An End-to-End Imagery-Based Modeling of Solving Geometric Analogy Problems

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Abstract

Geometric analogy problems remain an intriguing part of intelligence scales, which is closely correlated to many cognitive studies, such as perception, conception, memory, abstract and inductive reasoning. The problems not only target the most fundamental element — analogy-making — in human cognition, but also require integration of multiple components and stages: looking at the test booklet, thinking for a minute or two, and deciding the answer. Great efforts and achievements have been made to explain different individual aspects of this process. In this paper, we take a more holistic approach from the perspective of problem-solving, by modeling the entire process, from the moment the visual stimuli are received to the moment an answer is decided. Therefore, we explore how the final solution can be built upon visual inputs and necessary components that lie between the perceptual input and conceptual output. Particularly, we designed a novel similarity metric and a correspondence-finding method based on mapping and optimization. With these two basic blocks, we implemented a computational model, and report our initial results on a classical problem set.

Keywords: Analogy; Visual Imagery; Similarity

Introduction

If various cognitive gifts are the jewels in the crown of human intelligence, the analogy-making ability, as the core of cognition (Hofstadter, 2001), is undoubtedly one of the brightest ones. Analogy problems have always been an irreplaceable chapter in intelligence tests since they were first invented. The geometrically-flavored analogy problems are especially popular because they can be easily administered to people of different language, social, and cultural background. Figure 1 gives a simple example of geometric analogy problems. To solve this problem, a subject needs to select an answer from the five options so that the analogy — A is to B as C is to the answer — makes sense.

Imagine that a human subject solved the problem in the figure. She would probably describe it this way: in the first two images, the large circle surrounding the small circle moves down to surround the small square; thus, in the last two images, the large triangle surrounding the small square should move down to surround the small circle, which gives us Option 3 as the answer. This simple description perfectly explains what happens in the analogy, and most people would accept it as a reasonable answer. However, excessively relying on verbal protocols is inappropriate because the verbal description after the subject already solved the item tends to disguise the complexity of geometric analogy problems as a cognitive task. In the first place, the verbal description is more of a consequence of the solving process rather than the solving process per se. Second, the solving process might involve cognitive components that are not consciously accessible to the subject and thus barely reflected in the verbal description. Last but not least, the verbal description uses high-level concepts and ignores the potential difficulty of how these concepts are formed or chosen given the visual stimuli. This part is probably far more complicated and influential in geometric analogy tasks than one would expect (Barsalou, 1999; Hofstadter, 1979).

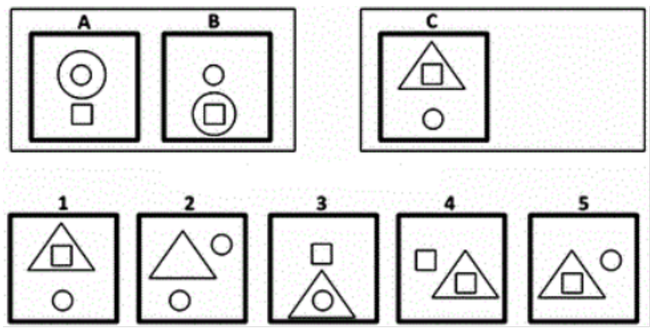

Figure 1: A simple example of geometric analogy problems (Lovett et al., 2009).

For these reasons, we take a more holistic end-to-end approach. An imagery-based model was designed and implemented to model the solving process of geometric analogy problems. We adopt one computational formulation of "visual imagery" as relying purely on pixel-level images, and operations over these images. While there are many other formulations of visual imagery that are possible, including those that incorporate higher-level features of various kinds, our work in this paper seeks to explore the extent to which pixel-based representations — i.e. those that contain information primarily about the raw spatial distribution of "ink" or brightness patterns in a visual image — might be sufficient for solving complex geometric analogy problems. In other words, what we propose here is one particular imagery-based strategy among many possible imagery-based strategies — and imagery-based strategies represent one particular class of representational strategies among many possible classes of strategies. Main contributions of this paper are:

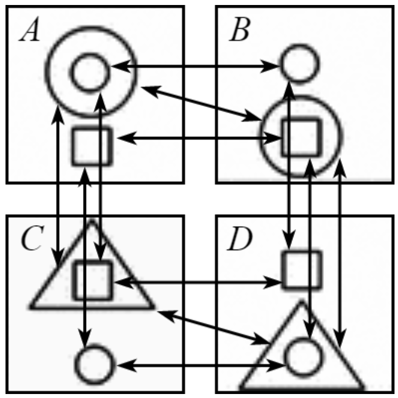

Figure 2: Correspondences between geometric objects.

- A new pixel-level similarity metric of geometric objects is designed to better serve the imagery-based reasoning process.
- An end-to-end imagery-based modeling of solving geometric analogy problems is implemented and evaluated against a classical problem set (Evans, 1968).

Intuitions Behind the Proposed Approach

Before we go into the technical details, it is a better idea to sketch the intuition behind the approach. Again, taking the item in Figure 1 as an example, recall that the verbal description of it entails high-level concepts such as "circle", "square", "triangle", "surrounding" and "moving down", and why and how these high-level concepts end up in the verbal description is not so self-evident, yet very crucial to the complete solving process.

Imagine that you see only one image, say the first image, of the item in Figure 1, without any contextual information. How would you describe it? Perhaps still using the same set of concepts. But, more probably, different people might describe it differently. For example, it looks like a symbol of lollipop. This leads us to consider the context-dependent nature of concept individuation. In our case of geometric analogy problems, the contextual information is the correspondence among geometric objects in the four images. Moreover, the concept individuation and the correspondence finding in solving process are better to be regarded as two viewpoints toward the same thing. For example, correspondences in our example problem can be depicted as in Figure 2, in which the horizontal ones are based on similarity and the vertical ones are based on spatial relation. One could say that the conceptual role of each object gives the correspondences, or, the other way around, that the correspondences determine the conceptual role of each object.

From a problem-solving perspective, when embedded into the incomplete analogy, a correct option would induce a self-consistent set of correspondences, or, equivalently, a self-consistent set of conceptual roles. This type of self-consistency can be formally verified by a process of consistency check: given the correspondences, two pathways exist between two diagonal images; for each starting object in each image, whichever pathway is followed, it should lead to the same ending object in the diagonal image. For example, in Figure 2, on one hand, the large circle in *A* corresponds with the large circle in *B*, which corresponds with the large triangle in *D*; on the other, the large circle in *A* corresponds with the large triangle in *C*, which corresponds with the large triangle in *D*. The choice of using diagonal images in consistency check is because each pathway contains the correspondences in both directions.

There are two general analogy-making theories. The first theory assumes a base domain and a target domain and, by comparing the relational structures in these two domains, mappings between them are inferred (Gentner, 1983). When the relational structures or domains are not clearly defined, analogy-making is usually performed through the second theory where a dynamic process is employed, in which structures and correspondences between structures adapt to each other and settle on an equilibrium (Barsalou, 1999; Hofstadter, 1979; Mitchell, 1993). For the purpose of end-to-end modeling of the solving geometric analogy problems, the second theory is preferable. In particular, the domains are not clearly defined and the desirable equilibrium is realized as a self-consistent set of correspondences. Note that given the proportional format of geometric analogy, the base and target domains are not clearly defined (i.e., central permutation property (Prade & Richard, 2009, 2010, 2013)); so are relational structures and mappings between structures. In the rest of this paper, we will discuss the technical details of the end-to-end modeling of solving geometric analogy problems, which bear resemblance to the second theory of analogy-making.

Similarity Metric

Imagery-based models are sensitive to the choice of similarity metrics. A basic formulation of similarity metric is the Jaccard index (Equation (1)), which measures the similarity between two finite sets (Kunda, McGreggor, & Goel, 2013). In our works, these two sets consist of black pixels representing two geometric objects. Another useful variant of the Jaccard index is the asymmetric Jaccard index (Equation (2)) that measures the extent to which one set is a subset/inside of the other set.

$$J(A,B)=\frac{|A\cap B|}{|A\cup B|}=\frac{|A\cap B|}{|A\cap B|+|A\setminus B|+|B\setminus A|} \quad (1)$$

$$\vec{J}(A,B)=\frac{|A\cap B|}{|A|}=\frac{|A\cap B|}{|A\cap B|+|A\setminus B|} \quad (2)$$

The Jaccard index works well for geometric objects that are ideally drawn, such as those generated through vector graphics. But it is not as effective for geometric objects that human subjects would see in real psychological tests and in daily life. These visual stimuli are subject to distortion and noise, which pose a problem for imagery models using the Jaccard index.

Table 1: Similarities between geometric objects in Figure 3. The Jaccard index is calculated using Equation (1). The soft Jaccard index is calculated using Equation (17) with α=0.03, d of one-norm and p=3.

	□vs ◸		□vs □	○vs ⌓		○vs ○
Jaccard	0.4318	>	0.3333	0.3913	>	0.2307
Soft Jaccard	0.2076	<	0.9564	0.6461	<	0.9449

For example, applying Equation (1) on the scanning image of Figure 1, which contains distortion and noise that are imperceptible to human vision, the Jaccard index between the two large circles in A and B is only 0.25185; the Jaccard index between the small square in A and the small circle in B is 0.48649 — these measurements violate the correspondences implied by the verbal description.

$$A_b = \operatorname*{argmin}_{x\in A} d(x,b) \text{ for each } b\in B \tag{3}$$

$$B_a = \operatorname*{argmin}_{x\in B} d(a,x) \text{ for each } a\in A \tag{4}$$

$$M_0 = \{(a,b)\in A\times B \mid a\in A_b \wedge b\in B_a\} \tag{5}$$

$$A_0 = \{a\in A \mid \exists b\in B \text{ s.t. } (a,b)\in M_0\} \tag{6}$$

$$B_0 = \{b\in B \mid \exists a\in A \text{ s.t. } (a,b)\in M_0\} \tag{7}$$

$$T_a = \{(b,a')\in B\times A \mid b\in B_a \wedge a'\in A_b\} \text{ for each } a\in A\backslash A_0 \tag{8}$$

$$T_b = \{(a,b')\in A\times B \mid a\in A_b \wedge b'\in B_a\} \text{ for each } b\in B\backslash B_0 \tag{9}$$

$$M_1 = \{(a,b,a')\in (A\backslash A_0)\times B\times A \mid (b,a')\in \operatorname*{argmin}_{(b,a')\in T_a} d(a,a')\} \tag{10}$$

$$M_2 = \{(b,a,b')\in (B\backslash B_0)\times A\times B \mid (a,b')\in \operatorname*{argmin}_{(a,b')\in T_b} d(b,b')\} \tag{11}$$

$$d_0 = \frac{1}{|M_0|} \sum_{(a,b)\in M_0} |d(a,b)|^p \tag{12}$$

$$d_1 = \frac{1}{|M_1|} \sum_{(a,b,a')\in M_1} |d(a,a')|^p \tag{13}$$

$$d_2 = \frac{1}{|M_2|} \sum_{(b,a,b')\in M_2} |d(b,b')|^p \tag{14}$$

$$D(A,B) = d_0 + d_1 + d_2 \tag{15}$$

$$\vec{D}(A,B) = d_0 + d_1 \tag{16}$$

$$S(A,B) = e^{-\alpha D(A,B)} \tag{17}$$

$$\vec{S}(A,B) = e^{-\alpha \vec{D}(A,B)} \tag{18}$$

Figure 3 and Table 1 give a clearer example of this issue. The first row of the table is the Jaccard indices of the objects in Figure 3. According to these values, the square on the left is more similar to the triangle than to another square on the right, and, similarly, the circle on the left is more similar to the semicircle than to another circle on the right. Note that the sides of the two squares differ by only 1 pixel, and so do the radii of the two circles.

We designed another similarity metric, which inherits the general idea of the Jaccard index and is more robust to distortion and noise. The new metric does not require strict recurrences of elements in the two sets; instead, two elements can be considered "recurring" to some extent depending on the distance between them. Therefore, we name it soft Jaccard index.

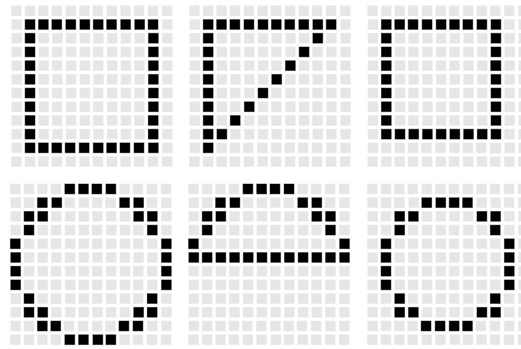

Figure 3: Geometric objects. Each cell in a grid denotes a pixel.

Given two sets $A=\{a_1,a_2,\cdots,a_m\}$ and $B=\{b_1,b_2,\cdots,b_n\}$ from a metric space with a metric d, the soft Jaccard index of A and B is defined by Equation 3 through 18. Note that the argmin gives a set of values that equally minimize the objective function. Equation 17 and 18 are the symmetric and asymmetric versions. Like the Jaccard index, the soft Jaccard index also consists of three terms — corresponding to $|A\cap B|$, $|A\backslash B|$ and $|B\backslash A|$ in the Jaccard index — subscripted by 0, 1 and 2 in the equations. The difference is that every term's contribution is calculated from the metric d instead of set cardinality. To compare with the Jaccard index, the soft Jaccard indices for the objects in Figure 3 are in the second row of Table 1. The soft Jaccard indices are more consistent with human perception than the Jaccard ones.

Correspondence Finding

The example analogy in Figure 1 can be characterized by a consistent set of correspondences in two analogical directions. In this section, we discuss how these correspondences can be found and used to interpret an analogy in a broader sense. We formulate conceptual correspondences in analogies as mathematical mappings. Thus, treatments of mathematical mappings could help understand analogy-making and modeling. We first consider two independent dimensions of mappings:

Qualitative vs Quantitative. A mapping can be derived from either qualitative relations or quantitative relations. A qualitative mapping depends on whether there is a good match between two qualitative relational structures. Thus, the validity of a qualitative mapping is considered binary. In contrast, a quantitative mapping is associated with a continuous score, say between 0 and 1, to indicate the extent of its validity. To let them work together, we give every qualitative mapping a score of 1 if it is valid or 0 if not.

Simple vs Complex. A mapping can also be derived either directly from geometric attributes of objects or from other mappings. Let us call them simple and complex mappings, respectively. In a sense, a complex mapping represents an isomorphism between two structures defined by two groups of mappings.

General Quantitative Mapping. Qualitative mappings are relatively easy to determine through structure matching, whereas quantitative mappings require additional considerations to coordinate multiple factors: (a) strong relations are preferable to weak ones; (b) the derived mapping should be unambiguous (i.e., injective) in that any two mapped objects should mutually be each other's best match; (c) the size of the mapping should be as large as possible to capture the largest isomorphism. Thus, we designed a template method to derive quantitative mappings as shown in Equation (19), where, given two sets $U=\{u_1,u_2,\cdots,u_m\}$ and $V=\{v_1,v_2,\cdots,v_n\}$ of objects, whether u_i and v_j are mapped to each other is denoted by $x_{ij}=1$ or 0, and $s_{ij}\in\mathbb{R}$ denotes a measurement of the relation between u_i and v_j, for example, similarity. The above factors are thus integrated into the optimization in Equation (19), where x_{ij} and t are variables. Note that, in this formulation, we assume larger values of s_{ij} indicate stronger relations. If smaller values of s_{ij} indicate stronger relations, the equations need to be accordingly negated.

$$\begin{aligned}
&\max \sum_{i,j} x_{ij} \\
\text{s.t. } &1 \geq \sum_{j} x_{ij} \text{ for all } i \\
&1 \geq \sum_{i} x_{ij} \text{ for all } j \\
&x_{ij}=x_{ji} \text{ for all } i,j \\
&(x_{ij}-0.5)(s_{ij}-t)>0 \text{ for all } i,j \\
&x_{ij}\in\{0,1\} \text{ for all } i,j \text{ ,and } t\in\mathbb{R}
\end{aligned} \tag{19}$$

Given the two dimensions of mappings and general quantitative mapping, we introduce the specific mappings:

Simple Quantitative: Shape Mapping. When the soft Jaccard index is used as the strength of relation in Equation (19), we obtain a mapping reflecting shape similarity. Since the soft Jaccard index gives values between 0 and 1, we use the minimum strength of the selected relations as the score of the mapping.

Simple Quantitative: Location Mapping. When Euclidean distance between objects is used as the strength of relation in Equation (19), we obtain a mapping based on the locations of objects. The score of this mapping is calculated as the normalized maximum strength of the selected relations.

Complex Quantitative Mappings. Let M_1:$A\rightarrow B$ and M_2: $C\rightarrow D$ be two injective mappings. A **delta shape mappings** is a complex quantitative mapping constructed from M_1 and M_2, representing the idea that the same shape change happens from A to B and from C to D. Similarly, a **delta location mappings** based on M_1 and M_2 represent the idea that the same location change happens from A to B and from C to D. These two complex quantitative mappings are thus between A and C and between B and D, orthogonal to the directions of M_1 and M_2.

Complex Quantitative: Delta Shape Mapping. When the difference between the soft Jaccard index of each M_1 pair and the soft Jaccard index of each M_2 pair is used as the strength of relation in Equation (19), we obtain the delta shape mapping. The score of this mapping is calculated from the maximum strength of the selected relations.

Complex Quantitative: Delta Location Mapping. When the difference between the distance of each M_1 pair and the distance of each M_2 pair is used as the strength of relation in Equation (19), we obtain the delta location mapping. The score of this mapping is calculated from the normalized maximum strength of the selected relations.

Simple Qualitative: Inside/Outside Mapping. Relations such as regional connection calculus were supposed to be used here. But for rapid prototyping, we use only the inside/outside relation. An inside/outside mapping exists if the relational structures of one set can strictly match the relational structure of the other set, and thus has a binary score of 0 or 1.

Complex Qualitative: Edge-Labeled Isomorphism Between Bipartite Multigraphs. Let $f_1,f_2,\cdots,f_n$ be injective mappings between sets A and B, and $f'_1,f'_2,\cdots,f'_n$ be injective mappings between sets C and D. These two groups of mappings form two edge-labeled bipartite multigraphs with labels in $\{1,2,\cdots,n\}$. We can derive two new mappings between A and C and between B and D from any label-preserving isomorphism between these two multigraphs. The score of the mapping is 1 if such isomorphism exists; otherwise 0.

There are cases when mappings are theoretically workable, but cumbersome. For example, when a geometric object is rotated or mirrored, we can certainly map every point of the object to where they are moved to. But a more efficient solution is to consider the transformation of the whole object. Therefore, we also include common affine transformations in our toolbox, and, using the soft Jaccard index, we score the validity of these transformations, as we did for mappings.

Experimental Studies

Besides the soft Jaccard index and the aforementioned mappings, we construct another conceptual layer in our modeling — interpretation — by assigning mappings or transformations to the two analogical directions, i.e., each interpretation is a combination of specific mappings or transformations. To solve a geometric analogy problem, each interpretation is scored for the geometric analogy obtained by inserting each option into the incomplete analogy, by aggregating the scores of its mappings or transformations. The interpretation and option of the highest score are selected as the answer to the problem. Following this outline, we implemented a computational model and ran it on a classical set of 20 geometric analogy problems (details found in (Lovett et al., 2009)), which was published in the 1942 edition of the *Psychological Test for College Freshmen of the American Council on Education*.

Table 2: Experimental Results. The second and third columns are two analogical directions. The last column shows the problems that were solved by that interpretation.

Interpretation	*A:B::C:?*	*A:C::B:?*	Mapping/Transformation	Consistency Check	Solved Problems
1	shape	inside/outside	Mapping	yes	3, 5, 7, 9, 11,17
2	shape	delta location	Mapping	no	1, 4
3	inside/outside	delta shape	Mapping	no	8
4	shape & location	isomorphism	Mapping	no	10, 20
5	shape = location	shape	Mapping	yes	15
6	density change	shape change	Transformation	no	13
7	duplicate	N/A	Transformation	no	16
8	affine	N/A	Transformation	no	2, 6, 12, 14, 18, 19

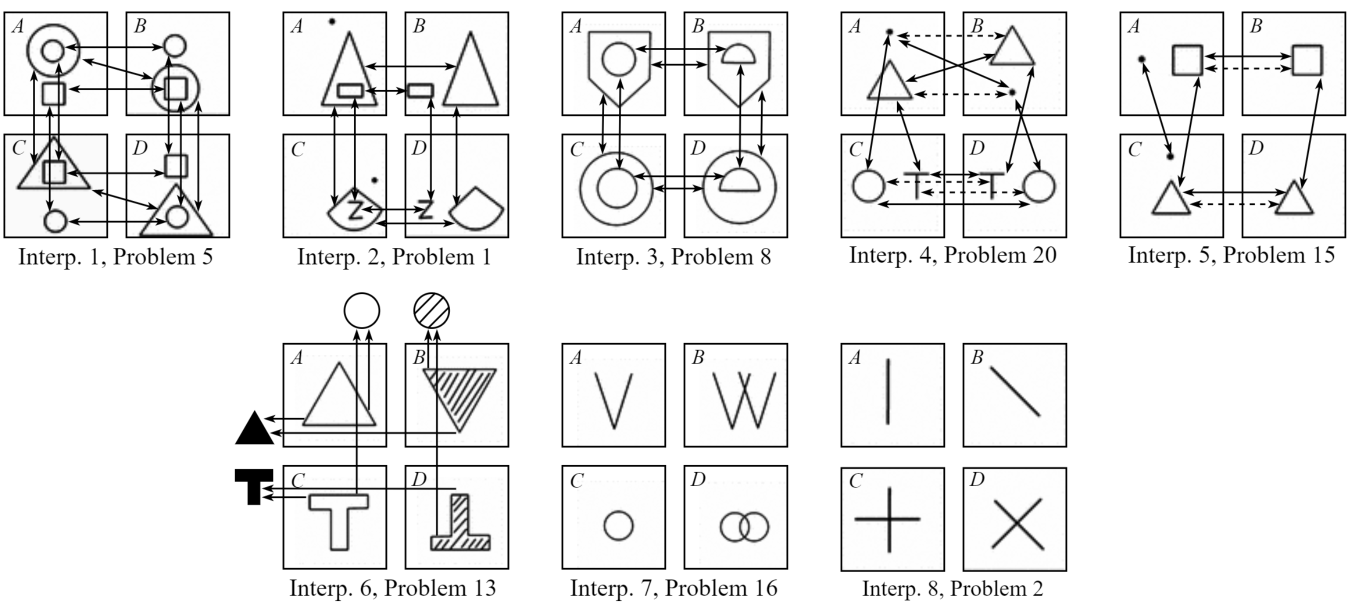

Figure 4: The typical problems solved by each interpretation.

The experimental results are summarized in Table 2. All the 20 geometric analogy problems were solved by 8 different interpretations. Table 2 lists each interpretation's mappings or transformations in the two analogical directions, and the solved problems. The first five interpretations are mapping interpretations, among which Interpretation 1 and 5 require a successful consistency check because their mappings in the two analogical directions are independently derived; in contrast, in Interpretation 2, 3 and 4, the complex mappings in one direction are built upon the simple mappings in the other direction with the consistency assumed to be true. Although the consistency holds in both cases, the corresponding analogies and how these analogies are processed are different. Interpretation 6, 7 and 8 are transformation interpretations, which apply to a large portion of the problems. This implies that, in addition to consistent mappings, visual imagery and mental transformation are another important facet of analogy-making.

It is worth pointing out that Problem 19 can be solved by two affine transformations — rotation and reflection — leading to different options. The rotation option won out marginally in our experiment, but the reflection option is more human-preferred.

To give a straightforward description of how the model works, we select for each interpretation a problem to describe the details. These problems and interpretations are visualized in Figure 4.

Interpretation 1: Problem 5 shows an analogy of topological variation between the two rows. Horizontally, two shape mappings $A \to B$ and $C \to D$ are constructed. Vertically, two inside/outside mappings $A \to C$ and $B \to D$ are constructed. These four mappings are consistent and characterize the repetition of the same topological change in the two rows.

Interpretation 2: Problem 1 shows an analogy of location change between the two rows. Horizontally, two shape mappings $A \to B$ and $C \to D$ are constructed. Vertically, two delta location mappings $A \to C$ and $B \to D$ are constructed on the basis of the horizontal shape mappings. The repetition of the same location change in the two rows is characterized by these four mappings.

Interpretation 3: Problem 8 shows an analogy of shape change between the two rows. This illustration is parallel to Interpretation 2's except that it describes shape change instead of location change, using inside/outside mapping instead of shape mapping in the horizontal direction.

Interpretation 4: Problem 20 shows an analogy of location exchange between the two rows. Note that location exchange is different from location change in that an object can only move to a previously-occupied place, and thus the movement is relative, whereas location change is the absolute movement in the global coordinate system. Horizontally, two types of mappings are constructed, where the dashed line indicates location mappings and the solid line indicates shape mappings. Vertically, an edge-labeled isomorphism is constructed on the basis of the horizontal mappings, where different mapping types serve as edge labels. The repetition of the same location exchange in the two rows is characterized by these mappings.

Interpretation 5: Problem 15 shows an analogy of adding or removing objects between the two rows. Horizontally, like Interpretation 4, shape mappings and location mappings are constructed but these two types of mappings are required to agree with each other. Vertically, shape mapping is constructed. Therefore, the same change of adding or removing objects in the two rows is described by these mappings. A consistency check is needed.

Interpretation 6: Problem 13 shows an analogy of texture change and shape change between the two rows. Horizontally, the texture change was supposed to be measured, but due to the lack of a general computational representation for texture, the density change of black pixel is used to approximate texture change. Vertically, shape change is represented by the change in soft Jaccard Index. The changes are supposed to be equal in rows and columns.

Interpretation 7: Problem 16 shows an analogy of duplication between the two rows. Horizontally, objects are duplicated in the same way (same location arrangement) from A to B and from C to D. The location arrangement is determined by repeatedly calculating the asymmetric soft Jaccard index at different relative locations. Vertically, a general quantitative mapping between the two sets of locations is computed using Euclidean distance as the strength of relations.

Interpretation 8: Problem 2 shows an analogy of affine transformation, a 45-degree rotation in this case, between the two rows. The soft Jaccard index is used to determine which affine transformation best matches the variation.

Related Work

Most previous models of solving geometric analogy problems employ predefined symbolic representation of visual stimuli and reply on certain kinds of matching mechanisms to find the structural similarity between the base and target domains (Evans, 1968; Carpenter, Just, & Shell, 1990; Lovett et al., 2009). This line of research focuses on high-level cognitive functions such as search strategy, selective attention and goal management. Another important type of models (Kunda et al., 2013; Yang, McGreggor, & Kunda, 2020) is based on visual imagery and mental transformation (Shepard & Metzler, 1971). These two types of models can be characterized by the "analytic" and "Gestalt" algorithms by Hunt. Our work in this paper resembles and differs from both of them in an obvious way, by combining their characteristics.

Another dichotomy of models of solving analogy problems involves the strategy from a perspective of problem-solving. In particular, two general strategies — constructive matching and response elimination — are commonly observed in human experiments (Snow, 1981; Bethell-Fox, Lohman, & Snow, 1984). Thus, models have been proposed to implement both strategies and to explain why specific strategies are used (Sternberg, 1977; Mulholland, Pellegrino, & Glaser, 1980) in certain circumstances. Our model falls into the class of response elimination.

Besides the problem set used in this work, there are other similar problem sets such the Raven's Progressive Matrices (Raven, Raven, & Court, 1998), in which the visual objects are more diverse (not just common nameable geometric shape) and thus requires more consideration on the modeling of perceptual processing. The research interest has also extended to the field of machine learning, especially computer vision, where two large-scale datasets have been proposed (Santoro, Hill, Barrett, Morcos, & Lillicrap, 2018; Zhang, Gao, Jia, Zhu, & Zhu, 2019), which consist of homogeneous computer-generated items of limited variation patterns. Because settings in psychometrics and machine learning are radically different, the implication of these works to the cognitive studies of analogy-making is still unclear.

Conclusion and Future Work

In this paper, we proposed a model of solving geometric analogy problems in an end-to-end manner, from perceptual input to conceptual output. For each problem, the model selects an option and an interpretation, which is based on the mappings or transformations in the two analogical directions. As a basic perceptual component of the model, we designed the soft Jaccard index that is robust to distortion and noise. As a basic conceptual component, we designed a formal correspondence-find method that integrates multiple factors.

Given the initial feasibility of the proposed model in the experimental studies, future research includes how the model relates to analogy-making theories and human behaviors. Another theoretical desideratum is to compare the soft Jaccard index to the Jaccard index and other similarity metrics, for example, according to our initial investigation, the soft Jaccard index shows an approximate triangle inequality, whereas the Jaccard index strictly follows the triangle inequality (Gilbert, 1972); the causes and implications of the difference would be interesting both mathematically and cognitively.

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