Mathematical invariants in people's probabilistic reasoning

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Abstract

Recent research has identified three invariants or identities that appear to hold in people's probabilistic decision making: the addition law identity, the Bayes rule identity, and the QQ identity (Costello and Watts, 2014, Fisher and Wolfe, 2014, Costello and Watts, 2016b, Wang and Busemeyer, 2013, Wang et al., 2014). Each of these identities represent specific agreement with the requirements of normative probability theory; strikingly, these identities seem to hold in people's probability judgments despite the presence of strong and systematic biases against the requirements of normative probability theory in those very same judgments. We assess the degree to which two formal models of probabilistic reasoning (the 'probability theory plus noise' model and the 'quantum probability' model) can explain these identities and biases in probabilistic reasoning.

Introduction

A fundamental goal of science is to find invariants: constant relationships that hold between different variables. While such invariants occur frequently in the 'hard' sciences, they are rare in behavioural science. Recent work, however, has identified three invariants that appear to hold in people's probabilistic reasoning: the addition law, Bayes rule and 'QQ' ('Quantum Question') identities (Costello and Watts, 2014, Fisher and Wolfe, 2014, Costello and Watts, 2016b, Wang and Busemeyer, 2013, Wang et al., 2014). Each identity describes a constant relationship that holds between different probabilistic judgments, and each represents specific agreement with the requirements of probability theory. Strikingly, these identities hold in people's probability judgments despite the presence of strong biases, or systematic deviations from probability theory, in those very same judgments. We assess two formal models of probabilistic reasoning (the probability theory plus noise model, Costello and Watts, 2016, and the quantum probability model, Wang et al. 2014) in terms of their ability to explain these invariant identities and biases.

Identities in probabilistic reasoning

We use the following notation. We take $P(A)$ to represent the normatively correct probability of event A. We take $P_*(A)$ to represent a subjective estimate of $P(A)$. The QQ identity involves the relationship between probability estimates when questions are presented in specific orders. We take $P_{BA}(A)$ to represent the subjective estimated probability of A when questions are presented in the order BA (when people are asked to estimate $P(A)$ immediately after being asked to estimate $P(B)$) and take $P_{AB}(A)$ to represent the estimate when questions are in the reverse order. Since a subsequent estimate for $P(B)$ cannot affect the results obtained from a prior estimate for $P(A)$, $P_*(A) = P_{AB}(A)$ and $P_*(B) = P_{BA}(B)$.

The QQ identity

Consider a situation where people are asked questions in two alternative orders AB or BA. This situation is commonly seen in polls; for example, in a Gallup poll conducted in September 1997, half of participants were asked the question "Do you think Al Gore is honest and trustworthy?" followed immediately by the question "Do you think Bill Clinton is honest and trustworthy?", while the other half of participants were asked the same questions in the reverse order (Moore, 2002). People's answers in such situations are often strongly influenced by the order of question presentation ($P_{AB}(A) \neq P_{BA}(A)$). In the Clinton-Gore questions, for example, 76% of participants answered 'yes' to the Gore question when it was asked first ($P_{AB}(A) = 0.76$), while 66% answered yes when it was asked second ($P_{BA}(A) = 0.66$): the prior presentation of the Clinton question produced a bias, reducing the likelihood of a 'yes' answer to the Gore question. Simultaneously, however, results (both from experimental studies and from polls) show that the following identity tends to hold reliably in sequential question answering:

$$P_{AB}(A \wedge B) + P_{AB}(\neg A \wedge \neg B) - P_{BA}(A \wedge B) - P_{BA}(\neg A \wedge \neg B) = 0$$

(where $A \wedge B$ represents a 'yes' answer to both A and B and $\neg A \wedge \neg B$ represents a 'no' answer to both questions). This expression has a value of -0.003 in answers to the Clinton-Gore questions, for example. This identity holds for questions across a wide range of different topics in 72 different national representative surveys in the US, and in laboratory studies of the effects of order in question answering, even though these surveys show significant bias due to question order (Wang et al., 2014). This is just as predicted by the quantum probability model, and is seen as providing 'the strongest form of support' for that model (Wang et al., 2014).

The Addition Law and Bayes Rule identities

A number of identities must hold in standard probability theory. One such identity is the addition law, which requires that

$$P(A) + P(B) - P(A \wedge B) - P(A \vee B) = 0 \qquad (1)$$

must hold for all events A and B. Two other 'expansion' identities require that

$$P(A \wedge B) + P(A \wedge \neg B) - P(A) = 0 \quad (2)$$

$$P(A \wedge B) + P(\neg A \wedge B) - P(B) = 0 \quad (3)$$

must hold for all events A and B. Consider experiments where we ask people to estimate various probabilities $P(A)$, $P(B)$, $P(A \wedge B)$, $P(A \vee B)$, $P(A \wedge \neg B)$, $P(B \wedge \neg A)$ (not in any fixed ordering), and combine those estimates as in the various identities. Results show that, when we combine people's probability estimates for a given pair of events A,B as in the addition law identity, the average value obtained is equal to probability theory's required value of 0. When we combine the same estimates for the same events A,B as in the two expansion identities, the average value is not equal to 0; instead, the average value is positive (typically around 0.25) and is similar for both of these expansion identities. In other words, people's probability estimates reliably agree with probability theory for the addition law identity, but deviate from probability theory for the two expansion identities.

The addition law identity applies to direct or marginal probabilities. Similar results hold for identities that involve conditional probabilities. One such identity is the additive form of Bayes rule, which requires that

$$P(B|A)P(A) - P(A|B)P(B) = 0 \quad (4)$$

must hold for all events A and B. Two parallel 'Bayes expansion' identities require that

$$P(A \wedge B) - P(A|B)P(B) = 0 \quad (5)$$

$$P(A \wedge B) - P(B|A)P(A) = 0 \quad (6)$$

must hold for all events A and B. Experimental results for these identities follow those seen above: for the Bayes Identity the average value in people's estimates is equal to 0, while for the two Bayes expansion identities, the average value is positive (typically around 0.12, half the value seen for expansion identities in Equations 2 and 3) and is similar for both of these expansion identities (see Table 1). Results for these identities don't just hold when averaging across events: they also hold separately for each individual pair of events A and B, and they hold for estimates about familiar everyday events, medical diagnoses, future political or economic outcomes, or personality-description scenarios (Costello and Watts, 2014, Fisher and Wolfe, 2014, Costello and Watts, 2016b).

These patterns of agreement with the addition law and Bayes rule identity and simultaneous violation of the expansion identities (with approximately the same positive value for Equations 2 and 3 and approximately half that value for Equations 5 and 6), are predicted by the probability theory plus noise model. Confirmation of these predictions has been taken as evidence that the probability theory plus noise model 'may provide a fully general account of the mechanisms by which people estimate probabilities' (Costello and Watts, 2016b).

The quantum probability model, then, accounts for the QQ identity and for biases due to order effects, while the noise model accounts for the addition law and Bayes rule identities and for biases in the expansion identities. Can either model explain all three sets of results? In the next section we show that the quantum model is in principle unable to explain the addition law and Bayes rule results. We then show that the noise model gives a natural account for all these results.

The quantum probability model

The quantum probability model (Wang and Busemeyer, 2013, Wang et al., 2014) assumes that people's probabilistic reasoning follows the mathematical rules used to calculate event probability in quantum theory. A fundamental aspect of quantum theory is that the probability of two quantum events can depend on the order in which those events are measured. This order dependence allows the quantum probability model to address various order effects seen in people's sequential inference and judgment.

Probability has a geometric interpretation in quantum theory, based on the projection of vectors. We avoid this geometric interpretation here and instead focus on explaining how quantum probability agrees with, and deviates from, standard probability theory. In quantum probability, an observable defines the set of all possible distinct outcomes for a given measurement: the set of possible answers to the question represented by that measurement. The primary theoretical distinction between quantum and standard probability lies in the idea of 'compatible' or 'incompatible' observables. Two observables are compatible if both observables can be measured simultaneously. If two observables are compatible, quantum probability theory reduces exactly to standard probability theory in all cases. This means that if two observables are compatible then all the probability theory identities described above have a value of 0, and there are no order effects in judgment.

Incompatible observables, by contrast, cannot be measured simultaneously, and measurement outcomes depend on the order of measurement. If all probabilities are measured with the same ordering then again quantum probability theory reduces exactly to standard probability theory (if all probabilities are of the form $P_{AB}(\dot{})$, for example, then all relationships between those probabilities match the requirements of standard probability theory and all probability theory identities hold). If probabilities are measured with different orderings, however, then quantum probability deviates from standard probability, producing biases in judgment and order effects in sequential question answering such as $P_{AB}(A) \neq P_{BA}(A)$ and $P_{AB}(A \wedge B) \neq P_{BA}(A \wedge B)$.

Addition law and Bayes rule identities

The addition law and Bayes rule identities apply in cases where questions are not presented in some specific order AB

Table 1: Predicted values of the noise model and the quantum model for a series of probability theory identities. Standard probability theory requires these identities to have a value of 0. Observed average values for these identities are from Costello and Watts (2016b), Experiment 1. Similar average values hold for each individual pair A,B in that experiment and in a range of other experiments.

	identity	noise model	quantum model			observed
			compatible	incompatible: order AB	incompatible: order BA	
(1)	$P(A)+P(B)-P(A\wedge B)-P(A\vee B)$	0	0	δ_A	δ_B	0.01
(2)	$P(A\wedge B)+P(A\wedge\neg B)-P(A)$	d	0	$-\delta_A$	0	0.26
(3)	$P(A\wedge B)+P(\neg A\wedge B)-P(B)$	d	0	0	$-\delta_B$	0.23
(4)	$P(B\|A)P(A)-P(A\|B)P(B)$	0	0	Δ_{AB}	Δ_{AB}	0.006
(5)	$P(A\wedge B)-P(A\|B)P(B)$	$d/2$	0	Δ_{AB}	0	0.12
(6)	$P(A\wedge B)-P(B\|A)P(A)$	$d/2$	0	0	$-\Delta_{AB}$	0.12

or BA, but instead are order independent. In this situation there are no order effects for simple probabilities (the probability of A is $P_*(A)=P_{AB}(A)$ and that of B is $P_*(B)=P_{BA}(B)$). Order effects for incompatible observables still apply when people are asked to estimate conjunctive or disjunctive probabilities such as $P(A\wedge B)$, $P(A\wedge\neg B)$ or $P(A\vee B)$. For such conjunctions or disjunctions the quantum probability model assumes a particular characteristic ordering for observables that depends on the causal link between those observables. Complex probabilities such as $P(A\wedge B)$ are estimated using this characteristic ordering. This means that the relationship between a simple probability $P_*(A)$ and the conjunctive probabilities $P(A\wedge B)$ and $P(A\wedge\neg B)$ will depend on this characteristic ordering. In particular, when the characteristic ordering of observables for conjunctions is AB we have

$$P_*(A)=P_{AB}(A)=P_{AB}(A\wedge B)+P_{AB}(A\wedge\neg B) \qquad (7)$$

as in standard probability theory (since the ordering of observables is the same for all three probabilities in this expression, quantum probability reduces to standard probability in this case). When the characteristic order of observables for conjunctions is BA, however, we have

$$P_*(A)=P_{AB}(A)=P_{BA}(A\wedge B)+P_{BA}(A\wedge\neg B)+\delta_A \qquad (8)$$

where δ_A is a 'quantum interference' term for observable A. This quantum interference term represents deviation from the requirements of probability theory in the estimate $P_*(A)$, and arises from the difference between probabilities measured in the orders AB and BA. Note that quantum interference is not an error term here: it is a constant that specifies the relationship between $P_*(A)$ and $P_{BA}(A\wedge B)+P_{BA}(A\wedge\neg B)$ for a given participant and a given pair of events AB. Parallel results arise for the probability of B, where with the characteric ordering BA we have

$$P_*(B)=P_{BA}(B)=P_{BA}(A\wedge B)+P_{BA}(\neg A\wedge B) \qquad (9)$$

as in probability theory, while with the ordering AB we have

$$P_*(B)=P_{BA}(B)=P_{AB}(A\wedge B)+P_{AB}(\neg A\wedge B)+\delta_B \qquad (10)$$

where δ_B is the interference term for the estimate $P_*(B)$.

From these expressions for $P_*(A)$ and $P_*(B)$ we derive the quantum probability model's predictions for values of the addition law and the two expansion identities (Equations 1, 2 and 3) in three separate situations: where observables are compatible, where the ordering of observables is AB, and where the ordering is BA. The first three lines of Table 1 shows these predictions. From Table 1 we see that, if observables are compatible, all three identities have a predicted value of 0 (contrary to experimental results). If observables are measured in the order AB or BA, however, one expansion identity has a predicted value of 0 and the addition law and the other expansion identity have values that deviate from 0 by exactly the same magnitude but with opposite signs (contrary to experimental results). The quantum probability model's predictions are inconsistent with the experimental results in all three situations.

In quantum probability theory a conditional probability $P(A|B)$ is necessarily measured in the order BA (with the given event occurring first and the conditional event occurring after). This means that the relationships

$$P_{BA}(A\wedge B)=P(A|B)P_{BA}(B)=P(A|B)P_*(B) \qquad (11)$$

$$P_{AB}(A\wedge B)=P(B|A)P_{AB}(A)=P(B|A)P_*(A) \qquad (12)$$

necessarily hold in quantum probability (since the probabilities in these expressions are all measured in the same order, and so follow the requirements of probability theory). We define

$$\Delta_{AB}=P_{AB}(A\wedge B)-P_{BA}(A\wedge B)$$

to represent the effect of order on conjunctive probability judgments $P_{AB}(A\wedge B)$ and $P_{BA}(A\wedge B)$. Then substituting from Equations 11 and 12 into the Bayes rule and 'Bayes expansion' identities (Equations 4, 5 and 6), we derive predictions in three separate situations, as before (see Table 1). Here we see that, if observables are compatible, all three identities have a predicted value of 0 (contrary to experimental results). If observables are measured in the order AB, one expansion identity has a predicted value of 0 and the Bayes rule and the other expansion identity have the same values, deviating from zero by Δ_{AB} (contrary to experimental results). If observables are measured in the order BA, one expansion identity has a predicted value of 0 and the Bayes rule and the other expansion identity have values that deviate from zero by exactly the same magnitude of Δ_B but with opposite signs (again, contrary to experimental results). The quantum probability model's predictions are inconsistent with the experimental results in all three situations.

The QQ identity

Consider our definition of Δ_{AB} to represent order effects for the conjunctive probability judgments $P_{AB}(A \wedge B)$ and $P_{BA}(A \wedge B)$. A necessary mathematical consequence of quantum probability is that exactly the same order effects apply to conjunctive probabilities $P_{BA}(\neg A \wedge \neg B)$ and $P_{AB}(\neg A \wedge \neg B)$, and so we have

$$P_{AB}(A \wedge B) - P_{BA}(A \wedge B) = \Delta_{AB} = P_{BA}(\neg A \wedge \neg B) - P_{AB}(\neg A \wedge \neg B)$$

and therefore the QQ identity holds for events A and B in the quantum probability model. Wang et al. (2014) estimate the size of the order effect in each of their 72 different polls or experimental studies via the related measure

$$Z = max \begin{cases} |P_{BA}(A \wedge B) - P_{AB}(A \wedge B)| + \\ \quad |P_{BA}(\neg A \wedge \neg B) - P_{AB}(\neg A \wedge \neg B)| \\ \\ |P_{BA}(A \wedge \neg B) - P_{AB}(A \wedge \neg B)| + \\ \quad |P_{BA}(\neg A \wedge B) - P_{AB}(\neg A \wedge B)| \end{cases}$$

(so that the overall order effect is equal to the summed absolute values of the order effects for $A \wedge B$ and $\neg A \wedge \neg B$, or for $A \wedge \neg B$ and $\neg A \wedge B$, whichever is greater). The greater the value of this measure, the larger the order effect. They find statistically significant order effects in most of these polls or studies, but reliable agreement with the QQ identity. The fact that this QQ identity appears to hold simultaneously with such order effects has been taken as clear evidence that 'human judgments follow quantum rules' (Wang et al., 2014).

The probability theory plus noise model

The probability theory plus noise model assumes that people estimate probabilities via a mechanism that is fundamentally rational (following standard frequentist probability theory), but is perturbed in various ways by the systematic effects or biases caused by purely random noise or error. This approach follows a line of research leading back at least to Thurstone (1927) and continued by various more recent researchers (see, e.g. Dougherty et al., 1999, Erev et al., 1994, Hilbert, 2012). This model explains a wide range of results on bias in people's direct and conditional probability judgments across a range of event types, and identifies various probabilistic expressions in which this bias is 'cancelled out' and for which people's probability judgments agree with the requirements of standard probability theory (see Costello and Watts, 2014, Costello and Mathison, 2014, Costello and Watts, 2016a,b,c).

In standard probability theory the probability of some event A is estimated by drawing a random sample of events, counting the number of those events that are instances of A, and dividing by the sample size. The expected value of these estimates is $P(A)$, the probability of A; individual estimates will vary with an approximately normal distribution around this expected value. The probability theory plus noise model assumes that people estimate the probability of some event A in exactly the same way: by randomly sampling items from their memory, counting the number that are instances of A, and dividing by the sample size. Since memory is subject to various forms of random error, the model assumes that items have some probability $d < 0.5$ of being counted incorrectly. Given this error, a randomly sampled item can be counted as A in two mutually exclusive ways: either the item truly is an instance of A and is counted correctly (this occurs with probability $P(A)(1-d)$, since $P(A)$ items are truly instances of A, and items have a $(1-d)$ chance of being read correctly), or else the item truly is not an instance of A and is counted incorrectly as A (this occurs with probability $(1-P(A))d$, since $(1-P(A))$ items are truly not instances of A, and items have a d chance of being read incorrectly). The expected value for a noisy estimate for the probability of A is thus

$$P_*(A) = P(A)(1-d) + (1-P(A))d = (1-2d)P(A) + d \quad (13)$$

and we expect individual estimates $p_*(A)$ to vary independently around this expected value. This equation represents a pattern of regressive bias moving probability estimates $P_*(A)$ away from the true, objectively correct probability $P(A)$. Reasoning just as above, the model similarly predicts an expected value for the conditional probability $P(A|B)$ of

$$P_*(A|B) = \frac{(1-2d)^2 P(A \wedge B) + d(1-2d)\,[P(A) + P(B)] + d^2}{(1-2d)P(B) + d} \quad (14)$$

These expressions account for various observed patterns of bias in people's direct and conditional probability judgment (see Costello and Watts, 2014, 2016b).

Addition law and Bayes rule identities

This model makes predictions about the values of various probability theory identities. If we substitute Equation 13 into the addition law identity, for example, we get an expected value of

$$P_*(A) + P_*(B) - P_*(A \wedge B) - P_*(A \vee B) = 0$$

and so this model predicts that this expression should have a value of 0, and just as seen in experimental results (Costello and Watts, 2014, 2016b, Fisher and Wolfe, 2014). Similarly, if we substitute Equation 13 and Equation 14 into the Bayes rule identity, we get an expected value of

$$P_*(B|A)P_*(A) - P_*(A|B)P_*(B) = 0$$

and again the model predicts a value of 0, just as just as seen in experimental results.

Agreement with probability theory for the addition law and the Bayes rule identity arises, in this model, despite significant regressive bias due to random noise in individual probability estimates making up these expressions. This is because in these identities the various biases due to random noise in those individual probability estimates all cancel out. For other probability theory identities, however, this model predicts no cancellation of regressive effects. For example, substituting the expression from Equation 13 into the two 'expansion' identities (Equation 2 and 3), we get

$$P_*(A \wedge B) + P_*(A \wedge \neg B) - P_*(A) = d$$

$$P_*(A \wedge B) + P_*(\neg A \wedge B) - P_*(B) = d$$

and the model predicts the same positive value for both identities, again just as observed in experimental results (Costello and Watts, 2016b, Fisher and Wolfe, 2014).

For the two 'Bayes expansion' identities (Equation 5 and 6) we get

$$\begin{aligned} &P_*(A \wedge B) - P_*(A|B)P_*(B) \\ &= d(1-d) - d(1-2d)\,[P(A) + P(B) - 2P(A \wedge B)] \end{aligned}$$

$$\begin{aligned} &P_*(A \wedge B) - P_*(B|A)P_*(A) \\ &= d(1-d) - d(1-2d)\,[P(A) + P(B) - 2P(A \wedge B)] \end{aligned}$$

Since probability theory requires that $0 \leq P(A) + P(B) - 2P(A \wedge B) \leq 1$ for all A and B, and since $d < 0.5$ by assumption, we see that

$$d^2 \leq d(1-d) - d(1-2d)\,[P(A) + P(B) - 2P(A \wedge B)] \leq d(1-d)$$

and values for both these identities will be distributed between d^2 and $d(1-d)$ in a way that depends on $P(A) + P(B) - 2P(A \wedge B)$. The average value for $P(A) + P(B) - 2P(A \wedge B)$ (across uniformly distributed probabilities that are constrained to be consistent with probability theory) is 0.5, and so the average value for this expression is equal to $d/2$, the centerpoint of this range. The model thus predicts the same average positive value for these identities; a value half that for the first two expansion identities. Again, this is just as seen in experimental results (Costello and Watts, 2016b).

The QQ identity and order effects

The probability theory plus noise model, as presented above, assumes that when $P(B)$ and $P(A)$ are estimated sequentially, the value given for $P(A)$ is not influenced by the prior value given for $P(B)$. This is because the model assumes that people estimate some probability $P(A)$ by drawing a sample of items *at random* from memory, and counting the proportion that are A. To allow sequential effects in the noise model, we can relax this assumption, and say that the chance of a given item being sampled from memory is influenced by the degree to which that item is already active or 'primed'. Since the estimation of probability $P(B)$ involves drawing a sample of items and counting the proportion that are B, those items that were counted as B are more active (are primed), and so are more likely to be included in the 'random' sample of items drawn when estimating $P(A)$, causing an order effect.

Suppose that the chance of an already primed item being sampled is s. Also suppose that $P(B)$ has just been estimated in a previous sample: $P_*(B)$ then represents the proportion of items in that previous sample that were read as B. A sample is now drawn to estimate $P(A)$. Each item drawn to make up that new sample has a probability $sP_*(B)$ of coming from the primed set of items that were already read as B, and a probability $1 - sP_*(B)$ of being drawn randomly from the set of all items in memory. For the $sP_*(B)$ items in our sample that were previously read as B, the probability of one of those items being read as A is $P_*(A|B)$; this is the conditional probability of an item being read as A, given that it was read as B. For the remaining items that were just sampled randomly from memory, the probability of one of those items being read as A is simply $P_*(A)$. Given that we have just given an estimate for the probability $P(B)$, then, the expected value for an immediately following estimate for $P(A)$ will be

$$P_{BA}(A) = sP_*(B)P_*(A|B) + (1 - sP_*(B))P_*(A)$$

and, substituting from Equations 13 and 14 and simplifying we get

$$P_{BA}(A) = P_*(A) + s(1-2d)^2[P(A \wedge B) - P(A)P(B)] \quad (15)$$

From Equation 15 we see that $P_{BA}(A) \neq P_*(A)$ and so $P_{BA}(A) \neq P_{AB}(A)$ will hold in this model in general, with the probability of a 'yes' answer to question A when that question comes first being different from the probability of a 'yes' answer when question A immediately follows question B. This model thus produces order effects in question answering, just as seen in experimental data.

Despite these order effects, the QQ identity also holds in this model. To see this, consider that, since $P_*(B)$ is the probability of answering 'yes' to a question B and $P_{BA}(A)$ is the probability of answering 'yes' to a question A that immediately follows a question B, the probability of answering 'yes' to both questions when presented in the order BA is

$$\begin{aligned} &P_{BA}(A \wedge B) = P_*(B)P_{BA}(A) \\ &= P_*(B)P_*(A) + P_*(B)s(1-2d)^2[P(A \wedge B) - P(B)P(A)] \end{aligned}$$

and the probability of answering 'yes' to both questions in the order AB is

$$\begin{aligned} &P_{AB}(S \wedge B) = P_*(A)P_{AB}(B) \\ &= P_*(A)P_*(B) + P_*(A)s(1-2d)^2[P(A \wedge B) - P(B)P(A)] \end{aligned}$$

and so

$$\begin{aligned} &P_{BA}(A \wedge B) - P_{AB}(A \wedge B) \\ &= s(1-2d)^2[P(A \wedge B) - P(B)P(A)][P_*(B) - P_*(A)] \end{aligned} \quad (16)$$

Using the same line of reasoning for the probability of answering 'no' to both questions, we get

$$\begin{aligned} &P_{AB}(\neg B \wedge \neg A) - P_{BA}(\neg B \wedge \neg A) \\ &= s(1-2d)^2[P(\neg B \wedge \neg A) - P(\neg B)P(\neg A)][P_*(\neg A) - P_*(\neg B)] \end{aligned}$$

Substituting from Equation 13 and rearranging we have

$$P_*(\neg A) - P_*(\neg B) = (1-2d)[1-P(A)] + d - (1-2d)[1-P(B)] - d$$
$$= P_*(B) - P_*(A)$$

and from standard probability theory we have

$$P(\neg A \wedge \neg B) - P(\neg A)P(\neg B) = P(A \wedge B) - P(B)P(A)$$

and so

$$P_{AB}(\neg A \wedge \neg B) - P_{BA}(\neg A \wedge \neg B)$$
$$= s(1-2d)^2[P(B \wedge A) - P(B)P(A)][P_*(B) - P_*(A)] \quad (17)$$

giving

$$P_{AB}(A \wedge B) + P_{AB}(\neg A \wedge \neg B) - P_{BA}(A \wedge B) - P_{BA}(\neg A \wedge \neg B) = 0$$

and this model satisfies the QQ identity.

Conclusions

Much research on people's probabilistic reasoning over the last 50 years has focused on the various significant biases seen in probability estimation and judgment. Invariants such as the addition law, the Bayes rule identity, and the QQ identity, which hold simultaneously with these biases, reveal an important fact: they show us that these biases are systematically and quantatitively related and can be explained mathematically. We can see this in the case of the QQ identity, where there are reliable order effects (biases) in responses which nonetheless cancel out when responses are combined in the identity. We also see this in the addition law and Bayes rule identities, where there are reliable biases in probability estimates which again, cancel out when those estimates are combined in those identities.

In this paper we've shown that, unlike the quantum probability model, the probability theory plus noise model is able explain the satisfaction of three invariants in people's probabilistic judgment (the addition law, Bayes rule and QQ identities) alongside the occurence of various forms of systematic bias in those same judgments. These results support the theoretical proposal in that account, which is that human probabilistic judgment is based on a rational process (one that follows frequentist probability theory) that is subject to random noise. It is important to stress that we are not suggesting that people's probability estimates are themselves rational. This is clearly not the case: there is very extensive evidence demonstrating that people's probability estimates are systematically biased away from the requirements of probability theory. We argue that these biases are a consequence of the influence of random noise on the probability estimates generated by an underlying rational process. While this noise is random, it has systematic, directional effects (our noisy model's expected averages for probability estimates are systematically biased away from the 'true' probability values, in a way that seems to match the biases seen in people's estimates) which are cancelled out in these three identities. This model gives a new and useful perspective on the various systematic biases seen in people's probabilistic reasoning.

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