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On the Infinitary Combinatorics of Small Cardinals and the Cardinality of the Continuum

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On the Infinitary Combinatorics of Small Cardinals and the Cardinality of the Continuum

A dissertation submitted in partial satisfaction
of the requirements for the degree
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by

Thomas Daniells Gilton

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Abstract of the Dissertation

On the Infinitary Combinatorics of Small Cardinals
and the Cardinality of the Continuum

by

Thomas Daniells Gilton
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2019
Professor Itay Neeman, Chair

This work is divided into two parts which are concerned, respectively, with the combinatorics of the cardinals $\aleph_1$ and $\aleph_2$.

The first part of the thesis contains the result due to the author and his advisor, Itay Neeman, that the Abraham-Rubin-Shelah Open Coloring Axiom is consistent with a large continuum; this answers a long-standing open question in forcing. Most of Part 1 appears in our submitted manuscript [35]. After surveying the relevant background in the first chapter, we proceed in the second chapter to define the notion of a Partition Product. This is a type of iteration built out of smaller ones in specific ways, roughly with memory conditions on the names and with isomorphism and coherence conditions on the various "memories." We will prove a number of useful facts about partition products in Chapter 2. In Chapter 3, we show how to construct so-called Preassignments of Colors in the context of partition products; this forms the technical heart of Part 1. And finally, in Chapter 4, we show how to construct partition products in L; in particular, we construct the partition product which yields the model witnessing our theorem.

Part 2 of the thesis addresses questions about a variety of combinatorial principles on $\aleph_2$. In each chapter in Part 2, we will be concerned with showing that some amount of Stationary Reflection holds at ω_2, more specifically showing that various amounts of stationary reflection

are compatible with other principles of wide interest. In Chapter 5, we provide an overview of these combinatorial principles and some of their history; we also spend some time collecting standard facts about Mitchell-type forcings which we will use in the subsequent chapters. Chapter 5 concludes with a proof that various Mitchell-type posets and their quotients are proper, a result which we assume is known, but which we have not encountered ourselves elsewhere.

Chapters 6 and 7 address questions arising from the recent paper *The Eightfold Way* by Cummings, Friedman, Magidor, Rinot, and Sinapova (see [24]). In Chapter 6, we answer an open question asked at the end of that paper by showing that it is consistent, from a Mahlo cardinal, that the Tree Property ($\mathsf{TP}(\omega_2)$) and Approachability (AP_{ω_1}) both fail at ω_2, while stationary reflection ($\mathsf{SR}(\omega_2)$) holds at ω_2. The authors of [24] obtained the consistency of this same configuration from a weakly compact cardinal; our result proves the consistency of this configuration from optimal assumptions. We remark here that we present the original proof discovered by the author of this thesis. Later, the author, working with John Krueger, provided a more streamlined proof of this same result; this proof will appear in the forthcoming paper [32]. Chapter 6 also includes an unrelated Easton-style lemma for preserving stationary subsets of countable cofinality; this result is due to the author and Omer Ben-Neria.

In Chapter 7, we show that for any Boolean combination, Φ, of $\mathsf{TP}(\omega_2)$ and AP_{ω_1}, Φ is consistent with a strong form of simultaneous stationary reflection on ω_2, namely that every stationary $S \subseteq \omega_2 \cap \operatorname{cof}(\omega)$ reflects almost everywhere. This strengthens some of the results from [24].

In Chapter 8, we return to the model from [34], making good on a promise from the postscript therein. In [34], the author and John Krueger originally sought to show that stationary reflection on ω_2 is consistent with a large continuum, and we built an involved mixed-support iteration to achieve such a model. However, we later learned from I. Neeman that such a model can be constructed by simply adding Cohen reals over the original Harrington-Shelah model ([39]). In Chapter 8 we will show that after a modification of our original preparatory iteration, we may obtain a model in which $\mathsf{SR}(\omega_2)$ and AP_{ω_1} both hold, in which $2^\omega > \omega_2$, and in which there are neither special Aronszajn trees on ω_2 nor weak Kurepa trees on ω_1. This is a configuration which cannot be obtained simply by adding Cohen reals over the original Harrington-Shelah model nor by the methods of disjoint stationary sequences from [32]. We hope that this demonstrates the usefulness of such a mixed support iteration.

In the final chapter, we provide a list of open questions which we would like to address in future work.

The dissertation of Thomas Daniells Gilton is approved.

Artem Chernikov

Andrew Scott Marks

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2019

to my parents, Michael and Kathleen, for their unconditional love;

to my beloved wife, Marian, for her indefatigable support;

and to my daughter Zoe:

weißt du eigentlich, wie lieb ich dich hab?

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Acknowledgments

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Vita

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Publications

(with J. Krueger) A Note on the Eightfold Way. To appear in *Proceedings of the American Mathematical Society*

(with J. Krueger) The Harrington-Shelah Model with Large Continuum. *Journal of Symbolic Logic*, 84 (2019), no. 2, 684-703

(with J. Krueger) Mitchell's Theorem Revisited. *Annals of Pure and Applied Logic*, Volume 168 (2017) no. 5, 922-1016

Part I

The Abraham-Rubin-Shelah Open Coloring Axiom and $2^{\aleph_0} > \aleph_2$

CHAPTER 1

Introduction to Part 1

The infinite has been a topic of perennial interest to mathematicians. One of the most fascinating and foundational discoveries in this regard is Georg Cantor's proof ([16]) that the set of real numbers, $\mathbb{R}$, has a strictly greater infinite size, or *cardinality*, than the set of natural numbers, $\mathbb{N}$. Cantor's proof was a watershed in our understanding of the infinite, as the ideas of his proof can be used to show that infinite sets come in a bewildering variety of sizes, known as *cardinal numbers*. In the form of an equation, Cantor's Theorem says that $2^{\aleph_0} > \aleph_0$. This, then, prompts the following natural question: just how much bigger is $2^{\aleph_0}$ than $\aleph_0$? The so-called *Continuum Hypothesis* (CH) is the assertion that $2^{\aleph_0}$ is as small as it can conceivably be, or in the form of an equation, that $2^{\aleph_0} = \aleph_1$ is true.

After Cantor's work, however, the eminent mathematicians Kurt Gödel and Paul Cohen discovered that it is not possible either to prove the CH or to prove its negation using ZFC, the standard axiom system for mathematics. In more detail, Gödel showed ([38]) how to build a model of ZFC, the so-called Constructible Universe, in which the CH holds. On the other hand, Cohen showed ([17]), in work that earned him the Fields Medal, how to use his technique of *Forcing* to construct models of ZFC in which the CH is false. One may summarize the state of play by saying that ZFC does not *decide* the value of $2^{\aleph_0}$. In light of these results, mathematicians in the field of Set Theory have found the following to be a fruitful question:

Question: Are there natural axioms which are consistent with ZFC and decide the value of $2^{\aleph_0}$?

This question has spawned an enormous line of research in set theory into the range of possible values of $2^{\aleph_0}$ consistent with various combinatorial principles of infinite cardinals.

In one direction, researchers have focused on showing that various combinatorial principles are consistent with the CH. Much of the research in this vein has appropriated Shelah's theory of Dee-completeness, a framework for building iterations which do not add any new reals, and hence preserve the CH; see chapters 5 and 8 of [67]. A number of interesting applications of this theory have been found, though we only mention a few. For instance, Abraham and Todorčević ([6]) and Todorčević ([77]) have shown that the P-Ideal Dichotomy, first introduced in [76], is consistent with the CH ; Abraham has shown (see [2]) that various coloring properties of Hajnal-Máté graphs are consistent with the CH; and Eisworth and others have shown that many varied and interesting topological principles are consistent with the CH (see [29], [28]). See also [61], and [7], and see [8] for a very different approach which involves adding some, but only a few, reals.

In another direction, researchers have sought to show that various combinatorial principles are consistent with or imply the failure of the CH. One line of attack for these problems is to show that some such principle is a consequence of the *Proper Forcing Axiom* (PFA), since PFA implies that $2^{\aleph_0} = \aleph_2$ (see [42]). For a classic treatment of the subject, see [13]; see also the introduction to [9] and the literature cited therein.

However, of particular interest to us, are principles (most of which are consequences of PFA) consistent with a *Large Continuum*, by which we mean any value of the continuum at least as big as $\aleph_3$. For examples of some notable results in this line of research, we mention the theorems of Abraham, Rubin, and Shelah that Baumgartner's Axiom ([14]), as well as the Semi-Open Coloring Axiom are consistent with arbitrarily large values of $2^{\aleph_0}$ (see [4]). Abraham and Shelah have studied isomorphism types of Aronszajn trees on ω_1, showing that an arbitrarily large continuum is consistent with the assertion that all Aronszajn trees on ω_1 are isomorphic on a club (see [5]). For a connection to measure theory, Judah, Shelah, and Woodin have shown (see [44]) that the Borel Conjecture (that every strong measure zero set is countable) is consistent with arbitrarily large values of the continuum, improving on an earlier result of Laver ([54]). More recently, Asperó and Mota have shown (see [9], [10]) that many different consequences of PFA are consistent with arbitrarily large values of $2^{\aleph_0}$ using their method of symmetric side conditions; the method of side conditions goes back to work of Todorčević (see [74]), as does the extension to put symmetry-type conditions on the models (see [75]).

We will focus our attention on so-called *Coloring Axioms*, which should be viewed as generalizations of Ramsey's Theorem ([64]) to pairs of countable ordinals, i.e., to colorings on ω_1. Given the remarkable success of Ramsey's Theorem in many diverse areas of mathematics, set theorists have found it natural to ask whether there are analogous results which hold for ω_1. The most straightforward generalization of Ramsey's Theorem is the assertion that any coloring of pairs of countable ordinals has an uncountable homogeneous set. However, as Sierpiński has shown (see [70]), this naive generalization is provably false, at least in ZFC (see [41] for a discussion of partition theorems in the context of the Axiom of Determinacy).

The failure of Ramsey's theorem to generalize straightforwardly to ω_1 has spawned a huge line of research into partition theorems for uncountable sets, though we only mention a few results here. One of the first results in this area is the theorem due to Dushnik and Miller ([27]) that $\omega_1 \to (\omega_1, \omega)^2$. Erdös and Rado (see [30]) later improved this to $\omega_1 \to (\omega_1, \omega+1)^2$, and they also showed that $\beth_n^+ \to (\aleph_1)^{n+1}_{\aleph_0}$ holds for all $n < \omega$. Todorčević has shown that $\omega_1 \to (\omega_1, \alpha)^2$ is consistent for all $\alpha < \omega_1$ (see [73]).

We are most concerned with obtaining consistent generalizations of Ramsey's Theorem to ω_1 by placing various topological restrictions on the colorings, resulting in so-called Coloring Axioms. The first such axiom to appear in the literature is due to Abraham, Rubin, and Shelah in their above-mentioned 1985 paper (see [4]); the definition is as follows, where we will use the notation $[A]^2$ to denote all two-element subsets of the set A.

Definition 1.0.1. *A function $\chi : [\omega_1]^2 \longrightarrow \{0,1\}$ is said to be an* open coloring *if it is continuous with respect to some second countable, Hausdorff topology on ω_1. $A \subseteq \omega_1$ is said to be χ-*homogeneous *if χ is constant on $[A]^2$.*

The Abraham-Rubin-Shelah Open Coloring Axiom*, abbreviated OCA_{ARS}, states that for* *any open coloring χ on ω_1, there exists a partition $\omega_1 = \bigcup_{n<\omega} A_n$ such that each A_n is χ-homogeneous.*

Abraham and Shelah ([11]) first studied a restricted version of this axiom during the course of their investigation into the relationship between Martin's Axiom and Baumgartner's Axiom ([14]). This restricted version is concerned just with monotonic subfunctions of injective, real-valued functions. The full version made its debut in [4], where the authors studied it alongside a number of other axioms about $\aleph_1$-sized sets of reals. In particular, they showed that OCA_{ARS} is consistent with ZFC.

A little later, Todorčević isolated the following axiom ([76]):

Definition 1.0.2. *The* Todorčević Open Coloring Axiom*, abbreviated OCA_T, states the following: let A be a set of reals, and suppose that $[A]^2 = K_0 \cup K_1$, where K_0 is open in $[A]^2$. Then either there is an uncountable $A_0 \subseteq A$ such that $[A_0]^2 \subseteq K_0$, or there is a partition $A = \bigcup_{n<\omega} A_n$ such that $[A_n]^2 \subseteq K_1$ for each $n < \omega$.*

If we restrict our attention to sets of reals A with size $\aleph_1$, we denote this axiom by $\mathsf{OCA}_T(\aleph_1)$.[1]

Both of these axioms are consequences of PFA, though their conjunction can be shown to be consistent with a direct iterated forcing argument. Further, they each imply that the CH is false. Indeed, OCA_{ARS} implies that any injective function $f : A \longrightarrow \mathbb{R}$, where $|A| = \aleph_1$, is a union of countably-many monotonic subfunctions; in particular, any such f has an $\aleph_1$-sized montonic subfunction. However, under the CH, there exists a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is not continuous on any uncountable set, and hence not monotonic on any uncountable set (see C_{62} of [71]). To make matters worse, under the CH, there is an injective, partial $f \subseteq \mathbb{R} \times \mathbb{R}$ with no uncountable monotonic subfunction (see [26]), and therefore even continuous colorings can fail to have large homogeneous subsets if the CH

[1] Note that $\mathsf{OCA}_T(\aleph_1)$ is a stronger axiom than SOCA from [4]. SOCA states that for a set of reals A of size $\aleph_1$ and partition $[A]^2 = K_0 \cup K_1$, where K_0 is open, there exists either an uncountable subset of A which is K_0-homogeneous or an uncountable subset of A which is K_1 homogeneous.

holds. With regards to OCA_T, this axiom implies that the bounding number $\mathfrak{b}$ is $\aleph_2$ (see [76]). Thus each of these axioms has some effect on the size of the continuum.

It is therefore of interest whether or not these axioms, individually or jointly, actually decide the value of the continuum. In the case of OCA_T, I. Farah has shown in an unpublished note that $\mathsf{OCA}_T(\aleph_1)$ is consistent with an arbitrarily large value of the continuum, though it is not known whether the full OCA_T is consistent with larger values of the continuum than $\aleph_2$. On the other hand, Moore has shown ([60]) that $\mathsf{OCA}_T + \mathsf{OCA}_{ARS}$, which is consistent, does decide that the continuum is exactly $\aleph_2$.

However, the question of whether OCA_{ARS} is powerful enough to decide the value of the continuum on its own, first asked in [4], has remained open. There are a number of difficulties in obtaining a model of OCA_{ARS} with a "large continuum," i.e., with $2^{\aleph_0} > \aleph_2$. Chief among these difficulties is to construct so-called *preassignments of colors*, which may very roughly be viewed as a way of diagonalizing out of obstructions to the desired forcings having the countable chain condition. More specifically, a preassignment of colors is a function which decides, in the ground model, whether the forcing will place a countable ordinal α inside *some* 0-homogeneous or *some* 1-homogeneous set, with respect to a fixed coloring. The authors of [11] first discovered the technique of preassigning colors and used this technique to prove the consistency of the restricted version of OCA_{ARS} mentioned above. The key to the consistency of OCA_{ARS} is to construct preassignments in such a way that the posets which add the requisite homogeneous sets, as guided by the preassignments, are c.c.c.

However, the known constructions of such "good" preassignments only work under the CH. Since forcing iterations whose strict initial segments satisfy the CH can only lead to a model where the continuum is at most $\aleph_2$, this creates considerable difficulties for obtaining models of OCA_{ARS} in which the continuum is, say, $\aleph_3$. In particular, the known techniques for obtaining models with a large continuum are likely to be ineffective in tackling this problem.

In this Part I of this thesis, we prove the following theorem, which is due to the author and his advisor, Itay Neeman.

Theorem 1.0.3. (Gilton, Neeman) *If* ZFC *is consistent, then so is* $\mathsf{ZFC}+\mathsf{OCA}_{ARS}+2^{\aleph_0}=\aleph_3$.

The key to our solution is to construct names for preassignments with a substantial amount of symmetry. Roughly, suppose that $\mathbb{P}$ is a "nice" iteration of $\aleph_1$-sized, c.c.c. posets, where the length of $\mathbb{P}$ is less than ω_2; note that $\mathbb{P}$ preserves the CH. We are able, for example, to construct a *single* $\mathbb{P}$-name $\dot{f}$ for a preassignment so that $\dot{f}$ can be interpreted by a host of different V-generics for $\mathbb{P}$ and still give rise to a c.c.c. product of posets. For instance, if $\dot{\chi}$ is a name for a continuous coloring in Cohen forcing for adding a single real, we can construct a single Cohen name $\dot{f}$ for a preassignment so that if $\langle c_\xi : \xi < \omega_3\rangle$ are pairwise mutually generic Cohen reals, then the product

$$\prod_{\xi<\omega_3} \mathbb{Q}(\dot{\chi}[c_\xi], \dot{f}[c_\xi])$$

is c.c.c. Here $\mathbb{Q}(\chi', f')$ denotes the poset to decompose ω_1 into countably-many χ'-homogeneous sets, as guided by f' (see Chapter 3 for the more precise definition). Building such a "symmetric" or "uniform" name $\dot{f}$ takes us well beyond the techniques of [4]. Moreover, constructing such "symmetric" names proves to be necessary, at least if we can only construct preassignments over models satisfying the CH, an assumption which seems to be at least practically necessary. This is since, assuming the GCH, there are only $\aleph_2$-many possible names (up to isomorphism) for preassignments of colors named by c.c.c. posets of size $\aleph_1$, and hence in the course of an iteration of length at least ω_3, the same *name* for a preassignment must show up unboundedly often (of course, with different interpretations).

We then combine such shorter iterations, which function as a type of alphabet, into much longer ones which we call *Partition Products*. A partition product can be viewed as a type of iteration with memory conditions on each coordinate; recall that memory iterations, roughly, provide restrictions on which regular suborders the names at each stage of the iteration can be drawn from. However, we put much more stringent restrictions on these memories. We demand that each "memory" is isomorphic to one of the "alphabet" or "canonical" partition products, and we demand also that the memories, when they overlap, do so in very particular ways. These conditions are meant to capture the behavior of intersections of various hulls in L (see Chapter 4), and we use these restrictions in order both to carry out counting arguments (see Lemma 3.2.10) and for the inductive construction of preassignments (see Lemma 3.3.4).

Finally, we force with a large partition product to construct a model of OCA_{ARS} wherein $2^{\aleph_0} = \aleph_3$. The general theme of our theorem, then, is the following: short iterations are necessary to preserve the CH and thereby construct effective preassignments; longer iterations, built out of these smaller ones in specific ways, can be used to obtain models with a large continuum.

The above method is general enough that it can be adapted to strengthen Theorem 1.0.3 to obtain the forcing axiom $\mathsf{FA}(\aleph_2, \text{Knaster}(\aleph_1))$; this forcing axiom asserts that for any Knaster poset $\mathbb{P}$ of size $\leq \aleph_1$ and any sequence $\langle D_i : i < \omega_2 \rangle$ of $\aleph_2$-many dense subsets of $\mathbb{P}$, there is a filter for $\mathbb{P}$ which meets each of the D_i. Thus we may obtain the following theorem:

Theorem 1.0.4. *If* ZFC *is consistent, then so is*

$$\mathsf{ZFC} + \mathsf{OCA}_{ARS} + 2^{\aleph_0} = \aleph_3 + \mathsf{FA}(\aleph_2, \text{Knaster}(\aleph_1)).$$

CHAPTER 2

The Basic Theory of Partition Products

In this chapter we define and explicate the notion of a partition product. In the first section, we present the definition itself and prove some fairly straightforward facts about it. In the second section, we define and put to good use the notion of a *shadow base*, which is a mechanism for keeping track of additional structure that comes with a partition product.

Roughly speaking, the class of partition products is a class of finite support iterations which are built in very specific ways, but which is rich enough to be closed under the following operations:

- products;
- products of iterations taken over a common initial segment;
- more general "partitioned products" of segments of the iterations taken over common earlier segments.

Each poset in this class may also be viewed as a type of memory iteration. We recall that memory iterations were invented by Shelah in [68]; for some further uses of memory iterations, see [69] (additional results on the null ideal) and [57] (results on cardinal characteristics). The very rough idea of iterating with memory is that when constructing an iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \leq \beta \rangle$, we demand that the $\mathbb{P}_\alpha$-name $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha \restriction B$-name, for some $B \subseteq \alpha$ so that $\mathbb{P}_\alpha \restriction B$ is a regular suborder of $\mathbb{P}_\alpha$. Thus $\dot{\mathbb{Q}}_\alpha$ is not allowed to refer to all of the information added by $\mathbb{P}_\alpha$ but only the information added by a certain regular suborder of $\mathbb{P}_\alpha$.

Our class of partition products may be viewed as memory iterations, where we place stringent requirements on how the memories behave. Indeed, we place isomorphism and coherence conditions on the memories; the former refers to the fact that the regular suborder from which a name is drawn must be isomorphic to one of a small class of “canonical” or “alphabet” partition products, and the latter refers to the fact that, when two memories overlap, they are required to do so according to a very specific recipe. This “recipe” is meant to capture the behavior of the intersections of hulls of various levels of L and ensures that these overlaps are definable. These features will in turn be used in order to carry out various counting arguments (in order to see that there are not too many “sufficiently simple” partition products) and in the construction of preassignments (which involves an induction on the order types of the overlaps of various memories).

2.1 Partition Products

2.1.1 Definition and Basic Facts

Our first goal in this section is to define the notion of a *partition product*. After the definition, we will provide comments which clarify the motivation described at the beginning of the chapter.

We begin by fixing some unbounded set $C \subseteq \omega_2$; this set will be specified in Section 5 (we don't need any further properties of it now). We define by recursion the notion of a *partition product* based upon a sequence $\underline{\mathbb{P}} \restriction \kappa = \langle \mathbb{P}_\delta : \delta \in C \cap \kappa \rangle$ of posets and a sequence $\underline{\dot{\mathbb{Q}}} \restriction \kappa = \langle \dot{\mathbb{Q}}_\delta : \delta \in C \cap \kappa \rangle$ of names, where $\kappa \in C \cup \{\omega_2\}$. Every such object will be a poset $\mathbb{R}$ consisting of various finite partial functions on some set X of ordinals. This set X will be definable from $\mathbb{R}$ and will be called the *domain* of $\mathbb{R}$. The definition is by recursion on κ, and we make the following recursive assumptions about the objects $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$:

(i) for each $\delta \in C \cap \kappa$, $\mathbb{P}_\delta$ (the so-called *canonical δ-partition product*) is a partition product based upon $\underline{\mathbb{P}} \restriction \delta$ and $\underline{\dot{\mathbb{Q}}} \restriction \delta$, and $\dot{\mathbb{Q}}_\delta$ is a $\mathbb{P}_\delta$-name for a poset. The domain of $\mathbb{P}_\delta$ is an ordinal, which we call ρ_δ, and $\rho_\delta \leq \delta^+$.

The definition of partition products below is such that for each $\delta \leq \kappa$, every partition product $\mathbb{R}$, with domain X, say, based upon $\underline{\mathbb{P}} \restriction \delta$ and $\underline{\dot{\mathbb{Q}}} \restriction \delta$ comes equipped with two additional functions $\mathrm{base}_{\mathbb{R}}$ and $\mathrm{index}_{\mathbb{R}}$ defined on X. These functions for $\mathbb{R}$ satisfy (among other properties to be specified later) the following:

(ii) for each $\xi \in X$, $\mathrm{index}_{\mathbb{R}}(\xi) \in C \cap \delta$, and $\mathrm{base}_{\mathbb{R}}(\xi)$ is a pair

$$\mathrm{base}_{\mathbb{R}}(\xi) = (b_{\mathbb{R}}(\xi), \pi^{\mathbb{R}}_{\xi}),$$

where $b_{\mathbb{R}}(\xi) \subseteq X \cap \xi$ and $\pi^{\mathbb{R}}_{\xi}$ is a bijection from $\rho_{\mathrm{index}_{\mathbb{R}}(\xi)}$ onto $b_{\mathbb{R}}(\xi)$.

For each $\delta \in C \cap \kappa$, we abbreviate $\mathrm{index}_{\mathbb{P}_\delta}$ by index_δ, and we abbreviate $\mathrm{base}_{\mathbb{P}_\delta}(\xi)$ by $\mathrm{base}_\delta(\xi) = (b_\delta(\xi), \pi^\delta_\xi)$, for each $\xi < \rho_\delta$. While (ii) above holds for all partition products, the next recursive assumption specifically concerns the canonical partition products $\mathbb{P}_\delta$:

(iii) for each $\xi < \rho_\delta$, $b_\delta(\xi)$ has ordertype $\rho_{\mathrm{index}_\delta(\xi)}$, and $\pi^\delta_\xi : \rho_{\mathrm{index}_\delta(\xi)} \longrightarrow b_\delta(\xi)$ is the order isomorphism.

Let us now pause for some clarificatory comments. The base function gives two pieces of data. The first is a set of ordinals which may be viewed as the "memory," i.e., the coordinates of a regular suborder from which the relevant name is drawn. The second relates to the index function, where the index function tells us which canonical partition product the "memory" is isomorphic to. The second piece of data from the base is the specific isomorphism from (i) the canonical partition product chosen by the value of the index function to (ii) the "memory" given by the base. We will later need these isomorphisms to be somewhat flexible: rather than simply requiring that the isomorphism is given by an order-preserving map from the canonical partition product to the base, we want to be able to rearrange the coordinates of the base (for instance, when "products" appear in the course of the iteration). The next part of the definition addresses this.

Given a partition product $\mathbb{R}$ with domain X based upon $\underline{\mathbb{P}} \restriction \delta$ and $\underline{\dot{\mathbb{Q}}} \restriction \delta$, for some $\delta \in C \cap (\kappa + 1)$, we say that a bijection $\sigma : X \longrightarrow X^*$ is an *acceptable rearrangement of* $\mathbb{R}$ if for all $\zeta, \xi \in X$, if $\zeta \in b_{\mathbb{R}}(\xi)$, then $\sigma(\zeta) < \sigma(\xi)$. The definition of partition products below is such that the following holds:

(iv) let $\delta \in C \cap \kappa$, let $\mathbb{R}$ be a partition product with domain X based upon $\underline{\mathbb{P}} \restriction \delta$ and $\underline{\dot{\mathbb{Q}}} \restriction \delta$, and suppose that $\sigma : X \longrightarrow X^*$ is an acceptable rearrangement of $\mathbb{R}$. Then σ lifts uniquely to an isomorphism (also denoted σ) from $\mathbb{R}$ to a partition product $\mathbb{R}^*$ on X^* based upon $\underline{\mathbb{P}} \restriction \delta$ and $\underline{\dot{\mathbb{Q}}} \restriction \delta$. We also have that any $\mathbb{R}$-name $\dot{\tau}$ lifts to a name in $\mathbb{R}^*$, which we denote by $\sigma(\dot{\tau})$, such that if G is generic for $\mathbb{R}$ and if G^* is the isomorphic generic induced by σ, then $\dot{\tau}[G] = \sigma(\dot{\tau})[G^*]$.

We call the partition product $\mathbb{R}^*$ in (iv) the *σ-rearrangement of* $\mathbb{R}$ and denote it by $\sigma[\mathbb{R}]$; we also refer to the $\mathbb{R}^*$-name $\sigma(\dot{\tau})$ as the *σ-rearrangement of* $\dot{\tau}$. Our definition of $\mathbb{R}^*$ and the lifted embedding, which we give later, are such that the next item holds:

(v) let δ, X, $\mathbb{R}$, and σ be as in (iv). Then for each $\xi \in X$, $\mathrm{base}_{\sigma[\mathbb{R}]}(\sigma(\xi)) = (\sigma[b_{\mathbb{R}}(\xi)], \sigma \circ \pi_\xi^{\mathbb{R}})$ and $\mathrm{index}_{\sigma[\mathbb{R}]}(\sigma(\xi)) = \mathrm{index}_{\mathbb{R}}(\xi)$.

In light of the requirement from (i) that $\rho_\delta \leq \delta^+$, for each $\delta \in C \cap \kappa$, let us fix surjections $\varphi_{\delta,\mu} : \delta \longrightarrow \mu$ for each $\mu < \rho_\delta$. We refer to this sequence of surjections as $\vec{\varphi}$, and we fix this notation until specified in Section 5. The next part of the definition specifies the coherence conditions we wish to impose on the memory overlaps.

Suppose that $\bar{\delta} \leq \delta$ are both in $C \cap \kappa$, $\bar{\mu} < \rho_{\bar{\delta}}$, and $\mu < \rho_\delta$. We say that a subset A of μ *matches* $\langle \delta, \mu \rangle$ to $\langle \bar{\delta}, \bar{\mu} \rangle$ if the following three conditions are satisfied:

(a) A is of the form $\varphi_{\delta,\mu}[\bar{\delta}]$;

(b) A is a countably closed subset of μ, i.e., closed under limit points less than μ of cofinality ω;

(c) if $\mu > \bar{\delta}$, then $\delta \in A$, $A \cap \delta = \bar{\delta}$, and, letting j denote the transitive collapse of A, we have that

$$j \circ \varphi_{\delta,\mu} \restriction \bar{\delta} = \varphi_{\bar{\delta},\bar{\mu}}.$$

We will now define what it means for two functions base and index on a set X to support a partition product, and after doing so, we will finally define a partition product.

Definition 2.1.1. *Let X be a set of ordinals, and let* base *and* index *be two functions with domain X. We say that* base *and* index *are functions which* support a partition product on X based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$ *if the following conditions are satisfied:*

1. *for each $\xi \in X$,* $\mathrm{index}(\xi) \in C \cap \kappa$ *and* $\mathrm{base}(\xi)$ *is a pair $(b(\xi), \pi_\xi)$, where $b(\xi) \subseteq X \cap \xi$ and $\pi_\xi : \rho_{\mathrm{index}(\xi)} \longrightarrow b(\xi)$ is an acceptable rearrangement of $\mathbb{P}_{\mathrm{index}(\xi)}$;*

2. *let $\xi \in X$, and set $\delta := \mathrm{index}(\xi)$. Then for all $\zeta \in b(\xi)$, setting $\zeta_0 := \pi_\xi^{-1}(\zeta)$, we have* $\mathrm{base}(\zeta) = (\pi_\xi[b_\delta(\zeta_0)], \pi_\xi \circ \pi^\delta_{\zeta_0})$ *and* $\mathrm{index}(\zeta) = \mathrm{index}_\delta(\zeta_0)$*;*

3. *let $\xi_1, \xi_2 \in X$. Suppose that $\mathrm{index}(\xi_1) \leq \mathrm{index}(\xi_2)$ and that there is some $\zeta \in b(\xi_1) \cap b(\xi_2)$. Set $\mu_1 := \pi_{\xi_1}^{-1}(\zeta)$ and $\mu_2 := \pi_{\xi_2}^{-1}(\zeta)$. Then $\pi_{\xi_1}[\mu_1] \subseteq \pi_{\xi_2}[\mu_2]$, and $\pi_{\xi_2}^{-1}[\pi_{\xi_1}[\mu_1]]$ matches $\langle \mathrm{index}(\xi_2), \mu_2 \rangle$ to $\langle \mathrm{index}(\xi_1), \mu_1 \rangle$.*

Definition 2.1.2. *We say that $\mathbb{R}$ is a* partition product *with domain X, based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$, with base and index functions* $\mathrm{base}_{\mathbb{R}}$ *and* $\mathrm{index}_{\mathbb{R}}$ *if*

1. $\mathrm{base}_{\mathbb{R}}$ *and* $\mathrm{index}_{\mathbb{R}}$ *support a partition product on X based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$ as in Definition 2.1.1;*

2. *$\mathbb{R}$ consists of all finite partial functions p with $\mathrm{dom}(p) \subseteq X$ so that for all $\xi \in \mathrm{dom}(p)$, $p(\xi)$ is a canonical $\pi^{\mathbb{R}}_\xi[\mathbb{P}_{\mathrm{index}_{\mathbb{R}}(\xi)}]$-name for an element of $\dot{\mathbb{U}}_\xi := \pi^{\mathbb{R}}_\xi(\dot{\mathbb{Q}}_{\mathrm{index}_{\mathbb{R}}(\xi)})$, i.e., the $\pi^{\mathbb{R}}_\xi$-rearrangement of the $\mathbb{P}_{\mathrm{index}_{\mathbb{R}}(\xi)}$-name $\dot{\mathbb{Q}}_{\mathrm{index}_{\mathbb{R}}(\xi)}$, as in (iv).*

$\mathbb{R}$ is ordered as follows: $q \leq_{\mathbb{R}} p$ iff $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, and for all $\xi \in \mathrm{dom}(p)$,

$$q \restriction b_{\mathbb{R}}(\xi) \Vdash_{\pi^{\mathbb{R}}_\xi[\mathbb{P}_{\mathrm{index}_{\mathbb{R}}(\xi)}]} q(\xi) \leq_{\dot{\mathbb{U}}_\xi} p(\xi).$$

The definition of a partition product refers not only to the sequences $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$, but additionally to the ordinal κ, and to the sequence of functions index_δ, base_δ, and $\varphi_{\delta,\mu}$ for $\delta \in C \cap \kappa$ and $\mu < \rho_\delta$. We suppress this dependence in the notation, viewing these additional objects as implicit in $\underline{\dot{\mathbb{Q}}} \restriction \kappa$.

We have one final bit of notation before making a number of additional remarks about the definition: given a partition product $\mathbb{R}$ with domain X, say, and given $X_0 \subseteq X$, we define $\mathbb{R} \restriction X_0$ to be the set $\{p \in \mathbb{R} : \mathrm{dom}(p) \subseteq X_0\}$, with the restriction of $\leq_{\mathbb{R}}$, which may or may not itself be a partition product.

Remark 2.1.3. Note that the definition of the ordering $\leq_{\mathbb{R}}$ in Definition 2.1.2 presupposes that for each $q \in \mathbb{R}$ and $\xi \in \mathrm{dom}(q)$, $q \restriction b_{\mathbb{R}}(\xi)$ is a condition in $\pi_\xi^{\mathbb{R}}[\mathbb{P}_{\mathrm{index}_{\mathbb{R}}(\xi)}]$. This holds as follows: fix $q \in \mathbb{R}$, $\xi \in \mathrm{dom}(q)$, and set $\delta := \mathrm{index}_{\mathbb{R}}(\xi)$. Let $\mathbb{S}$ abbreviate the poset $\pi_\xi^{\mathbb{R}}[\mathbb{P}_\delta]$. As $\delta < \kappa$, we know by recursion that $\mathbb{S}$ consists of all finite partial functions u on $b_{\mathbb{R}}(\xi)$ such that for each $\zeta \in \mathrm{dom}(u)$, $u(\zeta)$ is a canonical $\pi_\zeta^{\mathbb{S}}[\mathbb{P}_{\mathrm{index}_{\mathbb{S}}(\zeta)}]$-name for an element of $\pi_\zeta^{\mathbb{S}}(\dot{\mathbb{Q}}_{\mathrm{index}_{\mathbb{S}}(\zeta)})$. Now fixing $\zeta \in b_{\mathbb{R}}(\xi) \cap \mathrm{dom}(q)$, by (2) of Definition 2.1.1 and item (v), $\mathrm{base}_{\mathbb{R}}(\zeta) = \mathrm{base}_{\mathbb{S}}(\zeta)$ and $\mathrm{index}_{\mathbb{R}}(\zeta) = \mathrm{index}_{\mathbb{S}}(\zeta)$, and therefore, $q(\zeta)$ is indeed a canonical $\pi_\zeta^{\mathbb{S}}[\mathbb{P}_{\mathrm{index}_{\mathbb{S}}(\zeta)}]$-name for a condition in $\pi_\zeta^{\mathbb{S}}(\dot{\mathbb{Q}}_{\mathrm{index}_{\mathbb{S}}(\zeta)})$. Thus $q \restriction b_{\mathbb{R}}(\xi)$ is a condition in $\mathbb{S}$.

Note also that by similar reasoning, every condition in $\pi_\xi^{\mathbb{R}}[\mathbb{P}_{\mathrm{index}_{\mathbb{R}}(\xi)}]$ is a condition in $\mathbb{R}$, and in fact, $\mathbb{R} \restriction b_{\mathbb{R}}(\xi)$ equals $\pi_\xi^{\mathbb{R}}[\mathbb{P}_{\mathrm{index}_{\mathbb{R}}(\xi)}]$.

Remark 2.1.4. A partition product based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$ should be viewed (roughly) as an iteration into which we can fit many copies of the shorter posets $\mathbb{P}_\delta$ and $\mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta$, for $\delta \in C \cap \kappa$. In this way, the canonical partition products function as a kind of "alphabet" with which we build other partition products. In most of our intended applications, each name $\dot{\mathbb{Q}}_\delta$ will either be Cohen forcing for adding a single real or will be a $\mathbb{P}_\delta$-name for a poset to decompose ω_1 into countably-many homogeneous sets with respect to some open coloring $\dot{\chi}$.

Remark 2.1.5. A partition product is a somewhat flexible object in that we have a limited, but non-trivial, ability to rearrange coordinates. The reason we need these rearrangements to be *acceptable*, as defined above, is that if $\mathbb{R}$ is a partition product and $\zeta \in b_{\mathbb{R}}(\xi)$, then what happens at coordinate ξ depends on what happens at the earlier coordinate ζ, and therefore the image of ζ under a rearrangement must remain below the image of ξ. The ability to rearrange coordinates will be useful later on when we need to check (roughly) that there are not too many isomorphism types of sufficiently simple partition products (see Lemma 3.2.10).

Remark 2.1.6. We will prove Theorem 1.0.3 by forcing over L with a partition product $\mathbb{P}_{\omega_2}$ with domain ω_3. When we construct these objects in L, the set C in the definition will consist, roughly, of all uncountable $\kappa < \omega_2$ which look locally like ω_2, and the sequence $\vec{\varphi}$ will consist of canonical surjections in L. More specifically, we will show how to construct the sequences $\underline{\mathbb{P}} = \langle \mathbb{P}_\delta : \delta \in C \cup \{\omega_2\}\rangle$ and $\underline{\dot{\mathbb{Q}}} = \langle \dot{\mathbb{Q}}_\delta : \delta \in C\rangle$ in such a way that for each $\kappa \in C \cup \{\omega_2\}$, every partition product based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$ is c.c.c. In particular, our final partition product $\mathbb{P}_{\omega_2}$ will be c.c.c., which is the result that we need.

Every partition product is a dense subset of an iteration, as the next lemma shows.

Lemma 2.1.7. *Let $\mathbb{R}$ be a partition product with domain X. Then $\mathbb{R}$ is a dense subset of a finite support iteration on X.*

Proof. Let $\mathbb{R}^*$ be the finite support iteration based upon the sequence of names $\langle \dot{\mathbb{U}}_\xi : \xi \in X\rangle$, where the names are defined as in Definition 2.1.2 (2). Then $\mathbb{R}$ is a dense subset of $\mathbb{R}^*$; the proof is straightforward, using the fact that for each $\xi \in X$, $\dot{\mathbb{U}}_\xi$ is an $\mathbb{R} \restriction b_{\mathbb{R}}(\xi)$-name for a poset. $\square$

Remark 2.1.8. In studying partition products, we choose to work with this dense subset, rather than the iteration itself, to avoid various technicalities, especially with regards to restricting conditions.

We now want to understand further circumstances wherein we may restrict a partition product with domain X to various subsets of X and still obtain a partition product. This motivates the following key definition.

Definition 2.1.9. *Let $\mathbb{R}$ be a partition product, say with domain X, and let $B \subseteq X$. We say that B is* base-closed with respect to $\mathbb{R}$ *if for all $\xi \in B$, $b_{\mathbb{R}}(\xi) \subseteq B$.*

If the partition product $\mathbb{R}$ is clear from context, we will often drop the phrase "with respect to $\mathbb{R}$" in the above definition and simply say that $B \subseteq X$ is *base-closed*. We will also drop the "$\mathbb{R}$" from expressions such as $\mathrm{index}_{\mathbb{R}}$, $b_{\mathbb{R}}(\xi)$, and $\pi^{\mathbb{R}}_{\xi}$ if the context is clear.

Lemma 2.1.10. *Let $\mathbb{R}$ be a partition product with domain X, and let $\xi \in X$. Then $b(\xi)$ is base-closed. Also, for each $\zeta \in b(\xi)$, $\mathrm{index}(\zeta) < \mathrm{index}(\xi)$.*

Proof. Set $\delta := \mathrm{index}(\xi)$, let $\zeta \in b(\xi)$, and set $\zeta_0 := \pi_{\xi}^{-1}(\zeta)$. Then since π_ξ is an acceptable rearrangement of $\mathbb{P}_\delta$, condition (2) in Definition 2.1.1 and item (v) imply that $b(\zeta) = b_{\pi_\xi[\mathbb{P}_\delta]}(\zeta)$ and also that $b_{\pi_\xi[\mathbb{P}_\delta]}(\zeta)$ equals $\pi_\xi[b_\delta(\zeta_0)] \subseteq b(\xi)$. Thus $b(\zeta) \subseteq b(\xi)$.

To see that $\mathrm{index}(\zeta) < \delta$, we recall that $\mathrm{index}(\zeta) = \mathrm{index}_{\pi_\xi[\mathbb{P}_\delta]}(\zeta)$ which in turn equals $\mathrm{index}_\delta(\zeta_0)$. Since $\mathbb{P}_\delta$ is a partition product based upon $\underline{\mathbb{P}} \restriction \delta$ and $\underline{\dot{\mathbb{Q}}} \restriction \delta$, we must have $\mathrm{index}_\delta(\zeta_0) \in C \cap \delta$, and therefore $\mathrm{index}(\zeta) = \mathrm{index}_\delta(\zeta_0)$ is below δ. $\square$

The following lemma tells us that we may restrict the functions in a partition product to a base-closed subset and obtain a partition product which is also a regular suborder of the original.

Lemma 2.1.11. *Suppose that $\mathbb{R}$ is a partition product with domain X and that $B \subseteq X$ is base-closed. Then* base $\restriction B$ *and* index $\restriction B$ *support a partition product on B, and this partition product is exactly $\mathbb{R} \restriction B$. Moreover, if there is a $\beta \in C$ such that $\{\mathrm{index}(\xi) : \xi \in B\} \subseteq \beta$, then $\mathbb{R} \restriction B$ is a partition product based upon $\underline{\mathbb{P}} \restriction \beta$ and $\underline{\dot{\mathbb{Q}}} \restriction \beta$. Finally, $\mathbb{R} \restriction B$ is a complete subposet of $\mathbb{R}$.*

Proof. It is straightforward to check that base $\restriction B$ and index $\restriction B$ support a partition product on B, using the fact that B is base-closed and also to check that $\mathbb{R} \restriction B$ is the partition product supported by these functions. It is also straightforward to see that $\mathbb{R} \restriction B$ is based upon $\underline{\mathbb{P}} \restriction \beta$ and $\underline{\dot{\mathbb{Q}}} \restriction \beta$ if $\mathrm{index}(\xi) < \beta$, for all $\xi \in B$.

We now verify that the inclusion is a complete embedding of $\mathbb{R} \restriction B$ into $\mathbb{R}$. The only non-trivial property which we must check is the following: if $p \in \mathbb{R}$, $q \in \mathbb{R} \restriction B$, and $q \leq_{\mathbb{R}\restriction B} p \restriction B$, then q and p are compatible in $\mathbb{R}$. To see this, fix such p and q. We claim that

$r := q \cup p \upharpoonright (X \setminus \mathrm{dom}(q))$ is a condition in $\mathbb{R}$ which is below p and q. As it is clear that r is a condition, by (2) of Definition 2.1.2, we check that it is below both p and q. Fix $\xi \in \mathrm{dom}(r)$, and suppose that $r \upharpoonright \xi$ is below both $p \upharpoonright \xi$ and $q \upharpoonright \xi$. If ξ is not in $\mathrm{dom}(q) \cap \mathrm{dom}(p)$, then it is clear that $r \upharpoonright (\xi + 1)$ is a condition below both $p \upharpoonright (\xi + 1)$ and $q \upharpoonright (\xi + 1)$. So suppose that $\xi \in \mathrm{dom}(q) \cap \mathrm{dom}(p)$, and in particular, that $\xi \in B$. Since q extends $p \upharpoonright B$ in $\mathbb{R} \upharpoonright B$ and since the base and index functions for $\mathbb{R} \upharpoonright B$ are the restrictions of those for $\mathbb{R}$, we have that $q \upharpoonright b(\xi)$ forces in $\pi_\xi[\mathbb{P}_{\mathrm{index}(\xi)}]$ that $q(\xi) \leq_{\dot{\mathbb{U}}_\xi} p(\xi)$, where $\dot{\mathbb{U}}_\xi = \pi_\xi(\dot{\mathbb{Q}}_{\mathrm{index}(\xi)})$. Since $r \upharpoonright \xi$ extends $q \upharpoonright \xi$ and since $b(\xi) \subseteq \xi$, we know that $r \upharpoonright b(\xi)$ extends $q \upharpoonright b(\xi)$ in $\pi_\xi[\mathbb{P}_{\mathrm{index}(\xi)}]$. Therefore $r \upharpoonright b(\xi)$ also forces that $q(\xi)$ is below $p(\xi)$ in $\dot{\mathbb{U}}_\xi$. Since $r(\xi) = q(\xi)$, this finishes the proof. $\square$

If $\mathbb{R}$, X, and B are as in the previous lemma, and if G is generic for $\mathbb{R}$, we use $G \upharpoonright B$ to denote $\{p \upharpoonright B : p \in G\}$, which is generic for $\mathbb{R} \upharpoonright B$.

2.1.2 Rearranging Partition Products

Our next main goal is to prove the Rearrangement Lemma, which, as the name suggests, allows us to use an acceptable rearrangement to shift around the coordinates of a partition product and still obtain a partition product. More specifically, if we have an acceptable rearrangement of a partition product $\mathbb{R}$, then we can "compose" it with the base and index functions from $\mathbb{R}$, as stated in the next definition.

Definition 2.1.12. *Suppose that $\sigma : X \longrightarrow X^*$ is an acceptable rearrangement of $\mathbb{R}$, a partition product with domain X. We define the functions $\sigma[\mathrm{base}_{\mathbb{R}}]$ and $\sigma[\mathrm{index}_{\mathbb{R}}]$ on X^* as follows: fix $\xi \in X$. Then set $\sigma[\mathrm{index}_{\mathbb{R}}](\sigma(\xi)) = \mathrm{index}_{\mathbb{R}}(\xi)$, and set $\sigma[\mathrm{base}_{\mathbb{R}}](\sigma(\xi))$ to be the pair*

$$(b^*(\sigma(\xi)), \pi^*_{\sigma(\xi)}),$$

where $b^*(\sigma(\xi)) = \sigma[b_{\mathbb{R}}(\xi)]$, and where $\pi^*_{\sigma(\xi)} = \sigma \circ \pi^{\mathbb{R}}_\xi$.

The following item, known as the Rearrangement Lemma, shows that the objects as in

Definition 2.1.12 support a partition product isomorphic to the original one. The Rearrangement Lemma yields condition (iv) stated before Definition 2.1.1 above. It is proved for partition products based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$ by induction on κ, assuming it is already known for $\delta < \kappa$.

Lemma 2.1.13. (Rearrangement Lemma) *Suppose that $\mathbb{R}$ is a partition product with domain X and that $\sigma : X \longrightarrow X^*$ is an acceptable rearrangement of $\mathbb{R}$. Then $\sigma[\mathrm{base}_{\mathbb{R}}]$ and $\sigma[\mathrm{index}_{\mathbb{R}}]$ support a partition product on X^*. Moreover, letting $\sigma[\mathbb{R}]$ be this partition product, we have that there is a unique lift of σ to an isomorphism from $\mathbb{R}$ to $\sigma[\mathbb{R}]$.*

Proof. It is straightforward to check that the functions $\sigma[\mathrm{base}_{\mathbb{R}}]$ and $\sigma[\mathrm{index}_{\mathbb{R}}]$ satisfy all three conditions of Definition 2.1.1, since σ is an acceptable rearrangement. Thus we show that σ lifts to an isomorphism, also denoted σ, from $\mathbb{R}$ onto $\sigma[\mathbb{R}]$. Let $p \in \mathbb{R}$. Then we set $\sigma(p)$ to be the function with domain $\sigma[\mathrm{dom}(p)]$ such that for each $\xi \in \mathrm{dom}(p)$, $\sigma(p)(\sigma(\xi))$ equals the $\sigma \restriction b_{\mathbb{R}}(\xi)$-rearrangement of the name $p(\xi)$, as in (iv). This is well-defined by an inductive application of the Rearrangement Lemma to the acceptable rearrangement $\sigma \restriction b_{\mathbb{R}}(\xi)$ of the partition product $\mathbb{R} \restriction b_{\mathbb{R}}(\xi)$, which is based upon the sequence up to $\mathrm{index}_{\mathbb{R}}(\xi) < \kappa$. It is straightforward to see that $\sigma(p)$ is a condition in $\sigma[\mathbb{R}]$ and that this defines an isomorphism. $\square$

Remark 2.1.14. Given $\mathbb{R}$ and σ as in Lemma 2.1.13 and setting $\mathbb{R}^* := \sigma[\mathbb{R}]$, if G is generic for $\mathbb{R}$, then we use $\sigma(G)$ to denote the generic $\{\sigma(p) : p \in G\}$ for $\mathbb{R}^*$. Furthermore, given an $\mathbb{R}$-name $\dot{\tau}$, we recursively define $\sigma(\dot{\tau})$ to be the $\sigma[\mathbb{R}]$-name $\{\langle \sigma(p), \sigma(\dot{x}) \rangle : \langle p, \dot{x} \rangle \in \dot{\tau}\}$. It is straightforward to check that $\dot{\tau}[G] = \sigma(\dot{\tau})[\sigma(G)]$ for any generic G for $\mathbb{R}$. This name $\sigma(\dot{\tau})$ is the σ-rearrangement of $\dot{\tau}$ as in (iv) above.

Remark 2.1.15. Suppose that M and M^* are transitive, satisfy a sufficiently large fragment of $\mathsf{ZFC} - \mathsf{Powerset}$, and that $\sigma : M \longrightarrow M^*$ is an elementary embedding. Also, suppose that $\mathbb{R} \in M$ is a partition product, say with domain X, and that $\mathbb{R}$ is based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$. It is straightforward to check that $\pi := \sigma \restriction X$ provides an acceptable rearrangement of $\mathbb{R}$. There is now a potential conflict between the π-rearrangements of conditions in $\mathbb{R}$ and the images of these conditions under the embedding σ. However, these are the same if σ doesn't move any members of the "alphabet" $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$. The next lemma summarizes what we need about this situation and will be used crucially in the final proof of Theorem 1.0.3, which occurs in Chapter 4. For the next lemma, we will continue to use π to denote rearrangements, and we will keep σ as the elementary map.

Lemma 2.1.16. *Let $\sigma : M \longrightarrow M^*$, $\mathbb{R}$, X, κ, and π be as in Remark 2.1.15. Further suppose that for each $\delta \in C \cap \kappa$, σ is the identity on every element of $\mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta \cup \left\{\mathbb{P}_\delta, \dot{\mathbb{Q}}_\delta\right\}$. Then for each $p \in \mathbb{R}$, $\pi(p) = \sigma(p)$.*

Furthermore, setting $\mathbb{R}^* := \sigma(\mathbb{R})$, $\sigma[X]$ is a base-closed subset of $\mathbb{R}^*$, and $\mathbb{R}^* \restriction \sigma[X]$ equals $\pi[\mathbb{R}]$, the π-rearrangement of $\mathbb{R}$.

Additionally, suppose that G is V-generic for $\mathbb{R}$, G^* is V-generic for $\mathbb{R}^*$, and σ extends to a sufficiently elementary embedding $\sigma^* : M[G] \longrightarrow M^*[G^*]$. Suppose also that $\dot{\tau}$ is an $\mathbb{R}$-name (not necessarily in M) and $\pi(\dot{\tau})$ is the π-rearrangement of $\dot{\tau}$. Then $\pi(\dot{\tau})$ is an $\mathbb{R}^*$-name, and $\dot{\tau}[G] = \pi(\dot{\tau})[G^*]$. Finally, if $\dot{\mathbb{Q}}$ is an $\mathbb{R}$-name in M of M-cardinality $< \mathrm{crit}(\sigma)$ and names a poset contained in $\mathrm{crit}(\sigma)$, then $\sigma(\dot{\mathbb{Q}}) = \pi(\dot{\mathbb{Q}})$.

Proof. We only prove the second and third parts. For the second part, fix some $\xi \in X$. Then $b_{\mathbb{R}}(\xi)$ is in bijection, via a bijection in M, with some ρ_α, for $\alpha < \kappa$. However, ρ_α is below $\mathrm{crit}(\sigma)$, since σ is the identity on $\mathbb{P}_\alpha$. Therefore,

$$b_{\mathbb{R}^*}(\sigma(\xi)) = \sigma(b_{\mathbb{R}}(\xi)) = \sigma[b_{\mathbb{R}}(\xi)],$$

where the first equality holds by the elementarity of σ and the second since $\mathrm{crit}(\sigma) > |b_{\mathbb{R}}(\xi)|$. This implies that $\sigma[X]$ is base-closed, and therefore $\mathbb{R}^* \restriction \sigma[X]$ is a partition product by Lemma 2.1.11. By the first part of the current lemma, we see that every condition in $\mathbb{R}^* \restriction \sigma[X]$ is in the image of σ. However, $\pi(p) = \sigma(p)$ for each condition $p \in \mathbb{R}$, and consequently $\mathbb{R}^* \restriction \sigma[X]$ equals $\pi[\mathbb{R}]$, the π-rearrangement of $\mathbb{R}$.

For the third part, let G and G^* be as in the statement of the lemma. Also let $\pi(G)$ denote the π-rearrangement of the filter G, as defined in Remark 2.1.14. By same remark, we have that $\dot{\tau}[G] = \pi(\dot{\tau})[\pi(G)]$. We also see that $\pi(\dot{\tau})$ is an $\mathbb{R}^*$-name, since it is a $\pi[\mathbb{R}]$-name and since, by the second part of the lemma, $\pi[\mathbb{R}] = \mathbb{R}^* \restriction \sigma[X]$ and $\sigma[X]$ is base-closed. Furthermore, $\sigma[G]$ is a subset of G^*, by the elementarity of σ^*. However, by the first part of the current lemma, $\sigma[G] = \{\sigma(p) : p \in G\} = \{\pi(p) : p \in G\} = \pi(G)$, and therefore

$$\dot{\tau}[G] = \pi(\dot{\tau})[\pi(G)] = \pi(\dot{\tau})[G^*].$$

Finally, if $\dot{\mathbb{Q}} \in M$ and satisfies the assumptions in the statement of the lemma, then $\sigma(\dot{\mathbb{Q}}) = \sigma[\dot{\mathbb{Q}}]$, and $\sigma[\dot{\mathbb{Q}}] = \pi(\dot{\mathbb{Q}})$. This completes the proof of the lemma. $\square$

Before we give applications of the Rearrangement Lemma, we record our definition of an embedding.

Definition 2.1.17. *Suppose that $\mathbb{R}$ and $\mathbb{R}^*$ are partition products with respective domains X and X^*. We say that an injection $\sigma : X \longrightarrow X^*$* embeds *$\mathbb{R}$ into $\mathbb{R}^*$ if $\sigma : X \longrightarrow \mathrm{ran}(\sigma)$ is an acceptable rearrangement of $\mathbb{R}$, and if $\sigma[\mathrm{base}_{\mathbb{R}}] = \mathrm{base}_{\mathbb{R}^*} \restriction \mathrm{ran}(\sigma)$ and $\sigma[\mathrm{index}_{\mathbb{R}}] = \mathrm{index}_{\mathbb{R}^*} \restriction \mathrm{ran}(\sigma)$.*

It is straightforward to check that if σ is an embedding as in Definition 2.1.17, and if G^* is generic over $\mathbb{R}^*$, then the filter $\sigma^{-1}(G^*) := \{p \in \mathbb{R} : \sigma(p) \in G^*\}$ is generic over $\mathbb{R}$. We also remark that, in the context of the above definition, $\sigma[\mathbb{R}] = \mathbb{R}^* \restriction \mathrm{ran}(\sigma)$.

Lemma 2.1.18. *Suppose that $\mathbb{R}$ is a partition product with domain X and $B \subseteq X$ is base-closed. Then $\mathbb{R}$ is isomorphic to a partition product $\mathbb{R}^*$ with a domain X^* such that B is an initial segment of X^* and $\mathbb{R}^* \restriction B = \mathbb{R} \restriction B$.*

Proof. We define a map σ with domain X which will lift to give us $\mathbb{R}^*$. Let $\xi \in X$. If $\xi \in B$, then set $\sigma(\xi) = \xi$. On the other hand, if $\xi \in X \backslash B$, say that ξ is the γth element of $X \backslash B$, then we define $\sigma(\xi) = \sup(X) + 1 + \gamma$.

We show that σ is an acceptable rearrangement of $\mathbb{R}$, and then we may set $\mathbb{R}^* := \sigma[\mathbb{R}]$ by Lemma 2.1.13. So suppose that $\zeta, \xi \in X$ and $\zeta \in b(\xi)$; we check that $\sigma(\zeta) < \sigma(\xi)$. There are two cases. On the one hand, if $\xi \in B$, then $b(\xi) \subseteq B$, since B is base-closed, and therefore

$\zeta \in B$. Then $\sigma(\zeta) = \zeta < \xi = \sigma(\xi)$. On the other hand, if $\xi \notin B$, then either $\zeta \in B$ or not. If $\zeta \in B$, then $\sigma(\zeta) = \zeta < \sup(X) + 1 \leq \sigma(\xi)$, and if $\zeta \notin B$, then $\sigma(\zeta) < \sigma(\xi)$ since σ is order-preserving on $X \backslash B$. □

It will be helpful later on to know that we can apply Lemma 2.1.18 ω-many times, as in the following corollary.

Corollary 2.1.19. *Suppose that $\mathbb{R}$ is a partition product with domain X and that for each $n < \omega$, π_n is an acceptable rearrangement of $\mathbb{R}$. Suppose that $\langle B_n : n \in \omega \rangle$ is a $\subseteq$-increasing sequence of base-closed subsets of X where $B_0 = \varnothing$ and where $X = \bigcup_n B_n$. Then there is a partition product $\mathbb{R}^*$ which has domain an ordinal ρ^* and an acceptable rearrangement $\sigma : X \longrightarrow \rho^*$ of $\mathbb{R}$ which lifts to an isomorphism of $\mathbb{R}$ onto $\mathbb{R}^*$ and which also satisfies that for each $n < \omega$, $\sigma[B_n]$ is an ordinal and $\pi_n \circ \sigma^{-1}$ is order-preserving on $\sigma[B_{n+1} \backslash B_n]$.*

Proof. We aim to recursively construct a sequence $\langle \mathbb{R}_n : n < \omega \rangle$ of partition products, where $\mathbb{R}_n$ has domain X_n, and a sequence $\langle \sigma_n : n < \omega \rangle$ of bijections, where $\sigma_n : X \longrightarrow X_n$, so that

1. σ_n is an acceptable rearrangement of $\mathbb{R}$;
2. $\sigma_n[B_n]$ is an ordinal, and in particular, an initial segment of X_n;
3. for each $k < m < \omega$, $\sigma_k[B_k] = \sigma_m[B_k]$;
4. for each $n < \omega$, $\pi_n \circ \sigma_{n+1}^{-1}$ is order-preserving on $\sigma_{n+1}[B_{n+1} \backslash B_n]$.

Suppose that we can do this. Then we define a map σ on X, by taking $\sigma(\xi)$ to be the eventual value of the sequence $\langle \sigma_n(\xi) : n < \omega \rangle$; we see that this sequence is eventually constant by (3) and the assumption that $\bigcup_n B_n = X$. By (2) and (3), $\sigma[B_n]$ is an ordinal, for each $n < \omega$, and therefore the range of σ is an ordinal, which we call ρ^*. Furthermore, $\pi_n \circ \sigma^{-1}$ is order-preserving on $\sigma[B_{n+1} \backslash B_n]$ by (4), and since σ and σ_{n+1} agree on B_{n+1}. Finally, by (1) we see that σ is an acceptable rearrangement of $\mathbb{R}$, and we thus take $\mathbb{R}^*$ to be the partition product isomorphic to $\mathbb{R}$ via σ, by Lemma 2.1.13.

We now show how to create the above objects. Suppose that $\langle \mathbb{R}_m : m < n\rangle$ and $\langle \sigma_m : m < n\rangle$ have been constructed. If $n = 0$, we take $\mathbb{R}_0 = \mathbb{R}$ and σ_0 to be the identity; since $B_0 = \varnothing$, this completes the base case. So suppose $n > 0$. Apply Lemma 2.1.18 to the partition product $\mathbb{R}_{n-1}$ and the base-closed subset $\sigma_{n-1}[B_n]$ of X_{n-1} to create a partition product $\mathbb{R}_n$ on a set X_n which is isomorphic to $\mathbb{R}_{n-1}$ via the acceptable rearrangement $\tau_n : X_{n-1} \longrightarrow X_n$ and which satisfies that $\sigma_{n-1}[B_n]$ is an initial segment of X_n. Since $\sigma_{n-1}[B_{n-1}]$ is an ordinal, by (2) applied to $n-1$, and since $\sigma_{n-1}[B_n]$ is an initial segment of X_n, we see that τ_n is the identity on $\sigma_{n-1}[B_{n-1}]$. Also, by composing τ_n with a further function and relabelling if necessary, we may assume that $\pi_{n-1} \circ \tau_n^{-1}$ just shifts the ordinals in $\sigma_{n-1}[B_n \backslash B_{n-1}]$ in an order-preserving way and that $\tau_n \circ \sigma_{n-1}[B_n]$ is an ordinal. We now take σ_n to be $\tau_n \circ \sigma_{n-1}$, and we see that σ_n and $\mathbb{R}_n$ satisfy the recursive hypotheses. $\square$

Lemma 2.1.20. *Suppose that $\beta \in C \cap \kappa$ and that $\mathbb{R}$ is a partition product with domain X based upon $\underline{\mathbb{P}} \restriction (\beta + 1)$ and $\underline{\dot{\mathbb{Q}}} \restriction (\beta + 1)$. Then, letting $B := \{\xi \in X : \mathrm{index}(\xi) < \beta\}$ and $I := \{\xi \in X : \mathrm{index}(\xi) = \beta\}$, B is base-closed, and $\mathbb{R}$ is isomorphic to*

$$(\mathbb{R} \restriction B) * \prod_{\xi \in I} \dot{\mathbb{Q}}_\beta \left[\pi_\xi^{-1}\left(\dot{G}_B \restriction b(\xi)\right)\right],$$

where $\dot{G}_B$ is the canonical $\mathbb{R} \restriction B$-name for the generic filter.

Proof. To see that B is base-closed, fix $\xi \in B$. Then for all $\zeta \in b(\xi)$, $\mathrm{index}(\zeta) < \mathrm{index}(\xi) < \beta$ by Lemma 2.1.10, and so $\zeta \in B$. Thus by Lemma 2.1.18, we may assume that B is an initial segment of X, and hence I is a tail segment of X. Now let G_B be generic for $\mathbb{R} \restriction B$, and for each $\xi \in I$, let $G_{B,\xi}$ denote $\pi_\xi^{-1}(G_B \restriction b(\xi))$, which is generic for $\mathbb{P}_\beta$. The sequence of posets $\langle \dot{\mathbb{Q}}_\beta[G_{B,\xi}] : \xi \in I\rangle$ is in $V[G_B]$, and consequently the finite support iteration of $\langle \dot{\mathbb{Q}}_\beta[G_{B,\xi}] : \xi \in I\rangle$ in $V[G_B]$ is isomorphic to the (finite support) product $\prod_{\xi\in I} \dot{\mathbb{Q}}_\beta[G_{B,\xi}]$. Therefore, in V, $\mathbb{R}$ is isomorphic to the poset in the statement of the lemma. $\square$

Remark 2.1.21. The previous lemma shows that a partition product does indeed have product-like behavior, and it is part of the justification for our term "partition product."

2.1.3 Further Remarks on Matching

In this subsection we state and prove a few consequences of the matching conditions (a)-(c) above. These results will, in combination with the ability to rearrange a partition product, allow us to find isomorphism types of sufficiently simple partition products inside sufficiently elementary, countably-closed models (see Lemma 3.2.10).

Remark 2.1.22. As mentioned earlier, we will carry out the construction of partition products in L. The matching conditions (a)-(c), combined with Definition 2.1.1, are roughly meant to capture the idea that the base functions, up to some rearranging, behave like the ordinals in a countably-closed Skolem hull of some suitable level of L.

Lemma 2.1.23. *Let $\mathbb{R}$ be a partition product, say with domain X, based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$. Let $\xi_1, \xi_2 \in X$, set $\delta_i = \mathrm{index}(\xi_i)$, for $i = 1, 2$, and suppose that $\delta_1 \leq \delta_2$. Finally, let $A := \pi_{\xi_2}^{-1}[b(\xi_1) \cap b(\xi_2)]$. Then A is definable in $H(\omega_3)$ from $\vec{\varphi}$, the ordinals δ_1 and δ_2, and any cofinal $Z \subseteq A$.*

Proof. Let $Z \subseteq A$ be cofinal. For each $\alpha \in Z$, we have from Definition 2.1.1 (3) and condition (a) in the definition of matching that $A \cap \alpha = \varphi_{\delta_2,\alpha}[\delta_1]$. Therefore $A = \bigcup_{\alpha \in Z} \varphi_{\delta_2,\alpha}[\delta_1]$, which is a member of $H(\omega_3)$. $\square$

Corollary 2.1.24. *Let $\mathbb{R}$, X, ξ_1, ξ_2, and A be as in Lemma 2.1.23. Assume that for all $\xi \in C$, $\rho_\xi < \omega_2$. Suppose that the CH holds, and let $M \prec H(\omega_3)$ be countably closed so that M contains the objects $\underline{\mathbb{P}} \restriction \kappa$, $\underline{\dot{\mathbb{Q}}} \restriction \kappa$, $\vec{\varphi}$, and δ_1, δ_2. Then A is a member of M and the transitive collapse of M.*

Proof. First observe that A is a subset of ρ_{δ_2}, which is a member of M. Since $\rho_{\delta_2} < \omega_2$ and M contains ω_1 as a subset, $\rho_{\delta_2} \subseteq M$. In particular, $\sup(A)$ is an element of M.

Next, consider the case that $\sup(A)$ has countable cofinality. Then by the countable closure of M, we can find a cofinal subset Z of A inside M. By Lemma 2.1.23, we then conclude that $A \in M$.

Now suppose that $\sup(A)$ has uncountable cofinality. Recall from condition (b) of the definition of matching that A is countably closed in $\sup(A)$. Moreover, since $A \cap \alpha = \varphi_{\delta_2,\alpha}[\delta_1]$ for each $\alpha \in A$, we know that the sequence of sets $\langle \varphi_{\delta_2,\alpha}[\delta_1] : \alpha \in A \rangle$ is $\subseteq$-increasing. By the elementarity of M, we may find an ω-closed, cofinal subset Z of $\sup(A)$ such that $Z \in M$ for which the sequence of sets $\langle \varphi_{\delta_2,\alpha}[\delta_1] : \alpha \in Z \rangle$ is $\subseteq$-increasing. Combining this with the fact that $Z \cap A$ is also ω-closed and cofinal in $\sup(A)$, we have that

$$A = \bigcup_{\alpha \in A \cap Z} \varphi_{\delta_2,\alpha}[\delta_1] = \bigcup_{\alpha \in Z} \varphi_{\delta_2,\alpha}[\delta_1],$$

and hence A is in M, as $\bigcup_{\alpha \in Z} \varphi_{\delta_2,\alpha}[\delta_1]$ is in M by elementarity. Finally, since A is bounded in the ordinal $M \cap \omega_2$, A is fixed by the transitive collapse map. $\square$

2.2 Combining Partition Products

In this section, we develop the machinery necessary to combine partition products in various ways. This will be essential for later arguments where, in the context of working with a partition product $\mathbb{R}$, we will want to create another partition product $\mathbb{R}^*$ into which $\mathbb{R}$ embeds in a variety of ways. Forcing with $\mathbb{R}^*$ will then add plenty of generics for $\mathbb{R}$, with various amounts of agreement or mutual genericity.

The main result of this section is a so-called "grafting lemma" which gives conditions under which, given partition products $\mathbb{P}$ and $\mathbb{R}$, we may extend $\mathbb{R}$ to another partition product $\mathbb{R}^*$ in such a way that $\mathbb{R}^*$ subsumes an isomorphic image of $\mathbb{P}$; in this case $\mathbb{P}$ is, in some sense, "grafted onto" $\mathbb{R}$. One trivial way of doing this, we will show, is to take the partition product $\mathbb{P} \times \mathbb{R}$. However, the issue becomes somewhat delicate if we desire, as later on we often will, that $\mathbb{R}$ and the isomorphic copy of $\mathbb{P}$ in $\mathbb{R}^*$ have coordinates in common, and hence share some part of their generics. Doing so requires that we keep track of more information about the structure of a partition product, and we begin with the relevant definition in the first subsection.

2.2.1 Shadow Bases

Definition 2.2.1. *A triple $\langle x, \pi_x, \alpha\rangle$ is said to be a* shadow base *if the following conditions are satisfied: $\alpha \in C$, π_x has domain γ_x for some $\gamma_x \leq \rho_\alpha$, and $\pi_x : \gamma_x \longrightarrow x$ is an acceptable rearrangement of $\mathbb{P}_\alpha \restriction \gamma_x$.*

Moreover, if $\mathbb{R}$ is a partition product, say with domain X, we say that a shadow base $\langle x, \pi_x, \alpha\rangle$ is an $\mathbb{R}$-shadow base *if $x \subseteq X$ is base-closed and if π_x embeds $\mathbb{P}_\alpha \restriction \gamma_x$ into $\mathbb{R} \restriction x$.*

For example, if $\mathbb{R}$ is a partition product with domain X, then for any $\xi \in X$ the triple $\langle b(\xi), \pi_\xi, \mathrm{index}(\xi)\rangle$ is an $\mathbb{R}$- "shadow" base; this is part of the motivation for the term. In practice, a shadow base will be an initial segment, in a sense we will specify soon, of such a triple.

Definition 2.2.2. *Suppose that $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ are two shadow bases. We say that they* cohere *if the following holds: suppose that $\alpha \leq \beta$ and that there is some $\zeta \in x \cap y$. Define $\mu_x := \pi_x^{-1}(\zeta)$ and $\mu_y := \pi_y^{-1}(\zeta)$. Then*

1. *$\pi_x[\mu_x] \subseteq \pi_y[\mu_y]$; and*
2. *$\pi_y^{-1}[\pi_x[\mu_x]]$ matches $\langle \beta, \mu_y\rangle$ to $\langle \alpha, \mu_x\rangle$.*

A collection $\mathcal{B}$ of shadow bases is said to cohere *if any two elements of $\mathcal{B}$ cohere.*

Note that with this definition, item (3) of Definition 2.1.1 could be rephrased as saying that the two shadow bases $\langle b(\xi_1), \pi_{\xi_1}, \mathrm{index}(\xi_1)\rangle$ and $\langle b(\xi_2), \pi_{\xi_2}, \mathrm{index}(\xi_2)\rangle$ cohere.

Remark 2.2.3. It is straightforward to check that Corollary 2.1.24 holds for shadow bases too, in the following sense. Suppose that $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ are two coherent shadow bases, say with $\alpha \leq \beta$. Then $\pi_y^{-1}[x \cap y]$ is a member of any M as in the statement of Corollary 2.1.24, provided that α and β, as well as the additional parameters $\underline{\mathbb{P}} \restriction \beta$, $\underline{\dot{\mathbb{Q}}} \restriction \beta$, and $\vec{\varphi}$, are all in M.

Definition 2.2.4. *Given a shadow base $\langle x, \pi_x, \alpha\rangle$ and some $a \subseteq x$, we say that a is an* initial segment *of $\langle x, \pi_x, \alpha\rangle$ if a is of the form $\pi_x[\mu]$ for some $\mu \leq \mathrm{dom}(\pi_x)$.*

Given two shadow bases $\langle x_0, \pi_{x_0}, \alpha_0\rangle$ and $\langle x, \pi_x, \alpha\rangle$, we say that $\langle x_0, \pi_{x_0}, \alpha_0\rangle$ is an initial segment *of $\langle x, \pi_x, \alpha\rangle$ if $\alpha_0 = \alpha$, x_0 is an initial segment of $\langle x, \pi_x, \alpha\rangle$, and $\pi_x \restriction \mathrm{dom}(\pi_{x_0}) = \pi_{x_0}$.*

Remark 2.2.5. A simple but useful observation is that if $\langle x_0, \pi_{x_0}, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ are two coherent shadow bases, $\langle x_0, \pi_{x_0}, \alpha\rangle$ is an initial segment of $\langle x, \pi_x, \alpha\rangle$, and $(x \backslash x_0) \cap y = \varnothing$, then $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ cohere.

Lemma 2.2.6. *Suppose that $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ are coherent shadow bases and $\alpha \leq \beta$. Then $\pi_x^{-1}[x \cap y]$ is an ordinal $\leq \mathrm{dom}(\pi_x)$, and hence $x \cap y$ is an initial segment of $\langle x, \pi_x, \alpha\rangle$.*

Proof. Fix $\xi \in x \cap y$. By the definition of coherence and the fact that $\alpha \leq \beta$, we see that $\pi_x^{-1}(\xi) + 1 \subseteq \pi_x^{-1}[x \cap y]$. Thus

$$\pi_x^{-1}[x \cap y] = \bigcup_{\xi \in x \cap y} (\pi_x^{-1}(\xi) + 1),$$

and therefore $\pi_x^{-1}[x \cap y]$ is an ordinal. □

Lemma 2.2.7. *Suppose that $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ are two coherent shadow bases, where $\alpha \leq \beta$. Let $\zeta \in x \cap y$, and define $\mu_x := \pi_x^{-1}(\zeta)$ and $\mu_y := \pi_y^{-1}(\zeta)$. Then $\pi_y^{-1} \circ \pi_x$ is an order preserving map from μ_x into μ_y. In particular, $\mu_x \leq \mu_y$, and $\pi_x^{-1} \circ \pi_y$ is the transitive collapse of $\pi_y^{-1}[\pi_x[\mu_x]]$.*

Proof. By Definition 2.2.2 (1), we know that $\pi_x[\mu_x]$ is a subset of $\pi_y[\mu_y]$, and so $\pi_y^{-1} \circ \pi_x$ is indeed a map from μ_x into μ_y. Let us abbreviate $\pi_y^{-1} \circ \pi_x$ by j. Suppose that $\zeta < \eta < \mu_x$, and we show $j(\zeta) < j(\eta)$. Set $\zeta_y = j(\zeta)$ and $\eta_y = j(\eta)$. Since $\pi_x(\eta) = \pi_y(\eta_y) \in x \cap y$, Definition 2.2.2 (1) implies that $\pi_x[\eta] \subseteq \pi_y[\eta_y]$. Next, as $\zeta < \eta$, $\pi_x(\zeta) \in \pi_x[\eta]$, and so $\pi_y(\zeta_y) \in \pi_y[\eta_y]$. Finally, since π_y is a bijection we conclude that $\zeta_y \in \eta_y$, i.e., $j(\zeta) < j(\eta)$. □

As a result of the previous lemma, if two coherent shadow bases have the same "index", then their intersection is an initial segment of both.

Corollary 2.2.8. *Suppose that $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \alpha\rangle$ are two coherent shadow bases and that $\zeta \in x \cap y$. Then $\zeta_0 := \pi_x^{-1}(\zeta) = \pi_y^{-1}(\zeta)$, and in fact, $\pi_x \restriction (\zeta_0 + 1) = \pi_y \restriction (\zeta_0 + 1)$.*

Proof. Fix $\eta \in x \cap y$. Since both shadow bases have index α, we know from Lemma 2.2.7 that $\pi_x^{-1}(\eta) = \pi_y^{-1}(\eta)$. Since this holds for any $\eta \in x \cap y$, the result follows. $\square$

Remark 2.2.9. In the context of Corollary 2.2.8, we note that $\pi_x^{-1}[x \cap y] = \pi_y^{-1}[x \cap y]$ is an ordinal $\leq \rho_\alpha$, and if $x \neq y$, then this ordinal is strictly less than ρ_α.

We conclude this subsection with a very useful lemma.

Lemma 2.2.10. *Suppose that $\langle x, \pi_x, \alpha\rangle$, $\langle y, \pi_y, \beta\rangle$, and $\langle z, \pi_z, \gamma\rangle$ are shadow bases such that $\alpha, \beta \leq \gamma$. Suppose further that $x \cap y \subseteq z$, that $\langle x, \pi_x, \alpha\rangle$ and $\langle z, \pi_z, \gamma\rangle$ cohere, and that $\langle y, \pi_y, \beta\rangle$ and $\langle z, \pi_z, \gamma\rangle$ cohere. Then $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ cohere.*

Proof. By relabeling if necessary, we assume that $\alpha \leq \beta$. Let $\zeta \in x \cap y$, and we will show that (1) and (2) of Definition 2.2.2 hold. Define $\mu_x := \pi_x^{-1}(\zeta)$ and $\mu_y := \pi_y^{-1}(\zeta)$. As $x \cap y \subseteq z$, $\zeta \in z$, and therefore we may also define $\mu_z := \pi_z^{-1}(\zeta)$. Applying the coherence assumptions in the statement of the lemma, we conclude that

$$\pi_z^{-1}\left[\pi_x[\mu_x]\right] = \varphi_{\gamma,\mu_z}[\alpha] \text{ and } \pi_z^{-1}[\pi_y[\mu_y]] = \varphi_{\gamma,\mu_z}[\beta].$$

Since $\alpha \leq \beta$, it then follows that $\pi_x[\mu_x] \subseteq \pi_y[\mu_y]$.

We next show that $\pi_y^{-1}[\pi_x[\mu_x]] = \varphi_{\beta,\mu_y}[\alpha]$. By Lemma 2.2.7 applied to the shadow bases $\langle y, \pi_y, \beta\rangle$ and $\langle z, \pi_z, \gamma\rangle$, we conclude that $\pi_y^{-1} \circ \pi_z$, which we abbreviate as $j_{z,y}$, is the transitive collapse of $\pi_z^{-1}[\pi_y[\mu_y]]$. Furthermore, the definition of coherence also implies that $j_{z,y} \circ \varphi_{\gamma,\mu_z} \restriction \beta = \varphi_{\beta,\mu_y}$. Since $\alpha \leq \beta$ and since $\pi_z^{-1}[\pi_x[\mu_x]] = \varphi_{\gamma,\mu_z}[\alpha]$, we apply $j_{z,y}$ to conclude that $\pi_y^{-1}[\pi_x[\mu_x]] = \varphi_{\beta,\mu_y}[\alpha]$.

Now let $j_{y,x}$ denote the transitive collapse of $\pi_y^{-1}[\pi_x[\mu_x]]$; we check that $j_{y,x} \circ \varphi_{\beta,\mu_y} \restriction \alpha = \varphi_{\alpha,\mu_x}$. We also let $j_{z,x}$ be the transitive collapse of $\pi_z^{-1}[\pi_x[\mu_x]]$. From Lemma 2.2.7, we know that $j_{y,x} = \pi_x^{-1} \circ \pi_y$ and $j_{z,x} = \pi_x^{-1} \circ \pi_z$. Thus $j_{z,x} = j_{y,x} \circ j_{z,y}$. Since $j_{z,y} \circ \varphi_{\gamma,\mu_z} \restriction \beta = \varphi_{\beta,\mu_y}$ and $\alpha \leq \beta$, we conclude that $\varphi_{\alpha,\mu_x} = j_{y,x} \circ \varphi_{\beta,\mu_y} \restriction \alpha$, completing the proof. $\square$

2.2.2 Enriched Partition Products

In this subsection, we will consider in greater detail how shadow bases interact with partition products. We begin with the following definition.

Definition 2.2.11. *Let $\mathbb{R}$ be a partition product with domain X. A collection $\mathcal{B}$ of $\mathbb{R}$-shadow bases is said to be* $\mathbb{R}$-full *if for all $\xi \in X$, $\langle b(\xi), \pi_\xi, \text{index}(\xi)\rangle \in \mathcal{B}$. $\mathcal{B}$ is said to be an* $\mathbb{R}$-enrichment *if $\mathcal{B}$ is both coherent and $\mathbb{R}$-full.*

An enriched partition product *is a pair $(\mathbb{R}, \mathcal{B})$ where $\mathcal{B}$ is an enrichment of $\mathbb{R}$.*

The next definition is a strengthening of the notion of a base-closed subset which allows us to restrict an enrichment.

Definition 2.2.12. *Let $(\mathbb{R}, \mathcal{B})$ be an enriched partition product with domain X. A base-closed subset $B \subseteq X$ is said to* cohere *with $(\mathbb{R}, \mathcal{B})$ if for all triples $\langle x, \pi_x, \alpha\rangle$ in $\mathcal{B}$ and for every $\zeta \in B \cap x$, if $\zeta = \pi_x(\zeta_0)$, say, then $\pi_x[\zeta_0] \subseteq B$.*

Lemma 2.2.13. *Suppose that $(\mathbb{R}, \mathcal{B})$ is an enriched partition product with domain X and that $B \subseteq X$ coheres with $(\mathbb{R}, \mathcal{B})$. Let $\langle x, \pi_x, \alpha\rangle \in \mathcal{B}$, and define $\pi_{x\cap B}$ to be the restriction of π_x mapping onto $x \cap B$. Then $\langle x \cap B, \pi_{x\cap B}, \alpha\rangle$ is a shadow base.*

Additionally, if we define

$$\mathcal{B} \restriction B := \{\langle x \cap B, \pi_{x\cap B}, \alpha\rangle : \langle x, \pi_x, \alpha\rangle \in \mathcal{B}\},$$

then $(\mathbb{R} \restriction B, \mathcal{B} \restriction B)$ is an enriched partition product.

Proof. To see that $\langle x \cap B, \pi_{x\cap B}, \alpha\rangle$ is a shadow base, it suffices to show that $\pi_x^{-1}[x \cap B]$ is an ordinal. This holds since for each $\xi \in x \cap B$, by the coherence of B with $(\mathbb{R}, \mathcal{B})$, $\pi_x^{-1}(\xi) + 1 \subseteq \pi_x^{-1}[x \cap B]$.

Now we need to verify that $(\mathbb{R} \restriction B, \mathcal{B} \restriction B)$ is an enriched partition product. It is straightforward to see that $\mathcal{B} \restriction B$ is $(\mathbb{R} \restriction B)$-full, since B is base-closed and since the base and index functions for $\mathbb{R} \restriction B$ are exactly the restrictions of those for $\mathbb{R}$. Similarly, we see that each shadow base in $\mathcal{B} \upharpoonright B$ is in fact an $(\mathbb{R} \upharpoonright B)$-shadow base. Thus we need to check that any two elements of $\mathcal{B} \upharpoonright B$ cohere. Fix $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ in $\mathcal{B}$, and suppose that there exists $\zeta \in (x \cap B) \cap (y \cap B)$. Let $\mu_x < \rho_\alpha$ be such that $\zeta = \pi_{x\cap B}(\mu_x)$, and let $\mu_y < \rho_\beta$ be such that $\zeta = \pi_{y\cap B}(\mu_y)$. Then since B coheres with $(\mathbb{R}, \mathcal{B})$, $\pi_x \upharpoonright (\mu_x + 1) = \pi_{x\cap B} \upharpoonright (\mu_x + 1)$, and similarly $\pi_y \upharpoonright (\mu_y + 1) = \pi_{y\cap B} \upharpoonright (\mu_y + 1)$. Therefore conditions (1) and (2) of Definition 2.2.2 at ζ follow from their applications to $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ at ζ. $\square$

Definition 2.2.14. *Suppose that $\mathbb{P}$ and $\mathbb{R}$ are partition products and σ embeds $\mathbb{P}$ into $\mathbb{R}$. If $\langle x, \pi_x, \alpha\rangle$ is a $\mathbb{P}$-shadow base, we define $\sigma(\langle x, \pi_x, \alpha\rangle)$ to be the triple*

$$\langle \sigma[x], \sigma \circ \pi_x, \alpha\rangle.$$

If $\mathcal{B}$ is a collection of $\mathbb{P}$-shadow bases, we define $\sigma[\mathcal{B}] := \{\sigma(t) : t \in \mathcal{B}\}$.

The proof of the following lemma is routine.

Lemma 2.2.15. *Suppose that $\mathbb{P}$ and $\mathbb{R}$ are partition products, σ embeds $\mathbb{P}$ into $\mathbb{R}$, and $\mathcal{B}$ is a collection of $\mathbb{P}$-shadow bases. Then $\sigma[\mathcal{B}]$ is a collection of $\mathbb{R}$-shadow bases.*

The following technical lemma will be of some use later.

Lemma 2.2.16. *Suppose that $\mathbb{R}$ and $\mathbb{R}^*$ are partition products, σ_1, σ_2 are embeddings of $\mathbb{R}$ into $\mathbb{R}^*$, and $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ are two coherent $\mathbb{R}$-shadow bases, with $\alpha \leq \beta$. Let a be an initial segment of x such that $a \subseteq y$, $\sigma_1 \upharpoonright a = \sigma_2 \upharpoonright a$, and $\sigma_1[x \backslash a]$ is disjoint from $\sigma_2[y \backslash a]$. Then $\sigma_1(\langle x, \pi_x, \alpha\rangle)$ and $\sigma_2(\langle y, \pi_y, \beta\rangle)$ are coherent $\mathbb{R}^*$-shadow bases.*

Proof. From Lemma 2.2.15, we see that $\sigma_1(\langle x, \pi_x, \alpha\rangle)$ and $\sigma_2(\langle y, \pi_y, \beta\rangle)$ are $\mathbb{R}^*$-shadow bases. Furthermore, if $\zeta^* \in \sigma_1[x] \cap \sigma_2[y]$, then ζ^* must be in $\sigma_1[a] \cap \sigma_2[a]$, since $\sigma_1[x \backslash a] \cap \sigma_2[y \backslash a] = \varnothing$ and since $\sigma_1 \upharpoonright a = \sigma_2 \upharpoonright a$. As the injections σ_1 and σ_2 are equal on a, we then have that $\sigma_1^{-1}(\zeta^*) = \sigma_2^{-1}(\zeta^*) =: \zeta$. Thus $\zeta \in x \cap y$, and the coherence of the original triples at ζ implies the coherence of their images at ζ^*. $\square$

We next define a notion of embedding for enriched partition products.

Definition 2.2.17. *Suppose that* $(\mathbb{P}, \mathcal{B})$ *is an enriched partition product with domain* X*,* $(\mathbb{R}, \mathcal{D})$ *is an enriched partition product with domain* Y*, and* $\sigma : X \longrightarrow Y$ *is a function. We say that* σ embeds $(\mathbb{P}, \mathcal{B})$ into $(\mathbb{R}, \mathcal{D})$ *if* σ *embeds* $\mathbb{P}$ *into* $\mathbb{R}$*, as in Definition 2.1.17, and if* $\sigma[\mathcal{B}] \subseteq \mathcal{D}$*.*

We may now state and prove the Grafting Lemma; enrichments play a crucial role in its proof.

Lemma 2.2.18. (Grafting Lemma) *Let* $(\mathbb{P}, \mathcal{B})$ *and* $(\mathbb{R}, \mathcal{D})$ *be enriched partition products with respective domains* X *and* Y*. Suppose that* $\hat{X} \subseteq X$ *coheres with* $(\mathbb{P}, \mathcal{B})$ *and that there is a map* $\sigma : \hat{X} \longrightarrow Y$ *which embeds* $(\mathbb{P} \restriction \hat{X}, \mathcal{B} \restriction \hat{X})$ *into* $(\mathbb{R}, \mathcal{D})$*.Then there is an enriched partition product* $(\mathbb{R}^*, \mathcal{D}^*)$ *with domain* Y^* *such that* $Y \subseteq Y^*$*,* $\mathbb{R}^* \restriction Y = \mathbb{R}$*,* $\mathcal{D} \subseteq \mathcal{D}^*$*, and such that there is an extension* σ^* *of* σ *which embeds* $(\mathbb{P}, \mathcal{B})$ *into* $(\mathbb{R}^*, \mathcal{D}^*)$ *and which satisfies* $\sigma^* \left[X \backslash \hat{X}\right] = Y^* \backslash Y$*.*

Proof. We first define the map σ^* extending σ: if $\xi \in \hat{X}$, then set $\sigma^*(\xi) := \sigma(\xi)$. If $\xi \in X \backslash \hat{X}$, say ξ is the γth such element, then we set $\sigma^*(\xi) := \sup(Y) + 1 + \gamma$. Then σ^* is an acceptable rearrangement, since $\hat{X}$ is base-closed. Let $Y^* := Y \cup \operatorname{ran}(\sigma^*)$. Recalling that σ embeds $\mathbb{P} \restriction \hat{X}$ into $\mathbb{R}$, we know that $\sigma^*[\text{base}_{\mathbb{P}}] \restriction \operatorname{ran}(\sigma) = \text{base}_{\mathbb{R}} \restriction \operatorname{ran}(\sigma)$ and that $\sigma^*[\text{index}_{\mathbb{P}}] \restriction \operatorname{ran}(\sigma) = \text{index}_{\mathbb{R}} \restriction \operatorname{ran}(\sigma)$. Thus if we define $\text{base}^* := \text{base}_{\mathbb{R}} \cup \sigma^*[\text{base}_{\mathbb{P}}]$ and $\text{index}^* := \text{index}_{\mathbb{R}} \cup \sigma^*[\text{index}_{\mathbb{P}}]$, then base^* and index^* are functions.

Before we check that base^* and index^* support a partition product on Y^*, we need to check that $\mathcal{D} \cup \sigma^*[\mathcal{B}]$ consists of a coherent collection of shadow bases. To facilitate the discussion, we set $\mathcal{B}^* := \sigma^*[\mathcal{B}]$ and $\mathcal{D}^* := \mathcal{D} \cup \mathcal{B}^*$. So fix $\langle x, \pi_x, \alpha \rangle \in \mathcal{B}$ and $\langle y, \pi_y, \beta \rangle$ in $\mathcal{D}$, and we check that $\langle y, \pi_y, \beta \rangle$ and $\langle x^*, \pi_{x^*}, \alpha \rangle$ cohere, where $x^* := \sigma^*[x]$ and $\pi_{x^*} := \sigma^* \circ \pi_x$. By our assumption that σ embeds $(\mathbb{P} \restriction \hat{X}, \mathcal{B} \restriction \hat{X})$ into $(\mathbb{R}, \mathcal{D})$, we know that $\langle y, \pi_y, \beta \rangle$ and $\langle \sigma[x \cap \hat{X}], \sigma \circ \pi_{x \cap \hat{X}}, \alpha \rangle$ cohere. However, $\langle \sigma[x \cap \hat{X}], \sigma \circ \pi_{x \cap \hat{X}}, \alpha \rangle$ is an initial segment of $\langle x^*, \pi_{x^*}, \alpha \rangle$, as in Definition 2.2.4. Therefore by Remark 2.2.5, since $\sigma^*[X \backslash \hat{X}]$ is disjoint from y, we have that $\langle y, \pi_y, \beta \rangle$ and $\langle x^*, \pi_{x^*}, \alpha \rangle$ cohere.

We now check that base* and index* support a partition product on Y^*. Conditions (1) and (2) of Definition 2.1.1 for base* and index* follow because they hold for $\mathrm{base}_{\mathbb{R}}$ and $\mathrm{index}_{\mathbb{R}}$, as well as $\sigma^*[\mathrm{base}_{\mathbb{P}}]$ and $\sigma^*[\mathrm{index}_{\mathbb{P}}]$ individually, and since base* and index* are functions. Thus we need to verify condition (3). For this it suffices to check that it holds for $\xi_1 \in Y$ and $\xi_2 \in Y^* \backslash Y$. Rephrasing, we need to show that the triples $\langle b^*(\xi_1), \pi^*_{\xi_1}, \mathrm{index}^*(\xi_1)\rangle$ and $\langle b^*(\xi_2), \pi^*_{\xi_2}, \mathrm{index}^*(\xi_2)\rangle$ cohere. The first triple equals $\langle b_{\mathbb{R}}(\xi_1), \pi^{\mathbb{R}}_{\xi_1}, \mathrm{index}_{\mathbb{R}}(\xi_1)\rangle$ and so is in $\mathcal{D}$ since $\mathcal{D}$ is $\mathbb{R}$-full. The second triple is in $\mathcal{B}^*$, since it equals $\langle \sigma^*[b_{\mathbb{P}}(\hat{\xi}_2)], \sigma^* \circ \pi^{\mathbb{P}}_{\hat{\xi}_2}, \mathrm{index}_{\mathbb{P}}(\hat{\xi}_2)\rangle$, where $\sigma^*(\hat{\xi}_2) = \xi_2$. Consequently, both shadow bases are in $\mathcal{D}^*$ and are therefore coherent, by the previous paragraph. Thus condition (3) of Definition 2.1.1 is satisfied.

Thus base* and index* support a partition product on Y^*, which we call $\mathbb{R}^*$. Since the restrictions of base* and index* to Y equal $\mathrm{base}_{\mathbb{R}}$ and $\mathrm{index}_{\mathbb{R}}$, respectively, we have that $\mathbb{R}^* \restriction Y = \mathbb{R}$. Additionally, σ^* embeds $\mathbb{P}$ into $\mathbb{R}^*$, since base* and index* restricted to $\mathrm{ran}(\sigma^*)$ equal $\sigma^*[\mathrm{base}_{\mathbb{P}}]$ and $\sigma^*[\mathrm{index}_{\mathbb{P}}]$ respectively. Thus it remains to check that $\mathcal{D}^*$ is an enrichment of $\mathbb{R}^*$, and for this, it only remains to check that $\mathcal{D}^*$ is $\mathbb{R}^*$-full. However, $\mathcal{D}$ is $\mathbb{R}$-full, and since $\mathcal{B}$ is $\mathbb{P}$-full, $\mathcal{B}^*$ is full with respect to $\mathbb{R}^* \restriction \mathrm{ran}(\sigma^*)$. Thus $\mathcal{D}^*$ is $\mathbb{R}^*$-full. $\square$

Definition 2.2.19. *Let* $(\mathbb{P}, \mathcal{B})$*,* $(\mathbb{R}, \mathcal{D})$*,* $(\mathbb{R}^*, \mathcal{D}^*)$*,* $\hat{X}$*,* σ*, and* σ^* *be as in Lemma 2.2.18. We will say in this case that* $(\mathbb{R}^*, \mathcal{D}^*)$ *is* the extension of $(\mathbb{R}, \mathcal{D})$ by grafting $(\mathbb{P}, \mathcal{B})$ over σ*, and we will call* σ^* *the* grafting embedding.

Note that as a corollary, we get that the product of two partition products is isomorphic to a partition product; this fact could also be proven directly from the definitions.

Corollary 2.2.20. *Suppose that* $\mathbb{P}$ *and* $\mathbb{R}$ *are partition products with respective domains* X *and* Y*. Then* $\mathbb{P} \times \mathbb{R}$ *is isomorphic to a partition product* $\mathbb{R}^*$*.*

In fact, by Lemma 2.1.13 we may assume that $X \cap Y = \varnothing$*, that* $\mathbb{R}^*$ *is a partition product on* $X \cup Y$*, and that* $\mathbb{R}^* \restriction X = \mathbb{P}$ *and* $\mathbb{R}^* \restriction Y = \mathbb{R}$*. Finally, in this case, if* $\mathcal{B}$ *and* $\mathcal{D}$ *are enrichments of* $\mathbb{P}$ *and* $\mathbb{R}$ *respectively, then* $\mathcal{B} \cup \mathcal{D}$ *is an enrichment of* $\mathbb{R}^*$*.*

The following lemma gives a situation under which, after creating a single grafting embedding, we may extend a number of other embeddings without further grafting; it will be used later in constructing preassignments (see Lemma 3.3.4).

Lemma 2.2.21. *Let* $(\mathbb{P}, \mathcal{B})$ *and* $(\mathbb{R}, \mathcal{D})$ *be enriched partition products with domains* X *and* Y *respectively. Suppose that* X *can be written as* $X = X_0 \cup X_1$*, where both* X_0 *and* X_1 *cohere with* $(\mathbb{P}, \mathcal{B})$*. Let* $\mathcal{F}$ *be a finite collection of maps which embed* $(\mathbb{P} \restriction X_0, \mathcal{B} \restriction X_0)$ *into* $(\mathbb{R}, \mathcal{D})$*, and suppose that for each* $\sigma_0, \sigma_1 \in \mathcal{F}$*,*

$$\sigma_0[X_0 \cap X_1] = \sigma_1[X_0 \cap X_1].$$

Finally, fix a particular $\sigma_0 \in \mathcal{F}$*, let* $(\mathbb{R}^*, \mathcal{D}^*)$ *be the extension of* $(\mathbb{R}, \mathcal{D})$ *by grafting* $(\mathbb{P}, \mathcal{B})$ *over* σ_0*, and let* σ_0^* *be the grafting embedding. Then for all* $\sigma \in \mathcal{F}$*, the map*

$$\sigma^* := \sigma \cup (\sigma_0^* \restriction (X_1 \backslash X_0))$$

embeds $(\mathbb{P}, \mathcal{B})$ *into* $(\mathbb{R}^*, \mathcal{D}^*)$*.*

Proof. Fix $\sigma \in \mathcal{F}$. Before we continue, we note that σ^* and σ_0^* agree on all of X_1, since they agree on $X_0 \cap X_1$ by assumption and on $X_1 \backslash X_0$ by definition.

We first verify that σ^* provides an acceptable rearrangement of $\mathbb{P}$. So let $\zeta, \xi \in X$ so that $\zeta \in b_{\mathbb{P}}(\xi)$. If $\xi \in X_0$, then ζ is too, since X_0 is base-closed. Then $\sigma^*(\zeta) = \sigma(\zeta) < \sigma(\xi) = \sigma^*(\xi)$, since σ is an acceptable rearrangement of $\mathbb{P} \restriction X_0$. On the other hand, if $\xi \in X_1$, then $\zeta \in X_1$. Since $\sigma^* \restriction X_1 = \sigma_0^* \restriction X_1$, and $\sigma_0^* \restriction X_1$ is an acceptable rearrangement of $\mathbb{P} \restriction X_1$, we get that $\sigma^*(\zeta) < \sigma^*(\xi)$.

We may now see that σ^* embeds $\mathbb{P}$ into $\mathbb{R}^*$, as follows: let base* and index* be the functions which support $\mathbb{R}^*$. Then $\sigma^*[\text{index}_{\mathbb{P}}]$ and $\sigma^*[\text{base}_{\mathbb{P}}]$ agree with index* and base* on $\text{ran}(\sigma)$, since σ embeds $\mathbb{P} \restriction X_0$ into $\mathbb{R}$. Furthermore, $\sigma^*[\text{index}_{\mathbb{P}}]$ and $\sigma^*[\text{base}_{\mathbb{P}}]$ agree with index* and base* on $\sigma^*[X_1]$, since they are equal, respectively, to $\sigma_0^*[\text{index}_{\mathbb{P}}]$ and $\sigma_0^*[\text{base}_{\mathbb{P}}]$ restricted to $\sigma_0^*[X_1]$. Thus $\sigma^*[\text{index}_{\mathbb{P}}]$ and $\sigma^*[\text{base}_{\mathbb{P}}]$ are equal to the restriction of index* and base* to $\text{ran}(\sigma^*)$, and consequently, σ^* embeds $\mathbb{P}$ into $\mathbb{R}^*$.

We finish the proof of the lemma by showing that $\sigma^*[\mathcal{B}] \subseteq \mathcal{D}^*$. To see this, fix some $\langle x, \pi_x, \alpha \rangle \in \mathcal{B}$. We first claim that either $x \subseteq X_0$ or $x \subseteq X_1$. If this is false, then there exist $\alpha \in x \backslash X_0$ and $\beta \in x \backslash X_1$. Since $X_0 \cup X_1 = X$, we then have $\alpha \in X_1$ and $\beta \in X_0$. We suppose, by relabeling if necessary, that $\alpha_0 := \pi_x^{-1}(\alpha) < \pi_x^{-1}(\beta) =: \beta_0$. By the coherence of X_0 with $(\mathbb{P}, \mathcal{B})$, we conclude that $\pi_x[\beta_0] \subseteq X_0$. However, $\alpha = \pi_x(\alpha_0) \in \pi_x[\beta_0]$, and therefore $\alpha \in X_0$, a contradiction.

We now show that the shadow base $\langle x^*, \pi_{x^*}, \alpha \rangle$ is in $\mathcal{D}^*$, where $x^* := \sigma^*[x]$ and $\pi_{x^*} = \sigma^* \circ \pi_x$. On the one hand, if $x \subseteq X_0$, then the shadow base $\langle x, \pi_x, \alpha \rangle$ is in $\mathcal{B} \restriction X_0$, and therefore $\langle \sigma[x], \sigma \circ \pi_x, \alpha \rangle$ is a member of $\mathcal{D} \subseteq \mathcal{D}^*$. Since $\sigma = \sigma^* \restriction X_0$, $\langle x^*, \pi_{x^*}, \alpha \rangle = \langle \sigma[x], \sigma \circ \pi_x, \alpha \rangle$, completing this subcase. On the other hand, if $x \subseteq X_1$, then we see that $\langle x^*, \pi_{x^*}, \alpha \rangle = \langle \sigma_0^*[x], \sigma_0^* \circ \pi_x, \alpha \rangle$, since $\sigma^* \restriction X_1 = \sigma_0^* \restriction X_1$. It is therefore a member of $\mathcal{D}^*$, which finishes the proof. $\square$

CHAPTER 3

Constructing Preassignments of Colors

In this chapter we show how to construct the particular names for preassignments of colors that we need. Throughout this chapter, we make the following assumptions.

Assumption 3.0.1. The CH holds. $\kappa < \omega_2$ is in C, and for each $\xi \in C \cap \kappa$, ρ_ξ is below ω_2. Additionally, the κ-canonical partition product $\mathbb{P}_\kappa$ is defined, and in particular, $\mathbb{P}_\kappa$ is a partition product based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$. We also assume that the $\mathbb{P}_\kappa$-names $\dot{S}_\kappa$ and $\dot{\chi}_\kappa$ are defined and satisfy that $\dot{S}_\kappa$ names a countable basis for a second countable, Hausdorff topology on ω_1 and $\dot{\chi}_\kappa$ names a coloring open with respect to the topology generated by $\dot{S}_\kappa$. And finally, we assume that any partition product based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$ is c.c.c.

Remark 3.0.2. Our goal is to show, under Assumption 3.0.1, how to construct a $\mathbb{P}_\kappa$-name $\dot{\mathbb{Q}}_\kappa$ for a poset which decomposes ω_1 into countably-many $\dot{\chi}_\kappa$-homogeneous sets, in such a way that any partition product based upon $\underline{\mathbb{P}} \restriction (\kappa+1)$ and $\underline{\dot{\mathbb{Q}}} \restriction (\kappa+1)$ is c.c.c. In Chapter 4 we will use this as part of an inductive construction of a sequence $\underline{\mathbb{P}}$ which provides the right building blocks for our main theorem.

3.1 κ-Suitable Collections

We now consider how various copies of $\mathbb{P}_\kappa$ fit into a partition product $\mathbb{R}$, where $\mathbb{R}$ is based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$. Even though we have yet to construct the name $\dot{\mathbb{Q}}_\kappa$, we would still like to isolate the relevant behavior of copies of $\mathbb{P}_\kappa$ inside such an $\mathbb{R}$ which these copies *would* have if $\mathbb{R}$ *were* of the form

$$\mathbb{R} = \mathbb{R}^* \restriction \{\xi \in \mathrm{dom}(\mathbb{R}^*) : \mathrm{index}(\xi) < \kappa\},$$

for some partition product $\mathbb{R}^*$ based upon $\underline{\mathbb{P}} \restriction (\kappa+1)$ and $\underline{\dot{\mathbb{Q}}} \restriction (\kappa+1)$. This leads to the following definition.

Definition 3.1.1. *Let $\mathbb{R}$ be a partition product with domain X based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$. Let $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ be a set of pairs, where each $B_\iota \subseteq X$ is base-closed and where $\psi_\iota : \rho_\kappa \longrightarrow B_\iota$ is a bijection which embeds $\mathbb{P}_\kappa$ into $\mathbb{R}$. We say that the collection $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ is* κ-suitable with respect to $\mathbb{R}$ *if*

$$\{\langle B_\iota, \psi_\iota, \kappa\rangle : \iota \in I\} \cup \{\langle b(\xi), \pi_\xi, \mathrm{index}(\xi)\rangle : \xi \in X\}$$

is a coherent set of $\mathbb{R}$-shadow bases.

*Moreover, if $(\mathbb{R}, \mathcal{B})$ is an enriched partition product, we say that $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ is κ-*suitable with respect to $(\mathbb{R}, \mathcal{B})$ *if $\{\langle B_\iota, \psi_\iota, \kappa\rangle : \iota \in I\} \subseteq \mathcal{B}$ and if $\alpha \leq \kappa$ for all $\langle x, \pi_x, \alpha\rangle \in \mathcal{B}$.*

As the next lemma shows, κ-suitable collections give us subsets which cohere with the original partition product, since the indices of the triples in the enrichment do not exceed κ.

Lemma 3.1.2. *Suppose that $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ is κ-suitable with respect to an enriched partition product $(\mathbb{R}, \mathcal{B})$. Then for any $I_0 \subseteq I$, $\bigcup_{\iota \in I_0} B_\iota$ coheres with $(\mathbb{R}, \mathcal{B})$.*

Proof. Let $\langle x, \pi_x, \alpha\rangle \in \mathcal{B}$, and suppose that there exists $\zeta \in (\bigcup_{\iota \in I_0} B_\iota) \cap x$. Fix some $\iota \in I_0$ such that $\zeta \in B_\iota \cap x$. Then $\langle B_\iota, \psi_\iota, \kappa\rangle$ is in $\mathcal{B}$. Furthermore, $\alpha \leq \kappa$, by definition of κ-suitability with respect to $(\mathbb{R}, \mathcal{B})$. Since $\mathcal{B}$ is coherent, by definition of an enrichment, and since $\alpha \leq \kappa$, we have by Definition 2.2.2 that

$$\pi_x[\pi_x^{-1}(\zeta)] \subseteq \psi_\iota[\psi_\iota^{-1}(\zeta)].$$

Since $\mathrm{ran}(\psi_\iota) = B_\iota$, this finishes the proof. $\square$

We will often be interested in the following strengthening of the notion of an embedding, one which preserves the κ-suitable structure.

Definition 3.1.3. *Let* $\mathbb{R}$ *and* $\mathbb{R}^*$ *be two partition products, and let* $\mathcal{S} = \{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ *and* $\mathcal{S}^* = \left\{\langle B^*_\eta, \psi^*_\eta\rangle : \eta \in I^*\right\}$ *be* κ*-suitable collections with respect to* $\mathbb{R}$ *and* $\mathbb{R}^*$ *respectively. An embedding* σ *of* $\mathbb{R}$ *into* $\mathbb{R}^*$ *is said to be* $(\mathcal{S}, \mathcal{S}^*)$-suitable *if for each* $\iota \in I$*, there is some* $\eta \in I^*$ *such that* $\sigma \restriction B_\iota$ *isomorphs* $\mathbb{R} \restriction B_\iota$ *onto* $\mathbb{R}^* \restriction B^*_\eta$ *and* $\psi^*_\eta = \sigma \circ \psi_\iota$*. A collection* $\mathcal{F}$ *of embeddings is said to be* $(\mathcal{S}, \mathcal{S}^*)$-suitable *if each* $\sigma \in \mathcal{F}$ *is* $(\mathcal{S}, \mathcal{S}^*)$*-suitable.*

If σ *is* $(\mathcal{S}, \mathcal{S}^*)$*-suitable, we let* h_σ *denote the injection from* I *into* I^* *such that* σ *maps* B_ι *onto* $B^*_{h_\sigma(\iota)}$ *for each* $\iota \in I$*.*

The following technical lemmas will be used later in this chapter.

Lemma 3.1.4. *Suppose that* $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ *is* κ*-suitable with respect to an enriched partition product* $(\mathbb{R}, \mathcal{B})$ *and that the elements of* $\{B_\iota : \iota \in I\}$ *are pairwise disjoint. Then for any* $\langle x, \pi_x, \alpha\rangle \in \mathcal{B}$*,* $x \cap (\bigcup_{\iota \in I} B_\iota) = x \cap B_{\iota_0}$ *for a unique* $\iota_0 \in I$*.*

Proof. Suppose otherwise, and fix $\langle x, \pi_x, \alpha\rangle \in \mathcal{B}$ as well as distinct $\iota_0, \iota_1 \in I$ such that $x \cap B_{\iota_0} \neq \varnothing$ and $x \cap B_{\iota_1} \neq \varnothing$. Let $\zeta \in x \cap B_{\iota_0}$ and $\eta \in x \cap B_{\iota_1}$. Then $\zeta \neq \eta$, since $B_{\iota_0} \cap B_{\iota_1} = \varnothing$. Define $\zeta_0 := \pi_x^{-1}(\zeta)$ and $\eta_0 := \pi_x^{-1}(\eta)$. Since $\zeta_0 \neq \eta_0$, we suppose, by relabeling if necessary, that $\zeta_0 < \eta_0$. By definition of an enrichment, we know that $\langle B_{\iota_1}, \psi_{\iota_1}, \kappa\rangle$ and $\langle x, \pi_x, \alpha\rangle$ cohere, and since $\alpha \leq \kappa$ and $\eta \in B_{\iota_1} \cap x$, we conclude that $\pi_x[\eta_0] \subseteq B_{\iota_1}$. However, $\zeta_0 < \eta_0$, and so $\zeta = \pi_x(\zeta_0) \in \pi_x[\eta_0]$, which implies that $\zeta \in B_{\iota_1}$. This contradicts the fact that $B_{\iota_0} \cap B_{\iota_1} = \varnothing$. □

The next lemma gives a sufficient condition for creating an enrichment; it will be used in the construction of preassignments (see Lemma 3.3.4).

Lemma 3.1.5. *Suppose that* $\mathcal{S} = \{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ *is* κ*-suitable with respect to an enriched partition product* $(\mathbb{R}, \mathcal{B})$ *and that the elements of* $\{B_\iota : \iota \in I\}$ *are pairwise disjoint. Suppose further that* $\mathbb{R}^*$ *is a partition product with domain* X^* *and that* $\mathcal{S}^* = \left\{\langle B^*_\eta, \psi^*_\eta\rangle : \eta \in I^*\right\}$ *is* κ*-suitable with respect to* $\mathbb{R}^*$*. Finally, set* $\hat{X} := \bigcup_{\iota \in I} B_\iota$*, and suppose that there exists a finite collection* $\mathcal{F}$ *of* $(\mathcal{S}, \mathcal{S}^*)$*-suitable embeddings of* $\mathbb{R} \restriction \hat{X}$ *into* $\mathbb{R}^*$ *such that for any distinct*

$\iota_0, \iota_1 \in I$ and any (not necessarily distinct) $\pi_0, \pi_1 \in \mathcal{F}$,

$$B^*_{h_{\pi_0}(\iota_0)} \cap B^*_{h_{\pi_1}(\iota_1)} = \varnothing,$$

where for each $\pi \in \mathcal{F}$, h_π is the associated injection. Then

$$\mathcal{B}^* := \left\{\langle b^*(\xi), \pi^*_\xi, \mathrm{index}^*(\xi)\rangle : \xi \in X^*\right\} \cup \bigcup_{\pi \in \mathcal{F}} \pi\left[\mathcal{B} \restriction \hat{X}\right] \cup \left\{\langle B^*_\eta, \psi^*_\eta, \kappa\rangle : \eta \in I^*\right\}$$

is an enrichment of $\mathbb{R}^*$ and $\mathcal{S}^*$ is κ-suitable with respect to $(\mathbb{R}^*, \mathcal{B}^*)$.

Proof. We will first show that $\bigcup_{\pi \in \mathcal{F}} \pi\left[\mathcal{B} \restriction \hat{X}\right]$ is a coherent collection of $\mathbb{R}^*$-shadow bases. Since each $\pi \in \mathcal{F}$ is an embedding of $\mathbb{R} \restriction \hat{X}$ into $\mathbb{R}^*$, Lemma 2.2.15 implies that this is a set of $\mathbb{R}^*$-shadow bases. Thus we check coherence.

Fix $\pi_0, \pi_1 \in \mathcal{F}$ and $\langle x, \pi_x, \alpha\rangle, \langle y, \pi_y, \beta\rangle \in \mathcal{B} \restriction \hat{X}$, and assume, by relabeling if necessary, that $\alpha \leq \beta$. We show that $\langle x^*, \pi_{x^*}, \alpha\rangle$ and $\langle y^*, \pi_{y^*}, \beta\rangle$ cohere, where $x^* := \pi_0[x]$ and $\pi_{x^*} := \pi_0 \circ \pi_x$, and where $y^* := \pi_1[y]$, $\pi_{y^*} := \pi_1 \circ \pi_y$. By Lemma 3.1.4, and since x and y are subsets of $\hat{X}$, we may fix $\iota_0, \iota_1 \in I$ such that $x = x \cap \hat{X} = x \cap B_{\iota_0}$ and $y = y \cap \hat{X} = y \cap B_{\iota_1}$. There are two cases.

First suppose that $\iota_0 \neq \iota_1$. Then we must have that $x^* \cap y^* = \varnothing$. To see this, observe that

$$x^* = \pi_0[x] = \pi_0[x \cap B_{\iota_0}] \subseteq B^*_{h_{\pi_0}(\iota_0)},$$

and

$$y^* = \pi_1[y] = \pi_1[y \cap B_{\iota_1}] \subseteq B^*_{h_{\pi_1}(\iota_1)}.$$

Therefore $x^* \cap y^* = \varnothing$, as $B^*_{h_{\pi_0}(\iota_0)} \cap B^*_{h_{\pi_1}(\iota_1)} = \varnothing$, by assumption. We thus trivially have the coherence of $\langle x^*, \pi_{x^*}, \alpha\rangle$ and $\langle y^*, \pi_{y^*}, \beta\rangle$ in this case.

On the other hand, suppose that $\iota := \iota_0 = \iota_1$. Define $a \subseteq x$ to be the largest initial segment (see Definition 2.2.4) of $\langle x, \pi_x, \alpha\rangle$ on which π_0 and π_1 agree, and set $a^* := \pi_0[a] = \pi_1[a]$. In order to see that $\langle x^*, \pi_{x^*}, \alpha\rangle$ and $\langle y^*, \pi_{y^*}, \beta\rangle$ cohere, it suffices, in light of Lemma 2.2.16, to show that $\pi_0[x \backslash a]$ is disjoint from $\pi_1[y \backslash a]$. Towards this end, fix some $\zeta^* \in x^* \cap y^*$, and suppose for a contradiction that $\zeta^* \notin a^*$. Define $\mu_x := \pi^{-1}_{x^*}(\zeta^*)$, and observe that μ_x

is greater than the ordinal $\pi_x^{-1}[a]$, since $\zeta^* \notin a^*$. Using the abbreviation $\eta_i := h_{\pi_i}(\iota)$, for $i \in \{0,1\}$, we see that $\zeta^* \in B^*_{\eta_0} \cap B^*_{\eta_1}$, as $x^* = \pi_0[x \cap B_\iota] \subseteq B^*_{\eta_0}$, and as $y^* = \pi_1[y \cap B_\iota] \subseteq B^*_{\eta_1}$. Set $\zeta_0 := (\psi^*_{\eta_0})^{-1}(\zeta^*)$. Since the $\mathbb{R}^*$-shadow bases $\langle B^*_{\eta_0}, \psi^*_{\eta_0}, \kappa\rangle$ and $\langle B^*_{\eta_1}, \psi^*_{\eta_1}, \kappa\rangle$ cohere, Corollary 2.2.8 implies that $\psi^*_{\eta_0} \restriction (\zeta_0 + 1) = \psi^*_{\eta_1} \restriction (\zeta_0 + 1)$.

Now we observe that

$$\pi_{x^*}(\mu_x) = \pi_0(\pi_x(\mu_x)) = \zeta^* = \psi^*_{\eta_0}(\zeta_0) = \pi_0(\psi_\iota(\zeta_0)),$$

and therefore $\pi_x(\mu_x) = \psi_\iota(\zeta_0)$. Let us call this ordinal ζ. Since $\zeta \in B_\iota \cap x$, the coherence of $\langle x, \pi_x, \alpha\rangle$ with $\langle B_\iota, \psi_\iota, \kappa\rangle$ and the fact that $\alpha \leq \kappa$ imply that

$$\pi_x[\mu_x + 1] \subseteq \psi_\iota[\zeta_0 + 1].$$

As noted above, $\psi^*_{\eta_0} \restriction (\zeta_0 + 1) = \psi^*_{\eta_1} \restriction (\zeta_0 + 1)$, and therefore by the commutativity assumed in the statement of the lemma, π_0 and π_1 agree on $\psi_\iota[\zeta_0 + 1]$. In particular, they agree on $\pi_x[\mu_x + 1]$. Thus $\pi_x[\mu_x + 1]$ is an initial segment of $\langle x, \pi_x, \alpha\rangle$ on which π_0 and π_1 agree. Since $\zeta = \pi_x(\mu_x) \notin a$, this contradicts the maximality of a.

At this point, we have shown that $\bigcup_{\pi \in \mathcal{F}} \pi \left[\mathcal{B} \restriction \hat{X}\right]$ is a coherent collection of $\mathbb{R}^*$-shadow bases. We introduce the abbreviation

$$\mathcal{B}^*_0 := \left\{\langle b^*(\xi), \pi^*_\xi, \mathrm{index}^*(\xi)\rangle : \xi \in X^*\right\} \cup \left\{\langle B^*_\eta, \psi^*_\eta, \kappa\rangle : \eta \in I^*\right\}.$$

We know that $\mathcal{B}^*_0$ is a coherent set of $\mathbb{R}^*$-shadow bases, by the definition of κ-suitability. Therefore, to finish showing that $\mathcal{B}^*$ is an enrichment of $\mathbb{R}^*$, we now check that if $\langle y, \pi_y, \beta\rangle \in \mathcal{B}^*_0$, $\pi \in \mathcal{F}$, and $\langle x, \pi_x, \alpha\rangle \in \mathcal{B} \restriction \hat{X}$, then $\langle y, \pi_y, \beta\rangle$ and $\langle x^*, \pi_{x^*}, \alpha\rangle$ cohere, where $x^* := \pi[x]$ and $\pi_{x^*} = \pi \circ \pi_x$. By Lemma 3.1.4, let $\iota \in I$ be such that $x = x \cap \hat{X} = x \cap B_\iota$. Then $x^* = \pi[x] = \pi[x \cap B_\iota] \subseteq B^*_{h_\pi(\iota)}$. Now $\langle x, \pi_x, \alpha\rangle$ and $\langle B_\iota, \psi_\iota, \kappa\rangle$ cohere, and moreover, π isomorphs $\mathbb{R} \restriction B_\iota$ onto $\mathbb{R}^* \restriction B^*_{h_\pi(\iota)}$ and satisfies that $\psi^*_{h_\pi(\iota)} = \pi \circ \psi_\iota$. It is straightforward to see from this that $\langle x^*, \pi_{x^*}, \alpha\rangle$ and $\langle B^*_{h_\pi(\iota)}, \psi^*_{h_\pi(\iota)}, \kappa\rangle$ cohere. However, $\langle y, \pi_y, \beta\rangle$ and $\langle B^*_{h_\pi(\iota)}, \psi^*_{h_\pi(\iota)}, \kappa\rangle$ also cohere, by definition of κ-suitability. Since $\alpha, \beta \leq \kappa$, Lemma 2.2.10 therefore implies that $\langle y, \pi_y, \beta\rangle$ and $\langle x^*, \pi_{x^*}, \alpha\rangle$ cohere, which is what we wanted to show. $\square$

3.2 What suffices

Given a (possibly partial) 2-coloring χ on ω_1 and a function f from ω_1 into $\{0,1\}$, we use $\mathbb{Q}(\chi, f)$ to denote the poset to decompose ω_1 into countably-many χ-homogeneous sets which respect the function f. More precisely, a condition is a finite partial function q with $\mathrm{dom}(q) \subseteq \omega$ such that for each $n \in \mathrm{dom}(q)$, $q(n)$ is a finite subset of ω_1 on which f is constant, say with value i, and $q(n)$ is χ-homogeneous with color i, meaning that if $x, y \in q(n)$ and $\langle x, y\rangle \in \mathrm{dom}(\chi)$, then $\chi(x,y) = i$. The ordering is $q_1 \leq q_0$ iff $\mathrm{dom}(q_0) \subseteq \mathrm{dom}(q_1)$, and for each $n \in \mathrm{dom}(q_0)$, $q_0(n) \subseteq q_1(n)$.

Remark 3.2.1. Forcing with $\mathbb{Q}(\chi, f)$ adds reals over V. Say that G is generic for $\mathbb{Q}(\chi, f)$, giving the partition $\omega_1 = \bigcup_n A_n$ into χ-homogeneous sets. Then by an easy density argument, the map which sends $m < \omega$ to the unique n s.t. $m \in A_n$ is a new real.

Following [4], we refer to any such f as a *preassignment of colors*. Our main goal in this chapter is to come up with a $\mathbb{P}_\kappa$-name $\dot{f}$ for a particularly nice preassignment of colors for $\dot{\chi}_\kappa$, in the following sense:

Proposition 3.2.2. *There is a $\mathbb{P}_\kappa$-name $\dot{f}$ for a preassignment of colors so that for any partition product $\mathbb{R}$ based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$, any generic G for $\mathbb{R}$, and any finite collection $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ which is κ-suitable with respect to $\mathbb{R}$, the poset*

$$\prod_{\iota \in I} \mathbb{Q}\left(\dot{\chi}_\kappa\left[\psi_\iota^{-1}(G \restriction B_\iota)\right], \dot{f}\left[\psi_\iota^{-1}(G \restriction B_\iota)\right]\right)$$

is c.c.c. in $V[G]$.

Remark 3.2.3. Observe that in the previous proposition, the same name $\dot{f}$ is interpreted in a variety of ways, namely, by various generics for $\mathbb{P}_\kappa$ added by forcing with $\mathbb{R}$. Moreover, $\dot{f}$ is strong enough that the product of the induced homogeneous set posets is c.c.c. This is what we mean by referring to the name as "symmetric."

Before continuing with the main thread, let us see, in the following corollary, that the condition in the previous proposition suffices.

Corollary 3.2.4. *Let $\dot{f}_\kappa$ be a name witnessing Proposition 3.2.2, and set $\dot{\mathbb{Q}}_\kappa$ to be the $\mathbb{P}_\kappa$-name $\mathbb{Q}(\dot{\chi}_\kappa, \dot{f}_\kappa)$. Then any partition product based upon $\underline{\mathbb{P}} \restriction (\kappa+1)$ and $\underline{\dot{\mathbb{Q}}} \restriction (\kappa+1)$ is c.c.c.*

Proof of Corollary 3.2.4. Let $\mathbb{R}$ be a partition product based upon $\underline{\mathbb{P}} \restriction (\kappa+1)$ and $\underline{\dot{\mathbb{Q}}} \restriction (\kappa+1)$, and let X be the domain of $\mathbb{R}$. Set $\hat{X} := \{\xi \in X : \mathrm{index}(\xi) < \kappa\}$, and let $I := \{\xi \in X : \mathrm{index}(\xi) = \kappa\}$. By Lemma 2.1.20, $\mathbb{R}$ is isomorphic to

$$(\mathbb{R} \restriction \hat{X}) * \prod_{\xi \in I} \dot{\mathbb{Q}}_\kappa \left[\pi_\xi^{-1}\left(\dot{G} \restriction b(\xi)\right)\right],$$

and $\mathbb{R} \restriction \hat{X}$ is a partition product based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$. By Assumption 3.0.1, $\mathbb{R} \restriction \hat{X}$ is c.c.c. It is also straightforward to check that $\{\langle b(\xi), \pi_\xi\rangle : \xi \in I\}$ is κ-suitable, by the definition of $\mathbb{R}$ as a partition product based upon $\underline{\mathbb{P}} \restriction (\kappa+1)$ and $\underline{\dot{\mathbb{Q}}} \restriction (\kappa+1)$. Finally, from Proposition 3.2.2, we know each finitely-supported subproduct of

$$\prod_{\xi \in I} \dot{\mathbb{Q}}_\kappa \left[\pi_\xi^{-1}\left(G \restriction b(\xi)\right)\right]$$

is c.c.c. in $V[G \restriction \hat{X}]$, and hence the entire product is c.c.c. Since $\mathbb{R} \restriction \hat{X}$ is c.c.c. in V, this finishes the proof. □

We will prove Proposition 3.2.2 by working backwards through a series of reductions; the final proof of Proposition 3.2.2 occurs in Section 3.4. We first want to see what happens if a finite product $\prod_{l<m} \mathbb{Q}(\chi_l, f_l)$ is *not* c.c.c., where each χ_l is an open coloring on ω_1 with respect to some second countable, Hausdorff topology τ_l on ω_1 and $f_l : \omega_1 \longrightarrow \{0,1\}$ is an arbitrary preassignment. In light of this discussion, we are able to simplify the sufficient conditions for proving Proposition 3.2.2 by reducing the scope of our investigation to so-called *finitely generated* partition products (see Definition 3.2.6). With this simplification in place, we continue in the third subsection to isolate a property of the name $\dot{f}$, which we call

the *partition product preassignment property.* This will complete the final reduction, isolating exactly what we need to show in order to ensure that the desired posets are c.c.c. And finally, we show how to construct names with the partition product preassignment property.

Now consider a sequence $\langle \tau_l : l < m \rangle$ of second countable, Hausdorff topologies on ω_1 with respective open colorings $\langle \chi_l : l < m \rangle$ and preassignments $\langle f_l : l < m \rangle$. Let us define $\tau := \biguplus \tau_l$, a topology on $X := \biguplus_{l<m} \omega_1$, as well as $f := \biguplus f_l$ and $\chi := \biguplus \chi_l$. So, for example, if $x \in X$, then $f(x) = f_l(x)$, where l is unique s.t. x is in the lth copy of ω_1, and if $x, y \in X$ then $\chi(x, y)$ is defined iff x and y are distinct and belong to the same copy of ω_1, say the lth, and in this case, $\chi(x, y) = \chi_l(x, y)$. With this notation, we may view a condition in the product $\prod_{l<m} \mathbb{Q}(\chi_l, f_l)$ as a condition in $\mathbb{Q}(\chi, f)$. Note that χ is partial, and this is the only reason we allowed partial colorings in the definition of $\mathbb{Q}(\chi, f)$.

Now suppose that $\prod_{l<m} \mathbb{Q}(\chi_l, f_l)$ has an uncountable antichain. Then we claim that there exists an $n < \omega$, an uncountable subset A of X^n and a closed (in X^n) set $F \supseteq A$ so that

1. the function $\langle x(0), \ldots, x(n-1) \rangle \mapsto \langle f(x(0)), \ldots, f(x(n-1)) \rangle$ is constant on A, say with value $d \in 2^n$. Abusing notion we also denote this function by f;

2. no two tuples in A have any elements in common;

3. for every distinct $x, y \in F$, there exists some $i < n$ so that $\chi(x(i), y(i))$ is defined and $\chi(x(i), y(i)) \neq d(i)$.

To see that this is true, take an antichain of size $\aleph_1$ in the product $\prod_{l<m} \mathbb{Q}(\chi_l, f_l)$, and first thin it to assume that for each l, k all conditions contribute the same number of elements to the kth homogeneous set for χ_l. Now viewing the elements in the antichain as sequences arranging the members according to the coloring and homogeneous set they contribute to, call the resulting set A. Let n be the length of each sequence in A. We further thin A to secure (1). Next, thin A to become a Δ-system, and note that by taking n to be minimal, we secure (2). Now observe that, for each $x \in A$, if $i < j < n$ and $x(i)$ and $x(j)$ are part of the same homogeneous set for the same coloring χ_l, say with color c, then as χ_l is an open coloring, there exists a pair of open sets $U_{i,j} \times V_{i,j}$ in $\tau_i \times \tau_j$ such that

$$\langle x(i), x(j) \rangle \in U_{i,j} \times V_{i,j} \subseteq \chi_l^{-1}(\{c\}).$$

With this x still fixed, by intersecting at most finitely-many open sets around each $x(i)$, we may remove the dependence on coordinates $j \neq i$, and thereby obtain for each i, an open set U_i around $x(i)$ witnessing the values of χ. In particular, for any $i < j < n$ such that $x(i)$ and $x(j)$ are in the same homogeneous set for the same coloring, say χ_l, we have

$$\langle x(i), x(j) \rangle \in U_i \times U_j \subseteq \chi_l^{-1}(\{c\}),$$

where $c = \chi_l(x(i), x(j))$. By using basic open sets, of which there are only countably-many, we may thin A to assume that the sequence of open sets $\langle U_i : i < n \rangle$ is the same for all $x \in A$. As a result of fixing these open sets, and since A is an antichain, we see that (3) holds for the elements of A. Since χ is an open coloring, (3) also hold for F, the closure of A in X^n.

Remark 3.2.5. The conditions in the previous paragraph are equivalent to the existence of $n < \omega$, $d \in 2^n$, and a closed set $F \subseteq X^n$ so that (i) for any distinct $x, y \in F$, $\chi(x(i), y(i))$ is defined for all $i < n$, and for some $i < n$, $\chi(x(i), y(i)) \neq d(i)$; and (ii) for every countable $z \subseteq F$, there exists $x \in F \backslash z$ so that $f \circ x = d$. Indeed, it is immediate that (1)-(3) give (i) and (ii), and for the other direction, iterate (ii) to obtain the uncountable set A.

Any F as in Remark 3.2.5 is a closed subset of a second countable space, and so F is coded by a real. Thus if $\mathbb{R}$ is a partition product as in the statement of Proposition 3.2.2, then any $\mathbb{R}$-name $\dot{F}$ for such a closed set will only involve conditions intersecting countably-many support coordinates, since $\mathbb{R}$, by Assumption 3.0.1, is c.c.c. This motivates the following definition and subsequent remark.

Definition 3.2.6. *A partition product* $\mathbb{R}$ *with domain* X*, say, based upon* $\underline{\mathbb{P}} \restriction \kappa$ *and* $\underline{\dot{\mathbb{Q}}} \restriction \kappa$ *is said to be* finitely generated *if there is a finite,* κ*-suitable collection* $\{\langle B_\iota, \psi_\iota \rangle : \iota \in I\}$*, and* *a countable $Z \subseteq X$, such that*

$$X = Z \cup \bigcup_{\xi \in Z} b(\xi) \cup \bigcup_{\iota \in I} B_\iota.$$

In this case, we will refer to Z as the auxiliary part.

Note that in the above definition, if there is some $\xi \in Z \cap B_\iota$, then $b(\xi) \subseteq B_\iota$, since B_ι is base-closed. Thus it poses no loss of generality to assume that Z is disjoint from $\bigcup_\iota B_\iota$, and we will do so.

Remark 3.2.7. As shown by the arguments preceding Definition 3.2.6, Proposition 3.2.2 follows from its restriction to finitely generated partition products $\mathbb{R}$.

We further remark that Definition 3.2.6 refers implicitly to the following objects: index_δ, base_δ, and $\varphi_{\delta,\mu}$ for $\delta < \kappa$, as well as $\mathbb{P}_\kappa$, index_κ, base_κ, and $\varphi_{\kappa,\mu}$, which are needed in order to define a suitable collection.

It is also straightforward to see that grafting a finitely generated partition product over another such results in a partition product which is still finitely generated, as stated in the following lemma.

Lemma 3.2.8. *Let $(\mathbb{P}, \mathcal{B})$, $(\mathbb{R}, \mathcal{D})$, and σ be as in Lemma 2.2.18. Suppose that both $(\mathbb{P}, \mathcal{B})$ and $(\mathbb{R}, \mathcal{D})$ are finitely generated and that $(\mathbb{R}^*, \mathcal{D}^*)$ is the extension of $(\mathbb{R}, \mathcal{D})$ by grafting $(\mathbb{P}, \mathcal{B})$ over σ. Then $(\mathbb{R}^*, \mathcal{D}^*)$ is also finitely generated.*

One of the main advantages of looking at finitely generated partition products is that there are not too many of them, as made precise by the following two items. We remark here that the subsequent two lemmas are some of the key places where we use the involved definition of a partition product: this definition allows us to see that any appropriate model contains the isomorphism types of the overlaps of any two "memories" from a finitely-generated partition product. We may therefore reconstruct an isomorphic copy of this partition product in the model.

Lemma 3.2.9. *Let $M \prec H(\omega_3)$ be countably closed with $\underline{\mathbb{P}} \upharpoonright (\kappa+1)$, $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa \in M$. Then if $\mathbb{R}$ is a finitely generated partition product based upon $\underline{\mathbb{P}} \upharpoonright \kappa$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa$, then $\mathbb{R}$ is isomorphic to a partition product which has domain an ordinal ρ below $M \cap \omega_2$.*

Proof. Fix such an M, and let $\mathbb{R}$ be a finitely generated partition product based upon $\underline{\mathbb{P}} \upharpoonright \kappa$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa$, say with domain X. Let $\{\langle B_m, \psi_m \rangle : m < n\}$ be the κ-suitable collection and Z the auxiliary part, where we assume that Z is disjoint from the union of the B_m. Let us enumerate Z as $\langle \xi_k : k < \omega \rangle$ and set $\delta_k := \text{index}(\xi_k)$ for each $k < \omega$. Furthermore, we let π_k be the rearrangement of $\mathbb{P}_{\delta_k}$ associated to $\text{base}(\xi_k)$.

We intend to apply Corollary 2.1.19, and so we define a sequence $\langle \tau_m : m < \omega \rangle$ of rearrangements of $\mathbb{R}$ and base-closed subsets $\langle D_m : m < \omega \rangle$ of X. For each $m < n$, set τ_m to be the rearrangement which first shifts the ordinals in $X \backslash B_m$ past $\sup(B_m)$ and then acts as ψ_m^{-1} on B_m. For each $m \geq n$, say $m = n + k$, we set τ_m to be the rearrangement which first shifts the ordinals in $X \backslash (b(\xi_k) \cup \{\xi_k\})$ past ξ_k and then acts as π_k^{-1} on $b(\xi_k)$ and sends ξ_k to ρ_{δ_k}. We set $D_0 := \varnothing$, $D_{m+1} := \bigcup_{k \leq m} B_k$ for $m < n$, and $D_{n+1+k} := D_n \cup \bigcup_{l \leq k} (b(\xi_l) \cup \{\xi_l\})$ for $k < \omega$.

By Corollary 2.1.19, let σ be a rearrangement of $\mathbb{R}$ so that $\text{ran}(\sigma)$ is an ordinal ρ and so that for each $m < \omega$, $\sigma[D_m]$ is an ordinal and $\tau_m \circ \sigma^{-1}$ is order-preserving on $\sigma[D_{m+1} \backslash D_m]$. We then see that ρ equals $\sum_{m<\omega} \text{ot}(\sigma[D_{m+1} \backslash D_m])$. However, if $m < n$, then $\text{ot}(\sigma[D_{m+1} \backslash D_m])$ is no larger than ρ_κ, and if $m \geq n$, then $\text{ot}(\sigma[D_{m+1} \backslash D_m])$ is no larger than $\rho_{\delta_k} + 1$, where $m = n + k$. Therefore

$$\rho = \sum_{m<\omega} \text{ot}(\sigma[D_{m+1} \backslash D_m]) \leq \rho_\kappa \cdot n + \sum_{k<\omega} (\rho_{\delta_k} + 1).$$

By the elementarity and countable closure of M, the ordinal on the right hand side is an element of $M \cap \omega_2$. Since $M \cap \omega_2$ is an ordinal, ρ is also a member of $M \cap \omega_2$.

$\square$

Lemma 3.2.10. *Let $M \prec H(\omega_3)$ be countably closed containing $\underline{\mathbb{P}} \upharpoonright (\kappa+1)$, $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa$ and $\vec{\varphi}$ as members. Then, if $\mathbb{R}$ is a finitely generated partition product based upon $\underline{\mathbb{P}} \upharpoonright \kappa$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa$,* *then $\mathbb{R}$ is isomorphic to a partition product which belongs to M, as well as the transitive collapse of M.*

Proof. Let M be fixed as in the statement of the lemma, and let $\mathbb{R}$ be finitely generated. Let $\{\langle B_k, \psi_k\rangle : k < n\}$ be the κ-suitable collection and Z the auxiliary part associated to $\mathbb{R}$. By Lemma 3.2.9, we may assume that $\mathbb{R}$ is a partition product on some ordinal ρ and that $\rho \in M \cap \omega_2$. Since $M \cap \omega_2$ is an ordinal, $\rho \subseteq M$. Then $Z \subseteq M$, and so by the countable closure of M, Z is a member of M. Hence by the elementarity and countable closure of M, setting $\delta_\xi := \text{index}(\xi)$ for each $\xi \in Z$, the sequence $\langle \delta_\xi : \xi \in Z\rangle$ is in M.

Now fix $k < n$ and $\xi \in Z$, and note that by Remark 2.2.3, since δ_ξ and κ are in M, $\psi_k^{-1}[B_k \cap b(\xi)]$ is in M. Next consider the relation in μ, ν which holds iff $\pi_\xi(\mu) = \psi_k(\nu)$, and observe that by Lemma 2.2.7, this holds iff ν is the μth element of $\psi_k^{-1}[B_k \cap b(\xi)]$. Therefore, this relation is a member of M. By the countable closure of M, the relation in ξ, k, μ, ν which holds iff $\pi_\xi(\mu) = \psi_k(\nu)$ is in M too. Similarly, the relation (in ξ, ζ, μ, ν) which holds iff $\pi_\xi(\mu) = \pi_\zeta(\nu)$ and the relation (in k, l, μ, ν) which holds iff $\psi_k(\mu) = \psi_l(\nu)$ are also in M.

We now apply the elementarity of M to find a finitely generated partition product $\mathbb{R}^*$ with domain ρ which has the following properties, where base* and index* denote the functions supporting $\mathbb{R}^*$:

1. $\mathbb{R}^*$ has κ-suitable collection $\{\langle B_k^*, \psi_k^*\rangle : k < n\}$ and auxiliary part Z; moreover, for each $\xi \in Z$, $\text{index}^*(\xi) = \delta_\xi$;

2. for each $\mu, \nu < \rho$ and each $\xi, \zeta \in Z$, $\pi_\xi(\mu) = \pi_\zeta(\nu)$ iff $\pi^*_\xi(\mu) = \pi^*_\zeta(\nu)$, and similarly with one of the ψ_k (resp. ψ_k^*) replacing one or both of the π_i (resp. π_i^*).

It is also straightforward to see that $\mathbb{R}^*$ is a member of the transitive collapse of M, as it is an iteration of length below $M \cap \omega_2$ of posets of size $\leq \aleph_1$, and hence is not moved by the transitive collapse map.

We now define a bijection $\sigma : \rho \longrightarrow \rho$ which will be the rearrangement witnessing that $\mathbb{R}$ and $\mathbb{R}^*$ are isomorphic. Set $\sigma(\alpha) = \beta$ iff $\alpha = \beta$ are both in Z; or for some $\xi \in Z$,

$\alpha = \pi_\xi(\mu)$ and $\beta = \pi^*_\xi(\mu)$; or for some $k < n$, $\alpha = \psi_k(\mu)$ and $\beta = \psi^*_k(\mu)$. By (2), we see that σ is well-defined, i.e., there is no conflict when some of these conditions overlap. It is also straightforward to see that σ is an acceptable rearrangement of $\mathbb{R}$ and in fact, $\sigma[\text{base}] = \text{base}^*$ and $\sigma[\text{index}] = \text{index}^*$, so that σ is an isomorphism from $\mathbb{R}$ onto $\mathbb{R}^*$.

□

Recall that we are assuming the CH holds (Assumption 3.0.1). Thus for the rest of Chapter 3, we fix a structure M satisfying the conclusion of Lemma 3.2.10 such that $|M| = \aleph_1$. We write $M = \bigcup_{\gamma<\omega_1} M_\gamma$, for a continuous, increasing sequence of elementary, countable submodels M_γ, such that the relevant parameters are in M_0.

Remark 3.2.11. The crucial use of the CH is to fix the model M. We will use the decomposition $M = \bigcup_{\gamma<\omega_1} M_\gamma$ to partition a tail of ω_1 into the slices $[M_\gamma \cap \omega_1, M_{\gamma+1} \cap \omega_1)$. We will show that it suffices to define the preassignment one slice at a time, with the values of the preassignment on one slice independent of the others. As Lemma 3.2.13 below shows, the preassignment restricted to the slice $[M_\gamma \cap \omega_1, M_{\gamma+1} \cap \omega_1)$ only needs to anticipate "partition product names" which are members of M_γ. This idea that the preassignment need only work in the above slices goes back to Lemma 3.2 of [4]. Furthermore, the proof of our Lemma 3.2.12 is more or less the same as Lemma 3.2 of [4]; we are simply working in slightly greater generality in order to analyze products of posets rather than just a single poset.

We recall that $\dot{S}_\kappa$ names a countable basis for a second countable, Hausdorff topology on ω_1.

Lemma 3.2.12. *Suppose that $\dot{f}$ is a $\mathbb{P}_\kappa$-name for a function from ω_1 into $\{0,1\}$ which satisfies the following: for any finitely generated partition product $\mathbb{R}$, with κ-suitable collection $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ and auxiliary part Z, say, all of which are in M; for every γ sufficiently large so that $\mathbb{R}$, the κ-suitable collection, and Z are in M_γ; for any $\mathbb{R}$-name $\dot{F}$ in M_γ for a set of n-tuples in $X := \biguplus_{\iota\in I} \omega_1$, which is closed in $\left(\biguplus_\iota \dot{S}_\kappa[\psi_\iota^{-1}(\dot{G} \restriction B_\iota)]\right)^n$; for any generic G for $\mathbb{R}$; and for any x with*

$$x \in \dot{F}[G] \cap (M_{\gamma+1}[G] \backslash M_\gamma[G])^n,$$

there exist pairwise distinct tuples y, y' in $\dot{F}[G] \cap M_\gamma[G]$ so that for every $i < n$ and $\iota \in I$, if $x(i)$ is in the ι-th copy of ω_1, then so are $y(i)$ and $y'(i)$, and

$$\dot{\chi}_\kappa[\psi_\iota^{-1}(G \restriction B_\iota)](y(i), y'(i)) = \dot{f}[\psi_\iota^{-1}(G \restriction B_\iota)](x(i)).$$

Then $\dot{f}$ satisfies Proposition 3.2.2.

Proof. Let $\dot{f}$ be as in the statement of the lemma, and suppose that $\dot{f}$ failed to satisfy Proposition 3.2.2. By Remarks 3.2.5 and 3.2.7 there exist a finitely generated partition product $\mathbb{R}$, a condition $p \in \mathbb{R}$, an integer $n < \omega$, a sequence $d \in 2^n$, and an $\mathbb{R}$-name for a closed set $\dot{F}$ of n-tuples such that p forces that these objects satisfy Remark 3.2.5. We may assume that $\mathbb{R} \in M$ by Lemma 3.2.10. Since M is countably closed and contains $\mathbb{R}$, and since $\mathbb{R}$ is c.c.c. (by Assumption 3.0.1), we know that the name $\dot{F}$ belongs to M too. Thus we may find some $\gamma < \omega_1$ such that $\dot{F}$ and all other relevant objects are in M_γ.

Now let G be a generic for $\mathbb{R}$ containing the condition p. Let $S := \biguplus_\iota \dot{S}_\kappa[\psi_\iota^{-1}(G \restriction B_\iota)]$, let $f := \biguplus_\iota \dot{f}[\psi_\iota^{-1}(G \restriction B_\iota)]$, and let $\chi := \biguplus_\iota \dot{\chi}_\kappa[\psi_\iota^{-1}(G \restriction B_\iota)]$. By (ii) of Remark 3.2.5, we may find some $x \in F \cap (X \backslash M_\gamma[G])^n$, where $F := \dot{F}[G]$. We now want to consider how the models $\langle M_\beta : \gamma \leq \beta < \omega_1 \rangle$ separate the elements of x, and then we will apply the assumptions of the lemma to each $\beta \in [\gamma, \omega_1)$ such that $M_{\beta+1}[G] \backslash M_\beta[G]$ contains an element of x. Indeed, consider the finite set a of $\beta \in [\gamma, \omega_1)$ such that x contains at least one element in $M_{\beta+1}[G] \backslash M_\beta[G]$, and let $\langle \gamma_k : k < l \rangle$ be the increasing enumeration of a. Further, let x_k, for each $k < l$, be the subsequence of x consisting of all the elements of x inside $M_{\gamma_k+1}[G] \backslash M_{\gamma_k}[G]$.

We now work downwards from l to define a sequence of formulas $\langle \varphi_k : k \leq l \rangle$. We will maintain as recursion hypotheses that if $0 < k < l$, then (i) $\varphi_{k+1}(x_0, \ldots, x_k)$ is satisfied, and that (ii) the parameters of φ_{k+1} are in $M_{\gamma_0}[G]$. Let $\varphi_l(u_0, \ldots, u_{l-1})$ state that $u_0 {}^\frown \ldots {}^\frown u_{l-1} \in F$; then (i) and (ii) are satisfied. Now suppose that $0 < k < l$ and that φ_{k+1} is defined. Let F_k be the closure of the set of all tuples u such that $\varphi_{k+1}(x_0, \ldots, x_{k-1}, u)$ is satisfied. By (ii) and the fact that $x_0 {}^\frown \ldots {}^\frown x_{k-1} \in M_{\gamma_k}[G]$, we see that F_k is in $M_{\gamma_k}[G]$. Furthermore, $x_k \in F_k$. Therefore, by the assumptions of the lemma, we may find pairwise distinct tuples

$v_{k,L}, v_{k,R}$ in $M_{\gamma_k}[G] \cap F_k$ such that for every $i < n$ and $\iota \in I$, if $x_k(i)$ is in the ι-th copy of ω_1, then so are $v_{k,L}(i)$ and $v_{k,R}(i)$, and

$$\dot\chi_\kappa[\psi_\iota^{-1}(G \upharpoonright B_\iota)](v_{k,L}(i), v_{k,R}(i)) = \dot f[\psi_\iota^{-1}(G \upharpoonright B_\iota)](x_k(i)).$$

For each such i, fix a pair of disjoint, basic open sets U_i, V_i from $\dot S_\kappa[\psi_\iota^{-1}(G \upharpoonright B_\iota)]$ witnessing this coloring statement. By definition of F_k, we may find two further tuples $u_{k,L}, u_{k,R}$ such that for each $Z \in \{L, R\}$, $\varphi_{k+1}(x_0, \ldots, x_{k-1}, u_{k,Z})$ is satisfied, and such that the pair $\langle u_{k,L}(i), u_{k,R}(i)\rangle$ is in $U_i \times V_i$. Now define $\varphi_k(u_0, \ldots, u_{k-1})$ to be the following formula:

$$\exists w_{k,L}, w_{k,R} \left(\bigwedge_{Z \in \{L,R\}} \varphi_{k+1}(u_0, \ldots, u_{k-1}, w_{k,Z}) \wedge \bigwedge_i \Big(\langle w_{k,L}(i), w_{k,R}(i)\rangle \in U_i \times V_i \Big) \right).$$

Then (i) is satisfied, and since the only additional parameters are the basic open sets U_i and V_i, (ii) is also satisfied.

This completes the construction of the sequence $\langle \varphi_k : k \leq l\rangle$. Now using the fact that the sentence φ_0 is true and has only parameters in M_{γ_0}, we may work our way upwards through the sequence $\varphi_0, \varphi_1, \ldots, \varphi_l$ in order to find two tuples x_L, x_R of the same length as x such that $x_L, x_R \in F$, and such that for each $i < n$, $\langle x_L(i), x_R(i)\rangle \in U_i \times V_i$. In particular, for each $i < n$,

$$\dot\chi_\kappa[\psi_\iota^{-1}(G \upharpoonright B_\iota)](x_L(i), x_R(i)) = \dot f[\psi_\iota^{-1}(G \upharpoonright B_\iota)](x(i)),$$

where ι is such that $x(i)$ is in the ι-th copy of ω_1. However, recalling Remark 3.2.5 and the assumptions about the condition p, this contradicts the fact that $f \circ x = d$, and that there is some $i < n$ so that $\chi(x_L(i), x_R(i)) \neq d(i)$. $\square$

The following lemma gives a nice streamlining of the previous one and applies to any collection $\dot U$ of n-tuples in ω_1, not just collections $\dot F$ which are closed in the appropriate topology. The greater generality here is only apparent, as we can always take closures and obtain, because the colorings are open, the same result from its application to closed sets of tuples. However, it is technically convenient. Also, as a matter of notation, for each $\gamma < \omega_1$, we fix an enumeration $\langle \nu_{\gamma,n} : n < \omega\rangle$ of the slice $[M_\gamma \cap \omega_1, M_{\gamma+1} \cap \omega_1)$.

Lemma 3.2.13. *Suppose that $\dot{f}$ is a $\mathbb{P}_\kappa$-name for a function from ω_1 into $\{0,1\}$ satisfying the following: for any finitely generated partition product $\mathbb{R}$, say with κ-suitable collection $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ and auxiliary part Z, all of which are in M; for any γ sufficiently large such that M_γ contains $\mathbb{R}$, $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$, and Z; for any $l < \omega$; for any $\mathbb{R}$-name $\dot{U}$ in M_γ for a set of l-tuples in ω_1; and for any generic G for $\mathbb{R}$, if $\langle \nu_{\gamma,0}, \ldots, \nu_{\gamma,l-1}\rangle \in \dot{U}[G]$, then there exist pairwise distinct l-tuples $\vec{\mu}, \vec{\mu}'$ in $M_\gamma[G] \cap \dot{U}[G]$ so that for all $k < l$ and all $\iota \in I$,*

$$\dot{\chi}_\kappa[\psi_\iota^{-1}(G \restriction B_\iota)](\mu_k, \mu_k') = \dot{f}[\psi_\iota^{-1}(G \restriction B_\iota)](\nu_{\gamma,k}).$$

Then $\dot{f}$ satisfies Lemma 3.2.12.

Proof. We want to first observe that Lemma 3.2.12 follows from its restriction to sequences z which are bijections from some $n < \omega$ onto $\biguplus_\iota \{\nu_{\gamma,l} : l < m\}$, for some $m < \omega$. Towards this end, fix $\dot{F}$, G, and a tuple $x \in \dot{F}[G]$ as in the statement of Lemma 3.2.12. First, if x is not such a surjection, we may add additional coordinates to x to form a sequence x' which is a surjection onto $\biguplus_\iota \{\nu_{\gamma,l} : l < m\}$, for some $m < \omega$. Then we define the name $\dot{F}'$ as the product of $\dot{F}$ with the requisite, finite number of copies of ω_1, so that x' is a member of $\dot{F}'[G]$. Second, if x' contains repetitions, then we make the necessary shifts in x' to eliminate the repetitions and call the resulting sequence x''. We then consider the name $\dot{F}''$ of all tuples from $\dot{F}'$ which have the same corresponding shifts in their tuples as x''. $\dot{F}''$ still names a closed set and is still an element of M_γ. Thus $x'' \in \dot{F}''[G]$, and x'' is a bijection from some integer onto $\biguplus_\iota \{\nu_{\gamma,l} : l < m\}$, for some $m < \omega$. By applying the restricted version of Lemma 3.2.12 to x'' and $\dot{F}''$, we see that the desired result holds for x and $\dot{F}$.

To verify Lemma 3.2.12, fix $\dot{F}$, a generic G, and a sequence $x \in \dot{F}[G]$ as in the statement thereof, where we assume that x is a bijection from some n onto $\biguplus_\iota \{\nu_{\gamma,l} : l < m\}$, for some $m < \omega$. Define $\dot{U}$ to be the $\mathbb{R}$-name for the set of all tuples $\vec{\xi} = \langle \xi_0, \ldots, \xi_{m-1}\rangle$ in ω_1 such that $\vec{\xi}$ concatenated with itself $|I|$-many times is an element of $\dot{F}$, noting that $\dot{U}$ is still a member of M_γ. Since x is a bijection as described above, $\langle \nu_{\gamma,0}, \ldots, \nu_{\gamma,m-1}\rangle \in \dot{U}[G]$. Now apply the assumptions in the statement of the current lemma to find two pairwise distinct

m-tuples $\vec{\mu}, \vec{\mu}'$ in $M_\gamma[G] \cap \dot{U}[G]$ so that for all $l < m$ and $\iota \in I$,

$$\dot{\chi}_\kappa[\psi_\iota^{-1}(G \restriction B_\iota)](\mu_l, \mu_l') = \dot{f}[\psi_\iota^{-1}(G \restriction B_\iota)](\nu_{\gamma,l}).$$

Let y be the $|I|$-fold concatenation of $\vec{\mu}$ with itself, and let y' be defined similarly with respect to $\vec{\mu}'$. Then as $\vec{\mu}, \vec{\mu}' \in \dot{U}[G]$, we have y, y' are in $\dot{F}[G]$. And since $\vec{\mu}, \vec{\mu}'$ satisfy the appropriate coloring requirements, we have that y and y' satisfy the conclusion of Lemma 3.2.12. $\square$

Lemma 3.2.13 gives a sufficient condition for Proposition 3.2.2, and it thus implies that any partition product based upon $\underline{\mathbb{P}} \restriction (\kappa + 1)$ and $\underline{\dot{\mathbb{Q}}} \restriction (\kappa + 1)$ is c.c.c. In the next section, we consider how to obtain a $\mathbb{P}_\kappa$-name $\dot{f}$ as in Lemma 3.2.13.

3.3 How to get there

In this section, which forms the technical heart of Part 1, we show how to obtain a $\mathbb{P}_\kappa$-name $\dot{f}$ as in Lemma 3.2.13. In light of Remark 3.2.11 and Lemma 3.2.13, it suffices to define the name $\dot{f}$ separately for each of its restrictions to the slices $[M_\gamma \cap \omega_1, M_{\gamma+1} \cap \omega_1)$, and so let $\gamma < \omega_1$ be fixed for the remainder of this section. To simplify notation, we drop the γ-subscript from the enumeration $\langle \nu_{\gamma,n} : n < \omega \rangle$ of $[M_\gamma \cap \omega_1, M_{\gamma+1} \cap \omega_1)$, preferring instead to simply write $\langle \nu_n : n < \omega \rangle$. We note that the values of $\dot{f}$ on the countable ordinal $M_0 \cap \omega_1$ are irrelevant, by Remark 3.2.5.

In order to define the name $\dot{f}$, we recursively specify the $\mathbb{P}_\kappa$-name equal to $\dot{f}(\nu_k)$, which we call $\dot{a}_k$. Each $\dot{a}_k$ will be a canonical name, which we view as a function from a maximal antichain in $\mathbb{P}_\kappa$ into $\{0, 1\}$. We refer to these more specifically as *canonical color names.* By a *partial canonical color name* we mean a function from an antichain in $\mathbb{P}_\kappa$, possibly not maximal, into $\{0, 1\}$. When viewing such functions as names $\dot{a}$, we say that $\dot{a}[G]$, where G is generic for $\mathbb{P}_\kappa$, is defined and equal to i if there is some $p \in G$ which belongs to the domain of the function $\dot{a}$ and gets mapped to i. The upcoming definition isolates exactly what we need.

Definition 3.3.1. *Suppose that $\dot{a}_0, \ldots, \dot{a}_{l-1}$ are partial canonical color names. We say that they have the* partition product preassignment property at γ *if for every finitely generated partition product $\mathbb{R}$ with κ-suitable collection $\{\langle B_\iota, \psi_\iota \rangle : \iota \in I\}$, say, all of which are in M_γ; for every $\mathbb{R}$-name $\dot{U} \in M_\gamma$ for a collection of l-tuples in ω_1; and for every generic G for $\mathbb{R}$, the following holds: if $\langle \nu_0, \ldots, \nu_{l-1} \rangle \in \dot{U}[G]$, then there exist two pairwise distinct tuples $\vec{\mu}, \vec{\mu}' \in \dot{U}[G] \cap M_\gamma[G]$ so that for every $\iota \in I$ and $k < l$, if $\dot{a}_k[\psi_\iota^{-1}(G \restriction B_\iota)]$ is defined, then*

$$\dot{\chi}_\kappa[\psi_\iota^{-1}(G \restriction B_\iota)](\mu_k, \mu_k') = \dot{a}_k[\psi_\iota^{-1}(G \restriction B_\iota)].$$

Remark 3.3.2. In the context of Definition 3.3.1, we say that two sequences $\vec{\mu}$ and $\vec{\mu}'$ of length l *match* $\dot{a}_0, \ldots, \dot{a}_{l-1}$ *at* ι *with respect to* G, or *match at* B_ι *with respect to* G if for every $k < l$ such that $\dot{a}_k[\psi_\iota^{-1}(G \restriction B_\iota)]$ is defined,

$$\dot{\chi}_\kappa[\psi_\iota^{-1}(G \restriction B_\iota)](\mu_k, \mu_k') = \dot{a}_k[\psi_\iota^{-1}(G \restriction B_\iota)].$$

We say that two sequences $\vec{\mu}$ and $\vec{\mu}'$ *match* $\dot{a}_0, \ldots, \dot{a}_{l-1}$ *on* I *with respect to* G if for every $\iota \in I$, $\vec{\mu}$ and $\vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_{l-1}$ at ι with respect to G. If the filter G is clear from context, we drop the phrase "with respect to G." Furthermore, we will often want to avoid talking about the index set I explicitly, and so we will also say that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_{l-1}$ on $\mathcal{S} := \{\langle B_\iota, \psi_\iota \rangle : \iota \in I\}$, if for each $\langle B, \psi \rangle \in \mathcal{S}$, we have that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_{l-1}$ at B.

To prove Lemma 3.2.13, and in turn Proposition 3.2.2, we recursively construct the sequence $\langle \dot{a}_k : k < \omega \rangle$ in such a way that for each $l < \omega$, $\dot{a}_0, \ldots, \dot{a}_{l-1}$ have the partition product preassignment property at γ. More precisely, we show that if $\dot{a}_0, \ldots, \dot{a}_{l-1}$ are *total* canonical color names with the partition product preassignment property at γ, then there is a total name $\dot{a}_l$ so that $\dot{a}_0, \ldots, \dot{a}_l$ have the partition product preassignment property at γ.

For this in turn it is enough to prove that if $\dot{a}_0, \ldots, \dot{a}_{l-1}$ are total canonical color names, $\dot{a}_l$ is a partial canonical color name, $\dot{a}_0, \ldots, \dot{a}_l$ have the partition product preassignment property at γ, and $p \in \mathbb{P}_\kappa$ is incompatible with all conditions in the domain of $\dot{a}_l$, then there exist $p^* \leq_{\mathbb{P}_\kappa} p$ and $c \in \{0, 1\}$ so that $\dot{a}_0, \ldots, \dot{a}_l \cup \{p^* \mapsto c\}$ have the partition product preassignment property at γ. By a transfinite iteration of this process we can construct a sequence of names $\dot{a}_l^\xi$ with increasing domains, continuing until we reach a name whose domain is a maximal antichain. This final name is then total.

To prove the "one condition" extension above, we assume that it fails with $c = 0$ and prove that it then holds with $c = 1$. Our assumption is the following:

Assumption 3.3.3. $\dot{a}_0, \ldots, \dot{a}_{l-1}$ are total canonical color names, $\dot{a}_l$ is partial, $\dot{a}_0, \ldots, \dot{a}_l$ have the partition product preassignment property at γ, $p \in \mathbb{P}_\kappa$ is incompatible with all conditions in $\mathrm{dom}(\dot{a}_l)$, but for every $p^* \leq_{\mathbb{P}_\kappa} p$, $\dot{a}_0, \ldots, \dot{a}_l \cup \{p^* \mapsto 0\}$ do *not* have the partition product preassignment property at γ.

Our goal is to show that $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ do have the partition product preassignment property at γ. The following lemma is the key technical result which allows us to prove that $p \mapsto 1$ works in this sense and thereby continue the construction of the name $\dot{a}_l$. We note that the lemma is stated in terms of enriched partition products; the enrichments are used to propagate the induction hypothesis needed for its proof. After the statement of the lemma, and before its proof, we provide some expository comments which we believe will be helpful.

Lemma 3.3.4. *Let $(\mathbb{R}, \mathcal{B})$ be an enriched partition product with domain X which is finitely generated by a κ-suitable collection $\mathcal{S} = \{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ and auxiliary part Z, all of which belong to M_γ. Let $\bar{p}$ be a condition in $\mathbb{R}$, and let $\vec{\nu} := \langle \nu_0, \ldots, \nu_l\rangle$. Finally, let $\bar{\mathcal{S}} \subseteq \mathcal{S}$ be non-empty. Then there exist the following objects:*

(a) an enriched partition product $(\mathbb{R}^*, \mathcal{B}^*)$ with domain X^*, finitely generated by a κ-suitable collection $\mathcal{S}^*$ and an auxiliary part Z^*, all of which are in M_γ;

(b) a condition $p^* \in \mathbb{R}^*$;

(c) an $\mathbb{R}^*$-name $\dot{U}^*$ in M_γ for a collection of $l+1$-tuples in ω_1;

(d) a non-empty, finite collection $\mathcal{F}$ in M_γ of embeddings from $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^*, \mathcal{B}^*)$;

satisfying that for each $\pi \in \mathcal{F}$:

1. $p^* \leq_{\mathbb{R}^*} \pi(\bar{p})$;

and also satisfying that p^* forces the following statements in $\mathbb{R}^*$:

(2) $\vec{\nu} \in \dot{U}^*$;

(3) for any pairwise distinct tuples $\vec{\mu}, \vec{\mu}'$ in $\dot{U}^* \cap M_\gamma[\dot{G}^*]$, if $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l$ on $\mathcal{S}^*$, then there is some $\pi \in \mathcal{F}$ such that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\pi[\bar{\mathcal{S}}]$.

Let us pause to make a few motivational remarks which we believe will help explain the technicalities and paint a big picture of the proof. In rough but possibly helpful language, the partition product preassignment property asserts the following: in any finitely-generated partition product $\mathbb{R}$ with κ-suitable collection $\{\langle B_\iota, \psi_\iota \rangle : \iota \in I\}$, we have that if ν_l is in some subset of ω_1 which is a member of M_γ, then we can reflect the ordinal ν_l inside this M_γ-set to two ordinals μ and μ' so that μ and μ' get the correct color (namely 1) on every "branch" B_ι. What we have to work with is the fact that for every $p^* \leq_{\mathbb{P}_\kappa} p$, $p^* \mapsto 0$ fails to satisfy this property. This means that for each such extension p^*, we have a "0-counter-example partition product" $\mathbb{P}(p^*)$. This is some finitely-generated partition product so that no matter how we reflect ν_l, we fail to get 0 on at least one branch.

To show that $p \mapsto 1$ satisfies the desired property, we need to ensure that it is impossible to find a finitely-generated partition product and a condition therein forcing its failure. The way to do this is to take our starting partition product $\mathbb{R}$ and graft onto $\mathbb{R}$ various 0-counter-example partition products. This will result in some larger partition product $\mathbb{R}^*$ into which $\mathbb{R}$ embeds in multiple ways. We want to ensure that we can get the correct color along at least one such embedding.

In more detail, the proof will be an induction first on the (finite) number of branches and secondly on the agreement between the "branches" B_ι (we recall from our discussion of shadow bases that any two such branches (up to rearrangement) agree on an initial segment of ρ_κ and are disjoint past that. So there is a well-defined "agreement" on which to induct). The base case is when they are all pairwise disjoint. In this case, we will expand $\mathbb{R}$ by straightforwardly grafting counter-example partition products. However, when the branches have agreement, this becomes more challenging. To get into a prior case for which we can apply induction, we will first remove a "maximal" (in the well-defined agreement sense) branch. After applying induction to this collection with one fewer branch, we will need to restore many copies of the lost branch. But we must do so in such a way that the restored copies have a lower agreement than that of our original collection. The challenge of the proof is to show that this sketch can be carried out.

We recommend that the reader read the proof multiple times, each time working in slightly more generality. First we recommend that the reader consider the case where we simply have an $\mathrm{Add}(\omega, 1)$-name $\dot{\chi}$ for a continuous coloring on ω_1, where we are preassigning to only one ordinal below one condition, and where the "auxiliary parts" are all empty. Under this assumption, the base case (as described in the previous paragraph) is the only one to be considered, and may of the details in the following proof concerning grafting don't apply. Second, we recommend that the reader consider the case where we have an $\mathrm{Add}(\omega, 1) \times \mathrm{Add}(\omega, 1)$-name $\dot{\chi}$ for a coloring (also preassigning to only a single ordinal below a condition). In this case, a partition product involving the square of $\mathrm{Add}(\omega, 1)$ could give us a pair of generics $r_1 \times r_2$ and $r_1^* \times r_2^*$ for the square so that $r_1 = r_1^*$. Thus in this case, it is possible to have some amount of "agreement" between the generics added to the square in the full partition product. However, the agreement is on at most one coordinate, and so it simplifies the proof quite a bit. Finally, we recommend that the reader read the proof in full generality.

Proof. For the remainder of the proof, fix the objects $(\mathbb{R}, \mathcal{B})$, X, $\mathcal{S}$, Z, $\bar{p}$, and $\bar{\mathcal{S}}$ as in the statement of the lemma. We also set $J := \left\{\iota \in I : \langle B_\iota, \psi_\iota \rangle \in \bar{\mathcal{S}}\right\}$. Before we continue, let us introduce the following terminology: suppose that $p' \leq_{\mathbb{P}_\kappa} p$, $c \in \{0, 1\}$, and $\tilde{p} \in \mathbb{P}_\kappa$. We say that $\tilde{p}$ is *decisive* about the sequence of names $\dot{a}_0, \ldots, \dot{a}_l \cup \{p' \mapsto c\}$ if for each $k < l$, $\tilde{p}$ extends a unique element of $\mathrm{dom}(\dot{a}_k)$, and if $\tilde{p}$ either extends a unique element of $\mathrm{dom}(\dot{a}_l) \cup \{p'\}$ or is incompatible with all conditions therein. Note that any $\tilde{p}$ may be extended to a decisive condition, as $\mathrm{dom}(\dot{a}_k)$ is a maximal antichain in $\mathbb{P}_\kappa$, for each $k < l$.

For each $\iota \in I$ we set p_ι to be the condition $\psi_\iota^{-1}(\bar{p} \upharpoonright B_\iota)$ in $\mathbb{P}_\kappa$. By extending $\bar{p}$ if necessary, we may assume that for each $\iota \in I$, p_ι is decisive about $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$. Let us also define

$$J_p := \{\iota \in J : p_\iota \leq_{\mathbb{P}_\kappa} p\},$$

noting that for each $\iota \in J \backslash J_p$, p_ι is incompatible with p in $\mathbb{P}_\kappa$, since p_ι is decisive.

We will prove by induction that there exist objects as in (a)-(d) satisfying (1)-(3). The induction concerns properties of $\bar{\mathcal{S}}$, which we will refer to as the *matching core* of $\mathcal{S}$, in light of the requirement in (3) that the desired matching occurs on the image of $\bar{\mathcal{S}}$ under some $\pi \in \mathcal{F}$. By the definition of κ-suitability and Remark 2.2.9, for each distinct $\iota_0, \iota_1 \in I$, $\psi_{\iota_0}^{-1}[B_{\iota_0} \cap B_{\iota_1}] = \psi_{\iota_1}^{-1}[B_{\iota_0} \cap B_{\iota_1}]$ is an ordinal $< \rho_\kappa$. We will denote this ordinal by $\mathrm{ht}(B_{\iota_0}, B_{\iota_1})$; it is helpful to note that $\mathrm{ht}(B_{\iota_0}, B_{\iota_1}) = \max\{\alpha < \rho_\kappa : \psi_{\iota_0}[\alpha] = \psi_{\iota_1}[\alpha]\} = \sup\{\xi + 1 : \psi_{\iota_0}(\xi) = \psi_{\iota_1}(\xi)\}$. The induction will be first on the ordinal

$$\mathrm{ht}\left(\bar{\mathcal{S}}\right) := \max\{\mathrm{ht}(B_{\iota_0}, B_{\iota_1}) : \iota_0, \iota_1 \in J \wedge \iota_0 \neq \iota_1\}$$

and second on the finite size of $\bar{\mathcal{S}}$.

Case 1: $\mathrm{ht}\left(\bar{\mathcal{S}}\right) = 0$ (note that this includes as a subcase $|J| = 1$). For each $\iota \in J_p$, p_ι extends p in $\mathbb{P}_\kappa$, and so, by Assumption 3.3.3, $\dot{a}_0, \ldots, \dot{a}_l \cup \{p_\iota \mapsto 0\}$ do not have the partition product preassignment property at γ. For each $\iota \in J_p$, we fix the following objects as witnesses to this:

$(1)_\iota$ a partition product $\mathbb{R}_\iota^*$, say with domain X_ι^*, which is finitely generated by the κ-suitable collection $\mathcal{S}_\iota^* = \left\{\langle B_{\iota,\eta}^*, \psi_{\iota,\eta}^* \rangle : \eta \in I^*(\iota)\right\}$ and auxiliary part Z_ι^*, all of which are in M_γ;

$(2)_\iota$ a condition p_ι^* in $\mathbb{R}_\iota^*$;

$(3)_\iota$ an $\mathbb{R}_\iota^*$-name $\dot{U}_\iota^*$ in M_γ for a set of $l+1$-tuples in ω_1;

such that p^*_ι forces in $\mathbb{R}^*_\iota$ that

(4)$_\iota$ $\vec{\nu} \in \dot{U}^*_\iota$, and for any pairwise distinct tuples $\vec{\mu}, \vec{\mu}'$ in $\dot{U}^*_\iota \cap M_\gamma[\dot{G}^*_\iota]$, $\vec{\mu}$ and $\vec{\mu}'$ do *not* match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p_\iota \mapsto 0\}$ on $I^*(\iota)$.

For each $\eta \in I^*(\iota)$, let $p_{\iota,\eta}$ denote the $\mathbb{P}_\kappa$-condition $(\psi^*_{\iota,\eta})^{-1}\left(p^*_\iota \restriction B^*_{\iota,\eta}\right)$, and note that by extending the condition p^*_ι, we may assume that each $p_{\iota,\eta}$ is decisive about $\dot{a}_0, \ldots, \dot{a}_l \cup \{p_\iota \mapsto 0\}$. It is straightforward to check that since each such $p_{\iota,\eta}$ is decisive and since, by Assumption 3.3.3, $\dot{a}_0, \ldots, \dot{a}_l$ do have the partition product preassignment property at γ, we must have that

$$J^*(\iota) := \{\eta \in I^*(\iota) : p_{\iota,\eta} \leq_{\mathbb{P}_\kappa} p_\iota\} \neq \varnothing,$$

as otherwise we contradict (4)$_\iota$.

Let us introduce some further notation which will facilitate the exposition. For $\iota \in J \backslash J_p$, define $\mathbb{R}^*_\iota$ to be some isomorphic copy of $\mathbb{P}_\kappa$ with domain X^*_ι, say with isomorphism $\psi^*_{\iota,\iota}$; we will denote X^*_ι additionally by $B^*_{\iota,\iota}$ in order to streamline the notation in later arguments. For $\iota \in J \backslash J_p$, we set $\mathcal{S}^*_\iota := \left\{\langle B^*_{\iota,\iota}, \psi^*_{\iota,\iota}\rangle\right\}$ with index set $I^*(\iota) = \{\iota\}$ which we also denote by $J^*(\iota)$. Next, we define p^*_ι to be the image of p_ι under the isomorphism $\psi^*_{\iota,\iota}$ from $\mathbb{P}_\kappa$ onto $\mathbb{R}^*_\iota$, and we set $\dot{U}^*_\iota$ to be the $\mathbb{R}^*_\iota$-name for all $l+1$-tuples in ω_1. We remark here for later use that for each $\iota \in J$ and $\eta \in J^*(\iota)$,

$$(\psi^*_{\iota,\eta})^{-1}\left(p^*_\iota \restriction B^*_{\iota,\eta}\right) \leq_{\mathbb{P}_\kappa} (\psi_\iota)^{-1}(\bar{p} \restriction B_\iota).$$

Our next step is to amalgamate all of the above into one much larger partition product. Without loss of generality, by shifting if necessary, we may assume that the domains X^*_ι, for $\iota \in J$, are pairwise disjoint. Then, by Corollary 2.2.20, the poset $\mathbb{R}^*(0) := \prod_{\iota \in J} \mathbb{R}^*_\iota$ is a partition product with domain $\bigcup_{\iota \in J} X^*_\iota$. It is also a member of M_γ. Additionally, $\mathbb{R}^*(0)$ is finitely generated by the κ-suitable collection $\mathcal{S}^* := \bigcup_{\iota \in J} \mathcal{S}^*_\iota$ and auxiliary part $\bigcup_{\iota \in J_p} Z^*_\iota$. Let us abbreviate $\bigcup_{\iota \in J} B_\iota$ by X_0 and $\bigcup_{\iota \in J} X^*_\iota$ by X^*_0. We also let $p^*(0)$ be the condition in $\mathbb{R}^*(0)$ whose restriction to X^*_ι equals p^*_ι, and we let $\dot{U}^*$ be the $\mathbb{R}^*(0)$-name for the intersection of all the $\dot{U}^*_\iota$, for $\iota \in J$.

Now consider the product of indices

$$\hat{J} := \prod_{\iota \in J} J^*(\iota);$$

$\hat{J}$ is non-empty, finite, and an element of M_γ, since J and each $J^*(\iota)$ are. Let $\langle h_k : k < n \rangle$ enumerate $\hat{J}$. Each h_k selects, for every $\iota \in J$, an image of the $\mathbb{P}_\kappa$-"branch" B_ι inside $\mathbb{R}^*_\iota$. For each $k < n$, we define the map $\pi_k : X_0 \longrightarrow X_0^*$ corresponding to h_k by taking $\pi_k \restriction B_\iota$ to be equal to $\psi^*_{\iota, h_k(\iota)} \circ \psi_\iota^{-1}$, for each $\iota \in J$. This is well-defined since, by our assumption that $\mathrm{ht}\left(\bar{\mathcal{S}}\right) = 0$, we know that the sets B_ι, for $\iota \in J$, are pairwise disjoint. We also see that each π_k embeds $\mathbb{R} \restriction X_0$ into $\mathbb{R}^*(0)$, since it isomorphs $\mathbb{R} \restriction B_\iota$ onto $\mathbb{R}^*(0) \restriction B^*_{\iota, h_k(\iota)}$, for each $\iota \in J$. In fact, each π_k is $(\bar{\mathcal{S}}, \mathcal{S}^*)$-suitable by construction, and h_k is the associated injection h_{π_k} (see Definition 3.1.3). Finally, we want to see that $p^*(0)$ extends $\pi_k(\bar{p} \restriction X_0)$ for each $k < n$; but this follows by definition of π_k and our above observation that for each $\iota \in J$ and $\eta \in J^*(\iota)$,

$$(\psi^*_{\iota,\eta})^{-1}\left(p^*_\iota \restriction B^*_{\iota,\eta}\right) \leq_{\mathbb{P}_\kappa} (\psi_\iota)^{-1}(\bar{p} \restriction B_\iota).$$

Using Lemma 3.1.5, fix an enrichment $\mathcal{B}^*_0$ of $\mathbb{R}^*(0)$ such that $\mathcal{B}^*_0$ contains the image of $\mathcal{B} \restriction X_0$ under each π_k and such that $\left\{\langle B^*_{\iota,\eta}, \psi^*_{\iota,\eta}\rangle : \iota \in J \wedge \eta \in J^*(\iota)\right\}$ is κ-suitable with respect to $(\mathbb{R}^*(0), \mathcal{B}^*_0)$. Note that the assumptions of Lemma 3.1.5 are satisfied because the sets X^*_ι, for $\iota \in J$, are pairwise disjoint and $\{\pi_k : k < n\}$ is a collection of $(\bar{\mathcal{S}}, \mathcal{S}^*)$-suitable maps. blergg

Before continuing with the main argument, we want to consider an "illustrative case" in which we make the simplifying assumption that the domain of $\mathbb{R}$ is just X_0. The key ideas of the matching argument are present in this illustrative case, and after working through the details, we will show how to extend the argument to work in the more general setting wherein the domain of $\mathbb{R}$ has elements beyond X_0.

Proceeding, then, under the assumption that the domain of $\mathbb{R}_0$ is exactly X_0, we specify the objects from (a)-(d) satisfying (1)-(3). Namely, the finitely generated partition product $(\mathbb{R}^*(0), \mathcal{B}^*_0)$, generated by $\mathcal{S}^*$ and $\bigcup_{\iota \in J_p} Z^*_\iota$; the condition $p^*(0)$; the $\mathbb{R}^*(0)$-name $\dot{U}^*$; and the collection $\{\pi_k : k < n\}$ of embeddings are the requisite objects. From the fact that $\bar{p} = \bar{p} \restriction X_0$

we have that $p^*(0)$ is below $\pi_k(\bar{p})$ for each $k < n$. Since p^*_ι forces that $\vec{\nu} \in \dot{U}^*_\iota$ for each $\iota \in J$, we see that $p^*(0)$ forces that $\vec{\nu} \in \dot{U}^*$. Thus (3) remains to be checked.

Towards this end, fix a generic G^* for $\mathbb{R}^*(0)$ containing $p^*(0)$, and for each $\iota \in J$, set $G^*_\iota := G^* \restriction \mathbb{R}^*_\iota$. Also set $U^* := \dot{U}^*[G^*]$. Let us also fix two pairwise distinct tuples $\vec{\mu}$ and $\vec{\mu}'$ in $U^* \cap M_\gamma[G^*]$ which match $\dot{a}_0, \ldots, \dot{a}_l$ on $\mathcal{S}^*$. Our goal is to find some $k < n$ such that $\vec{\mu}$ and $\vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\pi_k[\bar{\mathcal{S}}]$. We will first show the following claim.

Claim 3.3.5. *For each $\iota \in J_p$, there is some $\eta \in J^*(\iota)$ such that*

$$\dot{\chi}_\kappa \left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta}) \right] (\mu_l, \mu'_l) = 1.$$

Proof of Claim 3.3.5. Recall that for each $\iota \in J_p$, by $(4)_\iota$ above, we know that the condition p^*_ι forces in $\mathbb{R}^*_\iota$ that for any two pairwise distinct tuples $\vec{\xi}, \vec{\xi}'$ in $\dot{U}^*_\iota \cap M_\gamma[\dot{G}^*_\iota]$, $\vec{\xi}$ and $\vec{\xi}'$ do *not* match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p_\iota \mapsto 0\}$ on $I^*(\iota)$. Fix some $\iota \in J_p$, and let $U^*_\iota := \dot{U}^*_\iota[G^*_\iota]$. Now observe that $\vec{\mu}$ and $\vec{\mu}'$ are in $U^*_\iota \cap M_\gamma[G^*_\iota]$: first, $U^* \subseteq U^*_\iota$; second, all of the posets under consideration are c.c.c. by Assumption 3.0.1, and therefore $M_\gamma[G^*]$ has the same ordinals as $M_\gamma[G^*_\iota]$. Since $\vec{\mu}, \vec{\mu}' \in U^*_\iota \cap M_\gamma[G^*_\iota]$, $\vec{\mu}, \vec{\mu}'$ fail to match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p_\iota \mapsto 0\}$ at some $\eta \in I^*(\iota)$. That is to say, one of the following holds:

(a) there is some $k \leq l$ such that

$$\dot{\chi}_\kappa \left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta}) \right] (\mu_k, \mu'_k) = 1 - \dot{a}_k \left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta}) \right]$$

(and in case $k = l$, $\dot{a}_k \left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta}) \right]$ is defined);

(b) or $(\{p_\iota \mapsto 0\}) \left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta}) \right]$ is defined and

$$\dot{\chi}_\kappa \left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta}) \right] (\mu_l, \mu'_l) = 1 - (\{p_\iota \mapsto 0\}) \left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta}) \right].$$

However, we assumed that $\vec{\mu}$ and $\vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l$ on $\mathcal{S}^*$. Therefore (a) is false and (b) holds. This implies in particular that $\psi^*_{\iota,\eta}(p_\iota) \in G^*_\iota \restriction B^*_{\iota,\eta}$ and that

$$(\{p_\iota \mapsto 0\}) \left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta}) \right] = 0.$$

Thus

$$\dot{\chi}_\kappa\left[\left(\psi^*_{\iota,\eta}\right)^{-1}(G^*_\iota \restriction B^*_{\iota,\eta})\right](\mu_l,\mu'_l) = 1 - (\dot{a}_l \cup \{p_\iota \mapsto 0\})\left[\left(\psi^*_{\iota,\eta}\right)^{-1}(G^*_\iota \restriction B^*_{\iota,\eta})\right] = 1.$$

Since $p^*_\iota \in G^*_\iota$, p^*_ι and $\psi^*_{\iota,\eta}(p_\iota)$ are compatible, and therefore p^*_ι, being decisive, extends $\psi^*_{\iota,\eta}(p_\iota)$. Thus $\eta \in J^*(\iota)$. $\square$

This completes the proof of the above claim. As a result, we fix some function h on J_p such that for each $\iota \in J_p$, $h(\iota) \in J^*(\iota)$ provides a witness to the claim for ι. Let $k < n$ such that $h = h_k \restriction J_p$. We now check that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\pi_k[\bar{\mathcal{S}}]$.

Observe that since $\vec{\mu}$ and $\vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l$ on $\mathcal{S}^*$, we only need to check that for each $\iota \in J$, if $p \in (\psi^*_{\iota,h_k(\iota)})^{-1}\left(G^* \restriction B^*_{\iota,h_k(\iota)}\right)$, then

$$\dot{\chi}_\kappa\left[\left(\psi^*_{\iota,h_k(\iota)}\right)^{-1}\left(G^*_\iota \restriction B^*_{\iota,h_k(\iota)}\right)\right](\mu_l,\mu'_l) = 1.$$

But this is clear: for $\iota \in J_p$, the conclusion of the implication holds, by the last claim and the choice of h_k. For $\iota \notin J_p$ the hypothesis of the implication fails, since $(\psi^*_{\iota,h_k(\iota)})^{-1}(p^*(0))$ extends p_ι which, for $\iota \notin J_p$, is incompatible with p.

We have now completed our discussion of the illustrative case when the domain of $\mathbb{R}$ consists entirely of X_0. We next work in full generality to finish with this case; we will proceed by grafting multiple copies of the part of $\mathbb{R}$ outside X_0 onto $\mathbb{R}^*(0)$. In more detail, recall that the maps π_k each embed $(\mathbb{R} \restriction X_0, \mathcal{B} \restriction X_0)$ into $(\mathbb{R}^*(0), \mathcal{B}^*_0)$. Thus we may apply Lemma 2.2.18 in M_γ, once for each $k < n$, to construct a sequence of enriched partition products $\langle(\mathbb{R}^*(k+1), \mathcal{B}^*_{k+1}) : k < n\rangle$ such that for each $k < n$, letting X^*_k denote the domain of $\mathbb{R}^*(k)$, $X^*_k \subseteq X^*_{k+1}$, $\mathbb{R}^*(k+1) \restriction X^*_k = \mathbb{R}^*(k)$, $\mathcal{B}^*_k \subseteq \mathcal{B}^*_{k+1}$, and such that π_k extends to an embedding, which we call π^*_k, of $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^*(k+1), \mathcal{B}^*_{k+1})$. We remark that by the grafting construction, for each $k < n$,

$$\pi^*_k[X \backslash X_0] = X^*_{k+1} \backslash X^*_k.$$

Let us now use $\mathbb{R}^*$ to denote $\mathbb{R}^*(n)$, X^* to denote the domain of $\mathbb{R}^*$, and $\mathcal{B}^*$ to denote $\mathcal{B}^*_n$. Also, observe that π^*_k embeds $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^*, \mathcal{B}^*)$, since it embeds $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^*(k+1), \mathcal{B}_{k+1})$

and since $\mathcal{B}_{k+1} \subseteq \mathcal{B}^*$ and $\mathbb{R}^*(k+1) = \mathbb{R}^* \upharpoonright X^*_{k+1}$. We claim that $(\mathbb{R}^*, \mathcal{B}^*)$ witnesses the lemma in this case.

We first address item (a). Since $(\mathbb{R}^*(0), \mathcal{B}^*_0)$ and $(\mathbb{R}, \mathcal{B})$ are both finitely generated and since $(\mathbb{R}^*, \mathcal{B}^*)$ was constructed from them by finitely-many applications of the Grafting Lemma, $(\mathbb{R}^*, \mathcal{B}^*)$ is itself finitely generated by Lemma 3.2.8. Moreover, as all of the partition products under consideration are in M_γ, the suitable collection and auxiliary part for $(\mathbb{R}^*, \mathcal{B}^*)$ are also in M_γ.

For (b), we define a sequence of conditions in $\mathbb{R}^*$ by recursion, beginning with $p^*(0)$. Suppose that we have constructed the condition $p^*(k)$ in $\mathbb{R}^*(k)$ such that if $k > 0$, then $p^*(k) \upharpoonright \mathbb{R}^*(k-1) = p^*(k-1)$ and $p^*(k)$ extends $\pi^*_{k-1}(\bar{p})$. To construct $p^*(k+1)$, note that $p^*(k)$ extends $\pi_k(\bar{p} \upharpoonright X_0)$, since $p^*(0)$ does, as observed before the illustrative case, and since $p^*(k) \upharpoonright \mathbb{R}^*(0) = p^*(0)$. Moreover,

$$\pi^*_k[X \backslash X_0] \cap \mathrm{dom}(p^*(k)) = \varnothing,$$

as $\mathrm{dom}(p^*(k)) \subseteq X^*_k$, and as $\pi^*_k[X \backslash X_0] \cap X^*_k = \varnothing$. Thus we see that

$$p^*(k+1) := p^*(k) \cup \pi^*_k \left(\bar{p} \upharpoonright (X \backslash X_0)\right)$$

is a condition in $\mathbb{R}^*(k+1)$ which extends $\pi^*_k(\bar{p})$. This completes the construction of the sequence of conditions, and so we now let p^* be the condition $p^*(n)$ in $\mathbb{R}^*$.

We take the same $\mathbb{R}^*(0)$-name $\dot{U}^*$ for (c). To address (d), we let $\mathcal{F} = \{\pi^*_k : k < n\}$, each of which, as noted above, is an embedding of $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^*, \mathcal{B}^*)$ and a member of M_γ.

This now defines the objects from (a)-(d), and so we check that conditions (1)-(3) hold. By the construction of p^* above, p^* extends $\pi^*_k(\bar{p})$ for each $k < n$, so (1) is satisfied. Moreover, we already know that $p^* \Vdash_{\mathbb{R}^*} \vec{\nu} \in \dot{U}^*$, since $p^*(0) \Vdash_{\mathbb{R}^*(0)} \vec{\nu} \in \dot{U}^*$ and since $\mathbb{R}^* \upharpoonright X^*_0 = \mathbb{R}^*(0)$. And finally, the proof of condition (3) is the same as in the illustrative case, using the fact that each π^*_k extends π_k. This completes the proof of the lemma in the case that $\mathrm{ht}(\bar{\mathcal{S}}) = 0$.

Case 2: $\mathrm{ht}\left(\bar{\mathcal{S}}\right) > 0$ (in particular, $\bar{\mathcal{S}}$ has at least 2 elements). We abbreviate $\mathrm{ht}\left(\bar{\mathcal{S}}\right)$ by δ in what follows. Fix $\iota_0, \iota_1 \in J$ which satisfy $\delta = \mathrm{ht}(B_{\iota_0}, B_{\iota_1})$, and set $\hat{J} := J \backslash \{\iota_0\}$.

The proof in this case will proceed in rough outline as follows; we will first remove the tail of the branch B_{ι_0} above δ, resulting in a partition product $\hat{\mathbb{R}}$ with one few element in its matching core. We apply induction to embed $\hat{\mathbb{R}}$ into a partition product $\mathbb{R}^*$. We then restore many copies of the branch B_{ι_0} to $\mathbb{R}^*$, resulting in $\mathbb{R}^{**}$; we also restore many copies of the auxiliary part of $\mathbb{R}$, letting $\mathbb{R}^{***}$ be the resulting partition product. Finally, we will see that the restored copies of B_{ι_0} in $\mathbb{R}^{***}$ form a suitable collection with smaller height than $\bar{\mathcal{S}}$, and this allows us to apply induction one last time to create our final partition product $\mathbb{R}^{****}$. We then show that we may embed $\mathbb{R}$ into $\mathbb{R}^{****}$ in many ways, so that the conclusion of the current lemma is satisfied.

By Lemma 3.1.2, $\hat{X}_0 := \bigcup_{\iota \in \hat{J}} B_\iota$ coheres with $(\mathbb{R}, \mathcal{B})$. Let $\hat{\mathbb{R}}$ be the partition product $\mathbb{R} \restriction \hat{X}_0$, and set $\hat{\mathcal{B}} := \mathcal{B} \restriction \hat{X}_0$, which, by Lemma 2.2.13, is an enrichment of $\hat{\mathbb{R}}$. Furthermore, $\hat{\mathbb{R}}$ is finitely generated with an empty auxiliary part and with $\hat{\mathcal{S}} := \left\{ \langle B_\iota, \psi_\iota \rangle : \iota \in \hat{J} \right\}$ as κ-suitable with respect to $(\hat{\mathbb{R}}, \hat{\mathcal{B}})$. We also let $\hat{p}$ be the condition $\bar{p} \restriction \hat{X}_0 \in \hat{\mathbb{R}}$. Finally, we let $\bar{\mathbb{R}} := \mathbb{R} \restriction \bigcup_{\iota \in J} B_\iota$, and $\bar{\mathcal{B}} = \mathcal{B} \restriction \bigcup_{\iota \in J} B_\iota$, so that $(\bar{\mathbb{R}}, \bar{\mathcal{B}})$ is also an enriched partition product.

Since $|\hat{\mathcal{S}}| < |\bar{\mathcal{S}}|$ and $\mathrm{ht}(\hat{\mathcal{S}}) \leq \mathrm{ht}(\bar{\mathcal{S}})$, we may apply the induction hypothesis to $(\hat{\mathbb{R}}, \hat{\mathcal{B}})$, the condition $\hat{p}$, the $\hat{\mathbb{R}}$-name for all $l+1$-tuples in ω_1, and with $\hat{\mathcal{S}}$ as the matching core. This produces the following objects:

$(a)^*$ an enriched partition product $(\mathbb{R}^*, \mathcal{B}^*)$ with domain X^*, say, finitely generated by a κ-suitable collection $\mathcal{S}^*$ and an auxiliary part Z^*, all of which are in M_γ;

$(b)^*$ a condition $p^* \in \mathbb{R}^*$;

$(c)^*$ an $\mathbb{R}^*$-name $\dot{W}^*$ in M_γ for a collection of $l+1$-tuples in ω_1;

$(d)^*$ a nonempty, finite collection $\mathcal{F}$ in M_γ of embeddings of $(\hat{\mathbb{R}}, \hat{\mathcal{B}})$ into $(\mathbb{R}^*, \mathcal{B}^*)$;

satisfying that for each $\pi \in \mathcal{F}$:

$(1)^*$ p^* extends $\pi(\hat{p})$ in $\mathbb{R}^*$;

and also satisfying that p^* forces the following statements in $\mathbb{R}^*$:

(2)* $\vec{\nu} \in \dot{W}^*$;

(3)* for any pairwise distinct $l+1$-tuples $\vec{\mu}$ and $\vec{\mu}'$ in $\dot{W}^* \cap M_\gamma[\dot{G}^*]$, if $\vec{\mu}$ and $\vec{\mu}'$ match $\dot{a}_0, \dots, \dot{a}_l$ on $\mathcal{S}^*$, then there is some $\pi \in \mathcal{F}$ such that $\vec{\mu}$ and $\vec{\mu}'$ match $\dot{a}_0, \dots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\pi[\hat{\mathcal{S}}]$.

Our next step is to restore many copies of the segment $\psi_{\iota_0}[\rho_\kappa \setminus \delta]$ of the lost branch B_{ι_0} in such a way that the restored copies form a κ-suitable collection with smaller height than δ; this will allow another application of the induction hypothesis. Towards this end, define

$$\mathcal{R} := \{\pi \circ \psi_{\iota_1}[\delta] : \pi \in \mathcal{F}\},$$

and, recalling that $\mathcal{F}$ is finite, let $x_0, \dots, x_{d-1}$ enumerate $\mathcal{R}$. We choose, for each $k < d$, a map $\pi_k \in \mathcal{F}$ so that $\pi_k \circ \psi_{\iota_1}[\delta] = x_k$.

We now work in M_γ to graft one copy of $\psi_{\iota_0}[\rho_\kappa \setminus \delta]$ onto $(\mathbb{R}^*, \mathcal{B}^*)$ over π_k, for each $k < n$. Indeed, since π_k embeds $(\hat{\mathbb{R}}, \hat{\mathcal{B}})$ into $(\mathbb{R}^*, \mathcal{B}^*)$, we may successively apply the Grafting Lemma to find an enriched partition product $(\mathbb{R}^{**}, \mathcal{B}^{**})$ on a domain X^{**} so that $\mathbb{R}^{**} \restriction X^* = \mathbb{R}^*$, $\mathcal{B}^* \subseteq \mathcal{B}^{**}$, and so that for each $k < d$, π_k extends to an embedding π_k^* of $(\bar{\mathbb{R}}, \bar{\mathcal{B}})$ into $(\mathbb{R}^{**}, \mathcal{B}^{**})$. Since $(\mathbb{R}^{**}, \mathcal{B}^{**})$ is finitely generated, by Lemma 3.2.8, we may let $\mathcal{S}^{**}$ denote the finite, κ-suitable collection for $(\mathbb{R}^{**}, \mathcal{B}^{**})$.

Let us make a number of observations about the above situation. First, we want to see that for each $\pi \in \mathcal{F}$, we may extend π to embed $(\bar{\mathbb{R}}, \bar{\mathcal{B}})$ into $(\mathbb{R}^{**}, \mathcal{B}^{**})$. Thus fix $\pi \in \mathcal{F}$, and let $k < d$ such that $\pi \circ \psi_{\iota_1}[\delta] = x_k$. We want to apply Lemma 2.2.21, and for this we need to see that π and π_k agree on $X_0 \cap B_{\iota_0}$. To verify this, we first claim that $X_0 \cap B_{\iota_0} = B_{\iota_1} \cap B_{\iota_0}$. Suppose that this is false, for a contradiction. Then there is some $\alpha \in X_0 \cap B_{\iota_0} \setminus B_{\iota_1}$. Fix $\iota \in J$ s.t. $\alpha \in B_\iota \cap B_{\iota_0}$. Then $\psi_{\iota_0}^{-1}[B_\iota \cap B_{\iota_0}] \leq \mathrm{ht}(\bar{\mathcal{S}}) = \delta$, and so $\alpha \in \psi_{\iota_0}[\delta]$. But $\psi_{\iota_0} \restriction \delta = \psi_{\iota_1} \restriction \delta$, and therefore $\alpha \in B_{\iota_1}$, a contradiction.

Thus $X_0 \cap B_{\iota_0} = B_{\iota_1} \cap B_{\iota_0}$. But $B_{\iota_1} \cap B_{\iota_0} = \psi_{\iota_1}[\delta]$, and therefore

$$\pi[B_{\iota_1} \cap B_{\iota_0}] = \pi \circ \psi_{\iota_1}[\delta] = x_k = \pi_k \circ \psi_{\iota_1}[\delta] = \pi_k[B_{\iota_1} \cap B_{\iota_0}].$$

Hence π and π_k agree on $X_0 \cap B_{\iota_0}$. By Lemma 2.2.21, the map

$$\pi^* := \pi \cup \pi_k^* \restriction (\psi_{\iota_0}[\rho_\kappa \backslash \delta])$$

is an extension of π which embeds $(\bar{\mathbb{R}}, \bar{\mathcal{B}})$ into $(\mathbb{R}^{**}, \mathcal{B}^{**})$. We make the observation that $\pi^*[B_{\iota_0}] = \pi_k^*[B_{\iota_0}]$, which will be useful later.

For each $k < d$, we use x_k^* to denote the image of B_{ι_0} under the map π_k^*. Let $\bar{\mathcal{S}}^{**} := \{\langle x_k^*, \pi_k^* \circ \psi_{\iota_0}, \kappa\rangle : k < d\}$. Then $\bar{\mathcal{S}}^{**} \subseteq \mathcal{S}^{**}$, and in particular, $\bar{\mathcal{S}}^{**}$ is κ-suitable. For $k \neq l$ we have

$$(\pi_k^* \circ \psi_{\iota_0})[\delta] = x_k \neq x_l = (\pi_l^* \circ \psi_{\iota_0})[\delta],$$

and hence $\mathrm{ht}(x_k^*, x_l^*) < \delta$. Therefore $\mathrm{ht}(\bar{\mathcal{S}}^{**}) < \delta$, since $\bar{\mathcal{S}}^{**}$ is finite.

We now have a collection $\mathcal{F}^* := \{\pi^* : \pi \in \mathcal{F}\}$ of embeddings of $(\bar{\mathbb{R}}, \bar{\mathcal{B}})$ into $(\mathbb{R}^{**}, \mathcal{B}^{**})$ and a finite, κ-suitable subcollection $\bar{\mathcal{S}}^{**}$ of $\mathcal{S}^{**}$ such that the height of $\bar{\mathcal{S}}^{**}$ is less than δ. But before we apply the induction hypothesis, we need to extend $(\mathbb{R}^{**}, \mathcal{B}^{**})$ to add generics for the full $\mathbb{R}$ and to also define a few more objects. Towards this end, we work in M_γ to successively apply the Grafting Lemma to each map π^* in $\mathcal{F}^*$ to graft $(\mathbb{R}, \mathcal{B})$ onto $(\mathbb{R}^{**}, \mathcal{B}^{**})$ over π^*. This results in a partition product $(\mathbb{R}^{***}, \mathcal{B}^{***})$ in M_γ with domain X^{***} so that $\mathbb{R}^{***} \restriction X^{**} = \mathbb{R}^{**}$, $\mathcal{B}^{**} \subseteq \mathcal{B}^{***}$, and so that each map $\pi^* \in \mathcal{F}^*$ extends to an embedding π^{***} of $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^{***}, \mathcal{B}^{***})$. By Lemma 3.2.8, $(\mathbb{R}^{***}, \mathcal{B}^{***})$ is still finitely generated, say with κ-suitable collection $\mathcal{S}^{***}$.

We now want to define a condition p^{***} in $\mathbb{R}^{***}$ by adding further coordinates to the condition $p^* \in \mathbb{R}^* \subseteq \mathbb{R}^{***}$ from $(a)^*$. By the grafting construction of $\mathbb{R}^{**}$, if $k < l < d$, then the images of $\psi_{\iota_0}[\rho_k \backslash \delta]$ under π_k^* and π_l^* are disjoint. Thus

$$p^{**} := p^* \cup \bigcup_{k<d} \pi_k^* \left(\bar{p} \restriction \psi_{\iota_0}[\rho_k \backslash \delta]\right)$$

is a condition in $\mathbb{R}^{**}$. Since by $(1)^*$, p^* extends $\pi(\hat{p})$ in $\mathbb{R}^*$ for each $\pi \in \mathcal{F}$, we conclude that p^{**} extends $\pi_k^*(\bar{p} \restriction \bigcup_{\iota \in J} B_\iota)$ for each $k < d$. Furthermore, if $\pi^* \in \mathcal{F}^*$, then for some $k < d$, π^* agrees with π_k^* on B_{ι_0}, as observed above. It is straightforward to see that this implies that p^{**} in fact extends $\pi^*(\bar{p} \restriction \bigcup_{\iota \in J} B_\iota)$ for each $\pi^* \in \mathcal{F}^*$. And finally, by the grafting construction of $\mathbb{R}^{***}$, we know that if π and σ are distinct embeddings in $\mathcal{F}$, then the images of $X \setminus \bigcup_{\iota \in J} B_\iota$ under π^{***} and σ^{***} are disjoint. Consequently,

$$p^{***} := p^{**} \cup \bigcup_{\pi \in \mathcal{F}} \pi^{***} \left(\bar{p} \upharpoonright \left(X \setminus \bigcup_{\iota \in J} B_\iota \right) \right)$$

is a condition in $\mathbb{R}^{***}$ which extends $\pi^{***}(\bar{p})$ for each $\pi \in \mathcal{F}$.

We are now ready to apply the induction hypothesis to the partition product $(\mathbb{R}^{***}, \mathcal{B}^{***})$, the condition $p^{***} \in \mathbb{R}^{***}$, and the matching core $\bar{\mathcal{S}}^{**}$, which has height below δ. This results in the following objects:

$(a)^{**}$ an enriched partition product $(\mathbb{R}^{****}, \mathcal{B}^{****})$ on a set X^{****} which is finitely generated, say with κ-suitable collection $\mathcal{S}^{****}$ and auxiliary part Z^{****}, all of which are in M_γ;

$(b)^{**}$ a condition p^{****} in $\mathbb{R}^{****}$;

$(c)^{**}$ an $\mathbb{R}^{****}$-name $\dot{U}^{****}$ in M_γ for a collection of $l+1$ tuples in ω_1;

$(d)^{**}$ a nonempty, finite collection $\mathcal{G}$ in M_γ of embeddings of $(\mathbb{R}^{***}, \mathcal{B}^{***})$ into $(\mathbb{R}^{****}, \mathcal{B}^{****})$;

satisfying that for each $\sigma \in \mathcal{G}$

$(1)^{**}$ p^{****} extends $\sigma(p^{***})$ in $\mathbb{R}^{****}$;

and such that p^{****} forces in $\mathbb{R}^{****}$ that

$(2)^{**}$ $\vec{\nu} \in \dot{U}^{****}$;

$(3)^{**}$ for any pairwise distinct tuples $\vec{\mu}, \vec{\mu}'$ in $M_\gamma[\dot{G}^{****}] \cap \dot{U}^{****}$ such that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l$ on $\mathcal{S}^{****}$, there is some $\sigma \in \mathcal{G}$ such that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\sigma[\bar{\mathcal{S}}^{**}]$.

This completes the construction of our final partition product. To finish the proof, we will need to define a number of embeddings from our original partition product $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^{****}, \mathcal{B}^{****})$ and check that the appropriate matching obtains. For $\sigma \in \mathcal{G}$ and $\pi \in \mathcal{F}$, we define the map $\tau(\pi, \sigma)$ to be the composition $\sigma \circ \pi^{***}$, which embeds $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^{****}, \mathcal{B}^{****})$.

We also observe that $p^{****} \leq \tau(\pi, \sigma)(\bar{p})$ for each such π and σ since p^{****} extends $\sigma(p^{***})$ in $\mathbb{R}^{****}$, and since p^{***} extends $\pi^{***}(\bar{p})$ in $\mathbb{R}^{***}$. Now define the $\mathbb{R}^{****}$-name $\dot{V}^*$ to be

$$\dot{U}^{****} \cap \bigcap_{\sigma \in \mathcal{G}} \dot{W}^* \left[\sigma^{-1}(\dot{G}^{****}) \upharpoonright X^* \right].$$

We observe that this is well-defined, since for each $\sigma \in \mathcal{G}$ and generic G^{****} for $\mathbb{R}^{****}$, $\sigma^{-1}(G^{****})$ is generic for $\mathbb{R}^{***}$, and hence its restriction to X^* is generic for $\mathbb{R}^*$. We also see that p^{****} forces that $\vec{\nu} \in \dot{V}^*$ because p^{****} forces $\vec{\nu} \in \dot{U}^{****}$, p^* is in $\sigma^{-1}(G^{****})$ for any generic G^{****} containing p^{****}, and p^* forces in $\mathbb{R}^*$ that $\vec{\nu} \in \dot{W}^*$.

We finish the proof of the lemma in this case by showing that the partition product $(\mathbb{R}^{****}, \mathcal{B}^{****})$, the condition $p^{****} \in \mathbb{R}^{****}$, the name $\dot{V}^*$, and the collection

$$\{\tau(\pi, \sigma) : \pi \in \mathcal{F} \ \wedge \ \sigma \in \mathcal{G}\}$$

of embeddings satisfy (1)-(3). We already know that p^{****} extends $\tau(\pi, \sigma)(\bar{p})$ for each π and σ and that $p^{****} \Vdash \vec{\nu} \in \dot{V}^*$. So now we check the matching condition. Towards this end, fix a generic H for $\mathbb{R}^{****}$ and two pairwise distinct tuples $\vec{\mu}, \vec{\mu}'$ in $\dot{V}^*[H] \cap M_\gamma[H]$ which match $\dot{a}_0, \ldots, \dot{a}_l$ on $\mathcal{S}^{****}$. We need to find some π and σ such that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\tau(\pi, \sigma)[\bar{\mathcal{S}}]$.

By $(3)^{**}$, we know that we can find some σ such that

(i) $\vec{\mu}$ and $\vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\sigma[\bar{\mathcal{S}}^{**}]$.

Let t denote the triple $\langle B_{\iota_0}, \psi_{\iota_0}, \kappa \rangle$. By construction of the maps π^*, for each $\pi \in \mathcal{F}$, there is some k so that $\pi^{***}(t) = \pi^*(t) = \pi^*_k(t) \in \bar{\mathcal{S}}^{**}$. Using (i) it follows that:

(ii) for every $\pi \in \mathcal{F}$, $\vec{\mu}$ and $\vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ at $\sigma \circ \pi^{***}(t) = \tau(\pi, \sigma)(t)$.

Now consider the filter $G^*_\sigma := \sigma^{-1}(H) \upharpoonright X^*$, which is generic for $\mathbb{R}^*$ and contains p^*. By Assumption 3.0.1, we know that all the posets under consideration are c.c.c., and therefore the models $M_\gamma[H]$ and $M_\gamma[G^*_\sigma]$ have the same ordinals, namely those of M_γ. Thus $\vec{\mu}, \vec{\mu}' \in M_\gamma[G^*_\sigma]$. Furthermore, by definition of $\dot{V}^*[H]$, we have that $\vec{\mu}, \vec{\mu}' \in \dot{W}^*[G^*_\sigma]$, and as a result

$\vec{\mu}, \vec{\mu}' \in M_\gamma[G^*_\sigma] \cap \dot{W}^*[G^*_\sigma]$. Thus by $(3)^*$, we can find some $\pi \in \mathcal{F}$ so that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\pi[\hat{\mathcal{S}}]$. Because π^{***} extends π, we may rephrase this to say that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\pi^{***}[\hat{\mathcal{S}}]$. Since σ embeds $(\mathbb{R}^{***}, \mathcal{B}^{***})$ into $(\mathbb{R}^{****}, \mathcal{B}^{****})$,

(iii) $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\tau(\pi, \sigma)[\hat{\mathcal{S}}]$.

Finally, (ii) and (iii) imply that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\tau(\pi, \sigma)[\bar{\mathcal{S}}]$, as $\bar{\mathcal{S}} = \hat{\mathcal{S}} \cup \{t\}$. This completes the proof of the lemma.

□

Corollary 3.3.6. *Under the assumptions of Lemma 3.3.4, suppose that $\dot{U}$ is an $\mathbb{R}$-name in M_γ for a set of $l+1$-tuples in ω_1 such that $\bar{p} \Vdash_{\mathbb{R}} \vec{\nu} \in \dot{U}$. Then the conclusion of Lemma 3.3.4 may be strengthened to say that $p^* \Vdash_{\mathbb{R}^*} \dot{U}^* \subseteq \bigcap_{\pi \in \mathcal{F}} \dot{U}[\pi^{-1}(\dot{G}^*)]$.*

Proof. Let $\dot{U}$ be fixed, and let $\dot{U}^*$ be as in the conclusion of Lemma 3.3.4. Define $\dot{U}^{**}$ to be the name $\dot{U}^* \cap \bigcap_{\pi \in \mathcal{F}} \dot{U}[\pi^{-1}(\dot{G}^*)]$, and observe that this name is still in M_γ. By condition (1) of Lemma 3.3.4, we know that p^* forces that $\bar{p}$ is in $\pi^{-1}(\dot{G}^*)$, for each $\pi \in \mathcal{F}$. Since each such $\pi^{-1}(\dot{G}^*)$ is forced to be V-generic for $\mathbb{R}$ and since $\bar{p} \Vdash_{\mathbb{R}} \vec{\nu} \in \dot{U}$, this implies that p^* forces that $\vec{\nu}$ is a member of $\dot{U}^{**}$. Finally, condition (3) of Lemma 3.3.4 still holds, since $\dot{U}^{**}$ is forced to be a subset of $\dot{U}^*$. □

Corollary 3.3.7. (Under Assumption 3.3.3) $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ *have the partition product preassignment property at γ.*

Proof. Suppose otherwise, for a contradiction. Then there exists a partition product $\mathbb{R}$, say with domain X, finitely generated by $\mathcal{S} = \{\langle B_\iota, \psi_\iota \rangle : \iota \in I\}$ and an auxiliary part Z, all of which are in M_γ; an $\mathbb{R}$-name $\dot{U}$ in M_γ; and a condition $\bar{p} \in \mathbb{R}$ (not necessarily in M_γ), such that $\bar{p}$ forces that $\vec{\nu} \in \dot{U}$, but also that for any pairwise distinct tuples $\vec{\mu}, \vec{\mu}'$ in $\dot{U} \cap M_\gamma[\dot{G}]$, there exists some $\iota_0 \in I$ such that $\vec{\mu}, \vec{\mu}'$ fail to match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ at ι_0. Apply Lemma 3.3.4 and Corollary 3.3.6 to these objects, with $\bar{\mathcal{S}} := \mathcal{S}$ and with the enrichment

$$\mathcal{B} := \{\langle b(\xi), \pi_\xi, \mathrm{index}(\xi) \rangle : \xi \in X\} \cup \mathcal{S},$$

to construct the objects as in the conclusions of Lemma 3.3.4 and Corollary 3.3.6. Also, fix a generic G^* for $\mathbb{R}^*$ which contains the condition p^*.

We now apply the fact that $\dot{a}_0, \ldots, \dot{a}_l$ have the partition product preassignment property at γ to the objects in the conclusion of Lemma 3.3.4: since $\vec{\nu} \in U^* := \dot{U}^*[G^*]$, we can find two pairwise distinct tuples $\vec{\mu}, \vec{\mu}'$ in $U^* \cap M_\gamma[G^*]$ which match $\dot{a}_0, \ldots, \dot{a}_l$ on I^*. Thus by (3) of Lemma 3.3.4, there is some embedding π of $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^*, \mathcal{B}^*)$ so that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\pi[\mathcal{S}]$. Now consider $G := \pi^{-1}(G^*)$, which is generic for $\mathbb{R}$ and contains the condition $\bar{p}$, since $p^* \leq_{\mathbb{R}^*} \pi(\bar{p})$. Since $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\pi[\mathcal{S}]$ and π is an embedding, $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\mathcal{S}$ with respect to the filter G. Finally, observe that $\vec{\mu}$ and $\vec{\mu}'$ are both in $\dot{U}[G] \cap M_\gamma[G]$: they are in $\dot{U}[G]$ by Corollary 3.3.6, since U^* is a subset of $\dot{U}[G]$. They are both in M_γ, hence in $M_\gamma[G]$, since by Assumption 3.0.1, $\mathbb{R}^*$ is c.c.c. However, this contradicts what we assumed about $\bar{p}$. □

3.4 Putting it together

Let us now put together the results from the previous three sections.

Lemma 3.4.1. *Suppose that $\dot{a}_0, \ldots, \dot{a}_{l-1}$ are total canonical color names which have the partition product preassignment property at γ. Then there is a total canonical color name $\dot{a}_l$ so that $\dot{a}_0, \ldots, \dot{a}_l$ have the partition product preassignment property at γ.*

Proof. We recursively construct a sequence $\dot{a}_l^\xi$ of names, taking unions at limit stages. If $\dot{a}_l^\xi$ has been constructed and $\mathrm{dom}(\dot{a}_l^\xi)$ is a maximal antichain in $\mathbb{P}_\kappa$, we set $\dot{a}_l = \dot{a}_l^\xi$. Otherwise, we pick some condition $p \in \mathbb{P}_\kappa$ incompatible with all conditions therein. If there is some extension $p^* \leq_{\mathbb{P}_\kappa} p$ so that $\dot{a}_0, \ldots, \dot{a}_{l-1}, \dot{a}_l^\xi \cup \{p^* \mapsto 0\}$ have the partition product preassignment property at γ, we pick some such p^* and set $\dot{a}_l^{\xi+1} := \dot{a}_l^\xi \cup \{p^* \mapsto 0\}$. Otherwise, Assumption 3.3.3 is satisfied, and hence by Corollary 3.3.7, $\dot{a}_0, \ldots, \dot{a}_{l-1}, \dot{a}_l^\xi \cup \{p \mapsto 1\}$ have the partition product preassignment property at γ. In this case we set $\dot{a}_l^{\xi+1} := \dot{a}_l^\xi \cup \{p \mapsto 1\}$. Note that the construction of the sequence $\dot{a}_l^\zeta$ halts at some countable stage, since $\mathbb{P}_\kappa$ is c.c.c., by

Assumption 3.0.1. $\square$

We now prove Proposition 3.2.2:

Proof of Proposition 3.2.2. Recall that for each $\gamma < \omega_1$, $\langle \nu_{\gamma,l} : l < \omega \rangle$ enumerates the slice $[M_\gamma \cap \omega_1, M_{\gamma+1} \cap \omega_1)$. By Lemma 3.4.1, we may construct, for each $\gamma < \omega_1$, a sequence of $\mathbb{P}_\kappa$-names $\langle \dot{a}_{\gamma,l} : l < \omega \rangle$ such that for each $l < \omega$, $\dot{a}_{\gamma,0}, \ldots, \dot{a}_{\gamma,l}$ have the partition product preassignment property at γ. We now define a function $\dot{f}$ by taking $\dot{f}(\nu_{\gamma,l}) = \dot{a}_{\gamma,l}$, for each $\gamma < \omega_1$ and $l < \omega$. The values of $\dot{f}$ on ordinals $\nu < M_0 \cap \omega_1$ are irrelevant, so we simply set $\dot{f}(\nu)$ to name 0 for each such ν. Then $\dot{f}$ satisfies the assumptions of Lemma 3.2.13 and hence satisfies Proposition 3.2.2. $\square$

CHAPTER 4

Constructing Partition Products in L

In this chapter, we show how to construct the desired partition products in L. In particular, we will construct the ω_2-canonical partition product $\mathbb{P}_{\omega_2}$, which will have domain ω_3. Forcing with $\mathbb{P}_{\omega_2}$ will provide the model which witnesses our theorem. We assume for this chapter that $V = L$. For each ordinal $\kappa \leq \omega_2$ and $\alpha \geq \kappa$, we let $\varphi_{\kappa,\alpha}$ denote the $<_L$-least surjection from κ onto α, if such exists.

Before we introduce some more definitions, let us fix a sufficiently large, finite fragment F of $\mathsf{ZFC} - \mathsf{Powerset}$ which is satisfied in $H(\omega_3)$. As a matter of notation, by the Gödel pairing function, we view each ordinal γ as coding a pair of ordinals, where $(\gamma)_k$, for $k \leq 1$, denotes the kth ordinal coded by γ; this will be useful for bookkeeping later.

4.1 Local ω_2's and Witnesses

Definition 4.1.1. *Let $\omega_1 < \kappa \leq \omega_2$, and let $\mathbb{A}$ be a sequence of elements of L_κ so that* $\mathrm{dom}(\mathbb{A}) \subseteq \kappa$*. We say that κ is a* local ω_2 with respect to $\mathbb{A}$ *if there is some $\delta > \kappa$ such that L_δ is closed under ω-sequences, contains $\mathbb{A}$ as an element, and such that*

$$L_\delta \models \kappa = \aleph_2 \wedge F \wedge \kappa \textit{ is the largest cardinal.}$$

If κ is a local ω_2 with respect to $\mathbb{A}$, we will refer to any such δ as above as a witness for κ with respect to $\mathbb{A}$ *or simply as a* witness for κ *if $\mathbb{A}$ is clear from context.*

We begin our discussion with the following straightforward lemma.

Lemma 4.1.2. *Suppose that L_δ is closed under ω-sequences, and let $p \in L_\delta$. Then* $\mathrm{Hull}^{L_\delta}(\omega_1 \cup \{p\})$ *is also closed under ω-sequences.*

The next lemma shows how a local ω_2 with respect to one parameter can project to another.

Lemma 4.1.3. *Suppose that δ is a witness for κ with respect to $\mathbb{A}$, and define $H := \mathrm{Hull}^{L_\delta}(\omega_1 \cup \{\mathbb{A}\})$. Suppose further that $H \cap \kappa = \bar{\kappa} < \kappa$. Then $\bar{\kappa}$ is a local ω_2 with respect to $\mathbb{A} \restriction \bar{\kappa}$, and* $\mathrm{ot}(H \cap \delta)$ *is a witness for $\bar{\kappa}$ with respect to $\mathbb{A} \restriction \bar{\kappa}$.*

Proof. Let $\pi : H \longrightarrow L_{\bar{\delta}}$ be the transitive collapse, so that $\pi(\kappa) = \bar{\kappa}$ and $\bar{\delta} = \mathrm{ot}(H \cap \delta)$. Since H is closed under ω-sequences, by Lemma 4.1.2, $L_{\bar{\delta}}$ is too. Since π is elementary, we will be done once we verify that $\pi(\mathbb{A}) = \mathbb{A} \restriction \bar{\kappa}$. Indeed, by the elementarity of π, $\pi(\mathbb{A})$ is a sequence with domain $\mathrm{dom}(\mathbb{A}) \cap \bar{\kappa}$. Furthermore, for each $i \in \mathrm{dom}(\mathbb{A})$, since $\mathbb{A}(i) \in L_\kappa$ and since L_δ satisfies that $\kappa = \aleph_2$, we have that $\mathbb{A}(i)$ has size $\leq \aleph_1$ in L_δ. Thus for each $i \in \mathrm{dom}(\mathbb{A}) \cap \bar{\kappa}$, $\mathbb{A}(i)$ is not moved by π, and consequently $\pi(\mathbb{A}) = \mathbb{A} \restriction \bar{\kappa}$. $\square$

If κ is a local ω_2 with respect to $\mathbb{A}$, we define the *canonical sequence of witnesses for κ with respect to* $\mathbb{A}$, denoted $\langle \delta_i(\kappa, \mathbb{A}) : i < \gamma(\kappa, \mathbb{A}) \rangle$. We set $\delta_0(\kappa, \mathbb{A})$ to be the least witness for κ. Suppose that $\langle \delta_i(\kappa, \mathbb{A}) : i < \gamma \rangle$ is defined, for some γ. If there exists a witness $\tilde{\delta}$ for κ such that $\tilde{\delta} > \sup_{i<\gamma} \delta_i(\kappa, \mathbb{A})$, then we set $\delta_\gamma(\kappa, \mathbb{A})$ to be the least such. Otherwise, we halt the construction and set $\gamma(\kappa, \mathbb{A}) := \gamma$. If we have $\gamma < \gamma(\kappa, \mathbb{A})$, then we also define $H(\kappa, \gamma, \mathbb{A})$ to be

$$H(\kappa, \gamma, \mathbb{A}) := \mathrm{Hull}^{L_{\delta_{\gamma(\kappa,\mathbb{A})}}}(\omega_1 \cup \{\mathbb{A}\}).$$

Remark 4.1.4. It is straightforward to check that if κ is a local ω_2 with respect to $\mathbb{A}$ and $\gamma < \gamma(\kappa, \mathbb{A})$, then because $L_{\delta_{\gamma(\kappa,\mathbb{A})}}$ is countably closed, being a witness for κ with respect to $\mathbb{A}$ is absolute between $L_{\delta_{\gamma(\kappa,\mathbb{A})}}$ and V. Thus the sequence $\langle \delta_i(\kappa, \mathbb{A}) : i < \gamma \rangle$, and consequently the ordinal γ, is definable in $L_{\delta_{\gamma(\kappa,\mathbb{A})}}$ as the longest sequence of witnesses for κ with respect to $\mathbb{A}$. Furthermore, in the case that $\kappa = \omega_2$, we see that $\gamma(\omega_2, \mathbb{A}) = \omega_3$.

For the rest of the section, we fix κ and $\mathbb{A}$; for the sake of readability, we will often drop explicit mention of the parameter $\mathbb{A}$ in notation of the from $\delta_\gamma(\kappa, \mathbb{A})$ and $H(\kappa, \gamma, \mathbb{A})$, preferring instead to write, respectively, $\delta_\gamma(\kappa)$ and $H(\kappa, \gamma)$.

Suppose that κ is such that $\gamma(\kappa)$ is a successor, say $\gamma + 1$, and further suppose that $H(\kappa, \gamma)$ contains κ as a subset. Then we refer to $\delta_\gamma(\kappa)$, the final element on the canonical sequence of witnesses for κ with respect to $\mathbb{A}$, as the *stable witness for κ with respect to $\mathbb{A}$*. It is stable in the sense that we cannot condense the hull further.

Lemma 4.1.5. *Suppose that $\gamma + 1 < \gamma(\kappa)$. Then $H(\kappa, \gamma) \cap \kappa \in \kappa$.*

Proof. Suppose otherwise. Then $\kappa \subseteq H(\kappa, \gamma)$. Since $\gamma + 1 < \gamma(\kappa)$, we know that $\hat{\delta} := \delta_{\gamma+1}(\kappa)$ exists, and in particular, $\delta_\gamma(\kappa) < \hat{\delta}$. Observe that $H(\kappa, \gamma)$ is a member of $L_{\hat{\delta}}$, and therefore we may find a surjection from ω_1 onto $H(\kappa, \gamma)$ in $L_{\hat{\delta}}$. Since $\kappa \subseteq H(\kappa, \gamma)$, this contradicts our assumption that $L_{\hat{\delta}}$ satisfies that κ is $\aleph_2$. □

If $\gamma + 1 < \gamma(\kappa)$, then the collapse of $H(\kappa, \gamma)$ moves κ. The level to which $H(\kappa, \gamma)$ collapses is then the stable witness for the images of κ and $\mathbb{A}$, as shown in the following lemma.

Lemma 4.1.6. *Suppose that $\gamma + 1 < \gamma(\kappa)$, and set $\bar{\kappa} := H(\kappa, \gamma) \cap \kappa$. Let j denote the collapse map of $H(\kappa, \gamma)$ and τ the level to which $H(\kappa, \gamma)$ collapses. Finally, set $\bar{\gamma} := j(\gamma)$. Then $\bar{\gamma} + 1 = \gamma(\bar{\kappa})$ and $\tau = \delta_{\bar{\gamma}}(\bar{\kappa})$ is the stable witness for $\bar{\kappa}$ and $\mathbb{A} \restriction \bar{\kappa}$.*

Proof. Let us abbreviate $H(\kappa, \gamma)$ by H. By Remark 4.1.4, we have that $\langle \delta_i(\kappa) : i < \gamma \rangle \in H(\kappa, \gamma)$; let $\langle \delta_i : i < \bar{\gamma} \rangle$ denote the image of this sequence under j. By the elementarity of j and the absoluteness of Remark 4.1.4, $\langle \delta_i : i < \bar{\gamma} \rangle$ is exactly equal to $\langle \delta_i(\bar{\kappa}) : i < \bar{\gamma} \rangle$, the canonical sequence of witnesses for $\bar{\kappa}$ with respect to $\mathbb{A} \restriction \bar{\kappa}$.

We next verify that $\tau = \delta_{\bar{\gamma}}(\bar{\kappa})$. By Lemma 4.1.3, we know that τ is a witness for $\bar{\kappa}$ with respect to $\mathbb{A} \restriction \bar{\kappa}$. Furthermore, τ is the least witness for $\bar{\kappa}$ above $\sup_{i<\bar{\gamma}} \delta_i(\bar{\kappa})$: suppose that there were a witness $\bar{\delta}$ for $\bar{\kappa}$ between $\sup_{i<\bar{\gamma}} \delta_i(\bar{\kappa})$ and τ. Then L_τ satisfies that $\bar{\delta}$ is a witness for $\bar{\kappa}$. By the elementarity of j^{-1}, setting $\delta := j^{-1}(\bar{\delta})$, we see that $L_{\delta_\gamma(\kappa)}$ satisfies that δ is a witness for κ. Since $L_{\delta_\gamma(\kappa)}$ is closed under ω-sequences, δ is in fact a witness for κ (with respect to $\mathbb{A}$). As δ is between $\sup_{i<\gamma} \delta_i(\kappa)$ and $\delta_\gamma(\kappa)$, this is a contradiction. Therefore τ is the least witness for $\bar{\kappa}$ with respect to $\mathbb{A} \upharpoonright \bar{\kappa}$ which is above $\sup_{i<\bar{\gamma}} \delta_i(\bar{\kappa})$. However, because L_τ is the collapse of H, we see that $\mathrm{Hull}^{L_\tau}(\omega_1 \cup \{\mathbb{A} \upharpoonright \bar{\kappa}\})$ is all of L_τ. Therefore τ is the stable witness for $\bar{\kappa}$ with respect to $\mathbb{A} \upharpoonright \bar{\kappa}$. $\square$

4.2 Building Partition Products and the Final Argument

In this section, we show how to construct the set C from the definition of a partition product, as well as the desired sequence of partition products $\underline{\mathbb{P}} = \langle \mathbb{P}_\delta : \delta \in C \cup \{\omega_2\} \rangle$ and names $\underline{\dot{\mathbb{Q}}} = \langle \dot{\mathbb{Q}}_\delta : \delta \in C \rangle$. The ω_2-canonical partition product $\mathbb{P}_{\omega_2}$ will force OCA_{ARS} and $2^{\aleph_0} = \aleph_3$, which proves Theorem 1.0.3. We will also show how to adapt our construction so that our model additionally satisfies $\mathsf{FA}(\aleph_2, \mathrm{Knaster}(\aleph_1))$; recall that this axiom asserts that we can meet any $\aleph_2$-many dense subsets of an $\aleph_1$-sized, Knaster poset.

Suppose that we've defined the set C up to an ordinal $\kappa \leq \omega_2$ as well as $\underline{\mathbb{P}} \upharpoonright \kappa$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa$ in such a way that the following recursive assumptions are satisfied, where $\mathbb{A} \upharpoonright \kappa$ denotes the "alphabet" sequence $\langle \langle \mathbb{P}_{\bar{\kappa}}, \dot{\mathbb{Q}}_{\bar{\kappa}} \rangle : \bar{\kappa} \in C \cap \kappa \rangle$:

(a) for each $\bar{\kappa} \in C \cap \kappa$, $\bar{\kappa}$ is a local ω_2 with respect to $\mathbb{A} \upharpoonright \bar{\kappa}$, $\mathbb{P}_{\bar{\kappa}}$ is a partition product based upon $\underline{\mathbb{P}} \upharpoonright \bar{\kappa}$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright \bar{\kappa}$, and $\dot{\mathbb{Q}}_{\bar{\kappa}}$ is a $\mathbb{P}_{\bar{\kappa}}$-name. In particular, conditions (i)-(v) from Chapter 2 are satisfied;

(b) every partition product based upon $\underline{\mathbb{P}} \upharpoonright \kappa$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa$ is c.c.c.

Let us suppose, by relabelling if necessary, that κ is the least local ω_2 with respect to $\mathbb{A} \upharpoonright \kappa$. We aim to define the partition product $\mathbb{P}_\kappa$ and, in the case that $\kappa < \omega_2$, to place κ in C and define the $\mathbb{P}_\kappa$-name $\dot{\mathbb{Q}}_\kappa$. We assume that $\mathbb{P}_\kappa \upharpoonright \gamma$ is defined, as well as the base and index functions $\mathrm{base}_\kappa \upharpoonright \gamma$ and $\mathrm{index}_\kappa \upharpoonright \gamma$. We divide into two cases.

Case 1: $\gamma + 1 = \gamma(\kappa)$, or $\gamma = \gamma(\kappa)$ is a limit.

If Case 1 obtains, then we halt the construction, setting $\rho_\kappa = \gamma$ and $\mathbb{P}_\kappa = \mathbb{P}_\kappa \upharpoonright \gamma$. If $\kappa < \omega_2$, then we need to define the name $\dot{\mathbb{Q}}_\kappa$. Suppose that the $(\gamma)_0$-th element under $<_L$ is a pair $\langle \dot{S}_\kappa, \dot{\chi}_\kappa \rangle$ of $\mathbb{P}_\kappa$-names, where $\dot{S}_\kappa$ names a countable basis for a second countable, Hausdorff topology on ω_1 and $\dot{\chi}_\kappa$ names a coloring on ω_1 which is open with respect to the topology generated by $\dot{S}_\kappa$. Then let $\dot{f}_\kappa$ be the $<_L$-least $\mathbb{P}_\kappa$-name satisfying Proposition 3.2.2, and set $\dot{\mathbb{Q}}_\kappa := \mathbb{Q}(\dot{\chi}_\kappa, \dot{f}_\kappa)$, so that by Corollary 3.2.4, any partition product based upon $\underline{\mathbb{P}} \upharpoonright (\kappa+1)$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright (\kappa+1)$ is c.c.c. If $(\gamma)_0$ does not code such a pair, then we simply let $\dot{\mathbb{Q}}_\kappa$ name Cohen forcing for adding a single real. It is clear in this case also, by Lemma 2.1.20, that any partition product based upon $\underline{\mathbb{P}} \upharpoonright (\kappa+1)$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright (\kappa+1)$ is c.c.c.

On the other hand, if $\kappa = \omega_2$, then the partition product $\mathbb{P}_{\omega_2}$ is defined. After completing the rest of the construction, we show that forcing with $\mathbb{P}_{\omega_2}$ provides the desired model witnessing our theorem.

Case 2: $\gamma + 1 < \gamma(\kappa)$.

In this case, we desire to continue the construction another step. Let $\bar{\kappa} := H(\kappa, \gamma) \cap \kappa$, which is below κ by Lemma 4.1.5; recall that we are suppressing explicit mention of the parameter $\mathbb{A} \upharpoonright \kappa$. We also let j be the transitive collapse map of $H(\kappa, \gamma)$ and set $\bar{\gamma} := j(\gamma)$. We halt the construction if either $\mathbb{P}_\kappa \upharpoonright \gamma$ is not a member of $H(\kappa, \gamma)$, or if it is a member of $H(\kappa, \gamma)$ and either $\bar{\kappa} \notin C$ or $\mathbb{P}_\kappa \upharpoonright \gamma$ is not mapped to $\mathbb{P}_{\bar{\kappa}} \upharpoonright \bar{\gamma}$ by j (we will later show that this does not in fact occur).

Suppose, on the other hand, that $\bar{\kappa} \in C$ and that $\mathbb{P}_\kappa \upharpoonright \gamma$ is a member of $H(\kappa, \gamma)$ which is mapped by j to $\mathbb{P}_{\bar{\kappa}} \upharpoonright \bar{\gamma}$. We shall specify the next name $\dot{\mathbb{U}}_\gamma$ as well as the values $\mathrm{base}_\kappa(\gamma)$ and $\mathrm{index}_\kappa(\gamma)$. By Lemma 4.1.6, we have that $\bar{\gamma} + 1 = \gamma(\bar{\kappa}, \mathbb{A} \upharpoonright \bar{\kappa})$. By recursion, this means that $\bar{\gamma} = \rho_{\bar{\kappa}}$, i.e., that $\mathbb{P}_{\bar{\kappa}} = \mathbb{P}_{\bar{\kappa}} \upharpoonright \bar{\gamma}$. We now pull these objects back along j^{-1}.

In more detail, we observe that, setting $\pi_\gamma := j^{-1}$, $\pi_\gamma \upharpoonright \rho_{\bar{\kappa}}$ provides an acceptable rearrangement of $\mathbb{P}_{\bar{\kappa}}$, since π_γ is order-preserving. In fact, the π_γ-rearrangement of $\mathbb{P}_{\bar{\kappa}}$ is exactly equal to $(\mathbb{P}_\kappa \upharpoonright \gamma) \upharpoonright \pi_\gamma[\rho_{\bar{\kappa}}]$, by Lemma 2.1.16; this Lemma applies since for each $\delta \in C \cap \bar{\kappa}$, π_γ is the identity on $\mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta \cup \left\{\mathbb{P}_\delta, \dot{\mathbb{Q}}_\delta\right\}$. By assumption (iv) of Chapter 2, we see that the π_γ-rearrangement of $\dot{\mathbb{Q}}_{\bar{\kappa}}$ is defined (though not necessarily an element of $L_{\delta_\gamma(\kappa)}$), and so we let $\dot{\mathbb{U}}_\gamma$ be the π_γ-rearrangement of $\dot{\mathbb{Q}}_{\bar{\kappa}}$. We now set $\mathrm{base}_\kappa(\gamma) := (\pi_\gamma[\rho_{\bar{\kappa}}], \pi_\gamma \upharpoonright \rho_{\bar{\kappa}})$ and set $\mathrm{index}_\kappa(\gamma) := \bar{\kappa}$. In particular, we observe that

$$b_\kappa(\gamma) = H(\kappa, \gamma) \cap \gamma$$

is an initial segment of the ordinals of $H(\kappa, \gamma)$.

Claim 4.2.1. $\mathrm{base}_\kappa \upharpoonright (\gamma + 1)$ *and* $\mathrm{index}_\kappa \upharpoonright (\gamma + 1)$ *support a partition product based upon* $\underline{\mathbb{P}} \upharpoonright \kappa$ *and* $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa$*.*

Proof of Claim 4.2.1. Condition (1) of Definition 2.1.1 follows from the comments in the above paragraph. Condition (2) holds at γ by the elementarity of π_γ and at all smaller ordinals by recursion. So we need to check condition (3), where it suffices to verify the matching condition for γ and some $\beta < \gamma$. So suppose that there is some $\xi \in b_\kappa(\beta) \cap b_\kappa(\gamma)$. We define $\bar{\kappa}^*$ to be $H(\kappa, \beta) \cap \kappa$, so that $\bar{\kappa}^* = \mathrm{index}_\kappa(\beta)$. We also let $j_{\kappa,\beta}$ denote the transitive collapse map of $H(\kappa, \beta)$ and $j_{\kappa,\gamma}$ the transitive collapse map of $H(\kappa, \gamma)$. Finally, let π_β denote $j^{-1}_{\kappa,\beta}$.

Now the models $H(\kappa, \beta)$ and $H(\kappa, \gamma)$ are both sufficiently elementary, in particular, with respect to the sequence of surjections $\vec{\varphi}$. Since κ is the largest cardinal in $H(\kappa, \gamma)$,

$$H(\kappa, \gamma) \cap \xi = \varphi_{\kappa,\xi}[H(\kappa, \gamma) \cap \kappa] = \varphi_{\kappa,\xi}[\bar{\kappa}],$$

and therefore $b_\kappa(\gamma) \cap \xi = \varphi_{\kappa,\xi}[\bar{\kappa}]$. Similarly, $b_\kappa(\beta) \cap \xi = \varphi_{\kappa,\xi}[\bar{\kappa}^*]$.

With this observation in mind, we now verify that (3) holds. Suppose that $\bar{\kappa}^* \leq \bar{\kappa}$, and let $\zeta_0 := \pi_\beta^{-1}(\xi)$ and $\zeta_1 := \pi_\gamma^{-1}(\xi)$. If $\bar{\kappa}^* = \bar{\kappa}$, then by the calculations in the previous paragraph, (3) holds trivially, since the models $H(\kappa, \beta)$ and $H(\kappa, \gamma)$ have the same intersection with $\xi + 1$. Thus we proceed under the assumption that $\bar{\kappa}^* < \bar{\kappa}$. Since the above paragraph shows that $\pi_\beta[\zeta_0] \subseteq \pi_\gamma[\zeta_1]$, we need to check that $A := \pi_\gamma^{-1}[\pi_\beta[\zeta_0]]$ matches $\langle \bar{\kappa}, \zeta_1 \rangle$ to $\langle \bar{\kappa}^*, \zeta_0 \rangle$.

Now $\pi_\beta[\zeta_0] = b_\kappa(\beta) \cap \xi$ has the form $\varphi_{\kappa,\xi}[\bar{\kappa}^*]$. Since $\bar{\kappa}^* < \bar{\kappa}$, we have that κ, ξ, and $\bar{\kappa}^*$ are all in $H(\kappa, \gamma)$. Thus so is $\pi_\beta[\zeta_0]$. Applying the elementarity of $j_{\kappa,\gamma} = \pi_\gamma^{-1}$, we see that $\pi_\gamma^{-1} \circ \varphi_{\kappa,\xi} \restriction \bar{\kappa}^* = \varphi_{\bar{\kappa},\zeta_1} \restriction \bar{\kappa}^*$, which shows that A has the form $\varphi_{\bar{\kappa},\zeta_1}[\bar{\kappa}^*]$. Therefore condition (a) in the definition of matching holds. Additionally, if we let σ denote the transitive collapse of A, then we see that $\sigma \circ \pi_\gamma^{-1}$ is the transitive collapse of $\pi_\beta[\zeta_0] = \varphi_{\kappa,\xi}[\bar{\kappa}^*]$, which is just $\pi_\beta^{-1} = j_{\kappa,\beta}$. However, the elementarity of π_β^{-1} implies that $\pi_\beta^{-1} \circ \varphi_{\kappa,\xi} \restriction \bar{\kappa}^* = \varphi_{\bar{\kappa}^*,\zeta_0}$, and therefore $\sigma \circ \varphi_{\bar{\kappa},\zeta_1} \restriction \bar{\kappa}^* = \varphi_{\bar{\kappa}^*,\zeta_0}$. And finally, to see that (b) holds, we first observe that $b_\kappa(\beta) \cap \xi$ is closed under limit points of cofinality ω below its supremum, because $H(\kappa, \beta)$ is closed under ω-sequences. Since $b_\kappa(\beta) \cap \xi$ is in $H(\kappa, \gamma)$, by applying $j_{\kappa,\gamma}$, we conclude that the collapse of $H(\kappa, \gamma)$, denoted L_τ, satisfies that A is closed under limit points of cofinality ω below its supremum. However, L_τ is closed under ω-sequences, and therefore A is in fact closed under limit points of cofinality ω below its supremum. Thus (b) is satisfied. Since the proof in the case that $\bar{\kappa} < \bar{\kappa}^*$ is entirely similar, this completes the proof of the claim. $\square$

We have now completed the construction of the desired sequence of partition products. Before we prove our main theorem, we need to verify that for each $\kappa \in C \cup \{\omega_2\}$, we obtain a partition product of the appropriate length, i.e., that the construction does not halt prematurely, as described at the beginning of Case 2.

Lemma 4.2.2. *For each $\kappa \in C \cup \{\omega_2\}$, $\rho_\kappa = \gamma(\kappa)$ if $\gamma(\kappa)$ is a limit or equals $\gamma(\kappa) - 1$ if $\gamma(\kappa)$ is a successor.*

Proof. Suppose that $\kappa \in C \cup \{\omega_2\}$ and that $\gamma + 1 < \gamma(\kappa)$. We need to show that $\mathbb{P}_\kappa \restriction \gamma$ is a member of $H(\kappa, \gamma)$, that $\bar{\kappa} \in C$, and that $\mathbb{P}_\kappa \restriction \gamma$ gets mapped by j, the collapse map of $H(\kappa, \gamma)$, to $\mathbb{P}_{\bar{\kappa}} \restriction \bar{\gamma}$, where $\bar{\kappa} = j(\kappa)$ and $\bar{\gamma} = j(\gamma)$. However, it is clear that $\mathbb{P}_\kappa \restriction \gamma$ is a member of $L_{\delta_\gamma(\kappa)}$, being definable in that model from the sequence $\langle \delta_i(\kappa) : i < \gamma \rangle$. It is also straightforward to verify that $\bar{\kappa}$ is the least local ω_2 with respect to the sequence $\mathbb{A} \restriction \bar{\kappa}$, and hence $\bar{\kappa} \in C$. Finally, the above construction of partition products is uniform, in the sense that $\mathbb{P}_\kappa \restriction \gamma$ is definable in $L_{\delta_\gamma(\kappa)}$ from κ, γ, and $\mathbb{A} \restriction \kappa$ by the same definition which defines $\mathbb{P}_{\bar{\kappa}} \restriction \bar{\gamma}$ in $L_{\delta_{\bar{\gamma}}(\bar{\kappa})}$ from $\bar{\kappa}$, $\bar{\gamma}$, and $\mathbb{A} \restriction \bar{\kappa}$. Thus $\mathbb{P}_\kappa \restriction \gamma$ is a member of $H(\kappa, \gamma)$ and gets mapped to $\mathbb{P}_{\bar{\kappa}} \upharpoonright \bar{\gamma}$ by j. $\square$

We finish by proving Theorem 1.0.3.

Proof of Theorem 1.0.3. We force over L with $\mathbb{P}_{\omega_2}$. By Lemma 4.2.2, $\mathbb{P}_{\omega_2}$ is a partition product with domain $\gamma(\omega_2)$, and by Remark 4.1.4, $\gamma(\omega_2) = \omega_3$ (we suppress mention of the parameter $\mathbb{A}$). Let us denote the sequence of names used to form $\mathbb{P}_{\omega_2}$ by $\langle \dot{\mathbb{U}}_\gamma : \gamma < \omega_3 \rangle$. Since $\mathbb{P}_{\omega_2}$ is a partition product based upon $\underline{\mathbb{P}} \upharpoonright \omega_2$ and $\underline{\dot{\mathbb{Q}}}$, it is c.c.c. Hence all cardinals are preserved. Since $\mathbb{P}_{\omega_2}$ has size $\aleph_3$ and is c.c.c., it forces that the continuum has size no more than $\aleph_3$. However, $\mathbb{P}_{\omega_2}$ adds $\aleph_3$-many reals, and to see this, we first recall that by Lemma 2.1.7, $\mathbb{P}_{\omega_2}$ is a dense subset of the finite support iteration of the names $\langle \dot{\mathbb{U}}_\gamma : \gamma < \omega_3 \rangle$. Next, each $\dot{\mathbb{U}}_\gamma$ either names Cohen forcing or one of the homogeneous set posets, and each of the latter adds a real by Remark 3.2.1. Thus $\mathbb{P}_{\omega_2}$ forces that the continuum has size exactly $\aleph_3$. We now want to see that $\mathbb{P}_{\omega_2}$ forces that OCA_{ARS} holds.

Towards this end, let $\langle \dot{S}, \dot{\chi} \rangle$ be a pair of $\mathbb{P}_{\omega_2}$-names, where $\dot{S}$ names a countable basis for a second countable, Hausdorff topology on ω_1 and $\dot{\chi}$ names a coloring which is open with respect to the topology generated by $\dot{S}$. Let $\gamma < \omega_3$ so that $\langle \dot{S}, \dot{\chi} \rangle$ is the $(\gamma)_0$-th element under $<_L$ and so that $\langle \dot{S}, \dot{\chi} \rangle$ is a $\mathbb{P}_{\omega_2} \upharpoonright (\gamma)_1$-name. Note that $\langle \dot{S}, \dot{\chi} \rangle$ is an element of $H(\omega_2, \gamma)$ since, by Remark 4.1.4, γ is, and also notice that $H(\omega_2, \gamma)$ satisfies that $\langle \dot{S}, \dot{\chi} \rangle$ is a $\mathbb{P}_{\omega_2} \upharpoonright \gamma$-name. Let j denote the transitive collapse map of $H(\omega_2, \gamma)$ and let $\pi := j^{-1}$ denote the anticollapse map. Set $\bar{\gamma} := j(\gamma)$ and $\kappa := j(\omega_2)$, and observe that by Lemma 4.1.6, j collapses $H(\omega_2, \gamma)$ onto $L_{\delta_{\bar{\gamma}}(\kappa)}$.

We will be done if we can show that forcing with $\dot{\mathbb{U}}_\gamma$ adds a partition of ω_1 into countably-many $\dot{\chi}$-homogeneous sets, and towards this end, let G be V-generic over $\mathbb{P}_{\omega_2}$. We use G_γ to denote the generic G adds for $\dot{\mathbb{U}}_\gamma[G \upharpoonright \gamma]$ over $V[G \upharpoonright \gamma]$. Set $\bar{G}$ to be $j\left[(G \upharpoonright \gamma) \cap H(\omega_2, \gamma)\right]$, and observe that $\bar{G}$ is generic for the poset $j(\mathbb{P}_{\omega_2} \upharpoonright \gamma) = \mathbb{P}_\kappa \upharpoonright \bar{\gamma} = \mathbb{P}_\kappa$ over $L_{\delta_{\bar{\gamma}}(\kappa)}$. Since $\mathbb{P}_\kappa$ is c.c.c. and $L_{\delta_{\bar{\gamma}}(\kappa)}$ is countably closed, $\bar{G}$ is also V-generic over $\mathbb{P}_\kappa$. In particular, π extends to an elementary embedding

$$\pi^* : L_{\delta_{\bar{\gamma}}(\kappa)}[\bar{G}] \longrightarrow L_{\delta_\gamma(\omega_2)}[G],$$

and since $\mathrm{crit}(\pi^*) > \omega_1$, we see that $\dot{S}[G] = j(\dot{S})[\bar{G}]$ and $\dot{\chi}[G] = j(\dot{\chi})[\bar{G}]$.

By the elementarity of j, $\langle j(\dot{S}), j(\dot{\chi})\rangle$ is the $(\bar{\gamma})_0$-th pair of $\mathbb{P}_\kappa$ names where the first coordinate names a countable basis for a second countable, Hausdorff topology on ω_1 and the second names a coloring which is open with respect to the topology generated by that basis. By the construction of $\dot{\mathbb{Q}}_\kappa$, this means that $\dot{\mathbb{Q}}_\kappa$ names the poset to decompose ω_1 into countably-many $j(\dot{\chi})$-homogeneous sets with respect to the preassignment $\dot{f}_\kappa$. Thus forcing with $\dot{\mathbb{Q}}_\kappa[\bar{G}]$ adds a decomposition of ω_1 into countably-many $j(\dot{\chi})[\bar{G}] = \dot{\chi}[G]$-homogeneous sets. We will be done if we can show that G adds a generic for $\dot{\mathbb{Q}}_\kappa[\bar{G}]$.

To see this, we recall from Case 2 of the construction that $\dot{\mathbb{U}}_\gamma$ is the $\pi \upharpoonright \rho_\kappa$-rearrangement of $\dot{\mathbb{Q}}_\kappa$. Moreover, as also described in Case 2, Lemma 2.1.16 applies. Thus $\dot{\mathbb{Q}}_\kappa[\bar{G}] = \dot{\mathbb{U}}_\gamma[G]$. G_γ is therefore $V[G \upharpoonright \gamma]$-generic for $\dot{\mathbb{Q}}_\kappa[\bar{G}]$, which finishes the proof. □

We wrap up by sketching a proof of Theorem 1.0.4.

Proof Sketch of Theorem 1.0.4. We first describe how to build the names on the sequence $\underline{\dot{\mathbb{Q}}}$. The only modification to the construction for the previous theorem is that if, in Case 1 above, $(\gamma)_0$ names a Knaster poset of size $\aleph_1$, then we set $\dot{\mathbb{Q}}_\kappa$ to be this Knaster poset. With this modification to the sequence $\underline{\dot{\mathbb{Q}}}$, we still maintain the recursive assumption that for each $\kappa \in C$, any partition product based upon $\underline{\mathbb{P}} \upharpoonright \kappa$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa$ is c.c.c.; this follows by Lemma 2.1.20, Lemma 2.1.13, and since the product of Knaster and c.c.c. posets is still c.c.c.

Now we want to see that forcing with this modified $\mathbb{P}_{\omega_2}$ gives the desired model. The proof that the extension satisfies OCA_{ARS} and $2^{\aleph_0} = \aleph_3$ is the same as before. To prove that it satisfies $\mathsf{FA}(\aleph_2, \mathrm{Knaster}(\aleph_1))$, suppose that $\dot{\mathbb{K}}$ is forced in $\mathbb{P}_{\omega_2}$ to be a Knaster poset of size $\aleph_1$. We may assume without loss of generality that $\dot{\mathbb{K}}$ is forced to be a subset of ω_1. Fix γ so that $(\gamma)_0$ codes $\dot{\mathbb{K}}$, making γ large enough so that $\dot{\mathbb{K}}$ is a $(\mathbb{P}_{\omega_2} \upharpoonright \gamma)$-name and so that all the dense sets we need to meet belong to $V[G \upharpoonright \gamma]$. Next, arguing as in the proof of Theorem 1.0.3, we have $\kappa < \omega_2$, $j\colon H(\omega_2, \gamma) \longrightarrow L_{\delta_{\bar{\gamma}}(\kappa)}$, and an extension

$$\pi^*\colon L_{\delta_{\bar{\gamma}}(\kappa)}[\bar{G}] \longrightarrow L_{\delta_\gamma(\omega_2)}[G \upharpoonright \gamma]$$

of the inverse π of j. By the modified Case 1 construction we have that $\dot{\mathbb{Q}}_\kappa = j(\dot{\mathbb{K}})$. By Case 2 in the construction of $\mathbb{P}_{\omega_2}$, $\dot{\mathbb{U}}_\gamma$ is the rearrangement of $\dot{\mathbb{Q}}_\kappa$ by $\pi \restriction \rho_\kappa$. However, by the final clause in Lemma 2.1.16, and since $\dot{\mathbb{Q}}_\kappa$ names a poset contained in $\omega_1 < \kappa = \mathrm{crit}(\pi)$, this rearrangement is exactly $\pi(\dot{\mathbb{Q}}_\kappa) = \dot{\mathbb{K}}$. So G_γ is generic for $\dot{\mathbb{K}}[G \restriction \gamma]$ over $V[G \restriction \gamma]$, and hence G_γ is a filter in $V[G]$ for $\dot{\mathbb{K}}[G \restriction \gamma]$ which meets the desired dense sets. $\square$

Part II

Stationary Reflection and Other Combinatorial Principles on $\aleph_2$

CHAPTER 5

Introduction to Part 2

In Part 2 of this thesis, we study a number of combinatorial principles at the second uncountable cardinal, $\aleph_2$. These results have a very different flavor than the results in Part 1, as they are concerned with making reflection and compactness properties of large cardinals hold at $\aleph_2$. All of the results in Part 2 are concerned with building models with various degrees of *Stationary Reflection*. Following the authors of [24], we are particularly interested in showing that these degrees of stationary reflection are compatible with other combinatorial principles of wide interest. In the first section of this chapter, we will review the relevant definitions and historical background. In the second section, we will collect, for reference, a number of useful facts about the forcings we will use in later chapters. In the final section, we show that Mitchell-type posets are proper (the proofs in that section are due to the author, though we suspect that the results are known already, possibly as "folklore").

5.1 Definitions and Background

Let us begin by defining the stationary reflection principles that we're interested in and surveying previous results.

Definition 5.1.1. *$C \subseteq \alpha$ is* club *if C is closed and unbounded in α.*

If $\mathrm{cf}(\alpha) \geq \omega_1$, then the collection of club subsets of α naturally forms a $\mathrm{cf}(\alpha)$-complete filter. Loosely speaking, sets which are positive measure with respect to this filter are the stationary sets.

Definition 5.1.2. *Suppose that* $\mathrm{cf}(\alpha) \geq \omega_1$*. $S \subseteq \alpha$ is* stationary *if $S \cap C \neq \varnothing$, for any club*

$C \subseteq \alpha$.

We are interested in which stationary sets have stationary initial segments.

Definition 5.1.3. *Suppose that* $\mathrm{cf}(\alpha) \geq \omega_1$. *A stationary* $S \subseteq \alpha$ *is said to* reflect *if there is some limit* $\beta < \alpha$ *so that* $S \cap \beta$ *is stationary in* β.

We remark here that ω_2 is the first ordinal at which stationary reflection is a non-trivial principle to study. Indeed, if $S \subseteq \omega_1$ is stationary, and without loss of generality consists only of limit ordinals, then there does not exist any limit $\alpha < \omega_1$ so that $S \cap \alpha$ is "stationary," since any ω-sequence of successor ordinals cofinal in α is disjoint from $S \cap \alpha$. A similar argument shows that no $S \subseteq \omega_2 \cap \mathrm{cof}(\omega_1)$ can reflect.

Definition 5.1.4. Stationary Reflection at ω_2*, abbreviated* $\mathsf{SR}(\omega_2)$*, is the statement that every stationary* $S \subseteq \omega_2 \cap \mathrm{cof}(\omega)$ *reflects.* Almost Everywhere Stationary Reflection*, abbreviated* $\mathsf{SR}(\omega_2)^*$*, is the statement that for any stationary* $S \subseteq \omega_2 \cap \mathrm{cof}(\omega)$*, there exists a club* $C \subseteq \omega_2$ *so that* S *reflects at* α*, for all* $\alpha \in C \cap \mathrm{cof}(\omega_1)$.

A notational remark is in order: in the paper [24], the authors use "RP" to denote what we are calling "Stationary Reflection." We depart from this notation in order to distinguish between stationary reflection for subsets of ω_2 and the higher-order reflection principles RP and WRP which concern reflection of stationary subsets of $P_{\omega_1}(\omega_2)$ (see [51]).

One feature of this line of research is the use (and in fact, necessity) of large cardinals. Let us define the ones that we'll use here.

Definition 5.1.5. *Let* κ *be a regular, uncountable cardinal.* κ *is said to be* Mahlo *if* $\{\alpha < \kappa : \alpha \text{ is inaccessible}\}$ *is stationary in* κ. κ *is said to be* Weakly Compact *if for any* $f : [\kappa]^2 \longrightarrow 2$*, there exists an* $H \in [\kappa]^\kappa$ *which is* f*-homogeneous.*

It is well-known that there are a wealth of equivalences to κ being weakly compact. See, for instance, [45] and [21]. Of particular interest to us is the following result (see Theorem 16.1 of [21]).

Proposition 5.1.6. *An inaccessible cardinal κ is weakly compact iff for every transitive set M with $|M| = \kappa$, $\kappa \in M$, and ${}^{<\kappa}M \subseteq M$, there is an elementary embedding $j : M \longrightarrow N$ where N is transitive, $|N| = \kappa$, ${}^{<\kappa}N \subseteq N$, $\mathrm{crit}(j) = \kappa$.*

One of the first results concerning stationary reflection is due to Baumgartner (see [15]) and states that it is consistent from a weakly compact cardinal that $\mathsf{SR}(\omega_2)$ holds. One particularly noteworthy feature of Baumgartner's proof is the following proposition, which we will have more to say about later.

Proposition 5.1.7. *Let $S \subseteq \alpha \cap \mathrm{cof}(\omega)$ be stationary, and let $\mathbb{R}$ be an ω_1-closed forcing. Then forcing with $\mathbb{R}$ preserves the stationarity of S.*

Given the large cardinal assumption of Baumgartner's result, it is natural to ask whether any such assumption is needed. It turns out that the answer is yes. Let us take a brief detour which will help us see why this is the case. We begin with the definition of a "square sequence."

Definition 5.1.8. *Let $\kappa \geq \omega_1$ be a cardinal. A sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ is said to be a $\square_\kappa$ sequence if the following conditions hold:*

1. *for all $\alpha < \kappa^+$, C_α is club in α;*

2. *for all $\beta < \kappa^+$ and all $\alpha \in \lim(C_\beta)$, $C_\beta \cap \alpha = C_\alpha$;*

3. *for all $\alpha < \kappa^+$, $\mathrm{ot}(C_\alpha) \leq \kappa$.*

$\square_\kappa$ is the assertion that such a sequence exists.

A sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$ is said to be a $\square^*_\kappa$ sequence if the following conditions hold:

1. *for all $\alpha < \kappa^+$, $\mathcal{C}_\alpha$ is a non-empty set of clubs in α, and $|\mathcal{C}_\alpha| \leq \kappa$;*

2. *for all $\beta < \kappa^+$, $C \in \mathcal{C}_\beta$, and $\alpha \in \lim(C)$, $C \cap \alpha \in \mathcal{C}_\alpha$;*

3. *for all $\alpha < \kappa^+$ and $C \in \mathcal{C}_\alpha$, $\mathrm{ot}(C) \leq \kappa$.*

$\square^*_\kappa$ is the assertion that a $\square^*_\kappa$ sequence exists.

A $\square_\kappa$ sequence may be viewed as a coherent system of singularizing all ordinals in the interval $[\kappa, \kappa^+)$. Jensen, in his landmark paper on the fine structure of L (see [43]) showed that in L, $\square_\kappa$ holds for all cardinals $\kappa \geq \omega_1$. For modern discussions of square principles in inner model theory, we refer the reader to [79].

It is well-known that square principles inhibit the amount of stationary reflection (for contemporary discussions, see [23] as well as [40]). In particular (see [25]), $\square_{\omega_1}$ implies that $\mathsf{SR}(\omega_2)$ fails. Furthermore, the failure of $\square_{\omega_1}$ entails that ω_2 is Mahlo in L (see [25]). Combining this, we see that if $\mathsf{SR}(\omega_2)$ holds, then ω_2 is Mahlo in L.

However, it turns out that this is in fact an equiconsistency. Indeed, Harrington and Shelah (see [39]) showed that it is consistent from a Mahlo cardinal that $\mathsf{SR}(\omega_2)$ holds. A key idea of their proof is that one can iterate to destroy the stationarity of any non-reflecting stationary $S \subseteq \omega_2 \cap \mathrm{cof}(\omega)$, after an initial Levy collapse of a Mahlo cardinal to ω_2. The technique of iterated club-adding will play a key role in the results of Part 2 of this thesis.

Thus the weakly compact cardinal used in Baumgartner's aforementioned theorem is more than necessary to obtain the consistency of $\mathsf{SR}(\omega_2)$. However, as Magidor pointed out in [56], in Baumgartner's model, we have a large amount of simultaneous stationary reflection; recall that $S, T \subseteq \alpha$ reflect *simultaneously* if there is some $\beta < \alpha$ so that $S \cap \beta$ and $T \cap \beta$ are both stationary in β. It turns out (see [56]) that the statement that every two stationary subsets of $\omega_2 \cap \mathrm{cof}(\omega)$ reflect implies that ω_2 is weakly compact in L.

In fact, as Magidor notes, in Baumgartner's model, there is an ω_2-complete filter $\mathcal{F}$ on ω_2 so that every stationary $S \subseteq \omega_2 \cap \mathrm{cof}(\omega)$ reflects on a set in $\mathcal{F}$. Magidor was able to show that, from a weakly compact cardinal, one can make the filter $\mathcal{F}$ the club filter on ω_2. Magidor also uses an iterated club adding technique, but with a very different flavor from that of Harrington and Shelah. In their paper [39], clubs are added to destroy the stationarity of various aberrant sets, whereas Magidor adds clubs (roughly) inside the stationary sets to bolster the amount of reflection.

Let us now turn to discuss and survey another principle of interest, one which concerns trees. We begin with some of the relevant definitions.

Definition 5.1.9. *Let $\kappa \geq \omega$ be regular. A κ-tree is a tree $(T, <_T)$ on κ so that every level of T has size $< \kappa$. A* cofinal branch *is a linearly ordered subset of T which meets every level. κ is said to have the* tree property *if every κ-tree has a cofinal branch; this statement is abbreviated $\mathsf{TP}(\kappa)$. A tree without a cofinal branch is said to be* Aronszajn.

One particularly strong way of witnessing that a tree is Aronszajn is by the existence of a specializing function.

Definition 5.1.10. *Say that $\kappa = \lambda^+$. A κ-tree T is* special *if there is a function $f : T \longrightarrow \lambda$ so that for any $x, y \in T$, if $x <_T y$, then $f(x) \neq f(y)$.*

Note that if T is a special Aronszajn tree in a model V, then T remains special (and hence Aronszajn) in any cardinal-preserving extension of V. A few other relevant properties of trees are the following.

Definition 5.1.11. *A κ-tree T is a* Souslin tree *if T has no κ-sized chains or antichains.*

*A κ-tree T is a κ-*Kurepa tree *if T has κ^+-many cofinal branches. If T is a tree on κ with width $\leq \kappa$ which has κ^+-many cofinal branches, then we say that T is a* weak κ- Kurepa tree*.*

Our discussion begins with the classic theorem of König that ω has the tree property ([46]). In contrast to ω, however, ω_1 fails to have the tree property, i.e., there is an ω_1-Aronszajn tree (the result is due, naturally enough, to Aronszajn and is described by Kurepa in [53]).

The investigation for ω_2 (and higher) is dependent on axioms beyond ZFC. For instance, as Specker has shown (see [72]), if $\kappa^{<\kappa}$ holds, then there is a special κ^+-Aronszajn tree. Jensen later showed ([43]) that for any cardinal κ, $\square^*_\kappa$ holds iff there is a special κ^+-Aronszajn tree; since $\kappa^{<\kappa}$ implies that a $\square^*_\kappa$ sequence exists, we can view Jensen's result as a generalization of Specker's.

A little later, William Mitchell showed [58] that from a Mahlo cardinal, it is consistent that no special Aronszajn trees exist on ω_2, and furthermore, from a weakly compact cardinal, that it is consistent that ω_2 has the tree property. Mitchell's proof was a watershed in forcing, and variations of the poset which he invented (the so-called Mitchel Forcing) will be used heavily later in this thesis. We will review Mitchell-type posets in the next section.

As can be seen from the above discussion of stationary reflection and the tree property, models in which these principles hold fail to be L-like, as they imply the failure of various square principles. A natural question to then ask is whether or not there are square-like combinatorial principles which can be investigated in ZFC alone. Remarkably enough, such principles exist, and the one most relevant to our investigations is *Approachability.*

Definition 5.1.12. *Let κ be a regular, uncountable cardinal, and let $\vec{a} = \langle a_\alpha : \alpha < \kappa \rangle$ be a sequence of bounded subsets of κ. We say that a limit ordinal $\beta < \kappa$ is* approachable with respect to $\vec{a}$ *if there exists an unbounded $A \subseteq \beta$ with $\mathrm{ot}(A) = \mathrm{cf}(\beta)$ so that*

$$\{A \cap \xi : \xi < \beta\} \subseteq \{a_\xi : \xi < \beta\}.$$

Given κ regular and uncountable, Shelah defined (see [65] and [66]) an ideal $I[\kappa]$ on κ as follows:

Definition 5.1.13. *$I[\kappa]$ consists of all $S \subseteq \kappa$ so that for some club $C \subseteq \kappa$ and some sequence $\vec{a}$ of bounded subsets of κ, every element of $S \cap C$ is approachable with respect to $\vec{a}$.*

Approachability *at κ is the statement that $\kappa \in I[\kappa]$, i.e., that $I[\kappa]$ is the trivial ideal.* Approachability at ω_2 *will be denoted* AP_{ω_1}.

Let us collect here some well-known facts about $I[\kappa]$, proofs of which may be found in [20].

1. If $\square_\mu^*$ holds, then $\mu^+ \in I[\mu^+]$;

2. if μ is regular, then $\mu^+ \cap \mathrm{cof}(< \mu) \in I[\mu^+]$;

3. suppose that $\mu < \kappa$ is regular and that $\kappa^{<\mu} = \kappa$. Then the restriction of $I[\kappa]$ to $\kappa \cap \operatorname{cof}(\mu)$ is generated by a single stationary set.

Remark 5.1.14. In item (3) above, it is worth noting that the maximal stationary set of cofinality μ points is constructed as follows: let $\vec{a} = \langle a_\alpha : \alpha < \kappa \rangle$ be an enumeration of $[\kappa]^{<\mu}$, and let S consist of all $\beta < \kappa$ which are approachable with respect to $\vec{a}$. Note that S is well-defined, modulo the club filter.

It is also worth noting that Shelah originally invented $I[\kappa]$ in order to clarify which stationary sets remain stationary after certain forcings. Even though, as stated in Proposition 5.1.7, countably-closed forcings preserve stationary subsets of cofinality ω ordinals, it is not in general true that μ^+-closed forcings preserve stationary subsets of $\lambda \cap \operatorname{cof}(\mu)$. However, stationary sets $S \subseteq \lambda \cap \operatorname{cof}(\mu)$ which are in $I[\lambda]$ do remain stationary after forcing with μ^+-closed forcings ([65]).

The relationship between the principles TP, RP, and AP at κ^{++} (for κ either regular or singular) has most extensively been studied in the recent paper [24], wherein the authors show, from appropriate large cardinal assumptions, that all eight Boolean combinations of the above three principles are consistent.

In Part 2, we will show how to improve their results in a number of ways. We recall that in [24], the authors showed that the configuration $\neg\mathsf{TP}(\omega_2) + \mathsf{SR}(\omega_2) + \neg\mathsf{AP}_{\omega_1}$ is consistent from a weakly compact cardinal. In their construction of such a model, they obtain $\mathsf{SR}(\omega_2)$ more-or-less for free, as in Baumgartner's model. However, they need to "work" to obtain the failure of $\mathsf{TP}(\omega_2)$, and to accomplish this, they use Kunen's forcing (see [52]) for adding a Suslin tree. We show how to improve their result to use the optimal large cardinal hypothesis:

Theorem 5.1.15. (Gilton) *It is consistent from a Mahlo cardinal that* $\neg\mathsf{TP}(\omega_2) + \mathsf{SR}(\omega_2) + \neg\mathsf{AP}_{\omega_1}$ *holds.*

The configuration above is indeed optimal since, as we noted at the beginning of this section, $\mathsf{SR}(\omega_2)$ entails that ω_2 is Mahlo in L.

In contrast to the construction from [24], we obtain the failure of $\mathsf{TP}(\omega_2)$ for free: we work with the least Mahlo cardinal, κ, in L, noting that κ is not weakly compact. By consistency strength considerations, we know that $\mathsf{TP}(\omega_2)$ must fail in any extension in which κ becomes ω_2. However, we do need to work to obtain $\mathsf{SR}(\omega_2)$. As in the Harrington-Shelah model mentioned above, we iterate club-adding in order to destroy the stationarity of any non-reflecting stationary subset of $\kappa \cap \mathrm{cof}(\omega)$. The proof of Theorem 5.1.15 occurs in Chapter 6.

We note here that Gilton and Krueger later (see [32]) simplified Gilton's original proof of Theorem 5.1.15. Instead of the Mitchell-style preparation forcing used in Gilton's proof, Gilton and Krueger use a countable-support iteration of proper forcings as a preparatory forcing. They show that there exists an object called a *Disjoint Stationary Sequence* on κ in the resulting extension; disjoint stationary sequences were originally invented by Krueger in [50]. The existence of a disjoint stationary sequence is preserved under a wide variety of forcings and implies that AP_{ω_1} fails.

We next improve a number of the results of [24] to include $\mathsf{SR}(\omega_2)^*$.

Theorem 5.1.16. (Gilton, Gilton and Ben-Neria) *Let Φ be any Boolean combination of AP_{ω_1} and $\mathsf{TP}(\omega_2)$. Then Φ is consistent with $\mathsf{SR}(\omega_2)^*$.*

We remark that the configuration $\mathsf{TP}(\omega_2) + \mathsf{AP}_{\omega_1} + \mathsf{SR}(\omega_2)^*$ is due to Gilton and Omer Ben-Neria, and that the configuration $\neg\mathsf{TP}(\omega_2) + \mathsf{AP}_{\omega_1} + \mathsf{SR}(\omega_2)^*$ already holds in Magidor's model [56], since the CH is satisfied therein. The idea for these constructions (except for $\neg\mathsf{TP}(\omega_2) + \mathsf{AP}_{\omega_1} + \mathsf{SR}(\omega_2)^*$ which is already taken care of) is to incorporate Magidor's iterated club-adding from [56] into the scheme of [24]. For the case $\mathsf{TP}(\omega_2) + \mathsf{AP}_{\omega_1} + \mathsf{SR}(\omega_2)^*$, this is done by building a Mitchell-style poset with quite a bit of collapsing and by using collapse absorption arguments. However, for the cases where AP_{ω_1} fails, we need to avoid undue collapsing, and for this we use Mitchell-type posets with "look-ahead." This idea goes back to Abraham's result ([1]) that it is consistent that the Tree Property holds at both $\aleph_2$ and $\aleph_3$ simultaneously. The looking-ahead is done by using a Laver diamond built from a supercompact cardinal. Our proofs of 5.1.16 occur in Chapter 7.

Finally, we make good on a promise from [34] and show that in a variation of that model, there are neither weak Kurepa trees on ω_1 nor special Aronszajn trees on ω_2; moreover we can also make AP_{ω_1} hold or fail. Originally, Gilton and Krueger sought to show that $\mathsf{SR}(\omega_2)$ is consistent with an arbitrarily large continuum. Towards this end, they built and investigated a mixed-support iteration which uses distributive, rather than closed forcings. However, they later learned from Itay Neeman that simply adding Cohen reals to the model of [39] preserves $\mathsf{SR}(\omega_2)$, and thus the result may be obtained through simpler means. Despite this, the technology of [34] could still be of use and represents an advance in our understanding of mixed support iterations. Furthermore, the mixed-support forcing allows us to show that a variety of other combinatorial principles are satisfied in the extension, principles which do not hold after simply adding Cohen reals to the Harrington-Shelah model.

Mixed-support iterations are a type of iteration that uses different supports on different coordinates. For instance, one might alternate between adding a Cohen real and collapsing a cardinal, using finite support for the Cohen coordinates and countably infinite support for the collapse coordinates. Indeed, Mitchell-type posets may be seen in this light. Mixed-support iterations have found use elsewhere too, for instance in Todorčević's investigation of partition properties on ω_1 (see [73]); in the result of Abraham and Shelah (see [5]), mentioned earlier, on isomorphism types of Aronszajn trees (though they actually used a mixed-support *product*); and in more modern research in [48], with applications to [37], [49], and [50]. In most of these applications, closed posets are used with the larger support, though in [48], strategic closure of certain two-step posets is used instead. However, in iterating club adding as in the Gilton-Krueger paper [34], such options are not available, due to the nature of the posets used to add the clubs. Thus the iteration schema worked-out therein and in Chapter 8 of this work proves to be a generalization of this line of research into mixed-support iterations.

5.2 A Quick Survey of Mitchell-Type Posets

In this section, we will review the definitions of various Mitchell-type posets and also collect a number of useful facts about them. These facts will be used throughout Part 2 of this thesis. We assume here that κ is a Mahlo cardinal, and we let A be the set of inaccessible cardinals below κ, so that A, by definition of Mahlo, is a stationary subset of κ. We begin with the original poset from Mitchell.

Definition 5.2.1. *The poset* $\mathbb{M}_{\text{Mitchell}}$ *consists of all pairs* (a, f) *where* $a \in \text{Add}(\omega, \kappa)$*, and where* f *is a countable partial function satisfying the following:*

1. $\text{dom}(f) \subseteq A$;
2. *for each* $\alpha \in \text{dom}(f)$*,* $f(\alpha)$ *is an* $\text{Add}(\omega, \alpha)$*-name for a condition in* $\text{Add}(\omega_1, 1)$*.*

We say $(b, g) \leq (a, f)$ *iff* $b \leq a$ *in* $\text{Add}(\omega, \kappa)$*,* $\text{dom}(f) \subseteq \text{dom}(g)$*, and for all* $\alpha \in \text{dom}(f)$*,* $b \upharpoonright \alpha \Vdash_{\text{Add}(\omega,\alpha)} g(\alpha) \leq_{\text{Add}(\omega_1,1)} f(\alpha)$*.*

We note that in (2) above, as well as in the definition of the ordering, the symbol $\text{Add}(\omega_1, 1)$ is the poset for adding a Cohen subset of ω_1, *as computed in the model* $V[\text{Add}(\omega, \alpha)]$.

As mentioned in the first subsection, Abraham ([1]) improved upon Mitchell's result to show that the tree property may hold at both ω_2 and ω_3 simultaneously. A key feature of his construction is the inclusion of various other forcings into the definition of the poset, making the Tree Property "robust" under further forcing; see [22] and [78] for additional uses of this idea.

Let us now review the definitions of the posets from [24], posets which use this "robustness" idea.

Definition 5.2.2. *The poset* $\mathbb{M}_0 \upharpoonright \beta$ *is defined by recursion on* $\beta \in A \cup \{\kappa\}$*, setting* $\mathbb{M}_0 := \mathbb{M}_0 \upharpoonright \kappa$*. Conditions in* $\mathbb{M}_0 \upharpoonright \beta$ *are triples* (a, q, r) *satisfying the following:*

1. $a \in \text{Add}(\omega, \beta)$;

2. $\operatorname{dom}(q) \subseteq A \cap \beta$, *and* $|\operatorname{dom}(q)| \leq \aleph_0$;

3. *for all* $\alpha \in \operatorname{dom}(q)$, $q(\alpha)$ *is an* $\operatorname{Add}(\omega, \alpha)$*-name for a condition in* $\operatorname{Col}(\omega_1, \alpha)^{V[\operatorname{Add}(\omega,\alpha)]}$;

4. $\operatorname{dom}(r) \subseteq A \cap \beta$, *and* $\operatorname{dom}(r)$ *is an Easton set of regular cardinals;*

5. *for all* $\alpha \in \operatorname{dom}(r)$, $r(\alpha)$ *is an* $(\mathbb{M}_0 \restriction \alpha)$*-name for a condition in* $\operatorname{Add}(\alpha, 1)^{V[\mathbb{M}_0 \restriction \alpha]}$.

Conditions in $\mathbb{M}_0 \restriction \beta$ *are ordered as follows: we set* $(a', q', r') \leq (a, q, r)$ *iff*

(a) $a' \leq a$ *in* $\operatorname{Add}(\omega, \beta)$;

(b) $\operatorname{dom}(q) \subseteq \operatorname{dom}(q')$, *and for all* $\alpha \in \operatorname{dom}(q)$, $a' \restriction \alpha \Vdash_{\operatorname{Add}(\omega,\alpha)} q'(\alpha) \leq q(\alpha)$;

(c) $\operatorname{dom}(r) \subseteq \operatorname{dom}(r')$, *and for all* $\alpha \in \operatorname{dom}(r)$, $(a', q', r') \restriction \alpha \Vdash_{\mathbb{M}_0 \restriction \alpha} r'(\alpha) \leq r(\alpha)$.

Now let A^* denote all successor points of A. $\mathbb{M}_1$ is defined in the same way as $\mathbb{M}_0$, except that in item (2), A^* is replaced by A.

Let us now collect some of the most useful facts about the posets $\mathbb{M}_i$.

Proposition 5.2.3. *Let* $\alpha \in (A \cap \lim(A)) \cup \{\kappa\}$, *and let* α^* *denote the successor of* α *in* A. *Then*

1. $\mathbb{M}_i \restriction \alpha$ *is* α*-Knaster, has size* α, *and forces that* $2^\omega = \alpha = \omega_2$;

2. *in the extension by* $\mathbb{M}_0 \restriction \alpha^*$, α^* *is a cardinal,* α *is no longer a cardinal, and* $2^\omega = \alpha^* > \omega_1$;

3. *in the extension by* $\mathbb{M}_1 \restriction \alpha^*$, α *and* α^* *are both cardinals, and* $\alpha = \omega_2 < 2^\omega = \alpha^*$.

In analyzing posets like the $\mathbb{M}_i$, it has proved to be effective to embed them into larger, simpler posets. This idea goes back to Laver's invention of termspace forcing; its application to Mitchell-type posets is due to Abraham (see [1]). As we will not need to use many of the details here, we state these result with less specificity than usual. We first record the definition of a forcing projection:

Definition 5.2.4. *Let $\mathbb{P}$ and $\mathbb{Q}$ be posets. We say that a function $\pi : \mathbb{P} \longrightarrow \mathbb{Q}$ is a* projection *if the following conditions are satisfied:*

1. $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$;
2. *if $p_1 \leq_{\mathbb{P}} p_0$, then $\pi(p_1) \leq_{\mathbb{Q}} \pi(p_0)$;*
3. *for all $p \in \mathbb{P}$, if $q \leq_{\mathbb{Q}} \pi(p)$, then there is some $\tilde{p} \leq_{\mathbb{P}} p$ s.t. $\pi(\tilde{p}) \leq_{\mathbb{Q}} q$.*

Proposition 5.2.5. *Let $\mathbb{M}$ denote $\mathbb{M}_0$ or $\mathbb{M}_1$. Then $\mathbb{M}$ is a forcing projection of a product $\mathrm{Add}(\omega, \kappa) \times \mathbb{B}$, where $\mathbb{B}$ is ω_1-closed in V.*

The idea is that $\mathbb{B}$ consists of all conditions in $\mathbb{M}_i$ ($i \in 2$), which are trivial on the Cohen part, with the same ordering as $\mathbb{M}_i$. Recalling Easton's Lemma, we obtain the following corollary:

Corollary 5.2.6. *All ω-sequences in $V[\mathbb{M}_i]$ are members of $V[\mathrm{Add}(\omega, \kappa)]$.*

It will prove to be very useful later to know that quotients or "tails" of the Mitchell forcing also look like Mitchell forcing. That is to say, we have that if $\alpha \in A$, then in $V[G_\alpha^{\mathbb{M}_i}]$, where $G_\alpha^{\mathbb{M}_i}$ is V-generic for $\mathbb{M}_i \restriction \alpha$, the poset $\mathbb{M}_i / G_\alpha^{\mathbb{M}_i}$ is isomorphic to a Mitchell-type poset. The following proposition gives us what we need.

Proposition 5.2.7. *Let $\alpha \in A$. Then there exists an $(\mathbb{M}_i \restriction \alpha)$-name $\dot{\mathbb{N}}_\alpha$ for a poset so that $\mathbb{M}_i$ is isomorphic to a dense subset of $(\mathbb{M}_i \restriction \alpha) * \dot{\mathbb{N}}_\alpha$ and so that in the extension by $\mathbb{M}_i \restriction \alpha$, $\mathbb{N}_\alpha$ is a forcing projection of a product $\mathbb{A} \times \mathbb{B}$, where $\mathbb{A}$ is ω_1-Knaster and where $\mathbb{B}$ is ω_1-closed.*

For the details, we refer the reader to [24].

5.3 Properness of Mitchell-Type Posets

In this section we briefly review the definition of a proper poset and show that the Mitchell-type posets surveyed in the previous section are proper.

The idea of a proper poset was invented by Shelah. Properness is a generalization of the properties of the countable chain condition and of countable closure, and it guarantees that the cardinal ω_1 is preserved. Moreover, this property is sufficiently robust that it is preserved under countable support iterations (see [42]). Let's now review the definitions.

Definition 5.3.1. *Let $\mathbb{P}$ be a poset and M a set. A condition $p \in \mathbb{P}$ is said to be an* $(M, \mathbb{P})$-generic condition *if p forces that $\dot{G} \cap M$ meets every dense $D \subseteq \mathbb{P}$ with $D \in M$.*

p is said to be a completely $(M, \mathbb{P})$-generic condition *if the cone of weaker conditions than p, i.e., $\{s \in \mathbb{P} : p \leq s\}$, is an $(M, \mathbb{P})$-generic filter.*

A poset $\mathbb{P}$ is said to be proper *if for all sufficiently large, regular θ there exists a club C in $P_{\omega_1}(\theta)$ so that for every $M \in C$ and $p \in \mathbb{P} \cap M$, there exists $p^* \leq p$ which is an $(M, \mathbb{P})$-generic condition.*

A key fact about proper posets is the following:

Proposition 5.3.2. *Suppose that $\mathbb{P}$ is proper. Then $\mathbb{P}$ preserves the stationarity of stationary sets of countable cofinality points.*

We also have that properness is preserved under projections. We have not seen this lemma in the literature ourselves, but as it is straightforward, we assume that we are not the original discoverers of it.

Lemma 5.3.3. *Suppose that $\pi : \mathbb{P} \longrightarrow \mathbb{Q}$ is a projection, and $\pi, \mathbb{P}, \mathbb{Q}$ are in an elementary submodel M. Then if p is an $(M, \mathbb{P})$-generic condition, $\pi(p)$ is an $(M, \mathbb{Q})$-generic condition. In particular, if $\mathbb{P}$ is proper, then $\mathbb{Q}$ is proper.*

Proof. Fix a condition p and a model M as in the statement of the lemma. First we prove a preliminary claim:

Claim: suppose $D \in M$ is a dense open subset of $\mathbb{Q}$. Then $D_0 := \pi^{-1}(D)$ is a dense open subset of $\mathbb{P}$ and $D_0 \in M$.

Proof. Since M is elementary, $D_0 \in M$. For density, fix a condition $p_0 \in \mathbb{P}$, and let $s \leq_{\mathbb{Q}} \pi(p_0)$ s.t. $s \in D$. Since π is a projection, fix $p_1 \leq_{\mathbb{P}} p_0$ s.t. $\pi(p_1) \leq_{\mathbb{Q}} s$. Since $s \in D$ is open, $\pi(p_1) \in D$ too. Thus $p_1 \in D_0$ is below p_0. $\square$

With the claim aside, we now argue by contradiction, in order to show that $\pi(p)$ is an $(M, \mathbb{Q})$-generic condition. Suppose, then, that there is a condition $q \leq_{\mathbb{Q}} \pi(p)$ and a dense open subset D of $\mathbb{Q}$ with $D \in M$ s.t.

$$(\dagger) \;\; q \Vdash D \cap M \cap \dot{G}_{\mathbb{Q}} = \varnothing.$$

Since π is a projection, let $\tilde{p} \leq_{\mathbb{P}} p$ s.t. $\pi(\tilde{p}) \leq_{\mathbb{Q}} q$.

Fix a V-generic G over $\mathbb{P}$ with $\tilde{p} \in G$, and let H be the π-induced generic for $\mathbb{Q}$ (i.e., the upwards closure of $\pi[G]$). Now since $\tilde{p} \in G$, we have $\pi(\tilde{p})$ and hence q are in H. Moreover, $\tilde{p}$ is an $(M, \mathbb{P})$-generic condition, as it is below the $(M, \mathbb{P})$-generic condition p. Thus since $D_0 \in M$ is a dense subset of $\mathbb{P}$, we know that $D_0 \cap M \cap G \neq \varnothing$, and consequently we may fix a condition s in this intersection. Now $s \in M \cap D_0$ implies that $\pi(s)$ is in $M \cap D$. Moreover, since $s \in G$, $\pi(s) \in H$. Thus

$$M \cap D \cap H \neq \varnothing$$

as witnessed by $\pi(s)$. However, this contradicts the fact that $(\dagger)$ holds and $q \in H$. $\square$

Corollary 5.3.4. *The posets $\mathbb{M}_i$ are proper, and if $\alpha < \kappa$ is inaccessible and $\bar{G}$ is V-generic for $\mathbb{M}_i \restriction \alpha$, then the tail $\mathbb{M}_i/\bar{G}$ is proper in $V[\bar{G}]$.*

We also collect one final lemma here which connects generic conditions to lifting embeddings (hence why generic conditions are often called "master conditions").

Lemma 5.3.5. *Suppose that M is an elementary submodel, $\mathbb{P} \in M$ a poset, and p an $(M, \mathbb{P})$-generic condition. Let $\pi_M : M \longrightarrow \bar{M}$ be the transitive collapse map. Then p forces that π_M lifts to an extension $\pi_M : M[G] \longrightarrow \bar{M}[\pi_M[G]]$.*

CHAPTER 6

$\neg\mathsf{TP}(\omega_2) + \neg\mathsf{AP}_{\omega_1} + \mathsf{SR}(\omega_2)$ from Optimal Assumptions

In this chapter, we answer Question (1) of [24], providing our original proof. In the first section, we will review the proof from [24] that forcing with $\mathbb{M}_1$ (see Definition 5.2.2) provides a model in which AP_{ω_1} fails; we will slightly rephrase their argument in order to be compatible with what we do afterwards. In the second section of this chapter, we will show how to iterate club adding in the $\mathbb{M}_1$-extension, in particular showing that this forcing preserves the failure of AP_{ω_1}. In the final section, we present an Easton-style lemma for preserving stationary sets which is due to the author and Omer Ben-Neria. It is not related to the arguments in sections 1 and 2 of this chapter.

Let us fix a Mahlo cardinal κ which is not weakly compact for the remainder of the chapter, and let us abbreviate $\mathbb{M}_1$ by $\mathbb{M}$.

6.1 The Preparatory Forcing and $\neg\mathsf{AP}_{\omega_1}$

In this section we show that in the extension by $\mathbb{M}$, $\kappa \notin I[\kappa]$. Recall from Proposition 5.2.3 that in $V[\mathbb{M}]$, $\kappa = 2^\omega = \omega_2$. Thus we may fix an enumeration $\vec{a} = \langle a_i : i < \kappa \rangle$ of all countable subsets of κ in $V[\mathbb{M}]$; this enumeration will be fixed for the remainder of the chapter.

In order to show that the failure of approachability is preserved after forcing with our subsequent club adding poset, we will need to phrase the argument in terms of certain useful models which appeared in [39]. We begin our discussion with their definition:

Definition 6.1.1. *A* rich model *is any elementary submodel N of some large enough $H(\theta)$ with the following properties:*

1. $\kappa_N := \sup(N \cap \kappa)$ *is an inaccessible cardinal below* κ, $\kappa_N \subseteq N$, *and* $|N| = \kappa_N$;

2. N *is closed under sequences of length* $< \kappa_N$;

3. the following parameters are in $N : \mathbb{M}$, $\dot{\mathbb{C}}$ *(our* $\mathbb{M}$*-name for the club-adding iteration, defined in the next section), and a fixed* $\mathbb{M}$*-name for* $\vec{a}$.

It is routine to see that since κ is Mahlo, for any large enough regular θ, there are stationarily-many rich models in $P_\kappa(H(\theta))$. Moreover, we also see that

$$T^* := \{\kappa_N : N \text{ is a rich model}\}$$

is a stationary subset of κ.

If N is a rich model, we will use $\bar{N}$ to denote the transitive collapse of N and π_N to denote the transitive collapse map from N to $\bar{N}$. The following lemma will be helpful later:

Lemma 6.1.2. *Suppose that* N *is a rich model and* $\bar{G}$ *is* V*-generic for* $\mathbb{M}_{\kappa_N}$. *Then* $\bar{N}[\bar{G}]$ *is closed under* $< \kappa_N$*-sequences from* $V[\bar{G}]$.

Proof. By definition of a rich model, N is closed under $< \kappa_N$-sequences from V, and hence so is $\bar{N}$. Moreover, $\pi_N(\mathbb{M}) = \mathbb{M}_{\kappa_N}$, and hence $\mathbb{M}_{\kappa_N} \in \bar{N}$. Since $\mathbb{M}_{\kappa_N}$ is κ_N-c.c., by Proposition 5.2.3, and since $\bar{N}$ is closed under $< \kappa_N$-sequences from V, standard arguments show that $\bar{N}[\bar{G}]$ is closed under $< \kappa_N$-sequences from $V[\bar{G}]$. □

We remark here that if G is V-generic over $\mathbb{M}$ and N is rich, then $\bar{G} := G \cap \mathbb{M}_{\kappa_N}$ is V-generic over $\mathbb{M}_{\kappa_N}$ and that $\pi_N^{-1}[\bar{G}] = \bar{G} \subseteq G$. Thus we may lift π_N to an isomorphism (also denoted π_N) $\pi_N : N[G] \longrightarrow \bar{N}[\bar{G}]$, noting that this is also the transitive collapse of $N[G]$. We will continue to use π_N to denote the original transitive collapse as well as the lifted map without comment.

Lemma 6.1.3. *If* N *is a rich model, then in* $V[\mathbb{M}_{\kappa_N}]$, $\vec{a} \restriction \kappa_N$ *enumerates* $[\kappa_N]^{\aleph_0}$.

Proof. Fix a rich model N as well as a V-generic G for $\mathbb{M}$, letting $\bar{G} := G \cap \mathbb{M}_{\kappa_N}$. Observe that by the elementarity of $N[G]$, $\vec{a} \restriction \kappa_N$ enumerates all countable subsets of κ_N which lie in $N[G]$. Since $\vec{a} \restriction \kappa_N = \pi_N(\vec{a}) \in \bar{N}[\bar{G}]$, we know that every a_i for $i < \kappa_N$ is a countable subset of κ_N in $\bar{N}[\bar{G}]$, and hence in $V[\bar{G}]$.

Thus we need to see that every countable subset of κ_N in $V[\bar{G}]$ appears on the sequence $\vec{a} \restriction \kappa_N$. By Lemma 6.1.2, we know that $\bar{N}[\bar{G}]$ is closed under $< \kappa_N$-sequences from $V[\bar{G}]$, and in particular, all countable subsets of κ_N in $V[\bar{G}]$ are elements of $\bar{N}[\bar{G}]$. Since π_N is the identity on bounded subsets of κ_N, we then conclude that every element of $([\kappa_N]^{\aleph_0})^{V[\bar{G}]}$ is in $N[G]$, and hence by elementarity appears on the sequence $\vec{a} \restriction \kappa_N$. $\square$

In order to show that $\kappa \notin I[\kappa]^{V[\mathbb{M}]}$, it suffices to show that there is a stationary set of points that all fail to be approachable w.r.t. $\vec{a}$. Indeed, if $\vec{b} = \langle b_i : i < \kappa \rangle$ is any other sequence of countable subsets of κ in $V[\mathbb{M}]$, then because $\vec{a}$ enumerates *all* countable subsets of κ in $V[\mathbb{M}]$, there is a club of points ν so that $\{b_i : i < \nu\} \subseteq \{a_i : i < \nu\}$.

The following standard branch lemmas, which are stated in less-than-full generality, will be crucial in showing that κ fails to be approachable.

Lemma 6.1.4. (Branch Lemmas) *Suppose that T is a tree of height δ, where δ has cofinality at least ω_1.*

1. (Silver) *If the levels of T have size less than 2^ω and $\mathbb{P}$ is an ω_1-closed poset, then $\mathbb{P}$ adds no new cofinal branches through T.*

2. (Unger) *If $\mathbb{P}$ is a forcing whose square $\mathbb{P} \times \mathbb{P}$ is c.c.c., then $\mathbb{P}$ adds no new cofinal branches through T.*

The following is essentially the same proof as in [24], rephrased in the context of rich models; working in the context of rich models will be useful later when we show that κ still fails to be approachable after our later club adding.

Proposition 6.1.5. *No ordinal $\alpha \in T^*$ is approachable w.r.t. $\vec{a}$ in $V[\mathbb{M}]$.*

Proof. Suppose otherwise, and fix an ordinal $\alpha \in T^*$ which is approachable w.r.t. $\vec{a}$ in $V[\mathbb{M}]$. By definition of T^*, we may fix a rich model N s.t. $\kappa_N = \alpha$. In what follows, we use α^* to denote the least V-inaccessible cardinal above α. Since κ_N is approachable w.r.t. $\vec{a}$ in $V[\mathbb{M}]$, there is an $E \subseteq \kappa_N$ of order-type ω_1 s.t.

$$\{E \cap \eta : \eta < \kappa_N\} \subseteq \{a_i : i < \kappa_N\}.$$

By Proposition 5.2.3, in the model $V[\mathbb{M}_{\alpha^*}]$, we have the following:

$$\kappa_N = \omega_2 < 2^\omega = \alpha^*.$$

In particular, E is not in $V[\mathbb{M}_{\alpha^*}]$, where κ_N is regular.

Next consider the tree

$$U := (2^{<\kappa_N})^{V[\mathbb{M}_{\kappa_N}]},$$

noting that U has width $(2^{<\kappa_N})^{V[\mathbb{M}_{\kappa_N}]} = \kappa_N$. By our assumption about E, if $\eta < \kappa_N$, then $E \cap \eta$ equals a_i for some $i < \kappa_N$. Since $\vec{a} \restriction \kappa_N \in V[\mathbb{M}_{\kappa_N}]$, it follows that $E \cap \eta$ is a member of $V[\mathbb{M}_{\kappa_N}]$. Consequently, the characteristic function of $E \cap \eta$,

$$c_{E \cap \eta} : \eta \longrightarrow 2,$$

is an element of the tree U. Since E is cofinal in κ_N, we conclude that the characteristic function of E is a cofinal branch through U, which exists in the model $V[\mathbb{M}]$. We will show that this is impossible.

As remarked earlier, $E \notin V[\mathbb{M}_{\alpha^*}]$, and therefore E is added by forcing with the poset $\mathbb{N}_{\alpha^*}$ (see Proposition 5.2.7). However, in the model $V[\mathbb{M}_{\alpha^*}]$, $\mathbb{N}_{\alpha^*}$ is a projection of $\mathbb{A} \times \mathbb{B}$, where the first is ω_1-Knaster and where the second is ω_1-closed. Since U has width $\kappa_N < (2^\omega)^{V[\mathbb{M}_{\alpha^*}]}$, we know from Lemma 6.1.4(1) that forcing with $\mathbb{B}$ does not add any new cofinal branches to U. Further forcing with $\mathbb{A}$ also fails to add cofinal branches through U, and since $\mathbb{N}_{\alpha^*}$ is a forcing projection of $\mathbb{A} \times \mathbb{B}$, we have that U does not have any new cofinal branches in $V[\mathbb{M}]$. However, $E \in V[\mathbb{M}]$, and so the characteristic function of E is a new, cofinal branch through U in $V[\mathbb{M}]$, a contradiction.

$\square$

Corollary 6.1.6. $T^* \notin I[\kappa]^{V[\mathbb{M}]}$.

Proof. This follows by the previous proposition and the fact that $\mathbb{M}$ preserves the stationarity of T^*, as $\mathbb{M}$ is κ-c.c. $\square$

6.2 Iterated Club Adding and the Failure of Approachability

In this section, we will construct the final model. We first define, in the extension by $\mathbb{M}$, an iteration of club adding with the intention of destroying the stationarity of any nonreflecting stationary subset S of $\kappa \cap \text{cof}(\omega)$. This is similar to the arguments from [39]. In the definition that follows, we make use of a tacit bookkeeping function to select names.

In $V[\mathbb{M}]$, let

$$\mathbb{C} = \langle \mathbb{C}_\alpha, \dot{\mathbb{R}}_\alpha : \alpha < \kappa^+ \rangle$$

denote the iteration with $< \kappa$-support, defined recursively as follows. Suppose that $\alpha < \kappa^+$ and that $\mathbb{C}_\alpha$ is defined. Let $\dot{S}_\alpha$ denote the next name for a non-reflecting stationary subset of $\kappa \cap \text{cof}(\omega)$, and let $\dot{\mathbb{R}}_\alpha$ denote the $\mathbb{C}_\alpha$-name for the poset of all closed, bounded subsets of κ in[1] $V[\mathbb{M}]$ which are disjoint from $\dot{S}_\alpha$, ordered by end-extension. We set $\mathbb{C}_{\alpha+1} := \mathbb{C}_\alpha * \dot{\mathbb{R}}_\alpha$, and we set $\mathbb{C} := \mathbb{C}_{\kappa^+}$.

Lemma 6.2.1. *$\mathbb{C}$ is κ^+-c.c. in $V[\mathbb{M}]$.*

Proof. For each $\alpha < \kappa^+$, $\dot{\mathbb{R}}_\alpha$ is forced to have size $(\kappa^{<\kappa})^{V[\mathbb{M}]} = \kappa$; hence $\dot{\mathbb{R}}_\alpha$ is trivially forced to be κ^+-c.c. Moreover, as the iteration $\mathbb{C}$ takes direct limits on the (stationary) set $\kappa^+ \cap \text{cof}(\kappa)$, we then have by standard arguments (for instance, see [12]) that $\mathbb{C}$ is κ^+-c.c. $\square$

We have two main tasks to complete in order to obtain the desired model:

(A) $\mathbb{C}$ is κ-distributive;

(B) $\kappa \notin I[\kappa]^{V[\mathbb{M} * \dot{\mathbb{C}}]}$.

[1] We'll show later that $\mathbb{C}$ is κ-distributive, and hence in hindsight, this requirement is unnecessary. However, it makes it easier to see immediately that $\mathbb{C}$ is κ^+-c.c. if we specifically require this.

(A) entails that $\mathbb{C}$ preserves ω_1 and κ, and therefore standard bookkeeping arguments and the fact that $\mathbb{C}$ is κ^+-c.c. imply that we indeed obtain a model of RP(κ). (B) entails that we haven't added any new clubs witnessing that κ is approachable. We will begin with (A), once we state the following simple, though necessary, lemma:

Lemma 6.2.2. *Let $\alpha < \kappa^+$, and let N be a rich model with $\alpha \in N$. Then in $V[\mathbb{M}_{\kappa_N}]$, $\pi_N(\mathbb{C}_\alpha)$ is an iteration with $< \kappa_N$ support adding clubs through the complements of $\langle \pi_N(\dot{S}_\beta) : \beta \in N \cap \alpha \rangle$.*

Proof. Let $\bar{G}$ be V-generic for $\mathbb{M}_{\kappa_N}$. Since $\bar{N}[\bar{G}]$ is closed under $< \kappa_N$-sequences from $V[\bar{G}]$ (by Lemma 6.1.2), the result follows by a straightforward absoluteness argument (see [32] for further details). $\square$

We will prove that $\mathbb{C}$ is κ-distributive by means of an induction on the following, where $\alpha \leq \kappa^+$:

> $(IH)_\alpha$: for all $\xi < \alpha$, $\mathbb{C}_\xi$ is κ-distributive, and if N is rich and $\alpha \in N$, then for all $\beta \in N \cap \alpha$, $\pi_N(\mathbb{C}_\beta)$ forces over $V[\mathbb{M}_{\kappa_N}]$ that $\pi_N(\dot{S}_\beta)$ is a nonstationary subset of κ_N.

What the induction hypothesis $(IH)_\alpha$ says beyond Lemma 6.2.2 is that from the perspective of $V[\mathbb{M}_{\kappa_N}]$, $\pi_N(\mathbb{C}_\alpha)$ is an iteration with $< \kappa_N$ support adding clubs through the complements of *non*stationary sets. Note that $(IH)_0$ is trivial, and that if α is a limit and $(IH)_\beta$ holds for all $\beta < \alpha$, then $(IH)_\alpha$ holds. Thus we'll only need to show that $(IH)_\alpha$ implies that $(IH)_{\alpha+1}$.

We therefore will assume that $(IH)_\alpha$ holds for the rest of the section, turning our attention to drawing out the implications of $(IH)_\alpha$. Part of our analysis here will also be used to show that (B) holds.

Lemma 6.2.3. *Let N be a rich model with $\alpha \in N$. Then $\pi_N(\mathbb{C}_\alpha)$ contains a κ_N-closed, dense subset in $V[\mathbb{M}_{\kappa_N}]$.*

Proof. Fix $\beta \in N \cap \alpha$, so that by our inductive assumption, $\pi_N(\mathbb{C}_\beta) \Vdash \pi_N(\dot{S}_\beta)$ is nonstationary. Let $\dot{C}_{\pi_N(\beta)}$ be a $\pi_N(\mathbb{C}_\beta)$-name for a club subset of κ_N disjoint from $\pi_N(\dot{S}_\beta)$, and let $\dot{D}_\beta$ name the set of conditions c in $\pi_N(\dot{\mathbb{R}}_\beta)$ such that $\max(c) \in \dot{C}_{\pi_N(\beta)}$. Then it is straightforward to check that $\dot{D}_\beta$ is forced to be a κ_N-closed, dense subset of $\dot{\mathbb{R}}_\beta$. Since this holds for all $\beta \in N \cap \alpha$, the result follows. $\square$

Proposition 6.2.4. *For any rich N with $\alpha \in N$ and any $p \in \mathbb{C}_\alpha \cap N[G]$, there is $q \leq_{\mathbb{C}_\alpha} p$ which is an $(N[G], \mathbb{C}_\alpha)$-completely generic condition.*

Proof. Fix a V-generic G for $\mathbb{M}$, and let $\bar{G}$ denote the restriction of G to $\mathbb{M}_{\kappa_N}$. From Lemma 6.2.3, we know that in $V[\bar{G}]$, $\pi_N(\mathbb{C}_\alpha)$ contains a κ_N-closed, dense subset. Furthermore, by definition of a rich model, we also know that $|\bar{N}[\bar{G}]| = \kappa_N$, and therefore in $V[\bar{G}]$, we may construct an $(\bar{N}[\bar{G}], \pi_N(\mathbb{C}_\alpha))$-generic filter K. Now define

$$\tilde{K} := \pi_N^{-1}[K],$$

so that $\tilde{K} \in V[G]$. We claim that $\tilde{K}$ has a lower bound q (a "flat condition"), which will be our desired condition. In what follows, as a matter of notation, if $s \in K$, we will denote $\pi_N^{-1}(s)$ by $\tilde{s}$. Indeed, for each $\xi \in N \cap \alpha$, set

$$q(\xi) := \bigcup \{\tilde{s}(\xi) : s \in K \ \wedge\ \xi \in \mathrm{dom}(\tilde{s})\} \cup \{\kappa_N\}.$$

If, by induction, $q \restriction \xi$ is a condition in $\mathbb{C}_\xi$ below $\tilde{s} \restriction \xi$ for each $\tilde{s} \in \tilde{K}$, then we see that $q \restriction \xi$ forces that $q(\xi) \in \dot{\mathbb{R}}_\xi$. Indeed, κ_N has cofinality ω_1 in $V[G]$, and since $\mathbb{C}_\xi$ is κ-distributive (by induction), $\mathbb{C}_\xi$ preserves the cofinality of κ_N. Thus we see that $q(\xi)$ is forced by $\mathbb{C}_\xi$ to be disjoint from $\dot{S}_\xi$.

We now verify that q is an $(N[G], \mathbb{C}_\alpha)$-completely generic condition: let $D \in N[G]$ be a dense subset of $\mathbb{C}_\alpha$. Then $\pi_N(D)$ is dense in $\pi_N(\mathbb{C}_\alpha)$ and a member of $\bar{N}[\bar{G}]$, and so there is a condition $s \in \pi_N(D) \cap K$. Then $\tilde{s} \in D \cap \tilde{K}$, and so $q \leq \tilde{s}$, as required. $\square$

Remark 6.2.5. *The proof of Proposition 6.2.4 also shows that if K is* any *$(\bar{N}[\bar{G}], \pi_N(\mathbb{C}_\alpha))$-generic filter in $V[G]$, then $\pi_N^{-1}[K]$ has a lower bound in $\mathbb{C}_\alpha$, and any such lower bound is an $(N[G], \mathbb{C}_\alpha)$-completely generic condition.*

Corollary 6.2.6. *$\mathbb{C}_\alpha$ is κ-distributive in $V[\mathbb{M}]$.*

Proof. Let $\dot{f}$ be a $\mathbb{C}_\alpha$-name for a function from ω_1 into the ordinals, and fix a condition $p \in \mathbb{C}_\alpha$. Let N be a rich model so that α, as well as $\mathbb{M}$-names for $\dot{f}$ and p, are in N. By Proposition 6.2.4, we can find an $(N[G], \mathbb{C}_\alpha)$-completely generic condition $q \leq p$. Since $\dot{f} \in N[G]$, the elementarity of $N[G]$ implies that for each $\nu < \omega_1$, the dense set D_ν of all conditions in $\mathbb{C}_\alpha$ which decide the value of $\dot{f}(\nu)$ is also a member of $N[G]$. Since q is completely generic, for each $\nu < \omega_1$, there is some condition p_ν with $q \leq p_\nu$ s.t. $p_\nu \in D_\nu$. Hence $q \in D_\nu$ too. Since this holds for all $\nu < \omega_1$, we have that q completely determines $\dot{f}$. $\square$

In particular, we now know that $\mathbb{C}_\alpha$ preserves ω_1 and κ. It remains to see that $(IH)_\alpha$ implies $(IH)_{\alpha+1}$; this is where we use the careful definition of $\mathbb{M}$.

Proposition 6.2.7. *Let N be a rich model with $\alpha \in N$. Then $\pi_N(\mathbb{C}_\alpha)$ forces over $V[\mathbb{M}_{\kappa_N}]$ that $\pi_N(\dot{S}_\alpha)$ is nonstationary.*

Proof. Let $\bar{G}$ be V-generic for $\mathbb{M}_{\kappa_N}$, and let H be $V[\bar{G}]$-generic over $\pi_N(\mathbb{C}_\alpha)$. We need to show that $\pi_N(\dot{S}_\alpha)[H]$ is a nonstationary subset of κ_N in $V[\bar{G} * H]$. By $(IH)_\alpha$, we know that $\pi_N(\mathbb{C}_\alpha)$ contains a κ_N-closed, dense subset, and moreover $\pi_N(\mathbb{C}_\alpha)$ has size κ_N. Since every condition in $\pi_N(\mathbb{C}_\alpha)$ has κ_N-many pairwise incompatible extensions, we conclude by standard forcing facts that $\pi_N(\mathbb{C}_\alpha)$ has a dense subset which is isomorphic to $\mathrm{Add}(\kappa_N, 1)$. We will abuse notation and use H to also denote the isomorphic generic for $\mathrm{Add}(\kappa_N, 1)$ under this isomorphism.

Let I be $V[\bar{G} * H]$-generic over $\mathbb{M}/(\bar{G} * H)$, and let $G := \bar{G} * H * I$, so that G is a V-generic filter over $\mathbb{M}$. Lift $\pi_N : N \longrightarrow \bar{N}$ to an extension $\pi_N : N[G] \longrightarrow \bar{N}[\bar{G}]$, and observe by Remark 6.2.5 that $\pi_N^{-1}[H]$ has a lower bound q in $\mathbb{C}_\alpha$, which is also an $(N[G], \mathbb{C}_\alpha)$-completely generic condition. Finally, we let J be $V[G]$-generic over $\mathbb{C}_\alpha$ containing q.

Now by definition of $\dot{S}_\alpha$, in $V[G*J]$ we have that $S_\alpha := \dot{S}_\alpha[J]$ is a nonreflecting stationary subset of κ. Since κ_N has cofinality ω_1 in $V[G]$ and $\mathbb{C}_\alpha$ is κ-distributive, κ_N still has cofinality

ω_1 in $V[G*J]$. Hence, $S \cap \kappa_N$ is a nonstationary subset of κ_N in the model $V[G*J]$. Again by the distributivity of $\mathbb{C}_\alpha$, $S_\alpha \cap \kappa_N$ is in $V[G]$ and is nonstationary there.

To finish, we will argue that $S_\alpha \cap \kappa_N$ is an element of $V[\bar{G}*H]$ and is nonstationary in that model.

Since $q \in J$ is a lower bound for $\pi_N^{-1}[H]$, we may lift π_N^{-1} to an isomorphism π_N^{-1} : $\bar{N}[\bar{G}*H] \longrightarrow N[G*J]$. By the elementarity of π_N, we see that $\pi_N(\dot{S}_\alpha)[H] = S_\alpha \cap \kappa_N$. In particular, $S_\alpha \cap \kappa_N$ is a member of $V[\bar{G}*H]$. By Corollary 5.3.4 we know that the poset $\mathbb{M}/(\bar{G}*H)$, being isomorphic to a dense subset of $\mathrm{Add}(\omega, \kappa_N^*) * \dot{\mathbb{N}}_{\kappa_N^*}$, is proper in the model $V[\bar{G}*H]$ and hence preserves the stationarity of stationary sets of cofinality ω ordinals. Since $S_\alpha \cap \kappa_N$ consists of points of cofinality ω and is nonstationary in $V[G]$, we conclude that $S_\alpha \cap \kappa_N$ is nonstationary in $V[\bar{G}*H]$, which is what we intended to show. $\square$

This completes the proof that $\mathsf{SR}(\omega_2)$ holds in $V[\mathbb{M}*\dot{\mathbb{C}}]$. We now finish by showing that the failure of approachability is preserved by the forcing $\mathbb{C}$.

Proposition 6.2.8. $T^* \notin I[\kappa]^{V[\mathbb{M}*\dot{\mathbb{C}}]}$.

Proof. We begin by noting that since $\mathbb{C}$ is κ-distributive over $V[\mathbb{M}]$, the sequence $\vec{a}$ in $V[\mathbb{M}]$ which enumerates $[\kappa]^{\aleph_0}$ is still an enumeration of all countable subsets of κ in $V[\mathbb{M}*\dot{\mathbb{C}}]$. The κ-distributivity of $\mathbb{C}$ further implies that no $\alpha \in T^*$ is approachable with respect to $\vec{a}$ in $V[\mathbb{M}*\dot{\mathbb{C}}]$, since this holds in $V[\mathbb{M}]$.

In order to finish showing that T^* is not in the approachability ideal, we need to verify that T^* remains stationary after forcing with $\mathbb{C}$.

To see this, let G be V-generic over $\mathbb{M}$, let $\dot{F}$ be a $\mathbb{C}$-name in $V[\mathbb{M}]$ for a club subset of κ, and let $p \in \mathbb{C}$ be a condition. Since $\mathbb{C}$ is κ^+-c.c., we may fix some α so that $\dot{F}$ is a $\mathbb{C}_\alpha$-name for a club subset of κ. Next, fix a rich model N so that $\mathbb{M}$-names for $\mathbb{C}_\alpha$, $\dot{F}$, and p are in N. By Proposition 6.2.4, we may build an $(N[G], \mathbb{C}_\alpha)$-completely generic condition $q \leq p$. By standard arguments, we see that

$$q \Vdash \kappa_N \in \dot{F},$$

and since $\kappa_N \in T^*$, this finishes the proof. $\square$

We may now wrap up the proof of the theorem.

Theorem 6.2.9. *It is consistent from a Mahlo cardinal that* $\neg\mathsf{TP}(\omega_2) + \neg\mathsf{AP}_{\omega_1} + \mathsf{SR}(\omega_2)$ *holds.*

Proof. We force with $\mathbb{M} * \dot{\mathbb{C}}$ (recall that $\mathbb{M} = \mathbb{M}_1$ from the previous chapter). We know that in the extension by this forcing, ω_1 is preserved, $\kappa = \omega_2$, and all cardinals above κ are preserved. AP_{ω_1} fails after forcing with $\mathbb{M}$, and by Proposition 6.2.8, $\mathbb{C}$ preserves the failure of AP_{ω_1}. Standard bookkeeping arguments show that $\mathbb{C}$ forces $\mathsf{SR}(\omega_2)$. And finally, since κ is not weakly compact, in the extension by $\mathbb{M} * \dot{\mathbb{C}}$, we must have the failure of $\mathsf{TP}(\omega_2)$. $\square$

6.3 An Easton-style Lemma for Preserving Stationary Sets

In this section, we present an Easton-style lemma for preserving stationary subsets of countable cofinality; this result is due to the author and Omer Ben-Neria. Recall that Easton's Lemma says that if $\mathbb{P}$ is a κ-c.c. poset (where κ is a regular, uncountable cardinal) and $\mathbb{Q}$ is a κ-closed poset, then $\mathbb{P}$ remains κ-c.c. after forcing with $\mathbb{Q}$, and $\mathbb{Q}$ is κ-distributive after forcing with $\mathbb{P}$.

In [78], Unger presents the following extraordinarily useful (for instance, see [62]) branch lemma about trees which has a similar flavor to Easton's Lemma.

Theorem 6.3.1. (Unger) *Suppose that κ and λ are regular cardinals so that $2^\kappa \geq \lambda$. Let $\mathbb{P}$ be κ^+-c.c. and $\mathbb{R}$ be κ^+-closed. Let $\dot{T}$ be a $\mathbb{P}$-name for a λ-tree. Then in the extension by $\mathbb{P}$, $\mathbb{R}$ cannot add a branch through T.*

The point of the theorem is that $\mathbb{R}$ is no longer κ^+-closed after forcing with $\mathbb{P}$ (at least if $\mathbb{P}$ is a non-trivial poset), so the standard branch lemmas don't apply to $\mathbb{R}$ in the $\mathbb{P}$-extension.

We present a lemma with a similar flavor, but which is related to preserving stationarity, rather than not adding branches.

Theorem 6.3.2. (Gilton, Ben-Neria) *Suppose that in V, $\mathbb{P}$ is c.c.c. and $\mathbb{Q}$ is ω_1-closed. Let κ be a regular, uncountable cardinal, and let $\dot{S}$ be a $\mathbb{P}$-name for a stationary subset of $\kappa \cap \mathrm{cof}(\omega)$. Then $\mathbb{P} \times \mathbb{Q}$ forces that $\dot{S}$ is stationary.*

Proof. We begin with the following useful claim.

Claim Let $\dot{\alpha}$ be a $\mathbb{P} \times \mathbb{Q}$-name for an ordinal, and let $q \in \mathbb{Q}$ be a condition. Then there exist an extension $r \leq_{\mathbb{Q}} q$ and a $\mathbb{P}$-name for an ordinal $\dot{\alpha}_{\mathbb{P}}$ so that

$$(0, r) \Vdash_{\mathbb{P} \times \mathbb{Q}} \dot{\alpha}_{\mathbb{P}} = \dot{\alpha}.$$

Proof. First let H be a V-generic filter for $\mathbb{Q}$ so that $q \in H$. By Easton's Lemma, we know that $\mathbb{P}$ is still c.c.c. in $V[H]$. Therefore there exists a maximal antichain $A = \{p_n : n < \omega\}$ of conditions in $\mathbb{P}$ so that for each $n < \omega$ there is an ordinal β_n so that

$$p_n \Vdash_{\mathbb{P}}^{V[H]} \check{\beta}_N = \dot{\alpha}.$$

Now as $\mathbb{Q}$ is ω_1-closed, we may find an extension r of q with $r \in H$ so that for each $n < \omega$,

$$(\dagger)\ r \Vdash^V \left(p_n \Vdash^{V[\dot{H}]} \check{\beta}_n = \dot{\alpha}\right).$$

Now let $\dot{\alpha}_{\mathbb{P}}$ be the $\mathbb{P}$-name $\left\{(\check{\beta}_n, p_n) : n < \omega\right\}$. We will show that

$$(0, r) \Vdash_{\mathbb{P} \times \mathbb{Q}} \dot{\alpha}_{\mathbb{P}} = \dot{\alpha}.$$

Towards this end, let $G \times H'$ be V-generic for $\mathbb{P} \times \mathbb{Q}$ with $r \in H'$. Since $\{p_n : n < \omega\}$ is a maximal antichain in $\mathbb{P}$, there exists some $n < \omega$ so that $p_n \in G$. By ($\dagger$) above, we have that $\dot{\alpha}[G \times H'] = \beta_n$. However, we also have that $\dot{\alpha}_{\mathbb{P}}[G] = \beta_n$. Thus

$$\dot{\alpha}[G \times H'] = \beta_n = \dot{\alpha}_{\mathbb{P}}[G],$$

which completes the proof. □

We may now continue with our proof of the theorem. Suppose for a contradiction that there exists a $\mathbb{P} \times \mathbb{Q}$-name $\dot{C}$ for a club subset of κ and a condition $(p, q) \in \mathbb{P} \times \mathbb{Q}$ so that

(p, q) forces $\dot{C} \cap \dot{S} = \varnothing$. Now fix a large enough regular cardinal θ so that $H(\theta)$ contains all of the parameters of interest. We may also fix an extension p^* of p and an elementary submodel M of $H(\theta)$ so that p^* forces that $\sup(M \cap \kappa) \in \dot{S}$. (Indeed, let $\langle M_i : i < \kappa \rangle$ be a continuous, $\in$-increasing sequence of elementary submodels of $H(\theta)$ so that M_0 contains all parameters of interest and so that each M_i has size $< \kappa$. Let G be V-generic over $\mathbb{P}$ with $p \in G$, and consider the club $\{\sup(M_i \cap \kappa) : i < \kappa\}$. Since $S := \dot{S}[G]$ is stationary in κ, we may find some i so that $\sup(M_i \cap \kappa) \in S$; in particular, $\sup(M_i \cap \kappa)$ has countable cofinality in both $V[G]$ and V. Let $p^* \leq p$ so that $p^* \Vdash \sup(M_i \cap \kappa) \in \dot{S}$.)

Set $\delta_M := \sup(M \cap \kappa)$. Also fix a sequence $\langle \delta_n : n < \omega \rangle$ of elements of M which is cofinal in δ_M. For each $n < \omega$, we let $\dot{\tau}_n$ denote the $(\mathbb{P} \times \mathbb{Q})$-name for $\min(\dot{C} \backslash \delta_n)$. By repeatedly applying the elementarity of M and the previous claim, we now construct a decreasing sequence $\langle q_n : n < \omega \rangle$ of extensions of q as well as a sequence of $\mathbb{P}$-names $\langle \dot{\tau}_n^* : n < \omega \rangle$ so that $q_n, \dot{\tau}_n^* \in M$, and $(p^*, q_n) \Vdash_{\mathbb{P} \times \mathbb{Q}} \dot{\tau}_n^* = \dot{\tau}_n$. Let q^* be a lower bound for the q_n. We observe here that since $\dot{\tau}_n^* \in M$ is a $\mathbb{P}$-name for an ordinal and $\mathbb{P}$ is c.c.c., M contains the set of all possible $\mathbb{P}$-interpretations of this name as an element. Consequently, the sup of all such possible interpretations is an ordinal in M below κ. Consequently, $p^* \Vdash_{\mathbb{P}} \dot{\tau}_n^* < \delta_M$.

Now fix a V-generic filter $G \times H$ for $\mathbb{P} \times \mathbb{Q}$ which contains (p^*, q^*). We will show that $\delta_M \in S \cap C$, a contradiction. By the choice of p^*, we know that $\delta_M \in S$. Furthermore, we also have that $\tau_n := \dot{\tau}_n[G \times H] = \dot{\tau}_n^*[G]$ is below δ_M for all n, and therefore $\sup_n \tau_n \leq \delta_M$. By choice of the cofinal sequence $\langle \delta_n : n < \omega \rangle$, we conclude that $\sup_n \tau_n = \delta_M$. However, each τ_n is an element of C and so δ_M is also an element of C, by closure. Thus $\delta_M \in S \cap C$, a contradiction. $\square$

CHAPTER 7

The Eightfold Way and Simultaneous Stationary Reflection

In this chapter, we will show, from large cardinal hypotheses, that if Φ is any Boolean combination of the principles $\mathsf{TP}(\omega_2)$ and AP_{ω_1}, then Φ is consistent with $\mathsf{SR}(\omega_2)^*$. Note that for each such configuration, at least a weakly compact is needed, since $\mathsf{SR}(\omega_2)^*$ has the consistency strength of a weakly compact, as discussed in Chapter 5.

We first remark that the configuration $\neg\mathsf{TP}(\omega_2)+\mathsf{AP}_{\omega_1}+\mathsf{SR}(\omega_2)^*$ is satisfied in Magidor's original model [56], since the CH holds in that model; recall that the CH implies $\omega_2 \in I[\omega_2]$ and that the tree property fails.

For the other three possibilities for the Boolean combination Φ, we will work to incorporate Magidor's iterated club adding into the framework of [24]. We recall that a key to Magidor's result is what we might loosely call an "absorption-preservation" argument. After forcing with $\mathbb{P}$, the Levy collapse of a weakly compact κ to become ω_2, Magidor iterates club adding to ensure that the desired reflection obtains. To see that the iterated club adding poset $\mathbb{R}$ is κ-distributive, Magidor takes (roughly) a weakly compact embedding $j : M \longrightarrow N$, noting that by standard absorption arguments, $j(\mathbb{P})$ is isomorphic to $\mathbb{P} * \dot{\mathbb{R}} * j(\mathbb{P})^{\mathrm{tail}}$, where the tail is ω_1-closed. Thus the tail preserves the stationarity of the sets dealt with in the course of forcing with $\mathbb{R}$. Consequently, one can build a "flat" master condition for the embedding, thereby lifting the embedding; the elementarity of this lifted embedding then gives the needed result.

The most straightforward of the remaining configurations for Φ is $\mathsf{TP}(\omega_2) + \mathsf{AP}_{\omega_1} +$

$\mathsf{SR}(\omega_2)^*$, which is due to Gilton and Ben-Neria. To obtain this configuration, we force with a version $\mathbb{M}$ of the Mitchell poset from [24] which involves slightly more aggressive cardinal collapsing; we then iterate club adding in the extension by $\mathbb{M}$. If we take a weakly compact embedding $j : M \longrightarrow N$ as in the above paragraph, then at "stage" κ, $j(\mathbb{M})$ adds a generic for $\mathrm{Col}(\omega_1, \kappa^+)$, and this collapse acts to absorb the club adding poset; the quotient of $j(\mathbb{M})$ by this initial segment is proper and so preserves the stationarity of stationary sets of cofinality ω ordinals. This will allow us to see that $\mathbb{R}$ is κ-distributive. The tree property is preserved by branch lemmas. We will provide more details about the club adding in this section than in the subsequent two sections.

However, the configurations involving $\neg\mathsf{AP}_{\omega_1}$ are less straightforward. Part of the challenge here is that we need to be more patient with our collapsing, in order to ensure the failure of AP_{ω_1}. A practical corollary of this is that we weren't able to find a natural way of incorporating collapses into a Mitchell-style poset forcing $\neg\mathsf{AP}_{\omega_1}$ in such a way that on the j-side, for some weakly compact embedding $j : M \longrightarrow N$, we obtain the appropriate generics for the iterated club adding.

In order to circumvent this challenge, we use a significantly larger large cardinal assumption, namely that of a supercompact. We then, following Abraham's Tree Property paper ([1]), incorporate a "look-ahead" into the Mitchell-type poset by means of a Laver function, constructed from the supercompactness of the cardinal which is to become ω_2. This look-ahead allows us to argue that we can always construct an embedding j so that $j(\mathbb{M})$ (where $\mathbb{M}$ is the Mitchell-type poset) adds the desired generics for the iterated club adding with a tail forcing that is sufficiently "well-behaved."

Let us begin with the first configuration mentioned above.

7.1 $\mathsf{TP}(\omega_2) + \mathsf{AP}_{\omega_1} + \mathsf{SR}(\omega_2)^*$

Let κ be a weakly compact cardinal. We recall that the *trace* of a stationary set S is the set of $\alpha < \sup(S)$ at which S reflects; this is denoted by $\mathrm{tr}(S)$. Further recalling that A denotes the (stationary) set of inaccessible cardinals below κ, we define the relevant Mitchell-type poset $\mathbb{M}$ as follows:

Definition 7.1.1. *Let $\mathbb{M}$ be the poset where conditions are pairs (a, f) satisfying the following:*

1. *$a \in \mathrm{Add}(\omega, \kappa)$;*
2. *f is a partial function with $\mathrm{dom}(f) \subseteq A$ and $|\mathrm{dom}(f)| \leq \aleph_0$;*
3. *for each $\alpha \in \mathrm{dom}(f)$, $f(\alpha)$ is an $\mathrm{Add}(\omega, \alpha)$-name for a condition in $\mathrm{Col}(\omega_1, \alpha^+)$;*

The ordering is as in Definition 5.2.2.

It is straightforward to see that the relevant results from Section 5.2 hold for $\mathbb{M}$, such as Proposition 5.2.3(1,2), Proposition 5.2.5, and Proposition 5.2.7.

In the extension by $\mathbb{M}$, we define a $< \kappa$-support iteration $\mathbb{R} = \langle \mathbb{R}_\beta, \dot{\mathbb{C}}_\beta : \beta < \kappa^+ \rangle$ of posets as follows: suppose that $\beta < \kappa^+$ and that $\mathbb{R}_\beta$ is defined. Let $\dot{S}_\beta$ be the next $\mathbb{R}_\beta$-name for a stationary subset of $\kappa \cap \mathrm{cof}(\omega)$, and set $\dot{\mathbb{C}}_\beta$ to be the $\mathbb{R}_\beta$-name for the poset to add an ω_1-club subset of $\mathrm{tr}(\dot{S}_\beta)$ by initial segments which come from $V[\mathbb{M}]$, ordered by end-extension.

We note first that each $\mathbb{R}_\beta$, for $\beta \leq \kappa^+$, is trivially ω_1-closed, since the trace of a stationary set doesn't contain any points of cofinality ω. We also may see that $\mathbb{R}$ is κ^+-c.c. Indeed, we recall that in the extension by $\mathbb{M}$, we have $\kappa^{<\kappa} = \kappa$. Thus for each $\beta < \kappa^+$, $\dot{\mathbb{C}}_\beta$ is forced to have size κ, and so $\dot{\mathbb{C}}_\beta$ is forced to trivially satisfy the κ^+-c.c. Since inverse limits are taken at each stage in $\kappa^+ \cap \mathrm{cof}(\kappa)$, standard arguments (see [12]) imply that $\mathbb{R}$ is κ^+-c.c. From this it follows that we may catch our tail and ensure that every stationary subset S of $\kappa \cap \mathrm{cof}(\omega)$ is dealt with at some stage below κ^+; similarly, every κ-tree T in the extension by $\mathbb{M} * \dot{\mathbb{R}}$ appears in the extension by $\mathbb{M} * \dot{\mathbb{R}}_\beta$, for some $\beta < \kappa^+$. Finally, to show that $\mathbb{R}$ is κ-distributive, it suffices to show that $\mathbb{R}_\beta$ is κ-distributive for each $\beta < \kappa^+$, and to see that $\mathsf{TP}(\omega_2)$ is satisfied in the extension by $\mathbb{M} * \dot{\mathbb{R}}$, it suffices to verify that $\mathsf{TP}(\omega_2)$ holds in each proper initial segment of the form $\mathbb{M} * \dot{\mathbb{R}}_\beta$, for some $\beta < \kappa^+$.

We will now prove by induction on $\beta < \kappa^+$ that in the extension by $\mathbb{M}$, $\mathbb{R}_\beta$ is κ-distributive. We will use the fact that $\mathbb{R}_\beta$ is κ-distributive in the proof that it preserves the tree property, but instead of repeating roughly the same argument twice, we combine these two facts into one proof, the first half of which will show distributivity, and the second half of which will use this to show the preservation of $\mathsf{TP}(\omega_2)$.

Thus fix a condition $(p, \dot{r}) \in \mathbb{M} * \dot{\mathbb{R}}_\beta$ as well as an $\mathbb{M} * \dot{\mathbb{R}}_\beta$-name $\dot{f}^*$ for a function from ω_1 into the ordinals. Also fix an $\mathbb{M} * \dot{\mathbb{R}}_\beta$-name $\dot{T}$ for a κ-tree. In V, let M^* be an elementary substructure of some large enough $H(\theta)$, where θ is regular, so that ${}^{<\kappa}M^* \subseteq M^*$, $|M^*| = \kappa$, and so that M^* contains $\dot{f}^*$, $\mathbb{M} * \dot{\mathbb{R}}_\beta$, and $\dot{T}$. Let M be the transitive collapse of M^*, noting that the collapse map preserves β, $\mathbb{M} * \dot{\mathbb{R}}_\beta$, and $\dot{T}$; let $\dot{f}$ be the image of $\dot{f}^*$ under the collapse map.

By the weak compactness of κ, since M is transitive, has size κ, and is closed under $< \kappa$-sequences, we may fix an elementary embedding $j : M \longrightarrow N$, where $\mathrm{crit}(j) = \kappa$.

We now fix a V-generic $G * H_\beta$ for $\mathbb{M} * \dot{\mathbb{R}}_\beta$ containing the condition $(p, \dot{r})$. We will find an extension $(p', \dot{r}')$ of $(p, \dot{r})$ so that p' forces that $\dot{r}'$ determines the values of $\dot{f}$ and which also forces that $\dot{T}$ has a cofinal branch.

Consider $j(\mathbb{M})$ on the N-side, and let κ^* denote the least N-inaccessible above κ. We know that this poset is isomorphic to a dense subset (with all terms computed in N) of

$$j(\mathbb{M}) \cong \mathbb{M} * (\mathrm{Add}(\omega, \kappa^*) \times \mathrm{Col}(\omega_1, \kappa^+)) * \dot{\mathbb{N}}_{\kappa^*},$$

where $\dot{\mathbb{N}}_{\kappa^*}$ is proper (and so preserves stationary subsets of cofinality ω ordinals). We next factor the collapse poset: $\dot{\mathbb{R}}_\beta$ is a member of N of size κ which is forced to be ω_1-closed and have $(2^\kappa)^N = (\kappa^+)^N$-many dense subsets. Thus N sees that after forcing with $\mathbb{M}$, $\mathrm{Col}(\omega_1, \kappa^+)$ factors as $\mathbb{R}_\beta * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}$ is forced to be ω_1-closed. Now extend $G * H_\beta$ to a V-generic $G^* := G * (\mathcal{A} \times (H_\beta * I)) * J$ for $j(\mathbb{M})$, where I is $V[G * H_\beta]$-generic for $\mathbb{Q}$, $\mathcal{A}$ is $V[G * H_\beta * I]$-generic for $\mathrm{Add}(\omega, \kappa^*)$ and J is $V[G * (\mathcal{A} \times (H_\beta * I))]$-generic for $\mathbb{N}_{\kappa^*}$.

Since $j[G] = G \subseteq G^*$, we may lift j to an extension $j : M[G] \longrightarrow N[G^*]$. Since $j \restriction \beta$ and H_β are both members of $N[G^*]$, we may define the function r^* in $N[G^*]$ as follows:

$\mathrm{dom}(r^*) := j[\beta]$, and for each $\alpha < \beta$,

$$r^*(j(\alpha)) := \bigcup \{c(\alpha) : c \in H_\beta \wedge \alpha \in \mathrm{dom}(c)\} \cup \{\kappa\}\,.$$

We claim that $r^* \in j(\mathbb{R}_\beta)$; observe that if this is the case, then we easily have that r^* extends each condition in $j[H_\beta]$.

We prove by induction on $\alpha < \beta$ that $r^* \restriction j(\alpha)$ is in $j(\mathbb{R}_\alpha)$. Since conditions in $j(\mathbb{R}_\beta)$ have $< j(\kappa)$ support and $|\mathrm{dom}(r^*)| \leq \kappa < j(\kappa)$, it suffices to show the successor case. So assume $r^* \restriction j(\alpha) \in j(\mathbb{R}_\alpha)$. We need to show that $r^* \restriction j(\alpha) \Vdash_{j(\mathbb{R}_\alpha)} r^*(j(\alpha)) \in j(\dot{\mathbb{C}}_\alpha)$; it suffices here to show that $r^* \restriction j(\alpha) \Vdash \kappa \in \mathrm{tr}(j(\dot{S}_\alpha))$.

Force with the poset $j(\mathbb{R}_\alpha)$ below $r^* \restriction j(\alpha)$ over $V[G^*]$ to obtain a generic filter $H^*_{j(\alpha)}$. Since $r^* \restriction j(\alpha)$ is a lower bound for $j[H_\alpha]$, where $H_\alpha := H_\beta \cap \mathbb{R}_\alpha$, we may lift j further to an elementary embedding $j : M[G * H_\alpha] \longrightarrow N[G^* * H^*_{j(\alpha)}]$. By the elementarity of j and the fact that $\mathrm{crit}(j) = \kappa$, we see that $j(S_\alpha) \cap \kappa = S_\alpha$. Thus to see that $\kappa \in \mathrm{tr}(j(S_\alpha))$, we need to verify that S_α is stationary in $N[G^* * H^*_{j(\alpha)}]$.

To begin, S_α is stationary in $N[G * H_\alpha]$; since the tail of $\mathbb{R}_\beta$ past stage α followed by $\dot{\mathbb{Q}}$ is ω_1-closed, S_α remains stationary in $N[G * H_\beta * I]$. S_α certainly remains stationary after adding the Cohen reals $\mathcal{A}$. Next, since $\mathbb{N}_{\kappa^*}$ is proper, S_α is still stationary in $N[G^*]$. Finally, by induction $\mathbb{R}_\alpha$ is κ-distributive, and so $j(\mathbb{R}_\alpha)$ is $j(\kappa)$-distributive. Hence $j(\mathbb{R}_\alpha)$ preserves the stationarity of S_α. Thus S_α is stationary in $N[G^* * H^*_{j(\alpha)}]$ as we intended to show.

This completes the construction of the condition r^*. We now consider the earlier embedding $j : M[G] \longrightarrow N[G^*]$. Since $\dot{f}$ is a name for a function with domain ω_1, the same is true of $j(\dot{f})$. Moreover, every value of $\dot{f}$ is decided by some condition in H_β, and so every value of $j(\dot{f})$ is decided by some condition in $j[H_\beta]$. Thus r^* decides all of the values of $j(\dot{f})$ since it bounds the filter $j[H_\beta]$. By the elementarity of j we may find in $M[G]$ an extension r' of r (recall that $(p, \dot{r})$ was our starting condition) in $\mathbb{R}_\beta$ which also decides all of the values of $\dot{f}$. Letting $p' \leq p$ in G force this, we have that $(p', \dot{r}')$ is a condition extending $(p, \dot{r})$ which forces that $\dot{f}$ is in the $\mathbb{M}$-extension. This completes the proof that $\mathbb{R}_\beta$ is κ-distributive.

Now let us see that $\dot{T}$ is forced to have a cofinal branch in the extension by $\mathbb{M} * \mathbb{R}_\beta$. Let

$j : M[G * H_\beta] \longrightarrow N[G^* * H^*_{j(\beta)}]$ be as before, where $r^* \in H^*_{j(\beta)}$. Since T has width $< \kappa$ and since j is an elementary embedding with $\mathrm{crit}(j) = \kappa$, we see that $j(T) \restriction \kappa = T$, and therefore T has a cofinal branch B in $N[G^* * H^*_{j(\beta)}]$. We will show that B is a member of $M[G * H_\beta]$. First, we know that B lives in $N[G^*]$ since $j(\mathbb{R}_\beta)$ is $j(\kappa)$-distributive (recall that we've shown that $\mathbb{R}_\beta$ is κ-distributive). Now consider the forcing which takes us from $N[G * H_\beta]$ to $N[G^*]$. This forcing is $(\mathrm{Add}(\omega, \kappa^*) \times \mathbb{Q}) * \dot{\mathbb{N}}_{\kappa^*}$, where $\mathbb{Q}$ is ω_1-closed. By Proposition 5.2.7, we know that $\dot{\mathbb{N}}_{\kappa^*}$ is forced to be a projection of a product $\mathbb{A} \times \mathbb{B}$, where $\mathbb{B}$ is ω_1-closed and where $\mathbb{A}$ is isomorphic to $\mathrm{Add}(\omega, j(\kappa))$.

Now in the model $N[G]$, we know that $2^\omega = \omega_2 = \kappa$. Since $\mathbb{R}_\beta$ is κ-distributive, this still holds in $N[G * H_\beta]$. Because T is a κ-tree, the Branch Lemmas (see Lemma 6.1.4) imply that the product $\mathrm{Add}(\omega, \kappa^*) \times \mathbb{Q}$ cannot add a cofinal branch through T. Furthermore, in the model $N[G * (\mathcal{A} \times (H_\beta * I))]$ we have that $2^\omega = \kappa^* > \omega_1$ and T is a tree whose height has cofinality ω_1 and all of whose levels have size at most ω_1. Thus Lemma 6.1.4 again implies that forcing with $\mathbb{A} \times \mathbb{B}$, and hence forcing with $\mathbb{N}_{\kappa^*}$, will not add a cofinal branch to T. Therefore B is a member of $N[G * H_\beta]$, and hence of $M[G * H_\beta]$, since these two models have the same subsets of κ.

This completes the proof that $\mathbb{M} * \dot{\mathbb{R}}$ forces $\mathsf{SR}(\omega_2)^* + \mathsf{TP}(\omega_2)$. We finish by showing that it forces that AP_{ω_1} holds. For this we will need a slight variation of the argument of Section 3.5 of [24] in order to see that AP_{ω_1} holds after forcing with $\mathbb{M}$. Once we verify this, we may conclude that $\mathbb{R}$ preserves AP_{ω_1} since $\mathbb{R}$ preserves all cardinals in the $\mathbb{M}$-extension.

Now in the $\mathbb{M}$-extension, we have that $2^\omega = \kappa$. By Remark 5.1.14, we know that there exists a maximal stationary set, S, of cofinality ω_1-points in $I[\kappa]^{\mathbb{M}}$, and moreover, S is defined, modulo clubs, as all $\beta < \kappa$ which are approachable with respect to the sequence $\vec{a} = \langle a_\alpha : \alpha < \kappa \rangle$, where $\vec{a}$ enumerates all countable subsets of κ in $V[\mathbb{M}]$. Next, every element of $\vec{a}$ appears in the smaller model $V[\mathrm{Add}(\omega, \kappa)] \subseteq V[\mathbb{M}]$. Since $\mathrm{Add}(\omega, \kappa)$ is c.c.c., for almost all $\alpha \in A$ (recall that A is the set of V-inaccessibles below κ), we have that $\vec{a} \restriction \alpha$ enumerates all countable subsets of α in $V[\mathrm{Add}(\omega, \alpha)]$; this model has the same countable sequences as $V[\mathbb{M} \restriction \alpha]$. Thus we may redefine S modulo clubs to consists of all $\alpha \in A$ so that there exists a cofinal $e \subseteq \alpha$ of order type ω_1 with $e \cap \eta \in V[\mathbb{M} \upharpoonright \alpha]$ for all $\eta < \alpha$.

Now in the definition of $\mathbb{M}$, we have that for each $\alpha \in A$, the forcing adds a $V[\mathbb{M} \upharpoonright \alpha]$-generic filter over $\mathrm{Col}(\omega_1, \alpha^+)$ (where this poset is computed in $V[\mathbb{M} \upharpoonright \alpha]$). However, forcing with $\mathrm{Col}(\omega_1, \alpha^+)$ certainly adds a generic for $\mathrm{Col}(\omega_1, \alpha)$. From this generic, we may define a cofinal ω_1-sequence e in α all of whose initial segments live in $V[\mathbb{M} \upharpoonright \alpha]$. As remarked in the previous paragraph, for almost all $\beta \in A$, $\vec{a} \upharpoonright \beta$ enumerates $[\beta]^{\aleph_0} \cap V[\mathbb{M} \upharpoonright \beta]$. Thus if $\alpha \in A$ is one of these β, then e provides a witness that $\alpha \in S$. Thus S contains almost all points in A, i.e., modulo clubs, $S = A$.

To finish, we need to see that A consists of almost all points of cofinality ω_1 in $V[\mathbb{M}]$. However, this is straightforward: if $\dot{C}$ is an $\mathbb{M}$-name for a club subset of κ, then since $\mathbb{M}$ is κ-c.c., there exists a club $D \subseteq \kappa$ in V so that $\Vdash_{\mathbb{M}} \check{D} \subseteq \dot{C}$. Now A is stationary in κ, since κ is Mahlo, and so $D \cap A \neq \varnothing$. Thus $\mathbb{M}$ forces that $\dot{C} \cap A \neq \varnothing$.

We now sum up what we have in the following theorem.

Theorem 7.1.2. (Ben-Neria, Gilton) *It is consistent from a weakly compact cardinal that* $\mathsf{TP}(\omega_2) + \mathsf{AP}_{\omega_1} + \mathsf{SR}(\omega_2)^*$ *holds.*

Proof. We force with the poset $\mathbb{M} * \dot{\mathbb{R}}$. We have seen that ω_1 and all cardinals $\lambda \geq \kappa$ are preserved by this forcing, and that κ becomes ω_2 in the extension. $\mathbb{M}$ forces that AP_{ω_1} holds, and since $\mathbb{R}$ preserves all cardinals, AP_{ω_1} holds in the final model. We also saw that $\mathsf{TP}(\omega_2)$ holds in the extension by $\mathbb{M} * \dot{\mathbb{R}}_\beta$, for each β, and it therefore holds in the final model, since $\mathbb{R}$ is κ^+-c.c. Finally, by carrying out an appropriate bookkeeping in the definition of $\mathbb{R}$ and using the fact that $\mathbb{R}$ is κ^+-c.c., we may see that every stationary subset of $\kappa \cap \mathrm{cof}(\omega) = \omega_2 \cap \mathrm{cof}(\omega)$ in the final model reflects to almost every point of cofinality ω_1. $\square$

7.2 $\mathsf{TP}(\omega_2) + \neg\mathsf{AP}_{\omega_1} + \mathsf{SR}(\omega_2)^*$

We now show how to obtain a model of $\mathsf{SR}(\omega_2)^* + \mathsf{TP}(\omega_2)$ in which AP_{ω_1} fails. As remarked at the beginning of this chapter, we will need to be more patient in the collapsing that we use for our Mitchell-type posets. In order to make this precise, we will greatly increase the large cardinal assumption that we use. Let's recall the relevant definitions (see [21]).

Definition 7.2.1. *A cardinal κ is said to be* λ-supercompact *iff there is a definable $j : V \longrightarrow M$ so that* $\mathrm{crit}(j) = \kappa$*, $j(\kappa) > \lambda$, and ${}^{\lambda}M \subseteq M$. κ is* supercompact *iff it is λ-supercompact for all λ.*

A key feature of supercompact cardinals is the following result, due to Laver (see [55]).

Theorem 7.2.2. (Laver) *Suppose that κ is supercompact. Then there exists a function $d : \kappa \longrightarrow V_\kappa$ so that for all $\lambda \geq \kappa$ and all $x \in H(\lambda^+)$, there is a supercompactness measure U on $P_\kappa(\lambda)$ so that $j_U(d)(\kappa) = x$.*

The function d in the above theorem can be thought of as a very strong Diamond-like sequence; indeed, the proof of the existence of such a function is similar to that of the various proofs of Diamond principles, as the function is constructed by providing local minimal counterexamples. The function d is often referred to as a Laver Diamond. Let us fix such a function d for the remainder of the chapter.

We now proceed to use d to construct Mitchell-type posets, following Abraham, which are similar to $\mathbb{M}_1$ from the paragraph after Definition 5.2.2. We recall that A denotes the set of inaccessible cardinals below κ, and A^* denotes $A \setminus \lim(A)$.

Definition 7.2.3. *We define the poset $\mathbb{M} \restriction \beta$ by recursion on $\beta \in A$, setting $\mathbb{M} := \mathbb{M} \restriction \kappa$. Conditions in $\mathbb{M} \restriction \beta$ consist of triples (a, f, g) where*

1. *$a \in \mathrm{Add}(\omega, \beta)$;*
2. *f is a partial function with $\mathrm{dom}(f) \subseteq A^* \cap \beta$ so that $|\mathrm{dom}(f)| \leq \aleph_0$;*
3. *for each $\alpha \in \mathrm{dom}(f)$, $f(\alpha)$ is an $\mathrm{Add}(\omega, \alpha)$-name for a condition in $\mathrm{Col}(\omega_1, \alpha)$;*
4. *g is a partial function with $\mathrm{dom}(g) \subseteq A \cap \beta$ so that $|\mathrm{dom}(g)| \leq \aleph_0$;*
5. *for all $\alpha \in \mathrm{dom}(g)$, $d(\alpha)$ is an $(\mathbb{M} \restriction \alpha)$-name for an ω_1-closed poset and $g(\alpha)$ is an $(\mathbb{M} \restriction \alpha)$-name for an element of $d(\alpha)$.*

We say that $(a', f', g') \leq (a, f, g)$ *iff* $a \subseteq a'$*;* $\mathrm{dom}(f) \subseteq \mathrm{dom}(f')$ *and* $\mathrm{dom}(g) \subseteq \mathrm{dom}(g')$*; for all* $\alpha \in \mathrm{dom}(f)$*,* $a' \restriction \alpha \Vdash_{\mathrm{Add}(\omega,\alpha)} f'(\alpha) \leq_{\mathrm{Col}(\omega_1,\alpha)} f(\alpha)$*; and for all* $\alpha \in \mathrm{dom}(g)$*,* $(a', f', g') \restriction \alpha \Vdash_{\mathbb{M}\restriction\alpha} g'(\alpha) \leq_{d(\alpha)} g(\alpha)$*.*

It is straightforward to see that the relevant results from Section 5.2 hold for $\mathbb{M}$, in particular, Proposition 5.2.3(1,3), Proposition 5.2.5, and Proposition 5.2.7.

In the extension by $\mathbb{M}$, we now define, as in the previous section, $\mathbb{R}$ to be the $< \kappa$-support iterated club adding to witness $\mathsf{SR}(\omega_2)^*$. Following the last section, we will show that for each $\beta < \kappa^+$, $\mathbb{R}_\beta$ is κ-distributive and preserves the tree property. From the κ^+-c.c. of $\mathbb{R}$, we then may conclude that $\mathbb{R}$ is κ-distributive and preserves the tree property. Suppose, by induction, that for all $\alpha < \beta$, $\mathbb{R}_\alpha$ is κ-distributive after forcing with $\mathbb{M}$.

We now fix $\mathbb{M} * \dot{\mathbb{R}}_\beta$-names $\dot{T}$ and $\dot{f}$, where $\dot{T}$ names a κ-tree and $\dot{f}$ names a function from ω_1 into the ordinals. Also fix a condition $(p, \dot{r})$ in $\mathbb{M} * \dot{\mathbb{R}}_\beta$. We will find an extension of $(p, \dot{r})$ which forces that $\dot{T}$ has a cofinal branch and which forces that $\dot{f}$ is in the extension by $\mathbb{M}$.

By properties of d, we may choose a supercompactness embedding $j : V \longrightarrow M$ so that $j(d)(\kappa) = \dot{\mathbb{R}}_\beta$, noting that $\dot{\mathbb{R}}_\beta = j(d)(\kappa)$ is a $j(\mathbb{M}) \restriction \kappa = \mathbb{M}$-name for an ω_1-closed poset. We then see that, letting κ^* denote the next inaccessible above κ in M, $j(\mathbb{M})$ is isomorphic to a dense subset of

$$j(\mathbb{M}) \restriction \kappa^* * \dot{\mathbb{N}}_{\kappa^*} \cong \mathbb{M} * (\dot{\mathbb{R}}_\beta \times \mathrm{Add}(\omega, \kappa^*)) * \dot{\mathbb{N}}_{\kappa^*}.$$

Note that since κ is an inaccessible limit of inaccessibles (i.e., in $A \backslash A^*$), the forcing $j(\mathbb{M})$ does not collapse κ until after the additional reals from $\mathrm{Add}(\omega, \kappa^*)$ are added.

Now let us fix a V-generic $G * H_\beta$ for $\mathbb{M} * \dot{\mathbb{R}}_\beta$ containing the starting condition $(p, \dot{r})$. Let $\mathcal{A}$ be generic for $\mathrm{Add}(\omega, \kappa^*)$ over $V[G * H_\beta]$, setting $I := G * (H_\beta \times \mathcal{A})$. Finally, let J be $V[I]$-generic over the tail $j(\mathbb{M})/I \cong \mathbb{N}_{\kappa^*}$. Set $G^* := I * J$.

Since $j[G] = G \subseteq G^*$, we may lift j to an elementary map $j : V[G] \longrightarrow M[G^*]$. $j \restriction \beta$ and H_β are members of $M[G^*]$, and hence we may construct, as in the previous section, the "flat" function r^* in $M[G^*]$ which we will show is the minimal lower bound for $j[H_\beta]$.

We need to show that $r^* \in j(\mathbb{R}_\beta)$; we will pass over the details that are very similar to those of the previous section. It suffices to show that if $r^* \upharpoonright j(\alpha) \in j(\mathbb{R}_\alpha)$ (and hence a lower bound for $j[H_\alpha]$), then $r^* \upharpoonright j(\alpha)$ forces that $\kappa \in \mathrm{tr}(j(\dot{S}_\alpha))$.

Let us lift $j : V[G^* * H_\alpha] \longrightarrow M[G^* * H^*_{j(\alpha)}]$, where $H^*_{j(\alpha)}$ is $V[G^*]$-generic for $j(\mathbb{R}_\alpha)$ and contains $r^* \upharpoonright j(\alpha)$. We then see that $S_\alpha = j(S_\alpha) \cap \kappa$, so it suffices to see that S_α remains stationary in $M[G^* * H^*_{j(\alpha)}]$.

However, this follows as before: the tail of the iterated club adding followed by $\mathrm{Add}(\omega, \kappa^*)$ certainly preserves the stationarity of S_α; and the poset $\mathbb{N}_{\kappa^*}$ is proper in $V[I]$, so S_α remains stationary in $M[G^*]$. Finally, $j(\mathbb{R}_\alpha)$ is $j(\kappa)$-distributive, and so S_α is stationary in $M[G^* * H^*_{j(\alpha)}]$.

This completes the construction of r^*. Since r^* decides all of the values of $j(\dot{f})$, by the elementarity of j we may find an extension of $(p, \dot{r})$ which forces that $\dot{f}$ is in the $\mathbb{M}$-extension. This completes the proof that $\mathbb{R}_\beta$ is κ-distributive.

As in the previous section, we also see that $\mathbb{M} * \dot{\mathbb{R}}_\beta$ forces that the κ-tree $\dot{T}$ has a cofinal branch. Indeed, if we let $j : V[G * H_\beta] \longrightarrow M[G^* * H^*_{j(\beta)}]$ be constructed as in the previous few paragraphs, we know that since $j(T) \upharpoonright \kappa = T$, T has a cofinal branch in $M[G^* * H^*_{j(\beta)}]$. Call this cofinal branch B. However, the forcing $\mathrm{Add}(\omega, \kappa^*) * \dot{\mathbb{N}}_{\kappa^*} * j(\dot{\mathbb{R}}_\beta)$, which takes us from $M[G * H_\beta]$ to $M[G^* * H^*_{j(\beta)}]$ cannot add this branch, and therefore B is a member of $M[G * H_\beta]$, and hence of $V[G * H_\beta]$.

This completes the proofs that $\mathsf{TP}(\omega_2)$ and $\mathsf{SR}(\omega_2)^*$ both hold in the extension by $\mathbb{M} * \dot{\mathbb{R}}$. We now need to show that AP_{ω_1} fails in this extension.

Suppose for a contradiction that $\kappa \in I[\kappa]$ is true in some extension by $\mathbb{M} * \dot{\mathbb{R}}$, and let $(p, \dot{r})$ be a condition forcing this. Let us temporarily step into some generic extension by this poset, $V[G * H]$, where $G * H$ contains $(p, \dot{r})$. Since $\kappa^{\aleph_0} = \kappa$ in this model, we may fix an enumeration $\vec{a} = \langle a_i : i < \kappa \rangle$ of all countable subsets of κ, and because $\kappa \in I[\kappa]$, we have that there is a club $C \subseteq \kappa$ so that every $\alpha \in C$ is approachable w.r.t. $\vec{a}$. Let $\dot{C}$ be a name for this club, and let us also assume, by extending if necessary, that $(p, \dot{r})$ forces that the above holds.

Next, construct an embedding $j : V[G * H] \longrightarrow M[G^* * H^*]$ as before, where $G * H$ is V-generic over $\mathbb{M} * \dot{\mathbb{R}}$. Note that $j(C) \cap \kappa = C$ and that $j(\vec{a}) \restriction \kappa = \vec{a}$. Consequently, κ is a limit point of $j(C)$ and hence a member of $j(C)$, and by definition of $j(C)$, we have that κ is approachable w.r.t. $\vec{a}$ in $M[G^* * H^*]$. Let $e \subseteq \kappa$ be a club of ordertype ω_1 witnessing this.

We now mimic the proof from [24] that $\mathbb{M}_1$ forces that AP_{ω_1} fails. Let U denote the tree $(2^{<\kappa})^{V[G]}$, noting that U has width κ. Note that the characteristic function χ_e of e gives a branch through U which is not a member of $V[G * H]$ (since $\kappa = \omega_2$ in that model); we show that this is impossible. Indeed, e lives in $M[G^*]$ by the $j(\kappa)$-distributivity of $j(\mathbb{R})$. But in the model $M[I]$, we have that $2^\omega > \kappa$, and hence $\mathbb{N}_{\kappa^*}$ cannot add a branch through U, being a projection of a product of an ω_1-closed forcing and Cohen forcing. Finally, $\mathrm{Add}(\omega, \kappa^*)$ cannot add this branch, and therefore χ_e (and hence e) lives in $V[G * H]$. This contradicts that $\kappa = \omega_2$ in that model.

This completes the proof that AP_{ω_1} fails in the extension by $\mathbb{M} * \dot{\mathbb{R}}$. Let us summarize in the following theorem.

Theorem 7.2.4. (Gilton) *It is consistent from large cardinals that* $\mathsf{TP}(\omega_2) + \neg\mathsf{AP}_{\omega_1} + \mathsf{SR}(\omega_2)^*$ *holds.*

7.3 $\neg\mathsf{TP}(\omega_2) + \neg\mathsf{AP}_{\omega_1} + \mathsf{SR}(\omega_2)^*$

In this section we will prove the consistency of the last of our Boolean combinations from the assumption that κ is a supercompact cardinal. More specifically, we will show that it is consistent that $\mathsf{SR}(\omega_2)^*$ holds, that AP_{ω_1} fails, and that there exists a Suslin tree on ω_2. Recall that a λ-tree T is Suslin iff T has no antichains of size λ (all of our trees have non-trivial splitting, and hence this implies that T has no chains of size λ). Note that this is equivalent to saying that the poset $(T, <_T)$ is λ-c.c.

In rough outline, we will force with the Mitchell-type poset $\mathbb{M}$ from the last section, and in the $\mathbb{M}$-extension, we will force with $\mathbb{P} * \dot{\mathbb{R}}$, where $\mathbb{P}$ is Kunen's forcing (see [52]) for adding a κ-Suslin tree and where $\dot{\mathbb{R}}$, in the $\mathbb{P}$-extension, is the Magidor-style iterated club adding.

We will need to show that if T is the generic tree added by forcing with $\mathbb{P}$, then T remains Suslin after forcing with $\mathbb{R}$ and that $\mathbb{R}$ is κ-distributive.

The proofs of these two facts will go hand-in-hand in an induction on the length of $\mathbb{R}$. With regards to showing that $\mathbb{R}$ preserves that T is Suslin, an appeal to Easton's Lemma won't work, since $\mathbb{R}$ is only κ-distributive, not κ-closed. So we need a different argument to show that $\mathbb{R}$ does not add any κ-sized antichains to T; this argument will involve lifting an elementary embedding $j : V \longrightarrow M$ and mimicking the proof that $\mathbb{P}$ forces that $\dot{T}$ is Suslin. And with regards to showing that $\mathbb{R}$ is κ-distributive, we also need to lift some such j, and this will involve (among other things) building a master condition for $j(\mathbb{P})$ on the M-side. To achieve this, we need to know that T has a cofinal branch in an appropriate extension of the M-side, and this will be achieved by forcing to add such a branch (i.e., forcing with $(T, <_T)$). However, we will need to know that the stationary sets which are dealt with in the course of the iteration $\mathbb{R}$ remain stationary on the M-side, and this requires knowing that $(T, <_T)$ is κ-c.c. (i.e., T is Suslin) after the initial segments of $\mathbb{R}$.

Having completed an overview of the argument, let us now fix an arbitrary V-generic G for $\mathbb{M}$. We begin by reviewing some standard facts about the forcing $\mathbb{P}$.

Definition 7.3.1. *In $V[G]$, let $\mathbb{P}$ consist of all trees $t \subseteq {}^{<\kappa}2$ of successor height $\alpha + 1$, for some $\alpha < \kappa$, which satisfy the following:*

1. *t is a normal tree, and all levels of t have size $< \kappa$;*
2. *t is homogeneous.*

The ordering is end-extension.

We recall that t is *homogeneous* if for any $s \in t$, the subtree of t above s equals t. In precise notation, t is homogeneous iff for all $s \in t$, $t_s = t$, where $t_s = \{u : s^\frown u \in t\}$. Note that this is equivalent to saying that for any two sequences s, u, we have that $s^\frown u \in t$ iff s and u are both in t; we will use this characterization of homogeneity frequently in what follows. Throughout this section, we will use $\dot{T}$ to denote the $\mathbb{P}$-name for the generic tree.

We recall the following standard fact about this forcing:

Lemma 7.3.2. *Suppose that $\gamma < \kappa$ is a limit, t is a normal, homogeneous γ-tree with all levels of size $< \kappa$, and $b \subseteq t$ is a cofinal branch. Then there exists a condition $t' \in \mathbb{P}$ so that $t \subseteq t'$.*

Proof. We define t' to add all tail segments of the branch b above every node of t. More precisely, we define

$$t' := t \cup \{s^\frown(b \backslash \alpha) : s \in t \wedge \alpha < \gamma\},$$

where for each $\alpha < \gamma$, $b\backslash\alpha$ denotes the unique u so that $(b \restriction \alpha)^\frown u = b$, i.e., the tail segment of b above α. It is straightforward to check that t' is a condition in $\mathbb{P}$ which extends t. $\square$

The condition t' constructed in the proof of the above lemma will be known as *the minimal extension of t by b.* A corollary of the above lemma is the following.

Corollary 7.3.3. $\mathbb{P}$ *is ω_1-closed.*

Standard arguments also show that the following is true:

Proposition 7.3.4. $\mathbb{P}$ *is κ-distributive and forces that $\dot{T}$ is a homogeneous Suslin tree on κ.*

A crucial tool in analyzing $\mathbb{P}$, and in proving the above proposition in particular, is to work with $\mathbb{P}$ followed by forcing with the generic tree (see [52]). The following summarizes what we need to know about this flavor of argument.

Lemma 7.3.5. *Let $\dot{\mathbb{Q}}$ be the $\mathbb{P}$-name for the forcing $(\dot{T}, <_{\dot{T}})$. Then $\mathbb{P} * \dot{\mathbb{Q}}$ has a κ-closed, dense subset. Hence, since $\kappa^{<\kappa} = \kappa$ in $V[G]$, $\mathbb{P} * \dot{\mathbb{Q}}$ is forcing equivalent to* $\mathrm{Add}(\kappa, 1)$.

We've now surveyed the properties of $\mathbb{P}$ that we will need. Next, in the extension by $\mathbb{M} * \dot{\mathbb{P}}$, let $\mathbb{R} = \langle \mathbb{R}_\beta, \dot{\mathbb{C}}_\beta : \beta < \kappa^+ \rangle$ be the Magidor iteration of club adding, where we use, as conditions in each iterand, closed and bounded subsets of κ which come from the extension by $\mathbb{M} * \dot{\mathbb{P}}$ (equivalently, from the extension by $\mathbb{M}$). We claim that forcing with $\mathbb{M} * \dot{\mathbb{P}} * \dot{\mathbb{R}}$ gives the desired model.

First, a few simple remarks about cardinal preservation. ω_1 is easily seen to be preserved. Furthermore, since $\kappa^{<\kappa} = \kappa$ in the extension by $\mathbb{M}$, and since $\mathbb{P}$ is κ-distributive in the $\mathbb{M}$-extension, the equation $\kappa^{<\kappa} = \kappa$ still holds after forcing with $\mathbb{P}$. Consequently, we may see that $\mathbb{R}$ is still κ^+-c.c. Thus all cardinals $\mu \geq \kappa^+$ are preserved. Furthermore, once we know that κ is preserved, the κ^+-c.c. of $\mathbb{R}$ implies that we may catch our tail and achieve a model in which $\mathsf{SR}(\omega_2)^*$ holds.

Let T denote the generic Suslin tree on κ added by forcing with $\mathbb{P}$. We will show by induction on $\beta < \kappa^+$ that $\mathbb{R}_\beta$ is κ-distributive in $V[G * T]$ and preserves that T is Suslin. Let us suppose as an induction hypothesis that the result holds for all $\alpha < \beta$, and we will show that it holds at β.

Towards this end, fix an $\mathbb{R}_\beta$-name $\dot{A}$ for a maximal antichain in T as well as a name $\dot{f}$ for a function from ω_1 into the ordinals. Finally, fix a condition $p \in \mathbb{R}_\beta$; we will find an extension of p which forces that $\dot{f}$ lives in $V[G * T]$ and which forces that $\dot{A}$ is bounded in T.

We first want to observe that in $V[G]$, $\mathbb{K} := \mathbb{P} * (\dot{\mathbb{Q}} \times \dot{\mathbb{R}}_\beta)$ is ω_1-closed on a dense subset. We know that $\mathbb{P} * \dot{\mathbb{Q}}$ is ω_1-closed (in fact, κ-closed) on a dense subset. Furthermore, $\mathbb{Q}$ is ω_1-distributive after forcing with $\mathbb{P}$, and therefore $\mathbb{R}_\beta$, which is ω_1-closed after forcing with $\mathbb{P}$, remains ω_1-closed after forcing with $\mathbb{P} * \dot{\mathbb{Q}}$. Thus $\mathbb{K} \cong \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}_\beta$ is ω_1-closed on a dense subset. We will unrepentantly abuse notation and use $\mathbb{K}$ to also denote this dense subset.

By the definition of d, we may select a supercompactness embedding $j : V \longrightarrow M$ so that $j(d)(\kappa) = \dot{\mathbb{K}}$, and we see that on the M-side, $j(\mathbb{M})$ is isomorphic to a dense subset of

$$j(\mathbb{M}) \cong \mathbb{M} * (\dot{\mathbb{K}} \times \mathrm{Add}(\omega, \kappa^*)) * \dot{\mathbb{N}}_{\kappa^*},$$

where κ^* is the least M-inaccessible above κ (see Propositions 5.2.3 and 5.2.7). We will use $j(\mathbb{M})^{\mathrm{tail}}$ to denote the $\mathbb{M} * \dot{\mathbb{K}}$-name for $\mathrm{Add}(\omega, \kappa^*) * \dot{\mathbb{N}}_{\kappa^*}$.

Now let us fix a $V[G * T]$-generic filter $B \times H_\beta$ over $\mathbb{Q} \times \mathbb{R}_\beta$, where $p \in H_\beta$ (p was our starting condition). B here denotes the generic branch through T. We let $I := T * (B \times H_\beta)$, the $V[G]$-generic for $\mathbb{K}$. Next, let J be generic for $j(\mathbb{M})^{\mathrm{tail}}$ over $V[G * I]$, and let G^* denote

$G*I*J$. Since $j[G] = G \subseteq G^*$, we may lift j to an elementary embedding $j : V[G] \longrightarrow M[G^*]$.

We first claim that there is a condition $t^* \in j(\mathbb{P})$ so that $T \subseteq t^*$. Indeed, T is a normal, homogeneous κ-tree in $M[G^*]$ all of whose levels have size $< j(\kappa)$. Since B is a cofinal branch through T in $M[G^*]$, we may find, by Lemma 7.3.2, a condition in $j(\mathbb{P})$ which extends T. Set $t^* \in j(\mathbb{P})$ to be the minimal extension of T by B.

Let T^* be $V[G^*]$-generic over $j(\mathbb{P})$ so that $t^* \in T^*$. Since $T = j[T]$ and T is an initial segment of t^*, we may extend j to an elementary embedding $j : V[G*T] \longrightarrow M[G^* * T^*]$.

We now claim that there is a condition $r^* \in j(\mathbb{R}_\beta)$ which is a lower bound for $j[H_\beta]$. Since $j[H_\beta]$ is a member of $M[G^*]$, we may define r^* as in the previous two sections to be the κ-flat function which we will show is the minimal lower bound for $j[H_\beta]$. This is obvious once we show that r^* is a condition in $j(\mathbb{R}_\beta)$. For this in turn, it suffices to see that if $\alpha < \beta$, then S_α is stationary in $M[G^* * T^*]$. So fix some such $\alpha < \beta$. By our inductive assumption, T is still Suslin after forcing with $\mathbb{R}_\alpha$, and hence $\mathbb{Q}$ is still κ-c.c. in the extension by $\mathbb{P} * \dot{\mathbb{R}}_\alpha$. Thus S_α is stationary in $V[G * T * (B \times H_\alpha)]$ (here $H_\alpha := H_\beta \cap \mathbb{R}_\alpha$). The tail of the forcing $\mathbb{R}_\beta$ from α onwards is still ω_1-closed after forcing with $\mathbb{Q}$, since $\mathbb{Q}$ is ω_1-distributive. Thus S_α is stationary in $V[G*I]$. Finally, the tail forcing $j(\mathbb{M})^{\text{tail}}$ is proper in $V[G*I]$, and hence it preserves the stationarity of S_α (recall that S_α consists of points of countable cofinality). Thus S_α is stationary in $V[G^*]$. Finally, $j(\mathbb{P})$ is ω_1-closed, and so we now see that S_α is stationary in $V[G^* * T^*]$ and hence in $M[G^* * T^*]$.

This completes the proof that r^* is a condition in $j(\mathbb{R}_\beta)$. Since r^* decides all of the values of $j(\dot{f})$, the elementarity of j implies that some extension of p (our original condition) decides all of the values of $\dot{f}$. Thus $\mathbb{R}_\beta$ is κ-distributive.

Recall that we had an $\dot{\mathbb{R}}_\beta$-name $\dot{A}$ for an antichain in T. We will finish our proof by showing that some extension of r^* forces $j(\dot{A})$ to be bounded in T^*. As r^* is a lower bound for $j[H_\beta]$, we may now lift j to an elementary embedding $j : V[G*T*H_\beta] \longrightarrow M[G^* * T^* * H^*]$, where H^* is $V[G^* * T^*]$-generic over $j(\mathbb{R}_\beta)$ containing r^*.

Let A be the interpretation of $\dot{A}$, and observe that by the elementarity of j, $j(A) \restriction \kappa = A$;

in particular, A is a member of $M[G^* * T^* * H^*]$. Recall that t^* is the minimal extension of T by the branch B. We will show that every node of t^* at level κ extends some element of A. This shows that A is in fact a maximal antichain in the generic tree T^*, and hence $j(A) = A$, since $j(A) \supseteq A$ is also a maximal antichain in T^*.

We recall that t^* is defined as follows:

$$t^* := t \cup \{s^\frown(B \backslash \gamma) : s \in T \wedge \gamma < \kappa\}.$$

By definition of t^*, to show that every element of t^* on level κ extends some element of A, it suffices to show that the following holds:

($*$) for each $s \in T$ and $\gamma < \kappa$, there is some $\eta > \gamma$ so that $s^\frown(B \upharpoonright [\gamma, \eta)) \in A$.

Note that ($*$) is a statement over $V[G*T*(B \times H_\beta)]$, and it suffices to verify that it holds in that model. However, ($*$) follows from a density argument and the genericity of the branch B, as we now show. We will temporarily work over the model $V[G*T]$. We argue that for each $s \in T$ and $\gamma < \kappa$, the following set $D_{s,\gamma}$ is dense in the product $\mathbb{Q} \times \mathbb{R}_\beta$, where

$$D_{s,\gamma} := \left\{(u, r) \in \mathbb{Q} \times \mathbb{R}_\beta : (\exists \eta > \gamma)\ (u, r) \Vdash s^\frown(\dot{B} \upharpoonright [\gamma, \eta)) \text{ extends a node in } \dot{A}\right\}.$$

So fix some condition (u, r) in $\mathbb{Q} \times \mathbb{R}_\beta$ as well as a node $s \in T$ and an ordinal $\gamma < \kappa$. Extend u if necessary so that $\mathrm{lh}(u) \geq \gamma$, noting that $u \Vdash_{\mathbb{Q}} \dot{B} \upharpoonright \gamma = u \upharpoonright \gamma$. Set $\bar{u} := u \upharpoonright [\gamma, \mathrm{lh}(u))$ (it is possible that this is the empty sequence). Since $u \in \mathbb{Q}$, u is an element of T. By the homogeneity of T, $\bar{u} \in T$. As $s \in T$, we have by another use of homogeneity that $s^\frown\bar{u} \in T$. Recalling that $\dot{A}$ is forced by $\mathbb{R}_\beta$ to be a maximal antichain in T, we may extend r to a condition r^* in $\mathbb{R}_\beta$ and find a (possibly trivial) extension $s^\frown\bar{u}^\frown\bar{u}^*$ of $s^\frown\bar{u}$ in T so that $r^* \Vdash s^\frown\bar{u}^\frown\bar{u}^*$ extends an element of $\dot{A}$. Now set $u^* := u^\frown\bar{u}^*$, and let η be the length of u^*. We observe that $u^* \Vdash \dot{B} \upharpoonright [\gamma, \eta) = \bar{u}^\frown\bar{u}^*$. From this it now follows that (u^*, r^*) forces that $s^\frown(\dot{B} \upharpoonright [\gamma, \eta))$ extends an element of $\dot{A}$, which completes the proof that $D_{s,\gamma}$ is dense. This in turn shows that ($*$) holds in $V[G * T * (B \times H_\beta)]$.

We have now shown that ($*$) holds, from which it follows that every node in t^* on level κ extends an element of A. Thus A is maximal in any extension of t^*, and consequently, A

is maximal in T^*. Since $j(A) \supseteq A$ is also maximal, $j(A) = A$. Thus $j(A)$ is bounded in T^*, and so by the elementarity of j, A is bounded in T, completing the proof that T is Suslin after forcing with $\mathbb{R}_\beta$.

This completes our inductive proof that for all $\beta < \kappa^+$, $\mathbb{R}_\beta$ is κ-distributive and preserves the fact that T is Suslin. From the κ^+-c.c. of the full iteration $\mathbb{R}$, it follows that $\mathbb{R}$ is κ-distributive and preserves that T is Suslin. Thus we have shown that in the extension by $\mathbb{M}*\dot{\mathbb{P}}*\dot{\mathbb{R}}$, $\mathsf{SR}(\omega_2)^*$ holds and $\mathsf{TP}(\omega_2)$ fails. The argument that AP_{ω_1} fails is almost exactly the same as in Section 7.2, replacing the ω_1-closure of "$\mathbb{R}$" in that context with the ω_1-closure of $\mathbb{P} * \dot{\mathbb{R}}$ in this context.

Let us summarize what we have in the following theorem.

Theorem 7.3.6. (Gilton) *It is consistent from large cardinals that* $\neg\mathsf{AP}_{\omega_1} + \mathsf{SR}(\omega_2)^*$ *and that there exists a Suslin tree on* ω_2*; in particular,* $\mathsf{TP}(\omega_2)$ *fails.*

CHAPTER 8

Combinatorics after forcing with a Suitable Mixed Support Iteration

In this chapter, we further study the mixed-support iteration from [34]. The main goal is to show that, after a slight modification to the preparation iteration therein, the final model satisfies that there exist neither weak Kurepa trees on ω_1 nor special Aronszajn trees on ω_2 (see Propositions 8.3.1 and 8.3.2). We will also show that we may get AP_{ω_1} to hold or fail in the final model. In the case where AP_{ω_1} fails, we will show that a Disjoint Stationary Sequence (see [32]) exists, but in the case where AP_{ω_1} holds, we need to use a more involved argument.

In the first few sections, we will review the main arguments from [34] and record some of the results here. We will skip the proofs of the more straightforward results (the interested reader should consult [34] for all of the details), though we will provide, often sketchy, proofs of the more interesting propositions from that paper.

In the second section, we review the proof that $\mathsf{SR}(\omega_2)$ holds in the final model, even after the modification to the preparation iteration. And in the last section, we prove that there are no special Aronszajn trees on ω_2, that there are no weak Kurepa trees on ω_1, and we indicate how to make AP_{ω_1} hold or fail.

8.1 Suitable Mixed Support Iterations

In this section we introduce and develop the basic properties of the type of mixed support iteration we're interested in. We will also isolate some properties of the posets which allow us to verify distributivity, and hence cardinal preservation. We will use these results in the next two sections. Throughout this chapter, we will use "odd" to denote the class of odd ordinals and "even" to denote the class of even ordinals.

8.1.1 Basic Facts

Broadly speaking, the iteration will alternate between adding Cohen reals and adding clubs disjoint from non-reflecting subsets of $\omega_2 \cap \mathrm{cof}(\omega)$. The support of a condition in our iteration will be finite on the Cohen part and size $< \omega_2$ on the club adding part. Here is the precise definition:

Definition 8.1.1. *Let $\alpha \leq \omega_3$. Let $\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle$ be a sequence of forcing posets and $\langle \dot{S}_\gamma : \gamma \in \alpha \cap \mathrm{odd} \rangle$ be a sequence so that for all odd $\gamma < \alpha$, $\dot{S}_\gamma$ is a nice $\mathbb{P}_\gamma$-name for a subset of $\omega_2 \cap \mathrm{cof}(\omega)$. Assume that for all $\beta \leq \alpha$, every member of $\mathbb{P}_\beta$ is a function whose domain is a subset of β, and define*

$$\mathbb{P}^c_\beta := \{p \in \mathbb{P}_\beta : \mathrm{dom}(p) \subseteq \mathrm{even}\}.$$

We say that the sequence of forcing posets is a suitable mixed support forcing iteration of length α based on the sequence of names *if the following conditions are satisfied:*

1. *$\mathbb{P}_0 = \{\varnothing\}$ is the trivial forcing;*

2. *if $\gamma < \alpha$ is even, then $p \in \mathbb{P}_{\gamma+1}$ iff p is a function whose domain is a subset of $\gamma + 1$ so that $p \restriction \gamma \in \mathbb{P}_\gamma$ and, if $\gamma \in \mathrm{dom}(p)$, then $p(\gamma) \in \mathrm{Add}(\omega)$;*

3. *if $\gamma < \alpha$ is odd, then $p \in \mathbb{P}_{\gamma+1}$ iff p is a function whose domain is a subset of $\gamma + 1$ so that $p \restriction \gamma \in \mathbb{P}_\gamma$ and, if $\gamma \in \mathrm{dom}(p)$, then $p(\gamma)$ is a nice $\mathbb{P}^c_\gamma$-name for a nonempty closed and bounded subset of ω_2 so that*

$$p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} p(\gamma) \cap \dot{S}_\gamma = \varnothing;$$

4. *if $\delta \leq \alpha$ is a limit ordinal, then $p \in \mathbb{P}_\delta$ iff p is a function whose domain is a subset of δ so that $|\,\mathrm{dom}(p) \cap \mathrm{even}\,| < \omega$, $|\,\mathrm{dom}(p) \cap \mathrm{odd}\,| < \omega_2$, and for all $\beta < \delta$, $p \restriction \beta \in \mathbb{P}_\beta$.*

The ordering is as follows: for all $\beta \leq \alpha$, $q \leq_{\mathbb{P}_\beta} p$ iff $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, and for all $\gamma \in \mathrm{dom}(p)$, if γ is even, then $p(\gamma) \subseteq q(\gamma)$, and if γ is odd, then

$$q \restriction (\gamma \cap \mathrm{even}) \Vdash_{\mathbb{P}^c_\gamma} q(\gamma) \textit{ is an end-extension of } p(\gamma).$$

Observe that this definition makes sense without assuming that the forcing preserves cardinals, if we interpret ω_2 as the ω_2 of the ground model. However, the only such iterations we will in fact study will preserve all cardinals.

Notation: We will often abbreviate $\Vdash_{\mathbb{P}_\beta}$ and $\leq_{\mathbb{P}_\beta}$ by, respectively, the more readable $\Vdash_\beta$ and $\leq_\beta$, where $\beta \leq \alpha$.

We required in Definition 8.1.1(3) that $p(\gamma)$ is a nice $\mathbb{P}^c_\gamma$-name, rather than a $\mathbb{P}_\gamma$-name, in order to prove the following absoluteness result.

Lemma 8.1.2. *Let M be a transitive model of $\mathsf{ZFC}-\mathsf{Powerset}$ so that $\omega_2 \in M$ and ${}^{\omega_1}M \subseteq M$. Suppose that $\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle$ is a sequence of forcing posets in M and $\langle \dot{S}_\gamma : \gamma \in \alpha \cap \mathrm{odd} \rangle$ is a sequence in M so that for each odd $\gamma < \alpha$, $\dot{S}_\gamma$ is a nice $\mathbb{P}_\gamma$-name for a subset of $\omega_2 \cap \mathrm{cof}(\omega)$. Then $\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle$ is a suitable mixed support iteration based on the sequence of names $\langle \dot{S}_\gamma : \gamma \in \alpha \cap \mathrm{odd} \rangle$ iff M satisfies that it is.*

The proof proceeds by verifying that all of the properties in Definition 8.1.1 are absolute between V and M, using the closure of M to see that M contains all of the relevant names.

Let us now fix a particular suitable mixed support forcing iteration $\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle$ based on $\langle \dot{S}_\gamma : \gamma \in \alpha \cap \mathrm{odd} \rangle$. As stated in the next lemma, the "Cohen part" is isomorphic to adding some number of Cohen reals.

Lemma 8.1.3. *Let $\beta \leq \alpha$. Then $\mathbb{P}^c_\beta$ is a regular suborder of $\mathbb{P}_\beta$, and $\mathbb{P}^c_\beta$ is isomorphic to $\mathrm{Add}(\omega, \mathrm{ot}(\beta \cap \mathrm{even}))$.*

Let us now introduce some more notation.

Definition 8.1.4. *Let $\beta \leq \alpha$. For $p, q \in \mathbb{P}_\beta$, we write $q \leq^*_\beta p$ to mean that $q \leq_\beta p$ and that $q \restriction \mathrm{even} = p \restriction \mathrm{even}$. For $p, q \in \mathbb{P}^c_\beta$, we write $q \leq^c_\beta p$ to mean that $q \leq_\beta p$. We will abbreviate $(\mathbb{P}_\beta, \leq^*_\beta)$ and $(\mathbb{P}^c_\beta, \leq^c_\beta)$ by, respectively, $\mathbb{P}^*_\beta$ and $\mathbb{P}^c_\beta$.*

For the next definition, we observe that if $p \in \mathbb{P}_\beta$ and $a \in \mathbb{P}^c_\beta$, then a and p are compatible in $\mathbb{P}_\beta$ iff a and $p \restriction \mathrm{even}$ are compatible in $\mathbb{P}^c_\beta$.

Definition 8.1.5. *Let $\beta \leq \alpha$. If $a \in \mathbb{P}^c_\beta$ and $p \in \mathbb{P}_\beta$, and if a and p are compatible in $\mathbb{P}_\beta$, we write $p + a$ to denote the function s so that $\mathrm{dom}(s) := \mathrm{dom}(a) \cup \mathrm{dom}(p)$, for all even $\gamma \in \mathrm{dom}(s)$, $s(\gamma) := a(\gamma) \cup p(\gamma)$, and for all odd $\gamma \in \mathrm{dom}(s)$, $s(\gamma) := p(\gamma)$.*

It is straightforward to check that under the assumptions of the above definition, $p + a$ extends both p and a and is, moreover, the greatest lower bound of them both.

The following comment helps clarify the definition of the ordering in Definition 8.1.1.

Lemma 8.1.6. *Let $\beta \leq \alpha$, $q \in \mathbb{P}_\beta$, $\dot{x}$ a $\mathbb{P}^c_\beta$-name, and $\varphi(x)$ a Δ_0-formula. Then*

$$q \Vdash_\beta \varphi(\dot{x}) \text{ iff } (q \restriction \mathrm{even}) \Vdash_{\mathbb{P}^c_\beta} \varphi(\dot{x}).$$

.

In particular, the property

$$q \restriction (\gamma \cap \mathrm{even}) \Vdash_{\mathbb{P}^c_\gamma} q(\gamma) \text{ is an end-extension of } p(\gamma)$$

of the ordering is equivalent to

$$q \restriction \gamma \Vdash_\gamma q(\gamma) \text{ is an end-extension of } p(\gamma).$$

The following proposition provides a type of interpolant condition between the Cohen and direct extension (i.e., $\leq^*$) parts.

Proposition 8.1.7. *Let $\beta \leq \alpha$. Suppose that $q \leq_\beta p$. Let $b := q \restriction \mathrm{even}$. Then there exists $q' \in \mathbb{P}_\beta$ so that*

$$q \leq_\beta q' \leq^*_\beta p$$

and

$$q \leq_\beta q' + b \leq_\beta q.$$

Crucial to our analysis of $\mathbb{P}_\alpha$ is to view $\mathbb{P}_\alpha$ as embedded in a type of product.

Definition 8.1.8. *Let $\beta \leq \alpha$. We define $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ as the forcing poset which consists of all pairs (a, p) where $a \in \mathbb{P}^c_\beta$ and $p \in \mathbb{P}_\beta$ so that a and p are compatible in $\mathbb{P}_\beta$, with the ordering $(a_1, p_1) \leq (a_0, p_0)$ iff $a_1 \leq^c_\beta a_0$ and $p_1 \leq^*_\beta p_0$.*

We observe that if $p \in \mathbb{P}_\beta$, then $(p \restriction \text{even}, p) \in \mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$. The next result shows that $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ is essentially a product forcing.

Lemma 8.1.9. *Let $\beta \leq \alpha$, and fix $(a, p) \in \mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ so that $a \leq^c_\beta p \restriction \text{even}$. Then $(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)/(a, p)$ is equal to the product forcing*

$$(\mathbb{P}^c_\beta/a) \times (\mathbb{P}^*_\beta/p).$$

Since there exist densely-many conditions (a, p) in $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ so that $a \leq^c_\beta p$, the previous lemma implies the following.

Lemma 8.1.10. *Let $\beta \leq \alpha$. Suppose that H is a V-generic filter on $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$. Then there is a condition $(a, p) \in H$ so that $a \leq^c_\beta p \restriction \text{even}$. Moreover, if (a, p) is any such condition in H, then letting $K := H \cap ((\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)/(a, p))$, we have that K is a V-generic filter on $(\mathbb{P}^c_\beta/a) \times (\mathbb{P}^*_\beta/p)$ and $V[H] = V[K]$.*

As usual in these types of analyses, there exist forcing projections from the product to the original poset.

Definition 8.1.11. *Let $\beta \leq \alpha$. Define the function $\tau_\beta : \mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta \longrightarrow \mathbb{P}_\beta$ by $\tau_\beta(a, p) := p + a$.*

The definition of τ_β makes sense by Definition 8.1.5. It is straightforward to see that τ_β is a surjective projection mapping.

We conclude this subsection with a few remarks about cardinal preservation.

Lemma 8.1.12. *Assume that $2^{\omega_1} = \omega_2$. Then*

1. *for all $\beta \leq \alpha$ with $|\beta| \leq \omega_2$, $|\mathbb{P}_\beta| \leq \omega_2$;*

2. *if* $\alpha = \omega_3$, *then* $\mathbb{P}_\alpha = \bigcup \{\mathbb{P}_\beta : \beta < \omega_3\}$ *has size* ω_3 *and is* ω_3*-c.c.;*

3. *if* $\alpha = \omega_3$, *then for all* $a \in \mathbb{P}^c_\alpha$, $\mathbb{P}^*_\alpha/a = \bigcup \left\{\mathbb{P}^*_\beta/a : \beta < \omega_3\right\}$ *has size* ω_3 *and is* ω_3*-c.c.*

8.1.2 Distributivity and Cardinal Preservation

The most challenging part of the consistency results is to verify that the suitable mixed support iteration of interest preserves ω_1 and ω_2. As we show below, it suffices to prove that the "term forcing" part $\mathbb{P}^*_\beta$ is ω_2-distributive for all $\beta < \omega_3$.

Let us briefly recall some of the details of the original Harrington-Shelah argument. We start with a model of GCH in which κ is a Mahlo cardinal. Let G be V-generic for the collapse $\mathrm{Col}(\omega_1, < \kappa)$. In $V[G]$, define a forcing iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_3, \beta < \omega_3\rangle$ so that for all $\alpha < \omega_3$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a forcing which destroys the stationarity of a nonreflecting stationary subset of $\omega_2 \cap \mathrm{cof}(\omega)$. Standard bookkeeping arguments show that all such stationary sets can be handled. To prove that the forcing is $\omega_2^{V[G]} = \kappa$-distributive, it suffices to verify this for all $\mathbb{P}_\alpha$ with $\alpha < \omega_3$. So fix an appropriate elementary substructure M which contains $\mathbb{P}_\alpha$, and let π denote the transitive collapse. It suffices to show that every condition in $M \cap \mathbb{P}_\alpha$ can be extended to a completely $(M, \mathbb{P}_\alpha)$-generic condition.

The fact that $\mathbb{P}_\alpha$ is an iteration of posets adding clubs disjoint from nonreflecting subsets of $\omega_2 \cap \mathrm{cof}(\omega)$ implies that $\pi(\mathbb{P}_\alpha)$ is an iteration of adding clubs disjoint from *nonstationary* subsets of the ordinal $M \cap \kappa$. Hence, $\pi(\mathbb{P}_\alpha)$ contains an $(M \cap \kappa)$-closed, dense subset, from which it follows that the tail of the collapse adds a $V[G \restriction (M \cap \kappa)]$-generic filter on $\pi(\mathbb{P}_\alpha)$. The image of this filter under π^{-1} then has a lower bound in $\mathbb{P}_\alpha$ and provides the desired completely generic condition.

In the case of our mixed support iteration, the situation is a bit more complicated. Our preparation will be a Mitchell-style poset $\mathbb{M}$; note that this is different from the countable support preparation iteration used in [34]. In the extension by $\mathbb{M}$, we define a suitable mixed support iteration which adds Cohen reals and clubs disjoint from nonreflecting stationary subsets of $\omega_2 \cap \mathrm{cof}(\omega)$.

Next let M be a model as above, where M contains the suitable mixed support iteration $\mathbb{P}_\alpha$ for some $\alpha < \omega_3$. The natural thing to try would be to argue that in $V[G \restriction (M \cap \kappa)]$, where G is V-generic over M, we have that $\pi(\mathbb{P}_\alpha)$ is a suitable mixed support forcing iteration adding clubs disjoint from nonstationary sets. However, we are only able to show that the product $\pi(\mathbb{P}^c_\alpha \otimes \mathbb{P}^*_\alpha)$ forces that the collapse of a nonreflecting set is nonstationary, rather than the smaller poset $\pi(\mathbb{P}_\alpha)$. Nevertheless, some technical arguments will show that this suffices to prove that $\mathbb{P}^*_\alpha$ is ω_2-distributive, and hence that $\mathbb{P}_\alpha$ preserves cardinals.

Let us now work through the relevant details from [34]. Recall that we have a fixed suitable mixed support iteration $\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle$ based on $\langle \dot{S}_\gamma : \gamma \in \alpha \cap \text{odd} \rangle$.

Proposition 8.1.13. *Let $\beta \leq \alpha$. If $\mathbb{P}^*_\beta$ is ω_2-distributive, then $\mathbb{P}_\beta$ preserves ω_1 and ω_2.*

Proof. Suppose that this were false, and fix a condition $p \in \mathbb{P}_\beta$ which forces that either ω_1^V or ω_2^V is no longer a cardinal. Set $a := p \restriction \text{even}$. Let H be a V-generic filter on $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ which contains the condition (a, p), and set $G := \tau_\beta[H]$. Then G is a generic filter on $\mathbb{P}_\beta$, and $p = p + a = \tau_\beta(a, p)$ is in G. Thus either ω_1^V or ω_2^V is no longer a cardinal in $V[H]$.

By Lemma 8.1.10, we know that $V[H] = V[K]$, where $K = K_1 \times K_2$ is a generic filter over $(\mathbb{P}^c_\beta/a) \times (\mathbb{P}^*_\beta/p)$. Since $\mathbb{P}^*_\beta$ is ω_2-distributive, by assumption, we have that ω_1^V and ω_2^V are still cardinals in $V[K_2]$. Moreover, $\mathbb{P}^c_\beta$ is still isomorphic to Cohen forcing in $V[K_2]$ and hence, being c.c.c., preserves all cardinals. Thus ω_1^V and ω_2^V are still cardinals in $V[K_2][K_1] = V[H]$, a contradiction. $\square$

The following proposition follows from Lemma 8.1.12.

Proposition 8.1.14. *Assume that $2^{\omega_1} = \omega_2$. Suppose that $\alpha = \omega_3$ and that for all $\beta < \alpha$, $\mathbb{P}^*_\beta$ is ω_2-distributive. Then so is $\mathbb{P}^*_{\omega_3}$.*

The following proposition will be crucial in later arguments. It is to be expected, based upon an analogy with Abraham's analysis of Mitchell-type forcings and our product $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$.

Proposition 8.1.15. *Let $\beta \leq \alpha$, and assume that $\mathbb{P}^*_\beta$ is ω_2-distributive. Let $\dot{x}$ be a $\mathbb{P}_\beta$-name for a set of ordinals of size less than ω_2. Then for all $p \in \mathbb{P}_\beta$, there exists $q \leq^*_\beta p$ and a nice $\mathbb{P}^c_\beta$-name $\dot{b}$ for a set of ordinals of size ω_1 so that q forces in $\mathbb{P}_\beta$ that $\dot{x} = \dot{b}$.*

For $A \subseteq \omega_2$, we let $\mathsf{CU}(A)$ denote the forcing which consists of closed, bounded subsets of A, ordered by end-extension. If A is unbounded, it is easy to check that $\mathsf{CU}(A)$ adds a club subset of ω_2 which is contained in A. One of the many consequences of 8.1.15 is that $\mathbb{P}_\alpha$ adds the desired filters for the club adding forcings, as stated in the next proposition.

Proposition 8.1.16. *Let $\gamma < \alpha$ be odd, and suppose that $\mathbb{P}^*_\gamma$ is ω_2-distributive. Then $\mathbb{P}_{\gamma+1}$ is forcing equivalent to $\mathbb{P}_\gamma * \mathsf{CU}(\omega_2 \backslash \dot{S}_\gamma)$.*

We now turn our attention to studying conditions under which $\mathbb{P}^*_\alpha$ is ω_2-distributive.

Lemma 8.1.17. *Let $\gamma < \alpha$ be odd. Assume that $\dot{C}$ is a $(\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma)$-name for a club subset of ω_2 which is disjoint from $\dot{S}_\gamma$. Let $p \in \mathbb{P}_\gamma$, and let $\dot{\zeta}$ be a $\mathbb{P}_\gamma$-name for an ordinal. If $(p \restriction \mathrm{even}, p)$ forces in $\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma$ that $\dot{\zeta} \in \dot{C}$, then p forces in $\mathbb{P}_\gamma$ that $\dot{\zeta}$ is not in $\dot{S}_\gamma$.*

Proof. Suppose for a contradiction that there is $q \leq_\gamma p$ which forces in $\mathbb{P}_\gamma$ that $\dot{\zeta} \in \dot{S}_\gamma$. Let $b := q \restriction \mathrm{even}$. By Proposition 8.1.7, we may fix $q' \in \mathbb{P}_\gamma$ so that $q \leq_\gamma q' \leq^*_\gamma p$ and $q \leq_\gamma q' + b \leq_\gamma q$.

Let H be V-generic over $\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma$ which contains (b, q'), and set $G := \tau_\gamma[H]$, which is V-generic over $\mathbb{P}_\gamma$. Finally, set $\zeta := \dot{\zeta}[G]$, $S_\gamma := \dot{S}_\gamma[G]$, and $C := \dot{C}[H]$, so that $C \cap S_\gamma = \varnothing$.

Since $q' \leq^*_\gamma p$ and $b \leq^c_\gamma p \restriction \mathrm{even}$, it follows that $(b, q') \leq (p \restriction \mathrm{even}, p)$, and so $(p \restriction \mathrm{even}, p) \in H$. Thus $\zeta \in C$, and since $C \cap S_\gamma = \varnothing$, $\zeta \notin S_\gamma$. On the other hand, $\tau_\gamma(b, q') = q' + b \in G$ and $q' + b \leq_\gamma q$, so $q \in G$. By choice of q, $\zeta \in S_\gamma$, a contradiction. $\square$

We need now to define a separative version of the term-forcing part.

Definition 8.1.18. *Let $\beta \leq \alpha$. Define the relation $\leq^{*,s}_\beta$ on $\mathbb{P}_\beta$ by letting $q \leq^{*,s}_\beta p$ if for all $r \leq^*_\beta q$, we have that r and p are compatible in $\mathbb{P}^*_\beta$. We will abbreviate $(\mathbb{P}_\beta, \leq^{*,s}_\beta)$ as $\mathbb{P}^{*,s}_\beta$.*

Note that $q \leq_\beta^* p$ implies that $q \leq_\beta^{*,s} p$. It is straightforward to verify that $\mathbb{P}_\beta^{*,s}$ is separative and that the identity function is a dense embedding of $\mathbb{P}_\beta^*$ into $\mathbb{P}_\beta^{*,s}$.

The next lemma provides some details about this separative ordering.

Lemma 8.1.19. *Let $\beta \leq \alpha$. Assume that $q \leq_\beta^{*,s} p$. Then*

1. $p \restriction \mathrm{even} = q \restriction \mathrm{even}$*;*
2. $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$*;*
3. *for all $\gamma \in \mathrm{dom}(p)$, $p \restriction (\gamma \cap \mathrm{even})$ forces in $\mathbb{P}_\gamma^c$ that one of $p(\gamma)$ and $q(\gamma)$ is an end-extension of the other.*

The following proposition provides the key sufficient condition for distributivity; we will provide a sketch of the proof.

Proposition 8.1.20. *Assume that for all odd $\gamma < \alpha$, $\mathbb{P}_\gamma^c \otimes \mathbb{P}_\gamma^*$ forces that $\dot{S}_\gamma$ is a nonstationary subset of ω_2. Then both $\mathbb{P}_\alpha^*$ and $\mathbb{P}_\alpha^{*,s}$ contain an ω_2-closed, dense subset.*

Proof. For each odd $\gamma < \alpha$. let $\dot{C}_\gamma$ be a $(\mathbb{P}_\gamma^c \otimes \mathbb{P}_\gamma^*)$-name for a club subset of ω_2 which is disjoint from $\dot{S}_\gamma$. For each $\beta \leq \alpha$, define D_β as the set of conditions $p \in \mathbb{P}_\beta$ so that for all odd $\gamma \in \mathrm{dom}(p)$, $(p \restriction (\gamma \cap \mathrm{even}), p \restriction \gamma)$ forces in $\mathbb{P}_\gamma^c \otimes \mathbb{P}_\gamma^*$ that $\max(p(\gamma)) \in \dot{C}_\gamma$. Note that for all $\xi \leq \beta \leq \alpha$, $D_\xi \subseteq D_\beta$ and that if $p \in D_\beta$, then $p \restriction \xi \in D_\xi$.

We will prove by induction on $\beta \leq \alpha$ that D_β is an ω_2-closed dense subset of both $\mathbb{P}_\beta^*$ and $\mathbb{P}_\beta^{*,s}$. Let $\beta \leq \alpha$ be fixed, and assume that this holds for all $\xi < \beta$, noting that as a result, $\mathbb{P}_\xi^*$ is ω_2-distributive, being forcing equivalent to an ω_2-closed poset.

We first verify closure. It suffice to show that any $\leq_\beta^{*,s}$-descending sequence of conditions in D_β of limit length $\delta < \omega_2$ has a lower bound in D_β. Thus fix such a sequence $\langle p_i : i < \delta \rangle$. Let $a := p_0 \restriction \mathrm{even}$ so that for all $i < \delta$, by Lemma 8.1.19(1), $a = p_i \restriction \mathrm{even}$.

Define the function q as follows: set $q \restriction \mathrm{even} = a$. Let

$$\mathrm{dom}(q) \cap \mathrm{odd} := \bigcup \{\mathrm{dom}(p_i) \cap \mathrm{odd} : i < \delta\}.$$

Now let γ be an odd ordinal in $\mathrm{dom}(q)$. By Lemma 8.1.19(3), $a \restriction \gamma$ forces in $\mathbb{P}^c_\gamma$ that $\{p_i(\gamma) : i < \delta\}$ is a family of closed, bounded subsets of ω_2 which are pairwise compatible under end-extension. Thus $a \restriction \gamma$ forces that the union of this set is bounded in ω_2 and closed below its supremum. We set $q(\gamma)$ to be a nice $\mathbb{P}^c_\gamma$-name for a nonempty, closed and bounded subset of ω_2 which is equal to $\bigcup\{p_i(\gamma) : i < \delta\} \cup \{\sup(\{\max(p_i(\gamma)) : i < \delta\})\}$ if $a \restriction \gamma$ is in the generic filter.

We now prove by induction on $\xi \leq \beta$ that $q \restriction \xi \in D_\xi$ and $q \restriction \xi \leq^*_\xi p_i \restriction \xi$ for all $i < \delta$. The only nontrivial case to consider is when $\xi = \gamma + 1$ for an odd ordinal γ.

So fix such a γ. Then $q \restriction \gamma \leq^*_\gamma p_i \restriction \gamma$ for all $i < \delta$. Thus by definition of D_β, each p_i with $\gamma \in \mathrm{dom}(p_i)$ satisfies that $(p_i \restriction (\gamma \cap \mathrm{even}), p_i \restriction \gamma) = (a \restriction \gamma, p_i \restriction \gamma)$ forces in $\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma$ that $\max(p_i(\gamma)) \in \dot{C}_\gamma$. Thus $(q \restriction (\gamma \cap \mathrm{even}), q \restriction \gamma) = (a \restriction \gamma, q \restriction \gamma)$ forces in $\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma$ that $\max(p_i(\gamma)) \in \dot{C}_\gamma$ for each $i < \delta$. As a result, and since $\dot{C}_\gamma$ names a club, we have that $(q \restriction (\gamma \cap \mathrm{even}), q \restriction \gamma) = (a \restriction \gamma, q \restriction \gamma)$ forces in $\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma$ that $\max(q(\gamma)) \in \dot{C}_\gamma$. By Lemma 8.1.17, we then see that $q \restriction \gamma$ forces that $q(\gamma)$ is disjoint from $\dot{S}_\gamma$. The inductive hypothesis then shows that $q \restriction (\gamma + 1) \in D_{\gamma+1}$ and $q \restriction (\gamma + 1) \leq^*_{\gamma+1} p_i \restriction (\gamma + 1)$ for all $i < \delta$.

The proof of density is similar, and the full details may be found in [34]. $\square$

The next lemma describes how we will use the preparation forcing (recall that our new preparation is itself a mixed-support iteration) in proofs of the main consistency results.

Lemma 8.1.21. *Assume that $2^{\omega_1} = \omega_2$. Suppose that $\mathbb{P}^{*,s}_\alpha$ contains an ω_2-closed dense subset. Let $G \times H$ be a generic filter on $\mathrm{Add}(\omega, \omega_2) \times \mathrm{Add}(\omega_2)$. Then in $V[G \times H]$, for any condition $(a, p) \in \mathbb{P}^c_\alpha \otimes \mathbb{P}^*_\alpha$ such that $a \leq^c_\alpha p \restriction \mathrm{even}$, there exists a V-generic filter K on $\mathbb{P}^c_\alpha \otimes \mathbb{P}^*_\alpha$ which contains (a, p), and moreover, $V[G \times H]$ is a generic extension of $V[K]$ by Cohen forcing.*

The proof of the above lemma is straightforward, and the key idea is the following: if D is the ω_2-closed, dense subset of $\mathbb{P}^{*,s}_\alpha$ from Proposition 8.1.20, then because $\mathbb{P}^{*,s}_\alpha$ is a separative of size ω_2 so that every extension has ω_2-many incompatible extensions, it is equivalent to $\mathrm{Add}(\omega_2, 1)$.

We have one final lemma to mention in this section.

Lemma 8.1.22. *Assume that for all $\beta < \alpha$, $\mathbb{P}_\beta$ preserves ω_1. Suppose that $\langle p_i : i < \delta \rangle$ is a $\leq^*_\alpha$-descending sequence of conditions, where $\delta \in \omega_2 \cap \mathrm{cof}(\omega_1)$. Then there is a condition q so that $q \leq^*_\alpha p_i$ for all $i < \delta$.*

The main idea for the proof of Lemma 8.1.22 is that we may find lowers bound for the club-parts, as the stationary sets which the clubs are supposed to be disjoint from consist entirely of cofinality ω ordinals.

8.2 Verifying $\mathsf{SR}(\omega_2)$

In this section, we define the preparation forcing and the suitable mixed support iteration in the extension by the preparation. Let $\mathbb{M}$ be the poset $\mathbb{M}_0$ from Definition 5.2.2.

Remark 8.2.1. We recall that for each inaccessible $\alpha < \kappa$, letting α^* denote the least inaccessible above α, $\mathbb{M} \upharpoonright \alpha^*$ is isomorphic to

$$\mathbb{M} \upharpoonright \alpha * (\mathrm{Add}(\omega, \alpha^*) \times \mathrm{Col}(\omega_1, \alpha) \times \mathrm{Add}(\alpha, 1)).$$

Let G be V-generic over $\mathbb{M}$, and in $V[G]$ we define a sequence of posets $\langle \mathbb{P}_\beta : \beta \leq \kappa^+ \rangle$. The sequence will be a suitable mixed support forcing iteration based upon a sequence of names $\langle \dot{S}_\gamma : \gamma \in \kappa^+ \cap \mathrm{odd} \rangle$, each of which is forced in the appropriate $\mathbb{P}_\gamma$ to be a nonreflecting subset of $\kappa \cap \mathrm{cof}(\omega)$. Definition 8.1.1 provides a recursive description of the iteration once we specify all of the names.

In order to verify that $\mathbb{P}_{\kappa^+}$ preserves all cardinals, we will assume two recursion hypotheses in $V[G]$. Fix a $\beta < \kappa^+$, and suppose that we've defined $\langle \mathbb{P}_\delta : \delta \leq \beta \rangle$ and $\langle \dot{S}_\gamma : \gamma \in \beta \cap \mathrm{odd} \rangle$.

> **Recursion Hypothesis I:** For all $\xi \leq \beta$, the poset $\mathbb{P}^*_\xi$ is ω_2-distributive (and therefore $\mathbb{P}_\xi$ preserves ω_1 and ω_2, by Proposition 8.1.13).

If we have the first recursion hypothesis for all $\beta < \kappa^+$, then $\mathbb{P}_{\kappa^+}$ preserves all cardinals: $\mathbb{P}_{\kappa^+}$ is κ^+-c.c. by Lemma 8.1.12, and $\mathbb{P}^*_{\kappa^+}$ is κ-distributive by Proposition 8.1.14 and hence preserves ω_1 and ω_2 by Proposition 8.1.13. Furthermore, by κ^+-c.c., any nice $\mathbb{P}_{\kappa^+}$-name $\dot{S}$ for a nonreflecting subset of $\kappa \cap \mathrm{cof}(\omega)$ is a nice $\mathbb{P}_\beta$-name, for some $\beta < \kappa^+$. Since each such initial segment has size κ and $2^\kappa = \kappa^+$ in $V[G]$, standard bookkeeping arguments show that we can arrange that any such name $\dot{S}$ is dealt with in the course of the iteration, i.e., that $\dot{S}$ appears as $\dot{S}_\gamma$ for some odd $\gamma < \kappa^+$. And finally, since $\mathbb{P}_{\gamma+1}$ is forcing equivalent to $\mathbb{P}_\gamma * \mathsf{CU}(\kappa \backslash \dot{S}_\gamma)$ by Proposition 8.1.16, we have that $\mathbb{P}_{\kappa^+}$ forces that $\dot{S}$ is nonstationary.

Maintaining the first recursion hypothesis requires that we work with a second recursion hypothesis and also introduce some more terminology.

Definition 8.2.2. *A set N in the ground model V is said to be* suitable *if $N \prec H(\theta)$, for some large enough, regular θ; $|N| < \kappa$; $\kappa_N := N \cap \kappa$ is an inaccessible cardinal; $|N| = \kappa_N$; ${}^{<\kappa_N}N \subseteq N$; and if $\mathbb{M} \in N$.*

*If $\beta < \kappa^+$, then we say that N is β-*suitable *if N is suitable, and if N contains $\mathbb{M}$-names for $\langle \mathbb{P}_\xi : \xi \leq \beta \rangle$ and $\langle \dot{S}_\gamma : \gamma \in \beta \cap \mathrm{odd} \rangle$.*

The second recursion hypothesis is the following:

> **Recursion Hypothesis II:** Suppose that N is β-suitable, and let π be the transitive collapse of $N[G]$. Then for all $\gamma \in N \cap \beta$, in the model $V[G \restriction \kappa_N]$, $\pi(\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma)$ forces that $\pi(\dot{S}_\gamma)$ is a nonstationary subset of κ_N.

To see that these recursion hypotheses hold for all $\beta < \kappa^+$, it suffices (see [34]) to prove the second recursion hypothesis only in the successor case $\beta + 1$ when β is odd, and then prove the first recursion hypothesis in an independent way. Both proofs will use the following lemma.

Remark 8.2.3. We remark here that the analogous version of this lemma in [34] (Lemma 3.7 there) does not carry through in its entirety to our setting. The reason is that we have added extra collapsing in our preparation (in order to ensure that AP_{ω_1} holds), and these collapsing posets will fail to be ω_1-closed after adding Cohen reals.

Lemma 8.2.4. *Suppose that both recursion hypotheses hold for all $\gamma < \beta$ and the second holds for β. Let N be β-suitable and $(a, p) \in N$ a condition in $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$. Let π be the transitive collapsing map of $N[G]$. Then in $V[G]$ there exists a $V[G \restriction \kappa_N]$-generic filter K on $\pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$ which contains $\pi(a, p)$.*

Furthermore, letting $J := \pi(\tau_\beta)[K]$, $K^+ := \pi^{-1}[K]$, and $J^+ := \pi^{-1}[J]$, we have that K^+ is an $(N[G], \mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$-generic filter containing (a, p), J^+ is an $(N[G], \mathbb{P}_\beta)$-generic filter, and $J^+ = \tau_\beta[K^+]$. Moreover, there exists $s \in \mathbb{P}_\beta$ so that for all $(b, q) \in K^+$, $s \leq^*_\beta q$.

Proof. We assume, by extending if necessary, that $a \leq_\beta p \restriction \text{even}$. Let $\pi(\langle \mathbb{P}_\xi : \xi \leq \beta \rangle) = \langle \mathbb{P}^\pi_\xi : \xi \leq \pi(\beta) \rangle$ and $\pi(\langle \dot{S}_\gamma \in \beta \cap \text{odd} \rangle) = \langle \dot{S}^\pi_\gamma : \gamma \in \pi(\beta) \cap \text{odd} \rangle$. We note here that by Lemma 8.1.2, $\langle \mathbb{P}^\pi_\xi : \xi \leq \pi(\beta) \rangle$ is a suitable mixed support forcing iteration based upon the sequence of names $\langle \dot{S}^\pi_\gamma : \gamma \in \pi(\beta) \cap \text{odd} \rangle$.

The second recursion hypothesis entails that in $V[G \restriction \kappa_N]$, for all $\gamma \in \pi(\beta) \cap \text{odd}$, $(\mathbb{P}^\pi_\gamma)^c \otimes (\mathbb{P}^\pi_\gamma)^*$ forces that $\dot{S}^\pi_\gamma$ is nonstationary in κ_N, and therefore by Proposition 8.1.20, $\pi(\mathbb{P}^{*,s}_\beta)$ contains a κ_N-closed, dense subset.

The preparation iteration $\mathbb{M}$ adds a $V[G \restriction \kappa_N]$-generic filter L over the poset $\text{Add}(\omega, \kappa_N) \times \text{Add}(\kappa_N, 1)$. By Lemma 8.1.21, in the model $V[G \restriction \kappa_N][L]$, we have a $V[G \restriction \kappa_N]$-generic filter K on $\pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$ which contains $\pi(a, p)$, and $V[G \restriction \kappa_N][L]$ is a generic extension of $V[G \restriction \kappa_N][K]$ by Cohen (or trivial) forcing.

We recall that $\tau_\beta : \mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta \longrightarrow \mathbb{P}_\beta$ is a surjective projection mapping. By absoluteness and the fact that π is an isomorphism, in $V[G \restriction \kappa_N]$, $\pi(\tau_\beta)$ is a surjective projection mapping from $\pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$ onto $\pi(\mathbb{P}_\beta)$. Let J, K^+, and J^+ be defined as in the statement of the lemma. It is straightforward to check that K^+ and J^+ are filters on $N[G] \cap (\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$ and $N[G] \cap \mathbb{P}_\beta$ respectively, and that $J^+ = \tau_\beta[K^+]$. It is also straightforward, using the isomorphism π, to check that K^+ is in fact an $N[G]$-generic filter over $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$, and similarly for J^+.

By Lemma 8.1.10, we may write $V[G \restriction \kappa_N][K] = V[G \restriction \kappa_N][K_1 \times K_2]$, where $K_1 \times K_2 := K \cap (\pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)/\pi(a, p))$ is a $V[G \restriction \kappa_N]$-generic filter on $(\pi(\mathbb{P}_\beta)^c/\pi(a)) \times (\pi(\mathbb{P}_\beta)^*/\pi(p))$. By Proposition 8.1.20, $\pi(\mathbb{P}_\beta)^*$ contains a κ_N-closed dense subset. By standard arguments it follows that there exists in $V[G \restriction \kappa_N][K]$ a $\pi(\leq^*_\beta)$-descending sequence $\langle q_i : i < \kappa_N\rangle$ below $\pi(p)$ which is dense in K_2. Let $r_i := \pi^{-1}(q_i)$ for all $i < \kappa_N$. Then $\langle r_i : i < \kappa_N\rangle$ is a $\leq^*_\beta$-descending sequence of conditions in $N[G] \cap \mathbb{P}^*_\beta$, below p, which is dense in $\pi^{-1}[K_2]$.

κ_N has cofinality ω_1 in $V[G]$, and since both recursion hypothesis hold for all $\gamma < \beta$, we know that $\mathbb{P}_\gamma$ preserves ω_1, for all $\gamma < \beta$. By Lemma 8.1.22, we may find a condition $s \in \mathbb{P}_\beta$ so that $s \leq^*_\beta r_i$ for all $i < \kappa_N$, and hence $s \leq^*_\beta r$ for all $r \in \pi^{-1}[K_2]$. It is easy to see that s satisfies the conclusion of the present lemma. □

Proposition 8.2.5. *Suppose that $\beta < \omega_3$ is odd, and assume both recursion hypotheses for all $\gamma \leq \beta$. Let N be $(\beta+1)$-suitable and π the transitive collapse map of $N[G]$. Then for all odd $\gamma \in N \cap (\beta+1)$, in the model $V[G \restriction \kappa_N]$, $\pi(\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma)$ forces that $\pi(\dot{S}_\gamma)$ is a nonstationary subset of κ_N.*

Proof. Since N is $(\beta+1)$-suitable, $\beta \in N$, and so, by the recursion hypotheses, we have the conclusion of the lemma for all odd $\gamma \in N \cap \beta$. So it suffices to verify that in $V[G \restriction \kappa_N]$, $\pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$ forces that $\pi(\dot{S}_\beta)$ is a nonstationary subset of κ_N.

Fix a condition $(a_0, p_0) \in \pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$, and we find an extension which forces that $\pi(\dot{S}_\beta)$ is nonstationary in κ_N. Assume, by extending if necessary, that $a_0 \leq p_0 \restriction \text{even}$ in $\pi(\mathbb{P}_\beta)^c$. By Lemma 8.1.9, we know that $\pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$ is equal to the product forcing $(\pi(\mathbb{P}_\beta)^c/a_0) \times (\pi(\mathbb{P}_\beta)^*/p_0)$.

Let K, J, K^+, J^+, and s be as in Lemma 8.2.4, where $(a_0, p_0) \in K$. We use J^+ to partially interpret the name $\dot{S}_\beta$ by letting S be the set of $\alpha < \kappa_N$ so that for some $u \in J^+$, $u \Vdash_\beta \check{\alpha} \in \dot{S}_\beta$. We claim that $S = \pi(\dot{S}_\beta)[J]$.

To see this, fix $\alpha < \kappa_N$. In $V[G]$, let D be the dense open set of conditions in $\mathbb{P}_\beta$ which decide whether or not α is a member of $\dot{S}_\beta$. By the elementarity of $N[G]$, $D \in N[G]$. Since J^+ is $N[G]$-generic, we may fix $w \in J^+ \cap D$. Let $w' := \pi(w)$, which is in $\pi(D)$. Since π is an isomorphism and by absoluteness, w' decides in $\pi(\mathbb{P}_\beta)$ whether or not $\pi(\alpha) = \alpha$ is in $\pi(\dot{S}_\beta)$ the same way as w decides whether α is in $\dot{S}_\beta$. From this it is easy to see that $S = \pi(\dot{S}_\beta)[J]$.

By the choice of $\dot{S}_\beta$, we have that $\mathbb{P}_\beta$ forces over $V[G]$ that $\dot{S}_\beta$ does not reflect to any ordinal in κ of cofinality ω_1. Now $\mathrm{cf}^{V[G]}(\kappa_N) = \omega_1$, and $\mathbb{P}_\beta$ preserves this by the recursion hypotheses. Hence $\mathbb{P}_\beta$ forces that there is a club $\dot{c}$ in κ_N of ordertype ω_1 disjoint from $\dot{S}_\beta \cap \kappa_N$. By the first recursion hypothesis for β, $\mathbb{P}^*_\beta$ is ω_2-distributive in $V[G]$, and therefore we may find $t \leq^*_\beta s$ and a $\mathbb{P}^c_\beta$-name $\dot{c}_0$ so that $t \Vdash_\beta \dot{c} = \dot{c}_0$. By the maximality principle for names, we may assume that $\dot{c}_0$ is a $\mathbb{P}^c_\beta$-name for a club of ordertype ω_1. As $\mathbb{P}^c_\beta$ is c.c.c., we may find a club subset d of κ_N in $V[G]$ so that $\mathbb{P}^c_\beta$ forces that $d \subseteq \dot{c}_0$. Then $t \Vdash_\beta d \cap \dot{S}_\beta = \varnothing$. From this, it is straightforward to see that $d \cap S = \varnothing$.

We now know that S is a nonstationary subset of κ_N in the model $V[G]$; we will show that this is true in the intermediate model in which S appears. Let us use $K_1 \times K_2 \times K_3$ to denote the $V[G \restriction \kappa_N]$-generic filter added by G over $\mathrm{Add}(\omega, \kappa_N^*) \times \mathrm{Col}(\omega_1, \kappa) \times \mathrm{Add}(\kappa_N, 1)$, noting that $V[G \restriction \kappa_N][K_1 \times K_2 \times K_3]$ equals $V[G \restriction \kappa_N^*]$, where κ_N^* is the least V-inaccessible cardinal above κ_N (recall Remark 8.2.1). Finally, let K_1^0 be the generic that K_1 adds to $\pi(\mathbb{P}_\beta)^c$, and let K_1^1 be such that (up to an abuse of notation) $K_1^0 \times K_1^1 = K_1$.

We have that S is a member of $V[G \restriction \kappa_N][K_1^0 \times K_3]$. Since $\mathbb{M}/(G \restriction \kappa_N^*)$ is proper in $V[G \restriction \kappa_N^*]$ by Corollary 5.3.4, we know that S is nonstationary in $V[G \restriction \kappa_N^*] = V[G \restriction \kappa_N][K_1 \times K_2 \times K_3]$. Cohen forcing is proper, so S is still nonstationary in $V[G \restriction \kappa_N][K_1^0 \times K_2 \times K_3]$. Now let us use $\mathbb{Q}$ to abbreviate $\mathrm{Col}(\omega_1, \kappa)$ as computed in $V[G \restriction \kappa_N]$; this is the same collapse as computed in the extension by $\mathrm{Add}^{V[G \restriction \kappa_N]}(\kappa_N, 1)$ as the latter poset is ω_1-closed. By Theorem 6.3.2 (the Easton-style lemma for stationary set preservation) applied in the model $V[G \restriction \kappa_N][K_3]$ with respect to a $\pi(\mathbb{P}_\beta)^c$-name for $\pi(\dot{S}_\beta)$ and with respect to the forcing $\mathbb{Q}$, we know that forcing with $\mathbb{Q}$ over $V[G \restriction \kappa_N][K_1^0 \times K_3]$ preserves stationary sets of cofinality ω ordinals. Thus S is nonstationary in $V[G \restriction \kappa_N][K_1^0 \times K_3]$.

To finish the proof, let $(a, p) \leq (a_0, p_0)$ in $K_1^0 \times K_3$ which forces in $\pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$ that $\pi(\dot{S}_\beta)$ is nonstationary in κ_N. $\square$

We now verify that the first recursion hypothesis holds for β, which will finish the proof that $\mathbb{P}_{\kappa^+}$ preserves cardinals.

Proposition 8.2.6. *Let $\beta < \kappa^+$, and assume the first and second recursion hypotheses hold* *for all $\gamma < \beta$ and that the second holds for β. Then $\mathbb{P}^*_\beta$ is ω_2-distributive.*

Proof. Let $p \in \mathbb{P}_\beta$ be a condition which forces in $\mathbb{P}^*_\beta$ that $\langle \dot{\alpha}_i : i < \omega_1 \rangle$ is a sequence of ordinals. We will find $s \leq^*_\beta p$ which decides, in the poset $\mathbb{P}^*_\beta$, the value of $\dot{\alpha}_i$ for all $i < \omega_1$.

Let N be β-suitable so that $N[G]$ contains p and $\langle \dot{\alpha}_i : i < \omega_1 \rangle$, and let π be the transitive collapse of $N[G]$. Let K, J, K^+, J^+, and s be as in Lemma 8.2.4, where $\pi(p \upharpoonright \text{even}, p) \in K$. Then $(p \upharpoonright \text{even}, p) \in K^+$.

Let $i < \omega_1$, and we will show that s decides the value of $\dot{\alpha}_i$. Indeed, letting D be the set of $(b, q) \in \mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ below $(p \upharpoonright \text{even}, p)$ so that q decides the value of $\dot{\alpha}_i$ in $\mathbb{P}^*_\beta$, we have that $D \in N[G]$ and that D is dense below $(p \upharpoonright \text{even}, p)$. Thus there exists $(b, q) \in D \cap K^+$. By Lemma 8.2.4, $s \leq^*_\beta q$, and since q decides the value of $\dot{\alpha}_i$, so does s. $\square$

8.3 Properties of Trees in the Final Model

We now aim to show that there exist neither weak Kurepa trees on ω_1 nor special Aronszajn trees on $\kappa = \omega_2$ in the extension by $\mathbb{M} * \dot{\mathbb{P}}_{\kappa^+}$. We begin with the first.

Proposition 8.3.1. *There are no weak Kurepa trees on ω_1 in $\mathbb{M} * \dot{\mathbb{P}}_{\kappa^+}$.*

Proof. Since $\mathbb{M} * \dot{\mathbb{P}}_{\kappa^+}$ is κ^+-c.c., it suffices to show that for each $\beta < \kappa^+$, there are no weak Kurepa trees on ω_1 in the extension by $\mathbb{M} * \dot{\mathbb{P}}_\beta$. So fix a $\beta < \kappa^+$.

Suppose for a contradiction that there is an $(\mathbb{M} * \dot{\mathbb{P}}_\beta)$-name $\dot{T}$ which is forced (without loss of generality) by the empty condition to be a weak Kurepa tree on ω_1. Before continuing with the main body of the argument, we want to reduce to the case where we have an $\mathbb{M} * \operatorname{Add}(\omega, \omega_1)$-name $\dot{U}$ for a weak Kurepa tree on ω_1.

Towards this end, let us momentarily step into an arbitrary $\mathbb{M}$-generic extension $V[G']$ of V. In $V[G']$, we have that $\dot{T}$ (abusing notation, this is the $\mathbb{P}_\beta$-name in $V[G']$ forced to be equal to $\dot{T}$) names an object of size ω_1. By Proposition 8.1.15, we can find $q \leq^*_\beta \varnothing$ and a nice $\mathbb{P}^c_\beta$-name $\dot{T}_c$ of size ω_1 so that $q \Vdash_{\mathbb{P}_\beta} \dot{T} = \dot{T}_c$.

Now let $X \subseteq \beta \cap \text{even}$ be of size ω_1 so that $\dot{T}_c$ is an $\text{Add}(\omega, X)$-name. Let $\mathcal{A}$ be $V[G']$-generic over $\mathbb{P}^c_\beta \cong \text{Add}(\omega, \text{ot}(\beta \cap \text{even}))$, and write $\mathcal{A}$ as $\mathcal{A}_X \times \mathcal{A}_{\beta \setminus X}$, where $\mathcal{A}_X$ is generic for $\text{Add}(\omega, X)$ and $\mathcal{A}_{\beta\setminus X}$ is generic for $\text{Add}(\omega, \beta \setminus X)$. Then we have that T is a member of $V[G' * \mathcal{A}_X]$. By Lemma 6.1.4 we know that forcing with $\text{Add}(\omega, \beta\setminus X)$ over $V[G' * \mathcal{A}_X]$ does not add any branches through T. Furthermore, as verified in the previous section, $\mathbb{P}^*_\beta$ is ω_2-distributive in $V[G']$, and since $\mathbb{P}^c_\beta$ is c.c.c., $\mathbb{P}^*_\beta$ remains ω_2-distributive after forcing with $\mathbb{P}^c_\beta$. Thus forcing with $\mathbb{P}^*_\beta$ over $V[G' * \mathcal{A}]$ does not add any branches through T. In particular, if we force with $\mathbb{P}^*_\beta$ over $V[G' * \mathcal{A}]$ below the condition q, letting H be the generic we obtain as a result, then we have that $\dot{T}[\mathcal{A} \times H] = \dot{T}_c[\mathcal{A}_X]$ is a weak Kurepa tree on ω_1. Consequently, all of the branches of T live in $V[G' * \mathcal{A}_X]$. Note that since $|X| = \aleph_1$, we have that T lives in a generic extension of $V[G']$ by $\text{Add}(\omega, \omega_1)$ and is a weak Kurepa tree there.

Now let us return back to V. We observe that the arguments of the previous few paragraphs demonstrate that if there exists an $(\mathbb{M} * \dot{\mathbb{P}}_\beta)$-name for a weak Kurepa tree on ω_1, then there exists an $\mathbb{M} * \text{Add}(\omega, \omega_1)$-name for such a tree. Since we are assuming that there is an $(\mathbb{M} * \dot{\mathbb{P}}_\beta)$-name for a weak Kurepa tree on ω_1, we may therefore fix an $\mathbb{M} * \text{Add}(\omega, \omega_1)$-name $\dot{U}$ for a weak Kurepa tree on ω_1.

We now proceed to derive a contradiction from this assumption. Let N be any suitable model with $\dot{U} \in N$. Let G be V-generic over $\mathbb{M}$, and let R be a $V[G]$-generic filter over $\text{Add}(\omega, \omega_1)$. Let π denote the transitive collapse $\pi : N[G] \longrightarrow \bar{N}[G \restriction \kappa_N]$, and let j denote the inverse of π. Since R is $V[G]$-generic over $\text{Add}(\omega, \omega_1)$, R is also $V[G \restriction \kappa_N]$-generic over $\text{Add}(\omega, \omega_1)$, and in $V[G \restriction \kappa_N][R]$, we have that $R = j[R]$. Thus we may lift $j : \bar{N}[G \restriction \kappa_N][R] \longrightarrow N[G][R]$. Let $U := \dot{U}[G * R]$, and observe that since U is (coded by) a subset of $\omega_1 < \kappa_N = \text{crit}(\pi)$, we have that $\pi(U) = U$. From this we conclude that $U \in \bar{N}[G \restriction \kappa_N][R]$.

We will next argue that forcing with the tail forcing $\mathbb{M}/(G \restriction \kappa_N)$ over the model $V[G \restriction \kappa_N][R]$ does not add any branches through U. But first a comment: note here that the Cohen generic reals added by R are *not* the same as those added by G at "stage" κ_N over $V[G \restriction \kappa_N]$. We need to use these "top" Cohen generic reals over the intermediate model

$V[G \restriction \kappa_N]$ to realize the name $\dot{U}$.

Returning to the main thread, we have that in $V[G \restriction \kappa_N]$, the tail forcing $\mathbb{M}/(G \restriction \kappa_N)$ is a forcing projection of a product $\mathbb{A} \times \mathbb{B}$, where $\mathbb{A}$ is isomorphic to $\mathrm{Add}(\omega, \kappa)$, and where $\mathbb{B}$ is ω_1-closed. Moreover, in the model $V[G \restriction \kappa_N]$, we have that $\mathrm{Add}(\omega, \omega_1)$ is c.c.c., $\dot{U}$ is an $\mathrm{Add}(\omega, \omega_1)$-name for a tree of width $\leq \omega_1$, and that $2^\omega = \kappa_N > \omega_1$. Thus by Theorem 6.3.1, we have that forcing with $\mathbb{B}$ over the model $V[G \restriction \kappa_N][R]$ does not add any cofinal branches through U. Finally, subsequent forcing with $\mathbb{A}$ does not add cofinal branches through U, and since $\mathbb{A} \times \mathbb{B}$ projects onto $\mathbb{M}/(G \restriction \kappa_N)$, we conclude that U does not have any new branches in $V[G * R]$.

Now we may obtain our contradiction. Recall that U is a weak Kurepa tree on ω_1 in the model $V[G * R]$. By definition, U then has at least κ-many branches in the model $V[G * R]$. By the results of the previous paragraph, all of these branches live in $V[G \restriction \kappa_N][R]$. However, in the model $V[G \restriction \kappa_N][R]$, we have that κ is inaccessible, and since U has size ω_1, there are at most $(2^{\omega_1})^{V[G \restriction \kappa_N][R]} < \kappa$-many branches through U in $V[G \restriction \kappa_N][R]$, contradicting the fact that U has κ-many branches in that model. $\square$

Proposition 8.3.2. *There does not exist a special Aronszajn tree on $\kappa = \omega_2$ in the extension by $\mathbb{M} * \dot{\mathbb{P}}_{\kappa^+}$.*

Proof. Since $\mathbb{M} * \dot{\mathbb{P}}_{\kappa^+}$ is κ^+-c.c., it suffices to show that for each $\beta < \kappa^+$, there does not exist a special Aronszajn tree on ω_2 in the $\mathbb{M} * \dot{\mathbb{P}}_\beta$-extension. Fix some $\beta < \kappa^+$.

Suppose for a contradiction that there exists an $\mathbb{M} * \dot{\mathbb{P}}_\beta$-name $(\dot{T}, \dot{f})$ forced (without loss of generality) by the empty condition to be a special Aronszajn tree on κ with specializing function $\dot{f} : \dot{T} \longrightarrow \omega_1$. Recalling that the set of β-suitable models is stationary, we may fix a β-suitable model N so that $(\dot{T}, \dot{f}) \in N$. Let G be a V-generic filter over $\mathbb{M}$, and let π denote the transitive collapse $\pi : N[G] \longrightarrow \bar{N}[G \restriction \kappa_N]$ of $N[G]$. Finally, let $j := \pi^{-1}$.

The proof will proceed by lifting j in a particular way and then showing that the tail forcing to complete the generic on the $\bar{N}$ side to the generic on the j-side cannot add branches through some initial segment of $\dot{T}$, a contradiction.

Let $K_1 \times K_2 \times K_3$ be the $V[G \restriction \kappa_N]$-generic added by G over $\mathrm{Add}(\omega, \kappa_N^*) \times \mathrm{Col}(\omega_1, \kappa) \times \mathrm{Add}(\kappa_N, 1)$, where κ_N^* is the least V-inaccessible above κ_N. By Proposition 8.1.20, we know that the $V[G \restriction \kappa_N]$-generic K_3 over $\mathrm{Add}(\kappa_N, 1)$ gives a filter which is $V[G \restriction \kappa_N]$-generic over the poset $\pi(\mathbb{P}^*_\beta/\varnothing)$, and in an abuse of notation, we will use K_3 to denote this $V[G \restriction \kappa_N]$-generic filter over $\pi(\mathbb{P}^*_\beta/\varnothing)$. We also have by Lemma 8.2.4 that the filter $j[K_3]$ has a lower bound in $\mathbb{P}^*_\beta/\varnothing$; let q be some such lower bound.

Now we wish to force with $(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)/\varnothing$ over $V[G]$; recall that by Lemma 8.1.9, $(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)/\varnothing$ is equal to $\mathbb{P}^c_\beta \times (\mathbb{P}^*_\beta/\varnothing)$. Moreover, since the Cohen part of q is empty, we have that $(\varnothing, q) \in \mathbb{P}^c_\beta \times (\mathbb{P}^*_\beta/\varnothing)$. Thus we may force with $\mathbb{P}^c_\beta \times (\mathbb{P}^*_\beta/\varnothing)$ below $(\varnothing, q)$ to obtain a generic $\mathcal{A}_\beta \times H_\beta$ for the product.

Next, observe that $\bar{\mathcal{A}}_\beta := \pi[\mathcal{A}_\beta]$ is $V[G \restriction \kappa_N][K_3]$-generic over $\pi(\mathbb{P}^c_\beta)$. Thus $\bar{\mathcal{A}}_\beta \times K_3$ is $V[G \restriction \kappa_N]$-generic over $\pi(\mathbb{P}^c_\beta) \times \pi(\mathbb{P}^*_\beta/\varnothing)$. Since $j[\bar{\mathcal{A}}_\beta] \subseteq \mathcal{A}_\beta$ and since $j[K_3] \subseteq H_\beta$ (because $q \in H_\beta$ is a lower bound for $j[K_3]$), we may lift $j : \bar{N}[G \restriction \kappa_N] \longrightarrow N[G]$ to an isomorphism $j : \bar{N}[G \restriction \kappa_N * (\bar{\mathcal{A}}_\beta \times K_3)] \longrightarrow N[G * (\mathcal{A}_\beta \times H_\beta)]$.

Let us next set T and f to be the interpretations of $\dot{T}$ and $\dot{f}$, respectively, by the generic $I_\beta := \tau_\beta[\mathcal{A}_\beta \times H_\beta]$ and let T^* and f^* be the interpretations of $\pi(\dot{T})$ and $\pi(\dot{f})$ by the generic $\pi(\tau_\beta)[\bar{\mathcal{A}}_\beta \times K_3]$. We have by the elementarity of j that $j(T^*, f^*) = (T, f)$. Furthermore, since T is an Aronszajn tree on κ in $V[G * I_\beta]$ with specializing function f and since ω_1 and κ are still cardinals in $V[G * (\mathcal{A}_\beta \times H_\beta)]$, we have that T is still an Aronszajn tree on κ with specializing function f in the model $V[G * (\mathcal{A}_\beta \times H_\beta)]$. By the elementarity of $N[G * (\mathcal{A}_\beta \times H_\beta)]$ and the fact that j is an isomorphism, we have that $\bar{N}[G \restriction \kappa_N][\bar{\mathcal{A}}_\beta \times K_3]$ satisfies that T^* is an Aronszajn tree on κ_N with specializing function f^*. This clearly still holds in $V[G \restriction \kappa_N][\bar{\mathcal{A}}_\beta \times K_3]$.

Since T^* is a special Aronszajn tree in $V[G \restriction \kappa_N][\bar{\mathcal{A}}_\beta \times K_3]$, it in particular does not have any cofinal branches. However, the elementarity of j and the fact that $\mathrm{crit}(j) = \kappa_N$ imply that $T^* = T \restriction \kappa_N$. Thus T^* does have a cofinal branch in the model $V[G * (\mathcal{A}_\beta \times H_\beta)]$. We will show that this is impossible by analyzing the forcing which takes us from $V[G \restriction \kappa_N * (\bar{\mathcal{A}}_\beta \times K_3)]$ to $V[G * (\mathcal{A}_\beta \times H_\beta)]$.

We first analyze the forcing which takes us from $V[G \restriction \kappa_N * (\bar{\mathcal{A}}_\beta \times K_3)]$ to $V[G * \bar{\mathcal{A}}_\beta]$. To begin, the forcing from $V[G \restriction \kappa_N]$ to $V[G]$ is $\mathbb{M}/(G \restriction \kappa_N)$, and this forcing is isomorphic to a dense subset of

$$(\mathrm{Add}(\omega, \kappa_N^*) \times \mathrm{Col}(\omega_1, \kappa) \times \mathrm{Add}(\kappa_N, 1)) * \dot{\mathbb{N}}_{\kappa_N^*},$$

where κ_N^* is the least V-inaccessible above κ_N. As we know, $\dot{\mathbb{N}}_{\kappa_N^*}$ satisfies that it is, in the extension by $\mathbb{M} \restriction \kappa_N^*$, a forcing projection of a product $\mathbb{A} \times \mathbb{B}$, where $\mathbb{A}$ is Cohen forcing for adding κ-many reals and where $\mathbb{B}$ is ω_1-closed.

As observed earlier, in the model $V[G \restriction \kappa_N][\bar{\mathcal{A}}_\beta \times K_3]$, T^* is a special Aronszajn tree on κ_N. Now in the model $V[G \restriction \kappa_N][\bar{\mathcal{A}}_\beta \times K_3]$, we have that $2^\omega = \kappa_N = \omega_2 > \omega_1$ and that T^* has width ω_1. Since K_2 is generic for a forcing which is ω_1-closed in $V[G \restriction \kappa_N][K_3]$, Theorem 6.3.1 implies that T^* has no cofinal branches in $V[G \restriction \kappa_N][\bar{\mathcal{A}}_\beta \times K_2 \times K_3]$. Adding further Cohen reals doesn't change this fact, so T^* still has no cofinal branches in

$$V[G \restriction \kappa_N][\bar{\mathcal{A}}_\beta \times K_1 \times K_2 \times K_3] = V[G \restriction \kappa_N^*][\bar{\mathcal{A}}_\beta].$$

Now let us analyze the forcing $\mathbb{A} \times \mathbb{B}$ in $V[G \restriction \kappa_N^*][\bar{\mathcal{A}}_\beta]$. From the perspective of the model $V[G \restriction \kappa_N^*]$, we have a $\pi(\mathbb{P}^c_\beta)$-name $\dot{T}^*_c$ for a tree of width $\leq \omega_1$ on the ordinal κ_N, that κ_N has cofinality ω_1, and that $\dot{T}^*_c$ has no cofinal branches. Additionally, $2^\omega = \kappa_N^* > \omega_1$. Since $\mathbb{B}$ is ω_1-closed in $V[G \restriction \kappa_N^*]$, we may again appeal to Theorem 6.3.1 to see that forcing with $\mathbb{B}$ over $V[G \restriction \kappa_N^*][\bar{\mathcal{A}}_\beta]$ does not add any cofinal branches through $\dot{T}^*_c[\bar{\mathcal{A}}_\beta] = T^*$. And finally, forcing with $\mathbb{A}$ also fails to add cofinal branches through T^*. Since forcing with $\mathbb{A} \times \mathbb{B}$ projects to forcing with $\mathbb{N}_{\kappa_N^*}$, we conclude that in $V[G * \bar{\mathcal{A}}_\beta]$, there are no cofinal branches through T^*.

Our final task is to consider the forcing to get from $V[G * \bar{\mathcal{A}}_\beta]$ to $V[G * (\mathcal{A}_\beta \times H_\beta)]$. Since we forced with $\mathbb{P}^c_\beta \times (\mathbb{P}^*_\beta/\varnothing)$ to get from $V[G]$ to $V[G * (\mathcal{A}_\beta \times H_\beta)]$, we have that the forcing to get from $V[G * \bar{\mathcal{A}}_\beta]$ to $V[G * (\mathcal{A}_\beta \times H_\beta)]$ is isomorphic to $\mathrm{Add}(\omega, \xi)$ for some ξ followed by $\pi(\mathbb{P}^*_\beta/\varnothing)$ (recall that $\mathbb{P}^c_\beta$ is isomorphic to $\mathrm{Add}(\omega, \mathrm{ot}(\beta \cap \mathrm{even}))$). As we have seen, forcing with $\mathrm{Add}(\omega, \beta)$ does not add branches through T^*, and thus T^* has no cofinal branches in $V[G * \mathcal{A}_\beta]$. What is more, by the arguments of the previous section, we know that $\pi(\mathbb{P}^*_\beta/\varnothing)$

is ω_2-distributive in the model $V[G]$, and since $\mathbb{P}^c_\beta$ is c.c.c., $\mathbb{P}^*_\beta$ remains ω_2-distributive in the model $V[G * \mathcal{A}_\beta]$. Consequently, forcing with $\mathbb{P}^*_\beta$ over $V[G * \mathcal{A}_\beta]$ does not add any branches through T^*, and therefore T^* does not have any cofinal branches in $V[G * (\mathcal{A}_\beta \times H_\beta)]$. However, this contradicts the fact that $T^* = T \restriction \kappa_N$ and that T is a $\kappa > \kappa_N$-tree, which imply that T^* does have a cofinal branch in $V[G * (\mathcal{A}_\beta \times H_\beta)]$.

$\square$

Theorem 8.3.3. *It is consistent from a Mahlo cardinal that* $\mathsf{SR}(\omega_2) + \mathsf{AP}_{\omega_1} + 2^\omega = \omega_3$ *hold and that there are neither weak Kurepa trees on* ω_1 *nor special Aronszajn trees on* ω_2*.*

Proof. We force with $\mathbb{R} := \mathbb{M} * \dot{\mathbb{P}}_{\kappa^+}$. In the extension by $\mathbb{R}$, we have that ω_1 is still a cardinal, κ is ω_2, and all cardinals above κ are preserved. As shown in [24], AP_{ω_1} holds in the $\mathbb{M}$-extension (recall that $\mathbb{M}$ is the poset $\mathbb{M}_0$ from that paper), and since $\mathbb{P}_{\kappa^+}$ preserves all cardinals, AP_{ω_1} still holds in the $\mathbb{R}$-extension. As verified in Section 8.2, $\mathsf{SR}(\omega_2) + 2^\omega = \omega_3$ holds after forcing with $\mathbb{R}$, and as verified in Propositions 8.3.1 and 8.3.2, there are neither weak Kurepa trees on ω_1 nor special Aronszajn trees on ω_2. $\square$

Remark 8.3.4. We can also obtain the above configuration, but with $\neg\mathsf{AP}_{\omega_1}$ instead of AP_{ω_1}. The idea is to use the forcing $\mathbb{M}_1$ from [24]. It is straightforward to see that, based upon the arguments of [32] and the fact that the tails of the Mitchell-type posets are proper, there exists a disjoint stationary sequence after forcing with $\mathbb{M}_1$. This implies the failure of AP_{ω_1}, and hence the nonexistence of special Aronszajn trees. Subsequent forcing with the suitable mixed support iteration will still preserve cardinals and will also preserve that there exists a disjoint stationary sequence, and hence that AP_{ω_1} fails. Finally, the argument for the non-existence of weak Kurepa trees on ω_1 is simpler than the one given here, as the preparatory forcing involves less collapsing.

CHAPTER 9

Some Open Questions

In this final chapter we record a number of questions which we think are of interest.

The main result of the thesis is that OCA_{ARS} is consistent with $2^{\aleph_0} = \aleph_3$. The methods used in the proof seem difficult to lift in a staightforward manner to obtain larger values of the continuum. Obtaining a value of $\aleph_4$ seems to require expansions of our current techniques, but any $\aleph_n$ for $n \geq 4$ should be obtainable once $\aleph_4$ is. We expect that such methods would be able to obtain $2^{\aleph_0} = \aleph_\alpha$, where $\alpha \in [3, \omega_2)$, since then any two hulls like those used in Chapter 4 would agree on the length of the $\aleph$-sequence.

Question 9.0.1. *How can we build a model of* OCA_{ARS} *in which* $2^{\aleph_0}$ *has the value* $\aleph_4$*? Or* $\aleph_n$ *for any* $n \geq 4$*? Or* $\aleph_{\omega+1}$*? What about various* $\aleph_\alpha$*, where* $\alpha < \omega_2$*?*

Of course, the most general version of the above question is the following. A test case for our methods would be obtaining $2^{\aleph_0} \geq \aleph_{\omega_2}$, since then, in contrast to the previous question, the sequence of alephs in two hulls as in Chapter 4 could be of different lengths.

Question 9.0.2. *Is* OCA_{ARS} *consistent with an arbitrarily large value of the continuum? And how can this be forced over an arbitrary ground model satisfying* GCH*, rather than* L *in particular?*

The methods used to obtain the Gilton-Neeman model might be useful in a number of other cases, specifically when we want to obtain a model where $2^{\aleph_0}$ is large, but where the key constructions to ensure that the forcings are well-behaved take place over models satisfying the CH. One of the main open questions in this area is whether or not the P-Ideal Dichotomy, abbreviated PID (see [77]), is consistent with a large continuum.

Question 9.0.3. *Is* PID *consistent with* $2^{\aleph_0} > \aleph_2$*?*

A restricted version of this problem would still be of interest:

Question 9.0.4. *Let* $\mathsf{PID}(\aleph_1)$ *denote the restriction of* PID *to ideals of countable sets on* ω_1*. Is* $\mathsf{PID}(\aleph_1)$ *consistent with a large continuum?*

A question which is very closely related to the previous one concerns ideals on ω_1 which are ω_1-generated. Recall that an ideal $\mathcal{I}$ of countable subsets of ω_1 is ω_1-generated if there is a sequence of countable sets $\langle X_\alpha : \alpha < \omega_1 \rangle$ which generates $\mathcal{I}$ (i.e., the elements of $\mathcal{I}$ are exactly the subsets of the unions of finitely-many of the X_α). There are two interesting ideal dichotomies related to such an ideal (see [3]). The first is the following: either ω_1 has an uncountable subset which is inside $\mathcal{I}$ or ω_1 can be decomposed into countably-many sets each of which is orthogonal to $\mathcal{I}$. The other ideal dichotomy, which "flips" the dichotomy as in the statement of PID, is the following: either ω_1 has an uncountable subset which is orthogonal to $\mathcal{I}$, or ω_1 can be decomposed into countably-many sets which are inside $\mathcal{I}$.

Question 9.0.5. *Let* $\mathcal{I}$ *be an* ω_1*-generated ideal on* ω_1*, and let* Φ *be either of the ideal dichotomies related to* $\mathcal{I}$ *in the previous paragraph. Is* Φ *consistent with* $2^{\aleph_0} > \aleph_2$*?*

Of course, we'd be remiss if we didn't mention the following question:

Question 9.0.6. *Is* OCA_T *consistent with* $2^{\aleph_0} > \aleph_2$*?*

We are further interested in seeing if many of the results from [4] can be extended to $\aleph_2$. Very likely, obtaining some such axiom at $\aleph_2$ requires obtaining the original version on $\aleph_1$ with a large continuum, and thus these questions are related to the ones discussed in this thesis. We frame this question as follows:

Question 9.0.7. *Can* SOCA *be generalized to* $\aleph_2$*? What about* OCA_{ARS} *or* OCA_T*?*

The second half of the thesis concerned itself with many combinatorial properties of $\aleph_2$. There are a host of questions related to this which are of interest to us. With regards to the

Eightfold Way paper, we'd first like to know how to obtain the configurations in Sections 7.2 and 7.3 from optimal assumptions (which almost assuredly is not a supercompact).

Question 9.0.8. *How can the configurations in Sections 7.2 and 7.3 be forced from a weakly compact cardinal?*

Another question in this vein is whether or not we can specialize trees in the context of forcing $\mathsf{SR}(\omega_2)^*$. Recall that the Special Aronszajn Tree Property on ω_2, denoted $\mathsf{SATP}(\omega_2)$, asserts that there exist Aronszajn trees on ω_2 and that all such are special. A question of interest to us is the following (the author and Omer Ben-Neria believe that we have a positive answer to the question, but we might be wrong):

Question 9.0.9. *Are* $\mathsf{SR}(\omega_2)^*$ *and* $\mathsf{SATP}(\omega_2)$ *consistent?*

Next, we'd like to know just how badly the approachability property can fail in such constructions. Let Θ denote the statement that $I[\omega_2]$ contains no stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$; recall that Mitchell ([59]) has shown that Θ is consistent from a greatly Mahlo cardinal (see [33] for an updated account of this argument). We would like to know the following:

Question 9.0.10. *Let* Φ *be any Boolean combination of* $\mathsf{SR}(\omega_2)$ *and* $\mathsf{TP}(\omega_2)$*. Is* $\Theta + \Phi$ *consistent?*

James Cummings has recently shown ([19]) that the tree property on ω_2 is consistent with the existence of a Kurepa tree on ω_1 with arbitrarily-many branches. It would be interesting to see whether this can be carried out in light of the full Eightfold Way.

Question 9.0.11. *Are all Boolean combinations of the principles from [24] consistent with a Kurepa tree on* ω_1 *with arbitrarily-many branches? What about with the non-existence of Kurepa or weak Kurepa trees on* ω_1*?*

As asked at the end of the Eightfold Way paper, in Question 3, we would like to know if their argument can be carried out on many different cardinals:

Question 9.0.12. *Can the Eightfold Way be carried out on two successive cardinals, say $\aleph_2$ and $\aleph_3$? How about on all $\aleph_n$, for $n \geq 2$?*

The Eightfold Way becomes more challenging at the double successor of a singular cardinal and requires larger large cardinals than at the double successor of a regular cardinal. We also echo Question 2 of their paper:

Question 9.0.13. *What is the large cardinal strength of each of the eight paths of the Eightfold Way on κ^{++}, where κ is a singular cardinal of cofinality ω? Can Gitik's results from [36] be extended to the other combinatorial principles studied in [24]? What about singular cardinals of uncountable cofinality?*

We've shown in this thesis that the Eightfold Way can be carried out with quite a bit of stationary reflection on ω_2. What about higher-order stationary reflection principles in $P_{\omega_1}(\omega_2)$? The first of these is WRP, which was introduced in [31]. And what about the principle RP (see [51] for a definition, noting that this is not the same "RP" as used in [24]), which strengthens WRP?

Question 9.0.14. *Can the four paths of the Eightfold Way in which $\mathsf{SR}(\omega_2)$ holds be extended to cover higher-order stationary reflection in $P_{\omega_1}(\omega_2)$?*

We also have seen in work related to this thesis that disjoint stationary sequences on ω_2 can be very useful for creating models in which AP_{ω_1} fails. Recently, Sean Cox (see [18]) has shown that $\neg\mathsf{AP}_{\omega_1}$ does not imply that there exists a disjoint stationary sequence on ω_2, by showing that PFA does not imply that a disjoint stationary sequence on ω_2 exists (recall that PFA does imply that AP_{ω_1} fails, see [47]). We would like to know the following:

Question 9.0.15. *What is the exact large cardinal strength of $\neg\mathsf{AP}_{\omega_1}$ and the non-existence of a disjoint stationary sequence on ω_2?*

The relationship between stationary reflection and square principles has been investigated very thoroughly in [40]. We restate Question 2 of their paper here, as it concerns the tension between reflection and incompactness:

Question 9.0.16. *Is the conjunction of* $\square(\omega_2, \omega)$ *and* $\mathrm{Refl}(\aleph_0, S^{\omega_2}_{\omega})$ *consistent?*

We also think that the following question, raised by Raghavan ([63]) is of much interest, as it relates many of the objects which we have been studying.

Question 9.0.17. *Is* $\mathsf{PID} + \mathsf{MA}_{\omega_1}$+there exists an ω_2 Aronszajn tree *consistent?*

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