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The problem of determining an analytical expression for the calculation of radiation dose is examined. We attempt to evaluate the primary dose rate due to an arbitrary spectrum of high-energy protons incident on a space vehicle. This problem was recently examined by Evans.[1]

The diameter of the vehicle is assumed to be much greater than the range of the high-energy protons and the curvature of the wall, therefore, may be neglected. Consequently the theory will be developed in terms of particles incident on a planar surface.

For the calculation of the depth dose, Evans presents a graphical method involving considerable labor. He also gives a crude approximate depth-dose theory, the results of which do not agree very well with those obtained by the more accurate graphical method.

We present an analytical method, the results of which agree very well with those obtained by the graphical method and yet involve little labor. With a simple two-term formula, the dose as a function of depth may be evaluated in a matter of minutes. The approximations in this formula are discussed and the results are compared with the graphical method.

We formulate the problem in terms of the angular flux, $\phi(\underset{\sim}{r}, E, \hat{\Omega})$, of particles incident on a planar surface. The quantity $\phi(\underset{\sim}{r}, E, \hat{\Omega})dE\, d\hat{\Omega}$ gives the number of particles per cm^2 per second at the point $\underset{\sim}{r}$ having the arbitrary energy E within dE and the arbitrary direction given by the unit vector $\hat{\Omega}$ within the solid angle $d\hat{\Omega}$. Thus if $\underset{\sim}{r}$ is the location of a surface element dS with an inward normal unit vector $\hat{n}$, then $\phi(\underset{\sim}{r}, E, \hat{\Omega})\, \hat{n} \cdot \hat{\Omega}\, dS\, dE\, d\hat{\Omega}$ gives the number of particles per second passing through the area element dS in the direction $\hat{\Omega}$ within $d\hat{\Omega}$ and having energy E within dE. This constitutes the surface source of particles incident upon the space vehicle.

Consider a vehicle composed of a thin outer shell of thickness a, surrounding a second medium whose arbitrary depth y is measured from the interface of the two media. (If desirable, the analysis can easily be extended to multilayer systems.) We wish to evaluate the energy deposition in a volume element $dV = dA\, dy$ located at depth y and having a surface area dA.

Only those incident particles having the direction $\hat{\Omega}$ headed toward the volume element dV can contribute to the primary dose rate in dV. In other words, if μ is the cosine of the angle between the point on the surface under consideration and the normal to the area dA, then $\hat{n} \cdot \hat{\Omega}$ must equal μ. Furthermore, only those particles within the cone of incidence given by the solid angle $d\hat{\Omega} = \mu dA/r^2$ can enter dV from the point $\underset{\sim}{r}$.

The energy of incident particles arriving at dV has been reduced in transit, by ionization and excitation of the atoms of the media. Let $E_2(y, E, \mu)$ be the energy of the protons in medium 2 upon arrival at dV; E_2 is a function of the depth y, the incident energy E, and the direction of incidence given by μ.

If the specific energy loss is given by dE_2/dr, then the energy deposited in traveling through dV in the direction $\hat{\Omega}$ is given by $(dE_2/dr)(dy/\mu)$. If the density of medium 2 is ρ_2, then the mass contained in dV is $\rho_2 dV = \rho_2 dA\, dy$. Therefore, the dose rate from all points on the surface due to an arbitrary incident spectrum is

$$D(y) = \int dE \int dS \frac{\mu}{r^2}\, \phi(\underset{\sim}{r}, E, \hat{\Omega})\, \frac{dE_2}{dr} \quad .$$

Let the surface element dS be designated by the angles μ and φ with respect to the origin at dA. The dose rate therefore may also be written as

$$D(y) = \frac{1}{\rho_2} \int_0^{2\pi} d\varphi \int_0^1 d\mu \int dE\, \phi(\underline{r}, E, \hat{\Omega}) \frac{dE_2(y, E, \mu)}{dr} \quad . \tag{1}$$

Equation (1) is the most general form of the primary dose rate due to a flux of arbitrary direction and energy incident on a planar surface.

For the special case of a uniform isotropic monoenergetic flux of protons of energy E_0, equation (1) simplifies to

$$D(y) = \frac{\Phi}{2\rho_2} \int_0^1 d\mu \frac{dE_2(y, E_0, \mu)}{dr} , \tag{2}$$

where the constant Φ gives the number of particles of energy E_0 per cm^2 per second, independent of direction.

Corresponding to the energy $E_2(y, E_0, \mu)$, the proton has a residual range $R(E_2) = R(y, E_0, \mu)$. Let $\overline{\mu}$ be the effective inverse linear relative stopping power of medium 2 with respect to medium 1. Since $\overline{\mu}$ is known to have a weak energy dependence, it is taken to be approximately independent of energy. If $R_1(E_0)$ and $R_2(E_0)$ are the ranges of a proton of energy E_0 when entirely in medium 1 or 2, then $\overline{\mu} = R_2(E_0)/R_1(E_0)$. The residual range in medium 2 is thus given by

$$R(E_2) = R(y, E_0, \mu) = R_2(E_0) - \frac{y}{\mu} - \overline{\mu} \frac{a}{\mu}$$

If we define

$$b(y) = \frac{a}{R_1(E_0)} + \frac{y}{R_2(E_0)} , \tag{3}$$

then

$$R(y, E_0, \mu) = R_2(E_0) \left[1 - \frac{b(y)}{\mu}\right] . \tag{4}$$

Note that the residual range R must be greater than or equal to zero, which means that only those particles within the cone given by

$$\mu \geq b(y) = \mu_{min} \tag{5}$$

can contribute to the dose rate.

One may now graphically evaluate the dose rate given by equation (2) by making use of expression (4) and the appropriate range-energy and specific energy-loss curves. The recipe for this graphical approach is given in Evans.[1]

To obtain an analytic representation of the dose rate, we have to assume an analytical form for the range-energy relation. The approximate straight-line behavior of a log-log plot of R(E) vs E implies the approximate analytical form

$$R(E) \approx pE^q, \tag{6}$$

where p and q are empirical constants giving the ordinate intercept and slope of log R vs log E.

Since the proton is assumed to experience at most small-angle elastic scatterings, the path length and the range are approximately equal. Therefore,

$$\frac{1}{\rho_2}\frac{dE_2}{dr} \approx \frac{1}{q}\frac{E_0}{\rho_2 R_2(E_0)} \frac{1}{\left[1 - \frac{b(y)}{\mu}\right]^{1-1/q}}. \tag{7}$$

From equations (2), (5), and (7), the dose rate is then given by

$$D(y) = \frac{\Phi}{2q}\frac{E_0}{\rho_2 R_2(E_0)} \int_{b(y)}^{1} \frac{d\mu}{\left[1 - \frac{b(y)}{\mu}\right]^{\alpha}} \tag{8}$$

where $\alpha = 1 - 1/q$.

The integral in equation (8) cannot be evaluated exactly. However, since the values of μ in the integral are such that $b(y)/\mu \leq 1$, one may expand the integrand in a power series, and then evaluate the integral term by term:

$$\left[1 - \frac{b(y)}{\mu}\right]^{-\alpha} = 1 + \alpha\,\frac{b(y)}{\mu} - \frac{\alpha(\alpha+1)}{2}\left(\frac{b(y)}{\mu}\right)^2 + \cdots . \tag{9}$$

To zeroth order, $[1 - b(y)/\mu]^{-\alpha} \approx 1$, the corresponding dose rate, $D^{(0)}(y)$, would be

$$D^{(0)}(y) = \frac{\Phi}{2q}\,\frac{E_0}{\rho_2 R_2}\,[1 - b(y)] . \tag{10}$$

Equation (10) is essentially the result of the approximate theory of Evans,[1] apart from a factor of $1/q$, which seems to have been omitted. Note that this zeroth order approximation neglects the y and μ dependence of the integrand, and it yields a linear dependence of the dose rate D on the depth y. The results obtained by the linear approximation are not very accurate in comparison with those obtained with the "exact" graphical method. (Cf. Fig. P-7, p. 31 of Ref. 1.)

To the next order of approximation

$$\left[1 - \frac{b(y)}{\mu}\right]^{-\alpha} = 1 + \alpha\,\frac{b(y)}{\mu} ,$$

the dose rate is

$$D^{(1)}(y) = \frac{\Phi}{2q}\,\frac{E_0}{\rho_2 R_2}\left[1 - b(y) + \alpha\, b(y)\log\frac{1}{b(y)}\right] . \tag{11}$$

This improves the theoretical dose rate and the log term introduces the necessary nonlinearity in the expression. However, a careful examination reveals that these successive orders of approximation converge rather slowly. The reason for this is that the major contribution to the integral in equation (8) comes when the denominator of the integrand is small; i.e., when μ is close to $b(y)$. But it is just in this limit that the series representation given by equation (9) converges very slowly.

To overcome this difficulty, we introduce a new variable x,

$$1 - \frac{b}{\mu} = x \geq 0.$$

With this change of variables, equation (8) becomes

$$\frac{D(y)}{\Phi} = \frac{1}{2q} \frac{E_0}{\rho_2 R_2(E_0)} b(y) \int_0^{1-b(y)} \frac{dx}{x^{a}(1-x)^2} . \qquad (12)$$

Since b(y) is never zero, x is always less than one, and we may now expand $(1-x)^{-2}$ in a power series. Furthermore, the major contribution to equation (12) occurs when x is close to zero, and the series expansion is particularly good in this limit. Hence, we may rewrite the integral in equation (12) as

$$\int_0^{1-b(y)} \frac{dx}{x^{a}(1-x)^2} = \frac{(1-b)^{1-a}}{1-a} \sum_{m=0}^{\infty} \frac{(m+1)(1-a)}{m+1-a} (1-b)^m . \qquad (13)$$

Let R_n be the remainder of the series in (13) after n terms:

$$R_n = \frac{1-a}{1-\frac{a}{n+1}}(1-b)^n + \frac{1-a}{1-\frac{a}{n+2}}(1-b)^{n+1} + \cdots .$$

Noting that $0 < a < 1$, one sees that the coefficients c_n of each of the terms in R_n are bounded as follows:

$$\frac{1-a}{1-\frac{a}{n+1}} \geq c_n > 1-a .$$

Therefore,

$$\frac{1-a}{1-\frac{a}{n+1}} \frac{(1-b)^n}{b} > R_n > (1-a) \frac{(1-b)^n}{b} . \qquad (14)$$

Equation (14) gives upper and lower bounds to the remainder of the series. In the limit of large n, the two bounds coalesce.

As an approximation to the remainder, we choose the arithmetic mean of the upper and lower bounds:

$$R_n \approx \frac{n+1-\frac{a}{2}}{n+1-a}\,(1-a)\,\frac{(1-b)^n}{b}\,. \tag{15}$$

It turns out that R_1 is a good approximation to the series remainder after the first term. Using equations (13) and (15) in (12), we have, in this approximation,

$$\frac{D(y)}{\Phi} = \frac{E_0}{2\rho_2 R_2(E_0)}\;\frac{[1-b(y)]^{1/q}}{2q(q+1)}\,[(3q+1)+(2q+1)(q-1)b(y)]\,. \tag{16}$$

Equation (16) is the analytical expression for the dose rate as a function of depth, which we have sought.

We compare the analytical result for the depth dose rate per unit flux given by equation (16) with the graphical method for the case treated by Evans.[1] This is the problem of a uniform isotropic monoenergetic flux of 40-MeV protons incident on liquid hydrogen enclosed in a copper shell 30 mils thick. (Evans treated a stainless steel shell, but we use copper because exact range-energy data are available for copper and we avoid the interpolation between curves necessary for stainless steel. At any rate, the difference between copper and stainless steel is small.)

The basic data are:

$$\frac{E_0}{\rho_2 R_2(E_0)} = 60\ \frac{\text{MeV}}{\text{g/cm}^2}\,; \qquad b(y) = 0.26 + 0.10^5 y; \qquad q = 1.8.$$

For this case equation (16) becomes

$$\frac{D(y)}{\Phi} = (21.9 + 1.15y)(0.74 - 0.10^5 y)^{1/1.8}\ \frac{\text{MeV}}{\text{g/cm}^2}\,. \tag{17}$$

The solid curve in Fig. 1 is the graph of the analytical expression (17). The dashed curves represent the results of the graphical method. There is a certain arbitrariness in the rounding off of peaks in the graphical method. The uncertainty in this rounding off is a crude approximation to the error involved in the neglect of the smearing-out effect of proton straggling. This uncertainty due to proton straggling is evidenced by two curves. The upper and lower dashed curves thus form an envelope within which the exact curve is to lie.

The analytical result given by equation (16) or (17) is seen to fall nicely within the envelope formed by the curves of the graphical method in Fig. 1. Aside from the uncertainty due to proton straggling, the agreement is quite good.

The basic approximations in expression (16) are the approximate range-energy relation given by equation (6) and the series remainder approximation given by equation (15). The former may be improved by choosing different constants p and q for different energy ranges, while the latter may be improved by using additional terms in the series (13) and the remainder R_n for $n > 1$.

Although these improvements may increase the accuracy of the analytical method, they come at the expense of simplicity, which is one of the great advantages of equation (16).

REFERENCE

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1 Robley D. Evans, Principles for the Calculation of Radiation Dose Rates in Space Vehicles. Arthur D. Little, Inc., Report No. 63270-05-01 (1961).

FIGURE CAPTION

Fig. 1. Dose rate in liquid H_2 surrounded by 30 mils of copper.

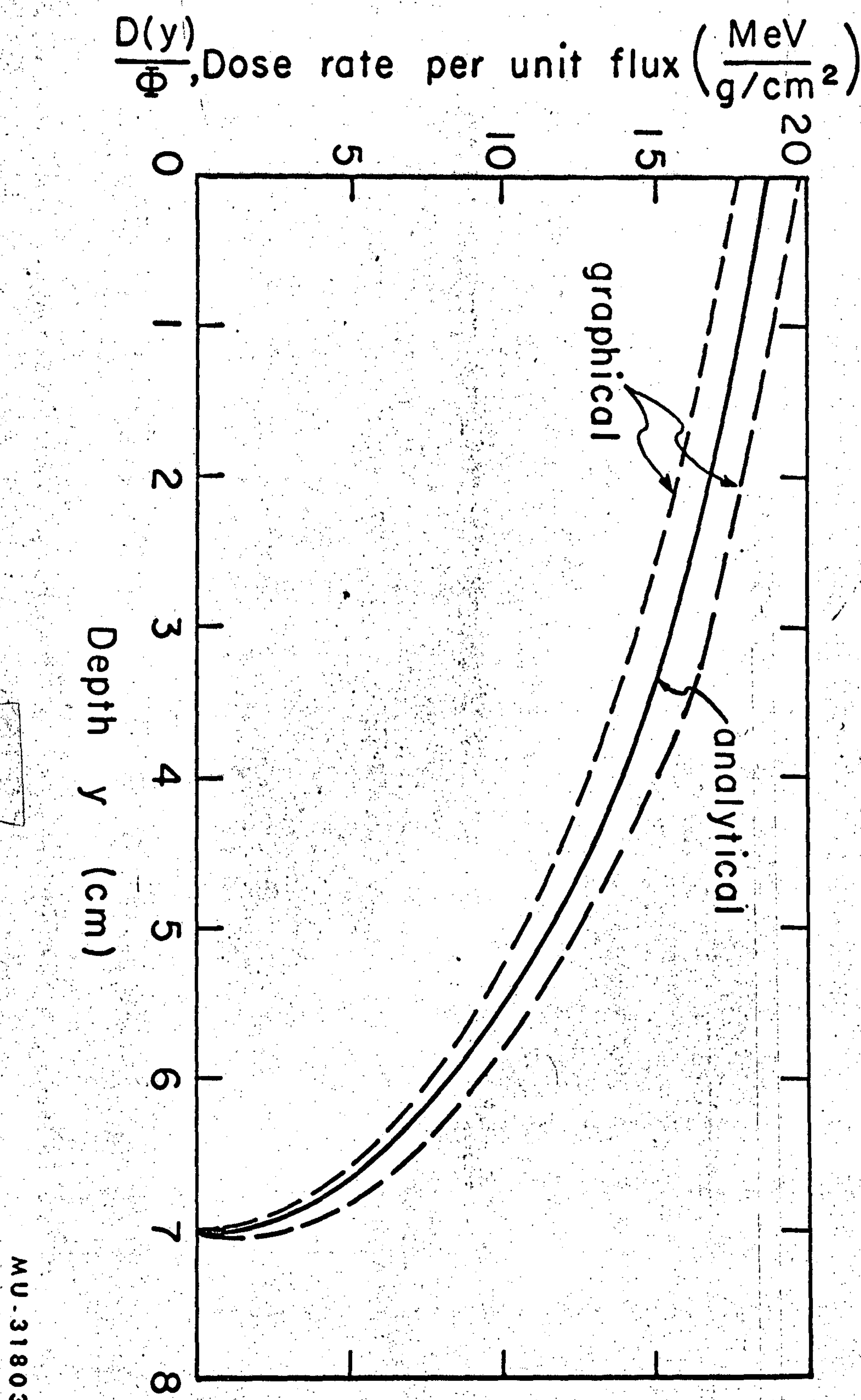
D(y)/Φ, Dose rate per unit flux (MeV/g/cm²)
0
5
10
15
20
graphical
analytical
0
1
2
3
4
5
6
7
8
Depth y (cm)
MU-31803

Fig. 1