Title
The divergence of the light field in optical media

Permalink
https://escholarship.org/uc/item/9dj653g5

Author
Preisendorfer, Rudolph W

Publication Date
1957-07-26

Visibility Laboratory
University of California
Scripps Institution of Oceanography
San Diego 52, California

THE DIVERGENCE OF THE LIGHT FIELD IN OPTICAL MEDIA

R. W. Preisendorfer

26 July 1957
Index Number NS 714-100

SIO REFERENCE 58-41

Bureau of Ships
Contract NObs-72092

Approved:

Seibert Q. Duntley

Seibert Q. Duntley, Director
Visibility Laboratory

Approved for Distribution:

Roger Revelle

Roger Revelle, Director
Scripps Institution of Oceanography

The Divergence of the Light Field in Optical Media+

by Rudolph W. Preisendorfer

Scripps Institution of Oceanography, University of California, La Jolla, Calif.

ABSTRACT

The most general relation for the divergence of the light vector is derived from the equation of transfer for an arbitrary optical medium, and is shown to yield a direct means of determining the volume absorption function in natural aerosols and hydrosols. It is also shown that a correct special form of the divergence relation (for the slab geometry) is derivable from the classical Schuster equations and their solutions.

INTRODUCTION

The object of this note is to exhibit two applications of the theory of the divergence $\nabla \cdot \underset{\sim}{H}$ of the light field vector $\underset{\sim}{H}$ (the vector irradiance). The results are completely general but will perhaps find greatest use in geophysical optics, in particular meteorological and hydrological optics. The first application yields a direct and simply realized experimental means of the determination of the volume absorption function a in an arbitrary optical medium. The principal result is that there now exists independent means of determining the three basic attenuating functions of an optical medium: The volume absorption function a, the volume scattering function σ, and the volume attenuation function α. The second application yields a theoretical critique of the classical (two-flow) Schuster Analysis of a light field. The net conclusion is that the differential

---

+Contribution from the Scripps Institution of Oceanography, New Series No.______

This paper represents results of research which has been supported by the Bureau of Ships, U. S. Navy.

equations of the (two-flow) Schuster Analysis and their solutions are consistent with respect to the divergence relation of the light field. In fact, it is shown that the equations and their solutions implicitly embody the divergence relation for the slab geometry.

THE GENERAL DIVERGENCE RELATION

In order to gain the proper perspective of the present results in the setting of general radiative transfer theory the discussion begins with the most general representation of a <u>radiative transfer process</u>[1] in an arbitrary optical medium, namely the equation of transfer for radiance[2] N:

$$[n^2(\underset{\sim}{x},t)/v(\underset{\sim}{x},t)]\, D[N(\underset{\sim}{x},\underset{\sim}{\xi},t)/n^2(\underset{\sim}{x},t)]/Dt = -\alpha(\underset{\sim}{x},t)\, N(\underset{\sim}{x},\underset{\sim}{\xi},t) + N_*(\underset{\sim}{x},\underset{\sim}{\xi},t) + N_\eta(\underset{\sim}{x},\underset{\sim}{\xi},t), \quad (1)$$

where

$$N_*(\underset{\sim}{x},\underset{\sim}{\xi},t) = \int_{\Xi} \sigma(\underset{\sim}{x},\underset{\sim}{\xi},\underset{\sim}{\xi}',t)\, N(\underset{\sim}{x},\underset{\sim}{\xi}',t)\, d\Omega(\underset{\sim}{\xi}'), \quad (2)$$

---

[1] Rudolph W. Preisendorfer, "A Mathematical Foundation for Radiative Transfer Theory," Doctoral Dissertation, U.C.L.A., May 1956.

[2] The radiometric terminology used here follows where possible (and extends where necessary) the terminology recommended by the Committee on Colorimetry, J. Opt. Soc. Am. **34**, 183–218 (1944), **34**, 245–266 (1944).

For the purposes of the present study, the manifold and complex mathematical features of the equation of transfer and its components are subordinate to their physical consequences. Hence in the interests of brevity it will be merely recalled that N is the radiance function associated with a fixed wavelength λ. $\underset{\sim}{x} = (x_1, x_2, x_3)$ is a three dimensional location vector and $\underset{\sim}{\xi} = (\xi_1, \xi_2, \xi_3)$ is a three dimensional unit direction vector in euclidean three space, and t denotes time. Ξ is the collection of all unit direction vectors and Ω is the solid angle measure function on Ξ e.g. $d\Omega = \sin\theta \, d\theta \, d\phi$ when spherical coordinates are used for Ξ. D/Dt is the lagrangian derivative operator. α is the volume attenuation function, σ is the volume scattering function, n is the index of refraction function and v is the velocity of light function ($v = c/n$) all these functions being associated with λ. N_* is the path function, and N_η is the emission function, the former representing radiant energy arising from scattering without change of wavelength λ, the latter representing radiant energy arising from scattering with change in wavelength to λ, and all other sources of radiant flux (of wavelength λ) within the optical medium. The term $(N_* + N_\eta)/\alpha$ is customarily called the source function of the medium. N_* and N_η evidentally have the same dimensions as αN (radiance per unit length). It follows from general radiative transfer theory that the functions α and σ are connected by the relation

$$\alpha(\underset{\sim}{x}, t) = a(\underset{\sim}{x}, t) + s(\underset{\sim}{x}, t), \tag{3}$$

where

$$s(\underset{\sim}{x}, t) = \int_{\Xi} \sigma(\underset{\sim}{x}, \underset{\sim}{\xi}, \underset{\sim}{\xi}', t) \, d\Omega(\underset{\sim}{\xi}). \tag{4}$$

Of central interest in what follows are the notions of vector irradiance $\underset{\sim}{H}$ and scalar irradiance h, defined, respectively, as follows:

$$\underset{\sim}{H}(\underset{\sim}{x},t) = \int_{\Xi} \underset{\sim}{\xi}\, N(\underset{\sim}{x},\underset{\sim}{\xi},t)\, d\Omega(\underset{\sim}{\xi}), \qquad (5)$$

$$h(\underset{\sim}{x},t) = \int_{\Xi} N(\underset{\sim}{x},\underset{\sim}{\xi},t)\, d\Omega(\underset{\sim}{\xi}). \qquad (6)$$

h is related to the radiant energy density u by the formula:

$$v(\underset{\sim}{x},t)\, u(\underset{\sim}{x},t) = h(\underset{\sim}{x},t).$$

In the photometric context the counterpart to $\underset{\sim}{H}(\underset{\sim}{x},t)$ is $\underset{\sim}{E}(\underset{\sim}{x},t)$, the vector illuminance, and to $h(\underset{\sim}{x},t)$ there corresponds $e(\underset{\sim}{x},t)$ the scalar illuminance, $\underset{\sim}{E}$ and e being defined in terms of the luminance function B. $\underset{\sim}{E}$ has been studied extensively for example by Gershun[3] and Moon and Spencer[4] under the name of the light vector, and essentially for the case of non-attenuating ($\alpha \equiv 0$) media in the steady state ($\partial B(\underset{\sim}{x},\underset{\sim}{\xi},t)/\partial t \equiv 0$).

---

[3] A. Gershun, "The Light Field," J. Math. Phys. 18, 51-151 (1939), translated by P. Moon and G. Timoshenko.

[4] P. Moon and D. E. Spencer, "The Theory of the Photic Field," J. Franklin Inst. 255, 33-50 (1953)

The case of $\alpha \neq 0$ has been studied by Boldyrev[5] illustrating the fact that the notion of $\underset{\sim}{H}$ can be extended to turbid media. Even though the present discussion is for the radiometric case, it is convenient to retain the descriptive term. light field for the vector-valued function $\underset{\sim}{H}$. $\underset{\sim}{H}$ has been found of use in astrophysical optics[6] and has a natural counterpart in neutron diffusion theory.[7]

Let $\underset{\sim}{n}$ be an arbitrary unit vector. Define $\Xi_+ = \{ \underset{\sim}{\xi} : \underset{\sim}{\xi} \cdot \underset{\sim}{n} \geq 0 \}$ Further, define

$$H(\underset{\sim}{x}, \underset{\sim}{n}, t) = \int_{\Xi_+} \underset{\sim}{\xi} \cdot \underset{\sim}{n} \; N(\underset{\sim}{x}, \underset{\sim}{\xi}, t) \, d\Omega(\underset{\sim}{\xi}), \tag{7}$$

so that in particular

$$H(\underset{\sim}{x}, -\underset{\sim}{n}, t) = \int_{\Xi_+} \underset{\sim}{\xi} \cdot (-\underset{\sim}{n}) \; N(\underset{\sim}{x}, \underset{\sim}{\xi}, t) \, d\Omega(\underset{\sim}{\xi}). \tag{8}$$

It follows that

$$\underset{\sim}{n} \cdot \underset{\sim}{H}(\underset{\sim}{x}, t) = H(\underset{\sim}{x}, \underset{\sim}{n}, t) - H(\underset{\sim}{x}, -\underset{\sim}{n}, t) \equiv \bar{H}(\underset{\sim}{x}, \underset{\sim}{n}, t). \tag{9}$$

---

[5] N. G. Boldyrev, "The Light Field in Diffusing Media," Trans. Opt. Inst., Leningrad, 6, 1-8 (1931).

[6] S. Chandrasekhar, Radiative Transfer (Clarendon Press, Oxford, 1950).

[7] B. Davison, Neutron Transport Theory (Clarendon Press, Oxford, 1957).

Thus for an arbitrary unit vector Ω, $\Omega \cdot H(x, t)$ is the net radiant flux $\bar{H}(x, \Omega, t)$, in the direction Ω, across a unit area normal to Ω, the flux taking place at x, at time t.

With x and t fixed, an integration of the equation of transfer (1) over Ξ yields, after some obvious reductions, the most general expression involving the divergence $\nabla \cdot H$ of H.

$$n^2(x,t)\,\nabla_x \cdot [H(x,t)/n^2(x,t)] + \int_\Xi [1/\rho(x,\xi,t)]\,\mu(x,\xi,t) \cdot \nabla_\xi N(x,\xi,t)\, d\Omega(\xi) + [1/v(x,t)]\,\partial h(x,t)/\partial t = -a(x,t)h(x,t) + h_0(x,t). \tag{10}$$

Here $\nabla_x \equiv \sum_{i=1}^{3} n_i\, \partial/\partial x_i$, $\nabla_\xi \equiv \sum_{i=1}^{3} \Omega_i\, \partial/\partial \xi_i$; Ω_i, $i = 1, 2, 3$ are mutually orthogonal unit direction vectors (the basis vectors) for an arbitrary but fixed coordinate system of euclidean three-space. $\rho(x, \xi, t)$ is the radius of curvature of the natural path of photons at (x, ξ) at time t, and may be represented in terms of ξ, and the index of refraction at x and t by the relation:

$$1/\rho(x,\xi,t) = [1/n(x,t)]\,[\,\xi \times \nabla n(x,t)\,]. \tag{11}$$

$\underset{\sim}{\nu}(\underset{\sim}{x},\underset{\sim}{\xi},t)$ is the principal unit normal vector to the natural path of photons at $(\underset{\sim}{x},\underset{\sim}{\xi})$ at time t, and is defined by

$$[1/\rho(\underset{\sim}{x},\underset{\sim}{\xi},t)]\,\underset{\sim}{\nu}(\underset{\sim}{x},\underset{\sim}{\xi},t)=[1/v(\underset{\sim}{x},t)]\,d\underset{\sim}{\xi}(t)/dt, \tag{12}$$

which is simply one of Frenet's three vector-equation representations of space curves; in the present context, the curves are natural space trajectories of photons. The function h_η is defined analogously to h :

$$h_\eta(\underset{\sim}{x},t) = \int_{\Xi} N_\eta(\underset{\sim}{x},\underset{\sim}{\xi},t)\,d\Omega(\underset{\sim}{\xi}). \tag{13}$$

The principal special forms of (10) are obtained by the following cumulative sequence of assumptions:

(i) <u>The optical medium has an index of refraction function constant with respect to $\underset{\sim}{x}$ and t</u>. Hence $\nabla\eta(\underset{\sim}{x},t)\equiv 0$ at all times t, and $1/\rho(\underset{\sim}{x},\underset{\sim}{\xi},t)\equiv 0$, so that (10) becomes:

$$\nabla\cdot\underset{\sim}{H}(\underset{\sim}{x},t) + (1/v)\,\partial h(\underset{\sim}{x},t)/\partial t = -a(\underset{\sim}{x},t)h(\underset{\sim}{x},t) + h_\eta(\underset{\sim}{x},t). \tag{14}$$

Here $\nabla_{\underset{\sim}{x}} = \nabla$, and v is a constant.

(ii) The <u>optical medium is in the steady state</u>. Hence all of the present functions are independent of t, and (14) becomes:

$$\nabla\cdot\underset{\sim}{H}(\underset{\sim}{x}) = -a(\underset{\sim}{x})\,\dot{h}(\underset{\sim}{x}) + h_\eta(\underset{\sim}{x}). \tag{15}$$

(iii) The optical medium is emission-free, i.e., $N_\eta(\underset{\sim}{x}, \underset{\sim}{\xi}) \equiv 0$,
so that (15) becomes:

$$\nabla \cdot \underset{\sim}{H}(\underset{\sim}{x}) = -a(\underset{\sim}{x})\, h(\underset{\sim}{x}). \tag{16}$$

(iv) The optical medium is non-absorbing, i.e., $a(\underset{\sim}{x}) \equiv 0$, so that (16) becomes:

$$\nabla \cdot \underset{\sim}{H}(\underset{\sim}{x}) = 0, \tag{17}$$

and the light field is solenoidal.

(15) and (16) are representative, without exception, of all practical geophysical settings. Of the two, (16) is by far the most common, and attention will therefore be restricted to this relation. However, results obtained from (15) can be extended with only trivial formal modifications to the cases in which $h_\eta \neq 0$.

THE ABSORPTION FUNCTION AND THE DIVERGENCE RELATION

The prototypes of the two optical media of central interest in geophysical optics are the atmosphere and the hydrosphere. The geometrical settings of these media are for all practical purposes adequately represented (locally) by the slab (plane-parallel) geometry using a terrestrially based coordinate system (See Fig. 1).

Fig. 1

Observe that the z-coordinate is measured positive as one progresses into each medium from the principal boundary (the xy plane). x, y, and angles θ, ϕ are measured in the same way in each medium, as shown.

For illustrative purposes, attention will be restricted to the hydrosphere, results obtained in this context, however, are immediately extendable, mutatis mutandis, to the meteorological context.

With respect to the given coordinate system for the hydrosphere, the divergence relation (16) may now be written,

$$\partial \bar{H}(x,y,z,\underset{\sim}{i})/\partial x + \partial \bar{H}(x,y,z,\underset{\sim}{j})/\partial y - \partial \bar{H}(x,y,z,\underset{\sim}{k})/\partial z = = -a(x,y,z)h(x,y,z). \tag{18}$$

Making the following assumption about the x and y components of $\underset{\sim}{H}$ ($\underset{\sim}{x}$):

$$\partial \bar{H}(x,y,z,\underset{\sim}{i})/\partial x = \partial \bar{H}(x,y,z,\underset{\sim}{j})/\partial y = 0 , \tag{18'}$$

which follows from the empirical observation that the radiance function N is the independent of x and y over appreciable distances in hydrosols (and aerosols), (18) reduces to

$$d\bar{H}(z,+)/dz = a(z)\, h(z), \tag{19}$$

where, in view of the above assumption, $\bar{H}(x,y,z,\underset{\sim}{k})$ now has been abbreviated to $\bar{H}(z,+) = H(z,+) - H(z,-)$, and a and h now have the functional dependence shown. The three quantities $H(z,+)$, $H(z,-)$, and $h(z)$ are experimentally measurable quantities over an interval of depths z, so that $d\bar{H}(z,+)/dz$ is readily determinable. Hence the formula,

$$a(z) = [1/h(z)]\, d\bar{H}(z,+)/dz \tag{20}$$

under the above assumption, yields a rigorous method of calculation of the volume absorption function a at each depth of the interval. If for example, the hydrosol is stratified (with respect to z) in any way, this fact is immediately uncovered by use of (20). Experimental determinations of a, based on (20), have recently been made for the case of a natural hydrosol using data from the Lake Pend Oreille experiments by J. E. Tyler. It was found that (20) yields a direct and simply used method for the determination of the absorption function, which is independent of methods customarily used in the determinations of α and σ. These experimental results will be illustrated in detail in a paper by J. E. Tyler.

The integrated form of (16) presents a formula which may yield a laboratory method for the determination of a; in any event, it is instructive to note that, by means of the divergence theorem of vector analysis,

$$-\bar{P}(S,-) \equiv \int_S \underset{\sim}{n}(\underset{\sim}{x}) \cdot \underset{\sim}{H}(\underset{\sim}{x})\, dA = \int_M \nabla \cdot \underset{\sim}{H}(\underset{\sim}{x})\, dV = -\int_M a(\underset{\sim}{x}) h(\underset{\sim}{x})\, dV, \quad (21)$$

i.e.,

$$\bar{P}(S,-) = \int_M a(\underset{\sim}{x}) h(\underset{\sim}{x})\, dV, \quad (22)$$

where $\bar{P}(S,-)$ is the <u>net inward</u> radiant flux over a surface S which bounds a subset M of the optical medium. A is the area measure function of S, $\underset{\sim}{n}(\underset{\sim}{x})$ is the unit outward normal to S at $\underset{\sim}{x}$, and V is the volume measure function of the optical medium. If a is constant over M, then (22) may be written:

$$\bar{P}(S,-) = a v U(M), \quad (23)$$

where $U(M)$ is the radiant energy content of M, and v is the speed of light in M.

SCHUSTER ANALYSIS OF A LIGHT FIELD AND THE DIVERGENCE RELATION

A (two-flow) Schuster Analysis of a light field (in the slab geometry context) is defined as the pair $(H(\cdot,+), H(\cdot,-))$ of irradiance functions. While more must be said to make the notion mathematically acceptable, this definition is adequate for the present purposes.

The practical applications of the Schuster Analysis of a light field have been studied extensively, and the major features of its history may be traced back to Schuster's orginal work by consulting a few key papers in the literature.[8,9]

A systematic investigation of the generalized Schuster Analysis of a light field has been made and will be given at a later time. The differential equations for the Schuster Analysis given below are sufficiently close to the classical forms to render them plausible for the present. The principal object of the present section is to show that the differential equations of the classical Schuster Analysis implicitly contain the correct

---

[8] S.Q.Duntley, "The Optical Properties of Diffusing Materials," J. Opt. Soc. Am. 32, 61-70 (1942).

[9] P. Kubelka, "New Contributions to the Optics of Intensely Light-Scattering Materials," J. Opt. Soc. Am. 38, 448-457 (1948).

form of the divergence relation (19) in the slab geometry context, and furthermore that the solutions of the differential equations are consistent with general radiative transfer theory in so far as they are consistent with the divergence relation (19).

Associated with the pair $(H(\cdot,+), H(\cdot,-))$ is the pair $(h(\cdot,+), h(\cdot,-))$ defined by

$$h(z,+) = \int_{\Xi_+} N(z,\cdot,\cdot)\, d\Omega , \tag{24}$$

and

$$h(z) = h(z,+) + h(z,-) . \tag{25}$$

Further, we define

$$D(z,+) = h(z,+)/H(z,+) , \tag{26}$$

$$D(z,-) = h(z,-)/H(z,-) , \tag{27}$$

as the <u>distribution functions</u> of the Schuster Analysis. The name arises from the fact that $D(\cdot,+)$ and $D(\cdot,-)$ give, among other things, quantitative measures of the shape of the <u>radiance distribution</u> $N(z,\cdot,\cdot)$ at depth z. If, for example, the radiance distribution at some depth z were collimated, and inclined at an angle $\theta > \pi/2$ then $D(z,-) = |\sec\theta|$. If, on the other hand, at some depth z, $N(z,\theta,\phi) =$ constant, for all $\theta \geq \pi/2$, $0 \leq \phi < 2\pi$ then $D(z,-) = 2$. Now, the classical Schuster Analysis essentially contains the assumption $D(z,+) = D(z,-) = 2$. This and the assumption that α and σ are independent of z yields the differential equation for the Analysis $(H(z,+), H(z,-))$:

$$-dH(z,+)/dz = -(Da+b)H(z,+) + bH(z,-),$$

$$dH(z,-)/dz = -(Da+b)H(z,-) + bH(z,+), \tag{28}$$

where b is the so-called back scattering coefficient.

The solution of the pair (28) may be written

$$H(z,+) = g_+ c_+ e^{kz} + g_- c_- e^{-kz}$$

$$H(z,-) = g_- c_+ e^{kz} + g_+ c_- e^{-kz} \tag{29}$$

where $g_+ = 1 + (Da/k)$, $g_- = 1-(Da/k)$, $k = [Da\,(Da+2b)]^{1/2}$, and c_+ and c_- are arbitrary constants of integration, fixed by specifying boundary conditions on the Analysis ($H(z,+)$, $H(z,-)$).

Adding the equations of (28) and recalling that $DH(z,+) = h(z,+)$, $DH(z,-) = h(z,-)$, and $h(z) = h(z,+) + h(z,-)$, the correct form of the divergence relation is obtained:

$$d\bar{H}(z,+)/dz = a\,h(z).$$

Furthermore, by adding and subtracting the members of (29):

$$H(z,+) + H(z,-) = 2c_+ e^{kz} + 2c_- e^{-kz},$$

$$H(z,+) - H(z,-) = 2(Da/k)c_+ e^{kz} - 2(Da/k)c_- e^{-kz},$$

Since $$h(z) = D\left[H(z,+) + H(z,-)\right],$$

and $$\bar{H}(z,+) = H(z,+) - H(z,-)$$

it follows from these relations and (30) that

$$d\bar{H}(z,+)/dz = 2Dac_+e^{kz} + 2Dac_-e^{-kz} = a\,h(z),$$

the divergence relation once again.

In closing it should be observed that the preceding discussion demonstrates the consistency of the classical Schuster Analysis of a light field with respect to the general divergence relation. This by no means is the crucial test of consistency with respect to the general laws of radiative transfer theory. The crucial test is associated with the assumptions $D(z,+) =$ constant and $D(z,-) =$ constant, $z \geq 0$, and these assumptions are already known, from experiments and numerical calculations, to be generally false. A general mathematical demonstration of this test (or some equivalent) and the question of possible limiting forms of $D(\cdot,+)$ and $D(\cdot,-)$ have yet to be resolved.

CAPTIONS

Figure 1

Illustrating the slab geometries for the atmosphere and the hydrosphere.

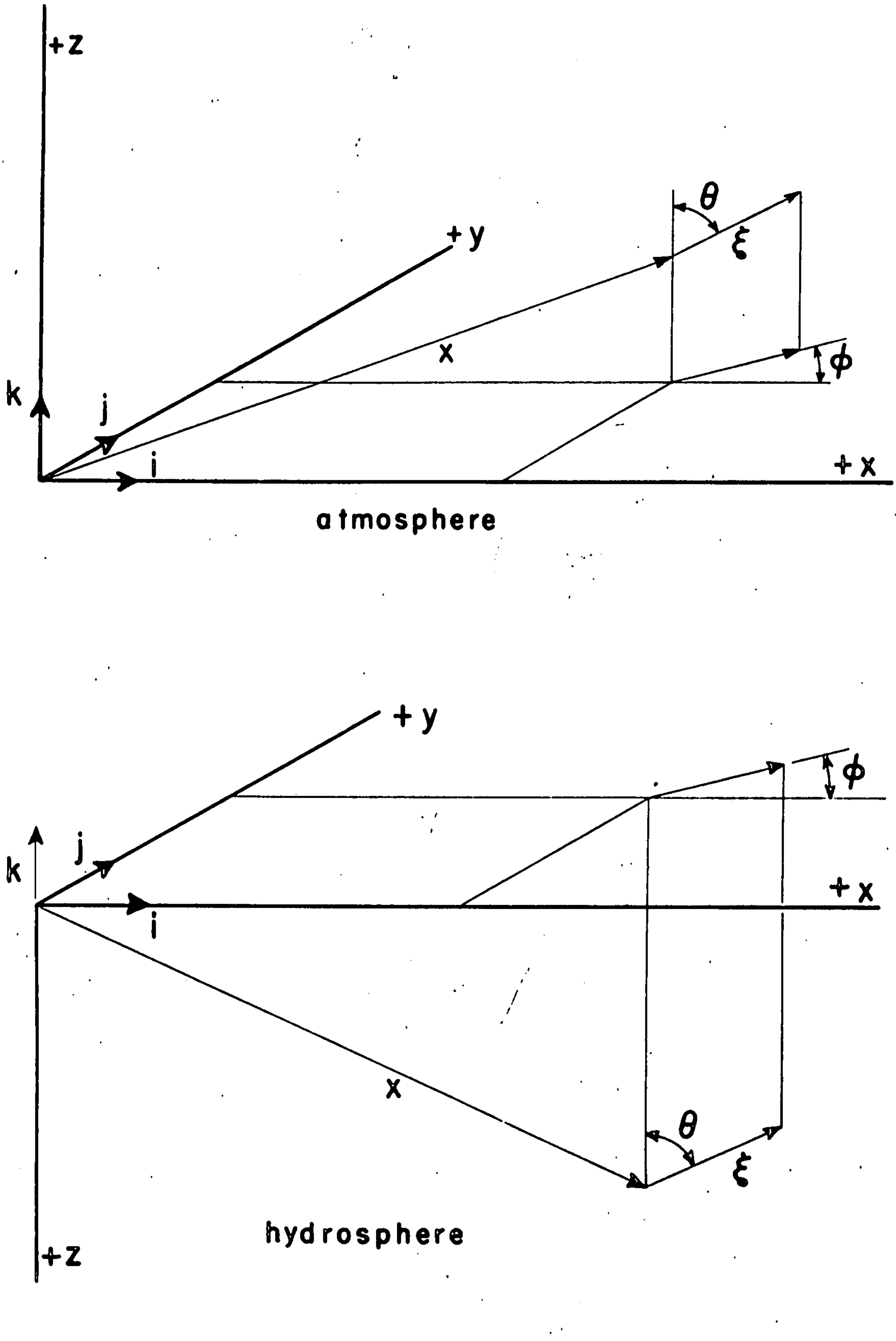

Figure 1. Rudolph W. Preisendorfer