ letters are therefore, respectively, m, $k+1$ and $m-k-1$ in case (p) and m, k and $m-k$ in case (q).

Apart from possibly vertex 2 (which is marked anyway) in case (p) or possibly an endpoint of the root edge (which needs not being marked) in case (q), any leaf in the tree corresponds to a sequence UD in the associated word. Requiring that all leaves be marked boils down to demanding that any D following a U be marked, i.e. the sequence $UD_\bullet$ is not allowed in the dressed words.

Altogether, a word coding for a tree in case (p) has the canonical form

$$D_\circ D_\bullet^{a_1} U^{b_1} D_\circ D_\bullet^{a_2} U^{b_2} \cdots D_\circ D_\bullet^{a_{k+1}} U^{b_{k+1}} \tag{4.1}$$

[3]Note that it may happen that the edge e be identical to the root edge itself, but with the opposite orientation, in which case the first letter of the word is a D. In all other cases the first letter is a U.

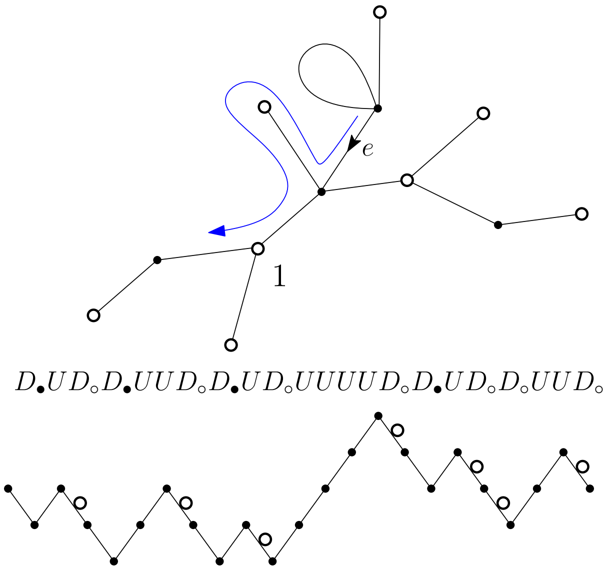

Figure 4.2: The coding of planar tight maps with one face with odd degree $2m$ (here $m = 23/2$) and one face of degree 1 by dressed words with letters U, $D_\circ$ and $D_\bullet$. As in Proposition 2.10 the map has $k+1$ (here $k = 7$) marked vertices, one of them distinguished as vertex 1. We indicated under the map the associated word, of length $2m-1$, and its visualization as a lattice path. We also indicated on the map the starting oriented edge e for the contour word.

where the a_i and b_i are nonnegative integers such that $a_1 + a_2 + \cdots + a_{k+1} = m - k - 1$ and $b_1 + b_2 + \cdots + b_{k+1} = m$. In other words, the a_i and b_i form weak compositions of $m - k - 1$ and m, respectively, into $k+1$ summands. There are respectively $\binom{m-1}{k}$ and $\binom{m+k}{k}$ such compositions, hence the number of trees in case (p) is $\binom{m-1}{k}\binom{m+k}{k} = p_k(m)$ as wanted.

Similarly, a word coding for a tree in case (q) has the canonical form

$$D_\bullet^{a_1}U^{b_1}D_\circ D_\bullet^{a_2}U^{b_2}D_\circ \cdots D_\bullet^{a_k}U^{b_k}D_\circ D_\bullet^{a_{k+1}}U^{b_{k+1}}U \tag{4.2}$$

where the a_i and b_i form weak compositions of $m-k$ and $m-1$, respectively, into $k+1$ summands[4]. There are respectively $\binom{m}{k}$ and $\binom{m+k-1}{k}$ such compositions, hence the number of trees in case (q) is $\binom{m}{k}\binom{m+k-1}{k} = q_k(m)$ as wanted.

This ends the combinatorial proof of Propositions 2.4 and 2.5.

Quasi-bipartite case. The proof of Proposition 2.10 is obtained along similar lines, see Figure 4.2 for an example. Indeed, we may transform bijectively a planar map with one face of odd degree $2m$ ($m \in \mathbb{Z}_{\geqslant 0} + 3/2$), one face of degree 1 and a distinguished vertex 1 into a plane tree with $m - 1/2$ edges with both a marked vertex 1 and a marked oriented edge e. This is done by

[4]Note that the case $a_1 = b_1 = 0$ corresponds to a word starting with a $D_\circ$ which, cyclically, comes after the last letter U. This situation corresponds to the case where the endpoint of the root edge further away from vertex 1 is a leaf and is marked. The case $a_1 > 0$ corresponds to the case where this vertex is a leaf and is unmarked and, finally, the case $a_1 = 0$, $b_1 > 0$ corresponds to the case where this vertex is not a leaf.

considering the unique vertex incident to the loop formed by the degree 1 face, by marking its incident edge e lying immediately to the left of that loop, with e oriented away from the vertex and finally erasing the loop. We can now use our coding of such pointed rooted trees by words with $2m-1$ letters. Taking the markings into account gives rise to a dressed word with exactly k occurrences of $D_\circ$, $m-1/2-k$ occurrences of $D_\bullet$ and $m-1/2$ occurrences of U, with no occurrence of the sequence $UD_\bullet$, hence with canonical form

$$D_\bullet^{a_1} U^{b_1} D_\circ D_\bullet^{a_2} U^{b_2} \cdots D_\circ D_\bullet^{a_{k+1}} U^{b_{k+1}} \tag{4.3}$$

where the a_i and b_i are nonnegative integers such that $a_1 + a_2 + \cdots + a_{k+1} = m - k - 1/2$ and $b_1 + b_2 + \cdots + b_{k+1} = m - 1/2$. In other words, the a_i and b_i form weak compositions of $m-k-1/2$ and $m-1/2$, respectively, into $k+1$ summands. There are respectively $\binom{m-1/2}{k}$ and $\binom{m+k-1/2}{k}$ such compositions, hence the number of maps at hand is $\binom{m-1/2}{k}\binom{m+k-1/2}{k} = \tilde{p}_k(m)$ as announced. This holds for $m \geqslant 3/2$. For $m = 1/2$, the value $\tilde{p}_k(1/2) = \delta_{k,0}$ is consistent with the fact that there is a unique planar map with two (distinguished) faces of degree 1 and one marked vertex labeled 1, which is its unique vertex so that the map cannot host any other marked vertices when $k > 0$.

4.2. Case of maps with two faces

Let us now provide a bijective proof of Proposition 2.6. To this end, we will first need to reinterpret slightly the objects counted by $p_k(m), q_k(m)$ that were discussed in the preceding section.

Definition 4.1. For given integers $a \geqslant 1$ and $b \geqslant 0$, an (a,b)-*forest* is a tight map with exactly two faces f, f_*, such that:

- f_* is a simple face[5] of degree $a+b$ and one distinguished incident vertex v_*,
- the a vertices following and including v_* in counterclockwise order around f_* are not marked.

We call the a vertices referred to above as the *unmarkable* vertices, and the other b vertices are called the *markable* vertices. In the illustrating figures, starting with Figure 4.3, the latter will be represented by white squares, while the former will be represented by crosses. Equivalently, by removing the $a+b$ edges incident to f_*, we may view an (a,b)-forest as a linearly ordered collection of $a+b$ rooted plane trees starting from the one rooted at v_*, whose leaves (non-root vertices of degree 1) are all marked, and such that the roots of the first a trees are unmarked while those of the remaining b trees may be marked or not.

The *size* of an (a,b)-forest is the degree of the face f. If we view it as a collection of trees as above, then this size is equal to $2e+a+b$ where e is the total number of edges in the trees composing the forest.

We now describe two simple bijections, illustrated on Figure 4.4, linking the numbers $p_k(m)$ and $q_k(m)$ to the forests discussed above. First, recall from Proposition 2.4 the interpretation of $p_k(m)$ as counting tight bipartite planar maps with one face f of degree $2m$ and $k+2$ distinct

[5] We say that a face is *simple* if its contour is a simple cycle, i.e. does not visit a same vertex several times.

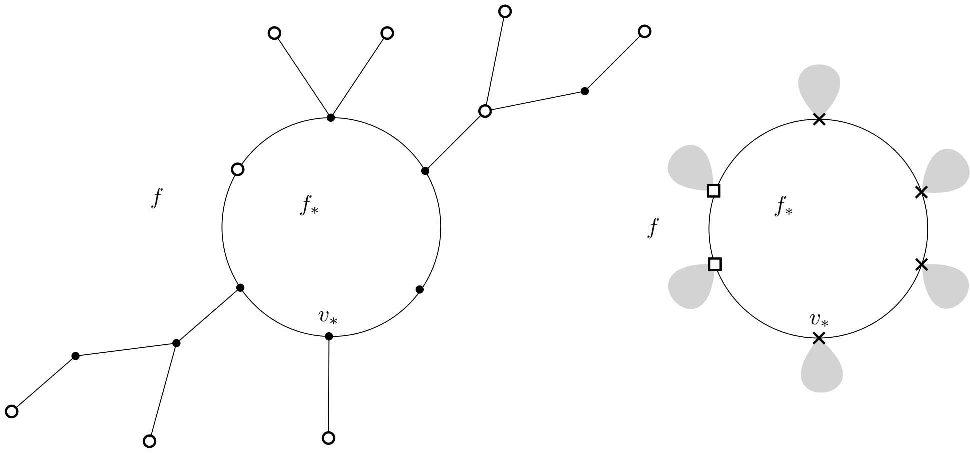

Figure 4.3: Left: an example of a $(4, 2)$-forest, with 9 marked vertices and size 28. The marked vertices are represented as white, as in Figure 1.1. Note that none of the first 4 vertices arriving in counterclockwise order after v_* are marked, while the following 2 comprise one marked and one unmarked vertex. Right: a schematic, generic representation of a $(4, 2)$-forest, where the grey blobs represent tree components (that may be reduced to a single root vertex) and white squares represent markable roots, while crosses represent unmarkable roots.

marked vertices, of which exactly two are labeled as 1 and 2. There is a natural operation consisting in cutting open the branch γ linking the distinguished vertices, into a simple face f_* of degree $2d$, where $d \geqslant 1$ is the graph distance between these vertices. In doing so, we duplicate the vertices lying on the path γ, except its extremities, into "left and right" copies (upon orienting γ from vertex 1 to vertex 2), and in case some of these vertices are marked, we always decide to transfer the mark to the left copy. The vertex initially distinguished and labeled as 1 is then renamed as v_* and seen as unmarked, while we remove the mark and label on the vertex initially labeled 2. The result is then a $(d + 1, d - 1)$-forest. Conversely, given a $(d + 1, d - 1)$-forest for some $d \geqslant 1$, we can glue together the r-th edge of f_* in counterclockwise order starting from v_* with the opposite $(2d - r + 1)$-th one, for $r \in \{1, 2, \ldots, d\}$, relabel v_* as vertex 1 and the diametrally opposite vertex of f_*, lying at distance d from v_*, as vertex 2, and finally, after gluing the vertices in pairs along the contour of f_*, transferring to the newly created vertices the marks carried by all markable vertices. By construction, every markable vertex is matched to an unmarkable vertex, and this operation is the inverse of the cutting procedure described above. Finally, these operations preserve the number k of marked (unlabeled) vertices, and are size-preserving in the sense that the degree $2m$ of the unique face of the map to be cut corresponds to the size of the resulting forest.

Similarly, recall from Proposition 2.5 that $q_k(m)$ enumerates rooted tight bipartite planar maps with one face f of degree $2m$ and $k + 1$ distinct marked vertices, exactly one of them being distinguished and labeled as vertex 1. Let v be the vertex incident to the root edge of such a map, and which is further away from vertex 1. Again, we cut open along the branch from vertex 1 to v, with length $d \geqslant 1$ say, creating a simple face f_* of degree $2d$. We transfer the marks along this path to the left copies of the vertices created in the cutting operation, and if v

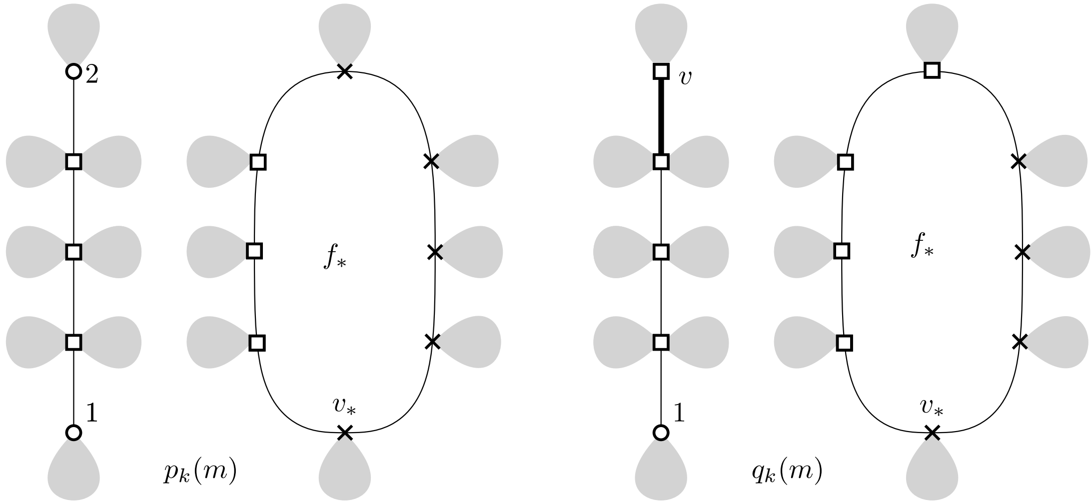

Figure 4.4: The cutting operation turning tight maps with one face into forests. On the left, tight maps counted by $p_k(m)$ are cut into $(d+1, d-1)$ forests, and on the right, rooted tight maps counted by $q_k(m)$ become (d, d)-forests.

happens to be marked, we keep this mark. Finally, we remove the mark on vertex 1 and rename it as v_*. This results in a (d, d)-forest, since now the vertex diametrically opposite to v_* in f_* is a markable vertex. This operation is clearly invertible by a similar gluing operation as above, and it preserves the number of marked unlabeled vertices as well as the degree of f. We may conclude with the following statement.

Proposition 4.2. *For integers $m \geqslant 1$ and $k \geqslant 0$, the number $p_k(m)$ (resp. $q_k(m)$) is the cardinality of the set of $(d+1, d-1)$-forests (resp. (d, d)-forests) with size $2m$ and k marked unlabeled vertices, where d can take any value in $\mathbb{Z}_{>0}$.*

Our bijective proof of Proposition 2.6 will consist in showing that a tight map with two faces f_1, f_2 of respective degrees $2m_1, 2m_2$, and with k unlabeled marked vertices and one extra distinguished marked vertex with label 1 can be decomposed uniquely and bijectively into a pair of tight maps consisting in

- a $(d_1 + 1, d_1 - 1)$-forest with k_1 marked vertices
- a (d_2, d_2)-forest with k_2 marked vertices

where $d_1, d_2 \geqslant 1$ and $k_1 + k_2 = k$. By Proposition 4.2 and the definition (2.5) of $p_k(m_1, m_2)$, this immediately implies Proposition 2.6.

Given an (a, b)-forest and an integer c such that $1 \leqslant c \leqslant \min(a-1, b)$, the *$c$-partial gluing* of the forest is the map obtained by gluing the r-th edge following v_* in counterclockwise order around f_* with the opposite $(a+b+1-r)$-th one, for $r \in \{1, 2, \ldots, c\}$, and transferring any mark on the markable vertices to the resulting glued vertices. The vertex inherited from v_* in the new map is distinguished and labeled as vertex 1, while the last vertex to be glued, lying at distance c from v_*, is distinguished and called v_{**}. Since $c \leqslant \min(a-1, b)$, some edges remain

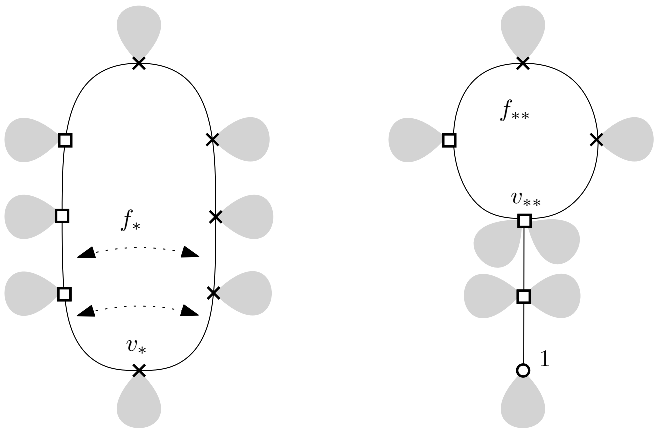

Figure 4.5: The 2-partial gluing of a $(5,3)$-forest yielding a $(2,2)^*$-forest.

unglued, and the resulting map still has two faces which we call f, f_{**}, where f_{**} is the "remnant" of f_*, of degree $a+b-2c$, and we do not change the name for the exterior face since it has the same contour information as the original face. Moreover, the assumption that $c \leqslant \min(a-1,b)$ implies that every markable vertex is glued to an unmarkable vertex.

The obtained map is then an $(a',b')^*$-forest with $a' = a-c-1$ and $b' = b-c+1$, according to the following definition, similar to Definition 4.1:

Definition 4.3. For integers $a \geqslant 0$ and $b \geqslant 1$, an $(a,b)^*$*-forest* is a tight map with exactly two faces f, f_{**}, such that:

- f_{**} is a simple face of degree $a+b$,
- there is an extra distinguished vertex labeled 1 incident to the face f, and we let v_{**} be the vertex incident to f_{**} that is closest to 1,
- the a vertices following and excluding v_{**} in counterclockwise order around f_{**} are not marked.

The partial gluing operation is clearly invertible, by cutting along the simple path of length c from v_{**} to the distinguished labeled vertex 1. See Figure 4.5 for an illustration.

Proof of Proposition 2.6. Let m_1, m_2, d_1, d_2 be positive integers, and k_1, k_2 be non-negative integers. Suppose we are given a (d_1+1, d_1-1) forest $\mathbf{f}_1$ with size $2m_1$ and k_1 marked vertices, and a (d_2,d_2) forest $\mathbf{f}_2$ with size $2m_2$ and k_2 marked vertices. We let $v_{*,1}, v_{*,2}$ be the distinguished vertices in these maps. As explained above, we wish to use these pieces to build a planar tight map with two faces of degrees $2m_1, 2m_2$, with $k = k_1 + k_2$ marked unlabeled vertices, and with one extra distinguished vertex. There are three possible situations, illustrated in Figure 4.6.

Suppose first that $d_1 = d_2 = d$. Then we can glue together the two simple boundaries of $\mathbf{f}_1$ and $\mathbf{f}_2$, in such a way that $v_{*,1}$ and $v_{*,2}$ are glued together into a single distinguished vertex labeled 1. The next d unmarkable vertices after $v_{*,1}$ (resp. the last $d-1$ markable vertices) of $\mathbf{f}_1$ are then glued to the last d markable vertices (resp. the $d-1$ unmarkable vertices following $v_{*,2}$)

of $\mathbf{f}_2$. The result is a tight map with two faces of degrees $2m_1, 2m_2$, and with $k_1 + k_2$ marked unlabeled vertices as well as a distinguished marked vertex labeled 1 lying on the boundary of both faces.

Suppose next that $d_1 > d_2$. In this case, we first perform the $(d_1 - d_2)$-partial gluing of $\mathbf{f}_1$, resulting in a $(d_2, d_2)^*$-forest (with distinguished vertex called $v_{**,1}$), which we glue along the simple face of $\mathbf{f}_2$ by identifying $v_{**,1}$ and $v_{*,2}$. Note that each of the d_2 markable vertices on either side of the gluing is matched with an unmarkable vertex on the other side. The resulting map has a distinguished labeled vertex incident to f_1 but not to f_2.

Finally, the case $d_2 > d_1$ is similar, except that we now perform the $(d_2 - d_1)$-partial gluing of $\mathbf{f}_2$ first, resulting in a $(d_1 - 1, d_1 + 1)^*$-forest, whose $d_1 + 1$ markable vertices and $d_1 - 1$ unmarkable vertices are matched with the $d_1 + 1$ unmarkable vertices and $d_1 - 1$ markable vertices of $\mathbf{f}_1$. The resulting map has a distinguished labeled vertex incident to f_2 but not to f_1.

The above construction can clearly be inverted by the following cutting operation. Start from a tight map $\mathbf{m}$ with two faces f_1, f_2, k marked vertices and one extra distinguished vertex labeled 1. We observe that such a map is unicyclic, and therefore contains a unique simple cycle γ, of length $2d \geqslant 2$ say. We cut along this cycle, separating f_1 and f_2. Formally, this means that we associate with $\mathbf{m}$ the two maps $\mathbf{m}_1, \mathbf{m}_2$ respectively obtained by removing all edges and vertices that are incident to f_2 but not f_1 on the one hand, and f_1 but not f_2 on the other hand. Note that for $i \in \{1, 2\}$, $\mathbf{m}_i$ is made of the initial face f_i and has an "exterior" simple face $f_{*,i}$ which is the remnant of the face f_{3-i}. All marked vertices of $\mathbf{m}$ that are not on γ are naturally transferred to either $\mathbf{m}_1$ or $\mathbf{m}_2$, and we need a convention to transfer the marked vertices lying on γ.

To this end, we distinguish the cases as above depending on whether the vertex labeled 1 is incident to both f_1, f_2, to f_1 but not f_2, or to f_2 but not f_1. In the first case, the cutting operation splits the labeled vertex 1 into two copies $v_{*,1}, v_{*,2}$, respectively belonging to $\mathbf{m}_1$ and $\mathbf{m}_2$, that we declare unmarkable. The d vertices following $v_{*,1}$ in counterclockwise order around $f_{*,1}$ are declared unmarkable, as well as the $d - 1$ vertices following $v_{*,2}$ in counterclockwise order around $f_{*,2}$, and all other vertices incident to $f_{*,1}$ and $f_{*,2}$ are declared markable. In this way, every vertex of γ has been split into a a markable/unmarkable pair in $\mathbf{m}_1$ and $\mathbf{m}_2$. We then transfer the marks that were located on the vertices γ to the unique associated markable duplicate. This gives the wanted pair $(\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{m}_1, \mathbf{m}_2)$ of $(d+1, d-1)$- and (d, d)-forests, of sizes $2m_1$ and $2m_2$, which receive k_1 and k_2 marked unlabeled vertices with $k_1 + k_2 = k$.

In the second case, we let v_{**} be the vertex incident to f_2 that is closest to the distinguished vertex 1. When cutting along the cycle γ, this vertex is separated into two copies, one called $v_{**,1}$ is incident to $f_{*,1}$ and is declared markable, as well as the $d-1$ vertices *preceding it* around $f_{*,1}$, the other called $v_{*,2}$ is incident to $f_{2,*}$ and is declared unmarkable, as well as the $d - 1$ vertices following it around $f_{*,2}$. The map $\mathbf{m}_2 =: \mathbf{f}_2$ is then a (d, d)-forest, while we further cut $\mathbf{m}_1$ along the simple path of length $d' \geqslant 1$ from $v_{**,1}$ to the distinguished vertex 1, hence creating a $(d+d'+1, d+d'-1)$-forest $\mathbf{f}_1$, attributing the marked vertices in the natural way (this operation is the reverse of the d'-partial gluing of the resulting forest).

The situation in the third case is similar, with a slightly different convention for the markable and unmarkable vertices, as illustrated in Figure 4.6. □

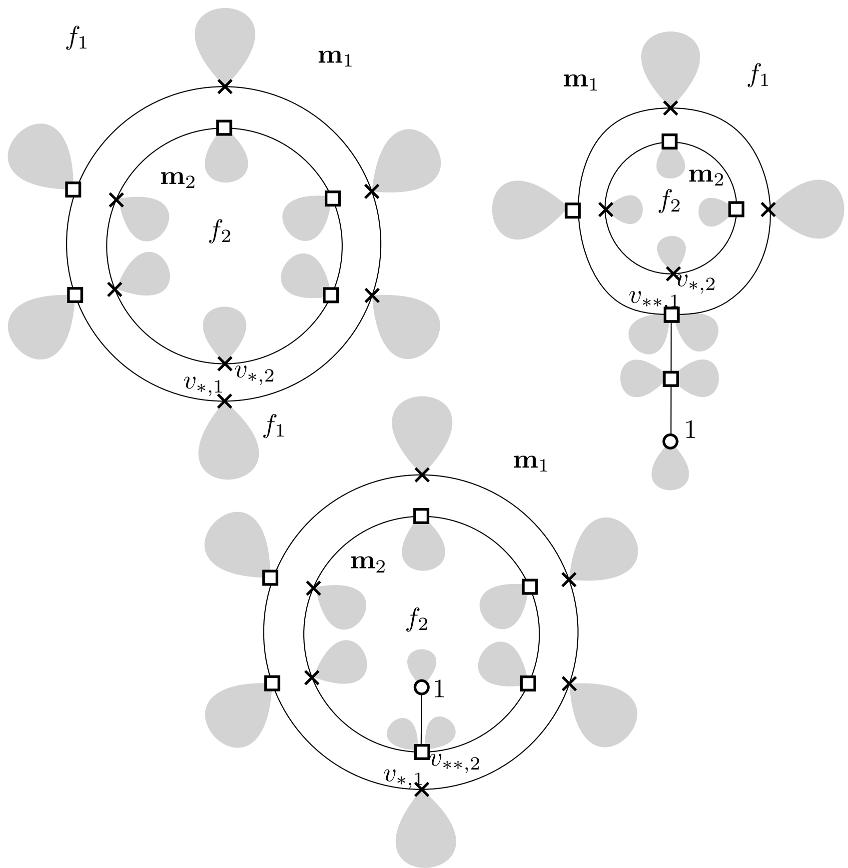

Figure 4.6: Decomposition of a tight map with two faces f_1, f_2, k marked unlabeled vertices and one extra distinguished labeled vertex, into two forests. The top left, top right, and bottom pictures represent respectively the situations where the distinguished labeled vertex belongs to the common boundary of f_1 and f_2, is incident to f_1 but not f_2, and is incident to f_2 but not f_1. We denote by $2d$ the length of the cycle separating f_1 and f_2. In the first case, the map is decomposed into a $(d+1, d-1)$ forest glued to a (d, d) forest. In the second case, the map is decomposed into the $(d_1 - d)$-partial gluing of a (d_1+1, d_1-1)-forest with a (d, d)-forest, and in the third case, it is instead a $(d+1, d-1)$-forest glued to the $(d_2 - d)$-partial gluing of a (d_2, d_2)-forest. Note that, on this picture, the counterclockwise order along the face $f_{*,2}$, which is the “exterior” simple face of $\mathbf{m}_2$, appears to be clockwise, since $f_{*,2}$ is the unbounded face in the plane embedding.

Quasi-bipartite case. We now consider the quasi-bipartite case where one assumes that $m_1, m_2 \in \mathbb{Z}_{\geqslant 0} + 1/2$, that is, $2m_1$ and $2m_2$ are odd integers, and aim at proving Proposition 2.9. A discussion parallel to the above applies, except that the separating cycle between the two faces of a tight map with faces of degrees $2m_1$ and $2m_2$ will have an odd length, say $2d-1$ for some $d \geqslant 1$.

In this situation, unfolding the above argument *mutatis mutandis*, there is now a canonical decomposition of a planar tight map with two faces of degrees $2m_1, 2m_2$, with k marked unlabeled vertices and one extra distinguished vertex labeled 1 into a pair formed of a $(d_1, d_1 - 1)$-forest and a $(d_2, d_2 - 1)$-forest, for some $d_1, d_2 \geqslant 1$, respectively with sizes $2m_1$ and $2m_2$ and with k_1 and k_2 marked vertices, where $k_1 + k_2 = k$. The situation is therefore more symmetric since the glued forests are of the same nature and have the same numbers of markable vertices.

By performing the $(d_i - 1)$-partial gluing of the $(d_i, d_i - 1)$-forest of size $2m_i$ with k_i marked vertices, we see that such objects are in bijection with tight maps with one face of degree $2m_i$, one face of degree 1, and k_i marked vertices, which are precisely counted by $\tilde{p}_{k_i}(m_i)$, as discussed in Section 4.1. Together with the above, this shows that $\tilde{p}_k(m_1, m_2)$ indeed enumerates the wanted quasi-bipartite planar tight maps with two faces, as stated in Proposition 2.9.

4.3. Case of pointed rooted maps via slices

We now aim at proving Proposition 2.7, interpreting $(n-1)!\, q_{n-1}(m_1, m_2, \ldots, m_n)$ as the number of pointed rooted planar tight bipartite maps with n labeled boundaries of respective lengths $2m_1, \ldots, 2m_n$, where $m_1, \ldots, m_n$ are integers not all equal to zero. To this end, we will need the slice decomposition developed in [BG12, BG14]. Here, we follow closely the presentation of [Bou19, Section 2.2] and adapt it to the tight setting.

A *slice* is a planar map with one distinguished *exterior face*, whose contour carries three distinguished (but not necessarily distinct) corners A, B and C appearing in this counterclockwise order around the map, that split the contour in three parts:

- the contour segment AB, called the *left boundary*[6], which is a geodesic path,
- the contour segment AC, called the *right boundary*, which is the unique geodesic path between its two endpoints, and intersects the left boundary only at A,
- the contour segment BC, called the *base*.

The length of the base (i.e. the number of edges on the corresponding contour segment, counted with multiplicity) is called the *width* of the slice. The length of the left boundary is called the *depth*, and the depth minus the length of the right boundary is called the *tilt*. The corner A is called the *apex*.

A slice of width 1 is called *elementary*. The tilt of an elementary slice is necessarily in $\{1, 0, -1\}$. By the uniqueness property of the right boundary, there is a unique elementary slice of tilt -1, called the *trivial* slice, which consists of a single edge with extremities $A = B$

[6] Note that, in the accepted denominations "left boundary" and "right boundary", the term "boundary" has a meaning different from that in the rest of the paper.

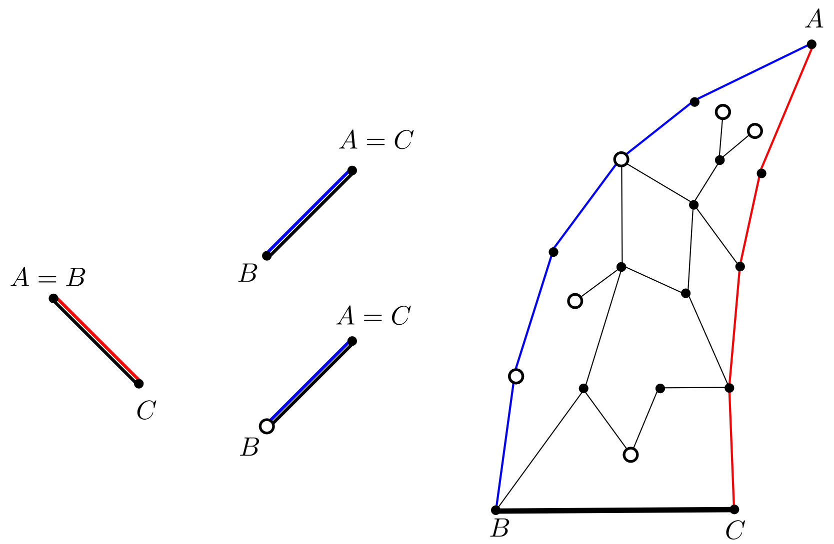

Figure 4.7: The different types of tight elementary slices, with marked vertices shown in white. Left: the trivial slice. Center: the empty slice, in its unmarked and marked versions. Right: a non-empty tight elementary slice.

and C. The trivial slice differs from the *empty* slice consisting in a single edge with extremities B and $A = C$, which has tilt $+1$. If we restrict our attention to bipartite maps, as is the case in this section, there are no slices with tilt 0.

Finally, a *tight slice* is a slice, elementary or not, that carries some marked vertices, in such a way that

- all vertices of degree 1 distinct from those incident to A, B and C are marked,
- *the right boundary carries no marked vertices*.

Note that the vertex incident to B may possibly be marked, but not those incident to A and C, even if those vertices have degree one. In particular, the empty slice comes with two tight versions, depending on whether the vertex incident to B is marked or not, and we will call the marked version the *marked empty slice*, which will play an important role later on. See Figure 4.7 for an illustration of the different types of tight elementary slices.

Pointed rooted maps and elementary slices. There is a simple one-to-one correspondence between pointed rooted planar bipartite maps on the one hand[7], and non-empty, bipartite elementary slices of tilt 1 on the other hand. Starting from a pointed rooted bipartite map $\mathbf{m}$ with root e and distinguished vertex v, we can perform the opening operation $O(\mathbf{m}, e)$ that opens the edge e into an exterior face of degree 2. We let b and c be the two corners incident to this new face, where c is closest from v. We then cut open the map along the leftmost geodesic[8] γ

[7]Here, as for tight maps, we use the slightly unusual convention that a rooted map is a map with a distinguished, non oriented edge.

[8]See for instance [Bou19, Figure 2.1] for a careful definition of the leftmost geodesic from a corner to a vertex.

from c to v, hence enlarging the exterior face. The resulting map is an elementary bipartite slice of tilt 1, if we let B be the corner inherited from b, A be the unique corner of the exterior face incident to v, and C be the corner immediately following B as we walk with the exterior face on the right (so that C is one of the two duplicates of c created after cutting). The fact that the right boundary AC is the unique geodesic between these two extremities comes from the fact that γ was chosen to be leftmost, and the slice is non-empty because it has at least one inner face, inherited from the map we started with. Conversely, starting from a non-empty elementary bipartite slice of tilt 1, we may glue "isometrically" together the left and right boundaries starting from the apex. This results in a bipartite map pointed at the vertex v incident to the apex, and with a face of degree 2 whose contour is made of the base and of the first edge of the left boundary incident to B, which are necessarily distinct since the slice is non-empty. We may finally glue these two edges together into a single edge e, at which we root the resulting map.

These two operations are inverse of one another. They specialize to a correspondence between pointed rooted planar *tight* bipartite maps and non-empty *tight* bipartite slices of tilt 1, if we take the convention that the marks of marked vertices in $\mathbf{m}$ that belong to the leftmost geodesic γ considered above should systematically be transferred to the left boundary of the slice.

Decomposing a slice into a path decorated with elementary slices. Next, we discuss the decomposition of a bipartite slice s of width $w \in \mathbb{Z}_{>0}$ and tilt $t \in \mathbb{Z}$ into a collection of elementary slices. We list the corners of the base as $c_0 = B, c_1, \ldots, c_w = C$, walking with the exterior face on the right. We let $\ell_i = d(c_0, A) - d(c_i, A)$ for $i \in \{0, 1, \ldots, w\}$, where d is the graph distance in s, and the distance between two corners is defined as the distance between their incident vertices. In particular, $\ell_w = t$ is the tilt of s, and $L = (\ell_0, \ell_1, \ldots, \ell_w)$ is a walk on $\mathbb{Z}$ with increments $\ell_i - \ell_{i-1} \in \{-1, +1\}$, that we will systematically identify with the lattice path made of the union of segments $[(i-1, \ell_{i-1}), (i, \ell_i)], 1 \leqslant i \leqslant n$ in the plane.

For every $i \in \{0, 1, \ldots, w\}$, we let γ_i be the leftmost geodesic from c_i to the apex A. In particular, γ_0 and γ_w are respectively the left and right boundary of s. For $i \in \{1, 2, \ldots, w\}$, we let v_i' be the first vertex common to γ_{i-1} and γ_i. Then the map s_i' delimited by these two geodesics is an elementary bipartite slice with base $c_{i-1}c_i$, with apex A_i' incident to v_i', and with tilt $t_i = \ell_i - \ell_{i-1}$. If $t_i = -1$, in which case we say that i is a *down step*, then s_i' is trivial, while if $t_i = +1$, in which case we call i an *up step*, then s_i' is non-trivial. It may however be the empty slice, precisely when the geodesic γ_{i-1} starts by following the base edge from c_{i-1} to c_i.

With a bipartite slice s with width w and tilt t, we have associated a lattice path L from $(0, 0)$ to (w, t) with increments ± 1, where each of the $(w+t)/2$ up steps i is decorated with a bipartite elementary slice s_i' of tilt 1, while all the $(w-t)/2$ down steps are decorated with the trivial slice, so that these last decorations are in fact irrelevant and can be omitted.

We can invert this decomposition: given a lattice path from $(0, 0)$ to (w, t) whose up steps i are decorated with bipartite elementary slices s_i' of tilt 1 (and where s_i' is the trivial slice if i is a down step), we may associate a slice of width w in the following way. For every down step i, we identify the segment $s_i = [(i-1, \ell_{i-1}), (i, \ell_i)]$ of the lattice path with the associated trivial slice s_i', hence color it in red as in Figure 4.7. Next, for every up step i, we consider an embedding of s_i' in the plane in which the base edge is the segment s_i, and so that the left boundary (resp. the right boundary) is represented as a curve, monotone in its two coordinates, that starts from $(i-1,\ell_{i-1})$ (resp. (i,ℓ_i)), is entirely contained in $[i-1,i]\times[\ell_{i-1},\infty)$ (resp. $[i-1,i]\times[\ell_i,\infty)$), and such that its r-th vertex, starting from the base, has its ordinate equal to $\ell_{i-1}+r-1$ (resp. ℓ_i+r-1). By convention, the edges of the left boundaries of the slices s'_i are declared blue, while the edges of the right boundaries are declared red. Note that if s'_i is an empty slice, then the above operation simply consists in coloring the segment s_i in blue. Then, every red element (either an edge lying on the right boundary of some slice, or the segment associated with a down step of the lattice path), lying in some square $[i-1,i]\times[\ell-1,\ell]$, attempts to be matched to the first available blue edge in some square $[j-1,j]\times[\ell-1,\ell]$ for some $j>i$, and all matched edges are glued together. After this gluing is performed, we obtain a bipartite slice of width w and tilt t, where the unmatched edges, i.e. the blue edges which are not preceded by red edges at the same ordinate, and the red edges that are not followed by a blue edge at the same ordinate, form respectively the left and right boundaries.

Let us now discuss how this decomposition behaves with respect to the tightness constraint. We first observe that it associates with a *tight* bipartite slice s of width $w\geqslant 1$ and tilt t a lattice path L from $(0,0)$ to (w,t) with ± 1 steps decorated with *tight* elementary bipartite slices. The only ambiguity that should be lifted is how we transfer the marks of marked vertices that belong to the union of leftmost geodesics γ_i defined above to exactly one of their duplicates. We choose the duplicate that belongs to the left boundary of the slice s'_i, where i is the maximal index such that the marked vertex at hand belongs to γ_{i-1}. Note that such a maximal index i always exists, since, by definition, tight slices carry no marked vertices on their right boundaries, and that the duplicate of the vertex is different from the apex of s'_i by maximality of i. With these conventions, all the slices $\mathrm{s}'_i, 1\leqslant i\leqslant w$, with transferred marks, are tight slices.

Moreover, there is an additional restriction on the family $L,(\mathrm{s}'_i,1\leqslant i\leqslant w)$ that guarantees that the original slice be tight, i.e. that it contains no undesired unmarked vertices of degree 1. Observe that, in the above correspondence, the vertex incident to a corner $c_i, i\in\{1,\dots,w-1\}$ of the base distinct from the extremities will have degree 1 precisely in the situation where i is a down step and $i+1$ is an up step decorated with the empty slice. Indeed, if the vertex v_i incident to c_i has degree 1, then the vertices v_{i-1} and v_{i+1} incident to the corners c_{i-1} and c_{i+1} are the same vertex, and therefore the geodesics γ_i and γ_{i+1} delimit the empty slice, since γ_i meets γ_{i+1} after one single step. Conversely, in the gluing procedure, if a down step i is immediately followed by an up step $i+1$, then the "red" segment $[(i-1,\ell_{i-1}),(i,\ell_{i-1}-1)]$ associated with the down-step i will be matched to the first edge of the left boundary of the slice s'_{i+1}. This will result in a vertex of degree 1 precisely when this first edge is equal to the base edge, and the only elementary slice of tilt 1 with this property is the empty slice. Consequently, in a tight slice, every such up step $i+1$ is decorated with a slice s'_i that is either non-empty, or is the *marked empty slice*, that is the empty slice with marked base vertex B.

By forgetting the redundant information of up steps that are decorated with unmarked empty slices, that is, by letting $(\mathrm{s}_j,1\leqslant j\leqslant k)$ be the sequence $(\mathrm{s}'_i,1\leqslant i\leqslant w)$ whose trivial and unmarked empty elements have been removed, we obtain the following result.

Proposition 4.4. *There is a one-to-one correspondence between tight bipartite slices of width w and tilt t on the one hand, and pairs of the form $(L, (\mathbf{s}_j, 1 \leqslant j \leqslant k))$ on the other hand, where:*

- *L is a lattice path from $(0,0)$ to (w,t) with ± 1 steps, that has k marked up steps, in such a way that every up step immediately following a down step is marked,*
- *for $1 \leqslant j \leqslant k$, $\mathbf{s}_j$ is either the marked empty slice or a tight bipartite elementary slice with at least one inner face.*

Now observe that if $\mathbf{s}$ is a non-empty bipartite slice with tilt 1, then the base edge is incident to an inner face. Calling $2m > 0$ the degree of this face, and after removing the base edge, we obtain a bipartite slice with width $2m-1$ and tilt 1. This simple operation preserves the tight characters of the maps at hand, which implies the following:

Corollary 4.5. *There is a one-to-one correspondence between non-trivial, non-empty tight bipartite elementary slices, whose inner face incident to the base edge has degree $2m > 0$ on the one hand, and pairs of the form $(L, (\mathbf{s}_j, 1 \leqslant j \leqslant k))$ on the other hand, where k is some non-negative integer and:*

- *L is a lattice path from $(0,0)$ to $(2m-1,1)$ with ± 1 steps, that has k marked up steps, in such a way that every up step immediately following a down step is marked,*
- *for $1 \leqslant j \leqslant k$, $\mathbf{s}_j$ is either the marked empty slice or a tight bipartite elementary slice with at least one inner face.*

Finally, by convention, we extend the above correspondence by associating with the marked empty slice the pair $(\{(0,0)\}, \varnothing)$ consisting of the trivial lattice path of length zero, with no marks.

By iterating the decomposition of this corollary, i.e. inductively replacing each non-empty elementary slice in the above decomposition by a lattice path with some marked up steps, and an ordered family of as many elementary slices, we obtain a plane tree (where the plane order is induced by the order of the up steps to which the slices are connected). See Figure 4.9 for an illustration. For integers $m_1, \ldots, m_n$ not all equal to 0, let $\mathcal{T}(m_1, \ldots, m_n)$ be the family of pairs $(\mathbf{t}, (L_i)_{1 \leqslant i \leqslant n})$ where:

- $\mathbf{t}$ is a rooted plane tree with vertices labeled by $\{1, 2, \ldots, n\}$, and, denoting by k_i the number of children of the vertex labeled i in $\mathbf{t}$,
- if $m_i > 0$ then L_i is a lattice path from $(0,0)$ to $(2m_i - 1, 1)$ with k_i marked up steps, such that all up steps immediately following a down step are marked,
- if $m_i = 0$ then L_i is the trivial path $\{(0,0)\}$, in which case necessarily $k_i = 0$.

Corollary 4.6. *For any non-negative integers $m_1, \ldots, m_n$ not all equal to 0, the iterated slice decomposition yields a one-to-one correspondence between pointed rooted planar tight maps with n labeled boundaries of respective lengths $2m_1, \ldots, 2m_n$, and the set $\mathcal{T}(m_1, \ldots, m_n)$.*

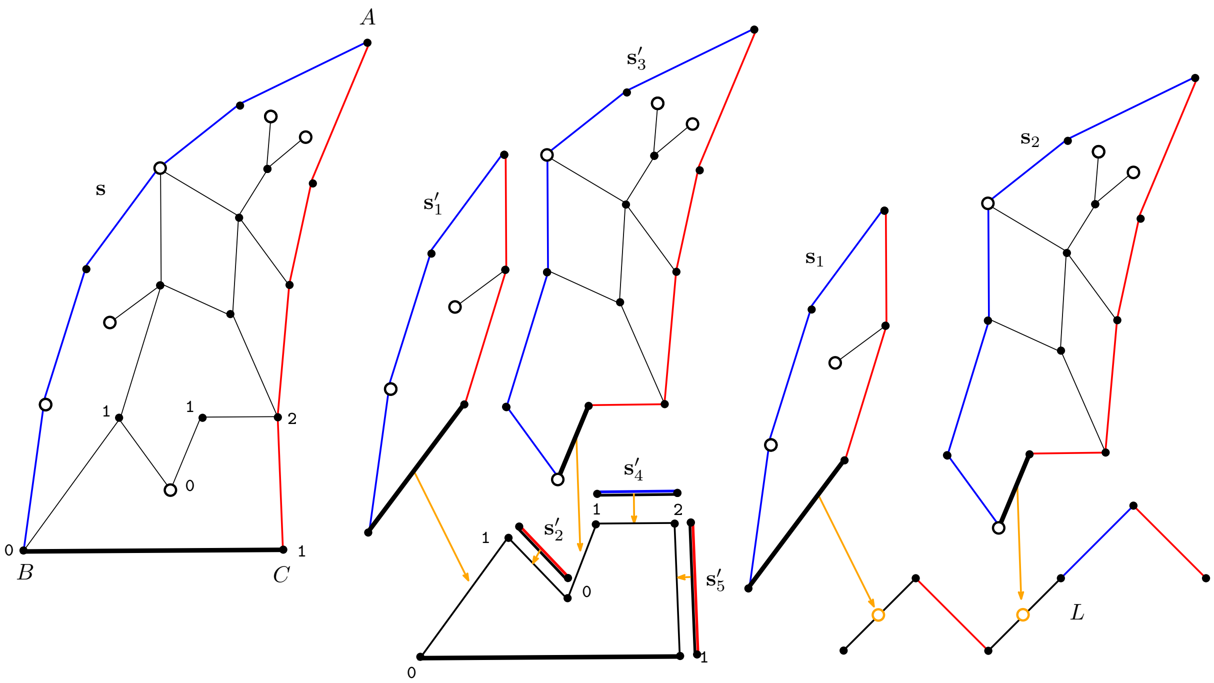

Figure 4.8: Illustration of the decomposition of a non-empty elementary tight slice of tilt 1 into a lattice path decorated with elementary slices. The middle picture shows the decomposition along the leftmost geodesics started from the corners of the inner face incident to the base. The right-hand side shows the result of the decomposition. Note that the trivial and empty slices $\mathbf{s}'_2, \mathbf{s}'_4$ and $\mathbf{s}'_5$ may be discarded, at the price of transferring their color to the corresponding step of the lattice path, while the two other slices are non-empty and therefore correspond to marked up steps. Note also that the second up step, which is consecutive to the first down step, is marked, as is required.

We are now in position to prove Proposition 2.7. Note that the number of lattice paths of length $2m-1$ from 0 to 1 that has exactly k marked up steps, including all up steps immediately following down steps, is precisely the number $q_k(m)$. Indeed, they are exactly counted by the words discussed in Section 4.1, with $m-1$ letters U, $m-k$ letters $D_\bullet$ and k letters $D_\circ$, with forbidden subword $UD_\bullet$ (up to flipping upside down the lattice paths to better match the interpretation of the letters U, D). In fact, this even holds for $m=k=0$ since in this case $q_k(0)=\delta_{k0}$, so we interpret $q_0(0)$ combinatorially as counting the unique marked empty slice.

We now proceed to enumerating the elements of the set $\mathcal{T}(m_1,\dots,m_n)$. By [BM14, Section 5, Equation (18)], for a given n-uple $(k_1,\dots,k_n)$ of non-negative integers, if $k_1+\cdots+k_n=n-1$ then there are exactly $(n-1)!$ rooted labeled plane trees on the vertex set $\{1,2,\dots,n\}$ such that vertex i has k_i children for all $i=1,\dots,n$, and there are no such tree otherwise. For any such tree $\mathbf{t}$, there are $\prod_{i=1}^n q_{k_i}(m_i)$ possible choices of marked lattice paths $L_i, 1\leqslant i\leqslant n$ such that $(\mathbf{t},(L_i)_{1\leqslant i\leqslant n})$ belongs to $\mathcal{T}(m_1,\dots,m_n)$. This finally explains

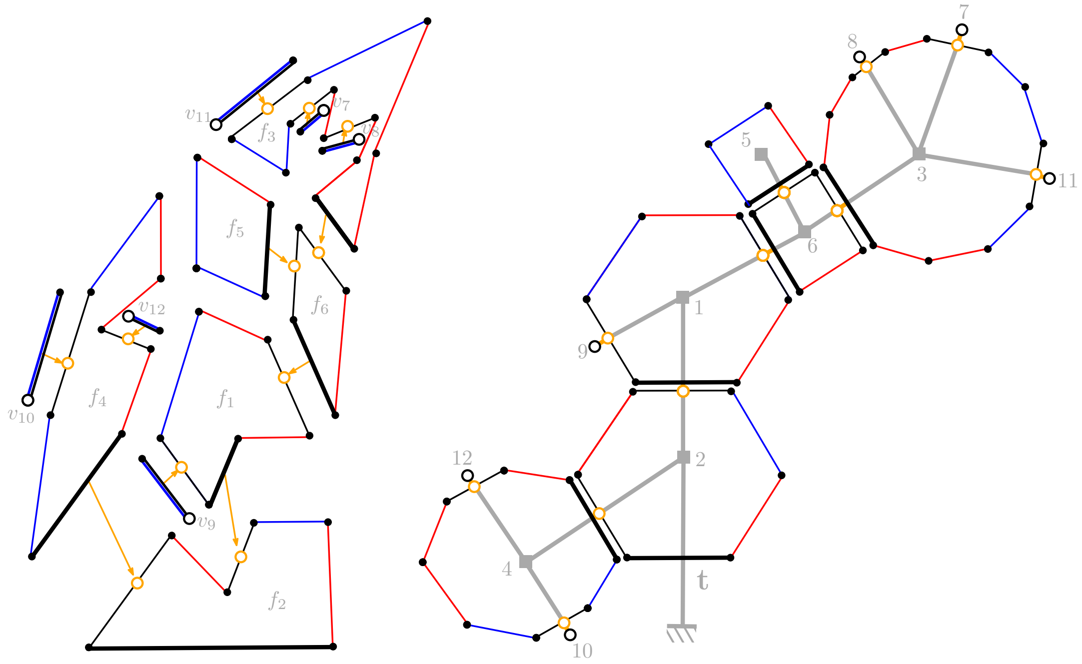

Figure 4.9: Iterating the slice decomposition yields a plane tree whose vertices are decorated by marked paths. On this picture, we do not represent the lattice paths, but encode them using a color code on polygons drawn around the vertices of $\mathfrak{t}$, where black/blue correspond to marked/unmarked up steps and red to down steps. Note that some of the terminal nodes of $\mathfrak{t}$ correspond to non-trivial paths with no marked steps, while some others correspond to the trivial path, represented as white dots, and are associated in turn to the marked vertices of the original tight slice.

the wanted formula

$$\begin{aligned} N_{0,n+2}(2m_1,\ldots,2m_n,2,0) &= (n-1)! \sum_{\substack{k_1,\ldots,k_n\geqslant 0\\ k_1+\cdots+k_n=n-1}} \prod_{i=1}^{n} q_{k_i}(m_i) \\ &= (n-1)!\, q_{n-1}(m_1,\ldots,m_n)\,, \end{aligned} \tag{4.4}$$

which concludes the proof.

4.4. General case

We will now prove Theorem 2.3 in all generality, as well as its quasi-bipartite analog, Theorem 2.8. As the case of maps with one face was already treated in Section 4.1, it suffices to treat the case where the first and second boundaries are faces, that is when $m_1, m_2 > 0$. To this end, we will combine the ideas of Sections 4.2 and 4.3 via the following:

Proposition 4.7. *Let m_1, m_2 be positive integers or half-integers and let $m_3, \ldots, m_n$ be non-negative integers ($n \geqslant 3$). Then, there is a bijection between the set of planar tight maps with n boundaries labeled from 1 to n with respective lengths $2m_1, 2m_2, \ldots, 2m_n$, and the set of pairs $(\mathbf{m}_{12}, \mathbf{s})$ such that there exists $k \in \{0, \ldots, n-3\}$ for which:*

- *$\mathbf{m}_{12}$ is a planar tight two-face map, with two faces of respective degrees $2m_1$ and $2m_2$, and $k+1$ distinct marked vertices, one of them distinguished,*
- *$\mathbf{s} = (\mathbf{s}_1, \ldots, \mathbf{s}_{k+1})$ is a $(k+1)$-tuple of slices such that:*
  - *for each $j = 1, \ldots, k+1$, $\mathbf{s}_j$ is either the marked empty slice or a tight bipartite elementary slice with at least one inner face, whose inner faces and marked vertices are labeled by integers in $\{3, \ldots, n\}$,*
  - *each $i \in \{3, \ldots, n\}$ appears in exactly one $\mathbf{s}_j$ and labels an inner face of degree $2m_i$ for $m_i > 0$, or a marked vertex for $m_i = 0$,*
  - *the label 3 appears in the first slice $\mathbf{s}_1$.*

Before proving this proposition, let us see how it implies Theorems 2.3 and 2.8. We start with the former: our purpose is to enumerate the pairs $(\mathbf{m}_{12}, \mathbf{s})$ of the proposition when m_1 and m_2 are integers. For a fixed $k \in \{0, \ldots, n-3\}$, the number of possible maps $\mathbf{m}_{12}$ is equal to $p_k(m_1, m_2)$ by Proposition 2.6, which we proved in Section 4.2. As for the number of possible $\mathbf{s}$, it is given by a slight variant of the reasoning in Section 4.3. Indeed, by recursively decomposing each slice $\mathbf{s}_j$ into a tree of lattice paths, we see that the set of possible $(k+1)$-tuples $\mathbf{s}$ is in bijection with the set $\mathcal{F}_{k+1}(m_3, \ldots, m_n)$ defined as the set of pairs $(\mathbf{f}, (L_i)_{3 \leqslant i \leqslant n})$ where:

- $\mathbf{f}$ is a plane forest with $k+1$ connected components, i.e. a $(k+1)$-tuple of rooted plane trees, whose vertices are labeled by $\{3, \ldots, n\}$, the label 3 appearing in the first component,
- denoting by k_i the number of children of the vertex labeled i in $\mathbf{f}$:
  - if $m_i > 0$ then L_i is a lattice path from $(0,0)$ to $(2m_i - 1, 1)$ with k_i marked up steps, such that all up steps immediately following a down step are marked,
  - if $m_i = 0$ then L_i is the trivial path $\{(0,0)\}$, in which case necessarily $k_i = 0$.

Note that $\mathcal{F}_1(m_3, \ldots, m_n)$ is nothing but the set $\mathcal{T}(m_3, \ldots, m_n)$ as defined in Section 4.3. By Proposition A.1 of Appendix A below, for a given n-uple $(k_3, \ldots, k_n)$, if $k_3 + \cdots + k_n = n-3-k$ then there are exactly $(n-3)!$ plane forests on the vertex set $\{3, \ldots, n\}$ with $k+1$ components, the first of which contains the label 3, and such that vertex i has k_i children for all $i = 3, \ldots, n$, and there are no such forests otherwise. For any such forest $\mathbf{f}$, there are $\prod_{i=3}^{n} q_{k_i}(m_i)$ possible choices of marked lattice paths $L_i, 3 \leqslant i \leqslant n$ such that $(\mathbf{f}, (L_i)_{3 \leqslant i \leqslant n})$ belongs to $\mathcal{F}_{k+1}(m_3, \ldots, m_n)$. This gives

$$\begin{aligned} \mathrm{Card}\,(\mathcal{F}_{k+1}(m_3, \ldots, m_n)) = (n-3)! \sum_{\substack{k_3, \ldots, k_n \geqslant 0 \\ k_3 + \cdots + k_n = n-3-k}} \prod_{i=3}^{n} q_{k_i}(m_i) \\ = (n-3)!\, q_{n-3-k}(m_3, \ldots, m_n). \end{aligned} \tag{4.5}$$

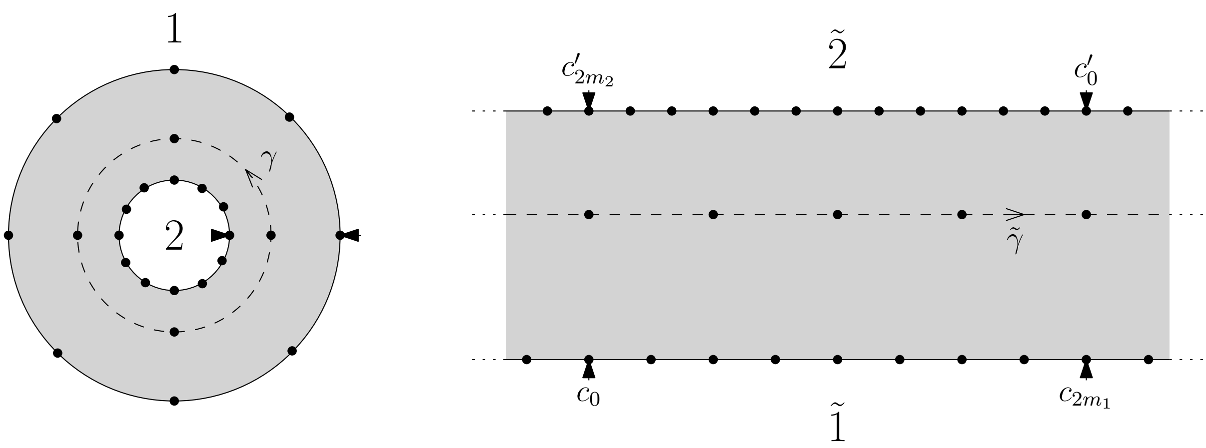

Figure 4.10: Sketch of an annular map $\mathbf{m}$ (left) and of its universal cover $\tilde{\mathbf{m}}$ (right). The external face 1 and the central face 2 of $\mathbf{m}$ lift respectively to the lower face $\tilde{1}$ and to the upper face $\tilde{2}$ of $\tilde{\mathbf{m}}$, which have infinite degrees (marked corners are represented by arrows). All the other faces form the grey region. The innermost minimal separating cycle γ lifts to a biinfinite geodesic $\tilde{\gamma}$.

Thus, multiplying by $p_k(m_1, m_2)$, summing over k and using (2.5), we get

$$\begin{aligned} N_{0,n}(2m_1, 2m_2, 2m_3 \ldots, 2m_n) &= (n-3)! \sum_{k=0}^{n-3} p_k(m_1, m_2) q_{n-3-k}(m_3, \ldots, m_n) \\ &= (n-3)!\, p_{n-3}(m_1, m_2, m_3, \ldots, m_n) \end{aligned} \tag{4.6}$$

as wanted. This concludes the proof of Theorem 2.3 assuming Proposition 4.7.

If we now assume that m_1, m_2 are half-integers, the only change we have to do in the above reasoning is that, for a fixed k, the number of possible maps $\mathbf{m}_{12}$ is now equal to $\tilde{p}_k(m_1, m_2)$ by Proposition 2.9 (also proved in Section 4.2). Thus, by (2.5) and (2.20), we now have

$$\begin{aligned} N_{0,n}(2m_1, 2m_2, 2m_3 \ldots, 2m_n) &= (n-3)! \sum_{k=0}^{n-3} \tilde{p}_k(m_1, m_2) q_{n-3-k}(m_3, \ldots, m_n) \\ &= (n-3)!\, \tilde{p}_{n-3}(m_1, m_2; m_3, \ldots, m_n) \end{aligned} \tag{4.7}$$

which establishes Theorem 2.8.

The remainder of this section is devoted to the proof of Proposition 4.7. It uses the slice decomposition of annular maps which was introduced in [BG14, Section 9.3], see also [Bou19, Section 2.2] for another exposition. Here we will give yet another, modernized, exposition which, following [BGM22], makes use of the key notion of *Busemann function*.

Decomposing an annular map into a pair of paths decorated with elementary slices. Let $\mathbf{m}$ be an *annular* map, that is a planar map with two distinguished faces labeled 1 and 2. For now, we do not assume that $\mathbf{m}$ is a tight map, but we assume that every face other than 1 and 2 has even degree. Denoting by $2m_1$ and $2m_2$ the respective degrees of 1 and 2, $\mathbf{m}$ is either bipartite when m_1 and m_2 are integers, or quasi-bipartite when they are half-integers.

Let us choose a representation of $\mathbf{m}$ in the complex plane such that face 1 is the unbounded face, and such that the origin (point of affix zero) lies in face 2. We then consider the preimage $\tilde{\mathbf{m}}$ of $\mathbf{m}$ by the mapping $z \mapsto \exp(2i\pi z)$: it is an infinite map which we call the *universal cover* of $\mathbf{m}$ (upon viewing 0 and ∞ as "punctures"). We refer to [BGM22] for a detailed discussion of the properties of the universal cover of a map drawn on the triply-punctured sphere, it can be adapted without difficulty to the simpler case considered here of a map drawn on the doubly-punctured sphere. For our purposes, we simply note that the faces 1 and 2 lift in $\tilde{\mathbf{m}}$ to unique faces $\tilde{1}$ and $\tilde{2}$, which have infinite degrees, while all the other faces and vertices of $\mathbf{m}$ lift to infinitely many preimages in $\tilde{\mathbf{m}}$ with the same finite degree. In particular, $\tilde{\mathbf{m}}$ is bipartite. See Figure 4.10 for an illustration.

Let s denote the *separating girth* of $\mathbf{m}$, that is the minimal length (number of edges) of a closed path in $\mathbf{m}$ winding around the origin. We call such path a *separating cycle* and say that it is *minimal* if it has length s. Let then γ be the *innermost* minimal separating cycle, which we define as follows. Consider the interiors of all minimal separating cycles. It is straightforward to check that their intersection is a simply connected region containing the origin, and that its boundary oriented clockwise is still a minimal separating cycle: this is the innermost minimal separating γ that we are looking for. We consider the path $\tilde{\gamma} = (\tilde{\gamma}(t))_{t\in\mathbb{Z}}$ obtained by following γ counterclockwise infinitely many times and lifting this biinfinite path in $\tilde{\mathbf{m}}$. The parametrization of $\tilde{\gamma}$ depends on a choice of a vertex $\tilde{\gamma}(0)$ whose projection belongs to γ, but we will see that the outcome of our construction does not depend on it. By [CdVE10, Proposition 2.5], $\tilde{\gamma}$ is a *biinfinite geodesic* in $\tilde{\mathbf{m}}$, which by definition means that

$$\tilde{d}(\tilde{\gamma}(t), \tilde{\gamma}(t')) = |t - t'| \tag{4.8}$$

for any $t, t' \in \mathbb{Z}$, where $\tilde{d}$ denotes the graph distance in $\tilde{\mathbf{m}}$. We now define the *Busemann function* $B_{\tilde{\gamma}}$ from the vertex set $\tilde{V}$ of $\tilde{\mathbf{m}}$ to $\mathbb{Z}$ by

$$B_{\tilde{\gamma}}(v) := \lim_{t\to\infty} \tilde{d}(v, \tilde{\gamma}(t)) - t, \qquad v \in \tilde{V}. \tag{4.9}$$

By the triangle inequality, one easily checks that the limit indeed exists and is attained for t large enough (for instance $B_{\tilde{\gamma}}(\tilde{\gamma}(s)) = -s$ for every s in $\mathbb{Z}$). Furthermore, for any two adjacent vertices v, v' we have $B_{\tilde{\gamma}}(v) - B_{\tilde{\gamma}}(v') = \pm 1$ (since $\tilde{\mathbf{m}}$ is bipartite), and every vertex v has a neighbor v' for which this difference is $+1$ (i.e. $B_{\tilde{\gamma}}$ has no local minimum). Let us denote by T the automorphism of $\tilde{\mathbf{m}}$ corresponding to the translation $z \mapsto z + 1$ of the complex plane (it corresponds to making one turn counterclockwise around the origin in $\mathbf{m}$): we then have $B_{\tilde{\gamma}}(Tv) = B_{\tilde{\gamma}}(v) - s$ as a consequence of the relation $T\tilde{\gamma}(t) = \tilde{\gamma}(t + s)$.

Now, let us denote by $(c_i)_{i\in\mathbb{Z}}$ and $(c'_i)_{i\in\mathbb{Z}}$ the successive corners incident to the faces $\tilde{1}$ and $\tilde{2}$, respectively, as we follow their contours walking with the face at hand to the right. This depends on a choice for the corners c_0 and c'_0, whose influence will be discussed later. We have $Tc_i = c_{i+2m_1}$ and $Tc'_i = c'_{i-2m_2}$ for all i.

We then set, for any $i \in \mathbb{Z}$, $\ell_i := B_{\tilde{\gamma}}(c_0) - B_{\tilde{\gamma}}(c_i)$ and $\ell'_i := B_{\tilde{\gamma}}(c'_0) - B_{\tilde{\gamma}}(c'_i)$ (these differences do not depend on the choice of $\tilde{\gamma}(0)$, as claimed). Observe that the definition of ℓ_i is analogous to that used in Section 4.3 for the decomposition of a slice into a path decorated with elementary slices: the distance to the apex $d(\cdot, A)$ is just replaced by the Busemann function $B_{\tilde{\gamma}}(\cdot)$.

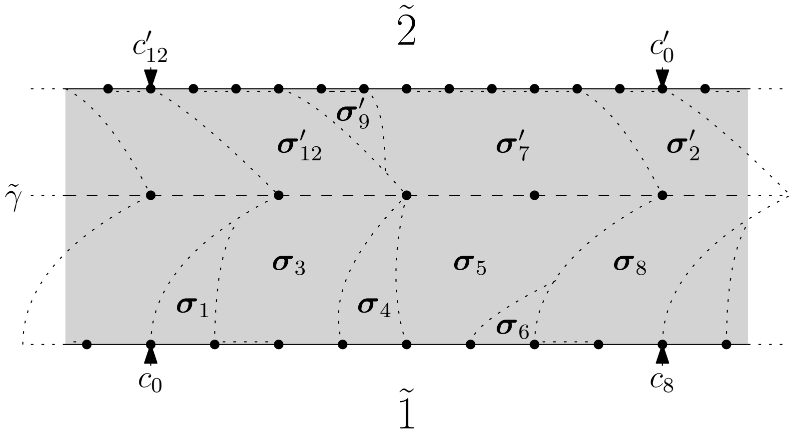

Figure 4.11: Sketch of the decomposition of the universal cover of an annular map into slices. The leftmost infinite geodesics (shown in dotted lines) eventually merge with the biinfinite geodesic $\tilde{\gamma}$ (shown in dashed lines). These geodesics delimit slices $\boldsymbol{\sigma}_i$ and $\boldsymbol{\sigma}'_i$, $i \in \mathbb{Z}$. In the situation displayed here, the slices $\boldsymbol{\sigma}_2$, $\boldsymbol{\sigma}_7$, $\boldsymbol{\sigma}'_1$, etc, are trivial slices (reduced to an edge).

The sequences $\ell = (\ell_i)_{i\in\mathbb{Z}}$ and $\ell' = (\ell'_i)_{i\in\mathbb{Z}}$ form infinite lattice paths (with increments ± 1) which are periodic: indeed, we have $\ell_{i+2m_1} = \ell_i + s$ and $\ell'_{i+2m_2} = \ell'_i - s$ for all i. Thus, these sequences are entirely determined by their data in a fundamental domain: $L := (\ell_0, \ell_1, \dots, \ell_{2m_1})$ and $L' := (\ell'_0, \ell'_1, \dots, \ell'_{2m_2})$ form lattice paths connecting $(0,0)$ to respectively $(2m_1, s)$ and $(2m_2, -s)$.

With each corner c_i, we associate the *leftmost* infinite geodesic $\gamma_i = (\gamma_i(t))_{t\geqslant 0}$ defined inductively as follows. We let $\gamma_i(0)$ be the vertex incident to c_i and, assuming that $\gamma_i(t)$ is known, we let $\gamma_i(t+1)$ be the leftmost vertex such that $B_{\tilde{\gamma}}(\gamma_i(t+1)) = B_{\tilde{\gamma}}(\gamma_i(t)) - 1$, using the edge $\gamma_i(t-1)\gamma_i(t)$ (or the corner c_i for $t=0$) as a reference. We define in the same way the leftmost infinite geodesic γ'_i starting at c'_i. We may check that each of these leftmost geodesics eventually merges with $\tilde{\gamma}$ (for γ'_i, this uses the fact that we chose γ to be the innermost minimal separating cycle).

Let $\boldsymbol{\sigma}_i$ (resp. $\boldsymbol{\sigma}'_i$) be the map delimited by γ_{i-1} and γ_i (resp. γ'_{i-1} and γ'_i), which we stop at their first common vertex v_i (resp. v'_i). It is an elementary bipartite slice with base $c_{i-1}c_i$ (resp. $c'_{i-1}c'_i$). As in Section 4.3, the slice is trivial whenever i corresponds to a down step of ℓ (resp. ℓ'), and is non-trivial otherwise (but may be empty). By periodicity, we have $\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_{i+2m_1}$ and $\boldsymbol{\sigma}'_i = \boldsymbol{\sigma}'_{i-2m_2}$ (in the sense of equality as maps). Note that every finite face, edge and vertex of $\tilde{\mathbf{m}}$ belongs to exactly one slice $\boldsymbol{\sigma}_i$ or $\boldsymbol{\sigma}'_i$ deprived of its right boundary.

Let us finally discuss the roles of the reference corners c_0 and c'_0. Changing the reference corner c_0 amounts to translating the lattice path ℓ, and to reparametrizing the sequence $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_i)_{i\in\mathbb{Z}}$ by a translation of i. In particular, both ℓ and $\boldsymbol{\sigma}$ are invariant if we change c_0 into one its translates $T^k c_0$, thus they only depend on the choice of a corner incident to face 1 in $\mathbf{m}$. Similarly, ℓ' and $\boldsymbol{\sigma}' = (\boldsymbol{\sigma}'_i)_{i\in\mathbb{Z}}$ only depend on the choice of a corner incident to face 2 in $\mathbf{m}$.

Let us now assume that the map $\mathbf{m}$ has a third distinguished element (face or vertex) labeled 3, which is indeed the case in the setting of Proposition 4.7. Then it is possible to choose corners incident to faces 1 and 2 in a canonical way. Namely, we pick a preimage $\tilde{3}$ of 3 in $\tilde{\mathbf{m}}$: $\tilde{3}$ belongs to precisely one slice $\boldsymbol{\sigma}_i$ or $\boldsymbol{\sigma}'_i$ deprived of its right boundary. If $\tilde{3}$ belongs to a $\boldsymbol{\sigma}_i$, then by changing the reference corner c_0 we can ensure that it belongs to $\boldsymbol{\sigma}_1$. Then, we choose c'_0 in such way that $B_{\tilde{\gamma}}(c'_0) = B_{\tilde{\gamma}}(c_0)$: this is possible because the function $B_{\tilde{\gamma}}(\cdot)$ has ± 1 increments along the contour of $\tilde{2}$ and decreases by s over a translation T, hence it is surjective. There might exist several such c'_0, in which case we may pick, say, the "rightmost" one. If $\tilde{3}$ belongs to a $\boldsymbol{\sigma}'_i$, we proceed in the same way upon exchanging the roles of c_0 and c'_0.

Lemma 4.8. *The above construction is a bijection between:*

- *the set of maps* $\mathbf{m}$ *with two marked faces of degrees* $2m_1, 2m_2 \in \mathbb{Z}_{>0}$ *and all other faces of even degree, with a third distinguished face or vertex, and with separating girth* $s \in \mathbb{Z}_{>0}$,
- *the set of quadruples* $(L, L', \boldsymbol{\sigma}, \boldsymbol{\sigma}')$*, where* L *and* L' *are lattice paths connecting* $(0,0)$ *to respectively* $(2m_1, s)$ *and* $(2m_2, -s)$*, where* $\boldsymbol{\sigma}$ *and* $\boldsymbol{\sigma}'$ *are sequences of elementary slices of respective periods* $2m_1$ *and* $2m_2$*, such that for every* $i = 1, \ldots, 2m_1$ *(resp.* $i = 1, \ldots, 2m_2$*) the tilt of* $\boldsymbol{\sigma}_i$ *and (resp.* $\boldsymbol{\sigma}'_i$*) is equal to* $\ell_i - \ell_{i-1}$ *(resp.* $\ell'_i - \ell'_{i-1}$*), and where either* $\boldsymbol{\sigma}_1$ *or* $\boldsymbol{\sigma}'_1$ *carries a distinguished face or vertex.*

To prove this lemma, we now describe the inverse procedure which consists in assembling an annular map $\mathbf{m}$ from a quadruple $(L, L', \boldsymbol{\sigma}, \boldsymbol{\sigma}')$.

Assembling an annular map from a pair of paths decorated with elementary slices. Using the lattice path L and the sequence of slices $\boldsymbol{\sigma}$ restricted to a period, we may use the inverse of the decomposition described in Section 4.3 to obtain a slice $\boldsymbol{\Sigma}$ of width $2m_1$ and tilt s. We then perform the operation described in [BG14, Section 7.1] and called *wrapping* in [Bou19, Section 2.2]. It consists in gluing "isometrically" the left and right boundaries of $\boldsymbol{\Sigma}$ together, but unlike the gluing performed in Section 4.3 to obtain a pointed rooted map, we now start from the two endpoints of the base of $\boldsymbol{\Sigma}$ which we identify together, and perform the gluing from there (see the illustrations in the aforementioned references). As the tilt s is positive, the s edges of the left boundary closest to the apex remain unglued. This produces an annular map $\mathbf{m}_1$ whose two distinguished faces have respective degrees $2m_1$ and s, the latter resulting from the unglued edges, the former resulting from the base, whose two identified endpoints yield a distinguished corner denoted c.

Similarly, the lattice path L' and the sequence of slices $\boldsymbol{\sigma}'$ yield a slice $\boldsymbol{\Sigma}'$ of width $2m_2$ and tilt $-s$, whose wrapping produces an annular map $\mathbf{m}_2$ with two distinguished faces of respective degrees $2m_2$ and s (since the tilt of $\boldsymbol{\Sigma}'$ is negative, it is now the s edges closest to the apex on the *right* boundary that form this latter face), the former having a distinguished corner denoted c'.

The annular map $\mathbf{m}$ is then obtained by gluing the two annular maps $\mathbf{m}_1$ and $\mathbf{m}_2$ together along the contours of their distinguished faces of degree s. The contour then becomes a separating cycle γ, which can be shown to have minimal length. Note that there are a priori s ways to glue $\mathbf{m}_1$ and $\mathbf{m}_2$ together, however only one way is compatible with the prescription that, in the universal cover $\tilde{\mathbf{m}}$ of $\mathbf{m}$, the corners c and c' admit respective lifts c_0 and c'_0 such that $B_{\tilde{\gamma}}(c_0) = B_{\tilde{\gamma}}(c'_0)$ where $\tilde{\gamma}$ is the infinite geodesic constructed by lifting γ.

We may check that the assembling procedure is indeed the inverse of the decomposition described above. The most subtle point is to show that any quadruple $(L, L', \boldsymbol{\sigma}, \boldsymbol{\sigma}')$ is left invariant if we assemble it then decompose the resulting annular map. This requires to check that the boundaries of the slices become, in the annular map, precisely the leftmost geodesics that we use in the decomposition. More details can be found in [BG14, Section 7].

Application to tight maps and end of the proof of Proposition 4.7. So far, all our discussion holds without the assumption that $\mathbf{m}$ is tight. Adding this constraint amounts to two restrictions on the corresponding quadruple $(L, L', \boldsymbol{\sigma}, \boldsymbol{\sigma}')$, which are similar to those encountered in Section 4.3. First, every $\boldsymbol{\sigma}_i$ and $\boldsymbol{\sigma}'_i$ must be a tight elementary slice. Here, we use the natural convention that the mark of a marked vertex is transferred to its copy in the unique slice deprived of its right boundary that contains it (in particular, marked vertices lying on the separating cycle γ have their marks transferred to $\boldsymbol{\sigma}_i$'s and not to $\boldsymbol{\sigma}'_i$'s). Second, the lattice paths ℓ and ℓ' (which are the periodic extensions of L and L') must be such that every up step immediately following a down step cannot be decorated with an unmarked empty slice.

To complete the proof of Proposition 4.7, we assume that $\mathbf{m}$ has $n \geqslant 3$ boundaries, labeled 1 to n. By the slice decomposition, the labels $3, \ldots, n$ get distributed among the $\boldsymbol{\sigma}_i$ and $\boldsymbol{\sigma}'_i$. Let $i_1 \leqslant \cdots \leqslant i_j$ be the indices i between 1 and $2m_1$ such that $\boldsymbol{\sigma}_i$ is neither the unmarked empty slice nor the trivial slice, i.e. contains at least one label. Let $i'_1 \leqslant \cdots \leqslant i'_{j'}$ be similarly the indices i between 1 and $2m_2$ such that $\boldsymbol{\sigma}'_i$ contains at least one label. We set $k = j + j' - 1$ and note that $0 \leqslant k \leqslant n-3$ since there are $n-2$ labels in total to distribute among the slices. Recall that, by the aforementioned prescription for choosing the reference corners, the label 3 is either in $\boldsymbol{\sigma}_1$ or $\boldsymbol{\sigma}'_1$. If it is in $\boldsymbol{\sigma}_1$, we set $(\mathrm{s}_1, \ldots, \mathrm{s}_{k+1}) := (\boldsymbol{\sigma}_{i_1=1}, \ldots, \boldsymbol{\sigma}_{i_j}, \boldsymbol{\sigma}'_{i'_1}, \ldots, \boldsymbol{\sigma}'_{i'_{j'}})$. Otherwise, we set $(\mathrm{s}_1, \ldots, \mathrm{s}_{k+1}) := (\boldsymbol{\sigma}'_{i'_1=1}, \ldots, \boldsymbol{\sigma}'_{i'_{j'}}, \boldsymbol{\sigma}_{i_1}, \ldots, \boldsymbol{\sigma}_{i_j})$. This defines the $(k+1)$-tuple of slices s of the proposition.

As for the two-face map $\mathbf{m}_{12}$, it is obtained as follows. Let $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ be the sequences obtained from respectively $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ by replacing every slice different from the unmarked empty slice and from the trivial slice with the marked empty slice. Then, $\mathbf{m}_{12}$ is the annular map obtained by assembling the quadruple $(L, L', \boldsymbol{\omega}, \boldsymbol{\omega}')$. It is by construction a tight two-face map with $k+1$ marked vertices, and we distinguish the marked vertex coming from the marked empty slice which replaces the slice containing the label 3.

We check that the mapping $\mathbf{m} \mapsto (\mathbf{m}_{12}, \mathrm{s})$ is a bijection by exhibiting the inverse bijection. Let $(\mathbf{m}_{12}, \mathrm{s})$ be a pair as in the proposition, and let $(L, L', \boldsymbol{\omega}, \boldsymbol{\omega}')$ be the quadruple obtained by decomposing the two-face map $\mathbf{m}_{12}$, its distinguished marked vertex playing the role of the third distinguished element labeled 3 used in the construction of page 38. By construction the sequences $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ consist only of empty or trivial slices, with a number $k+1$ of marked empty slices. Replacing these marked empty slices with $\mathrm{s}_1, \ldots, \mathrm{s}_{k+1}$, we obtain two sequences $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ such that the assembling of the quadruple $(L, L', \boldsymbol{\sigma}, \boldsymbol{\sigma}')$ gives the tight map $\mathbf{m}$ we are looking for. This ends the proof of Proposition 4.7. □

Remark 4.9. It might seem more direct to attempt to enumerate the quadruples $(L, L', \boldsymbol{\sigma}, \boldsymbol{\sigma}')$. But, since the paths L and L' have a height variation depending on the separating girth s, this leads to an expression involving a sum over s. The trick of "recombining" L and L' into a two-face map allows to circumvent this issue.

5. Bijective proofs for non necessarily bipartite maps

In this section, we explain how the bijective approach of Section 4 may be extended so as to enumerate planar tight maps which are not necessarily bipartite.

Recall from Section 2.2 that a *petal* is a face of degree one. The key idea to extend our construction to the non bipartite case is to realize that petals play a role very similar to marked vertices. Mimicking the organization of Section 4, we will first enumerate tight maps with a single non-petal face in Section 5.1, then tight maps with just two non-petal faces in Section 5.2, and finally tight maps with an arbitrary number of non-petal faces in Section 5.4. A prerequisite to this latter enumeration will be that of tight non necessarily bipartite slices in Section 5.3.

Before we start our discussion, let us recall from Section 2.2 the definition of the polynomials

$$p_{k,e}(m) := \frac{1}{(k!)^2} \prod_{i=1}^{k} \left(m^2 - \left(i - \frac{e}{2}\right)^2 \right) = \binom{m + \frac{e}{2} - 1}{k} \binom{m - \frac{e}{2} + k}{k} \tag{5.1}$$

for $k \in \mathbb{Z}_{\geqslant 0}$. For $m - \frac{e}{2} \in \mathbb{Z}$, we may interpret $p_{k,e}(m)$ as counting words of the form (4.1) where the a_i and b_i are nonnegative integers such that $a_1 + a_2 + \cdots + a_{k+1} = m + \frac{e}{2} - k - 1$ and $b_1 + b_2 + \cdots + b_{k+1} = m - \frac{e}{2}$ (i.e. the word has length $2m$, $k+1$ occurrences of $D_\circ$ and e more D's than U's). This interpretation holds a priori only for $m \geqslant \max\left(k + 1 - \frac{e}{2}, \frac{e}{2}\right)$ but this domain may be extended to $m \geqslant \max\left(1 - \frac{e}{2}, \frac{e}{2} - k\right)$ since $p_{k,e}(m)$ vanishes in the additional domain.

Remark 5.1. Even though we will not use it in the sequel, let us mention that, by a variation of the arguments of Section 4.1, we may show that, for m as above, $p_{k,e}(m)$ is the number of two-face tight maps with one face of degree $2m$, one simple face of degree $|e|$ and $k+1$ marked vertices, one of them distinguished, with the condition that for $e < 0$ no marked vertex is incident to the face of degree e. Note that such maps are closely related with the notion of $(a,b)^*$-forests of Definition 4.3: for $e > 0$, $p_{k,e}(m)$ counts $(0,e)^*$-forests with size $2m$ and $k+1$ marked vertices including the distinguished vertex.

5.1. Petal trees

We recall that a *petal tree* is a planar map having an *exterior* face of arbitrary degree, and such that every other face is a petal. A tight petal tree is just a petal tree with marked vertices, which is tight as a map, see again Figure 2.1.

We also recall, for $r, s \in \mathbb{Z}_{\geqslant 0}$, $\epsilon \in \mathbb{Z}$ and $m \in \mathbb{Z}/2$, the definition of the quasi-polynomial

$$\pi^{(\epsilon)}_{r,s}(m) := \begin{cases} \binom{r+s}{s} p_{r+s,s+1+\epsilon}(m) & \text{if } m - \frac{s+1+\epsilon}{2} \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Our goal in this section is to prove Proposition 2.14, showing that, for $m \in \mathbb{Z}_{>0}/2$ and $\epsilon \in \{-1, 0, 1\}$, $\pi^{(\epsilon)}_{r,s}(m)$ enumerates tight petal trees with one exterior face of degree $2m$, $s + 1 + \epsilon$ petals (excluding the exterior face when $m = 1/2$), $1 + \epsilon$ of which are distinguished, and $r + 1 - \epsilon$ marked vertices, $1 - \epsilon$ of which are distinguished. The reason for the ϵ-dependence in the statement is that we must distinguish two elements among marked vertices and petals. For this reason, there are three situations to consider: we may distinguish two marked vertices ($\epsilon = -1$), two petals ($\epsilon = 1$), or one of each type ($\epsilon = 0$).

We note that, upon "labeling" the undistinguished elements, Proposition 2.14 is equivalent to the more symmetric statement:

Proposition 5.2. *For $0 \leqslant e \leqslant k+2$ and $m \in \mathbb{Z}_{>0}/2$, the number of tight petal trees with an exterior face of degree $2m$, with e petals and $k+2-e$ marked vertices, all labeled, is given by:*

$$N_{0,k+3}(2m, \underbrace{1,\ldots,1}_{e}, \underbrace{0,\ldots,0}_{k+2-e}) = \begin{cases} k!\,p_{k,e}(m) & \text{if } m - \frac{e}{2} \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

More precisely, Proposition 5.2 is recovered from Proposition 2.14 by setting $k = r + s$ and $e = s+1+\epsilon$ (hence $k+2-e = r+1-\epsilon$). The petal trees considered in Proposition 2.14 have s (respectively r) non-distinguished petals (respectively marked vertices) which we may label in $s!$ (respectively $r!$) ways. Using $s!r!\binom{r+s}{s} = (r+s)! = k!$, we obtain (5.3) for $1+\epsilon \leqslant e \leqslant k+1+\epsilon$, hence for the whole range $0 \leqslant e \leqslant k+2$ by letting in ϵ vary in $\{-1, 0, 1\}$.

Proof of Proposition 2.14. Let us first discuss how we may code petal trees using words, or equivalently lattice paths, as we did in Section 4.1 for ordinary trees. Note first that petal trees have two types of edges: *"tree-type"* edges whose both sides are incident to the exterior face and *"petal-type"* edges with one edge side incident to a petal and the other to the exterior face. In particular, the tree-type edges have distinct endpoints and they form a plane tree (i.e. a map with a single face), which we call the *"wood"* of the petal tree. Here the coding that we shall use applies to *rooted* petal trees, i.e. petal trees where we distinguish a corner (the root corner) *in the exterior face* (note that this also induces a rooting of the wood tree). Starting from this root corner and following the contour of the exterior face going counterclockwise around the tree (i.e. with the exterior face on the right), we record a letter U (respectively D) for each tree-type edge visited while going away from (respectively towards) the root and a letter E for each visited petal-type edge. The obtained three-letter word may alternatively be visualized as a lattice path with three types of elementary steps: up, down and horizontal, associated respectively to the letters U, D and E. This path is a Motzkin path of length equal to the degree $2m'$ of the exterior face, i.e. it goes from $(0,0)$ to $(2m', 0)$ and *stays above the x-axis*. Indeed, the ordinates of the path are *non-negative* since they record graph distances to the *root vertex* incident to the root corner. The number s' of occurrences of the letter E in the coding word is nothing but the number of petals, and is such that $m' - \frac{s'}{2}$, which is the number of U's (or equivalently of D's), is a nonnegative integer.

Let us now consider a rooted petal tree with marked vertices, *where the root vertex is unmarked*, and such that all non-root leaves are marked. Every non-root vertex being bijectively associated with its parent tree-type edge in the wood tree, hence with a letter D, we record the markings as in Section 4.1 by replacing each D by a $D_\circ$ if the associated vertex is marked and by a $D_\bullet$ otherwise. We end up with a word made of four letters, U, $D_\circ$, $D_\bullet$ and E, associated with a "dressed" Motzkin path, with k' occurrences of $D_\circ$ and $m' - \frac{s'}{2} - k'$ occurrences of $D_\bullet$

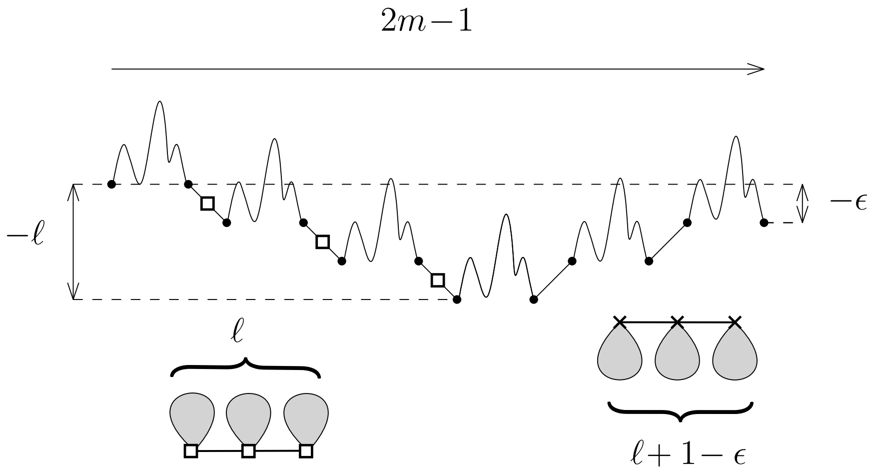

Figure 5.1: Schematic picture of the decomposition of a lattice path of length $2m-1$, height difference $-\epsilon$, and minimal height $-\ell$ into a sequence of $2\ell+1-\epsilon$ Motzkin paths, where the squares indicate that the underlying descending step may be marked (if associated with a $D_\circ$) or not (if associated with a $D_\bullet$). For $j=1,2,\cdots,\ell$, the j-th Motzkin path is followed by the first elementary down step reaching height $-j$. These ℓ Motzkin paths code for an ordered set $\mathcal{F}_\square$ of ℓ rooted petal trees whose root vertex are marked or not according to the nature of the following elementary down step. For $j=\ell+2,\ell+3,\cdots,2\ell+1-\epsilon$, the j-th Motzkin path is preceded by the last elementary up step reaching height $j-1-2\ell$. Together with the $(\ell+1)$-th Motzkin path, these Motzkin paths code for an ordered set $\mathcal{F}_\times$ of $\ell+1-\epsilon$ rooted petal trees whose root vertex is unmarked. The ordered sets $\mathcal{F}_\square$ and $\mathcal{F}_\times$ are drawn here with additional edges connecting the successive roots of their petal tree components, represented schematically by grey blobs with a black boundary, with a root vertex drawn as a square if markable and a cross if not.

if the petal tree has k' marked vertices. As before, requiring that all non-root leaves be marked simply amounts to forbidding the sequence $UD_\bullet$ in the coding words.

Returning to the setting of Proposition 2.14 and assuming that $m-\frac{s+1+\epsilon}{2}\in\mathbb{Z}$, recall that $p_{r+s,s+1+\epsilon}(m)$ counts three-letter words of the form (4.1) with $r+s+1$ occurrences of $D_\circ$, $m-r-\frac{s+1-\epsilon}{2}$ occurrences of $D_\bullet$ and $m-\frac{s+1+\epsilon}{2}$ occurrences of U, so that the height difference is $-(s+1+\epsilon)$. As already mentioned, the property holds for the extended range $m\geqslant\max\left(1-\frac{s+1+\epsilon}{2},1-r-\frac{s+1-\epsilon}{2}\right)$, hence for all positive m in the current setting where $r,s\geqslant 0$ and $\epsilon\in\{-1,0,1\}$. We now transform the three-letter word into a four-letter word coding for a *sequence of petal trees* as follows: we first remove the first letter $D_\circ$, which leaves us with $r+s$ occurrences of $D_\circ$, and pick s of them that we transform into E's. This transformation can be done in $\binom{r+s}{s}$ ways, hence the obtained four-letter words are counted by $\binom{r+s}{s}p_{r+s,s+1+\epsilon}(m)$. By construction, these words have s occurrences of E, r occurrences of $D_\circ$, $m-\frac{s+1+\epsilon}{2}$ occurrences of U and $m-r-\frac{s+1-\epsilon}{2}$ occurrences of $D_\bullet$. These correspond to lattice paths of length $2m-1$ and height difference $-\epsilon$, say from $(0,0)$ to $(2m-1,-\epsilon)$, hence with minimal height $-\ell$ for some $\ell\geqslant\max(0,\epsilon)$. Any such path is canonically decomposed into a sequence of $2\ell+1-\epsilon$ dressed Motzkin paths (possibly of length 0), obtained by cutting out

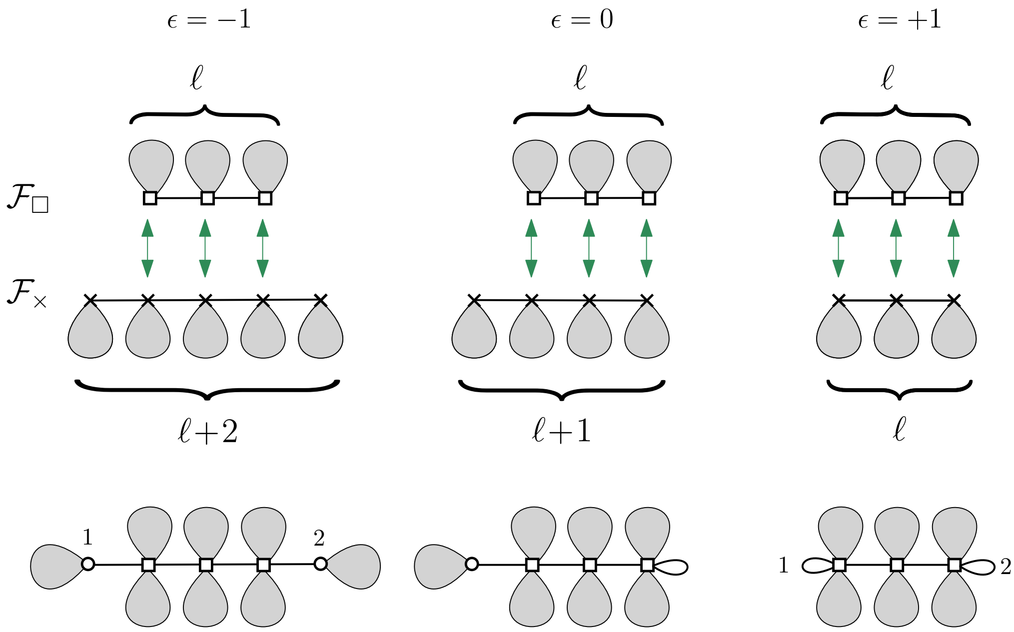

Figure 5.2: The gluing of a sequence $\mathcal{F}_\square$ of ℓ petal trees with markable roots and a sequence $\mathcal{F}_\times$ of $\ell+1-\epsilon$ petal trees with unmarked roots into a tight petal tree with $1-\epsilon$ distinguished marked vertices (represented by circles) and $1+\epsilon$ distinguished petals (represented by small empty loops), for $\epsilon=-1$ (left), 0 (center) and 1 (right).

the first elementary down steps reaching height $-j$ for $j=1,2,\ldots,\ell$ and the last elementary up steps reaching height $-j'+1$ for $j'=\ell,\ell-1,\ldots,\epsilon+1$, see Figure 5.1. Each Motzkin path component codes for a marked petal tree whose root vertex is unmarked, and we decide to mark it if this Motzkin path is followed by a $D_\circ$ in the original path. By doing so, we both ensure that the total number of marked vertices in the petal tree sequence is r and that the four-letter word can be recovered bijectively from the petal tree sequence (since we know the nature $D_\bullet$, $D_\circ$ or U of all the removed steps). Note that by construction, only the first ℓ petal trees may have their root vertex marked. All in all, $\binom{r+s}{s}p_{r+s,s+1+\epsilon}(m)$ enumerates pairs $(\mathcal{F}_\square,\mathcal{F}_\times)$ made of a sequence $\mathcal{F}_\square$ of ℓ petal trees with marked vertices whose root vertex is markable, and a sequence $\mathcal{F}_\times$ of $\ell+1-\epsilon$ petal trees with marked vertices whose root vertex is unmarked, for some arbitrary integer $\ell \geqslant \max(0,\epsilon)$, with a total of s petals, r marked vertices, and $m-\ell-\frac{s+1-\epsilon}{2}$ tree-type edges, and with all the non-root leaves in the petal trees marked. As a final step, the pairs $(\mathcal{F}_\square,\mathcal{F}_\times)$ are transformed into the desired tight petal trees by attaching the roots of the ℓ (respectively $\ell+1-\epsilon$) petal trees in the sequence $\mathcal{F}_\square$ (respectively $\mathcal{F}_\times$) by additional edges and gluing them along a spine of length $\ell-\epsilon$ as displayed in Figure 5.2. Note that a markable root vertex is always matched to an unmarked one: the markings of root vertices may thus be transferred without ambiguity after gluing, with no risk of double markings along the spine. Let us detail the three possibilities $\epsilon=-1,0,1$.

For $\epsilon=-1$, the sequence $\mathcal{F}_\times$ has two more petal trees than $\mathcal{F}_\square$ so that the root vertices of its two extremal petal trees remain unmatched. We decide to mark these vertices and distinguish them as vertices 1 and 2: the resulting object is a tight petal tree with s petals and $r+2$ marked vertices, two of which distinguished. The construction is clearly reversible by cutting along the branch between vertices 1 and 2, and holds for any $\ell \geqslant 0$ (for $\ell = 0$, vertices 1 and 2 are incident). The degree of the exterior face is easily found equal to $2(m - \ell - \frac{s+2}{2}) + s + 2(\ell + 1) = 2m$ as wanted.

For $\epsilon = 0$, the sequence $\mathcal{F}_\times$ has one more petal tree than $\mathcal{F}_\square$ so that the root vertex of its first petal tree remains unmatched. We decide to mark this vertex and to add an additional, distinguished petal to mark the corner inbetween the last two glued petal trees at the end of the spine: the resulting object is a tight petal tree with $s + 1$ petals, one of which distinguished, and $r + 1$ marked vertices, one of which distinguished. The construction is clearly reversible by cutting along the branch between the distinguished marked vertex and the distinguished petal, eventually removing this petal, and holds for any $\ell \geqslant 0$ (for $\ell = 0$, the distinguished vertex is incident to the distinguished petal). The degree of the exterior face is $2(m - \ell - \frac{s+1}{2}) + s + 1 + 2\ell = 2m$ as wanted.

Finally, for $\epsilon = 1$, the sequences $\mathcal{F}_\times$ and $\mathcal{F}_\square$ have the same number $\ell \geqslant 1$ of petal tree components and their gluing is made reversible by adding a petal inbetween the glued petal trees at each extremity of the spine, that we distinguish as petals 1 and 2: the resulting object is a tight petal tree with $s + 2$ petals, two of which distinguished, and r marked vertices. The construction is clearly reversible by cutting along the branch between petals 1 and 2, eventually removing these petals, and holds for any $\ell \geqslant 1$ (for $\ell = 1$, the distinguished petals are incident to the same vertex). The degree of the exterior face is $2(m - \ell - \frac{s}{2}) + s + 2 + 2(\ell - 1) = 2m$ as wanted. This ends the proof of Proposition 2.14. $\square$

5.2. Petal necklaces

We now consider *petal necklaces*, namely planar maps having two distinguished faces of arbitrary degrees, and such that any other face is a petal. Again, a tight petal necklace is just a petal necklace with marked vertices, which is tight as a map. We have the following enumeration result:

Proposition 5.3. *For r, s non-negative integers not both zero and $m_1, m_2 \in \mathbb{Z}_{>0}/2$, let*

$$\Pi_{r,s}(m_1, m_2) := N_{0,r+s+2}(2m_1, 2m_2, \underbrace{0, \ldots, 0}_{r}, \underbrace{1, \ldots, 1}_{s}) \tag{5.4}$$

be the number of tight petal necklaces with two distinguished faces of degrees $2m_1$ and $2m_2$, with r marked vertices and s petals, all labeled. Then, for $r \geqslant 1$ we have

$$\Pi_{r,s}(m_1, m_2) = (r-1)!s! \sum_{\substack{r_1,r_2,s_1,s_2 \geqslant 0 \\ r_1+r_2=r-1 \\ s_1+s_2=s}} \left(\pi^{(-1)}_{r_1,s_1}(m_1)\pi^{(1)}_{r_2,s_2}(m_2) + \pi^{(0)}_{r_1,s_1}(m_1)\pi^{(0)}_{r_2,s_2}(m_2)\right) \tag{5.5}$$

and, for $s \geqslant 1$,

$$\Pi_{r,s}(m_1, m_2) = r!(s-1)! \sum_{\substack{r_1,r_2,s_1,s_2 \geqslant 0 \\ r_1+r_2=r \\ s_1+s_2=s-1}} \left(\pi^{(0)}_{r_1,s_1}(m_1)\pi^{(1)}_{r_2,s_2}(m_2) + \pi^{(1)}_{r_1,s_1}(m_1)\pi^{(0)}_{r_2,s_2}(m_2)\right) \tag{5.6}$$

where $\pi^{(\epsilon)}_{r,s}$ is the univariate quasi-polynomial defined in (2.26).

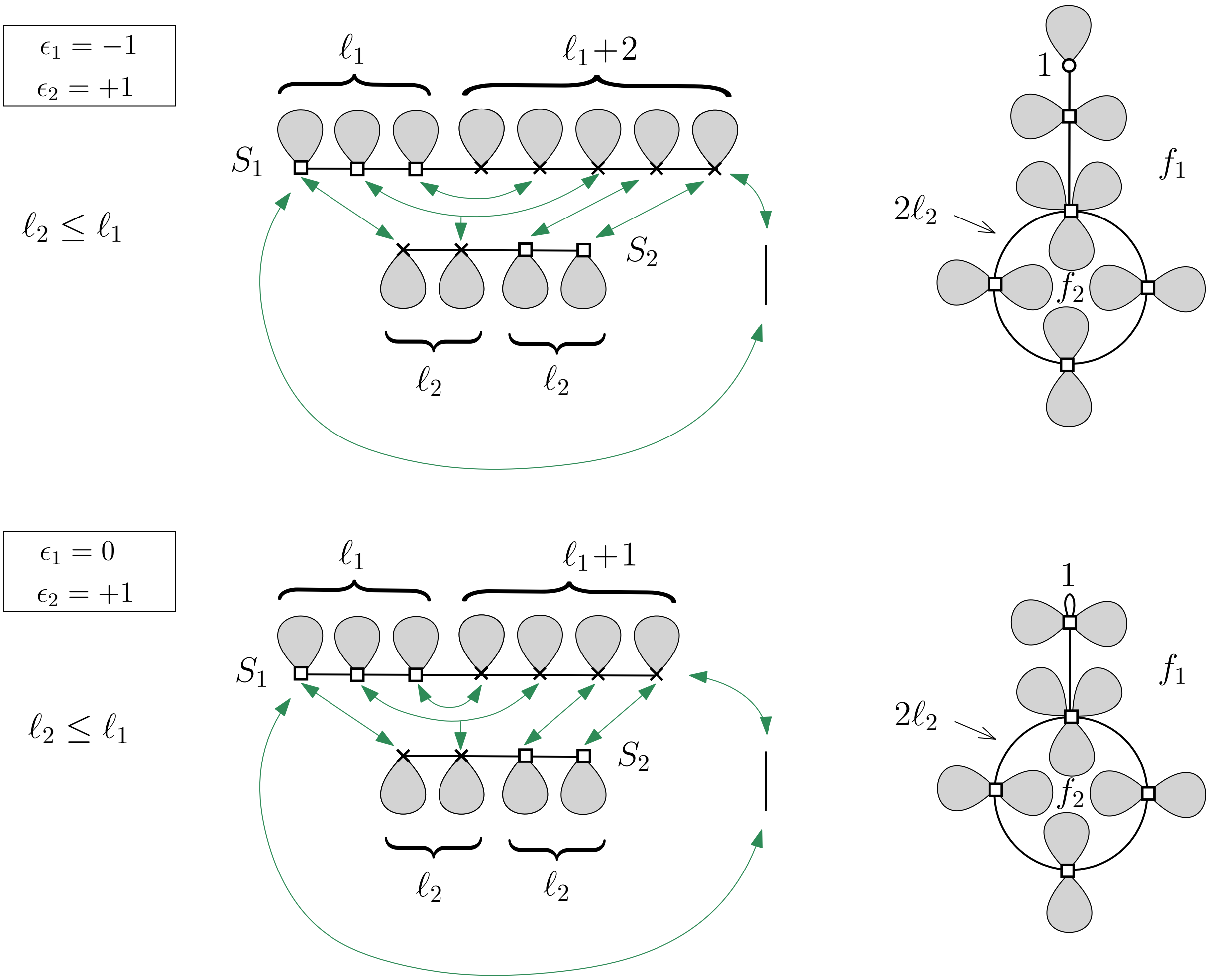

Figure 5.3: The partial and mutual gluings of the segments S_1 (oriented from left to right) and S_2 (oriented from right to left) connecting the petal tree components of the sequences $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, here for $\ell_2 \leqslant \ell_1$. The vertex identifications are indicated by arrows. Top: in the case $\epsilon_1 = -1$, $\epsilon_2 = +1$, the partial gluing of S_1 amounts to pulling in its $(\ell_1 + 1)$-th vertex, creating a branch of length $\ell_1 - \ell_2 + 1$ in f_1 with a distinguished vertex 1 at its end. Bottom: in the case $\epsilon_1 = 0$, $\epsilon_2 = +1$, the partial gluing of S_1 amounts to pulling in its ℓ_1-th edge, creating a branch of length $\ell_1 - \ell_2$ in f_1 with a distinguished petal 1 at its end. In both cases, the mutual gluing generates a loop of even length $2\ell_2$.

This proposition admits a "partially unlabeled" equivalent, which will be useful later on.

Proposition 5.4. *Given $r_0, s_0 \in \mathbb{Z}_{\geqslant 0}$ and $m_1, m_2 \in \mathbb{Z}_{>0}/2$, the number of tight petal necklaces with two distinguished labeled faces of degrees $2m_1, 2m_2$, one distinguished marked vertex, r_0 other unlabeled marked vertices, and s_0 unlabeled petals, is equal to*

$$\pi^{(0)}_{r_0,s_0}(m_1,m_2) := \sum_{\substack{r_1,r_2,s_1,s_2\geqslant 0\\ r_1+r_2=r_0\\ s_1+s_2=s_0}} \left(\pi^{(-1)}_{r_1,s_1}(m_1)\pi^{(1)}_{r_2,s_2}(m_2) + \pi^{(0)}_{r_1,s_1}(m_1)\pi^{(0)}_{r_2,s_2}(m_2)\right), \tag{5.7}$$

while the number of tight petal necklaces with two distinguished labeled faces of degrees $2m_1$, $2m_2$, one distinguished petal, s_0 other unlabeled petals, and r_0 unlabeled marked vertices, is equal to

$$\pi^{(1)}_{r_0,s_0}(m_1,m_2) := \sum_{\substack{r_1,r_2,s_1,s_2\geqslant 0\\ r_1+r_2=r_0\\ s_1+s_2=s_0}} \left(\pi^{(0)}_{r_1,s_1}(m_1)\pi^{(1)}_{r_2,s_2}(m_2) + \pi^{(1)}_{r_1,s_1}(m_1)\pi^{(0)}_{r_2,s_2}(m_2)\right). \tag{5.8}$$

Propositions 5.3 and 5.4 are indeed equivalent to one another, since the petal necklaces considered in the latter have three distinguished boundaries hence no symmetries. Namely, we pass from (5.7) to (5.5) by taking $r = r_0 + 1$ and $s = s_0$, and from (5.8) to (5.6) by taking $r = r_0$ and $s = s_0 + 1$. In both cases, we multiply by $r_0!s_0!$ to label the unlabeled marked vertices/petals in all possible ways. For later use we record the following:

Remark 5.5. Given a petal necklace $\mathbf{m}_{12}$ with a distinguished marked vertex/petal as considered in Proposition 5.4, it is possible to label the marked vertices and petals in a *canonical* way from 1 to $r_0 + s_0 + 1$, with the label 1 assigned to the distinguished marked vertex/petal. The precise labeling procedure is irrelevant, as long as it is deterministic[9].

Proof of Proposition 5.4. Our proof is a direct generalization of that of Proposition 2.6 given in Section 4.2. Still, rather than recoursing to (a, b)- and $(a, b)^*$-forests or generalizations thereof, we will instead use the related notion of pairs of petal tree sequences[10], as introduced in the proof of Proposition 2.14. More precisely, we have seen there that, for r_1, s_1 nonnegative integers, $m_1 \in \mathbb{Z}_{>0}/2$, and $\epsilon_1 = -1, 0, 1$, the quantity $\pi^{(\epsilon_1)}_{r_1,s_1}(m_1)$ defined in (2.26) enumerates pairs $(\mathcal{F}_\square, \mathcal{F}_\times)$ of sequences of respectively ℓ_1 and $\ell_1 + 1 - \epsilon_1$ rooted petal trees for some arbitrary integer $\ell_1 \geqslant \max(0, \epsilon_1)$. The petal trees have a total of r_1 marked vertices, with all their non-root leaves marked, a total of s_1 petals, and a total of $m_1 - \ell_1 - \frac{s_1+1-\epsilon_1}{2}$ tree-type edges. Finally, only the petal trees in $\mathcal{F}_\square$ may have their root vertex marked. The enumeration statement

[9] For instance, calling f_1 and f_2 the two distinguished faces, a possible algorithm consists in picking the vertex v on the boundary between f_1 and f_2 lying on the same branch (possibly of length 0) in the petal necklace as the distinguished marked vertex/petal. Following the contour of f_1 clockwise, starting from the leftmost corner at v in this face, and doing then the same for f_2, we label the marked vertices and petals, including the distinguished element, by successive integers from 1 to $r_0 + s_0 + 1$ at the *first visit* of such marked vertex (encountered via an incident corner) or petal (encountered via a petal-type edge). We then perform a cyclic permutation of the labels so that the distinguished element receives the label 1.

[10] There is indeed a clear correspondence between (a, b)-forests and collections of trees with a unmarkable roots and b markable ones, and this could be generalized to our setting, with petal trees replacing trees.

above holds even if $m_1 - \frac{s+1+\epsilon_1}{2}$ is a half-integer, as the number of sequence pairs is trivially 0 in this case, and so is $\pi^{(\epsilon_1)}_{r_1,s_1}(m_1)$ by definition.

To construct a petal necklace, we first merge the two sequences $\mathcal{F}_\square$ and $\mathcal{F}_\times$ into a single sequence $\mathcal{F}_1$ of $2\ell_1 + 1 - \epsilon_1$ petal trees, which we transform into a single connected object by adding edges connecting the successive roots of the petal tree components and forming a linear segment S_1 of length $2\ell_1 - \epsilon_1$ – see Figure 5.3 for a schematic representation. This first connected object will code for the face of degree $2m_1$ of our necklace. The coding of the face of degree $2m_2$ is performed via a second sequence $\mathcal{F}_2$, taken now in the set enumerated by $\pi^{(\epsilon_2)}_{r_2,s_2}(m_2)$. The sequence $\mathcal{F}_2$ has the same properties, mutatis mutandis, as the sequence $\mathcal{F}_1$, with now $2\ell_2+1-\epsilon_2$ petal trees, and a total of r_2 marked vertices and s_2 petals. Again $\mathcal{F}_2$ is transformed into a connected object by adding a linear segment S_2 of $2\ell_2 - \epsilon_2$ edges connecting its successive petal tree roots. To obtain the desired necklace, we will simply, by some "partial gluing" process reminiscent of that for (a, b)-forests, squeeze the largest of the two segments S_1 and S_2 so as to get two segments of the same length $\min(2\ell_1 - \epsilon_1, 2\ell_2 - \epsilon_2)$ which can then be glued together, head to tail, into a map with a spine that, after closing by some additional edge, forms the cycle separating the two distinguished faces of the necklace. Let us discuss in detail how we perform the partial and mutual gluing processes. An important property of both processes is that we always identify a markable vertex to an unmarked one. By transferring the possible marking of their markable copy, this guarantees that all the vertices obtained by gluing are markable and marked or not without ambiguity (and without double markings).

Consider first the case where, say $\epsilon_1 = -1$ and $\epsilon_2 = +1$ so that S_2 has length $2\ell_2 - 1$, with some $\ell_2 \geqslant 1$ and S_1 has length $2\ell_1 + 1$ for some $\ell_1 \geqslant 0$. Assume first that $\ell_2 \leqslant \ell_1$: we may then squeeze S_1 by "pulling in" its $(\ell_1 + 1)$-th vertex, namely by gluing the $(\ell_1 + 1 - i)$-th (markable) vertex along S_1 to the $(\ell_1 + 1 + i)$-th (unmarked) one for $i = 1, \ldots, \ell_1 - \ell_2 + 1$. This results in a segment S'_1 of the same length as S_2, with ℓ_2 markable vertices followed by ℓ_2 unmarked ones, together with a branch (attached to the ℓ_2-th vertex of S'_1) of length $\ell_1 - \ell_2 + 1$ with all its vertices markable but the last one, corresponding to the former $(\ell_1 + 1)$-th vertex along S_1. This vertex was unmarked and we decide to mark and distinguish it, say with the label 1. We finally glue the two segments S'_1 and S_2, head to tail, into a linear spine that we close with an additional edge into a cycle of length $2\ell_2$ separating two faces f_1 and f_2 – see Figure 5.3-top. Again every markable vertex of S_2 is matched to an unmarked vertex of S'_1 and vice versa, so that all the vertices along the cycle are markable. It is easily checked that f_1 and f_2 have respective degrees $2m_1$ and $2m_2$ and we end up with a petal necklace *with an additional marked vertex* incident to f_1 but not to f_2. For $\ell_2 \geqslant \ell_1 + 2$, a similar construction consisting now in squeezing the segment S_2 by pulling in its $(\ell_2 + 1)$-th vertex generates a necklace where the separating cycle has length $2(\ell_1 + 1)$ and where the additional marked vertex 1 is incident to f_2 but not to f_1. Finally, if $\ell_2 = \ell_1 + 1$, S_1 and S_2 have the same length and no squeezing is necessary: the faces f_1 and f_2 are then separated by a cycle of length $2\ell_2 = 2(\ell_1 + 1)$ and the vertex 1 (obtained by gluing the $(\ell_1 + 1)$-th vertex of S_1 to the $(\ell_2 + 1)$-th vertex of S_2) lies along this cycle, i.e. is incident to both faces. The construction is clearly reversible by a cutting procedure along the separating cycle and along the branch leading from this cycle to vertex 1. We deduce that $\pi^{(-1)}_{r_1,s_1}(m_1)\pi^{(1)}_{r_2,s_2}(m_2)$ counts petal necklaces with s_1 unlabeled petals and r_1 unlabeled marked vertices in its distinguished face f_1 of degree $2m_1$, s_2 unlabeled petals and r_2 unlabeled marked vertices in its distinguished face f_2

of degree $2m_2$, and with an additional marked vertex (distinct from all the others) distinguished as vertex 1 and lying anywhere in the map (the map hence has a total of $r_1+r_2+1 \geqslant 1$ marked vertices). The necklaces are easily seen to be tight, since the petal tree components have no unmarked leaves and no unmarked leaf was created in the process. Still, as apparent from the above discussion, a restriction applies to these necklaces since, by construction, the length of the cycle separating their distinguished faces is always even. It is easily seen that the missing set of necklaces is enumerated by $\pi^{(0)}_{r_1,s_1}(m_1)\pi^{(0)}_{r_2,s_2}(m_2)$: indeed, repeating the above construction on the corresponding pairs of petal tree sequences $\mathcal{F}_1$ (with $2\ell_1+1$ petal trees) and $\mathcal{F}_2$ (with $2\ell_2+1$ petal trees), we get necklaces whose separating cycle now has the odd length $2\min(\ell_1,\ell_2)+1$ (for some arbitrary $\ell_1,\ell_2 \geqslant 0$), again with an additional marked vertex distinguished as vertex 1. We finally obtain (5.7) by summing over r_1 and r_2 with $r_1+r_2=r_0$, and over s_1 and s_2 with $s_1+s_2=s_0$, to account for the dispatching of the marked vertices and petals among f_1 and f_2.

An alternative construction consists in starting from $\epsilon_1=0$ and $\epsilon_2=+1$ so that S_2 still has length $2\ell_2-1$, with some $\ell_2 \geqslant 1$ while S_1 has length $2\ell_1$ for some $\ell_1 \geqslant 0$. As before, we assume first that $\ell_2 \leqslant \ell_1$: we may then squeeze S_1, now by "pulling in" its ℓ_1-th connecting edge, namely by gluing the (ℓ_1+1-i)-th (markable) vertex along S_1 to the (ℓ_1+i)-th (unmarked) one for $i=1,\ldots,\ell_1-\ell_2+1$. This results in a segment S'_1 of the same length as S_2, with ℓ_2 markable vertices followed by ℓ_2 unmarked ones, together with a branch (attached to the ℓ_2-th vertex of S'_1) of length $\ell_1-\ell_2$ with all its vertices markable, having now a petal at its end corresponding to the former ℓ_1-th edge along S_1. This newly created petal is adjacent to f_1 but not to f_2 and we decide to distinguish it with the label 1. Gluing S'_1 and S_2 and closing, we end up with a cycle of even length $2\ell_2$ separating two faces f_1 and f_2, see the bottom of Figure 5.3. For $\ell_2 \geqslant \ell_1+1$, squeezing now the segment S_2 by pulling in its ℓ_2-th edge generates a necklace with a separating cycle of odd length $2\ell_1+1$ and where the additional petal is now adjacent to f_2. To get the missing necklaces, namely those with an odd separating cycle and a distinguished petal adjacent to f_1, and those with an even separating cycle and a distinguished petal adjacent to f_2, we have, as clear by symmetry, to supplement the above family of necklaces enumerated by $\pi^{(0)}_{r_1,s_1}(m_1)\pi^{(1)}_{r_2,s_2}(m_2)$ by that enumerated by $\pi^{(1)}_{r_1,s_1}(m_1)\pi^{(0)}_{r_2,s_2}(m_2)$. We finally obtain (5.8) by summing over r_1 and r_2 with $r_1+r_2=r_0$, and over s_1 and s_2 with $s_1+s_2=s_0$, to account for the dispatching of the marked vertices and petals among f_1 and f_2. □

Remark 5.6. The consistency of the two expressions (5.5) and (5.6) for $\Pi_{r,s}(m_1,m_2)$ when $r,s \geqslant 1$ may be checked directly by setting $s! = (s-1)!(s_1+s_2)$ in (5.5), $r! = (r-1)!(r_1+r_2)$ in (5.6), and upon using the identifications

$$\sum_{\substack{r_1,r_2,s_1,s_2\geqslant 0\\ r_1+r_2=r-1\\ s_1+s_2=s}} s_1\pi^{(-1)}_{r_1,s_1}(m_1)\pi^{(1)}_{r_2,s_2}(m_2) = \sum_{\substack{r_1,r_2,s_1,s_2\geqslant 0\\ r_1+r_2=r\\ s_1+s_2=s-1}} r_1\pi^{(0)}_{r_1,s_1}(m_1)\pi^{(1)}_{r_2,s_2}(m_2) \tag{5.9}$$

$$\sum_{\substack{r_1,r_2,s_1,s_2\geqslant 0\\ r_1+r_2=r-1\\ s_1+s_2=s}} s_2\pi^{(-1)}_{r_1,s_1}(m_1)\pi^{(1)}_{r_2,s_2}(m_2) \overset{(*)}{=} \sum_{\substack{r_1,r_2,s_1,s_2\geqslant 0\\ r_1+r_2=r-1\\ s_1+s_2=s}} s_2\pi^{(1)}_{r_1,s_1}(m_1)\pi^{(-1)}_{r_2,s_2}(m_2) \tag{5.10}$$

$$= \sum_{\substack{r_1,r_2,s_1,s_2\geqslant 0\\ r_1+r_2=r\\ s_1+s_2=s-1}} r_2\pi^{(1)}_{r_1,s_1}(m_1)\pi^{(0)}_{r_2,s_2}(m_2) \tag{5.11}$$

$$\sum_{\substack{r_1,r_2,s_1,s_2\geqslant 0\\ r_1+r_2=r-1\\ s_1+s_2=s}} s_1\pi^{(0)}_{r_1,s_1}(m_1)\pi^{(0)}_{r_2,s_2}(m_2) = \sum_{\substack{r_1,r_2,s_1,s_2\geqslant 0\\ r_1+r_2=r\\ s_1+s_2=s-1}} r_1\pi^{(1)}_{r_1,s_1}(m_1)\pi^{(0)}_{r_2,s_2}(m_2) \tag{5.12}$$

$$\sum_{\substack{r_1,r_2,s_1,s_2\geqslant 0\\ r_1+r_2=r-1\\ s_1+s_2=s}} s_2\pi^{(0)}_{r_1,s_1}(m_1)\pi^{(0)}_{r_2,s_2}(m_2) = \sum_{\substack{r_1,r_2,s_1,s_2\geqslant 0\\ r_1+r_2=r\\ s_1+s_2=s-1}} r_2\pi^{(0)}_{r_1,s_1}(m_1)\pi^{(1)}_{r_2,s_2}(m_2)\,. \tag{5.13}$$

Here all equalities follow from the identity $s\pi^{(\epsilon)}_{r,s}(m) = (r+1)\pi^{(1+\epsilon)}_{r+1,s-1}(m)$ for $s \geqslant 1$, apart from that marked with a $(*)$ which follows from the identity $\pi^{(1)}_{r,s}(m) = \pi^{(-1)}_{r,s}(m) + \pi^{(-1)}_{r-1,s}(m)$ for any $r \geqslant 0$ (with the convention $\pi^{(-1)}_{-1,s}(m) = 0$), see (B.10)-(B.13) in Appendix B for similar so-called transmutation relations.

5.3. Tight slices

As in the case of bipartite maps, a key ingredient in the derivation of a general formula for the number of tight general non-bipartite maps is the enumeration of tight slices. Recall the basic definitions pertaining to (tight) slices from Section 4.3, and note that we do not assume anymore that the face degrees be even integers. In this general context, it is still true that the only elementary slice of tilt -1 is the trivial slice, but a major difference is that the set of elementary slices of tilt 0 is now non-empty.

As discussed in Section 4.3, a (tight) slice s with width $w \geqslant 1$ and tilt t can be decomposed into a collection of w (tight) elementary slices. This discussion applies without change in our general context: we cut the slice along the leftmost geodesics started from the consecutive corners $c_0, c_1, \ldots, c_w$ incident to the base, and record the lattice path $L = (\ell_0, \ell_1, \ldots, \ell_w)$ where $\ell_i = d(c_0, A) - d(c_i, A)$, and A is the apex of s. The only difference is that the tilt $t_i = \ell_i - \ell_{i-1}$ of the slice s'_i with base $c_{i-1}c_i$ can take any value $\{-1,0,1\}$ rather than only $\{-1,1\}$. As before, the tightness condition requires that, for $i \in \{2,\ldots,w\}$, if $t_i = +1$ and $t_{i-1} = -1$, then the slice s'_i is either the marked empty slice, or a tight elementary slice with at least one inner face. For $i \in \{1,\ldots,w\}$, if $t_i = -1$ then the slice s'_i is necessarily trivial, while if $t_i = 0$ then the slice s'_i automatically has at least one inner face of odd degree. We may forget the redundant information of all down steps, as well as up steps decorated with unmarked empty slices, by letting $\mathrm{s}_j, 1 \leqslant j \leqslant k$ be the sequence $(\mathrm{s}'_i, 1 \leqslant i \leqslant w)$ whose trivial and unmarked empty elements have been removed. This gives:

Proposition 5.7. *There is a one-to-one correspondence between tight slices of width w and tilt t on the one hand, and pairs of the form $(L, (\mathrm{s}_j, 1 \leqslant j \leqslant k))$ on the other hand, where:*

- *$L = (\ell_0, \ldots, \ell_w)$ is a lattice path from $(0,0)$ to (w,t) with increments in $\{-1,0,1\}$, that has k marked steps $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant w$ with $\ell_{i_j} - \ell_{i_j-1} \in \{0,1\}$, in such a way that every up step immediately following a down step is marked, and every horizontal step is marked,*

- *for* $1 \leqslant j \leqslant k$, $\mathbf{s}_j$ *is a tight elementary slice of tilt* $\ell_{i_j} - \ell_{i_j-1}$, *and in the case where this tilt is* 1, $\mathbf{s}_j$ *is either the marked empty slice, or has at least one inner face.*

We obtain the following generalization of Corollary 4.5.

Corollary 5.8. *For* $m \in \mathbb{Z}_{>0}/2$, *there is a one-to-one correspondence between non-empty tight elementary slices of tilt* $\epsilon \in \{0,1\}$, *whose inner face incident to the base edge has degree* $2m$ *on the one hand, and pairs of the form* $(L, (\mathbf{s}_j, 1 \leqslant j \leqslant k))$ *on the other hand, where* k *is a non-negative integer, and:*

- $L = (\ell_0, \dots, \ell_{2m-1})$ *is a lattice path from* $(0,0)$ *to* $(2m-1, \epsilon)$ *with increments in* $\{-1, 0, 1\}$, *that has* k *marked steps* $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant 2m-1$ *with* $\ell_{i_j} - \ell_{i_j-1} \in \{0,1\}$, *in such a way that every up step immediately following a down step is marked, and every horizontal step is marked,*

- *for* $1 \leqslant j \leqslant k$, $\mathbf{s}_j$ *is a tight elementary slice of tilt* $\ell_{i_j} - \ell_{i_j-1}$, *and in the case where this tilt is* 1, $\mathbf{s}_j$ *is either the marked empty slice, or has at least one inner face.*

By iterating the decomposition of this corollary, we obtain an encoding of tight elementary slices by plane trees, where every vertex may be associated with a slice of tilt in $\{0,1\}$. Let $m_1, \dots, m_n$ be integers or half integers, not all equal to 0, and $\epsilon \in \{0,1\}$. Let $\mathcal{T}_\epsilon(m_1, \dots, m_n)$ be the family of pairs $(\mathbf{t}, (L_i)_{1 \leqslant i \leqslant n})$, satisfying the following conditions.

- $\mathbf{t}$ is a rooted plane tree with n vertices labeled by $\{1, 2, \dots, n\}$.

- Each vertex receives a type in $\{0,1\}$, and the root has type ϵ. We let $\epsilon_i \in \{0,1\}$ be the type of vertex i and $(w_1^{(i)}, \dots, w_{k_i}^{(i)})$ be the types of the k_i children of vertex i in $\mathbf{t}$ in planar order.

- For $i \in \{1, 2, \dots, n\}$ such that $m_i > 0$, $L_i = (\ell_0^{(i)}, \dots, \ell_{2m_i-1}^{(i)})$ is a lattice path from $(0,0)$ to $(2m_i - 1, \epsilon_i)$ with increments in $\{-1, 0, 1\}$, and with k_i marked steps $1 \leqslant j_1 < \cdots < j_{k_i} \leqslant 2m_i - 1$, such that $\ell_{j_r}^{(i)} - \ell_{j_r-1}^{(i)} = w_r^{(i)}$ for every $r \in \{1, \dots, k_i\}$. In particular, only up or horizontal steps may be marked. We further require that every up step of L_i immediately following a down step is marked, and every horizontal step of L_i is marked.

- For $i \in \{1, 2, \dots, n\}$ such that $m_i = 0$, we have $\epsilon_i = 1$, $k_i = 0$ and L_i is the trivial path $\{(0,0)\}$.

This leads to the following generalization of Corollary 4.6.

Corollary 5.9. *For* $\epsilon \in \{0,1\}$ *and every choice of integers or half-integers* $m_1, \dots, m_n$ *not all equal to* 0, *the iterated slice decomposition yields a bijection between the set of elementary tight slices with* n *labeled boundaries of respective lengths* $2m_1, \dots, 2m_n$ *with tilt* ϵ *and the set* $\mathcal{T}_\epsilon(m_1, \dots, m_n)$.

It turns out that, in order to prove Theorem 2.12 in Section 5.4, we will need an extension of this corollary associating sequences of slices with sequences of trees, namely forests. Still, as a warm-up, let us first consider the enumeration of $\mathcal{T}_\epsilon(m_1,\ldots,m_n)$, which is done based on that of *two-type trees*, obtained as a special case of Proposition A.3 of Appendix A. Note that the types 1 and 0 considered in the present section correspond to types A and B respectively in the notation of the appendix. We count the elements $(\mathbf{t},(L_i)_{1\leqslant i\leqslant n})$ of $\mathcal{T}_\epsilon(m_1,\ldots,m_n)$ by fixing the types $\epsilon_1,\ldots,\epsilon_n$ of the vertices $1,\ldots,n$ of $\mathbf{t}$ (so that $\epsilon_i = w_0^{(i)}$ in the notation of Appendix A) with the constraint that the root vertex should have type ϵ, as well as the sequence of types $w_1^{(i)},\ldots,w_{k_i}^{(i)}$ of the consecutive children (in planar order) of vertex i. These sequences must satisfy the consistency conditions (A.1) and (A.2) of Appendix A, which in this context are rewritten as

$$\sum_{i=1}^n \epsilon_i = \epsilon + \sum_{i=1}^n r_i \qquad \text{and} \qquad \sum_{i=1}^n \bar\epsilon_i = \bar\epsilon + \sum_{i=1}^n s_i\,, \tag{5.14}$$

where we let $\bar\eta = 1-\eta$ for $\eta\in\{0,1\}$, and where $r_i = \sum_{j=1}^{k_i} w_j^{(i)}$ and $s_i = \sum_{j=1}^{k_i} \bar w_j^{(i)}$ are the numbers of type-1 and type-0 children of vertex i. These consistency relations simply express in two different ways the number of type 1 (resp. 0) vertices in the tree $\mathbf{t}$. Note that the number of type-0 (resp. type-1) vertices that are the children of type-1 (resp. type-0) vertices (those are respectively the numbers b^A and a^B considered in Appendix A) is given by

$$\sum_{i=1}^n \epsilon_i s_i \qquad \left(\text{resp. } \sum_{i=1}^n \bar\epsilon_i r_i\right)\,. \tag{5.15}$$

Finally, by Proposition A.3 (in the notation therein, we have $a^O=\epsilon$ and $b^O=\bar\epsilon$, expressing the fact that we are counting "forests" made of only one tree with root of type ϵ), we see that the number of possible trees $\mathbf{t}$ contributing to $\mathcal{T}_\epsilon(m_1,\ldots,m_n)$, with types given by $\epsilon_1,\ldots,\epsilon_n$ and given a consistent type array $(w_j^{(i)})_{j=1,\ldots,k_i}^{i=1\ldots,n}$, is equal to

$$\left(\epsilon\sum_{i=1}^n \epsilon_i s_i + \bar\epsilon\sum_{i=1}^n \bar\epsilon_i r_i\right)\left(\sum_{i=1}^n \epsilon_i - 1\right)!\left(\sum_{i=1}^n \bar\epsilon_i - 1\right)! \tag{5.16}$$

where $\sum_{i=1}^n \epsilon_i$ and $\sum_{i=1}^n \bar\epsilon_i$ are the numbers of vertices of types 1 and 0 respectively. In the case where either one of these numbers is equal to 0, in accordance with (A.6), we should replace the whole formula by $(n-2)!$.

It now remains to enumerate, for a given consistent array $(w_j^{(i)})_{j=1,\ldots,k_i}^{i=1,\ldots,n}$, and for a given tree $\mathbf{t}$ as above, the number of possible lattice paths L_i, $1\leqslant i\leqslant n$ so that $(\mathbf{t},(L_i)_{1\leqslant i\leqslant n})$ is an element of $\mathcal{T}_\epsilon(m_1,\ldots,m_n)$. As before, we let $r_i=\sum_{j=1}^{k_i} w_j^{(i)}$ and $s_i=\sum_{j=1}^{k_i}\bar w_j^{(i)} = k_i - r_i$. If $m_i=0$ then by definition of $\mathcal{T}_\epsilon(m_1,\ldots,m_n)$, this requires $L_i=\{(0,0)\}$, $r_i=s_i=0$ and $\epsilon_i=1$: by (2.27), this is precisely counted by $\pi_{r_i,s_i}^{(\epsilon_i)}(0)=\delta_{r_i,0}\delta_{s_i,0}\delta_{\epsilon_i,1}$. If $m_i>0$, L_i should be a lattice path from $(0,0)$ to $(2m_i-1,\epsilon_i)$, with increments in $\{-1,0,1\}$, in which k_i up or horizontal steps are marked, say $1\leqslant j_1<\cdots<j_{k_i}\leqslant 2m_i-1$, with increments at these steps respectively given by $w_1^{(i)},\ldots,w_{k_i}^{(i)}$, and in such a way that all horizontal steps are marked, as well as all the up steps immediately following a down step. As explained in the proof of Proposition 2.14

(and up to changing the up steps into down steps and vice versa), the number of such paths is equal to $p_{r_i+s_i,s_i+1+\epsilon_i}(m_i)$ if $m_i - \frac{s_i+1+\epsilon_i}{2} \in \mathbb{Z}$, and to zero otherwise. In particular, this number depends on the array $(w_j^{(i)})_{j=1,\ldots,k_i}^{i=1,\ldots,n}$ only through the values of r_i, s_i. Noting that for a given value of r_i, s_i, there are $\binom{r_i+s_i}{s_i}$ possible choices of $(w_j^{(i)}, 1 \leqslant j \leqslant k_i)$ inducing these values, and recalling that $\pi_{r_i,s_i}^{(\epsilon_i)} = \binom{r_i+s_i}{s_i} p_{r_i+s_i,s_i+1+\epsilon_i}(m_i)$, we finally obtain the following result.

Proposition 5.10. *For $\epsilon = 0, 1$ and for every choice of integers or half-integers $m_1, \ldots, m_n$ not all equal to 0, the cardinality of $\mathcal{T}_\epsilon(m_1, \ldots, m_n)$, and hence the number of elementary tight slices with n labeled boundaries of respective lengths $2m_1, \ldots, 2m_n$ and tilt ϵ, is given by*

$$\mathrm{Card}\left(\mathcal{T}_\epsilon(m_1, \ldots, m_n)\right) = \sum_{\substack{\epsilon_1,\ldots,\epsilon_n=0,1 \\ r_1,\ldots,r_n,s_1,\ldots,s_n \geqslant 0 \\ \epsilon_1+\cdots+\epsilon_n=\epsilon+r_1+\cdots+r_n \\ \bar\epsilon_1+\cdots+\bar\epsilon_n=\bar\epsilon+s_1+\cdots+s_n}} \left(\epsilon \sum_{i=1}^n \epsilon_i s_i + \bar\epsilon \sum_{i=1}^n \bar\epsilon_i r_i\right) \left(\sum_{i=1}^n \epsilon_i - 1\right)! \left(\sum_{i=1}^n \bar\epsilon_i - 1\right)! \prod_{i=1}^n \pi_{r_i,s_i}^{(\epsilon_i)}(m_i)\,, \tag{5.17}$$

where the first three factors in the sum should be replaced by $(n-2)!$ whenever the argument in either of the two factorials equals -1.

This proposition has interesting consequences for certain evaluations of lattice count polynomials. Proceeding similarly to Section 4.3, we may indeed associate bijectively tight maps with elementary tight slices of tilt in $\{0,1\}$. Precisely, starting from an elementary tight slice of tilt 0, we may glue isometrically the left and right boundaries to obtain a tight map with one marked petal (delimited by the base edge of the slice) and one marked vertex (given by the apex). This construction can be inverted by cutting along the leftmost geodesic from the marked petal to the pointed vertex. Similarly, starting from an elementary tight slice of tilt 1, we may glue isometrically the left and right boundaries *starting from the base*[11] to obtain a tight map with two marked petals (one being delimited by the base edge of the slice, and the other one by the edge of the left boundary incident to the apex). The inverse construction is more involved, but is in fact a particular case of the slice decomposition of annular maps.

These bijections imply that for every $m_1, \ldots, m_n \in \mathbb{Z}_{\geqslant 0}/2$ not all equal to zero, and for $\epsilon \in \{0,1\}$, we have

$$N_{0,n}(2m_1, \ldots, 2m_n, 1, \epsilon) = \mathrm{Card}(\mathcal{T}_\epsilon(m_1, \ldots, m_n))\,. \tag{5.18}$$

Finally, from the discussion of [BG12, Appendix A], we obtain the identity

$$N_{0,n}(2m_1, \ldots, 2m_n, 2, 0) = \mathrm{Card}\left(\mathcal{T}_1(m_1, \ldots, m_n)\right) + \sum_{\{1\} \subset I \subsetneq \{1,\ldots,n\}} \mathrm{Card}\left(\mathcal{T}_0\left((m_i)_{i \in I}\right)\right) \mathrm{Card}\left(\mathcal{T}_0\left((m_i)_{i \notin I}\right)\right)\,. \tag{5.19}$$

[11] This means in particular that the two endpoints of the base are glued together. If we proceed as in Section 4.3 and glue the left and right boundaries of a slice of tilt 1 starting from the apex, we do obtain a pointed rooted tight map, but this is not the most general such map, since by construction the two ends of the root edge (inherited from the base) are at different distances from the distinguished vertex (inherited from the apex).

5.4. General maps

We are now ready to get a general enumeration formula for planar tight maps. Our approach will be parallel to that of Section 4.4, and we start by stating the following analog of Proposition 4.7:

Proposition 5.11. *Let $m_1, \dots, m_n$ be non-negative integers or half-integers ($n \geqslant 3$) such that $m_1, m_2 > 0$. Then, there is a bijection between the set of planar tight maps with n boundaries labeled from 1 to n with respective lengths $2m_1, 2m_2, \dots, 2m_n$, and the set of pairs $(\mathbf{m}_{12}, \mathbf{s})$ such that there exist $r_0, s_0 \geqslant 0$ with $r_0 + s_0 \in \{0, \dots, n-3\}$ for which the following holds.*

- *$\mathbf{m}_{12}$ is a tight petal necklace with two distinguished faces of degrees $2m_1, 2m_2$, with one extra distinguished element being either a marked vertex or a petal, and with r_0 other marked vertices and s_0 other petals.*
- *$\mathbf{s} = (\mathbf{s}_1, \dots, \mathbf{s}_{r_0+s_0+1})$ is a $(r_0 + s_0 + 1)$-tuple of slices such that:*
  - *each $\mathbf{s}_j$ ($j = 1, \dots, r_0 + s_0 + 1$) is a tight elementary slice of tilt 0 or 1 containing at least one inner face or marked vertex,*
  - *the inner faces and marked vertices of these slices are labeled by integers in $\{3, \dots, n\}$,*
  - *each $i \in \{3, \dots, n\}$ appears in exactly one $\mathbf{s}_j$ and labels an inner face of degree $2m_i$ for $m_i > 0$, or a marked vertex for $m_i = 0$,*
  - *the label 3 appears in the first slice $\mathbf{s}_1$.*
- *$\mathbf{m}_{12}$ and $\mathbf{s}$ are compatible in the sense that, if we label the marked vertices and petals of $\mathbf{m}_{12}$ in a canonical way from 1 to $r_0 + s_0 + 1$ as in Remark 5.5, and for $j = 1, \dots, r_0 + s_0 + 1$ we set $w_j^{(0)} = 1$ (resp. $w_j^{(0)} = 0$) if label j is on a marked vertex (resp. petal), then $\mathbf{s}_j$ has tilt $w_j^{(0)}$. Note that $r_0 = \sum_{j=2}^{r_0+s_0+1} w_j^{(0)}$.*

The proof of this proposition follows exactly the same lines as that of Proposition 4.7 in Section 4.4. In particular, Lemma 4.8 admits a direct non-bipartite extension in which the maps $\mathbf{m}$ may have all their faces of arbitrary degrees, the lattice paths L, L' may contain horizontal steps, and consistently the sequences $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$ may contain slices of tilt 0. Further details are left to the reader. We deduce the following enumerative result:

Proposition 5.12. *For $n \geqslant 3$ and for non-negative integers or half-integers $m_1, \dots, m_n$ with, say, $m_1, m_2 > 0$, we have*

$$N_{0,n}(2m_1, \dots, 2m_n) = \sum_{\substack{\epsilon_3,\dots,\epsilon_n = 0,1 \\ r_1,\dots,r_n,s_1,\dots,s_n \geqslant 0 \\ \sum_{i=3}^n \epsilon_i = \sum_{i=1}^n r_i + 1 \\ \sum_{i=3}^n (1-\epsilon_i) = \sum_{i=1}^n s_i}} \left(\sum_{i=1}^n r_i\right)! \left(\epsilon_3(s_1+s_2) + \sum_{i=3}^n \epsilon_i s_i\right) \left(\sum_{i=1}^n s_i - 1\right)! \tag{5.20}$$
$$\times \left(\pi^{(-1)}_{r_1,s_1}(m_1)\pi^{(1)}_{r_2,s_2}(m_2) + \pi^{(0)}_{r_1,s_1}(m_1)\pi^{(0)}_{r_2,s_2}(m_2)\right) \prod_{i=3}^n \pi^{(\epsilon_i)}_{r_i,s_i}(m_i)$$

$$+\sum_{\substack{\epsilon_3,\dots,\epsilon_n=0,1\\ r_1,\dots,r_n,s_1,\dots,s_n\geqslant 0\\ \sum_{i=3}^n\epsilon_i=\sum_{i=1}^n r_i\\ \sum_{i=3}^n(1-\epsilon_i)=\sum_{i=1}^n s_i+1}}\left(\sum_{i=1}^n s_i\right)!\left((1-\epsilon_3)(r_1+r_2)+\sum_{i=3}^n(1-\epsilon_i)r_i\right)\left(\sum_{i=1}^n r_i-1\right)!$$
$$\times\left(\pi^{(0)}_{r_1,s_1}(m_1)\pi^{(1)}_{r_2,s_2}(m_2)+\pi^{(1)}_{r_1,s_1}(m_1)\pi^{(0)}_{r_2,s_2}(m_2)\right)\prod_{i=3}^n\pi^{(\epsilon_i)}_{r_i,s_i}(m_i)$$

where $\pi^{(\epsilon)}_{r,s}$ is the univariate quasi-polynomial defined in (2.26) and where it is understood that $(\epsilon_3(s_1+s_2)+\sum_{i=3}^n\epsilon_i s_i)(\sum_{i=1}^n s_i-1)!$ is equal to 1 if all the s_i are zero, and that $((1-\epsilon_3)(r_1+r_2)+\sum_{i=3}^n(1-\epsilon_i)r_i)(\sum_{i=1}^n r_i-1)!$ is equal to 1 if all the r_i are zero.

Proof. We need to enumerate the compatible pairs $(\mathbf{m}_{12},\mathbf{s})$ of Proposition 5.11. Note first that, given r_0 and s_0, the number of possible petal necklaces $\mathbf{m}_{12}$ is given by Proposition 5.4. Turning now to the number of possible $\mathbf{s}$ compatible with a given petal necklace $\mathbf{m}_{12}$, it is given by a direct extension of Corollary 5.9 as follows: by decomposing recursively each elementary tight slice $\mathbf{s}_j$ (with tilt $w_j^{(0)}$) into a tree of lattice paths, we see that the set of possible (r_0+s_0+1)-tuples $\mathbf{s}$ is in bijection with the set $\mathcal{F}_{(w_1^{(0)},\dots,w^{(0)}_{r_0+s_0+1})}(m_3,\dots,m_n)$ defined as the set of pairs $(\mathbf{f},(L_i)_{1\leqslant i\leqslant n})$ satisfying the following conditions.

- $\mathbf{f}$ is a plane forest with r_0+s_0+1 connected components, i.e. a (r_0+s_0+1)-tuple of rooted plane trees, and with a total of $n-2$ vertices labeled by $\{3,\dots,n\}$, the label 3 appearing in the first component.
- Each vertex receives a type in $\{0,1\}$, and the root vertex of the j-th tree component in the forest has type $w_j^{(0)}$. For $i\in\{3,\dots,n\}$, we let $\epsilon_i\in\{0,1\}$ be the type of vertex i, k_i be its number of children, and $(w_1^{(i)},\dots,w_{k_i}^{(i)})$ be the types of these children (numbered in planar order in the rooted tree component at hand). We also set $r_i:=\sum_{j=1}^{k_i}w_j^{(i)}$ the numbers of those children which are of type 1 and $s_i:=k_i-r_i$ the numbers of those children which are of type 0.
- For $i\in\{3,\dots,n\}$:
  - if $m_i>0$, $L_i=(\ell_0^{(i)},\dots,\ell^{(i)}_{2m_i-1})$ is a lattice path from $(0,0)$ to $(2m_i-1,\epsilon_i)$ with increments in $\{-1,0,1\}$, and with k_i marked steps $1\leqslant j_1<\cdots<j_{k_i}\leqslant 2m_i-1$, such that $\ell^{(i)}_{j_r}-\ell^{(i)}_{j_r-1}=w_r^{(i)}$ for every $r\in\{1,\dots,k_i\}$. In particular, only up or horizontal steps may be marked, and we further require that every horizontal step of L_i is marked, as well as every up step immediately following a down step,
  - if $m_i=0$, we have $\epsilon_i=1$, $k_i=r_i=s_i=0$, and L_i is the trivial path $\{(0,0)\}$.

We may now obtain the number of elements of $\mathcal{F}_{(w_1^{(0)},\dots,w^{(0)}_{r_0+s_0+1})}(m_3,\dots,m_n)$ from the results of Appendix A for the enumeration of two-type forests, where the types A and B therein correspond to types 1 and 0 in the present setting. For fixed ϵ_i and k_i, $i=3,\dots,n$ and for fixed $(w_j^{(i)})^{i=3,\dots,n}_{j=1,\dots,k_i}$, the number of forests $\mathbf{f}$ is non-zero only if the two consistency relations (A.1) and (A.2) are satisfied, namely

$$\sum_{i=3}^{n}\epsilon_i = w_1^{(0)} + r_0 + \sum_{i=3}^{n} r_i \quad \text{and} \quad \sum_{i=3}^{n}\bar{\epsilon}_i = \bar{w}_1^{(0)} + s_0 + \sum_{i=3}^{n} s_i \tag{5.21}$$

where, as before, we use the shorthand notation $\bar{\eta} := 1-\eta$ for $\eta \in \{0,1\}$. Again, these identities simply express in two different ways the number of vertices of type 1 (first identity) and of type 0 (second identity), corresponding respectively to the quantities denoted by a and b in Appendix A. When these conditions are satisfied, we may use the constrained enumeration result (A.6), with the correspondence $a^O = w_1^{(0)} + r_0$, $b^O = \bar{w}_1^{(0)} + s_0$, $a^B = \sum_{i=3}^{n}\bar{\epsilon}_i r_i$ and $b^A = \sum_{i=3}^{n}\epsilon_i s_i$: if the vertex 3 is of type 1, i.e. $\epsilon_3 = 1$, the number of forests $\mathbf{f}$ is given by

$$\begin{cases} \left(s_0 + \sum\limits_{i=3}^{n}\epsilon_i s_i\right)\left(r_0 + \sum\limits_{i=3}^{n} r_i\right)!\left(s_0 + \sum\limits_{i=3}^{n} s_i - 1\right)! & \text{if } w_1^{(0)} = 1, \\ \left(\sum\limits_{i=3}^{n}\bar{\epsilon}_i r_i\right)\left(r_0 + \sum\limits_{i=3}^{n} r_i - 1\right)!\left(s_0 + \sum\limits_{i=3}^{n} s_i\right)! & \text{if } w_1^{(0)} = 0, \end{cases} \tag{5.22}$$

with, in the first line, the convention $(s_0+\sum_{i=3}^{n}\epsilon_i s_i)\times(s_0+\sum_{i=3}^{n}s_i-1)! = 1$ if $s_0+\sum_{i=3}^{n}s_i = 0$, i.e. when s_0 and all the s_i's for $i \geqslant 3$ are zero. Note that, in the second line, the quantity $r_0 + \sum_{i=3}^{n} r_i - 1$ is always nonnegative since, from the first consistency relation, it is equal to $\sum_{i=4}^{n}\epsilon_i$. By symmetry, if the vertex 3 is of type 0, i.e. $\epsilon_3 = 0$, the number of forests $\mathbf{f}$ is given by

$$\begin{cases} \left(r_0 + \sum\limits_{i=3}^{n}\bar{\epsilon}_i r_i\right)\left(s_0 + \sum\limits_{i=3}^{n} s_i\right)!\left(r_0 + \sum\limits_{i=3}^{n} r_i - 1\right)! & \text{if } w_1^{(0)} = 0, \\ \left(\sum\limits_{i=3}^{n}\epsilon_i s_i\right)\left(s_0 + \sum\limits_{i=3}^{n} s_i - 1\right)!\left(r_0 + \sum\limits_{i=3}^{n} r_i\right)! & \text{if } w_1^{(0)} = 1, \end{cases} \tag{5.23}$$

with, in the first line, the convention $(r_0+\sum_{i=3}^{n}\bar{\epsilon}_i r_i)\times(r_0+\sum_{i=3}^{n}r_i-1)! = 1$ if $r_0+\sum_{i=3}^{n}r_i = 0$, i.e. when r_0 and all the r_i's for $i \geqslant 3$ are zero. Again, in the second line, the quantity $s_0 + \sum_{i=3}^{n} s_i - 1$ is always nonnegative since, from the second consistency relation, it is equal to $\sum_{i=4}^{n}\bar{\epsilon}_i$. Both cases $\epsilon_3 = 1$ or 0 above may be summarized into the enumeration formulas

$$\begin{cases} \left(\epsilon_3 s_0 + \sum\limits_{i=3}^{n}\epsilon_i s_i\right)\left(s_0 + \sum\limits_{i=3}^{n} s_i - 1\right)!\left(r_0 + \sum\limits_{i=3}^{n} r_i\right)! & \text{if } w_1^{(0)} = 1, \\ \left(\bar{\epsilon}_3 r_0 + \sum\limits_{i=3}^{n}\bar{\epsilon}_i r_i\right)\left(r_0 + \sum\limits_{i=3}^{n} r_i - 1\right)!\left(s_0 + \sum\limits_{i=3}^{n} s_i\right)! & \text{if } w_1^{(0)} = 0, \end{cases} \tag{5.24}$$

with the conventions that $(\epsilon_3 s_0+\sum_{i=3}^{n}\epsilon_i s_i)(s_0+\sum_{i=3}^{n}s_i-1)! = 1$ if s_0 and all the s_i's for $i \geqslant 3$ are zero and that $(\bar{\epsilon}_3 r_0 + \sum_{i=3}^{n}\bar{\epsilon}_i r_i)(r_0 + \sum_{i=3}^{n} r_i - 1)! = 1$ if r_0 and all the r_i's for $i \geqslant 3$ are zero.

It remains to enumerate, for a given array $(w_j^{(i)})_{j=1,\ldots,k_i}^{i=3,\ldots,n}$ satisfying the consistency relations (5.21), and for a given two-type forest $\mathbf{f}$ as above, the number of families of lattice paths $(L_i)_{3\leqslant i\leqslant n}$ so that $(\mathbf{f},(L_i)_{3\leqslant i\leqslant n})$ is an element of $\mathcal{F}_{(w_1^{(0)},\ldots,w_{r_0+s_0+1}^{(0)})}(m_3,\ldots,m_n)$. Repeating the counting argument in the proof of Proposition 5.10, this number is simply equal to $\prod_{i=3}^n \pi^{(\epsilon_i)}_{r_i,s_i}(m_i)$. Combining with (5.24), and recalling the consistency relations (5.21), we obtain in the case $w_1^{(0)}=1$

$$\mathrm{Card}\left(\mathcal{F}_{(w_1^{(0)}=1,w_2^{(0)},\ldots,w^{(0)}_{r_0+s_0+1})}(m_3,\ldots,m_n)\right)=\sum_{\substack{\epsilon_3,\ldots,\epsilon_n=0,1\\ r_3,\ldots,r_n,s_3,\ldots,s_n\geqslant 0\\ \sum_{i=3}^n \epsilon_i=r_0+\sum_{i=3}^n r_i+1\\ \sum_{i=3}^n \bar{\epsilon}_i=s_0+\sum_{i=3}^n s_i}}\left(r_0+\sum_{i=3}^n r_i\right)!\left(\epsilon_3 s_0+\sum_{i=3}^n \epsilon_i s_i\right)\left(s_0+\sum_{i=3}^n s_i-1\right)!\prod_{i=3}^n \pi^{(\epsilon_i)}_{r_i,s_i}(m_i)\,, \tag{5.25}$$

and in the case $w_1^{(0)}=0$

$$\mathrm{Card}\left(\mathcal{F}_{(w_1^{(0)}=0,w_2^{(0)},\ldots,w^{(0)}_{r_0+s_0+1})}(m_3,\ldots,m_n)\right)=\sum_{\substack{\epsilon_3,\ldots,\epsilon_n=0,1\\ r_3,\ldots,r_n,s_3,\ldots,s_n\geqslant 0\\ \sum_{i=3}^n \epsilon_i=r_0+\sum_{i=3}^n r_i\\ \sum_{i=3}^n \bar{\epsilon}_i=s_0+\sum_{i=3}^n s_i+1}}\left(s_0+\sum_{i=3}^n s_i\right)!\left(\bar{\epsilon}_3 r_0+\sum_{i=3}^n \bar{\epsilon}_i r_i\right)\left(r_0+\sum_{i=3}^n r_i-1\right)!\prod_{i=3}^n \pi^{(\epsilon_i)}_{r_i,s_i}(m_i)\,. \tag{5.26}$$

Note that these expressions do not depend on the precise sequence $(w_1^{(0)},\ldots,w^{(0)}_{r_0+s_0+1})$ but only on the values of $r_0=\sum_{j=2}^{r_0+s_0+1} w_j^{(0)}$, of $s_0=\sum_{j=2}^{r_0+s_0+1}\bar{w}_j^{(0)}$ and of $w_1^{(0)}$. This is expected since permuting the terms of the sequence $(w_j^{(0)})_{j=2,\ldots,n}$ at fixed r_0 and s_0 simply amounts to changing the order of the last r_0+s_0 trees in the forest $\mathbf{f}$, a harmless operation as far as enumeration is concerned. We may therefore use the counting formulas (5.7) and (5.8) for petal necklaces $\mathbf{m}_{12}$ with fixed r_0, s_0 and with a fixed value $w_1^{(0)}=1$ and $w_1^{(0)}=0$ respectively. The expression (5.20) for $N_{0,n}(2m_1,\ldots,2m_n)$ is obtained by inserting (5.25) into (5.7) and (5.26) into (5.8), adding these two contributions and finally summing over r_0 and s_0, which trivially amounts to replacing in (5.25) and (5.26) each occurrence of r_0 by r_1+r_2 and each occurrence of s_0 by s_1+s_2, and summing over $r_1,r_2,s_1,s_2\geqslant 0$. This ends the proof of Proposition 5.12. □

Even though it is not apparent, the right-hand side of (5.20) turns out to be, as expected, symmetric upon permuting the m_i's for i in the whole set $\{1,\ldots,n\}$. This property is shown in Appendix B where, after some algebraic manipulations, this quantity is given a manifestly symmetric form, see for instance (B.4) or (B.17). Using this result, we arrive at the following unified theorem encompassing all the main theorems of the paper (Theorems 2.3, 2.8 and 2.12):

Theorem 5.13. *For $n\geqslant 3$ and $m_1,m_2,\ldots,m_n\in\mathbb{Z}_{\geqslant 0}/2$, not all equal to zero, the number $N_{0,n}(2m_1,2m_2,\ldots,2m_n)$ of planar tight maps with n boundaries labeled from 1 to n with re-*

spective lengths $2m_1, 2m_2, \ldots, 2m_n$ *is given by the symmetric quasi-polynomial*

$$
\begin{aligned}
&N_{0,n}(2m_1, 2m_2, \ldots, 2m_n) = \\
&\qquad \sum_{\left(\substack{\epsilon_1,\ldots,\epsilon_n \\ r_1,\ldots,r_n \\ s_1,\ldots,s_n}\right) \in I_n} \left(\sum_{i=1}^n r_i\right)! \left(\sum_{i=1}^n \epsilon_i s_i\right) \left(\sum_{i=1}^n s_i - 1\right)! \prod_{i=1}^n \pi^{(\epsilon_i)}_{r_i,s_i}(m_i) \\
&\qquad +(n-3)! \sum_{\substack{(r_1,\ldots,r_n)\in\mathbb{Z}^n_{\geqslant 0} \\ r_1+\cdots+r_n=n-3}} \sum_{1\leqslant j<\ell\leqslant n} \pi^{(0)}_{r_j,0}(m_j)\pi^{(0)}_{r_\ell,0}(m_\ell) \prod_{\substack{i=1 \\ i\neq j,\ell}}^n \pi^{(1)}_{r_i,0}(m_i) \qquad (5.27) \\
&\qquad +(n-3)! \sum_{\substack{(r_1,\ldots,r_n)\in\mathbb{Z}^n_{\geqslant 0} \\ r_1+\cdots+r_n=n-3}} \pi^{(-1)}_{r_1,0}(m_1) \prod_{i=2}^n \pi^{(1)}_{r_i,0}(m_i).
\end{aligned}
$$

with $\pi^{(\epsilon)}_{r,s}(m)$ *as in* (2.26) *and* I_n *as in* (2.29)*. Note that the sum in the last line is equal to the symmetric polynomial* $(n-3)!p_{n-3}(m_1, \ldots, m_n)$ *when all* m_i *are integers, and to zero otherwise, hence it is symmetric.*

Proof. Since $N_{0,n}(2m_1, 2m_2, \ldots, 2m_n)$ is (by definition) symmetric in $m_1, m_2, \ldots, m_n$ and so is the right-hand side of (5.27), we may assume without loss of generality that $m_1 > 0$ and either $m_2 > 0$ or $m_2 = \cdots = m_n = 0$. In the first case, Proposition 5.12 and the identification of the right-hand sides of (5.20) and (5.27) proved in Proposition B.1 of Appendix B allows to conclude. In the second case, both sides of the equation reduce to the univariate quasi-polynomial $(n-3)!\pi^{(-1)}_{n-3,0}(m_1)$, equal to $(n-3)!p_{n-3}(m_1)$ if m_1 is a positive integer, and to zero otherwise. □

Let us now explain how to recover Theorems 2.3, 2.8 and 2.12 from Theorem 5.13. As in Remark 2.13, let us denote by k the number of half-integers among $m_1, \ldots, m_n$. From the general property that $\pi^{(\epsilon_i)}_{r_i,s_i}(m_i)$ is non-zero only if $m_i - \frac{s_i+1+\epsilon_i}{2} \in \mathbb{Z}$, we see that the sum in the third line of the right-hand side of (5.27) is non-zero if and only if all the m_i's are integers, i.e. the map is bipartite ($k = 0$), in which case it reduces precisely to $(n-3)!p_{n-3}(m_1, \ldots, m_n)$. Similarly, the sum in the second line is non-zero if and only if exactly two of the m_i's, say for $i = i_1$ and $i = i_2$ ($1 \leqslant i_1 < i_2 \leqslant n$), are half-integers, i.e. the map is quasi-bipartite ($k = 2$), in which case it reduces to $(n-3)!\tilde{p}_{n-3}(m_{i_1}, m_{i_2}; m_1, \ldots, \check{m}_{i_1}, \ldots, \check{m}_{i_2}, \ldots, m_n)$ where $\check{m}_{i_\ell}$ means that the argument m_{i_ℓ} is omitted. Therefore, for $k = 1$ and $k \geqslant 3$, the only possibly non-zero term is the sum in the first line, which matches precisely the right hand side of (2.28). This proves Theorem 2.12 (where in practice, only even values of k yield a non-zero result, as it should). In order to recover Theorems 2.3 and 2.8, it only remains to check that the sum in the first line of the right-hand side of (5.27) vanishes in the bipartite and quasi-bipartite cases. Note that if m_i is an integer, $\pi^{(\epsilon_i)}_{r_i,s_i}(m_i)$ is non-zero only if s_i and $1 - \epsilon_i$ have the same parity, a property which, for $s_i \geqslant 0$ and $1 - \epsilon_i \in \{0, 1\}$ implies that $s_i \geqslant 1 - \epsilon_i$. If all the m_i's are integers, the required constraint $\sum_{i=1}^n (1-\epsilon_i) = \sum_{i=1}^n s_i + 2$ in the definition of the set I_n cannot be fulfilled, hence the sum vanishes. If exactly two of the m_i's, say for $i = i_1$ and $i = i_2$, are half-integers, this same constraint imposes that $\epsilon_{i_1} = \epsilon_{i_2} = 0$ and $s_{i_1} = s_{i_2} = 0$ while $s_i = 1-\epsilon_i$ for all $i \neq i_1, i_2$, hence $(\sum_{i=1}^{n} \epsilon_i s_i) = \sum_{i \neq i_1, i_2} (1-\epsilon_i)\epsilon_i = 0$ and the first sum again vanishes. To summarize, for an even value of k, exactly one of the three lines in the right-hand side of (5.27) is not identically zero. Each line corresponds to one of the mutually exclusive situations $k \geqslant 4$ (first line), $k = 2$ (second line) and $k = 0$ (third line), corresponding to Theorem 2.12, 2.8 and 2.3 respectively.

6. Conclusion

In this paper, we have provided an explicit expression for the planar lattice count quasi-polynomial $N_{0,n}(2m_1, \ldots, 2m_n)$, by extending the slice decomposition to the case of planar tight maps. Note that, in contrast with previous work such as [BG14, Bud22b], our approach is not based on generating functions but rather on direct bijective enumeration techniques.

We note that our most general formula, stated in Theorem 5.13, still involves a complicated sum over a no less complicated set I_n. As discussed in the Appendix B, there are in fact many identities that can be used to rewrite it, and it is not impossible that it admits a significantly simpler expression yet to be unveiled. A similar remark applies to our extension of Tutte's slicings formula given in Theorem 3.2.

We believe that the methodology of Sections 4 and 5 is quite robust and may be adapted to other map enumeration problems. A first problem one may think of is a model of maps with *continuous* edge lengths, whose set can be associated with a natural volume measure. The volumes of such measures have been considered by Kontsevich [Kon92], and, as noted by Norbury, they correspond to the homogeneous top degree part of the lattice count polynomials, see [Nor10, Theorem 3]. Combining with our Theorem 2.3, we obtain the following result (see Norbury's paper for the definition of $V_{0,n}$).

Proposition 6.1. *The genus zero volume polynomial $V_{0,n}$ reads explicitly*

$$V_{0,n}(b_1, \ldots, b_n) = \frac{(n-3)!}{2^{2n-7}} \sum_{\substack{k_1, \ldots, k_n \geqslant 0 \\ k_1 + \cdots + k_n = n-3}} \left(\frac{b_1^{k_1} \cdots b_n^{k_n}}{k_1! \cdots k_n!} \right)^2 . \tag{6.1}$$

This amounts to the known expression for the genus zero intersection numbers, see for instance [LZ04, Proposition 4.6.10] and [Nor10, Corollary 1]. Still, a direct construction of maps with continuous edge lengths, using continuous paths and mimicking the above discrete paths encoding for slices, should also be possible and interesting, and we plan to investigate this question in future work. A second extension is motivated by the work of Budd [Bud22b] who generalized Norbury's results to the case of *irreducible maps*, i.e. maps with a girth constraint. The slice decomposition of these maps was discussed in [BG14] in the non-tight case, and we expect it to be adaptable to the tighness constraint. Finally, combining these two ideas, namely, passing to maps with continuous edge lengths *and* adding an irreducibility constraint, one obtains *irreducible metric maps* which have been considered by Budd [Bud22a], who showed that their volumes are related to the Weil–Petersson volumes of hyperbolic surfaces. We plan to investigate the slice decomposition of these maps, which might shed new light on these questions.

In another direction, the bijective techniques presented in this paper should pave the road to the study of continuum limits of random planar tight maps, when the number of faces tends to infinity. We believe in particular that, as soon as the face degrees are well-behaved in a certain sense, the Gromov–Hausdorff limit of appropriately renormalized planar tight maps, seen as discrete metric spaces, should be given by the *Brownian sphere*. To this purpose, the recent approach by Marzouk [Mar18, Mar22], dealing with limits of planar non necessarily tight maps with prescribed face degrees, should be particularly relevant.

Note that our approach is currently restricted to the planar (i.e. genus 0) case. This is a current limitation of the slice decomposition. We hope however that this limitation will be challenged by further investigations. A first result in this vein is the bijective study of pairs of pants (planar maps with three boundaries) done in [BGM22], and the fact that general surfaces can be decomposed into pairs of pants gives some support to our hope.

A. Enumeration of one- and two-type labeled plane forests

This appendix lists the forest enumeration results that we need in this paper. We consider *plane forests* (sequences of rooted plane trees) which are *labeled* (distinct labels are assigned to the vertices). We start with the case of forests with one type of vertices.

Proposition A.1. *Let $n, k_1, \ldots, k_n$ be non-negative integers such that $k_1 + \cdots + k_n < n$. Then, there are exactly $(n-1)!$ plane forests with n vertices labeled $\{1, \ldots, n\}$, such that vertex i has k_i children for all $i = 1, \ldots, n$ and such that vertex 1 appears in the first tree. (Such forests consist necessarily of $k_0 := n - k_1 - \cdots - k_n$ trees.)*

Proof. The case of trees ($k_0 = 1$) is given explicitly in [BM14, Section 5, Equation (18)]. It implies the general case since, for $k_0 > 1$, there is a straightforward bijection between the set of forests at hand and the set of labeled plane trees such that vertex 1 has $k_1 + k_0 - 1$ children, the number of children of the other vertices being unmodified. $\square$

Remark A.2. Proposition A.1 can alternatively be proved directly by exhibiting a bijection between the set of forests at hand and the set of cyclic orders on $\{1, \ldots, n\}$. Such a bijection is obtained by simply listing the vertex labels of a labeled plane forest in depth-first order, giving a linear, hence a cyclic, order on $\{1, \ldots, n\}$. Conversely, from a cyclic order and the data of the k_i's, we construct a conjugacy class of Łukasiewicz words—see e.g. [Sta99, Section 5.3]—whose letters are labeled and which contains exactly one word coding for a plane forest having vertex 1 in the first tree.

We now turn our attention to labeled plane forests with two types of vertices, say A and B. Our purpose is to enumerate such *two-type* forests in which, for every vertex, we prescribe not only its type but also the sequence formed by the types of all its children, read in the planarity order. Similar counting problems, for an arbitrary number of types, have been previously considered in the literature—see e.g. [BS13, BM14, CL16] and references therein—but since the general formulas are quite complicated we provide a self-contained derivation for two types. Our approach is closely related with that of Chottin [Cho81] who treated the case of two-type trees.

Handling forests with several components however involves an extra difficulty, which we circumvent by specializing the general multitype approach of Bacher and Schaeffer [BS13]. Note that the latter two references consider plane forests which are unlabeled, but adding vertex labels does not fundamentally change the problem since plane forests have no symmetries.

To state our result we need some definitions and notation. Let us consider a two-type plane forest whose vertices are labeled $\{1, \ldots, n\}$. For every vertex $i = 1, \ldots, n$, we denote by k_i its number of children, and we let $w^{(i)} = (w_0^{(i)}, w_1^{(i)}, \ldots, w_{k_i}^{(i)}) \in \{A, B\}^{k_i+1}$ be the sequence such that $w_0^{(i)}$ is the type of i and such that, for every $j = 1, \ldots, k_i$, $w_j^{(i)}$ is the type of the j-th child of i in the planarity order. We also define a sequence $w^{(0)} = (w_0^{(0)}, w_1^{(0)}, \ldots, w_{k_0}^{(0)})$ where k_0 is the number of trees of the forest, $w_0^{(0)}$ is a third type denoted O and, for $j = 1, \ldots, k_0$, $w_j^{(0)}$ is the type of the j-th *root*, i.e. of the root vertex of the j-th tree of the forest. In this sense, 0 can be seen as the label of a *super-root* of type O, which is the parent of all the roots (which have the usual types A or B).

The collection $\boldsymbol{w} = (w^{(0)}, w^{(1)}, \ldots, w^{(n)})$ is called the *type array* of the forest. Denoting by $[\cdot]$ the Iverson bracket ($[P]$ is equal to 1 if P is true, and to 0 otherwise), we have necessarily

$$\sum_{i=1}^{n} \left[w_0^{(i)} = A\right] = \sum_{i=0}^{n} \sum_{j=1}^{k_i} \left[w_j^{(i)} = A\right] := a \tag{A.1}$$

as seen by expressing in two different ways the number a of type A vertices. Similarly, the number b of type B vertices is given by

$$\sum_{i=1}^{n} \left[w_0^{(i)} = B\right] = \sum_{i=0}^{n} \sum_{j=1}^{k_i} \left[w_j^{(i)} = B\right] := b. \tag{A.2}$$

Note that, by adding these two equations, we obtain the relation $n = k_0 + k_1 + \cdots + k_n$ already seen in the context of one-type forests in Proposition A.1. An array $\boldsymbol{w} = (w_j^{(i)})_{j=0,\ldots,k_i}^{i=0,\ldots,n}$ with $w_0^{(0)} = O$ and $w_j^{(i)} \in \{A, B\}$ for i, j not both zero is said *consistent* if it satisfies (A.1) and (A.2).

Proposition A.3. *Let $\boldsymbol{w}$ be a consistent array, and let a and b be as defined in* (A.1) *and* (A.2)*, respectively. Define furthermore the integers*

$$a^O := \sum_{j=1}^{k_0} \left[w_j^{(0)} = A\right], \qquad a^B := \sum_{i=1}^{n} \sum_{j=1}^{k_i} \left[w_0^{(i)} = B\right] \left[w_j^{(i)} = A\right], \tag{A.3}$$

$$b^O := \sum_{j=1}^{k_0} \left[w_j^{(0)} = B\right], \qquad b^A := \sum_{i=1}^{n} \sum_{j=1}^{k_i} \left[w_0^{(i)} = A\right] \left[w_j^{(i)} = B\right], \tag{A.4}$$

namely a^O and b^O correspond to the number of type A and type B roots, respectively, while a^B is the number of type A vertices with a type B parent, and vice versa for b^A.

Then, we have the following enumerative formulas.

- *(General enumeration) The number of two-type labeled plane forests of type array $\boldsymbol{w}$—thus containing a type A and b type B vertices—is equal to*

$$\begin{cases} (a^O b^O + a^O b^A + a^B b^O)(a-1)!(b-1)! & \text{if } a > 0 \text{ and } b > 0, \\ a^O (a-1)! & \text{if } a > 0 \text{ and } b = 0, \\ b^O (b-1)! & \text{if } a = 0 \text{ and } b > 0, \\ 1 & \text{if } a = 0 \text{ and } b = 0. \end{cases} \tag{A.5}$$

- *(Constrained enumeration) Assume that, say, $a > 1$ and $w_0^{(1)} = A$, i.e. vertex 1 has type A. Then, the number of two-type labeled plane forests of type array $\boldsymbol{w}$ such that vertex 1 is in the first tree is equal to*

$$\begin{cases} (b^O + b^A)(a-1)!(b-1)! & \text{if } b > 0 \text{ and } w_1^{(0)} = A, \\ a^B (a-1)!(b-1)! & \text{if } b > 0 \text{ and } w_1^{(0)} = B, \\ (a-1)! & \text{if } b = 0. \end{cases} \tag{A.6}$$

(The first and second cases correspond to a first root of type A and B, respectively.)

Proof. Let us first note that the cases where a or b vanish follow immediately from Proposition A.1 (for the general enumeration, we perform a circular permutation of the trees to lift the constraint that vertex 1 is in the first tree, giving the extra factor a^O or b^O). Hence, we assume from now on that a and b are both positive.

We claim that the general enumeration formula follows from the constrained one. Indeed, we may partition the set of forests of type array $\boldsymbol{w}$ according to the index $j = 1, \ldots, k_0$ of the tree containing vertex 1. Upon doing a circular permutation of the trees, we deduce from (A.6) that, for $w_0^{(1)} = A$, the number of forests with a given value of j is equal to $(b^O + b^A)(a-1)!(b-1)!$ if $w_j^{(0)} = A$, and to $a^B(a-1)!(b-1)!$ if $w_j^{(0)} = B$. As the first (resp. second) case occurs for a^O (resp. b^O) values of j, summing over j gives the first line of (A.5). The case $w_0^{(1)} = B$ is deduced by exchanging the roles of A and B.

It remains to prove the constrained enumeration formula. We will do so by giving an algorithm to construct any forest of type array $\boldsymbol{w}$, where it will be manifest that the number of possibilities is given by (A.6). The algorithm is easier to visualize if, instead of working with labels $0, 1, \ldots, n$, we relabel the vertices as $O, A_1, \ldots, A_a, B_1, \ldots, B_b$ to make their type apparent (as the type array $\boldsymbol{w}$ is fixed, this may be done by fixing a bijection between $\{0, 1, \ldots, n\}$ and $\{O, A_1, \ldots, A_a, B_1, \ldots, B_b\}$ hence does not change the counting problem). The general idea, illustrated on Figure A.1, is to proceed in several stages, by first "assembling" the type B vertices together, before dealing with the types A and O vertices. Let us describe the different stages of the algorithm in detail.

I. We start with isolated vertices $O, A_1, \ldots, A_a, B_1, \ldots, B_b$, to which we attach sequences of *dangling A-edges* and *B-edges*: these dangling edges will be connected later to type A and B vertices, respectively. The type array $\boldsymbol{w}$ tells us precisely the sequence which we have to attach to each vertex. We define a total order on the dangling edges by listing

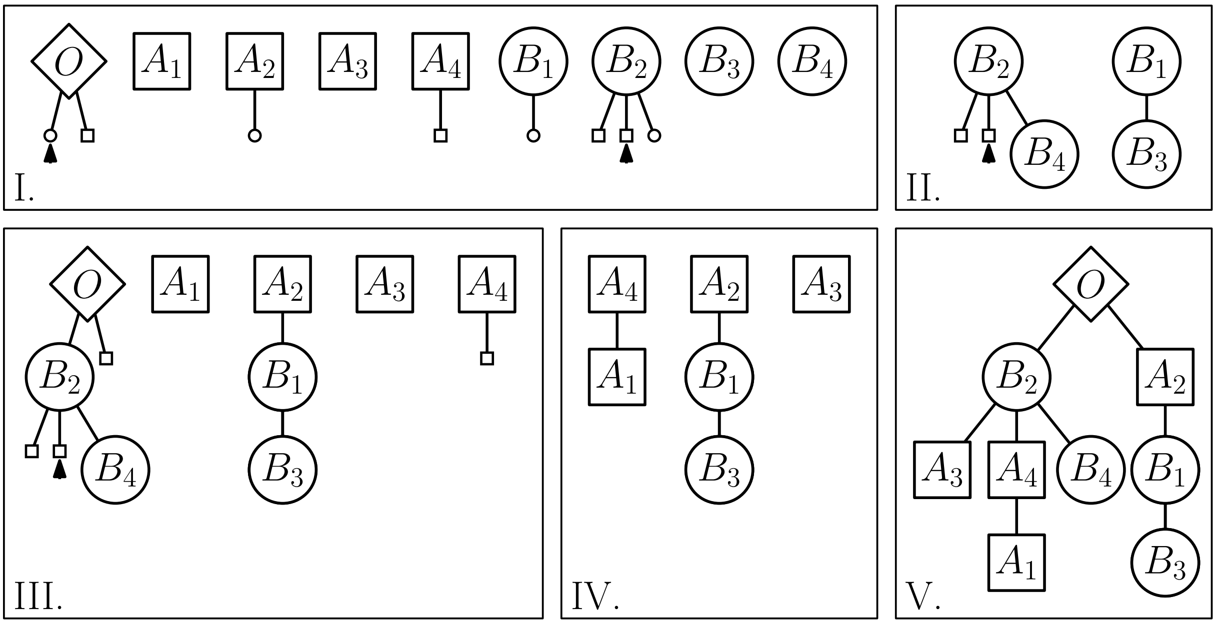

Figure A.1: Construction of a two-type forest in the case $w_0^{(1)} = B$. See the main text for a precise description of the stages I-V. We display the type A (resp. B) vertices and dangling A-edges (resp. B-edges) with squares (resp. circles). The distinguished dangling edges are indicated with arrows.

those incident to O, then those incident to A_1, etc. We distinguish the *first* edge incident to O: it has type $w_0^{(1)}$. We then distinguish another dangling edge of the opposite type:

- if $w_0^{(1)} = A$, then we distinguish a B-edge incident either to O or to a type A vertex: there are $b^O + b^A$ possible choices,
- if $w_0^{(1)} = B$, then we distinguish an A-edge incident to a type B vertex: there are a^B possible choices.

II. We form a plane forest with the vertices $B_1, \dots, B_b$, by attaching them together via their incident B-edges. By Proposition A.1, there are $(b-1)!$ ways to do so (with B_1 in the first tree). A simple computation shows that the resulting forest is made of $b^O + b^A$ trees. If $w_0^{(1)} = B$, we permute the trees circularly so that the tree containing the distinguished A-edge comes first.

III. We attach the roots of the forest constructed at stage II to the B-edges (in number $b^O + b^A$) dangling from $O, A_1, \dots, A_a$. Precisely, we attach the root of the first tree to the distinguished B-edge, and we then proceed circularly using the order on dangling edges defined at stage I. Note that all the B-edges have now been matched to type B vertices. We end up with a sequence of *supernodes*, which are trees with roots $O, A_1, \dots, A_a$, subtrees made of type B vertices, and dangling A-edges. Observe that the distinguished A-edge always belongs to the supernode with root O (precisely, the first edge incident to O is either the distinguished A-edge when $w_0^{(1)} = A$, or leads to a subtree which contains it when $w_0^{(1)} = B$).

IV. Viewing the supernodes as compound vertices, we form a plane forest with a^O trees by assembling the supernodes with roots $A_1, \dots, A_a$ via their dangling A-edges. Note that the number of "children" of a supernode is prescribed by its number of dangling A-edges. By Proposition A.1, there are $(a-1)!$ ways to do so (with A_1 in the first tree).

V. We then complete the construction by attaching these trees to the A-edges of the supernode with root O, starting with the first tree which we attach to the distinguished A-edge, and then proceeding circularly using the order of stage I. All dangling edges have now been matched, and vertex 1 is in the first tree by construction.

It is plain from stage I that we obtain a forest of type array $\boldsymbol{w}$. Furthermore, each such forest is obtained in precisely one way, as we may check that it is obtained for a unique choice of the second distinguished dangling edge at stage I [12] and of the one-type forests at stages II and IV. □

Remark A.4. The factor $a^O b^O + a^O b^A + a^B b^O$ in (A.5) corresponds to a sum over the three Cayley trees on the set $\{O, A, B\}$. For k types of vertices there would be as many terms as Cayley trees on a set with $k+1$ elements [BS13].

B. Symmetrizing the planar lattice count quasi-polynomials

Let us fix an integer $n \geqslant 3$ and denote by $\Pi(m_1, \dots, m_n)$ the right-hand side of (5.20). It is manifest that it is a quasi-polynomial in $2m_1, \dots, 2m_n$ of degree $n-3$, since $\pi_{r,s}^{(\epsilon)}(m)$ is a univariate quasi-polynomial in $2m$ of degree $r+s$. The purpose of this appendix is to show that $\Pi(m_1, \dots, m_n)$ is in fact *symmetric* in $m_1, \dots, m_n$, which we will do by rewriting it in a manifestly symmetric form. Note that the expression (5.20) displays a symmetry in $m_4, \dots, m_n$ only. Here we assume only that $m_i \in \mathbb{Z}/2$, $i = 1, \dots, n$ without further restriction.

It is useful to introduce compact notations for the high-dimensional sums appearing in (5.20). Let I be the finite subset of $\{0,1\}^n \times \mathbb{Z}^n_{\geqslant 0} \times \mathbb{Z}^n_{\geqslant 0}$ defined by

$$I := \left\{ \begin{pmatrix} \epsilon_1, \dots, \epsilon_n \\ r_1, \dots, r_n \\ s_1, \dots, s_n \end{pmatrix} : \begin{array}{c} \sum_{i=1}^n \epsilon_i = \sum_{i=1}^n r_i + 1 \\ \sum_{i=1}^n (1-\epsilon_i) = \sum_{i=1}^n s_i + 2 \end{array} \right\}. \tag{B.1}$$

Given a tuple in I, we set

$$r := \sum_{i=1}^n r_i, \qquad s := \sum_{i=1}^n s_i, \tag{B.2}$$

[12] Precisely, if $w_1^{(0)} = A$ then the distinguished B-edge corresponds to the edge closest to B_1 attached to a parent not of type B, and if $w_1^{(0)} = B$ then the distinguished A-edge corresponds to the parent edge of the oldest type A ancestor of A_1.

and note that $r+s=n-3$ by the definition of I. Let $I_{r\geqslant 1}$, $I_{r=0}$, $I_{s\geqslant 1}$, and $I_{s=0}$, be the subsets of I consisting of tuples such that $r\geqslant 1$, $r=0$, $s\geqslant 1$, and $s=0$, respectively. Note that the set I_n introduced in (2.29) corresponds in our present notations to $I_{s\geqslant 1}$. Note also that $I_{r=0}$ consists of tuples such that $r_i=0$ for all i, exactly one ϵ_i is equal to 1, and $s_1+\cdots+s_n=s=n-3$. Similarly, $I_{s=0}$ consists of tuples such that $s_i=0$ for all i, exactly two ϵ_i are equal to 0, and $r_1+\cdots+r_n=r=n-3$.

For $e_1,e_2\in\{0,1\}$, we let $I^{(e_1,e_2)}$ be the subset of I consisting of tuples such that $\epsilon_1=e_1$ and $\epsilon_2=e_2$. We also allow for the value $e_1=-1$, which means that we consider tuples such that $\epsilon_1=-1$, keeping the sum condition in (B.1) unchanged. The notations $I^{(e_1,e_2)}_{r\geqslant 1}$, etc, should hopefully be self-explanatory. Note that $I^{(-1,1)}_{s=0}$ consists of tuples such that $s_i=0$ for all i, $\epsilon_1=-1$ and $\epsilon_i=1$ for all $i\geqslant 2$, and $r_1+\cdots+r_n=r=n-3$. We finally use, for $\epsilon\in\{0,1\}$, the shorthand notation $\bar\epsilon:=1-\epsilon$ (which we shall never use for $\epsilon=-1$).

Armed with all these notations, we can rewrite $\Pi(m_1,\dots,m_n)$[13] as

$$\begin{aligned}\Pi(m_1,\dots,m_n) = &\sum_{I^{(-1,1)}_{s\geqslant 1}\cup I^{(0,0)}_{s\geqslant 1}} r!(s-1)!\left(\epsilon_3(s_1+s_2)+\sum_{j=3}^n \epsilon_j s_j\right)\prod_{i=1}^n \pi^{(\epsilon_i)}_{r_i,s_i}(m_i)\\ &+\sum_{I^{(0,1)}_{r\geqslant 1}\cup I^{(1,0)}_{r\geqslant 1}} (r-1)!s!\left(\bar\epsilon_3(r_1+r_2)+\sum_{j=3}^n \bar\epsilon_j r_j\right)\prod_{i=1}^n \pi^{(\epsilon_i)}_{r_i,s_i}(m_i)\\ &+\sum_{I^{(-1,1)}_{s=0}\cup I^{(0,0)}_{s=0}\cup I^{(0,1)}_{r=0}\cup I^{(1,0)}_{r=0}} r!s!\prod_{i=1}^n \pi^{(\epsilon_i)}_{r_i,s_i}(m_i).\end{aligned} \tag{B.3}$$

Here, the first (resp. second) sum corresponds to the first (resp. second) sum in (5.20) when at least one of the s_i's (resp. one of the r_i's) is non zero, hence when $s\geqslant 1$ (resp. $r\geqslant 1$); the third sum accounts for the conventional values in (5.20):

$$\left(\epsilon_3(s_1+s_2)+\sum_{i=3}^n \epsilon_i s_i\right)\left(\sum_{i=1}^n s_i-1\right)! \to 1 = s!$$

when all the s_i's are zero (or equivalently $s=0$) and

$$\left(\bar\epsilon_3(r_1+r_2)+\sum_{i=3}^n \bar\epsilon_i r_i\right)\left(\sum_{i=1}^n r_i-1\right)! \to 1 = r!$$

when all the r_i's are zero (or equivalently $r=0$).

The main result of this appendix is:

[13] Here we use the shorthand notation $\sum_J(\cdot)$ for $\sum_{\begin{pmatrix}\epsilon_1,\dots,\epsilon_n\\ r_1,\dots,r_n\\ s_1,\dots,s_n\end{pmatrix}\in J}(\cdot)$.

Proposition B.1. *The quasi-polynomial* $\Pi(m_1,\ldots,m_n)$ *is symmetric in* $m_1,\ldots,m_n$ *and admits the expression*

$$\Pi(m_1,\ldots,m_n) = \sum_{I_{s\geqslant 1}} r!(s-1)!\left(\sum_{j=1}^{n}\epsilon_j s_j\right)\prod_{i=1}^{n}\pi_{r_i,s_i}^{(\epsilon_i)}(m_i) \quad + \sum_{I_{s=0}^{(-1,1)}\cup I_{s=0}} r!s!\prod_{i=1}^{n}\pi_{r_i,s_i}^{(\epsilon_i)}(m_i). \quad \text{(B.4)}$$

Note that the sum over $I_{s=0}^{(-1,1)}$ *is equal to* $(n-3)!\sum_{r_1+\cdots+r_n=n-3}\pi_{r_1,0}^{(-1)}(m_1)\prod_{i=2}^{n}\pi_{r_i,0}^{(1)}(m_i)$ *which is equal to the symmetric polynomial* $(n-3)!p_{n-3}(m_1,\ldots,m_n)$ *when all* m_i *are integers, and to zero otherwise, hence it is symmetric like the rest.*

The expression above for $\Pi(m_1,\ldots,m_n)$ is precisely the right-hand side of (5.27). Indeed, we already noticed that $I_{s\geqslant 1}$ is nothing but the set I_n defined in (2.29), hence the sum over $I_{s\geqslant 1}$ gives the first term in the right-hand side of (5.27). Furthermore, the set $I_{s=0}$ consists of tuples such that $s_i=0$ for all i, exactly two ϵ_i are equal to 0, and $r_1+\cdots+r_n=r=n-3$, hence the sum over $I_{s=0}$ corresponds to the second term in (5.27). Finally, the sum over $I_{s=0}^{(-1,1)}$ corresponds to the third term in (5.27).

In order to prove Proposition B.1, we will first record the following:

Proposition B.2. *The univariate polynomials* $\pi_{r,s}^{(\epsilon)}(m)$ *satisfy the relations*

$$s\pi_{r,s}^{(\epsilon)}(m) = (r+1)\pi_{r+1,s-1}^{(\epsilon+1)}(m), \quad \text{(B.5)}$$

$$\pi_{r,s}^{(1)}(m) = \pi_{r,s}^{(-1)}(m) + \pi_{r-1,s}^{(-1)}(m), \quad \text{(B.6)}$$

$$s\pi_{r,s}^{(1)}(m) = (r+1)\pi_{r+1,s-1}^{(0)}(m) + r\pi_{r,s-1}^{(0)}(m), \quad \text{(B.7)}$$

valid for all $r,s\geqslant 0$ *and all integer* ϵ*, with the convention that* $\pi_{-1,s}^{(\epsilon)}=\pi_{r,-1}^{(\epsilon)}=0$.

Proof. The first relation follows immediately from the mere definition (2.26) of $\pi_{r,s}^{(\epsilon)}(m)$. For the two other ones, we make use of the “dilaton-like” equation

$$kp_{k,e+2}(m) = kp_{k,e}(m) + (k-e)p_{k-1,e}(m) \quad \text{(B.8)}$$

which can be checked from the definition (2.25) of $p_{k,e}(m)$ and is valid for all $k\geqslant 0$, with the convention that $p_{-1,e}(m)=0$. This dilaton-like equation implies immediately (B.6), and to get (B.7) we first apply (B.5) at $\epsilon=1$ then the dilaton-like equation to go back from $\pi^{(2)}$'s to $\pi^{(0)}$'s. □

Lemma B.3 (Transmutation relations)**.** *For any* $j=1,\ldots,n$*, we have*

$$\sum_{I_{s\geqslant 1}} r!(s-1)!(\bar{\epsilon}_j s_j)\prod_{i=1}^{n}\pi_{r_i,s_i}^{(\epsilon_i)}(m_i) = \sum_{I_{r\geqslant 1}}(r-1)!s!(\epsilon_j r_j)\prod_{i=1}^{n}\pi_{r_i,s_i}^{(\epsilon_i)}(m_i). \quad \text{(B.9)}$$

We also have the four identities

$$\sum_{I^{(-1,1)}_{s\geqslant 1}} r!(s-1)!\,\epsilon_3 s_1 \prod_{i=1}^{n} \pi^{(\epsilon_i)}_{r_i,s_i}(m_i) = \sum_{I^{(0,1)}_{r\geqslant 1}} (r-1)!s!\,\epsilon_3 r_1 \prod_{i=1}^{n} \pi^{(\epsilon_i)}_{r_i,s_i}(m_i), \tag{B.10}$$

$$\begin{aligned}\sum_{I^{(-1,1)}_{s\geqslant 1}} r!(s-1)!\,\epsilon_3 s_2 \prod_{i=1}^{n} \pi^{(\epsilon_i)}_{r_i,s_i}(m_i) &\overset{(*)}{=} \sum_{I^{(1,-1)}_{s\geqslant 1}} r!(s-1)!\,\epsilon_3 s_2 \prod_{i=1}^{n} \pi^{(\epsilon_i)}_{r_i,s_i}(m_i) \\ &= \sum_{I^{(1,0)}_{r\geqslant 1}} (r-1)!s!\,\epsilon_3 r_2 \prod_{i=1}^{n} \pi^{(\epsilon_i)}_{r_i,s_i}(m_i),\end{aligned} \tag{B.11}$$

$$\sum_{I^{(1,0)}_{s\geqslant 1}} (r-1)!s!\,\bar\epsilon_3 r_1 \prod_{i=1}^{n} \pi^{(\epsilon_i)}_{r_i,s_i}(m_i) = \sum_{I^{(0,0)}_{r\geqslant 1}} r!(s-1)!\,\bar\epsilon_3 s_1 \prod_{i=1}^{n} \pi^{(\epsilon_i)}_{r_i,s_i}(m_i), \tag{B.12}$$

$$\sum_{I^{(0,1)}_{s\geqslant 1}} (r-1)!s!\,\bar\epsilon_3 r_2 \prod_{i=1}^{n} \pi^{(\epsilon_i)}_{r_i,s_i}(m_i) = \sum_{I^{(0,0)}_{r\geqslant 1}} r!(s-1)!\,\bar\epsilon_3 s_2 \prod_{i=1}^{n} \pi^{(\epsilon_i)}_{r_i,s_i}(m_i). \tag{B.13}$$

Finally, for $j \geqslant 3$ we have

$$\sum_{I^{(-1,1)}_{s\geqslant 1}} r!(s-1)!(\epsilon_j s_j) \prod_{i=1}^{n} \pi^{(\epsilon_i)}_{r_i,s_i}(m_i) = \sum_{I^{(1,1)}_{r\geqslant 1}} (r-1)!s!(\bar\epsilon_j r_j) \prod_{i=1}^{n} \pi^{(\epsilon_i)}_{r_i,s_i}(m_i). \tag{B.14}$$

Proof. In the left-hand side of (B.9), the only contributing tuples are those with $\epsilon_j = 0$. By applying the relation (B.5) to the factor $s_j \pi^{(0)}_{r_j,s_j}(m_j)$ appearing in the product, and by performing the change of variables $\bar\epsilon_j \to \epsilon_j$, $r_j + 1 \to r_j$, $s_j - 1 \to s_j$ (leaving all other elements of the tuples unchanged), we obtain precisely the nonzero terms of the right-hand side.

The proofs of the identities (B.10) and (B.12) are entirely similar, applying now (B.5) to the factor $s_1 \pi^{(\epsilon_1)}_{r_1,s_1}(m_1)$ (with $\epsilon_1 = -1, 0$) appearing in the left-hand side of (B.10) (with $\epsilon_1 = -1$) and in the right-hand side of (B.12) (with $\epsilon_1 = 0$). For (B.11) and (B.13), we proceed in the same way (now with $\epsilon_2 = -1, 0$), after noting that that it is possible to first replace $I^{(-1,1)}_{s\geqslant 1}$ by $I^{(1,-1)}_{s\geqslant 1}$ in (B.11) (as indicated by the $\overset{(*)}{=}$ sign) using (B.6) and appropriate changes of variables.

For (B.14), we apply (B.7) to the factor $s_j \pi^{(1)}_{r_j,s_j}(m_j)$ in the left-hand side (as only the tuples with $\epsilon_j = 1$ contribute), and we apply (B.6) to the factor $\pi^{(1)}_{r_1,s_1}(m_1)$ in the right-hand side. Appropriate changes of variables then show that the difference vanishes. □

Proof of Proposition B.1. Note that the left-hand sides of (B.10)-(B.13) all appear in (B.3), up to ϵ_3 or $\bar\epsilon_3$ prefactors. Changing them into the corresponding right-hand sides, and using $\epsilon_3 + \bar{\epsilon}_3 = 1$, allows to write (B.3) as

$$\begin{aligned}\Pi(m_1,\dots,m_n) &= \sum_{I^{(-1,1)}_{s\geqslant 1}} r!(s-1)!\left(\sum_{j=3}^n \epsilon_j s_j\right)\prod_{i=1}^n \pi^{(\epsilon_i)}_{r_i,s_i}(m_i)\\ &+ \sum_{I^{(0,0)}_{s\geqslant 1}} r!(s-1)!\left(s_1+s_2+\sum_{j=3}^n \epsilon_j s_j\right)\prod_{i=1}^n \pi^{(\epsilon_i)}_{r_i,s_i}(m_i)\\ &+ \sum_{I^{(0,1)}_{r\geqslant 1}\cup I^{(1,0)}_{r\geqslant 1}} (r-1)!s!\left(\sum_{j=1}^n \bar{\epsilon}_j r_j\right)\prod_{i=1}^n \pi^{(\epsilon_i)}_{r_i,s_i}(m_i)\\ &+ \sum_{I^{(-1,1)}_{s=0}\cup I^{(0,0)}_{s=0}\cup I^{(0,1)}_{r=0}\cup I^{(1,0)}_{r=0}} r!s!\prod_{i=1}^n \pi^{(\epsilon_i)}_{r_i,s_i}(m_i).\end{aligned} \tag{B.15}$$

By the transmutation relation (B.14), we may replace the sum over $I^{(-1,1)}_{s\geqslant 1}$ in the first line by a sum over $I^{(1,1)}_{r\geqslant 1}$ in the third line. This gives a sum over $I^{(0,1)}_{r\geqslant 1}\cup I^{(1,0)}_{r\geqslant 1}\cup I^{(1,1)}_{r\geqslant 1}$ which we can rewrite as a sum over $I_{r\geqslant 1}$ minus a sum over $I^{(0,0)}_{r\geqslant 1}$. We claim that this latter sum will almost cancel the sum over $I^{(0,0)}_{s\geqslant 1}$ in the second line. Indeed, by writing $s_1+s_2+\sum_{j=3}^n \epsilon_j s_i = s - \sum_{j=3}^n \bar{\epsilon}_j s_j$ in the sum over $I^{(0,0)}_{s\geqslant 1}$, and $\sum_{j=1}^n \bar{\epsilon}_j r_j = r - \sum_{j=3}^n \epsilon_j r_j$ in the sum over $I^{(0,0)}_{r\geqslant 1}$, we see using the transmutation relation (B.9) that their difference evaluates to

$$\left(\sum_{I^{(0,0)}_{s\geqslant 1}} - \sum_{I^{(0,0)}_{r\geqslant 1}}\right) r!s!\prod_{i=1}^n \pi^{(\epsilon_i)}_{r_i,s_i}(m_i) = \left(\sum_{I^{(0,0)}_{r=0}} - \sum_{I^{(0,0)}_{s=0}}\right) r!s!\prod_{i=1}^n \pi^{(\epsilon_i)}_{r_i,s_i}(m_i). \tag{B.16}$$

Observe that the sum over $I^{(0,0)}_{s=0}$ precisely cancels the one appearing in the last line of (B.15), and the sums over $I^{(0,0)}_{r=0}$, $I^{(0,1)}_{r=0}$ and $I^{(1,0)}_{r=0}$ combine to form a sum over $I_{r=0}$. We arrive at the expression

$$\begin{aligned}&\Pi(m_1,\dots,m_n) =\\ &\qquad \sum_{I_{r\geqslant 1}} (r-1)!s!\left(\sum_{j=1}^n \bar{\epsilon}_j r_j\right)\prod_{i=1}^n \pi^{(\epsilon_i)}_{r_i,s_i}(m_i) \quad + \sum_{I^{(-1,1)}_{s=0}\cup I_{r=0}} r!s!\prod_{i=1}^n \pi^{(\epsilon_i)}_{r_i,s_i}(m_i)\end{aligned} \tag{B.17}$$

which is interesting on its own, since it is already symmetric in $m_1,\dots,m_n$, see again the remark below (B.4). To obtain the wanted final expression, we write $\sum_{j=1}^n \bar{\epsilon}_j r_j = r - \sum_{j=1}^n \epsilon_j r_j$ in the sum over $I_{r\geqslant 1}$ and apply again the transmutation relation (B.9) and the identity (B.16), changing the sum over $I_{r\geqslant 1}$ and that over $I_{r=0}$ into the sum over $I_{s\geqslant 1}$ and that over $I_{s=0}$ of (B.4), leaving the sum over $I^{(-1,1)}_{s=0}$ unchanged. $\square$

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