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Global Exponential Stabilization on the n-Dimensional Sphere

Pedro Casau, Christopher G. Mayhew, Ricardo G. Sanfelice and Carlos Silvestre

***Abstract*— In this paper, we show that the existence of centrally synergistic potential functions on the n-dimensional sphere, denoted by $\mathbb{S}^n$, is a sufficient condition for the global asymptotic stabilization of a point in $\mathbb{S}^n$. Additionally, if these functions decrease exponentially fast during flows and are bounded from above and from below by some polynomial function of the tracking error, then the reference point can be globally exponentially stabilized. We construct two kinds of centrally synergistic functions: the first kind consists of a finite family of potential functions on $\mathbb{S}^n$ while the second kind consists of an uncountable number of potential functions on $\mathbb{S}^n$. While the former generates a simpler jump logic, the latter is optimal in the sense that it generates flows with minimal length.**

I. Introduction

The problem of designing controllers for systems with rotational degrees of freedom is at the core of robotics research, spanning a multitude of applications, including the stabilization/trajectory tracking of spacecrafts [5], [22], [11], unmanned air vehicles [13], [23], [14], autonomous underwater vehicles [21], [1], [8], as well as the stabilization of robotic manipulators [10] and the 3D pendulum [4]. Such mechanical systems are often described by elements of the n-dimensional sphere, i.e., the set of vectors in $n+1$ Euclidean space with unit-norm, denoted by $\mathbb{S}^n$. For example: a joint in a planar robotic manipulator is characterized by its angular displacement, which is an element of $\mathbb{S}^1$– the unit circle – and the attitude of rigid-body vehicles in 3D space is characterized by a rotation matrix, which is a collection of three orthogonal vectors in $\mathbb{S}^2$.

The controllers described in the aforementioned papers typically rely on continuous feedback strategies to stabilize a given reference trajectory or point. However, due to topological obstructions it is impossible to accomplish this objective globally, that is, for every initial condition [2], [6, Theorem 4.1]. To overcome these issues, some authors have proposed discontinuous feedback strategies (e.g. [15], [7]); however, as shown in [20], such solutions suffer from chattering and are not robust to small measurement noise.

Recent advances in the theory of hybrid control brought forth a number of results on the robustness of hybrid systems, namely, it has been shown that, if a system satisfies the so-called hybrid basic conditions and a given compact set is uniformly globally stable then it is robustly uniformly globally stable with respect to small measurement noise (c.f. [9]). This appealing property of hybrid systems has nurtured substantial development of hybrid control techniques for systems with rotational degrees of freedom, namely, rigid-body stabilization by hybrid feedback [16], [19] and stabilization of the 3D pendulum [17].

In this paper, we propose a solution to the global exponential stabilization of a reference point r in $\mathbb{S}^n$. For this purpose, we extend the concept of centrally synergistic potential functions that was introduced in [18]. These functions induce a *gradient-like* vector field on the sphere that, with an appropriate switching strategy, renders a given reference point globally asymptotically stable for the closed-loop hybrid system. Moreover, if the function and its derivative satisfy some appropriate bounds, then the reference point is globally exponentially stabilized. We also develop two novel centrally synergistic potential functions and we compare the trajectories of the two induced closed-loop hybrid systems by means of a numerical study. The synergistic potential functions that we develop render a given reference point globally exponentially stable. In particular, the second class of functions that we introduce ensures that, for initial conditions near undesired equilibrium points, the system follows the path of least distance to the given reference point, i.e., it induces a geodesic flow.

The paper is organized as follows. In Section II, we present the notation used throughout the paper. In Section III, we develop the concept of synergistic potential functions and, in Section IV, we design such functions. In Section V, we perform a numerical study in order to compare the trajectories associated with each class of synergistic potential functions and, in Section VI we provide some concluding remarks. Due to space constraints, the proofs of the results in this paper will appear elsewhere.

II. Preliminaries & Notation

A. Preliminaries on Differential Geometry and the n-Dimensional Sphere

Given smooth manifolds M and N, $C^n(M,N)$ denotes the set of functions from M to N that are continuously differentiable up to order n. For each $p \in M$, T_pM denotes the *tangent space to M at p* and, given $F \in C^1(M,N)$, $d_pF : \mathsf{T}_pM \to \mathsf{T}_{F(p)}N$ denotes the *push-forward of F at p* [12, Chapter 3]. A point $p \in M$ is said to be a *regular point*

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of F if d_pF is surjective and it is said to be a *critical point of* F otherwise. The set of critical points of F is denoted by $\operatorname{Crit} F$.

A *vector field on* M is a map that assigns a tangent vector to each $p \in M$. Given a function $f \in C^1(\mathbb{R}^n, \mathbb{R})$ and a Riemannian metric g on M, the *gradient of* f, denoted by ∇f, is the unique vector field satisfying $g(\nabla f, X) = X(f)$ for every vector field X [12, p. 343]. A tangent vector is a derivation so it acts on continuously differentiable functions defined on the manifold.

Let $\mathbb{R}^n$ be endowed with the Euclidean metric $\langle u, v\rangle = u^\top v$, defined for each $u, v \in \mathsf{T}_p\mathbb{R}^n \cong \mathbb{R}^n$, $p \in \mathbb{R}^n$. Then, given a function $f \in C^1(\mathbb{R}^n, \mathbb{R})$, the *gradient of* f is the vector field given by $\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x) & \ldots & \frac{\partial f}{\partial x_n}(x)\end{bmatrix}^\top$, for each $x \in \mathbb{R}^n$. Moreover, the norm of a vector $v \in \mathsf{T}_p\mathbb{R}^n$ is defined as $|v| := \sqrt{\langle v, v\rangle}$. The function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, given by $f(x) = \langle x, x\rangle$, satisfies $d_x f \neq 0$ for each $x \in \mathbb{R}^{n+1}\backslash\{0\}$, hence every point $x \in \mathbb{R}^{n+1}$, except for the origin, is a regular point of f. It follows from [12, Corollary 5.14] that the n-dimensional sphere, given by $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \langle x, x\rangle = 1\}$, is a embedded smooth manifold of $\mathbb{R}^n$. Considering the Euclidean metric, it follows from [12, Proposition 5.38] that $\mathsf{T}_x\mathbb{S}^n = \{v \in \mathbb{R}^{n+1} : \langle \nabla f(x), v\rangle = 0\}$. Any vector $v \in \mathbb{R}^{n+1}$ can be projected onto $\mathsf{T}_x\mathbb{S}^n$ using the map $\Pi(x) := I_{n+1} - xx^\top$, where I_{n+1} denotes the $(n+1)\times(n+1)$ identity matrix. Given a function $V \in C^1(\mathbb{S}^n, \mathbb{R})$, its set of critical points is the set of points given by

$$\operatorname{Crit} V := \{x \in \mathbb{S}^n : \Pi(x)\nabla V(x) = 0\}.$$

B. Hybrid Systems & Exponential Stability

A hybrid system $\mathcal{H} = (C, F, D, G)$ in $\mathbb{R}^n$ is defined as follows:

$$\mathcal{H} : \begin{cases} x \in C & \dot{x} \in F(x) \\ x \in D & x^+ \in G(x) \end{cases},$$

where $C \subset \mathbb{R}^n$ is the flow set, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the flow map, $D \subset \mathbb{R}^n$ denotes the jump set, and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the jump map. A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_j + 1] \times \{j\}),$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_J$. It is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap [0, T] \times \{0, 1, ..., J\}$ is a compact hybrid domain.

Every solution $(t, j) \mapsto x(t, j)$ to a hybrid system is defined on a hybrid time domain $\operatorname{dom} x \subset \mathbb{R}_{\geq 0} \times \mathbb{N}_0$, where $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers and $\mathbb{N}_0$ denotes the set of natural numbers and zero. A solution to a hybrid system is said to be *maximal* if it cannot be extended by flowing nor jumping, *complete* if its domain is unbounded, and *precompact* if it is complete and bounded (the reader is referred to [9, Chapter 2] for more information on solutions to hybrid systems).

Let $\mathcal{A} \subset \mathbb{R}^n$ denote a compact set and $|x|_{\mathcal{A}} := \min_{y\in\mathcal{A}}|x - y|$. The set $\mathcal{A} \subset \mathbb{R}^n$ is said to be: *stable* for $\mathcal{H}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that every maximal solution x to $\mathcal{H}$ with $|x(0, 0)|_{\mathcal{A}} \leq \delta$ satisfies $|x(t, j)|_{\mathcal{A}} \leq \epsilon$ for all $(t, j) \in \operatorname{dom} x$; *attractive* for $\mathcal{H}$ if each maximal solution x to $\mathcal{H}$ is complete and $\lim_{t+j\to\infty}|x(t, j)|_{\mathcal{A}} = 0$; *asymptotically stable* for $\mathcal{H}$ if it is both stable and attractive for $\mathcal{H}$. The set $\mathcal{A}$ is globally exponentially stable for the hybrid system $\mathcal{H}$ if each maximal solution x is complete and if there exist strictly positive real numbers k, λ such that for each solution x, from each initial condition $x(0, 0) \in \mathbb{R}^n$ the following holds:

$$|x(t, j)|_{\mathcal{A}} \leq ke^{-\lambda(t+j)}|x(0, 0)|_{\mathcal{A}},$$

for each $(t, j) \in \operatorname{dom} x$ (c.f. [24]).

III. Global Exponential Stabilization on $\mathbb{S}^n$ Using Synergistic Potential Functions

Each vector field on $\mathbb{S}^n$ can be represented by

$$\dot{x} = \Pi(x)v(x) \quad x \in \mathbb{S}^n, \tag{1}$$

for some $v \in C^1(\mathbb{S}^n, \mathbb{R}^{n+1})$. Therefore, the problem of stabilizing a reference point $r \in \mathbb{S}^n$ typically amounts to the design of a continuous control law v. However, it has been shown in [2] that it is impossible to globally asymptotically stabilize a reference point on a compact manifold by means of continuous feedback. To overcome these limitations of continuous feedback, we make use of the concept of centrally synergistic potential functions on $\mathbb{S}^n$ introduced in [18, Section 3], and we demonstrate that, if such functions exist, then it is possible to globally asymptotically stabilize a given reference point $r \in \mathbb{S}^n$ by means of hybrid feedback. We also show that, under some additional conditions, it is possible to achieve global exponential stabilization of the given reference point. The concept of centrally synergistic potential functions of $\mathbb{S}^n$ introduced next is pivotal for the work developed in this paper.

Definition 1. *Given $r \in \mathbb{S}^n$ and a compact set $Q \subset \mathbb{R}^m$, for some $m > 0$, we say that $V \in C^1(\mathbb{S}^n \times Q, \mathbb{R})$ is a centrally synergistic potential function relative to r, if it is positive definite relative to*

$$\mathcal{A}_V := \{r\} \times Q \tag{2}$$

and if there exists $\delta > 0$ such that

$$\mu_V(x, y) := V(x, y) - \min_{z\in Q} V(x, z) > \delta,$$

for each $(x, y) \in \mathcal{E}(V)$, where

$$\mathcal{E}(V) := \{(x, y) \in \mathbb{S}^n \times Q : x \in \operatorname{Crit}(V_y)\backslash\{r\}\} \tag{3}$$

with $V_y(x) := V(x, y)$. We also say that V has synergy gap exceeding δ. □

Notice that Definition 1 differs only slightly from the concept introduced in [18, Section 3], in the sense that we require $Q \subset \mathbb{R}^m$ to be compact but not necessarily finite, making the concept used in this paper slightly more general in this regard. To abbreviate notation, we define the minimum and the minimizer of a centrally synergistic function as follows:

$$\nu_V(x) := \min_{y\in Q} V(x, y) \tag{4a}$$

$$\varrho_V(x) := \arg\min_{y\in Q} V(x, y), \tag{4b}$$

respectively. In the sequel, we make use of centrally synergistic potential functions in order to create a hybrid controller that, given $r \in \mathbb{S}^n$, globally asymptotically stabilizes $\mathcal{A}_V$.

Let $\mathcal{X} := \mathbb{S}^n \times Q$, then, given a centrally synergistic function relative to $r \in \mathbb{S}^n$ with synergy gap exceeding δ, denoted by $V \in C^1(\mathcal{X}, \mathbb{R})$, we define the hybrid controller with state $y \in Q$, input $x \in \mathbb{S}^n$ and output $v \in \mathbb{R}^{n+1}$, given by

$$\left.\begin{aligned} v &= -\nabla V_y(x) \\ \dot{y} &= 0 \end{aligned}\right\} (x, y) \in C_V := \{(x, y) \in \mathcal{X} : \mu_V(x, y) \leq \delta\}$$

$$y^+ \in \varrho_V(x) \quad (x, y) \in D_V := \{(x, y) \in \mathcal{X} : \mu_V(x, y) \geq \delta\}, \tag{5}$$

where $V_y(x) := V(x, y)$ for each $y \in Q$. The interconnection between the kinematic model (1) and the controller (5) is the closed-loop hybrid system $\mathcal{H}_V := (C_V, F_V, D_V, G_V)$, said to be the *hybrid system associated with* V, given by

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= F_V(x, y) = \begin{pmatrix} -\Pi(x)\nabla V_y(x) \\ 0 \end{pmatrix} && (x, y) \in C_V \\ \begin{pmatrix} x^+ \\ y^+ \end{pmatrix} &\in G_V(x, y) = \begin{pmatrix} x \\ \varrho_V(x) \end{pmatrix} && (x, y) \in D_V. \end{aligned} \tag{6}$$

The existence of centrally synergistic potential functions relative to r is a sufficient condition for the global asymptotic stability of $\mathcal{A}_V$ in (2) for the hybrid system (6).

Theorem 2. *Given $r \in \mathbb{S}^n$ and a compact set $Q \subset \mathbb{R}^m$, if there exists $\delta > 0$ such that $V \in C^1(\mathbb{S}^n \times Q, \mathbb{R})$ is a centrally synergistic potential function relative to r with synergy gap exceeding δ, then the set $\mathcal{A}_V$ given in* (2) *is globally asymptotically stable for the hybrid system $\mathcal{H}_V = (C_V, F_V, D_V, G_V)$ given by* (6)*.*

The next theorem states that if a synergistic potential function satisfies some additional conditions then we are able to assert global exponential stability of the set (2) for the hybrid system (6).

Theorem 3. *Given $r \in \mathbb{S}^n$ and a compact set $Q \subset \mathbb{R}^m$, if $V \in C^1(\mathbb{S}^n \times Q, \mathbb{R})$ is a centrally synergistic potential function relative to r with synergy gap exceeding δ satisfying the following for some $p, \underline{\alpha}, \overline{\alpha}, \lambda > 0$*

$$\underline{\alpha}|x - r|^p \leq V(x, y) \leq \overline{\alpha}|x - r|^p \qquad \forall (x, y) \in C_V \cup D_V, \tag{7a}$$

$$\langle \nabla V(x, y), F_V(x, y) \rangle \leq -\lambda V(x, y) \quad \forall (x, y) \in C_V, \tag{7b}$$

then the set $\mathcal{A}_V$ is globally exponentially stable for the hybrid system $\mathcal{H}_V = (C_V, F_V, D_V, G_V)$ given by (6)*.*

In the next section, we show how to construct different centrally synergistic potential functions on $\mathbb{S}^n$.

IV. Constructing Synergistic Potential Functions

In Section IV-A, we show that it is possible to devise a centrally synergistic potential function that enables global exponential stabilization of a reference point using a finite set $Q \subset \mathbb{R}^m$. In Section IV-B, we devise a centrally synergistic function using a connected subset Q of $\mathbb{S}^n$. While the first strategy has a simpler jump logic, the second strategy generates flows that follow the path of least distance from the initial condition to the given reference point.

A. Synergistic Potential Functions on $\mathbb{S}^n$ with Finite Q

Given $r \in \mathbb{S}^n$, we define the function $h_r : \mathbb{S}^n \to \mathbb{R}$ as

$$h_r(x) = 1 - r^\top x. \tag{8}$$

We note that h_r returns the height of x above the plane tangent to $\mathbb{S}^n$ at r and thus is commonly referred to as the height function. We briefly recall some basic properties of h_r.

Lemma 4. *Given $r \in \mathbb{S}^n$, h_r defined in* (8) *satisfies*

$$\begin{aligned} \nabla h_r(x) &= -r \\ \operatorname{Crit} h_r &= \{-r, r\} \end{aligned}$$

and thus,

$$\begin{aligned} \arg\min_{x \in \mathbb{S}^n} h_r(x) &= r & \arg\max_{x \in \mathbb{S}^n} h_r(x) &= -r \\ h_r(r) &= 0 & h_r(-r) &= 2. \end{aligned}$$

In particular, h_r is positive definite on $\mathbb{S}^n$ relative to r.

Inspired by the synergistic potential functions presented in [19], we define synergistic potential function on $\mathbb{S}^n$ by angular warping. We define the *warping function relative to* r as follows:

$$\mathcal{W}_X(x) := e^{X h_r(x)} x, \tag{9}$$

for each $x \in \mathbb{S}^n$, with $X \in \mathbb{R}^{(n+1)\times(n+1)}$ skew-symmetric, i.e. satisfying $X^\top = -X$. Note that $\mathcal{W}_X(r) = r$.

Lemma 5. *The push-forward of* (9) *at x is given by*

$$d_x \mathcal{W}_X = e^{X h_r(x)} \big(I_{n+1} - X x r^\top\big).$$

The following result states that the function $\mathcal{W}_X$ is a diffeomorphism from $\mathbb{S}^n$ to itself provided that $\sigma_{\max}(X) < 1$, where $\sigma_{\max}(X)$ denotes the maximum singular value of X.

Lemma 6. *If $\sigma_{max}(X) < 1$ then $\mathcal{W}_X : \mathbb{S}^n \to \mathbb{S}^n$ defined in* (9) *is a diffeomorphism. Moreover, the inverse of its push-forward at x is given by*

$$(d_x \mathcal{W}_X)^{-1} = \left(I_{n+1} + \frac{X x r^\top}{1 - r^\top X x}\right) e^{-X h_r(x)}$$

Let us define a potential function on $\mathbb{S}^n$ as

$$U_X := h_r \circ \mathcal{W}_X, \tag{10}$$

with $X \in \mathbb{R}^{(n+1)\times(n+1)}$ skew-symmetric and satisfying $\sigma_{\max}(X) < 1$. In order to prove that there exists a finite family of such functions that is centrally synergistic we must first find the set of critical values of U_X.

Lemma 7. *Given $X \in \mathbb{R}^{(n+1)\times(n+1)}$ skew-symmetric and satisfying $\sigma_{max}(X) < 1$, the critical values of the function $U_X \in C^1(\mathbb{S}^n, \mathbb{R})$, given by* (10)*, are $\operatorname{Crit} U_X = \mathcal{W}_X^{-1}(\operatorname{Crit} h_r) = \{r, \mathcal{W}_X^{-1}(-r)\}$.*

From the result presented below it is possible to conclude that (10) is positive definite relative to r.

Lemma 8. *If $\sigma_{max}(X) < 1$ then there exists $k > 0$ such that the following set of inequalities are satisfied:*

$$\alpha_-|r-x|^2 \leq U_X(x) \leq \alpha_+|r-x|^2 \qquad \forall x \in \mathbb{S}^n, \tag{11a}$$

$$|\Pi(x)\nabla U_X|^2 \geq k(1 + r^\top \mathcal{W}_X(x))U_X(x) \qquad \forall x \in \mathbb{S}^n, \tag{11b}$$

for each $x \in \mathbb{S}^n$, with U_X given by (10)*, and*

$$\alpha_\pm := \frac{1}{2}\left(1 \pm \frac{\sigma_{max}(X)}{1-\sigma_{max}(X)}\right).$$

Using the function (10) and the results derived in this section, we are able to design a function that is centrally synergistic relative to r.

Proposition 9. *Given $r \in \mathbb{S}^n$, let $Q := \{-1, 1\}$ and $X \in \mathbb{R}^{(n+1)\times(n+1)}$ denote a skew-symmetric matrix satisfying $\sigma_{max}(X) < 1$ and $Xr \neq 0$. There exists $\delta > 0$ such that*

$$U(x,y) := U_{yX}(x), \tag{12}$$

with U_{yX} given by (10) *for each $y \in Q$, is a centrally synergistic potential function relative to r with synergy gap exceeding δ.*

It follows directly from the previous result and Theorem 2 that, given $r \in \mathbb{S}^n$, the set $\mathcal{A}_U = \{r\} \times \{-1, 1\}$ is globally asymptotically stable for the closed-loop hybrid system (6) when the centrally synergistic potential function (12) is considered. From Lemma 8 and Theorem 3 it also follows that $\mathcal{A}_U$ is globally exponentially stable.

Theorem 10. *Given $r \in \mathbb{S}^n$, $Q := \{-1, 1\}$ and $X \in \mathbb{R}^{(n+1)\times(n+1)}$ skew-symmetric satisfying $\sigma_{max}(X) < 1$, $Xr \neq 0$ and a centrally synergistic potential function relative to r with synergy gap exceeding δ for some $\delta > 0$, denoted by U and given by* (12)*, the set $\mathcal{A}_U := \{r\} \times Q$ is globally exponentially stable for the hybrid system associated with U given by* (6).

This strategy for the global exponential stabilization of the system (1) disregards the rotational symmetry of elements of $\mathbb{S}^n$. In order to take advantage of this symmetry, we developed the centrally synergistic potential function presented in the next section.

B. Centrally Synergistic Potential Functions on $\mathbb{S}^n$ with Connected Q

Let $k > 0$, $\mathcal{V} = \mathbb{S}^n \times \mathbb{S}^n \setminus \{(r, r)\}$, and define $V : \mathcal{V} \to \mathbb{R}$ for all $(x, y) \in \mathcal{V}$ as

$$\begin{aligned} V(x,y) &= \frac{h_r(x)}{h_r(x) + kh_y(x)} \\ &= \frac{1 - r^\top x}{1 - r^\top x + k(1 - y^\top x)}. \end{aligned} \tag{13}$$

Note that V is undefined on (r, r), as $h_r(x) + kh_y(x) = 0$ if and only if $x = y = r$. Next, we present some basic properties of V.

Lemma 11. *The function $V : \mathcal{V} \to \mathbb{R}$ satisfies*

$$\arg\min_{(x,y)\in\mathcal{V}} V(x,y) = V^{-1}(0) = \{(r,y) \in \mathcal{V}\}$$

$$\arg\max_{(x,y)\in\mathcal{V}} V(x,y) = V^{-1}(1) = \{(x,x) \in \mathcal{V}\}$$

Moreover, V is positive definite on $\mathcal{V}$ relative to $\{(r, y) \in \mathcal{V}\}$ and for each $x \in \mathbb{S}^n$,

$$\arg\min_{y\in\mathbb{S}^n} V(x,y) = \begin{cases} -x & \text{if } x \neq r \\ \mathbb{S}^n \setminus \{r\} & \text{if } x = r \end{cases}$$

$$\min_{y\in\mathbb{S}^n} V(x,y) = \frac{1 - r^\top x}{2k + 1 - r^\top x}.$$

We now provide some differential properties of V, which follow from elementary calculation and some tedious manipulation.

Lemma 12. *The function $V : \mathcal{V} \to [0, 1]$ defined in* (13) *satisfies*

$$\nabla_x V(x,y) = \frac{kV(x,y)y - (1 - V(x,y))r}{1 - r^\top x + k(1 - y^\top x)} \tag{14a}$$

$$|\Pi(x)\nabla_x V(x,y)|^2 = \frac{2kV(x,y)(1 - V(x,y))\big(1 - r^\top y\big)}{\big(1 - r^\top x + k(1 - y^\top x)\big)^2} \tag{14b}$$

□

Corollary 13. *Given $y \in \mathbb{S}^n$, define $V_y : \mathbb{S}^n \to \mathbb{R}$ for each $x \in \mathbb{S}^n$ as $V_y(x) = V(x, y)$, with V given by* (13)*. Then,*

$$\operatorname{Crit} V_y = \begin{cases} \{r, y\} & \text{if } r \neq y \\ \mathbb{S}^n & \text{otherwise.} \end{cases}$$

Given $\gamma \in \mathbb{R}$ satisfying

$$-1 \leq \gamma < 1,$$

we define the set $Q(r, \gamma) \subset \mathbb{S}^n$ as

$$Q(r,\gamma) = \big\{y \in \mathbb{S}^n : r^\top y \leq \gamma\big\}. \tag{15}$$

The boundary of $Q(r, \gamma)$, denoted $\partial Q(r, \gamma)$, is

$$\partial Q(r,\gamma) = \big\{y \in \mathbb{S}^n : r^\top y = \gamma\big\}.$$

Define the functions $\alpha, \sigma : [-1, 1] \to \mathbb{R}$ for each $v \in [-1, 1]$ as

$$\alpha(v) = \gamma v - \sqrt{(1 - v^2)(1 - \gamma^2)}$$

$$\sigma(v) = \gamma\sqrt{1 - v^2} + v\sqrt{1 - \gamma^2}.$$

Theorem 14. *Given $r \in \mathbb{S}^n$ and $\gamma \in [-1, 1)$, let $Q(r, \gamma)$ be given by* (15)*. Then, considering the definitions* (4b) *and* (4a)*, the following holds for the function V given in* (13)

$$\varrho_V(x) = \begin{cases} Q(r,\gamma) & \text{if } x = r \\ -x & \text{if } r^\top x \geq -\gamma \\ \sigma\big(r^\top x\big)\frac{\Pi(x)r}{|\Pi(x)r|} + \alpha\big(r^\top x\big)x & \text{if } -1 < r^\top x < -\gamma \\ \partial Q(r,\gamma) & \text{if } r^\top x = -1. \end{cases} \tag{16a}$$

$$\nu_V(x) = \begin{cases} \dfrac{1 - r^\top x}{1 - r^\top x + 2k} & \text{if } r^\top x \geq -\gamma \\ \dfrac{1 - r^\top x}{1 - r^\top x + k(1 - \alpha(r^\top x))} & \text{if } r^\top x < -\gamma \end{cases} \tag{16b}$$

for each $x \in \mathbb{S}^n$. □

Moreover, from Corollary 13 and from definition (3) it follows

$$\mathcal{E}(V) = \{(x,x) \in \mathbb{S}^n \times \mathbb{S}^n : x \in Q(r,\gamma)\},$$

for V given by (13). In the sequel, let

$$\mathcal{X} := \mathbb{S}^n \times Q(r,\gamma)$$

denote the reduced state space and let the function $\Delta : \mathbb{S}^n \to \mathbb{R}$ be given by

$$\Delta(x) := 1 - \nu_V(x).$$

for all $x \in \mathbb{S}^n$. We now note that both μ_V and Δ are continuous on $\mathcal{X}$ and $\mathbb{S}^n$, respectively, and that Δ agrees with μ_V on the set $\mathcal{E}(V)$ (when the duplicated argument is ignored).

Theorem 15. *The functions $\mu_V : \mathcal{X} \to \mathbb{R}$ and $\Delta : \mathbb{S}^n \to \mathbb{R}$ are continuous and for each $x \in Q(r,\gamma)$ (and therefore each $(x,x) \in \mathcal{E}(V)$),*

$$\mu_V(x,x) = \Delta(x).$$

Furthermore, $\Delta(x) > 0$ for all $x \in \mathbb{S}^n$, and in particular,

$$\min_{x\in Q(r,\gamma)} \Delta(x) = \min_{x\in\mathbb{S}^n} \Delta(x) = \frac{1+\gamma}{2/k+1+\gamma}$$

□

We conclude that, for any given $r \in \mathbb{S}^n$ and $\gamma \in [-1,1)$, the function (13) is a centrally synergistic potential function relative to r with synergy gap exceeding δ, for any

$$\delta \in \left(0, \frac{1+\gamma}{2/k+1+\gamma}\right).$$

While this is enough to establish global asymptotic stability of $\mathcal{A}$ for (6), according to Theorem 2, we also prove that V satisfies (7), so that global exponential stability is guaranteed by Theorem 3.

It follows from the fact that $\mathcal{E}(V) = V^{-1}(1)$ (c.f. Lemma 11) and from $\mathcal{E}(V) \subset D_V$, with D_V given in (5), that $V^* \in \mathbb{R}$, given by

$$V^* = \max_{(x,y)\in C_V} V(x,y),$$

with C_V given in (5) satisfies $0 < V^* < 1$. We note that V^* exists since V is continuous on the compact set C_V. The next theorem follows naturally from these considerations

Theorem 16. *The function $V \in C^1(\mathbb{S}^n \times Q(r,\gamma))$ given in* (13) *satisfies* (7) *with*

$$\underline{\alpha} = \frac{1}{2(1+k+\sqrt{1+2k\gamma+k^2}}, \tag{17a}$$

$$\overline{\alpha} = \frac{1}{2(1+k-\sqrt{1+2k\gamma+k^2}}, \tag{17b}$$

$$\lambda = \frac{2k(1-V^*)(1-\gamma)}{\left(1+k+\sqrt{1+2k\gamma+k^2}\right)^2}. \tag{17c}$$

Therefore, the set (2) *is globally exponentially stable for* (6)*.*

The novel kind of centrally synergistic potential functions on $\mathbb{S}^n$ that we introduced in this section is of particular interest, not only because $\{r\} \subset \mathbb{S}^n$ is globally exponentially stable for the hybrid system (6), but also because the map ϱ_V, that is used to select a new controller during jumps, is such that the flow generated by the vector field $\Pi(x)\nabla V_{\varrho_V(x_0)}(x)$, for each $x_0 \in D_v$, converges to r through the path of minimum distance.

For every compact manifold M and for every pair of points $p, q \in M$, there exists a path $c : \mathbb{R} \to M$ with $c(a) = p$ and $c(b) = q$ for some $a, b \in R$ such that the length of $c : \mathcal{I} \to M$, with $\mathcal{I} := [a,b]$, given by

$$L(c) = \int_a^b \left|\frac{dc}{dt}(\tau)\right| d\tau,$$

is minimal among all other paths with endpoints p and q. The path c is a *minimal geodesic* (c.f. [3]). If a path $p(t) : [0,\infty) \to M$ is Lebesgue integrable, then its length is given by

$$L^\infty(p) = \int_0^\infty \left|\frac{dp}{dt}(\tau)\right| d\tau.$$

It is possible to verify that, for every solution (x,y) of (6), $x(t,j) = x(t,j+1)$ for each $(t,j),(t,j+1) \in \text{dom } (x,y)$. Therefore, $x{\downarrow_t}(t) := x(t,J(t))$, with

$$J(t) := \max\{j : (t,j) \in \text{dom } (x,y)\},$$

is defined for each $t \in [0, \sup_t \text{dom } (x,y))$, for each solution (x,y) to (6), and it is absolutely continuous, hence $L^\infty(x{\downarrow_t})$ is well-defined. The next lemma states that solutions to the hybrid system (6), with V given by (13), have minimal length.

Lemma 17. *Consider the function $V \in C^1(\mathbb{S}^n \times Q(r,\gamma))$ given in* (13) *and the hybrid system associated with V in* (6)*. For each solution (x,y) to* (6) *with initial condition $(x_0,y_0) \in D_V$, we have that $L^\infty(x{\downarrow_t}) = L(c_{x_0,r})$, where $c_{x_0,r}$ denotes the minimal geodesic from x_0 to r.*

This property also holds for almost all initial conditions using the continuous controller generated by the gradient of the height function h_r. However, the main contribution of the controller presented in this section is that it exponentially stabilizes any given reference point globally, which no continuous controller is able to do. Moreover, even if we disambiguate the controller output at the unstable critical point $x = -r$ by adding some discontinuity, there are no guarantees of robustness to small measurement noise in that scenario.

In the next section, we present a simulation scenario that compares the two different centrally synergistic functions devised in this paper.

V. Numerical Study

In this section, we compare the length of solutions to (6) using the centrally synergistic potential functions U and V, given by (12) and (13), respectively, using as a reference point the vector $r = \begin{bmatrix}0 & 0 & -1\end{bmatrix}^\top$. The function V used in these simulations has parameters $\gamma = 0.5$ and $k = 1$, thus, from Theorem 15, it has synergy gap exceeding δ for some $\delta \in (0, 3/7)$. In particular, we chose $\delta = 3/14$. The function U makes use of the parameter

$$X = \begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & 0 \\ -0.25 & 0 & 0 \end{bmatrix},$$

which satisfies the constraint $\sigma_{\max}(X) < 1$ and $Xr \neq 0$.

We consider a set of initial conditions that lie in the jump set of the hybrid system associated with V. From the analysis of Figure 1 it is possible to conclude that each flow of the

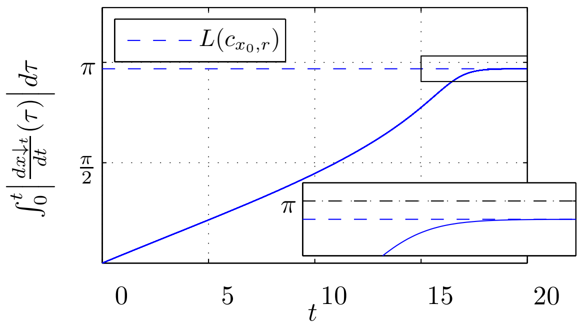

Fig. 1. Length of $x\downarrow_t$ for 20 solutions (x, y) to the hybrid system associated with V with different initial conditions x_0 at the same distance to r and $c_{x_0,r}$ denotes the minimal geodesic with endpoints x_0 and r.

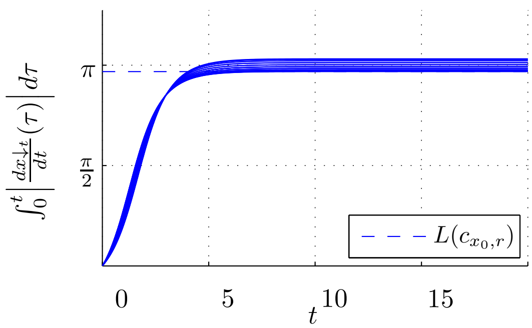

Fig. 2. Length of $x\downarrow_t$ for 20 solutions (x, y) to the hybrid system associated with U with different initial conditions x_0 at the same distance to r and $c_{x_0,r}$ denotes the minimal geodesic with endpoints x_0 and r.

hybrid system associated with V has the same length and converges to the length of the minimal geodesic between x_0 and r, as expected. The same is not true for the paths generated by gradient vector field generated by U, as shown in Figure 2. This illustrates that solutions to (6) associated with (12) have some directional bias, unlike the solutions to (6) associated with (13) which exploit the symmetry of $\mathbb{S}^n$ as highlighted in Lemma 17.

It is clear that, for the same set of initial conditions, there is a directional bias in the stabilization of the system which is suboptimal, in the sense that the system evolves along paths whose length are greater than the length of the minimal geodesic.

VI. Conclusion

In this paper, we have expanded the concept of centrally synergistic potential functions present in the literature and we have studied the stability properties of the closed-loop system generated by the gradient vector field and a jump logic that enforces jumps near the critical points. We proved that the closed-loop system is globally asymptotically stable if such functions exist. In addition, we proved that it is globally exponentially stable if the given centrally synergistic potential function satisfy certain bounds. We also constructed two different classes of synergistic potential functions, the second of which is able to select a controller that generates flows along geodesics if the initial condition lies in the jump set of the associated hybrid system. A numerical study was used to evaluate the properties of the proposed functions.

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