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On Blowup of Jang's Equation and Constant Expansion Surfaces

DISSERTATION

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for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

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ABSTRACT OF THE DISSERTATION

On Blowup of Jang's Equation and Constant Expansion Surfaces

by

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Doctor of Philosophy in Mathematics

University of California, Irvine, 2022

Professor Richard M. Schoen, Chair

In 1978, the physicist P.S. Jang introduced a quasilinear elliptic equation in an attempt to generalize Geroch's approach to the positive mass conjecture of general relativity. The first existence and regularity result of Jang's equation was obtained by R. Schoen and S.-T. Yau through the capillary regularization procedure and stability-based a priori estimates. Yet, the solutions produced by this procedure may blow up in some black hole regions.

Schoen–Yau showed that the graph of a blowup solution to Jang's equation is asymptotic to cylinders over apparent horizons. J. Metzger showed that such cylindrical asymptotics are exponential, and he estimated the asymptotic rate by certain spectral properties of apparent horizons, followed by Q. Han, M. Khuri, and W. Yu. Their estimates involve delicate barrier construction and require the assistance of regularized solutions. We provide a simple proof of the sharp estimates that also apply to general blowup solutions.

We prove the first analytic and geometric result of regularized solutions to Jang's equation in black hole regions by applying two natural geometric treatments: translation and dilation. First, we show that the graphs of properly translated solutions converge subsequentially to constant expansion surfaces. Second, we characterize the limits of properly rescaled solutions. Third, we investigate the structure of black hole regions that arise in the Schoen–Yau regularization procedure. Finally, we discuss a special case of low-speed blowup behavior.

Chapter 1

Introduction

1.1 Geometry of spacetime

In special relativity, a flat spacetime is modeled by Minkowski spacetime $\mathbb{R}^{1,3}$ endowed with the non-degenerate symmetric quadratic form

$$\mathbf{g}_0 = -dt^2 + \sum_{i=1}^{3}(dx^i)^2, \tag{1.1.1}$$

where $t = x^0$ is the temporal coordinate, and x^i's are the spatial coordinates for $i = 1, 2, 3$. Since the metric $\mathbf{g}_0$ has one negative eigenvalue and 3 positive eigenvalues, we say that $\mathbf{g}$ has **signature** $(-, +, +, +)$. In view of this structure, we have the following decomposition of the tangent space of $\mathbb{R}^{1,3}$. Let $v = (v^0, v^1, v^2, v^3) \in \mathbb{R}^{1,3}$ be a vector.

1. If $\mathbf{g}_0(v, v) < 0$, then v is **time-like**, and is interpreted as the 4-velocity of a massive object.

2. If $\mathbf{g}_0(v, v) = 0$, then v is **null** or **light-like**, and is interpreted as the 4-velocity of a light ray (photon).

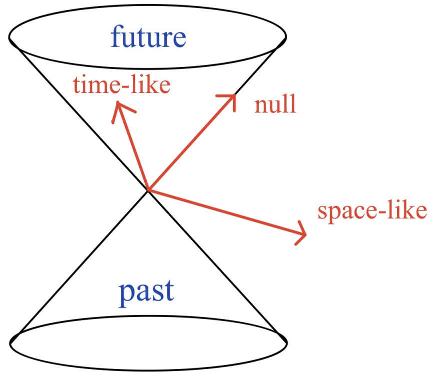

Figure 1.1: Light cone with examples of time-like, null and space-like vectors.

3. If $\mathbf{g}_0(v, v) > 0$, then v is **space-like** , and is interpreted as a tangent vector of a Riemannian submanifold of spacetime.

Furthermore, the **light cone**, a 3-dimensional hypersurface comprising all null vectors, decomposes the set of all time-like vectors into two open connected subdomains, called the **future** (with $v^0 > 0$) and the **past** (with $v^0 < 0$). This gives the spacetime causal structure, which Riemannian geometry does not possess.

In general relativity, a curved spacetime is modeled by a 4-dimensional Lorentzian manifold $(\mathcal{S}, \mathbf{g})$, where $\mathcal{S}$ is a smooth 4-dimensional manifold and $\mathbf{g}$ is a nondegenerate symmetric quadratic form with signature $(-,+,+,+)$. At each point $p \in \mathcal{S}$, there exists a "orthonormal" basis e_0, e_1, e_2, e_3 with respect to $\mathbf{g}$ for the tangent space $T_p\mathcal{S}$ such that $\mathbf{g}(e_0, e_0) = -1$ and $\mathbf{g}(e_i, e_i) = 1$ for $i = 1, 2, 3$. Analogous to Minkowski spacetime, any tangent vector is time-like, null, or space-like. Likewise, we call a submanifold N^k of spacetime $(\mathcal{S}, \mathbf{g})$ time-like, null, or space-like if all tangent vectors on N^k are time-like, null, or space-like, respectively. Finally, we always assume that a spacetime $(\mathcal{S}, \mathbf{g})$ is **time-orientable**; that is, there exists a global continuous unit time-like vector field η, i.e., $\mathbf{g}(\eta, \eta) = -1$, which designates causal relations (the future and past light cones) at every point in the spacetime $(\mathcal{S}, \mathbf{g})$.

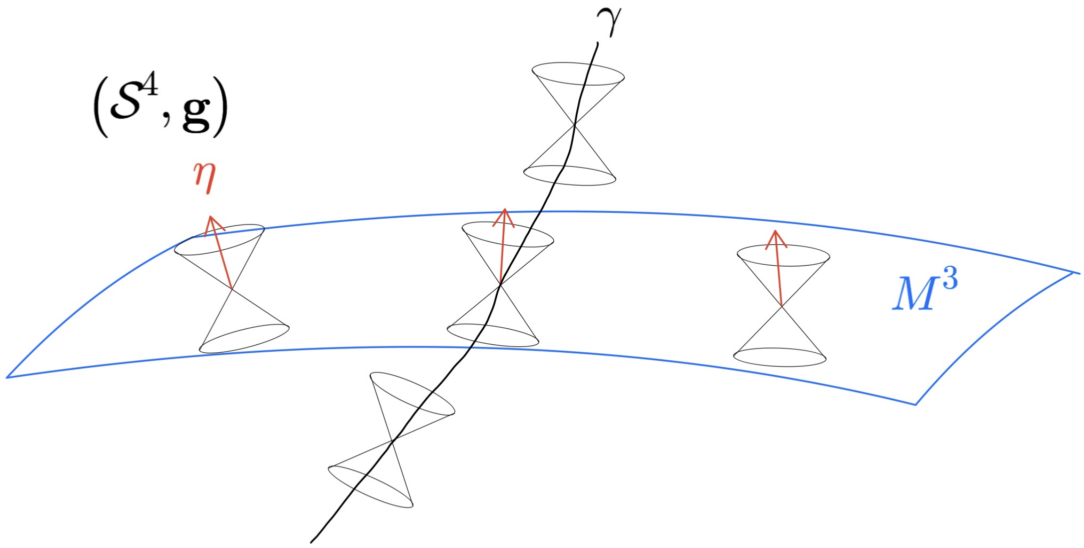

Figure 1.2: Time-like curve γ and space-like hypersurface M in spactime $(\mathcal{S}, \mathbf{g})$ oriented by time-like vector field η

Let us recall the basic Riemannian geometry constructions which apply to the Lorentzian setting. Let $(\mathcal{S}, \mathbf{g})$ be a 4-Lorentzian manifold. For simplicity, we also denote the Lorentzian metric $\mathbf{g}$ by $\langle \cdot, \cdot \rangle$. We assume that the indices $1 \leq i, j, k, \ell \leq 3$ and $0 \leq a, b, c, d \leq 3$. In addition, we take Einstein summation convention, i.e., when an index appears twice in a single term, it automatically implies summation of that term over all the values of the index. First of all, the metric naturally extends to all tensor bundles. For instance, if $\mathbf{S}, \mathbf{T}$ are $(0, 2)$-tensors, then

$$\langle \mathbf{S}, \mathbf{T} \rangle = \mathbf{g}^{ac}\mathbf{g}^{bd}\mathbf{S}_{ab}\mathbf{T}_{cd}.$$

The metric $\mathbf{g}$ uniquely defines a torsion-free and $\mathbf{g}$-compactible affine connection $\mathbf{D}$, called the **Levi-Civita connection**. In local coordinates x^0, x^1, x^2, x^3, we write $\partial_a = \frac{\partial}{\partial x^a}$ for simplicity. We define the **Christoffel symbol** $\mathbf{\Gamma}^c_{ab}$ by

$$\mathbf{D}_{\partial_a}\partial_b = \mathbf{\Gamma}^c_{ab}\partial_c,$$

where one can compute

$$\mathbf{\Gamma}^c_{ab} = \frac{1}{2}\mathbf{g}^{cd}\Big(\frac{\partial \mathbf{g}_{ad}}{\partial x^b} + \frac{\partial \mathbf{g}_{bd}}{\partial x^a} - \frac{\partial \mathbf{g}_{ab}}{\partial x^d}\Big).$$

The Riemann curvature tensor is defined as for any vector fields X, Y, Z,

$$\mathbf{R}_{X,Y}Z = \mathbf{D}_X\mathbf{D}_Y Z - \mathbf{D}_Y\mathbf{D}_X Z - \mathbf{D}_{[X,Y]}Z,$$

where $[X, Y] = XY - YX$ is the Lie bracket. In coordinates, we write

$$\mathbf{R}_{\partial_c,\partial_d}\partial_b = \mathbf{R}^a{}_{bcd}\partial_a,$$

where one can compute

$$\mathbf{R}^a{}_{bcd} = \partial_c\mathbf{\Gamma}^a_{db} - \partial_d\mathbf{\Gamma}^a_{cb} + \big(\mathbf{\Gamma}^e_{db}\mathbf{\Gamma}^a_{ce} - \mathbf{\Gamma}^e_{cb}\mathbf{\Gamma}^a_{de}\big).$$

We define the contractions of the Riemannian curvature tensor:

$$\text{Ricci Tensor :} \quad \mathbf{Ric}(\partial_b, \partial_d) = \mathbf{Ric}_{bd} = \mathbf{R}^a{}_{bad},$$

$$\text{Scalar Curvature :} \quad \mathbf{R} = \mathbf{g}^{bd}\mathbf{Ric}_{bd}.$$

1.2 Theory of General Relativity

Einstein's general relativity is a theory of gravity compatible with special relativity. Unlike Newton's theory, gravity is a consequence of the curvature of the spacetime rather than being considered as a force. There are three fundamental hypotheses in the theory of general relativity (cf. [44] Section 4.3).

(H1) The spacetime is a 4-dimensional time-orientable Lorentzian manifold.

(H2) A freely falling test massive body travels along time-like geodesics.

(H3) Einstein's equation holds:

$$\mathbf{G} := \mathbf{Ric} - \frac{1}{2}\mathbf{R}\mathbf{g} = 8\pi\mathbf{T}, \tag{1.2.1}$$

where $\mathbf{G}$ is called the **Einstein curvature tensor**, and $\mathbf{T}$ is a symmetric $(0,2)$-tensor, called the **stress-energy-momentum tensor**, representing a continuous matter distribution in the spacetime.

When $\mathbf{T} = 0$, (1.2.1) is called the **vacuum Einstein equation**, and can be reduced to $\mathbf{Ric} = 0$. Historically, Einstein discovered the vacuum equation before writing down the full equation.

1.2.1 Dominant Energy Condition

For any observer in the spacetime with future-directed time-like 4-velocity u, $-\mathbf{T}(u,\cdot)^\sharp$ represent the **energy-momentum 4-current density** of matter as seen by the observer. Here the musical isomorphism $(\cdot)^\sharp : T^*\mathcal{S} \to T\mathcal{S}$ is computed with respect to the Lorentzian metric $\mathbf{g}$. In a local orthonormal frame $u = e_0, e_1, e_2, e_3$ where the observer is stationary,

$$-\mathbf{T}(u,\cdot)^\sharp = T_{00}e_0 - \sum_{i=1}^{3} T_{0i}e_i,$$

where $T_{ab} = \mathbf{T}(e_a, e_b)$ for any $0 \le a, b \le 3$. In particular, the component $\mathbf{T}(u,u)$ represents the **energy density** of matter and the component $-T_{0i}e_i$ represents the **momentum density** of matter in e_i-direction measured by the observer. We say that $(\mathcal{S}, \mathbf{g})$ (or $\mathbf{T}$) satisfies the **dominant energy condition** if for any time-like vector u, $-\mathbf{T}(u,\cdot)^\sharp$ is a future-directed, null or time-like vector, i.e.,

$$T_{00} \geq \sqrt{\sum_{i=1}^{3} (T_{0i})^2}. \tag{1.2.2}$$

This means that the speed of energy flow of matter is always less than the speed of light. We can see from Einstein's equation (1.2.1) that the dominant energy condition is a certain positivity condition on the Einstein curvature tensor $\mathbf{G}$. There are other energy conditions which are usually considered in different contexts involving pressures of matter, e.g., **weak energy condition**, $\mathbf{T}(u,u) \geq 0$, and **strong energy condition**, $\mathbf{T}(u,u) \geq -\frac{1}{2}\mathrm{tr}_{\mathbf{g}}\mathbf{T}$ for any future-directed time-like vector u (cf. [44] Section 9.2).

1.2.2 Schwarzschild Spacetime

A few months after Einstein published his vacuum field equation (with $\mathbf{T} = 0$), the solution corresponding to the exterior gravitational field of a static, spherically symmetric isolated body was discovered by Karl Schwarzschild. The Schwarzschild solution is an important example to consider when discussing the notion of (total) mass and its related properties, e.g., the positive mass theorem and Penrose inequality.

For $m > 0$, define the Schwarzschild spacetime with mass m to be

$$\Big(\mathcal{S} := \mathbb{R} \times \big(\mathbb{R}_+ \times \mathbb{S}^2\big), \mathbf{g}_m := -\big(1 - \frac{2m}{r}\big)dt^2 + \big(1 - \frac{2m}{r}\big)^{-1}dr^2 + r^2 d\xi^2_{\mathbb{S}^2}\Big), \tag{1.2.3}$$

where $d\xi^2_{\mathbb{S}^2}$ denotes the standard round metric on $\mathbb{S}^2$. Note that r should be regarded as a radial coordinate rather than a distance function to the singularity at origin in any sense. In the weak field regime ($r \to \infty$), the behavior of a test mass in the Schwarzschild spacetime $(\mathcal{S}, \mathbf{g}_m)$ agrees with the behavior of a test mass in the Newtonian theory of gravity of an isolated point mass m at the origin (cf. [44] Section 6.2). Thus, we interpret the parameter m as the **total mass** of the Schwarzschild spacetime $(\mathcal{S}, \mathbf{g}_m)$. If $m < 0$, the metric $\mathbf{g}_m$ is incomplete; if $m = 0$, $\mathbf{g}_m = \mathbf{g}_0$ is simply the Minkowski metric, which can be viewed as a special case of the Schwarzschild solution.

Under the coordinate transformation $r = \rho(1 + \frac{m}{2\rho})^2$, the Schwarzshild metric can be written as a warped product

$$\mathbf{g}_m = -\left(\frac{1 - \frac{2m}{\rho}}{1 + \frac{2m}{\rho}}\right)^2 dt^2 + \left(1 + \frac{m}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\xi^2_{\mathbb{S}^2}\right). \tag{1.2.4}$$

Note that $d\rho^2 + \rho^2 d\xi^2_{\mathbb{S}^2}$ is the Euclidean metric in spherical coordinates. The induced Riemannian metric g_m on the time-slice $\{t = 0\}$, often called the **Riemannian Schwarzschild metric**, in isotropic[1] coordinates (t, x^1, x^2, x^3) takes the form

$$g_m = \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij} dx^i dx^j. \tag{1.2.5}$$

Thus, Riemannian Schwarzschild metric g_m is conformally flat.

1.3 Initial Data Sets

1.3.1 Initial Value Problem for General Relativity

Let $(\mathcal{S}^4, \mathbf{g})$ be a time-orientable spacetime governed by Einstein's equation (1.2.1). Suppose that there exists a space-like hypersurface $M^3 \subset \mathcal{S}$ which intersects every inextendible time-like curve, interpreted as a maximal worldline of a massive body, exactly once. Such hypersurface M is called a **Cauchy surface** and is thought of as a "snapshot $\{t = t_0\}$" of the spacetime $(\mathcal{S}^4, \mathbf{g})$. A spacetime $(\mathcal{S}, \mathbf{g})$ that possesses a Cauchy surface is said to be **globally hyperbolic** (cf. [44] Section 8.3). In fact, we can foliate globally hyperbolic

[1]Refer to [44] p. 93 in Section 5.1 for definition.

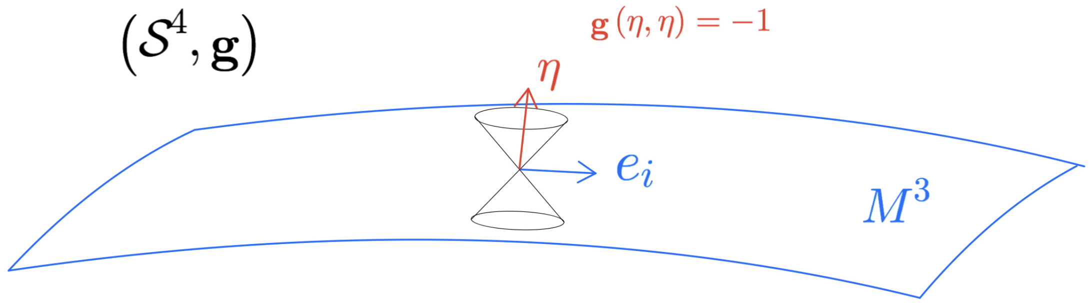

Figure 1.3: A Cauchy surface in spacetime. η is a unit time-like normal vector field and e_i is a tangential space-like vector field.

$(\mathcal{S}, \mathbf{g})$ by Cauchy surfaces, M_t, parametrized by a global time function t with $M_0 = M$ (cf. [44] Theorem 8.3.14). Thus, there exists a global unit vector field η (i.e., $\mathbf{g}(\eta, \eta) = -1$) in spacetime normal to the hypersurfaces M_t, interpreted as the "flow of time" experienced by a stationary observer.

In a well-posed initial formulation of general relativity, one is interested in finding the unique solution $(\mathcal{S}, \mathbf{g})$, called **maximal Cauchy development**, to Einstein's equation (1.2.1) satisfying certain suitable initial conditions imposed on a given Cauchy surface M. Einstein's equation (1.2.1), in certain choice of gauge, i.e., choice of coordinates, is a quasilinear wave equation (cf. [44] Section 4.4). In the initial value problem of linear wave equation, one places initial conditions on displacement and velocity. In analogy, in initial value formulation of general relativity, where the gravitational field is represented by $\mathbf{g}$, given a Cauchy surface M as the initial time-slice, one places the initial conditions on gravitational field $\mathbf{g}$:

$$\mathbf{g}\big|_M = g \quad \text{(initial "displacement")}, \quad \big(L_\eta \mathbf{g}\big)\big|_M = 2k \quad \text{(initial "velocity")}. \tag{1.3.1}$$

Here L_η denotes the Lie derivative along η, and $k = \mathbf{g}(\mathbf{D}_{(\cdot)}\eta, \cdot)$ denotes the second fundamental form of M with respect to η where $\mathbf{D}$ is the Levi-Civita connection with respect to $\mathbf{g}$. Note that g is a Riemannian metric and h is a symmetric $(0, 2)$-tensor on M.

Definition 1.1. A triple (M, g, k) is called an **initial data set** if M is a complete smooth

3-manifold without boundary, equipped with a symmetric positive-definite $(0,2)$-tensor g as Riemannian metric and a symmetric $(0,2)$-tensor k representing the second fundamental form of M in the maximal Cauchy development $(\mathcal{S}, \mathbf{g})$.

Pick a Lorentz frame adapted to M, i.e., $e_0 = \eta$ a unit time-like normal to M, and e_1, e_2, e_3 tangent to M. As before, indices i, j, k, ℓ range from 1 to 3. Note that the tensor k and the index k that appears in subscript or superscript carry totally different meanings and should be treated individually. Let $\mathrm{R}^i{}_{jkl}$ and $\mathbf{R}^i{}_{jkl}$ denote the Riemannian curvature tensor of (M, g) and $(\mathcal{S}, \mathbf{g})$, respectively. Let ∇ denote the Levi-Civita connection of (M, g). The Gauss-Codazzi equations on the initial data set (M, g, k) provide the following relationships:

$$\begin{aligned} &\text{(Gauss Equation)} \quad \mathbf{R}_{ijkl} = \mathrm{R}_{ijkl} + k_{ik}k_{jl} - k_{il}k_{jk}, \\ &\text{(Codazzi Equation)} \quad \nabla_j k^i{}_k - \nabla_k k^i{}_j = \mathbf{R}^i{}_{0jk}. \end{aligned}$$

Taking trace of the Gauss equation with respect to g twice, we get

$$\begin{aligned} \mathrm{R} + (\mathrm{tr}_g k)^2 - |k|_g^2 &= \sum_{i,j=1}^{3} \mathbf{R}_{ijij} \\ &= \left(\sum_{i,j=1}^{3} \mathbf{R}_{ijij} + \sum_{j=1}^{3} \mathbf{g}^{00}\mathbf{R}_{0j0j} \right) - \sum_{j=1}^{3} \mathbf{g}^{00}\mathbf{R}_{0j0j} \\ &= \sum_{j=1}^{3} \mathbf{Ric}_{jj} + \mathbf{Ric}_{00} \\ &= \mathbf{R} + 2\mathbf{Ric}_{00} = 2\mathbf{G}_{00}. \end{aligned}$$

Here R denotes the scalar curvature of g on M. Taking trace of the Codazzi equation, we have

$$\nabla_i \left(k^i{}_k - \mathrm{tr}_g(k)\delta^i{}_k \right) = \mathbf{Ric}_{0k}.$$

Not every initial data set (M, g, k) gives physically suitable initial conditions for general relativity. Recall that we always assume the dominant energy condition defined in Section 1.2.1 holds for the matter $\mathbf{T}$ in the right hand side of Einstein's equation (1.2.1). Since we assume that the Cauchy surface M is embedded in $(\mathcal{S}, \mathbf{g})$, the Gauss and Codazzi equations on M together with Einstein's equation (1.2.1) give **constraint equations**:

$$\begin{aligned} &\text{(Hamiltonian Constraint)} \quad \mathbf{T}(\eta,\eta) = \mu := \frac{1}{16\pi}\big(\mathrm{R}_g - |k|_g^2 + (\mathrm{tr}_g k)^2\big), \\ &\text{(Momentum Constraint)} \quad \mathbf{T}(\eta,\cdot)\big|_M = J := \frac{1}{8\pi}\mathrm{div}\big(k - \mathrm{tr}_g(k) g\big), \end{aligned} \tag{1.3.2}$$

Here the scalar function μ agrees with the **local mass density** of matter $\mathbf{T}$ and the vector $-J^\sharp$ agrees with **local current density** of matter $\mathbf{T}$ observed by a stationary observer in the initial data set (M, g, k). Thus, if $\mathbf{T}$ satisfies the dominant energy condition (1.2.2), then in particular on initial data set (M, g, k)

$$\mu \geq |J|_{\mathbf{g}}. \tag{1.3.3}$$

By slight abuse of language, we still say that the initial data set satisfies the **dominant energy condition** if (1.3.3) holds true.

An important special choice of initial data set satisfying the dominant energy condition is when the Cauchy surface M is totally geodesic, i.e., $k = 0$. Then such M is called a **time-symmetric slice**, since time reflection about M is an isometry of the maximal Cauchy development $(\mathcal{S}, \mathbf{g})$ generated by (M, g). Furthermore, the dominant energy condition (1.3.3) becomes a positivity condition on scalar curvature

$$\mathrm{R}_g \geq 0. \tag{1.3.4}$$

1.3.2 Asymptotic Flatness

Since gravity is attractive, it is physically reasonable to believe that matter is concentrated in some bounded regions, e.g., galaxies. When we study the structure of a galaxy distance from others, we may approximate it by an isolated system. Asymptotic flatness characterizes the property that in an isolated system the gravitational field becomes weak and thus the spacetime is asymptotic to the flat Minkowski spacetime near infinity.

Definition 1.2 ([39]). An initial data set (M, g, k) is **asymptotically flat (with ℓ ends)** if there is a compact subset $K \subset M$ such that $M \setminus K$ consists of finite number of connected components $M_1, \ldots, M_\ell$, called **infinite ends**, each of which is diffeomorphic to $\mathbb{R}^3 \setminus \overline{B}$ for a closed ball $\overline{B}$ in $\mathbb{R}^3$ such that under these diffeomorphisms

$$g_{ij} - \delta_{ij} = \mathrm{O}^2(|x|^{-1}), \quad k_{ij} \in \mathrm{O}^2(|x|^{-2}), \quad \sum_{i=1}^{3} k_{ii} = \mathrm{O}(|x|^{-3}),$$

and

$$\mathrm{R}_g = \mathrm{O}^1(|x|^{-4}).$$

Here by $f = \mathrm{O}^k(|x|^{-p})$ we mean that

$$\sup_{M \setminus K} \sum_{|I|=0}^{k} |x|^{p+|I|} |\partial_I f| < \infty,$$

where $\partial_I = \partial_{x^{i_1}} \partial_{x^{i_2}} \cdots \partial_{x^{i_j}}$ for multi-index $I = (i_1, i_2, \ldots, i_j)$ and $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the Euclidean distance in these coordinates.

For instance, the time-symmetric slice $\{t = 0\}$ of Schwarzschild spacetime in Section 1.2.2 is asymptotically flat.

1.3.3 Mass

Defining an energy satisfying a conservation law in general relativity is very different from pre-relativistic theories. The strategy of integrating local energy density over the background space no longer works. The primary reason is that gravitational field $\mathbf{g}$ describes the spatial property as well as the dynamical aspect of the spacetime $(\mathcal{S}, \mathbf{g})$. While Einstein's equivalence principle asserts that there is no observer who can be insulated by the influence of gravity, and thus there is no canonical gauge-free decomposition of $\mathbf{g}$ into a background part and a dynamical part. This leads to lack of local energy in general relativity. Moreover, integrating the local energy of matter $\mathbf{T}$ over a space-like hypersurface is not enough, since the gravitational field also contributes to the total energy. For instance, $\mathbf{T}$ is everywhere zero in time-slice $t = 0$ of Schwarzschild spacetime with metric g_m defined in (1.2.5), but the total energy should be m. However, it is possible to define the notion of **total energy** of an isolated system measured by an observer at infinity.

Motivated by the comparison between Schwarzschild spacetime and Newtonian model in weak field regime, if the Riemannian metric g on time-slice is asymptotic to Schwarzschild at an infinite end, i.e.,

$$g_{ij} = \left(1 + \frac{\mathbf{m}}{2|x|}\right)^4 \delta_{ij} + O(|x|^{-2}), \tag{1.3.5}$$

one may expect the total energy measured at this infinite end to be $\mathbf{m}$. More generally, for an asymptotically flat initial data set (cf. Definition 1.2) R. Arnowitt, S. Deser and C.W. Misner [5] introduced the total energy at any infinite end M_p, now often called the **ADM-energy**, defined by the flux integral over a coordinate 2-sphere near infinity

$$E_{\mathrm{ADM}}(M_p, g) = \frac{1}{16\pi} \lim_{r\to\infty} \sum_{i,j=1}^{3} \int_{|x|=r} (\partial_{x^i} g_{ij} - \partial_{x^j} g_{ii}) \frac{x^j}{|x|} \, d\mathcal{H}^2.$$

The ADM-formulation coincides with the weak field approximation in the asymptotically

Schwarzchild case.

Proposition 1.3. *If (M, g) is asymptotically Schwarzschild, i.e., (1.3.5) holds, then* $E = \mathbf{m}$.

Proof. Write

$$g_{ij} = \left(1 + \frac{\mathbf{m}}{2|x|}\right)^4 \delta_{ij} + \varepsilon_{ij},$$

where $\varepsilon_{ij} = \mathrm{O}^1(|x|^{-2})$. Then

$$\partial_{x^i} g_{ij} = 4\left(1 + \frac{\mathbf{m}}{2|x|}\right)^3 \cdot \frac{\mathbf{m}}{2}\left(-\frac{x^i}{|x|^3}\right)\delta_{ij} + \partial_{x^i}\varepsilon_{ij},$$
$$\partial_{x^j} g_{ii} = 4\left(1 + \frac{\mathbf{m}}{2|x|}\right)^3 \cdot \frac{\mathbf{m}}{2}\left(-\frac{x^j}{|x|^3}\right)\delta_{ii} + \partial_{x^j}\varepsilon_{ii}.$$

Therefore, the integrand becomes

$$\sum_{i,j=1}^{3} \left(\partial_{x^i} g_{ij} - \partial_{x^j} g_{ii}\right)\frac{x^j}{|x|} = 4\mathbf{m}\left(1 + \frac{\mathbf{m}}{2|x|}\right)^3 \left(\frac{1}{|x|^2}\right) + \mathrm{O}(|x|^{-3}).$$

Integrate over the sphere $|x| = \sigma$, we have

$$\begin{aligned}\sum_{i,j=1}^{3} \int_{|x|=r} \left(\partial_{x^i} g_{ij} - \partial_{x^j} g_{ii}\right)\frac{x^j}{|x|}\, d\mathcal{H}^2 &= \left\{4\mathbf{m}\left(1 + \frac{\mathbf{m}}{2\sigma}\right)^3\left(\frac{1}{\sigma^2}\right) + \mathrm{O}(\sigma^{-3})\right\} 4\pi\sigma^2 \\ &= 16\pi\mathbf{m}\left(1 + \frac{\mathbf{m}}{2\sigma}\right)^3 + \mathrm{O}(\sigma^{-1}).\end{aligned}$$

Finally, let $\sigma \to \infty$ and divide 16π, we get $E = \mathbf{m}$. □

Furthermore, the ADM-energy is gauge invariant [7]. Likewise, the **ADM linear momentum**

$$P^i_{\mathrm{ADM}}(M_p, g) = \frac{1}{8\pi}\lim_{r\to\infty}\sum_{j=1}^{3}\int_{|x|=r} |x|^{-1}\left(x^j k^i_j - x^i k^i_j\right)\, d\mathcal{H}^2,$$

is well-defined [33]. The **ADM 4-energy-momentum vector** $\left(P^a_{\mathrm{ADM}}\right) = (E_{\mathrm{ADM}}, P^i_{\mathrm{ADM}})$, treated as a 4-vector in Minkowski spacetime, is invariant under coordinate transformations preserving asymptotic flatness. Finally, we define the **ADM mass** m_{ADM} by

$$m_{\mathrm{ADM}} = \sqrt{-P^a_{\mathrm{ADM}}(P_{\mathrm{ADM}})_a}.$$

We will refer to the **positivity of mass** as $E_{\mathrm{ADM}} \geq |P^i_{\mathrm{ADM}}|$, i.e., $-P^a_{\mathrm{ADM}}(P_{\mathrm{ADM}})_a \geq 0$, and refer to the **positivity of energy** as $E_{\mathrm{ADM}} \geq 0$.

The following density theorem allows one to conformally deform the initial data set, taking an arbitrarily small cost of the ADM-energy, such that the dominant energy condition holds strictly.

Proposition 1.4 (Density theorem, [39] Lemma 1 cf. also [40])**.** *Let (M, g, k) be an initial data set. Given $\varepsilon > 0$, there is a function $u > 0$ on M such that*

$$u = 1 + \frac{A_k}{r} + \mathrm{O}(r^{-2}), \quad |\partial u| = \mathrm{O}(r^{-2}), \quad |\partial\partial u| = \mathrm{O}(r^{-3})$$

on M_k and $A_k < \varepsilon$ so that $(M, u^4 g, u^2 k)$ is an initial data set with mass density $\overline{\mu}$ and current density $\overline{J}$ satisfying

$$\overline{\mu} > |\overline{J}|.$$

1.3.4 Null Expansions and Trapped Surfaces

Following the settings in the beginning of Section 1.3, suppose (M, g, k) is an initial data set in spacetime $(\mathcal{S}, \mathbf{g})$ carrying a future-directed normal η of M in $\mathcal{S}$ such that $\mathbf{g}(\eta, \eta) = -1$, and such that at any $p \in M$, $k(X, Y) = \mathbf{g}(\mathbf{D}_X \eta, Y)$ for all $X, Y \in T_p M$. Let $\Omega^3 \subseteq M$ be an open region in M (not necessarily bounded), $\Sigma^2 = \partial\Omega$ be a smooth embedded two-sided surface, and let ν be the unit normal vector field on Σ pointing out of Ω in M. Let h denote the second fundamental form of Σ in M with respect to ν so that $h_p(X, Y) := g(\nabla_X \nu, Y)$ for all $X, Y \in T_p\Sigma$ for all $p \in \Sigma$, and let H denote the mean curvature with respect to ν.

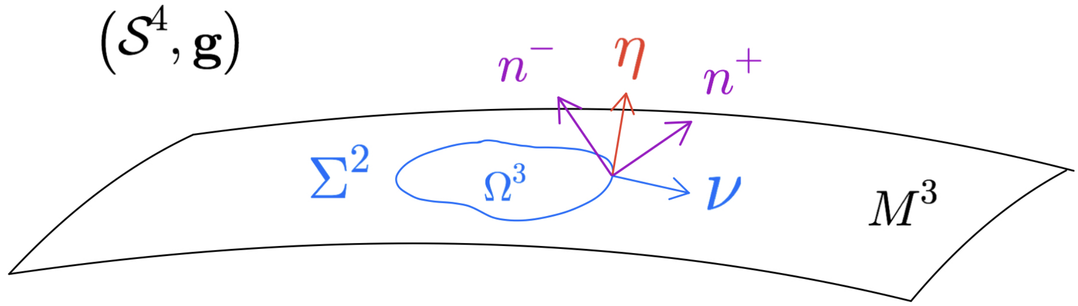

Figure 1.4: Two canonical null normal vector fields $n^\pm$ on Σ.

Now we think of Σ as a space-like 2-surface embedded in $\mathcal{S}$. There are two independent canonical future-directed null normal vector fields $n^+ := \eta + \nu$ and $n^- := \eta - \nu$ on Σ in $\mathcal{S}$ (see Figure 1.4). Then we can define the null second fundamental form $\mathbf{h}^\pm$ of Σ in $\mathcal{S}$ with respect $n^\pm$ by $\mathbf{h}^\pm_p(X,Y) := \mathbf{g}(\mathbf{D}_X n^\pm, Y) = (k \pm h)(X,Y)$ for all $X, Y \in T_p\Sigma$, $p \in \Sigma$.

Definition 1.5. We define the **outward(+)/inward(−) null expansion** to be the mean curvature of Σ with respect to $n^\pm$,

$$\theta^\pm[\Sigma] := \mathrm{tr}_\Sigma \mathbf{h} = \mathrm{K}[\Sigma] \pm \mathrm{H}[\Sigma], \tag{1.3.6}$$

where $\mathrm{K}[\Sigma] = \mathrm{tr}_\Sigma k$ is the trace of k restricted on Σ and $\mathrm{H}[\Sigma] = \mathrm{div}_\Sigma \nu$ is the mean curvature with respect to the outward unit normal ν on Σ.

Recall that mean curvature is the first variation of volume form. Thus, the null outward/inward expansion, respectively, measures the "expansion" of area of outgoing/ingoing light shells, $\left\{\Sigma^\pm_s\right\}_{s\in[0,\varepsilon)}$, emanating from Σ, up to first order. Here the outward/inward light shells are $\Sigma^\pm_s = \left\{\exp_y(sn^\pm(y)) : y \in \Sigma\right\}$ for $s \in [0, \varepsilon)$.

The existence of black holes is one of most fascinating prediction of Einstein's theory of general relativity. Roughly speaking, a **black hole region** of a spacetime is a region in which the gravitational field is so strong such that even a light ray emanating from the black

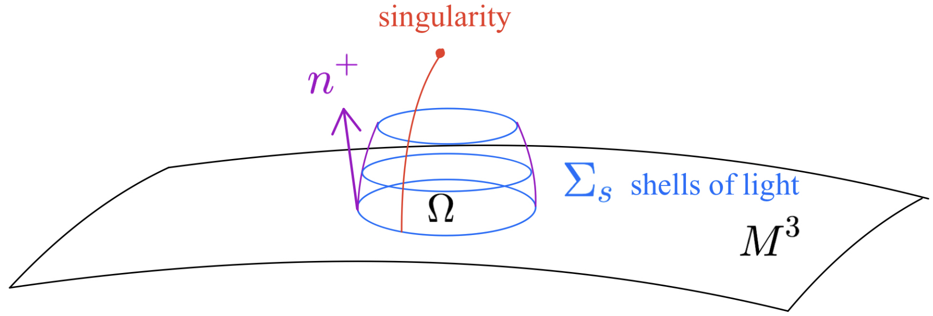

Figure 1.5: Illustration of the Penrose-Hawking singularity theorem.

hole region can not escape to its complement at any future time, while from every point of the complement, a light ray is able to escape to infinity. The boundary of the black hole region is called the **event horizon**. From the definition, it seems very unlikely to define black hole region on a initial data set without knowing the global structure of spacetime. Penrose proposed the idea of locating black hole regions with trapped surfaces.

Definition 1.6. A 2-surface Σ in an initial data set (M, g, k) is said to be **trapped** if both $\theta^+[\Sigma] < 0$ and $\theta^-[\Sigma] < 0$ hold true.

We typically expect $\theta^-[\Sigma] < 0$ because the inward light shells shrink, while $\theta^+[\Sigma] < 0$ is saying that even outward light shells also shrink in area measure. This captures the idea of "light not able to escape." For instance, all coordinate spheres with $0 < \rho < \frac{m}{2}$ at time-slice $t = 0$ in Schwarzschild spacetime (1.2.5) are trapped surfaces. In fact, the Penrose-Hawking singularity theorem states that under appropriate energy condition on matter, there exists a light ray emanating from a closed trapped surface Σ that eventually runs into a singularity (cf. [44] Theorem 9.5.3 and 9.5.4, also [22] Proposition 4.4.3). See Figure 1.5 for the illustration. Furthermore, under certain global assumptions, one may show that the trapped region Σ is indeed lies inside the black hole region (cf. [44] Proposition 12.2.2). The region Ω enclosed by Σ is called a **trapped region**, which is interpreted as the intersection of a part of black hole region with the time-slice (M, g, k).

For ease of exposition, we also define one-sided conditions. The surface Σ is called **outer trapped** or **outer untrapped**, if $\theta^+[\Sigma] < 0$ or $\theta^+[\Sigma] > 0$, respectively, without condition imposed on $\theta^-[\Sigma]$. If the borderline case $\theta^+[\Sigma] = 0$ holds, then Σ is called a **marginally outer trapped surface** (**MOTS**). Analogously, **outer trapped**, **outer untrapped**, and **marginally inner trapped** surfaces (MITS) are defined with $\theta^-[\Sigma]$. We call Σ an **apparent horizon** if it is either a MOTS or MITS. A compact apparent horizon can be interpreted as the cross-section of the event horizon in the initial data set.

In time-symmetric slice $(M, g, k = 0)$, an apparent horizon is just a *minimal surface* satisfying mean curvature $\mathrm{H} = 0$. In this case, MOTS can be realized by a variational problem of area, and the existence and regularity theory is well-developed. Solutions can be constructed by minimization or min-max procedure.

1.3.5 Stability Operator for Null Expansion

In this subsection, we extend the definition of initial data set (M^{n+1}, g, k) to all dimensions $n \geq 1$ by assuming that (M^{n+1}, g) is a $(n+1)$-dimensional Riemannian manifold carrying a symmetric $(0, 2)$-tensor k. Let $\Sigma^n \subset M^{n+1}$ be a smooth embedded two-sided hypersurface in an initial data set (M^n, g, k) and let ν be the normal vector field assigned to Σ. Let Φ_τ be a smooth one-parameter family of diffeomorphisms of M for $\tau \in (-\varepsilon, \varepsilon)$ so that Φ_0 is the identity map. Then $\Sigma_\tau := \Phi_\tau(\Sigma)$ defines variations of Σ such that $\frac{d}{d\tau}|_{\tau=0}\Phi_\tau|_\Sigma = X + \varphi\nu$, where X is a tangential vector field and φ is a smooth function on Σ. We have the following variation formulas (cf. [31] Lemma 5.1 and [1] section 2.2)

$$\frac{d}{d\tau}\Big|_{\tau=0} \mathrm{H}[\Sigma_\tau] = \langle \nabla^\Sigma \mathrm{H}[\Sigma], X\rangle - \Delta^\Sigma\varphi - \big(|h|^2_\Sigma + \mathrm{Ric}(\nu,\nu)\big)\varphi, \tag{1.3.7}$$

$$\frac{d}{d\tau}\Big|_{\tau=0} \mathrm{K}[\Sigma_\tau] = \langle \nabla^\Sigma \mathrm{K}[\Sigma], X\rangle + 2k(\nu, \nabla^\Sigma\varphi) + \nabla_\nu\big(\mathrm{tr}_M(k)\big)\varphi - (\nabla_\nu k)(\nu,\nu)\varphi, \tag{1.3.8}$$

where ∇^Σ and Δ^Σ denote respectively the gradient operator and non-positive Laplacian operator on Σ equipped with induced metric, $|h|^2_\Sigma$ denotes the square norm of the second fundamental form of Σ in M with respect to ν, and Ric and D denote ambient Ricci curvature and Levi-Civita connection in M. Now let $\xi := \big(k(\nu,\cdot)^\sharp\big)^\top \in \Gamma(\mathrm{T}\Sigma)$, we have

$$(D_\nu\, k)(\nu,\nu) = -\mathrm{H}[\Sigma]\, k(\nu,\nu) + \langle h, k\rangle_\Sigma + (\mathrm{div}_M\,(k))(\nu) - \mathrm{div}_\Sigma\,(\xi).$$

Using the Gauss equation and the definition of local density mass μ in constraint equations (1.3.2), we can compute

$$\mathrm{Ric}(\nu,\nu) = \mu + \frac{1}{2}\Big(- \mathrm{R}_\Sigma + |k|_g - \big(\mathrm{tr}_g\, k\big)^2 - |h|^2_\Sigma + \mathrm{H}^2_\Sigma\Big),$$

and using definition of local current density J of (M, g, k) in (1.3.2) we have

$$\big(\mathrm{div}_M\,(k)\big)(\nu) = J(\nu) + D_\nu(\mathrm{tr}_\Sigma\, k).$$

Combining all above identities, we obtain

$$\begin{aligned}\frac{d}{d\tau}\Big|_{\tau=0}\theta^\pm[\Sigma_\tau] &= \langle \nabla^\Sigma\, \theta^\pm[\Sigma], X\rangle - \Delta^\Sigma\varphi \pm 2\langle \xi, \nabla^\Sigma\varphi\rangle \\ &\quad + \Big(\mathcal{P}^\pm \pm \mathrm{div}_\Sigma\xi - |\xi|^2 - \frac{1}{2}\theta^\pm[\Sigma]\,\big(\theta^\pm[\Sigma] \mp 2\mathrm{tr}_M\,(k)\big)\Big)\varphi,\end{aligned} \tag{1.3.9}$$

where $\mathcal{P}^\pm = \frac{1}{2}\mathrm{R}_\Sigma - \frac{1}{2}|h \pm k|^2_\Sigma - \mu \mp J(\nu)$. We define the *stability operator of expansion* by

$$\mathcal{L}^\pm_\Sigma\varphi = -\Delta^\Sigma\varphi \pm 2\langle \xi, \nabla^\Sigma\varphi\rangle + \Big(\mathcal{P}^\pm \pm \mathrm{div}_\Sigma\xi - |\xi|^2 - \frac{1}{2}\theta^\pm[\Sigma]\,\big(\theta^\pm[\Sigma] \mp 2\mathrm{tr}_M\,(k)\big)\Big)\varphi. \tag{1.3.10}$$

If $\varphi > 0$, we have a simpler expression

$$\begin{aligned}\varphi^{-1}\mathcal{L}_\Sigma\varphi &= \mathrm{div}_\Sigma(\pm\xi - \nabla^\Sigma \log\varphi) - |\pm\xi - \nabla^\Sigma\log\varphi|^2_\Sigma \\ &\quad + \mathcal{P}^\pm - \frac{1}{2}\theta^\pm[\Sigma]\,\big(\theta^\pm[\Sigma] \mp 2\mathrm{tr}_M\,(k)\big).\end{aligned} \tag{1.3.11}$$

Notice that the linear operator $\mathcal{L}_\Sigma$ is not self-adjoint due to the first-order derivative contributed by k. Thus, apparent horizons do not arise as stationary points of an elliptic variational problem in initial data set (M, g, k). As discussed in [3], when Σ is closed, the Krein-Rutman theorem in general elliptic operator theory implies that the *principal eigenvalue* $\lambda_1 = \lambda_1(\mathcal{L}_\Sigma)$ is real and that there is a smooth positive eigenfunction β defined on Σ satisfying $\mathcal{L}_\Sigma \beta = \lambda_1 \beta$. Recall that the principal eigenvalue of $\mathcal{L}_\Sigma$ is the eigenvalue of $\mathcal{L}_\Sigma$ having the minimal real part. Moreover, λ_1 is *simple*, that is, the dimension of the eigenspace corresponding to λ_1 is one. For more details refer to [3] Section 4. As a generalization of stability of MOTS defined in [2, 3], a constant expansion surface Σ is said to be **stable** if the principal eigenvalue λ_1 of $\mathcal{L}_\Sigma$ is nonnegative. A more general stability for surfaces related to null expansion is defined in [12].

1.4 Jang's Equation

1.4.1 Initial Data Sets of Minkowski Spacetime

One of the fundamental question in general relativity is whether or not the total mass of an isolated system is positive if the local mass of matter is positive, called *positive mass theorem*. More precisely, positive mass theorem states that if an initial data set satisfies the dominant energy condition, then the total mass is nonnegative and vanishes only when the initial data set is that for Minkowski spacetime. A weaker version involving only ADM-energy is called positive energy theorem. Since the rigidity part of positive energy/mass theorem characterizes the initial data sets in Minkowski spacetime, P.S. Jang [24] consider the following two equivalent problems as he attempted to generalize Geroch's argument in time-symmetric slices to general initial data sets.

Proposition 1.7 ([24] Theorem I). *An initial data set (M, g, k) is that for Minkowski space-*

time if and only if there exist a function f and a flat metric g_{ij}^{flat} defined on M satisfying the overdetermined system of equations

$$\begin{cases} g_{ij} = g_{ij}^{\text{flat}} - \nabla_i f \nabla_j f \\ k^{ij} = \dfrac{\nabla^i \nabla^j f}{\sqrt{1+|\nabla f|^2}} \end{cases} \tag{1.4.1}$$

The key observation of Proposition 1.7 is that a Riemannian manifold (M^3, g) is a space-like hypersurface in Minkowski space $\mathbb{R}^{1,3}$ if and only if M is a normal graph of a function f defined on a space-like Euclidean hyperplane $\mathbb{R}^3$ in $\mathbb{R}^{1,3}$ with metric g given by

$$g_{ij} = g_{ij}^E - \partial_i f \partial_j f,$$

where g_{ij}^E is the flat metric on the chosen Euclidean hyperplane and $|\nabla f|^2 < 1$ since (M, g) is space-like. By pulling f and g^E back to M, we then obtain the function and flat metric stated in Proposition 1.7. For the detail of full derivation, refer to [24, Appendix]. Note that the metric equation in 1.4.1 is equivalent to

$$g_{ij}^{\text{flat}} = g_{ij} + \nabla_i f \nabla_j f.$$

Note that the metric $g + df \otimes df$ on right hand side, often called the **Jang's deformation of** g, is precisely the induced metric of the graph of $t = f(x)$ in Riemannian manifold $M \times \mathbb{R}$ with product metric $g + dt^2$. As a corollary of Proposition 1.7, we have two equivalent embedding problems.

Corollary 1.8. *An initial data set (M, g, k) is that for Minkowski spacetime if and only if there exist a function f on M such that the graph of $t = f(x)$ in $(M \times \mathbb{R}, g + dt^2)$ has flat induced metric and prescribed second fundamental form k.*

The system of equations (1.4.1) is overdetermined and is usually unsovable. Thus, Jang considered the trace equation involving the defect of second fundamental form on the graph of $t = f(x)$ in $(M \times \mathbb{R}, g + dt^2)$ in Corollary 1.8:

$$\sum_{i,j} \Big(g^{ij} - \frac{f^i f^j}{\sqrt{1 + |\nabla f|^2}} \Big) \Big(\frac{\nabla_i \nabla_j f}{\sqrt{1 + |\nabla f|^2}} - k_{ij} \Big) = 0,$$

where $f^i = g^{ij} f_j$ and the first factor is exactly the inverse of induced metric on the graph of $t = f(x)$ in $(M \times \mathbb{R}, dt^2 + g)$. This equation is called **Jang's equation**. We let

$$\mathrm{H}[f] := \sum_{i,j} \Big(g^{ij} - \frac{f^i f^j}{\sqrt{1 + |\nabla f|^2}} \Big) \frac{\nabla_i \nabla_j f}{\sqrt{1 + |\nabla f|^2}} = \mathrm{div}_M \Big(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \Big)$$

denote the mean curvature of graph(f) with respect to downward unit normal and let

$$\mathrm{K}[f] := \sum_{i,j} \Big(g^{ij} - \frac{f^i f^j}{\sqrt{1 + |\nabla f|^2}} \Big) k_{ij} = \mathrm{tr}_{\mathrm{Graph}(f)} k$$

be the trace of the tensor k on the tangent space of graph(f), where k is extended to $M \times \mathbb{R}$ trivially in the vertical direction, i.e., $k(\partial_t, \cdot) = 0$ and $\nabla_t k = 0$. Then Jang equation is actually marginally a MITS equation

$$\mathrm{H}[f] - \mathrm{K}[f] = 0$$

in the new initial data set $(M \times \mathbb{R}, dt^2 + g, k)$.

1.4.2 Schoen–Yau Regularized Solutions

Jang's approach to proof of general positive energy theorem has not been developed because of the lack of existence and regularity theory. The first existence and regularity result was proved by Schoen–Yau [39] in which they gave the first complete proof of positive energy theorem in a very different approach from one of Geroch and Jang.

The main analytic difficulty with Jang's equation is the lack of an a priori estimate of $\sup_M |f|$. To study the existence and regularity properties of Jang equation, Schoen–Yau in [39, Section 4] (also cf. [16] for $\dim M \geq 3$) introduced an elliptic regularization procedure of Jang's equation by adding a capillary term. Combining the existence and regularity theory of prescribed mean curvature equation together with continuity method, they showed [39, Lemma 3] the following existence and regularity result for regularized solutions.

Proposition 1.9 ([39] Lemma 3)**.** *For every $s > 0$ there exists a unique smooth solution f_s of regularized equation*

$$\Big(g^{ij} - \frac{f_s^i f_s^j}{1 + |\nabla f_s|^2}\Big)\Big(\frac{\nabla_i \nabla_j f_s}{\sqrt{1 + |\nabla f_s|^2}} - k_{ij}\Big) = s f_s. \tag{1.4.2}$$

satisfying $\lim_{x\to\infty} f_s(x) = 0$ *at each infinite end.*

The key initial estimates to proceed the standard elliptic theory for f_s are as follows. Thanks to the extra capillary term, Schoen–Yau proved by maximum principle argument that there are constants $\mu_1 = \max_M |\mathrm{tr}_g k|$ and $\mu_2 = \mu_2(|\mathrm{Ric}|_{C^0(M)}, |k|_{C^1(M)})$ such that

$$|s f_s| \leq \mu_1 \quad \text{and} \quad |s \nabla f_s| \leq \mu_2 \quad \text{in } M. \tag{1.4.3}$$

As we see from (1.4.3) that the bound for (weighted) Hölder norm of f_s is typically getting worse as $s \to 0^+$. Therefore, Schoen–Yau further proved the following geometric estimates for the general Jang's equation (1.4.5) including the regularized equations (1.4.2) satisfying bounds (1.4.3).

Proposition 1.10 ([39], Proposition 1 and 2)**.** *Let $F \in C^1(M)$ and μ_1, μ_2 be constants so that*

$$\sup_M |F| \leq \mu_1, \quad \sup_M |\nabla F| \leq \mu_2. \tag{1.4.4}$$

Suppose f is a C^2 solution to

$$\mathrm{H}[f] - \mathrm{K}[f] = F(x). \tag{1.4.5}$$

Then

(1) *There exists $c_1 = c_1(M, g, k, \mu_1, \mu_2)$ such that the second fundamental form h of* $\mathrm{Graph}(f)$ *is uniformly bounded:*

$$|h|^2 \leq c_1, \tag{1.4.6}$$

(2) *There is $\rho = \rho(M, g, k, \mu_1, \mu_2) > 0$ such that for every $X_0 \in \mathrm{Graph}(f)$ and (y^1, y^2, y^3, y^4) normal coordinates in $M \times \mathbb{R}$ on which $T_{X_0}\mathrm{Graph}(f)$ is the $y^1y^2y^3$-space, the local defining function $w(y)$ for $\mathrm{Graph}(f)$ is defined on $\{y = (y^1, y^2, y^3) : |y| \leq \rho\}$ with*

$$\mathrm{Graph}(f) \cap B^4(X_0; \frac{\rho}{2}) \subseteq \{(y, w(y)) : |y| \leq \rho\}.$$

Furthermore, for any $\alpha \in (0, 1)$ there is a constant $c_2 = c_2(M, g, k, \mu_1, \mu_2, \alpha) > 0$ such that

$$\|w\|_{3,\alpha;\{y:|y|\leq\rho\}} \leq c_2.$$

Here, $B^4(X_0, r)$ denotes the geodesic ball in $(M \times \mathbb{R}, g + dt^2)$ and $\|w\|_{3,\alpha;\{y:|y|\leq\rho\}}$ denotes the $C^{3,\alpha}$-Holder norm in the Euclidean ball $\{y : |y| \leq \rho\}$ on the tangent space.

(3) *There are constants c_3, c_4 depending on M, g, k, μ_1, μ_2 such that the following Harnack-type inequalities hold*

$$\sup_{\mathrm{Graph}(f)\cap B^4(x_0;\frac{\rho}{2})} \langle \nu, -\partial_t \rangle \leq c_3 \inf_{\mathrm{Graph}(f)\cap B^4(X_0;\frac{\rho}{2})} \langle \nu, -\partial_t \rangle;$$

$$\sup_{\mathrm{Graph}(f)\cap B^4(x_0;\frac{\rho}{2})} |\overline{\nabla} \log\langle \nu, -\partial_t \rangle| \leq c_4.$$

Here, ν is the downward pointing normal of $\mathrm{Graph}(f)$ *in* $M \times \mathbb{R}$ *and* $\overline{\nabla}$ *denotes the Levi-Civita connection on* $\mathrm{Graph}(f)$*.*

One key ingredient in the proof of Proposition 1.10 and further applications is the stability inequality derived from spectral property of stability operator $\mathcal{L}$. Let $G = \mathrm{Graph}(f)$ denote the graph of $t = f(x)$, let $\nu = \big(1 + |\nabla f|^2\big)^{-1/2}(\nabla f - \partial_t)$ denote the downward unit normal to $\mathrm{Graph}(f)$, and let $\beta = \langle \nu, -\partial_t\rangle = \big(1 + |\nabla f|^2\big)^{-1/2}$ denote the vertical component of ν. We then decompose $-\partial_t = X + \beta\nu$ where $X = -\beta^2(\nabla f + |\nabla f|^2\partial_t)$ is a bounded tangent vector field. Note that since equation (1.4.5) is insensitive to vertical translations, ∂_t gives a Jacobi field on the graph of solution f to (1.4.5). Use the variation formula of null expansion θ^- (1.3.9), we get

$$0 = X(F) - \Delta^G\beta - 2\langle \xi, \overline{\nabla}\beta\rangle + \left(\mathcal{P}^- - \mathrm{div}_G\xi - |\xi|^2 - \frac{1}{2}F\big(F + 2\mathrm{tr}_M(k)\big)\right)\beta.$$

Since $\beta > 0$, we may divide both sides by β and use the expression (1.3.11). Then we obtain

$$\begin{aligned} 0 = \beta^{-1}X(F) &- \mathrm{div}_G(\xi + \overline{\nabla}\log\beta) - |\xi + \overline{\nabla}\log\beta|^2_G \\ &+ \frac{1}{2}\mathrm{R}_G - \frac{1}{2}|h - k|^2 - \mu + J(\nu) - \frac{1}{2}F\big(F + 2\mathrm{tr}_M(k)\big). \end{aligned} \tag{1.4.7}$$

Note that the tangential derivative is bounded:

$$|\beta^{-1}X(F)| = \beta|\nabla f(F)| \leq |\nabla F| \leq \mu_2.$$

Multiply (1.4.7) by a test function φ^2, integrate over G, integrate the divergence term by parts [2] together with pointwise Cauchy–Schwartz inequality

$$2|\xi + \overline{\nabla}\log\beta|\,|\overline{\nabla}\phi||\phi| - |\xi + \overline{\nabla}\log\beta|^2\phi^2 \leq |\overline{\nabla}\phi|^2,$$

[2] The boundary integral will decay to zero at infinity due to the decay rate estimate of f derived by barrier argument.

and absorb terms involving F by constant $C(F, \nabla F)$, then we obtain

$$\int_G (\mu - J(\nu))\varphi^2 + \frac{1}{2}|h-k|^2\varphi^2 \le \int_G |\overline{\nabla}\varphi|^2 + \frac{1}{2}\mathrm{R}_G\varphi^2 + C(F,\nabla F)\varphi^2, \tag{1.4.8}$$

where the constant $C(F, \nabla F)$ depends also on (M, g, k), and $C = 0$ if $F = 0$. This inequality is analogous to the stability inequality for minimal surfaces. Schoen–Yau modified the stability argument in [36] to derive the pointwise curvature estimate for G. Note that this is where the dominant energy condition comes into the analysis of Jang's equation. For solutions of Jang's equation, i.e., $F = 0$, we can drop the positive curvature term and get

$$\int_G 2(\mu - J(\nu)) \le \int_G 2|\overline{\nabla}\varphi|^2 + \mathrm{R}_G. \tag{1.4.9}$$

This inequality is closely related to spectral property of the conformal Laplacian and plays an important role of reduction argument (cf. Section 1.5.1).

The regularizes solutions 1.9 and a priori estimates Proposition 1.10 make the establishment of existence and regularity of Jang's equation, and yet the solutions may blow up in some black hole regions enclosed by apparent horizons.

Proposition 1.11 (cf. [39] Proposition 4, also see [16] for $3 \le \dim M \le 7$)**.** *There exists a positive sequence $s_j \to 0$ and disjoint open sets $\Omega_+, \Omega_-, \Omega_0$ with the following properties:*

(1) f_{s_j} diverges to $\pm\infty$ on $\Omega_\pm$ respectively and f_{s_j} converges to a smooth function f_0 on Ω_0 which satisfies Jang equation $\mathrm{H}[f_0] - \mathrm{K}[f_0] = 0$ and drops off at the rate $f_0 \in \mathrm{O}^3(|x|^{-1/2})$ at each infinity of M.

(2) The sets Ω_+ and Ω_- have compact closures and $M = \overline{\Omega}_+ \cup \overline{\Omega}_- \cup \overline{\Omega}_0$. Each connected component $\Sigma_\pm$ of $\partial\Omega_\pm$ is a closed properly embedded smooth apparent horizon in M satisfying $\mathrm{H}[\Sigma_\pm] \pm \mathrm{K}[\Sigma_\pm] = 0$ where $\mathrm{H}[\Sigma_\pm]$ is computed with respect to the unit normal on $\partial\Omega_\pm$ pointing out of $\Omega_\pm$. No two connected components of Ω_+ (respectively Ω_-) can

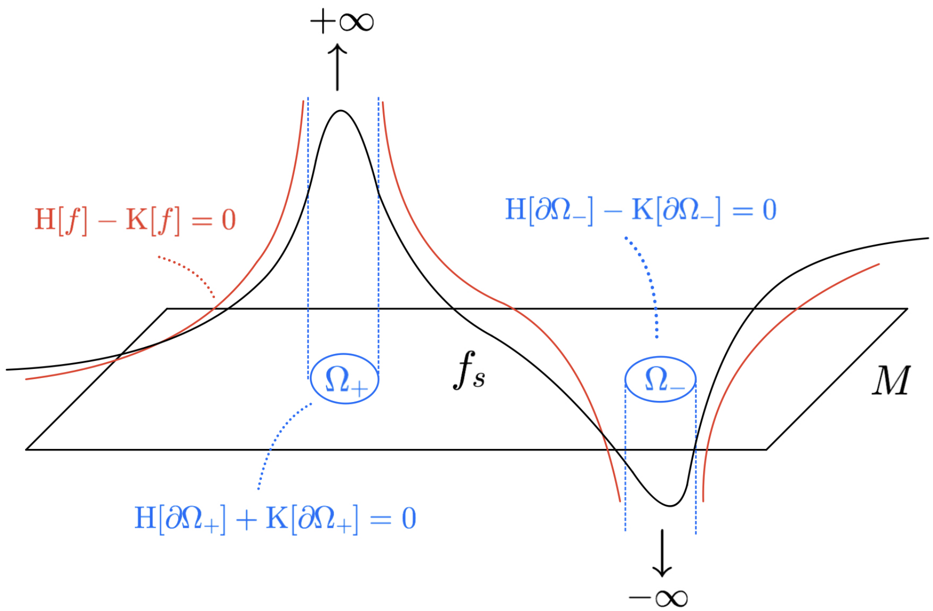

Figure 1.6: Blowup solution to Jang equation and regularized solutions

share a common boundary.

(3) Graph(f_{s_j}) *converges smoothly to a hypersurface S in $M \times \mathbb{R}$. Each component of S is either a component of* Graph(f_0, Ω_0) *or a cylinder $\Sigma \times \mathbb{R}$ over a component Σ of $\partial\Omega_+ \cap \partial\Omega_-$. Any two components of S are separated by a positive distance.*

The analysis of boundary $\partial\Omega_0 \cap \Omega_\pm$ in Proposition 1.11 is based on the following argument. Applying the uniform local $C^{3,\alpha}$ estimate in Proposition 1.10 to the sequence Graph$(f - a_j)$ as $a_j \to \pm\infty$, the hypersurfaces Graph$(f_0 - a, \Omega_0)$ converge to the cylinder $(\partial\Omega_\pm \cap \partial\Omega_0) \times \mathbb{R}$ uniformly in the sense of $C^{2,\alpha}_{loc}$. As a corollary, we have the information about asymptotic behavior of $G_0 :=$ Graphf_0 near $\partial\Omega_0$.

Corollary 1.12 (Rough convergence to cylinder, Schoen–Yau [39] Corollary 2)**.** *Let $\Sigma \subset \partial\Omega_+ \cap \partial\Omega_0$ (resp. $\Sigma \subset \partial\Omega_- \cap \partial\Omega_0$) be a boundary component and let $\mathcal{O}$ be an open neighborhood of Σ which does not intersect with other components of $\partial\Omega_0$, then for T sufficiently large, the 3-manifold $G_0 \cap \big(\mathcal{O} \times [T, \infty)\big)$ can be represented in the form $\sigma = w(y, t)$ for a smooth positive function w defined on $\Sigma \times [T, \infty)$ (resp. $\Sigma \times (-\infty, -T])$, where σ denotes* *the distance function to $\Sigma \times \mathbb{R}$ in $M \times \mathbb{R}$. Moreover, for any $\varepsilon > 0$, there exists $T_\varepsilon \geq T$ such that*

$$w(y,t) + |Dw(y,t)| + |D^2 w(y,t)| + [D^2 w]_\alpha < \varepsilon \tag{1.4.10}$$

for all $y \in \Sigma$ and $t \geq T_\varepsilon$ (resp. $t \leq -T_\varepsilon$). Here D denotes the covariant derivative on $\Sigma \times \mathbb{R}$.

As a consequence of Corollary 1.12, the stability inequality (1.4.9) propagates to boundary of Ω_0 through argument of separation of variable on cylinder. Assuming the strict dominant energy condition, which is a generic condition by Proposition 1.4, one can show that the first eigenvalue of the conformal Laplacian is positive. Thus, there exists a metric on $\partial\Omega_0$ admitting positive Gauss curvature. Then Gauss–Bonnet theorem implies that boundary components of Ω_0 are 2-spheres.

Proposition 1.13 ([39])**.** *Assume the dominant energy condition holds strictly, i.e., $\mu - |J| \geq \delta > 0$. The closed smooth apparent horizons arise as components of Ω_0 in Proposition 1.11 are 2-spheres.*

Following a similar argument with (1.4.9) replaced by (1.4.8) with $C = 0$, one can show that for boundary component Σ of Ω_0 the symmetrized stability operator of expansion $\mathcal{L}_\Sigma^{\mathrm{sym}}\varphi := -\Delta\varphi + \big(\frac{1}{2}\mathrm{R}_\Sigma - \frac{1}{2}|h+k|^2 - \mu - J(\nu)\big)\varphi$ on Σ has non-negative spectrum. Andersson–Metzger proved by a delicate barrier argument that boundary components of Ω_0 are stable in the sense of $\mathcal{L}_\Sigma$, which is a stronger stability than symmetrized stability [19, Lemma 2.2].

Proposition 1.14 (cf. [4])**.** *The closed smooth apparent horizons appear as components of $\partial\Omega_\pm, \partial\Omega_0$ in Proposition 1.11 are stable.*

1.5 Application to the Positive Mass Theorem

1.5.1 Positive Mass Theorem

In 1981, Schoen and Yau [39] proved the positive energy theorem (PET) for general initial data sets by reducing the problem to the time-symmetric case, which they had proved in 1979 [37] using area minimizing hypersurfaces. For the simplicity, we assume that the initial data set has only one infinite end.

Theorem 1.15 (Riemannian PET, Schoen–Yau [37])**.** *Let (M^3, g) be an asymptotically flat Riemannian manifold satisfying $R_g \geq 0$. Then $E_{\mathrm{ADM}} \geq 0$ and equality holds if and only if (M^3, g) is isometric to $(\mathbb{R}^3, \delta)$.*

Recall that the dominant energy condition is equivalent to $R_g \geq 0$ in time-symmetric slice.

Theorem 1.16 (Spacetime PET, Schoen–Yau [39])**.** *Let (M^3, g, k) be an asymptotically flat initial data set satisfying the dominant energy condition. Then $E_{\mathrm{ADM}} \geq 0$ and equality holds if and only if (M^3, g, k) can be embedded in Minkowski spacetime $\mathbb{R}^{1,3}$.*

The full PMT was obtained by M. Eichmair, L.-H. Huang, D. Lee, and R. Schoen via a reduction argument based on a density theorem [17, Theorem 18] analogous to Proposition 1.4 and the boost argument of D. Christodoulou and N. OḾurchadha [13].

Theorem 1.17 (Spacetime PMT, Eichmair–Huang–Lee–Schoen [17])**.** *Let $3 \leq n < 8$ and let (M, g, k) be an n-dimensional asymptotically flat initial data set that satisfies the dominant energy condition. Then*

$$E \geq |P|$$

where (E, P) is the ADM-energy-momentum 4-vector of (M, g, k).

1.5.2 Reduction Argument of the Positive Energy Theorem

To focus on the reduction argument using Jang's equation in [39] without introducing too much technicality, we assume that there exists a smooth entire solution to Jang's equation, i.e., no blowup occurs.

Proof. Note that the induced metric on $G = \text{Graph}(f)$ (Jang's deformation of g) is $\overline{g} = g + df \otimes df$ and we may pull it back to M. Since $f \in \mathrm{O}^3(|x|^{-\frac{1}{2}})$, we have $df \otimes df \in \mathrm{O}^2(|x|^3)$ and hence $\overline{g}$ is still asymptotically flat. Furthermore, it follows directly from decay rate analysis that $E_{\mathrm{ADM}}(\overline{g}) = E_{\mathrm{ADM}}(g)$.

In lieu of the dominant energy condition, the stability inequality (1.4.9) implies that

$$6 \int_M |\overline{\nabla}\varphi|^2 dV_{\overline{g}} \leq 8 \int_M |\overline{\nabla}\varphi|^2 \, dV_{\overline{g}} + \int_M \mathrm{R}_{\overline{g}} \varphi^2 \, dV_{\overline{g}}. \tag{1.5.1}$$

The right hand side of (1.5.1) is exactly the integral form associate with the conformal Laplacian $L(\overline{g}) = \Delta_{\overline{g}}\varphi - \frac{1}{8} R_{\overline{g}}\varphi$. It follows from standard methods that the following equation is solvable.

Lemma 1.18 ([39] Lemma 4)**.** *There exists a solution $u > 0$ satisfying*

$$\Delta_{\overline{g}} u - \frac{1}{8} R_{\overline{g}} u = 0 \quad \textit{on } M, \tag{1.5.2}$$

and

$$u = 1 + \frac{A}{r} + \mathrm{O}(r^{-2}) \quad \textit{as } r \to \infty, \tag{1.5.3}$$

where A is a nonpositive constant.

The asymptotic form (1.5.3) can be derived from potential theory using Green's function of Laplacian. Yet, to see that $A \leq 0$, we need to use (1.5.1) again. Substitute φ by u in (1.5.1), integrate by parts, and use the equation (1.5.2), then we have for any large $\sigma > 0$

$$\begin{aligned}
6\int_{B_\sigma} |\overline{\nabla} u|^2 dV_{\overline{g}} &\leq 8\int_{B_\sigma} |\overline{\nabla} u|^2 \, dV_{\overline{g}} + \int_{B_\sigma} \mathrm{R}_{\overline{g}} u^2 \, dV_{\overline{g}} \\
&= 8\int_{\partial B_\sigma} \frac{\partial u}{\partial x^j} \frac{x^j}{|x|} dA_{\overline{g}} \\
&= -32\pi A + \mathrm{O}(\sigma^{-1}),
\end{aligned}$$

where B_σ is a coordinate ball with Euclidean radius σ. By taking $\sigma \to \infty$, we get

$$A \leq \frac{3}{16\pi} \int_M |\overline{\nabla} u|^2 \leq 0. \tag{1.5.4}$$

It follows from (1.5.2) and Proposition A.1 that the conformal metric $u^4\overline{g}$ has zero scalar curvature. Finally, apply Riemannian positive energy theorem to $(M, u^4\overline{g})$ and use Proposition 1.3, we obtain

$$0 \leq E_{\mathrm{ADM}}\big(u^4\overline{g}\big) = 2A + E_{\mathrm{ADM}}(\overline{g}) \leq E_{\mathrm{ADM}}(g).$$

When $E_{\mathrm{ADM}}(g) = 0$ holds, we find that $A = 0$. The inequality (1.5.4) implies that $u \equiv 1$ and hence $\mathrm{R}_{\overline{g}} = 0$. Apply rigidity part of Riemannian positive energy theorem to $(M, \overline{g})$, we get $\overline{g}_{ij} = \delta_{ij}$ in certain coordinates (y^1, y^2, y^3). Furthermore, integrate (1.4.7) on large coordinate ball $|y| \leq \sigma$ and integrate the divergence term by parts, we find

$$\int_{|y|\leq\sigma} \big(\mu - J(\nu)\big) + |h-k|^2 dA_\delta \leq -\int_{|y|=\sigma} \langle \xi + \nabla \log \beta, \nu \rangle dA_\delta.$$

Note that the scalar curvature term vanishes. In view of the dominant energy condition and decay rates of k and f, taking $\sigma \to \infty$ implies $h = k$ on M. In conclusion, we have $\overline{g} = \delta$ and $h = k$. The embedding problem considered by Jang, Proposition 1.7, implies that (M, g, k) is an initial data set of Minkowski spacetime. □

Since, in general, the solution f to Jang's equation may blowup in some black hole regions, the cylindrical ends near apparent horizons definitely require extra care. First of all, Schoen and Yau blow down these cylindrical ends to finite cones with zero scalar curvature over apparent horizons using conformal deformations. Since by Proposition 1.13 the apparent horizons are 2-spheres, and these cones are topologically punctured balls. They showed that, in appropriate coordinates, these cones are uniformly equivalent to Euclidean punctured balls. Next, they conformally deform the entire new manifold such that the scalar curvature vanishes as the model case. Finally, they blow up these punctured balls by Green's function of Laplacian to infinite ends, and estimate the contributions of these new ends to the ADM-energy are ε-small.

1.5.3 General Cases

The Riemannian PET theorem has been extended to higher dimensions in different ways. The minimal surface argument of Schoen and Yau to prove Riemannian PET in dimension 3 [37] extends to dimension up to 7 by a dimension reduction argument (see [38] and [43]). The dimension restriction is to prevent the singularity of area minimizing surfaces. In 2017, Schoen and Yau [42] extended their argument to all dimensions by minimizing slicing argument. This method has a subtle connection with the non-existence of a metric admitting positive scalar curvature on the torus in dimension $n \leq 7$ in [38].

The technical difficulties of the reduction argument using Jang's equation shown in Section 1.5.2 in high dimensions are twofold. The apparent horizons that arise in the blowup of Jang's equation in high dimensions may have potential singularities and potentially complicated topology. The stability-based regularity of apparent horizons in [39] is available up to dimension 5. The singularity issue for dimensions up to 7 was resolved by Eichmair [16] through his early work [14, 15] on the almost minimizing property of Jang's equation.

In the same paper, Eichmair overcame the topological issue through the conformal darning method. In view of the result [42], it is natural to expect the extension of the Jang reduction argument to dimension $n > 7$.

An independent approach to PMT using the Dirac operator method for spin manifolds was done by Witten [45]. See also [34]. This method works in all dimensions without reduction to the Riemannian case, while the spin structure is necessary and non-generic in high dimensions. Another independent approach to PMT addressing the singularity of minimizing hypersurfaces in all dimensions was given by Lohkamp [25, 26, 27, 28, 29]. Recently, in 2021, Sakovich [35] used the Jang reduction argument to prove the PMT in the asymptotically hyperbolic setting.

Chapter 2

Sharp Exponential Asymptotic Estimates of Jang's Equation

2.1 Introduction

Schoen–Yau showed that the graph of a blowup solution to Jang's equation is asymptotic to cylinders over apparent horizons. J. Metzger proved that such cylindrical asymptotics are exponential and gave upper and (partial) lower estimates of the asymptotic rate in terms of certain spectral properties of apparent horizons; Q. Han and M. Khuri gave a full lower estimate; and W. Yu obtained the sharp upper and lower estimates. Their estimates involve delicate barrier construction and require the assistance of regularized solutions. In Chapter 2, we will give a simple proof of the sharp estimates which also apply to general blowup solutions (not necessarily limits of regularized solutions).

Now we recall Schoen-Yau's rough asymptotic estimates, Corollary 1.12. For the sake of simplicity, we will refer to $\Sigma \subset \partial\Omega_+ \cap \partial\Omega_0$ and ν as the outward unit normal on Σ throughout this present paper, and all arguments can be adapted to the case $\Sigma \subset \partial\Omega_- \cap \partial\Omega_0$

correspondingly. According to Proposition 1.11, Σ is a MOTS. We can assume that every point in $\mathcal{O}$ in Corollary 1.12 is passed by a unique geodesic orthogonal to Σ. Let $\overline{\sigma} > 0$ be a number less than the minimum of injectivity radii of all points on Σ. We introduce the normal coordinates y^1, y^2, σ adapted to Σ on $\mathcal{O}$ via the map

$$\Upsilon : \Sigma \times (-\overline{\sigma}, \overline{\sigma}) \to \mathcal{O} : (y, \sigma) \to \exp_y \big(\sigma \nu(y)\big) \tag{2.1.1}$$

where y^1, y^2 are coordinates on Σ. We denote basis vectors by $\partial_i = \frac{\partial}{\partial y^i}$ for $1 \leq i \leq 2$ and $\partial_\sigma = \frac{\partial}{\partial \sigma}$. By properties of exponential map, we have $\langle \partial_i, \partial_\sigma \rangle(p) = 0$ for $1 \leq i \leq 2$ and $\nabla_{\partial_\sigma} \partial_\sigma(p) = 0$ for all $p \in \mathcal{O}$. In normal coordinates, the metric g in $\mathcal{O}$ can be written as

$$\sum_{i,j=1}^{n} \gamma_{ij}(y, \sigma) dy^i dy^j + d\sigma^2 = g(y, \sigma).$$

where $\gamma_{ij} = \langle \partial_i, \partial_j \rangle$. We define the parallel surfaces $\Sigma_\sigma = \{\sigma \equiv \text{const}\}$ of distance σ away from Σ, then

$$\begin{aligned} g\big|_{\Sigma_\sigma} &= \sum_{i,j=1}^{n} \gamma_{ij}(y, \sigma) dy^i dy^j, \\ \partial_\sigma \gamma_{ij}(y, \sigma) &= 2h_{ij}(y, \sigma), \\ \partial_\sigma^2 \gamma_{ij}(y, \sigma) &= 2\big(h_i^{\ k} h_{kj} - R_{j\sigma i\sigma}\big)(y, \sigma), \end{aligned} \tag{2.1.2}$$

where $h_{ij}(y, \sigma) := \langle \nabla_{\partial_i} \partial_\sigma, \partial_j \rangle(y, \sigma)$ is the second fundamental form of Σ_σ with respect to ∂_σ. The normal coordinates nicely capture the geometry of Σ in M. Likewise, we parallelly extend the normal coordinates (y, σ) on $\mathcal{O}$ to normal coordinates (y, σ, t) on $\mathcal{O} \times \mathbb{R}$. For any $\varepsilon < \overline{\sigma}$, the graph $G_0 \cap \big(\mathcal{O} \times [T_\varepsilon, \infty)\big)$ in Corollary 1.12 can be express as

$$\Big\{ \big(y, w(y, t), t\big) : y \in \Sigma, t \geq T_\varepsilon \Big\}$$

in normal coordinates for a positive function w defined on $\Sigma \times [T_\varepsilon, \infty)$. More precisely, $w(y, t)$

satisfies for $y \in \Sigma, t \geq T_\varepsilon$,

$$f_0(y, w(y,t)) = t, \qquad w(f_0(y,s), y) = s. \tag{2.1.3}$$

Let $\mathcal{C}$ denote the cylinder $\Sigma \times \mathbb{R}_+$ and let D be the Levi-Civita connection on $\mathcal{C}$. It is easy to see that the stability operator on $\mathcal{C}$ is $\mathcal{L}_\mathcal{C} = -\partial_t^2 + \mathcal{L}_\Sigma$. Observe that G_0 (with respect to upward normal) and $\Sigma \times \mathbb{R}$ both satisfy MOTS equation $\mathrm{H} + \mathrm{K} = 0$, and that $G_0 \cap \big(\mathcal{O} \times [T, \infty)\big)$ is asymptotic to $\Sigma \times \mathbb{R}$ in $C^{2,\alpha}$ topology. By direct computation using the properties of normal coordinates (2.1.2) (also c.f. [32]), one can show that if (1.4.10) holds for sufficiently small $\varepsilon > 0$, then w satisfies

$$(-\partial_t^2 + \mathcal{L}_\Sigma) w(y,t) = Q(y, w, Dw, D^2 w),$$

where Q is of the form

$$Q(y, w, Dw, D^2w) = w * w + w * Dw + Dw * Dw + w * D^2 w + Dw * Dw * D^2 w,$$

where * denotes certain contraction with a bounded tensor depending only on the geometry of Σ in $(\mathcal{O}, g, k)$ but independent of variable t. Therefore, w satisfies all the settings in Theorem 2.4. Moreover, the coefficients of $\mathcal{L}$ and Q purely depend on the geometry of (M, g, k) near Σ.

J. Metzger improved the rough asymptotic estimate, Corollary 1.12, to upper and (partial) lower exponential decay estimates assuming Σ is strictly stable.

Theorem 2.1 (Weak exponential decay, Metzger [32] Theorem 4.2 and 4.4)**.** *Assume the situation of Corollary 1.12. Suppose in addition that Σ is strictly stable with principal eigenvalue $\lambda > 0$. Then for all $0 < \mu < \lambda$ there exists $\bar{\varepsilon} = \bar{\varepsilon}(\mu) > 0$ depending only on the geometry near Σ and μ such that if (1.4.10) holds with $\varepsilon = \bar{\varepsilon}$, then there exists a con-*

stant $c_5 = c_5(\mu) > 0$ *depending only on the local geometry near* Σ, μ, *and* λ *such that for* $(y,t) \in \Sigma \times [T_{\bar{\varepsilon}}, \infty)$,

$$w(y,t) + |Dw(y,t)| + |D^2 w(y,t)| \leq c_5 e^{-\sqrt{\mu}(t - T_{\bar{\varepsilon}})}. \tag{2.1.4}$$

Moreover, for any $\mu > \lambda$ *there is no constant* $C > 0$ *such that for* $(y,t) \in \Sigma \times [T_{\bar{\varepsilon}}, \infty)$,

$$w(y,t) + |Dw(y,t)| + |D^2 w(y,t)| \leq C e^{-\sqrt{\mu}(t - T_{\bar{\varepsilon}})}. \tag{2.1.5}$$

Q. Han and M. Khuri [21] gave both upper and lower asymptotic estimates for the generalized Jang's equation, which was introduced by H. Bray and M. Khuri [9, 10] in an attempt to prove the spacetime Penrose inequality. In particular, when the static potential $\phi \equiv 1$, their result improves Metzger's lower estimate (2.1.5). Translating the setting using the conversion equation (2.1.3), the lower blowup rate estimate of Han-Khuri reads as follows.

Theorem 2.2 ([21], Theorem 1.1 for the case $\phi \equiv 1$)**.** *Assume the situation of Theorem 2.1. There exist constants* μ *and* C *such that for* $(y,t) \in \Sigma \times [T_{\bar{\varepsilon}}, \infty)$,

$$w(y,t) \geq C e^{-\sqrt{\mu}(t - T_{\bar{\varepsilon}})}. \tag{2.1.6}$$

Despite the fact that they did not discuss the dependence of μ due to complexity of generalized Jang's equation, we know $\mu \geq \lambda$ by (2.1.4).

W. Yu in his doctoral thesis further improved Metzger's estimate to the sharp estimate by using more involved barrier construction.

Theorem 2.3 (Sharp exponential decay, Yu [46] Theorem 4)**.** *Assume the situation of Theorem 2.1. There exists* $\varepsilon_0 > 0$ *depending only the geometry near* Σ *such that if (1.4.10) holds with* $\varepsilon = \varepsilon_0$, *then there exist constants* c_6, c_7 *depending only on the local geometry of* Σ *in* *initial data set* (M, g, k) *such that for* $(y, t) \in \Sigma \times [T_{\varepsilon_0}, \infty)$,

$$w(y,t) + |Dw(y,t)| + |D^2 w(y,t)| \leq c_6 e^{-\sqrt{\lambda}(t - T_{\varepsilon_0})}, \tag{2.1.7}$$

and

$$w(y,t) \geq c_7 e^{-\sqrt{\lambda}(t - T_{\varepsilon_0})}. \tag{2.1.8}$$

Proof. We apply Theorem 2.4 by substituting w with $w(\cdot, t + T_{\varepsilon_0})$ defined on $\mathcal{C}$ to get a simpler proof. □

All the upper estimates were obtained by delicate barrier construction using the stability condition of apparent horizon Σ. To deliver asymptotic estimates to blowup solutions, this barrier argument requires the assistance of finite regularized solutions. We will investigate the asymptotic estimates of a general elliptic equation on a cylinder without the need for regularized solutions. Furthermore, we can keep track of the constants' dependence on the geometry near Σ more explicitly and easily than Yu did.

2.2 Asymptotic Rate of Elliptic Equation on Cylinder

Let $n \geq 1$ and let (Σ^n, γ) be a compact smooth n-dimensional Riemannian manifold without boundary. Let D denote the Levi-Civita connection on Σ. Let

$$\mathcal{L} = -a^{ij} D_i D_j + b^i D_i + c$$

be a uniformly elliptic differential operator with coefficient functions satisfying $a^{ij} \in C^{1,\alpha}(\Sigma)$ positive-definite, $b^i, c \in C^{0,\alpha}(\Sigma)$. Define cylinder $\mathcal{C} = \Sigma \times [0, \infty)$ equipped with the product metric $\gamma + dt^2$. We let t be the $(n+1)$-th coordinate and still let D denote the covariant derivative on $\mathcal{C}$. Suppose w is a positive C^3 solution on $\mathcal{C}$ to the quasilinear equation

$$(-\partial_t^2 + \mathcal{L})w = Q(y, w, Dw, D^2 w), \tag{2.2.1}$$

where the quadratic source term $Q : \Sigma \times \mathbb{R} \times T^*\Sigma \times \left(T^*\Sigma \otimes T^*\Sigma\right) \to \mathbb{R}$ is a differentiable function satisfying

$$Q(y, w, Dw, D^2 w) = w * w + w * Dw + Dw * Dw + w * D^2 w + Dw * Dw * D^2 w, \tag{2.2.2}$$

where * denotes certain contraction with a bounded tensor independent of variable t. This equation is saying that the linearized equation vanishes. Moreover, we assume that w satisfies the rough decay condition

$$\lim_{T\to\infty} |w|_{2,\alpha,\Sigma\times[T,\infty)} = 0, \tag{2.2.3}$$

where $|w|_{2,\alpha,\Sigma\times[T,\infty)}$ denotes the unweighted Hölder norm on $\Sigma \times [T, \infty)$. Since $\mathcal{L}$ and Q are insensitive to translation in t, it follows from (2.2.3) that we may further assume that

$$|w|_{2,\alpha,\mathcal{C}} \leq \varepsilon_0 < 1 \tag{2.2.4}$$

by replacing $w(\cdot, t)$ with $w(\cdot, t + T_0)$ for a sufficiently large T_0.

Theorem 2.4. *Suppose that $\mathcal{L}$ has principal eigenvalue $\lambda > 0$. There exist constants ε_0, c_8, $c_9 > 0$ depending only on Σ, γ, a^{ij}, b^i, c, Q and λ such that if w is a positive function defined on $\mathcal{C}$ satisfying (2.2.1), (2.2.3), and (2.2.4) with ε_0, then for any $(y, t) \in \mathcal{C}$*

$$|w(t, y)| + |Dw(t, y)| + |D^2 w(y, t)| \leq c_8 e^{-\sqrt{\lambda}t}, \tag{2.2.5}$$

and

$$|w(t, y)| \geq c_9 e^{-\sqrt{\lambda}t}. \tag{2.2.6}$$

2.3 Proof of Main Result

For any $t \geq 1$, we define cylinder $\mathcal{C}_t = \Sigma \times (t-1, t+1)$. For any $(y_1, s_1), (y_2, s_2) \in \mathcal{C}_t$, let $\rho(y_1, y_2)$ denote the distance in Σ induced by γ, let $d_t(s_1) = \min\{|s_1 - t + 1|, |s_1 - t - 1|\}$ denote the minimum distance of (y_1, s_1) to $\partial \mathcal{C}_t = \Sigma \times \{t \pm 1\}$, and let $d_t(s_1, s_2) = \min\{d_t(s_1), d_t(s_2)\}$. For any $\alpha \in (0,1)$, we define the weighted Hölder norm $\|w\|_{2,\alpha,\mathcal{C}_t}$ on $\mathcal{C}_t$ by

$$\|w\|_{2,\alpha,\mathcal{C}_t} = \sup_{1 \leq i,j \leq n+1} \sup_{(y,s) \in \mathcal{C}_t} \big(|w(y,s)| + d_t(s)|D_i w(y,s)| + d_t(s)^2 |D_i D_j w(y,s)|\big) + [w]^*_{2,\alpha,\mathcal{C}_t},$$

where the weighted semi-norm $[w]_{2,\alpha,\mathcal{C}_t}$ is defined by

$$[w]^*_{2,\alpha,\mathcal{C}_t} = \sup_{1 \leq i,j \leq n+1} \sup_{(y_1,s_1) \neq (y_2,s_2)} d_t(s_1,s_2)^{2+\alpha} \frac{|D_i D_j w(y_1,s_1) - D_i D_j w(y_2,s_2)|}{\big(\rho(y_1,y_2)^2 + |s_1 - s_2|^2\big)^{\frac{\alpha}{2}}}.$$

In addition, we define the weighted Hölder norm $\|w\|^{(2)}_{0,\alpha,\mathcal{C}}$ by

$$\|w\|^{(2)}_{0,\alpha,\mathcal{C}_t} = \sup_{(y,s) \in \mathcal{C}_t} d_t(s)^2 |w(y,s)| + \sup_{(y_1,s_1) \neq (y_2,s_2)} d_t(s_1,s_2)^{2+\alpha} \frac{|w(y_1,s_1) - w(y_2,s_2)|}{\big(\rho(y_1,y_2)^2 + |s_1 - s_2|^2\big)^{\frac{\alpha}{2}}}.$$

Proposition 2.5. *Suppose that $\mathcal{L}$ has principal eigenvalue $\lambda > 0$. For any $0 < \mu < \lambda$, there exist constants $\bar{\varepsilon} = \bar{\varepsilon}(\mu)$, $c_{10} = c_{10}(\mu) > 0$ depending only on Σ, γ, a^{ij}, b^i, c, Q, λ and μ such that if w is a positive function defined on $\mathcal{C}$ satisfying (2.2.1), (2.2.3), and (2.2.4) with $\varepsilon_0 < \bar{\varepsilon}$, then for any $(y,t) \in \mathcal{C}$*

$$w \leq c_{10} e^{-\sqrt{\mu} t}. \tag{2.3.1}$$

Proof. By Krein-Rutman Theorem, there exists a *positive* smooth eigenfunction β defined on Σ of $\mathcal{L}$ corresponding to λ. We may assume by scaling β that

$$\min_{\Sigma} \beta = 1.$$

Take $A = e^{\sqrt{\mu}}$. We claim that for all $(y,t) \in \mathcal{C}$

$$w(y,t) \le Ae^{-\sqrt{\mu}t}\beta(y).$$

We first note from (2.2.4) that for all $(y,0) \in \Sigma \times [0,1]$

$$\frac{w(y,t)}{\beta(y)} - Ae^{-\sqrt{\mu}t} < 1 - 1 = 0.$$

and from rough decay condition (2.2.3) that

$$\lim_{t\to\infty} \sup_{y\in\Sigma} \Big(\frac{w(y,t)}{\beta(y)} - Ae^{-\sqrt{\mu}t}\Big) = 0.$$

Suppose the claim is not true, then there exist $y_0 \in \Sigma$ and $t_0 > 1$ such that

$$\frac{w(y_0,t_0)}{\beta(y_0)} - Ae^{-\sqrt{\mu}t_0} = \max_{\mathcal{C}} \Big(\frac{w(y,t)}{\beta(y)} - Ae^{-\sqrt{\mu}t}\Big) =: B > 0.$$

This is equivalent to a global almost exponential bound of w

$$w(y,t) \le Ae^{-\sqrt{\mu}t}\beta(y) + B\beta(y) \tag{2.3.2}$$

in which the equality holds at (y_0,t_0). Define $F(y,t) := w(y,t) - Ae^{-\sqrt{\mu}t}\beta(y) - B\beta(y)$. Then $F(y,t)$ achieves maximum at (y_0,t_0). By derivatives tests, we have

$$\begin{cases} F(y_0,t_0) = 0, \\ DF(y_0,t_0) = 0, \\ D^2F(y_0,t_0) \le 0. \end{cases} \tag{2.3.3}$$

It follows immediately from derivative tests (2.3.3) that

$$(-\partial_t^2 + \mathcal{L})F(y_0, t_0) = (-\partial_t^2 - a^{ij} D_i D_j) w(y_0, t_0) + 0 + 0 \geq 0. \tag{2.3.4}$$

On the other hand, equation (2.2.1) together with positivity of B, β and equality of (2.3.2) gives

$$\begin{aligned}(-\partial_t^2 + \mathcal{L})F(y_0, t_0) &= Q(y_0, t_0) - (\lambda - \mu) A e^{-\sqrt{\mu} t_0} \beta(y_0) - \lambda B \beta(y_0) \\ &\leq Q(y_0, t_0) - (\lambda - \mu)\Big(A e^{-\sqrt{\mu} t_0} \beta(y_0) + B\beta(y_0)\Big) \\ &= Q(y_0, t_0) - (\lambda - \mu) w(y_0, t_0).\end{aligned} \tag{2.3.5}$$

By abuse of notation, $Q(y, t)$ means $Q\big(y, w(y, t), Dw(y, t), D^2 w(y, t)\big)$.

We will exploit the structure of Q to bound $Q(y_0, t_0)$ by $w(y_0, t_0)$. In view of the structure of Q (2.2.2) and (2.2.4), there exists a constant C_1 such that for any $t \geq 1$

$$\|Q\|^{(2)}_{0,\alpha,\mathcal{C}_t} \leq \varepsilon_0 C_1 \|w\|_{2,\alpha,\mathcal{C}_t}. \tag{2.3.6}$$

For any $t \geq 1$, by interior Schauder estimate there exists constant C_2 depending only on a^{ij}, b^i, c, Σ and γ such that

$$\|w\|_{2,\alpha,\mathcal{C}_t} \leq C_2\big(\|w\|_{0,\mathcal{C}_t} + \|Q\|^{(2)}_{0,\alpha,\mathcal{C}_t}\big)$$

Plugging (2.3.6) into the source term, we get

$$\begin{aligned}\|w\|_{2,\alpha,\mathcal{C}_t} &\leq C_2\big(\|w\|_{0,\mathcal{C}_t} + \varepsilon_0 C_1 \|w\|_{2,\alpha,\mathcal{C}_t}\big) \\ &\leq C_2 \|w\|_{0,\mathcal{C}_t} + \frac{1}{2}\|w\|_{2,\alpha,\mathcal{C}_t}.\end{aligned}$$

provided that ε_0 in (2.2.4) is sufficiently small such that $\varepsilon_0 C_1 C_2 \leq \frac{1}{2}$. This implies that for any $t \geq 1$

$$\|w\|_{2,\alpha,\mathcal{C}_t} \leq 2C_2\|w\|_{0,\mathcal{C}_t}. \tag{2.3.7}$$

Plugging global almost exponential bound (2.3.2) of w into (2.3.7) to control the Hölder norm $\|w\|_{2,\alpha,\mathcal{C}_{t_0}}$ by $w(y_0, t_0)$:

$$\begin{aligned}
\|w\|_{2,\alpha,\mathcal{C}_{t_0}} &\leq 2C_2\|w\|_{0,\mathcal{C}_{t_0}} \leq 2C_2 \sup_{(y,t)\in\mathcal{C}_{t_0}} \left(Ae^{-\sqrt{\mu}t}\beta(y) + B\beta(y)\right) \\
&\leq 2C_2 \max_{y\in\Sigma} \beta(y)\left(Ae^{-\sqrt{\mu}(t_0-1)} + B\right) \\
&\leq 2C_2 \max_{\Sigma} \beta(y)e^{\sqrt{\mu}}\left(Ae^{-\sqrt{\mu}t_0}\beta(y_0) + B\beta(y_0)\right) \\
&= \left(2C_2 \max_{\Sigma} \beta(y)e^{\sqrt{\mu}}\right) w(y_0, t_0).
\end{aligned} \tag{2.3.8}$$

In the second inequality to the last, we use that fact that $\min \beta = 1$. Since β is a positive solution to $\mathcal{L}\beta = \lambda\beta$ on Σ, Harnack estimate implies that there exists C_3 depending only on a^{ij}, b^i, c, Σ, γ and λ such that

$$\max_{\Sigma} \beta \leq C_3. \tag{2.3.9}$$

Combined (2.3.6), (2.3.8), and (2.3.9), we get a pointwise estimate of quadratic term Q

$$|Q(y_0, t_0)| \leq \|Q\|^{(2)}_{0,\alpha,\mathcal{C}_{t_0}} \leq 2\varepsilon_0 C_1 C_2 C_3 e^{\sqrt{\mu}} w(y_0, t_0), \tag{2.3.10}$$

and hence (2.3.5) implies that

$$(-\partial_t^2 + \mathcal{L})F(y_0, t_0) \leq \left[2\varepsilon_0 C_1 C_2 C_3 e^{\sqrt{\mu}} - (\lambda - \mu)\right] w(y_0, t_0).$$

Since $\lambda > \mu$, we may take $\varepsilon_0 > 0$ in (2.3.6) smaller such that $\varepsilon_0 < (2C_1C_2C_3e^{\sqrt{\mu}})^{-1}(\lambda - \mu)$, then we have

$$(-\partial_t^2 + \mathcal{L})F(y_0, t_0) < 0,$$

which contradicts to (2.3.4). Therefore, if $\varepsilon_0 < \overline{\varepsilon}(\mu) := (2C_1C_2C_3e^{\sqrt{\mu}})^{-1}\min\{1, (\lambda - \mu)\}$ in (2.2.4), then for all $(y, t) \in \mathcal{C}$

$$w(y,t) \leq Ae^{-\sqrt{\mu}t}\beta(y) \leq \big(AC_3\big)e^{-\sqrt{\mu}t}. \tag{2.3.11}$$

Finally, we take $c_{10} = AC_3$. Note that $\overline{\varepsilon}(\mu), c_{10}(\mu)$ depends only on a^{ij}, b^i, c, Q, Σ, γ, λ and μ. □

Now we will use the weak exponential decay to get the sharp decay.

Proof of Theorem 2.4. Let $T > 1$ and take $A = e^{\sqrt{\lambda}T}$. We claim that for all $(y, t) \in \mathcal{C}$

$$w(y,t) \leq A\big(2 - \frac{1}{1+t}\big)e^{-\sqrt{\lambda}t}\beta(y).$$

We first note from (2.2.4) that for all $(y, 0) \in \Sigma \times [0, T]$

$$\frac{w(y,t)}{\beta(y)} - A\big(2 - \frac{1}{1+t}\big)e^{-\sqrt{\lambda}t} < 1 - 1 = 0$$

and from rough decay condition (2.2.3) that

$$\limsup_{t\to\infty}\sup_{y\in\Sigma}\Big(\frac{w(y,t)}{\beta(y)} - A\big(2 - \frac{1}{1+t}\big)e^{-\sqrt{\lambda}t}\Big) = 0.$$

Suppose the claim is not true, then there exist $y_0 \in \Sigma$ and $t_0 > T$ such that

$$\frac{w(y_0,t_0)}{\beta(y_0)} - A\big(2 - \frac{1}{1+t_0}\big)e^{-\sqrt{\lambda}t_0} = \max_{\mathcal{C}}\Big(\frac{w(y,t)}{\beta(y)} - A\big(2 - \frac{1}{1+t}\big)e^{-\sqrt{\lambda}t}\Big) =: B > 0.$$

This is equivalent to the global bound

$$w(y,t) \le A\big(2 - \frac{1}{1+t}\big)e^{-\sqrt{\lambda}t}\beta(y) + B\beta(y) \tag{2.3.12}$$

in which the equality holds at (y_0, t_0). Define $F(y,t) := w(y,t) - A\big(2 - \frac{1}{1+t}\big)e^{-\sqrt{\lambda}t}\beta(y) - B\beta(y)$. It follows directly from derivative tests that

$$(-\partial_t^2 + \mathcal{L})F(y_0, t_0) \ge 0. \tag{2.3.13}$$

Choose $\mu = \frac{1}{2}\lambda$ such that $0 < \mu < \lambda < 4\mu$. Let $\varepsilon = \frac{1}{2}\bar{\varepsilon}(\mu)$ and $c_{10}(\mu)$ be defined as in the end of the proof of Proposition 2.5 depending only on $a^{ij}, b^i, c, Q, \Sigma, \gamma$ and λ. By Proposition 2.5, $w \le c_{10}(\mu)e^{-\sqrt{\mu}t}$ for all $(y,t) \in \mathcal{C}$. Combine this with Schauder estimate (2.3.7), for $t \ge 1$

$$\|w\|_{2,\alpha,\mathcal{C}_t} \le 2C_2\|w\|_{0,\mathcal{C}_t} \le 2C_2\|c_{10}e^{-\sqrt{\mu}s}\|_{0,\mathcal{C}_t} = \big(2C_2c_{10}e^{\sqrt{\mu}}\big)e^{-\sqrt{\mu}t},$$

where we let $C_4 = 2C_2c_{10}e^{\sqrt{\mu}}$. Combine this with (2.2.2) to improve (2.3.6)

$$|Q(y,t)| \le C_1C_4^2e^{-2\sqrt{\mu}t}, \tag{2.3.14}$$

Since $t_0 \ge T > 1$, equation (2.2.1) gives

$$\begin{aligned}(-\partial_t^2 + \mathcal{L})F(y_0,t_0) &= Q(y_0,t_0) - \frac{2A}{(1+t_0)^3}e^{-\sqrt{\lambda}t_0}\beta(y_0) - \frac{2A\sqrt{\lambda}}{(1+t_0)^2}e^{-\sqrt{\lambda}t_0}\beta(y_0) - B\lambda\beta(y_0)\\ &\le C_4^2C_1e^{-2\sqrt{\mu}t_0} - \frac{2\sqrt{\lambda}}{(1+t_0)^2}e^{-\sqrt{\lambda}t_0}\\ &\le \frac{1}{(1+t_0)^2}e^{-\sqrt{\lambda}t_0}\big[C_4^2C_1e^{-(2\sqrt{\mu}-\sqrt{\lambda})t_0}(1+t_0)^2 - 2\sqrt{\lambda}\big]\end{aligned}$$

In the first inequality, we use the fact that $A > 1$ and drop two negative terms involving faster decay and B with which we do not have nice control. Since $2\sqrt{\mu} = \sqrt{2\lambda} > \sqrt{\lambda}$, there exists $T_0 > 1$ such that for all $t \geq T_0$

$$0 < e^{-(2\sqrt{\mu}-\sqrt{\lambda})t}(1+t)^2 < \frac{2\sqrt{\lambda}}{C_4^2 C_1}.$$

Take $T \geq T_0$, then

$$(-\partial_t^2 + \mathcal{L})F(y_0, t_0) < 0,$$

which contradicts to (2.3.13). Therefore, if $T \geq T_0$ and $A \geq e^{\sqrt{\mu}T}$, then for all $(y,t) \in \mathcal{C}$

$$w(y,t) \leq A\big(1 + \frac{t}{1+t}\big)e^{-\sqrt{\lambda}t}\beta(y) \leq \big(2AC_3\big)e^{-\sqrt{\lambda}t}.$$

Together with (2.3.7), we may take $c_8 = 4AC_2C_3$ such that (2.2.5) holds true.

Using (2.3.14) and analogous minimum principle argument, one can show that there exists $c_9 > 0$ sufficiently small such that

$$w \geq c_9\big(1 + \frac{1}{t+1}\big)e^{-\sqrt{\lambda}t}\beta \geq c_9 e^{-\sqrt{\lambda}t}.$$

□

Theorem2.3 Sharp exponential decay. We apply Theorem 2.4 by substituting w with $w(\cdot, t + T_{\varepsilon_0})$ defined on $\mathcal{C}$ to get a simpler proof. □

Chapter 3

Solutions and Constant Expansion Surfaces in Black Hole

3.1 Notation

3.1.1 Level Sets

Let u be a function defined on M and let $C \in \mathbb{R}$ be a number. Denote the **super-level set** of u by

$$E_C^+(u) := \{x \in M : u(x) > C\}$$

and denote the **sub-level set** of u by

$$E_C^-(u) := \{x \in M : u(x) < C\}.$$

3.1.2 Normal Coordinates

Recall the setting of normal coordinates introduced in Chapter 2. Let $\Sigma \subset M$ be a smooth embedded two-sided 2-dimensional surface assigned with unit normal vector field ν. We will use the couple (Σ, ν) to denote the aforementioned data. Let $\overline{\sigma} > 0$ be a number less than the minimum of injectivity radii of all points on Σ. We introduce the normal coordinates (y, σ) adapted to Σ on a neighborhood $\mathcal{O}$ of Σ via the map

$$\Upsilon : \Sigma \times (-\overline{\sigma}, \overline{\sigma}) \to \mathcal{O} : (y, \sigma) \to \exp_y \big(\sigma\nu(y)\big).$$

We denote **half geodesic tubular neighborhood with thickness** δ around Σ on the $\pm\nu$-side, respectively, by

$$\mathcal{N}_\delta^{\pm}(\Sigma, \nu) := \{\Upsilon(y, \pm\sigma) : x \in \Sigma, 0 \leq \sigma < \delta)\},$$

and the **(full) tubular neighborhood with thickness** 2δ around Σ by

$$\mathcal{N}_\delta(\Sigma) := \{y \in M : \mathrm{dist}(x, \Sigma) < \delta\}.$$

Sometimes we will analyze the properties of constant expansion surfaces near another. It would be useful to consider graphs in normal coordinates. For $w \in C^\infty(\Sigma)$ with $|w| < \delta$, we let $\mathfrak{Graph}(w) = \{\Upsilon(y, w(y)) : y \in \Sigma\}$ denote the graph of w in normal coordinates.

3.1.3 Past Directed Null Expansion

For the sake of simplicity, we will always use an unconventional past directed expansion $\theta[\Sigma] := \mathrm{H}[\Sigma] - \mathrm{K}[\Sigma]$ with a specified choice of unit space-like normal vector field throughout this chapter. For a function f defined on M, we let $\theta[f]$ denote the past directed null expansion computed with respect to the *downward* normal vector field on $\mathrm{Graph}(f) \subset M \times \mathbb{R}$. Similarly, for a function w defined on Σ, we let $\theta[w]$ denote the past directed null expansion of $\mathfrak{Graph}(w)$ computed with respect to $\partial_\sigma^\perp / |\partial_\sigma^\perp|$ in normal coordinates (y, σ), where $\partial_\sigma^\perp$ is the projection of ∂_σ onto the normal bundle of $\mathfrak{Graph}(w)$.

3.2 Limits of Regularized Solutions in Black Hole Regions

3.2.1 Capillary Blowdown Limit

Recall that for every $s \in (0, 1]$ there exists a unique smooth regularized solution f_s such that

$$(g^{ij} - \frac{f_s^i f_s^j}{1 + |\nabla f_s|^2})\Big(\frac{\nabla_i \nabla_j f_s}{\sqrt{1 + |\nabla f_s|^2}} - k_{ij}\Big) = s f_s.$$

satisfying $\lim_{x\to\infty} f_s(x) = 0$ at each infinite end. The capillary term $u_s := s f_s$ in regularized equations will play an important role in our analysis. In [39] R. Schoen and S.T. Yau proved by maximum principle argument that there are constants $\mu_1 = \max_M |\mathrm{tr}_g k|$ and $\mu_2 = \mu_2(|\mathrm{Ric}|_{C^0(M)}, |k|_{C^1(M)})$ such that in M

$$|u_s| = |s f_s| \leq \mu_1, \qquad |\nabla u_s| = |s \nabla f_s| \leq \mu_2.$$

Let $s_j \to 0^+$ be any decreasing sequence such that $f_0 := \lim_{s\to 0^+} f_s$ is a smooth function, and let Ω_0, Ω_+ and Ω_- be disjoint black hole regions as stated in Proposition 1.11. By Arzela–Ascoli theorem, a subsequence of functions u_{s_j} converges uniformly on M to a Lipschitz function $u \in C^{0,1}(M)$ satisfying

$$\begin{cases} u = 0 & \text{in } \Omega_0, \\ u \geq 0 & \text{in } \Omega_+, \\ u \leq 0 & \text{in } \Omega_-. \end{cases} \tag{3.2.1}$$

We call u a *capillary blowdown limit* of regularized solutions f_s. Let Ω be a connected component of Ω_+, which is bounded by Proposition 1.11. For simplicity, throughout the present paper we will prove most of the propositions only for connected components of Ω_+ and all statements corresponding to Ω_- hold analogously. From now on, we will fix the selection of decreasing sequence $s_j \to 0+$, the Lipschitz blowdown limit $u := \lim u_{s_j}$, and the connected component $\Omega \subset \Omega_+$ of black hole regions.

Recall that by definition $f_{s_j} \to +\infty$ in Ω. In order to study the limit behaviour of f_{s_j} as $j \to \infty$, it is necessary to translate down these regularized solutions in an appropriate manner. It is natural to consider a sequence of reference points $\{x_j\}$ in $\overline{\Omega}$ to keep track of the evolution of regularized solutions. For every j, we define the translated solution according to the reference point x_j to be

$$\tilde{f}_{s_j}^{(x_j)}(\cdot) := f_{s_j}(\cdot) - f_{s_j}(x_j) \quad \text{so that} \quad \tilde{f}_{s_j}^{(x_j)}(x_j) = 0.$$

Thus, the regularized equation (1.4.2) reads

$$\theta[\tilde{f}_{s_j}^{(x_j)}] = s_j f_{s_j}, \tag{3.2.2}$$

since the left hand side of regularized equation is invariant under vertical translation. For every sequence $s_j \to 0^+$, the local estimates in Proposition 1.10 and Arzela–Ascoli theorem allow us to find a convergent subsequence of $\text{Graph}(\tilde{f}_{s_j}^{(x_j)})$ on the left hand side of (3.2.2) if we select suitable reference points; the observation (1.4.3) and Arzela–Ascoli theorem allow us to find a convergent subsequence of capillary terms (expansion functions) on right hand

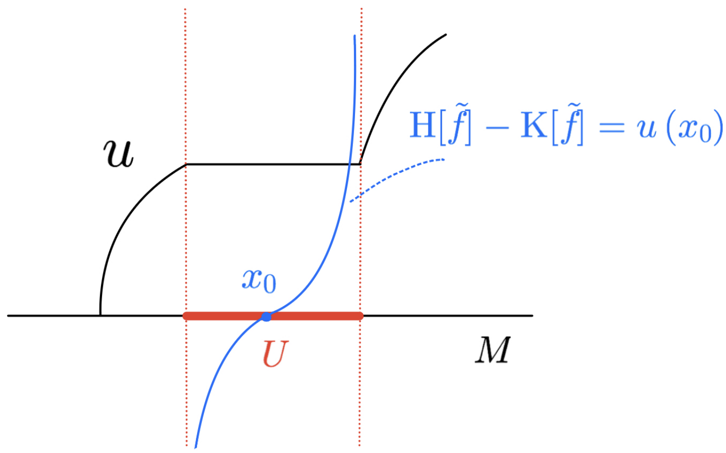

Figure 3.1: A graphical limit $\tilde{f}$ of properly translated regularized solutions $f_{s_j}^{(x_0)}$ lies in the cylinder over a level-set of capillary blowdown limit u.

side of (3.2.2) in closure of black hole region Ω.

The following basic lemma shows that any non-empty subsequential limit must take place in a certain level-set of the capillary blowdown limit u.

Lemma 3.1. *Suppose the reference point sequence $\{x_j\} \subset \Omega$ converges to $x_0 \in \overline{\Omega}$. Set $\Theta := u(x_0)$ as the value. Then*

(1) $\Theta = \lim u_{s_j}(x_j)$.

(2) If $x \in E_\Theta^+(u)$, then $\lim \tilde{f}_{s_j}^{(x_j)}(x) = +\infty$; If $x \in E_\Theta^-(u)$, then $\lim \tilde{f}_{s_j}^{(x_j)}(x) = -\infty$. Therefore, any subsequential limit of graph($\tilde{f}_{s_j}^{(x_j)}$) lies in $E_\Theta(u) \times \mathbb{R}$ provided it exists (cf. Figure 3.1).

Proof. (1) It follows immediately from the uniform convergence and equicontinuity of u_{s_j} in $\overline{\Omega}$.

(2) Suppose $x \in E_\Theta^+(u)$, then $a := u(x) - \Theta > 0$. Since $\lim u_{s_j}(x) = u(x)$ and $\lim u_{s_j}(x_j) = \Theta$

uniformly, for any sufficiently large j

$$u_{s_j}(x) > u(x) - \frac{a}{4} = \Theta + \frac{3a}{4}$$

and

$$u_{s_j}(x_j) < \Theta + \frac{a}{4}.$$

If follows that for any sufficiently large j

$$\begin{aligned}\tilde{f}_{s_j}^{(x_j)}(x) &= \frac{1}{s_j}\big(u_{s_j}(x) - u_{s_j}(x_j)\big)\\ &> \frac{1}{s_j}[(\Theta + \frac{3a}{4}) - (\Theta + \frac{a}{4})]\\ &= \frac{a}{2s_j} \to +\infty.\end{aligned}$$

If $x \in E_\Theta^-(u)$, then $\lim_{j\to\infty} \tilde{f}_{s_j}^{(x_j)}(x) = -\infty$ holds analogously. □

3.2.2 The Shape of Limit of Regularized Solutions in Black Hole Regions

In this subsection, we aim to characterize the geometry of the limits of translated regularized solutions.

In the following theorem, we show that any limit graph of caps of f_{s_j} satisfies the *constant expansion equation*, which is an analogue of the constant mean curvature equation in a spacetime setting.

Theorem 3.2 (Shape of cap)**.** *Let $\Theta := \max_{\overline{\Omega}} u \geq 0$. There exists a sequence of reference points $\{x_j\} \subset \Omega$, a subsequence $\{j'\} \subset \mathbb{N}$, and a non-empty maximal domain $U \subset u^{-1}(\Theta) \cap \overline{\Omega}$*

such that $\tilde{f}^{(x_{j'})}_{s_{j'}}$ *converges smoothly to a function* $\tilde{f}$ *in* U *satisfying the constant expansion equation:*

$$\theta[\tilde{f}] = \Theta \quad \text{and } \tilde{f}(x) \to -\infty \text{ as } U \ni x \to \partial U. \tag{3.2.3}$$

Each connected component $\tilde{\Sigma}$ *of* ∂U *is a closed properly embedded smooth surface in* $u^{-1}(\Theta) \cap \overline{\Omega}$ *with constant expansion* $\theta[\tilde{\Sigma}] = \Theta$ *computed with respect to the unit normal of* $\tilde{\Sigma}$ *pointing into* U*.*

Remark 3.3. (1) U is called the **maximal domain** of solution $\tilde{f}$ to constant expansion equation (3.2.3) in the sense that $\tilde{f}$ blows up on approach to ∂U and hence $\tilde{f}$ can not extend to any smooth solution to (3.2.3) defined in a proper superset of U.

(3) u has a constant value Θ in $\overline{U}$.

(2) In general, Θ could be 0, i.e., u is identically 0 in the black hole region Ω. This corresponds to a very special slow-speed blowup scenario. We will discuss more properties of Ω in Section 3.5 when this special case occurs.

Proof. Recall that $\overline{\Omega}$ is compact. For every $j \in \mathbb{N}$, pick reference point $x_j \in \overline{\Omega}$ such that $f_{s_j}(x_j) = \max_{\overline{\Omega}} f_{s_j}$. We can select a convergent subsequence $x_{j'}$ with $x_0 \in \overline{\Omega}$. Observe that $\tilde{f}^{(x_{j'})}_{s_{j'}}$ is a solution to (3.2.2) and $(x_{j'}, 0) \in \mathrm{Graph}(\tilde{f}^{(x_{j'})}_{s_{j'}})$ converges to $(x_0, 0)$. By the local $C^{3,\alpha}$-estimate in Proposition 1.10 in a neighborhood of $(x_0, 0)$ and Arzela–Ascoli theorem, we may assume by passing to a further subsequence that $\mathrm{Graph}(\tilde{f}^{(x_{j'})}_{s_{j'}})$ converges to a properly embedded submanifold in $C^{2,\alpha}_{loc}$-sense. Let $\tilde{S}$ denote the connected component of the limit submanifold containing $(x_0, 0)$. Since $\tilde{f}^{(x_{j'})}_{s_{j'}} \leq 0$ in $\overline{\Omega}$ for every j', it follows from the Harnack inequality in Proposition 1.10 that the component $\tilde{S}$ is a graph of a $C^{2,\alpha}_{loc}$ function $\tilde{f} \leq 0$ defined in an open neighborhood U of x_0 and approaching to $-\infty$ on approach to ∂U. Observe that $\lim \tilde{f}^{(x_{j'})}_{s_{j'}}(x) = -\infty$ if $x \in M \backslash \Omega_+$, so U is contained in Ω and x_0 is away from $\partial\Omega$. Combining Lemma 3.1 together with the $C^{2,\alpha}_{loc}$ convergence of $\tilde{f}^{(x_{j'})}_{s_{j'}}$ and uniform convergence of $u_{j'}$ on two sides of equations (3.2.2), U is a subset of $u^{-1}(\Theta) \cap \Omega$ and hence $\tilde{f}$ satisfies equation (3.2.3) in U. By standard elliptic theory, $\tilde{f}$ is smooth.

Note that the equation (3.2.3) is invariant under vertical translation. For any $a \in R$, $\tilde{f} + a$ satisfies equation (3.2.3). By local estimates in Proposition 1.10 and Arzela–Ascoli theorem, there is a sequence $a_i \to +\infty$ such that Graph$(\tilde{f} + a_i)$ converge to a three dimensional submanifold in $M \times \mathbb{R}$ in $C^{2,\alpha}$-sense. Remark that we only need to consider $a \to +\infty$ since $\tilde{f} \leq 0$. By the Harnack inequality in Proposition 1.10, each component of the limit submanifold is a cylinder over a closed surface in ∂U, denoted by $\tilde{\Sigma} \times \mathbb{R}$. Since $\tilde{f} + a_i$ satisfies equation (3.2.3) for all i, $C^{2,\alpha}_{loc}$-convergence implies that $\tilde{\Sigma}$ with compatible unit normal satisfies the same constant expansion equation: $\theta[\tilde{\Sigma}] = \Theta$. □

Corollary 3.4. *Let $\Theta \geq 0$. Suppose Z is a connected component of $u^{-1}(\Theta) \cap \overline{\Omega}$ in which u attains local maximum (resp. minimum). Namely, there exists an open neighborhood O of Z such that for all $x \in O \backslash Z$.*

$$u(x) < \Theta \quad \textit{(resp. } u(x) > \Theta\textit{)}.$$

Then there exists a sequence of reference points $\{x_j\} \subset Z$, a subsequence $\{j'\} \subset \mathbb{N}$ and a non-empty maximal domain $U \subset Z$ such that $\tilde{f}^{(x_{j'})}_{s_{j'}}$ converges smoothly to a function $\tilde{f}$ in U satisfying constant expansion equation:

$$\theta[\tilde{f}] = \Theta \quad \textit{and } \tilde{f}(x) \to -\infty \textit{ (resp. } +\infty\textit{) as } U \ni x \to \partial U. \tag{3.2.4}$$

Each connected connected component $\tilde{\Sigma}$ of ∂U is a closed properly embedded smooth surface in Z with constant expansion $\theta[\tilde{\Sigma}] = \Theta$ computed with respect to the unit normal of $\tilde{\Sigma}$ pointing inside of U (resp. pointing outside of U).

Remark 3.5. In Corollary 3.4, the assumption that u attains its strict local maximum Θ in Z is equivalent to that Z is component of $\partial E^-_\Theta(u) \backslash \partial E^+_\Theta(u)$.

Proof. We only point out the key steps for the case when u attains a strict local maximum in Z. Note Z is a closed subset of compact set Ω, so Z is compact. We may assume $\overline{O}$ is compact. For every $j \in \mathbb{N}$, pick reference point $x_j \in \overline{O}$ so that $f_{s_j}(x_j) = \max_{\overline{O}} f_{s_j}$. By Lemma 3.1, $\tilde{f}_{s_j}^{(x_j)}(x) < 0$ for all $x \in \partial O$ for all suficiently large j. Since $\tilde{f}_{s_j}^{(x_j)}(x_j) = 0$ for all j, $\tilde{f}_{s_j}^{(x_j)}$ attains maximum at interior point $x_j \in O$ and $\nabla \tilde{f}_{s_j}^{(x_j)}(x_j) = 0$ for large j. Apply the argument of previous proof, there exists a subsequence $x_{j'}$ converging to x_0 and a solution $\tilde{f}$ to (3.2.4) defined in the maximal domain $U \subset Z$ containing x_0. This implies that x_0 is an interior point of Z and $x_{j'} \in Z$ for large j. Other results follow analogously as in the previous proof. □

Specifically, we can pick one fixed reference point $x_0 \in \overline{\Omega}$ and investigate the local limiting behavior of translated regularized solutions to (1.4.2) around x_0.

Theorem 3.6 (Local convergence)**.** *Let $x_0 \in \Omega$. Consider the sequence of translated functions $\tilde{f}_{s_j}^{(x_0)}$ satisfying $\tilde{f}_{s_j}^{(x_0)}(x_0) = 0$ for all j. There exists a subsequence $\{j'\} \subset \mathbb{N}$ such that one of the following statement is true.*

(1) (Graphical convergence) There exists a maximal domain $U_{x_0} \subset u^{-1}(u(x_0)) \cap \overline{\Omega}$ containing x_0 such that $\tilde{f}_{s_{j'}}^{(x_0)}$ converges smoothly to a function $\tilde{f}_0^{(x_0)}$ satisfying

$$\theta[\tilde{f}_0^{(x_0)}] = u(\overline{U}_{x_0}) \quad \textit{and } |\tilde{f}_0^{(x_0)}| \to \infty \textit{ on approach to } \partial U_{x_0}.$$

In particular,

$$\lim_{j' \to \infty} |\nabla f_{s_{j'}}(x_0)| = \lim_{j' \to \infty} |\nabla \tilde{f}_0^{(x_0)}(x_0)| < +\infty,$$

and

$$\lim_{j'\to\infty} \frac{\nabla f_{s_{j'}}(x_0)}{\sqrt{1+|f_{s_{j'}}(x_0)|^2}} \quad \textit{exists and has length} < 1.$$

Each component Σ *of* ∂U_{x_0} *is a closed smooth surface satisfying*

$$\theta[\Sigma] = u(\overline{U}_{x_0}).$$

Here, $\theta[\Sigma]$ *is computed with respect to the unit normal vector field* ν *which coincides with*

$$\nu(y) = \lim_{j'\to\infty} \frac{\nabla f_{s_{j'}}(y)}{\sqrt{1+|\nabla f_{s_{j'}}(y)|^2}} \quad \textit{for all } y \in \Sigma.$$

(2) (Cylindrical convergence) There exists a closed smooth surface $\Sigma_{x_0} \subset u^{-1}(u(x_0)) \cap \overline{\Omega}$ *passing through* x_0 *such that* $\mathrm{Graph}(\tilde{f}^{(x_0)}_{s_{j'}})$ *converges to* $\Sigma_{x_0} \times \mathbb{R}$ *smoothly. In particular,*

$$\lim_{j'\to\infty} |\nabla f_{s_{j'}}(x_0)| = +\infty,$$

and

$$\lim_{j'\to\infty} \frac{\nabla f_{s_{j'}}(x_0)}{\sqrt{1+|\nabla f_{s_{j'}}(x_0)|^2}} \quad \textit{exists and has length} = 1.$$

The surface Σ_{x_0} *satisfies*

$$\theta[\Sigma_{x_0}] = u(\Sigma_{x_0}).$$

Here, $\theta[\Sigma_{x_0}]$ *is computed with respect to the unit normal vector field* ν *which coincides* *with*

$$\nu(y)=\lim_{j'\to\infty}\frac{\nabla f_{s_{j'}}(y)}{\sqrt{1+|\nabla f_{s_{j'}}(y)|^2}}\quad \textit{for all } y\in\Sigma_{x_0}.$$

As a consequence of the convergence, there exists $\delta>0$ such that

(a) $\lim_{j'\to\infty}\tilde{f}^{(x_0)}_{s_{j'}}(x)=+\infty$ *for* $x\in\mathcal{N}^+_\delta(\Sigma_{x_0},\nu)$,

(b) $\lim_{j'\to\infty}\tilde{f}^{(x_0)}_{s_{j'}}(x)=-\infty$ *for* $x\in\mathcal{N}^-_\delta(\Sigma_{x_0},\nu)$.

Proof. Since $\tilde{f}^{(x_0)}_{s_j}$ satisfies (3.2.2) and $(x_0,0)\in\mathrm{Graph}(\tilde{f}^{(x_0)}_{s_j})$ for all j, by the local estimate in Proposition 1.10 we conclude that there exists a subsequence $\{j'\}$ such that graph$(\tilde{f}^{(x_0)}_{s_{j'}})$ converges to a properly embedded submanifold in $M\times\mathbb{R}$ in $C^{2,\alpha}_{loc}$-sense. Denote the component of the limit submanifold containing $(x_0,0)$ by $\tilde{S}$. By the Harnack-type inequality in Proposition 1.10, $\tilde{S}$ is either graphical or cylindrical.

Notice that $\nabla\tilde{f}^{(x_0)}_{s_j}(x)=\nabla f_{s_j}(x)$ for all $j\in\mathbb{N},x\in M$, and the vector

$$\frac{\nabla f_{s_j}}{\sqrt{1+|\nabla f_{s_j}|^2}}$$

is the horizontal component of the downward unit normal vector field on Graph$(\tilde{f}^{(x_0)}_{s_j})$. For the graphical case, the results follow analogously as the proof of Theorem 3.2. For the cylindrical case, by Lemma 3.1 and $C^{2,\alpha}_{loc}$ convergence we have $\theta[\Sigma_{x_0}]=u(x_0)$. Lastly, to check the compatibility of the unit normal of ∂U_{x_0} or respectively Σ_{x_0} at y, we may just pick y as new reference point for the subsequence $f_{s_{j'}}$, then the limiting behavior of $f^{(x_0)}_{s_{j'}}$ near y, local estimate and Harnack inequality imply that any subsequence of Graph$(\tilde{f}^{(y)}_{s_{j'}})$ converges cylindrically to the component of $\partial U_{x_0}\times\mathbb{R}$ containing $(y,0)$ or respectively $\Sigma_{x_0}\times\mathbb{R}$ and the original sequence converges in the same way. □

As an immediate application of the local convergence, we can show the existence of smooth closed constant expansion surface in any level set of u containing a regular point.

Corollary 3.7. *Suppose x_0 is a regular point of u at value Θ. Then there exists a closed smooth embedded surface Σ_{x_0} in $u^{-1}(\Theta) \cap \overline{\Omega}$ containing x_0 with constant expansion $\theta[\Sigma_{x_0}] = \Theta$. The unit normal vector field ν of Σ_{x_0} chosen as in Theorem 3.6 coincides with $\nabla u(x_0)/|\nabla u(x_0)|$ at x_0.*

Proof. From assumption, $\nabla u(x_0)$ exists and $\nabla u(x_0) \neq 0$. By local linear approximation of u around x_0, we can conclude that $x_0 \in \partial E^-_\Theta(u) \cap \partial E^+_\Theta(u)$ and the tangent space $T_{x_0}u^{-1}(\Theta) = \{\nabla u(x_0)\}^\perp$. Since x_0 is not an interior point of $u^{-1}(\Theta)$, Corollary 3.6 implies that graphical convergence is impossible and there exists a closed smooth CES Σ_{x_0} containing x_0 with expansion Θ. The direction of unit normal to Σ_{x_0} at x_0 is determined by local linear approximation of u, Lemma 3.1 and local limit behavior of $f^{(x_0)}_{s_j}$ in Theorem 3.6 (2). □

3.2.3 Stability of Constant Expansion Surfaces

We will end this section by showing the stability of all closed smooth embedded CESs in M that arise as boundary components of maximal domains or base sections of cylinders in subsection 3.2. The stability result for MOTS Proposition 1.14 was first proved by Andersson and Metzger in [4]. In the excellent survey paper [1], a simplified geometric argument was provided, but the constructed barrier functions did not work well. In the communication with Michael Eichmair, one of the authors of [1], he suggested a different model function to fix the glitch. The proof here essentially follows the idea for MOTS in [1] with Eichmair's modification.

Proposition 3.8 (Stability of CES)**.** *The closed smooth CESs which arise in Theorem 3.2, Corollary 3.4, and Theorem 3.6 (1) as boundaries of maximal domains and in Theorem 3.6 (2) as bases of cylinders are stable.*

Proof. Suppose (Σ, ν) is a *unstable* closed smooth surface in (M, g, k) with constant expansion Θ and $\lambda_1(\mathcal{L}_\Sigma) = -\alpha^2 < 0$ for some $\alpha > 0$. We will construct barrier functions in an open neighborhood of Σ. By Krein-Rutman theorem, there exists a strict positive function $\phi \in C^\infty(\Sigma)$ such that $\mathcal{L}_\Sigma \phi = -\alpha^2 \phi$. If ν is extended by parallel transportation and k is extended trivially in vertical direction, then the stability operator of $(\Sigma \times \mathbb{R}, \nu)$ with respect to (M, g, k) is $\mathcal{L}_{\Sigma\times\mathbb{R}} = -\partial_t^2 + \mathcal{L}_\Sigma$. If we feed the stability operator with a test function of the form $T(t)\phi(x)$ where $T \in C^2(\mathbb{R})$, then

$$\mathcal{L}_{\Sigma\times\mathbb{R}}\big(T(t)\phi(x)\big) = -(T'' + \alpha^2 T)\phi(x).$$

In the model case $\alpha = 1$, consider the smooth function $\eta(t) = \big(\arctan(t+1) - \arctan(1)\big)$. By numerical analysis, η has the following properties: (1) Range$(\eta) = (-\frac{3\pi}{4}, \frac{\pi}{4})$ and η is strictly increasing with $\eta(0) = 0$. (2) $\eta'' + \eta$ has a unique real root $t_r \approx 0.6456$. In particular, for $t \in (-\infty, 1/2]$, $\eta''(t) + \eta(t) < 0$ and the maximum is $\eta''(1/2) + \eta(1/2) \approx -0.0866$. For general $\alpha > 0$, we may consider $T(t) = \eta(\alpha t)$. Then

$$\mathcal{L}_{\Sigma\times\mathbb{R}}\big(T(t)\phi(x)\big) \geq -\big(\eta''(1/2) + \eta(1/2)\big)\alpha^2 \min_\Sigma \phi > 0 \quad \text{for} \quad t \in (-\infty, 1/(2\alpha)]. \tag{3.2.5}$$

For any sufficiently small $\varepsilon > 0$, the hypersurface

$$\big\{ \exp_{(x,t)} \big(\varepsilon T(t)\phi(x)\nu(x)\big) \in M \times \mathbb{R} : (x,t) \in \Sigma \times (-\infty, 1/(2\alpha)]\big\}$$

is a smooth hypersurface with boundary in $M \times \mathbb{R}$ whose expansion is *strictly* greater than Θ everywhere. Notice that T is monotone, so the hypersurface can be expressed as the graph of a function $f_* : V \to (-\infty, 1/(2\alpha))$ where $V = \big\{ \exp_x(s\phi(x)\nu(x)) \in M : s \in (-3\pi\varepsilon/4, \varepsilon\eta(1/2)), x \in \Sigma\big\}$ is a open neighborhood of Σ such that $0 < f_* < 1/(2\alpha)$ in the part of V with $0 < s < \varepsilon\eta(1/2)$ and $f_* \to -\infty$ as $s \to -3\pi\varepsilon/4^+$. Moreover, f_* satisfying $\theta[f_*] > \Theta$ is a sub-solution to equation $\theta[f] = \Theta$. Analogously, we may construct a supersolution f^* satisfying $\theta[f^*] < \Theta$ associated with the hypersurface

$$\Big\{ \exp_{(t,x)}\big(-\varepsilon T(-t)\phi(x)\nu(x)\big) : (x,t) \in \Sigma \times [-1/(2\alpha), +\infty)\Big\}$$

defined in an open neighborhood of Σ.

If the CES (Σ, ν) that arises in the regularization procedure as in the assumption is unstable, then the barrier functions constructed in the first paragraph prevent the translated regularized solutions from blowing up exactly at Σ. This contradicts the formation of such a CES. □

3.3 Characterization of Capillary Blowdown Limit

3.3.1 Capillary Blowdown Limit as Viscosity Solution

We begin by replacing f_s by u_s/s in regularized equations (1.4.2). Then u_s satisfies

$$(g^{ij} - \frac{u_s^i u_s^j}{s^2 + |\nabla u_s|^2})\Big(\frac{\nabla_i \nabla_j u_s}{\sqrt{s^2 + |\nabla u_s|^2}} - k_{ij}\Big) = u_s. \tag{3.3.1}$$

Let u be a blowdown limit of regularized solutions to Jang's equation. Now we make some heuristic assumptions that u is C^2 and $\nabla u_{s_j} \to \nabla u$ for one sequence $s_j \to 0^+$. In the region $\{x : \nabla u(x) \neq 0\}$, the sequence of regularized equations (3.3.1) converges to the geometric equation

$$\mathrm{div}_M\Big(\frac{\nabla u}{|\nabla u|}\Big) - \mathrm{tr}_g(k) + k\Big(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\Big) = u. \tag{3.3.2}$$

In addition, Corollary 3.7 is another clue that u satisfies (3.3.2) in $\{x : \nabla u(x) \neq 0\}$. This geometric equation can be interpreted as follows: Any regular level set of classical solution,

u, has constant expansion equal to the evaluation of u. By simple calculations, (3.3.2) is equivalent to

$$-\mathrm{div}_M(\nabla u) + \nabla^2 u\Big(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\Big) + |\nabla u|\Big\{u + \mathrm{tr}_g(k) - k\Big(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\Big)\Big\} = 0. \tag{3.3.3}$$

It is obvious that the equation is singular in the set $\{\nabla u = 0\}$, which is inevitable according to the existence of interior of level set at extremal value by Theorem 3.2. It is necessary to find a weaker notion of a solution. Since u has already been a Lipschitz continuous function by construction, inspired by the work on level-set formulation of mean curvature flow done by L.C. Evans and J. Spruck [18], we may expect viscosity solution is suitable notion of weak solution. Before we define viscosity solutions to (3.3.3) on manifolds, we recall several terminologies introduced in [6].

Definition 3.9. (1) Let $f : M \to [-\infty, \infty)$ a lower semi-continuous function. Define the *second order superjet* of f at x by

$$J^{2,+}f(x) = \{(d\varphi(x), d^2\varphi(x)) : \ \varphi \in C^2(M;\mathbb{R}),\ f - \varphi \text{ attains a local maximum at } x\}.$$

(2) Let $f : M \to (-\infty, \infty]$ a upper semi-continuous function. Define the *second order subjet* of f at x by

$$J^{2,-}f(x) = \{(d\varphi(x), d^2\varphi(x)) : \ \varphi \in C^2(M;\mathbb{R}),\ f - \varphi \text{ attains a local minimum at } x\}.$$

Remark 3.10. Let $x \in M$, $\zeta \in T_x^*M$, $A \in \mathcal{L}^2_{sym}(T_xM)$. Then the followings are equivalent:

(1) $(\zeta, A) \in J^{2,+}f(x)$

(2) $f(\exp_x(\eta)) \le f(x) + \langle \zeta, \eta\rangle_x + \frac{1}{2}\langle A\eta, \eta\rangle_x + o(|\eta|_x^2)$

(3) $(\zeta, A) \in J^{2,+}(f \circ \exp_x)(0_x)$ where 0_x is the origin in T_xM

(4) $(\zeta, A) \in -J^{2,-}(-f)(x)$

Let $x_n \to x$, $\zeta_n \in T^*_{x_n}M$ and $A_n \in \mathcal{L}^2_{sym}(T_{x_n}M)$. We denote by $\zeta_n \to \zeta \in T^*_x M$ if $\langle \zeta_n, V\rangle_{x_n} \to \langle \zeta, V\rangle_x$ for all smooth vector field V near x and we denote by $A_n \to A \in \mathcal{L}^2_{sym}(T_x M)$ if $\langle AV, V\rangle_{x_n} \to \langle AV, V\rangle_x$ for all smooth vector field V near x.

Definition 3.11. (1) Let $f : M \to [-\infty, \infty)$ a lower semi-continuous function. Define

$$\overline{J^{2,+}}f(x) = \{(\zeta, A) \in T^*_x M \times \mathcal{L}^2_{sym}(T_x M) : \exists x_n \to x, \exists (x_n, A_n) \in J^{2,+}f(x_n) \text{ such that } (x_n, f(x_n), \zeta_n, A_n) \to (x, f(x), \zeta, A)\}$$

(2) Let $f : M \to (-\infty, \infty]$ a upper semi-continuous function. Define

$$\overline{J^{2,-}}f(x) = \{(\zeta, A) \in T^*_x M \times \mathcal{L}^2_{sym}(T_x M) : \exists x_n \to x, \exists (x_n, A_n) \in J^{2,-}f(x_n) \text{ such that } (x_n, f(x_n), \zeta_n, A_n) \to (x, f(x), \zeta, A)\}$$

Now we are ready to define viscosity solutions to (3.3.3). Let $x \in M$, $r \in \mathbb{R}$, $\zeta \in T_x M$, $A \in \mathcal{L}^2_{sym}(T_x M)$. Define

$$\mathcal{F}(x, r, \zeta, A) := -\mathrm{tr}_g A(x) + \langle A\frac{\zeta}{|\zeta|}, \frac{\zeta}{|\zeta|}\rangle_x + |\zeta|_x\{r + \mathrm{tr}_g k(x) - k(\frac{\zeta}{|\zeta|}, \frac{\zeta}{|\zeta|})(x)\}$$

and its degenerate form

$$\mathcal{G}(x, \zeta, A) := -\mathrm{tr}_g A(x) + \langle A\zeta, \zeta\rangle_x.$$

Definition 3.12. $u \in C^0(M) \cap L^\infty(M)$ is a ***viscosity subsolution*** of equation (3.3.3) if for all $x \in M$ either for all $(\zeta \neq 0, A) \in \overline{J^{2,+}}u(x)$

$$\mathcal{F}(x, u(x), \zeta, A) \leq 0,$$

or for all $(0, A) \in \overline{J^{2,+}}u(x)$ there exists $\xi \in T_xM$ with $|\xi|_x \leq 1$

$$\mathcal{G}(x, \xi, A) \leq 0.$$

Similarly, $u \in C^0(M) \cap L^\infty(M)$ is a ***viscosity supersolution*** of equation (3.3.3) if for all $x \in M$ either for all $(\zeta \neq 0, A) \in \overline{J^{2,-}}u(x)$

$$\mathcal{F}(x, u(x), \zeta, A) \geq 0,$$

or for all $(0, A) \in \overline{J^{2,-}}u(x)$ there exists $\xi \in T_xM$ with $|\xi|_x \leq 1$

$$\mathcal{G}(x, \xi, A) \geq 0.$$

$u \in C^0(M) \cap L^\infty(M)$ is a ***viscosity solution*** of equation (3.3.3) if u is both a viscosity subsolution and supersolution.

In the following theorem, we apply the argument in the proof of existence of weak mean curvature flow in viscosity sense using elliptic regularization by L.C. Evans and J. Spruck [18] to show that any blowdown limit of regularized solutions is a viscosity solution.

Theorem 3.13. *Let u be a capillary blowdown limit of f_s. Then u is a viscosity solution to the geometric equation (3.3.3).*

Proof. Let $\varphi \in C^2(M)$ and suppose $u - \varphi$ has a *strict* local maximum at a point $x_0 \in M$. Choose $u_{s_j} \to u$ uniformly near x_0, then $u_{s_j} - \varphi$ has a local maximum at a point x_j with $x_j \to x_0$ as $j \to \infty$. Since u_{s_j} and φ are twice differentiable, we have

$$\nabla u_{s_j}(x_j) = \nabla \varphi(x_j),$$
$$\nabla^2 u_{s_j}(x_j) \leq \nabla^2 \varphi(x_j).$$

Thus, equation (3.3.1) implies for all j at x_j

$$- \operatorname{tr}_g \nabla^2 \varphi + \nabla^2 \varphi\Big(\frac{\nabla \varphi}{\sqrt{s_j^2 + |\nabla \varphi|^2}}, \frac{\nabla \varphi}{\sqrt{s_j^2 + |\nabla \varphi|^2}}\Big)$$
$$+ \sqrt{s_j^2 + |\nabla \varphi|^2}\Big\{u_{s_j} + \operatorname{tr}_g k - k\left(\frac{\nabla \varphi}{\sqrt{s_j^2 + |\nabla \varphi|^2}}, \frac{\nabla \varphi}{\sqrt{s_j^2 + |\nabla \varphi|^2}}\right)\Big\} \leq 0. \tag{3.3.4}$$

Suppose $\nabla\varphi(x_0) \neq 0$. Then $\nabla\varphi(x_j) \neq 0$ for all sufficiently large j. Passing to limit, we get

$$\mathcal{F}\big(x_0, u(x_0), \nabla\varphi(x_0), \nabla^2\varphi(x_0)\big) \leq 0.$$

Suppose $\nabla\varphi(x_0) = 0$. Set $\eta_j := \frac{\nabla\varphi(x_j)}{\sqrt{s_j^2+|\nabla\varphi(x_j)|^2}} \in T_{x_j}M$ such that (3.3.4) becomes

$$- \operatorname{tr}_g \nabla^2 \varphi(x_j) + \nabla^2 \varphi\big(\eta_j, \eta_j\big)(x_j)$$
$$+ \sqrt{s_j^2 + |\nabla \varphi(x_j)|^2}\big\{u_{s_j}(x_j) + \operatorname{tr}_g k(x_j) - k\big(\eta_j, \eta_j\big)(x_j)\big\} \leq 0.$$

Since $|\eta|_{x_j} \leq 1$, we may assume up to subsequence $\eta_j \to \eta \in T_{x_0}M$ with $|\eta|_{x_0} \leq 1$. Letting $j \to \infty$, since u and k are bounded we obtain

$$\mathcal{G}(x_0, \eta, \nabla^2\varphi(x_0)) \leq 0.$$

If $u - \varphi$ has a local maximum which may not be strict, we repeat the argument above with

$$\tilde{\varphi}(x) = \varphi + d(x, x_0)^4$$

satisfying $\nabla\tilde{\varphi}(x_0) = \nabla\varphi(x_0)$ and $\nabla^2\tilde{\varphi}(x_0) = \nabla^2\varphi(x_0)$ in place of φ. Here, d is the distance function defined on (M, g). Therefore, u is a viscosity subsolution.

It follows analogously that u is a viscosity supersolution. □

3.3.2 A Priori Estimates of Foliation of Stable Constant Expansion Surfaces

In this subsection, we will prove the a priori estimate of foliation of stable constant expansion surfaces. The proof will follow the stability argument leading to the a priori estimates of the regularized Jang's equation in [39] and the one of stable minimal hypersurfaces in [36]. Here, we only comment on the key ingredients adapted to the assumptions that we consider.

Recall that (M, g, k) is an asymptotically flat initial data set satisfying the dominant energy condition. Given positive constants $\mathcal{T}$ and B, suppose Σ assigned with unit normal ν is a closed smooth stable CES with constant expansion $\Theta_0 \in [-\mathcal{T}, \mathcal{T}]$ having the second fundamental form $|h_\Sigma|^2 \leq B$. Suppose $\Psi : (a, b) \times \Sigma \to M$ is a smooth foliation of closed stable CES initiated from Σ with expansion in the range $[-\mathcal{T}, \mathcal{T}]$. Let $\Sigma_\tau = \Psi(\tau, \Sigma)$ and let $\nu_\tau = \Psi_*(\partial_\tau)/|\Psi_*(\partial_\tau)|$ where Ψ_* is the pushforward of Ψ, then Ψ satisfies the following properties:

(1) $\Psi(\tau_0, \cdot) = \mathrm{Id}_\Sigma(\cdot)$ on Σ for some $\tau_0 \in (a, b)$ and $\nu_{\tau_0} = \nu$.

(2) The expansion $\Theta_\tau := \theta[\Sigma_\tau]$ of Σ_τ with respect to unit normal ν_τ is a constant in $[-\mathcal{T}, \mathcal{T}]$ for any $\tau \in (a, b)$.

(3) $\lambda_1(\mathcal{L}_{\Sigma_\tau}) \geq 0$ for any $\tau \in (a, b)$.

Now fix arbitrary $\tau \in (a, b)$. Let e_1, e_2, e_3 be a local orthonormal frame for Σ_τ with e_1, e_2 tangent to Σ_τ and e_3 normal to Σ_τ. Let $\omega_1, \omega_2, \omega_3$ be the corresponding dual orthonormal coframe of one-form. The structure equations of M are given by

$$d\omega_i = -\sum_{j=1}^{3} \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0;$$

$$d\omega_{ij} = -\sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,l=1}^{3} R_{ijkl}\, \omega_k \wedge \omega_l.$$

Let ∇ and $\overline{\nabla}$ denote the Levi-Civita connections on M and Σ_τ respectively. In this subsection, the indices range 1,2 and constant C may change from time to time but depend only on the initial data set (M, g, k), given constants $\mathcal{T}$ and B. The first important ingredient of the a priori estimate is Simon's inequality. By virtue of the asymptotically flatness of (M, g, k), the background Riemannian curvature tensor and its covariant derivatives are bounded. This is a key assumption to derive the lower bound of Laplacian of second fundamental form h_{ij} of Σ_τ as in [39] on page 236

$$\Delta h_{ij} \geq \overline{\nabla}_i \overline{\nabla}_j \mathrm{H} - (\sum_{m,k} h_{mk}^2) + \mathrm{H} \sum_m h_{im} h_{mj} - C(|h| + 1)\delta_{ij}.$$

Following the same computation in [39] on page 236-237, one can obtain the Simon's inequality (cf. [39] (2.16))

$$\begin{aligned} |h|\Delta|h| \geq{}& c(2) \sum_{i,j,k} (\overline{\nabla}_k h_{ij})^2 - |h|^4 - |\mathrm{H}||h|^3 \\ &+ \sum_{i,j} h_{ij} \overline{\nabla}_i \overline{\nabla}_j \mathrm{H} - C|\overline{\nabla}\mathrm{H}|^2 - C(|h|^2 + 1) \end{aligned} \tag{3.3.5}$$

where $c(2)$ is a constant that depends only on $\dim(\Sigma) = 2$.

The second important ingredient is stability inequality. In [39], they derive the stability inequality by observing that vertical translations generate a Jacobi field. Now in our setting we assume the stability directly. Let $\beta > 0$ be a smooth eigenfunction of $\mathcal{L}_{\Sigma_\tau}$ corresponding to non-negative principle eigenvalue λ_1. Using (1.3.11), we have

$$0 \leq \lambda_1 = \frac{\mathcal{L}_{\Sigma_\tau}\beta}{\beta} = -\mathrm{div}_{\Sigma_\tau}(\xi + \overline{\nabla}\log\beta) - |\xi + \overline{\nabla}\log\beta|^2_{\Sigma_\tau} + \frac{1}{2}\mathrm{R}_\Sigma - \frac{1}{2}|h-k|^2_{\Sigma_\tau} - \mu + J(\nu) - \frac{1}{2}\Theta_\tau(\Theta_\tau + 2\mathrm{tr}_g k). \tag{3.3.6}$$

Note that the dominant energy condition implies $-\mu + J(\nu) \leq 0$. Let $\varphi \in C^\infty(\Sigma)$. Multiplying (3.3.6) by φ^2, integrating by part and applying Young's inequality to the first term, we find

$$0 \leq \int_{\Sigma_\tau} |\overline{\nabla}\varphi|^2 + \frac{1}{2}\int_{\Sigma_\tau}\Big\{\mathrm{R}_{\Sigma_\tau} - |h-k|^2_{\Sigma_\tau} - \Theta_\tau(\Theta_\tau + 2\mathrm{tr}_g k)\Big\}\varphi^2. \tag{3.3.7}$$

Using Guass equation and cancelling out H^2 terms in R_Σ and Θ_τ^2, we get

$$\int_{\Sigma_\tau} |h|^2\varphi^2 \leq \int_{\Sigma_\tau}|\overline{\nabla}\varphi|^2 + C\int_{\Sigma_\tau}(|h|+1)\varphi^2. \tag{3.3.8}$$

Combining (3.3.5) and (3.3.8) together with the control $|\overline{\nabla}\mathrm{H}[\Sigma]|^2 = |\overline{\nabla}\mathrm{K}[\Sigma]|^2 \leq C(|h|^2+1)$ on constant expansion surface, following the argument in [39] replacing φ by $|h|\varphi^2$ and then absorbing $|h|^3\varphi^4$ by $|h|^4\varphi^4$ and φ^2, we may derive

$$\int_{\Sigma_\tau}|h|^4\varphi^4 \leq \int_{\Sigma_\tau}|\overline{\nabla}\varphi|^4 + C\int_{\Sigma_\tau}\varphi^4. \tag{3.3.9}$$

The third ingredient is the local area bound for Σ_τ. We will follow the calibration argument in [39] on page 243 with minor modification. Observe that in the region sweep by the foliation Ψ we have

$$\mathrm{div}_M(\nu_\tau) = \Theta_\tau + \mathrm{tr}_g(k) - k(\nu_\tau, \nu_\tau) \tag{3.3.10}$$

where $|\Theta_\tau| \leq \mathcal{T}$. Let $x_0 \in \Sigma_\tau$, $B_\sigma(x_0)$ be the geodesic ball in (M, g) centered at x_0 and let W be the region enclosed by Σ and Σ_τ. Let $0 < \rho_0 \leq 1$ such that $\rho_0 \leq \mathrm{inj}(M, g)$. Integrating identity (3.3.10) over the region $W \cap B_\sigma(x_0)$ for $0 < \sigma \leq \rho_0$ and applying divergence theorem, we obtain

$$\text{Area}(\Sigma_\tau \cap B_\sigma(x_0)) \le \text{Area}(\Sigma \cap B_\sigma(x_0)) + \text{Area}(\partial B_\sigma(x_0) \cap W) + C\mathcal{T}\sigma^3.$$

Since we have $|h_\Sigma|^2 \le B$ for the initial sheet, there exists a constant ρ_1 depending on $M, \Sigma, g, k, B, \mathcal{T}$ such that for $0 < \sigma \le \rho_1$

$$\text{Area}(\Sigma_\tau \cap B_\sigma(x_0)) \le C\sigma^2. \tag{3.3.11}$$

With the area bound (3.3.11) the results of Hoffman and Spruck [23] imply that there is a number $\rho_2 \le \rho_1$ such that the Michael-Simon type Sobolev inequality holds:

$$\Big(\int_{\Sigma_\tau} \varphi^2\Big)^{1/2} \le C\int_{\Sigma_\tau} |\overline{\nabla}\varphi| + |\varphi||\mathrm{H}|. \tag{3.3.12}$$

for any Lipschitz φ vanishing outside of $\Sigma_\tau \cap B_{\rho_2}(x_0)$. Using the bounds for expansion, k and area (3.3.11) together with Hölder inequality, we obtain

$$\Big(\int_{\Sigma_\tau} \varphi^2\Big)^{1/2} \le C\int_{\Sigma_\tau} |\overline{\nabla}\varphi|$$

and hence for arbitrary $p > 2$

$$\Big(\int_{\Sigma_\tau} |\varphi|^p\Big)^{1/p} \le C\int_{\Sigma_\tau} |\overline{\nabla}\varphi|^2. \tag{3.3.13}$$

Fixing the geodesic distance cutoff function to x_0 depending on ρ_2, (3.3.9) and (3.3.11) imply

$$|h_{\Sigma_\tau}|^2 \in L^2\big(B_{\rho_2/2}(x_0)\big). \tag{3.3.14}$$

Let $q = |h_{\Sigma_\tau}|^2 + 1$. Following the argument in [39], q is a positive weak subsolution to certain elliptic equation. De Georgi-Nash-Moser iteration technique together with the L^2-

bound (3.3.11) and (3.3.14) for q now gives pointwise curvature bound for extrinsic curvature

$$\sup_{\Sigma_\tau} |h_{\Sigma_\tau}|^2 \leq C. \tag{3.3.15}$$

Note that the Sobolev inequality (3.3.13) for large $p > 2$ is sufficient for iteration technique for dimension 2. Also, (3.3.14) where $2 > \frac{1}{2}\dim(\Sigma_\tau) = 1$ guarantees the structural conditions are satisfied.

Lastly, following the argument in [39], (3.3.15) implies the uniform local $C^{3,\alpha}$ estimate. We conclude the results of this subsection in the following proposition.

Proposition 3.14. *Let (M, g, k) be an asymptotically flat initial data set satisfying dominant energy condition. Given positive constants $\mathcal{T}$ and B, suppose Σ assigned with unit normal ν is a closed smooth stable CES with constant expansion $\Theta_0 \in [-\mathcal{T}, \mathcal{T}]$ having the second fundamental forms $|h_\Sigma|^2 \leq B$. Suppose $\Psi : (a, b) \times \Sigma \to M$ is a smooth foliation of closed stable CES initiated from Σ with expansion in the range $[-\mathcal{T}, \mathcal{T}]$. Given $\alpha \in (0, 1)$, then there exist constants ρ and C_α depending on $M, \Sigma, g, k, \mathcal{T}, B$ such that for any $\tau \in (a, b)$, for every $x_0 \in \Sigma_\tau$ if (x^1, x^2, x^3) normal coordinates in M on which $T_{x_0}\Sigma_\tau$ is the x^1x^2-space, then the local defining function $w(x)$ for Σ_τ is defined on $\{x = (x^1, x^2) : |x| \leq \rho\}$ with*

$$\Sigma_\tau \cap B^3(x_0; \frac{\rho}{2}) \subseteq \text{Graph}(w)$$

and satisfies

$$\|w\|_{3,\alpha,\{x:|x|\leq\rho\}} \leq C_\alpha.$$

3.3.3 Existence of Smooth Solutions

Proposition 3.15. *Suppose* (Σ, ν) *is a closed smooth embedded strictly stable CES in* (M, g, k) *with* $\theta[\Sigma] \equiv \tau_0$ *in* (M, g, k). *Then there exists a constant* $\varepsilon > 0$ *and a smooth CES foliation* $\Psi : (\tau_0 - \varepsilon, \tau_0 + \varepsilon) \times \Sigma \to M$ *satisfying the following properties. Let* Σ_τ *denote the sheet* $\Psi(\tau, \Sigma)$. *We have*

(1) $\Psi(\tau_0, \cdot) = \mathrm{Id}_\Sigma(\cdot)$ *on* Σ.

(2) $\theta[\Sigma_\tau] \equiv \tau$ *for all* $\tau \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon)$.

(3) (Local uniqueness) If $\tilde{\Sigma}$ *is a closed smooth CES in* $\Psi\big((\tau_0 - \varepsilon, \tau_0 + \varepsilon) \times \Sigma\big)$ *and can be expressed as a graph of* $w \in C^\infty(\Sigma)$ *in normal coordinates around* Σ, *then* $\tilde{\Sigma} = \Sigma_{\tilde{\tau}}$ *for some* $\tilde{\tau} \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon)$.

Proof. We begin with proving local existence of smooth foliation by using implicit function theorem. Let $\Upsilon : \Sigma \times (-\delta, \delta) \to M : (y, \sigma) \mapsto \exp_y(\sigma \nu_y)$ be the normal coordinates around Σ with respect to the unit normal ν. For a function $w \in C^\infty(\Sigma)$, denote the graph $\{\exp_y\big(w(y)\nu_y\big) : y \in \Sigma\}$ of w in normal coordinates by $\mathfrak{Graph}(w)$. We also let $\theta[w]$ simply denote the expansion of $\mathfrak{Graph}(w)$ in the unit normal $\partial_\sigma^\perp / |\partial_\sigma^\perp|$ where $\partial_\sigma^\perp$ is the projection of ∂_σ onto the normal space of $\mathfrak{Graph}(w)$. Observe that the operator

$$\mathcal{T} : C^\infty(\Sigma) \times \mathbb{R} \to C^\infty(\Sigma)$$

defined by

$$\mathcal{T}(w, \tau) = \theta[w] - \tau$$

is a Frechet smooth mapping and $\mathcal{T}(0, \tau_0) = 0$. The linearization of $\mathcal{T}$ with respect to the first argument at $(0, \tau_0)$ is given by

$$(D_1\mathcal{T})|_{(0,\tau_0)}(w') = \mathcal{L}_\Sigma w'$$

for $w' \in C^\infty(\Sigma)$. Since $\lambda_1(\mathcal{L}_\Sigma) > 0$, the linearization operator $D_1\mathcal{T}(0, \tau_0)$ is an isomophism from $C^\infty(\Sigma)$ onto $C^\infty(\Sigma)$. By implicit function theorem, there exists $\varepsilon > 0$ and a unique Frechet smooth mapping

$$\mathcal{S} : (\tau_0 - \varepsilon, \tau_0 + \varepsilon) \longrightarrow C^\infty(\Sigma) \tag{3.3.16}$$

such that

$$\mathcal{S}(\tau_0) = 0$$

and for $\tau \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon)$

$$\mathcal{T}(\mathcal{S}(\tau), \tau) = 0. \tag{3.3.17}$$

Define the smooth one-parameter family of embeddings

$$\Psi : (\tau_0 - \varepsilon, \tau_0 + \varepsilon) \times \Sigma \longrightarrow M$$

by

$$\Psi(\tau, y) = \exp_y\big(\mathcal{S}(\tau)(y)\nu_y\big)$$

for $\tau \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon)$, $y \in \Sigma_0$. Denote the sheet $\Psi(\tau, \Sigma)$ by Σ_τ. It follows that $\{\Sigma_\tau : \tau \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon)\}$ is a smooth one-parameter family of closed smooth embedded surfaces with constant expansion τ. Thus, (1) and (2) has been established. The local uniqueness property (3) follows from the contraction principle in the proof of implicit function theorem.

It remains to show that Φ is a foliation. Observe that $\Psi(\tau, \cdot)$ satisfies the evolution equation

$$\frac{d}{d\tau}\Psi = \psi_\tau \nu_{\Sigma_\tau} \tag{3.3.18}$$

where $\psi_\tau \in C^\infty(\Sigma_\tau)$ satisfies

$$\mathcal{L}_{\Sigma_\tau}\psi_\tau = 1. \tag{3.3.19}$$

To see that Ψ is a foliation, it suffices to show the velocity function $\psi_\tau > 0$ for all $\tau \in (T_-, T_+)$. Toward contradiction, suppose $\psi_\tau(x) \leq 0$ for some $\tau \in (T_-, T_+)$ and $x \in \Sigma_\tau$. Let $\beta_\tau > 0$ denote the (unique up to scaling) eigenfunction of $\mathcal{L}_{\Sigma_\tau}$ associated with the principal eigenvalue $\lambda_1(\mathcal{L}_{\Sigma_\tau})$. There exists $b_\tau \geq 0$ such that $\min_{\Sigma_\tau}(\psi_\tau + b_\tau\beta_\tau) = 0$. At minimum point, by (3.3.19) we obtain

$$0 \geq -\Delta_{\Sigma_\tau}(\psi_\tau + b_\tau\beta_\tau) = \mathcal{L}_{\Sigma_\tau}(\psi_\tau + b_\tau\beta_\tau) = 1 + b_\tau\lambda_1(\mathcal{L}_{\Sigma_\tau})\beta_\tau \geq 1.$$

This is a contradiction. ☐

Corollary 3.16 (Maximal smooth stable foliation)**.** *Suppose (Σ, ν) is a closed smooth embedded strictly stable CES in (M, g, k) with $\theta[\Sigma] \equiv \tau_0$ in (M, g, k). Then there exists an open interval (T_-, T_+) containing τ_0 and a smooth CES foliation $\Psi : (T_-, T_+) \times \Sigma \to M$ satisfying the following properties:*

(1) $\Psi(\tau_0, \cdot) = \mathrm{Id}_\Sigma(\cdot)$ *on* Σ.

(2) $\theta[\Sigma_\tau] \equiv \tau$ *for all* $\tau \in (T_-, T_+)$.

(3) $\lambda_1(\mathcal{L}_{\Sigma_\tau}) > 0$ *for* $\tau \in (T_-, T_+)$.

Furthermore, if $|T_+| < \infty$ (resp. $|T_-| < \infty$), then Σ_τ converges to a smooth marginally stable CES Σ_{T_+} (resp. Σ_{T_-}) as $\tau \to T_+$ (resp. $\tau \to T_-$).

Proof. It is known that the principal eigenvalue depends (Lipschitz) continuously on the coefficients of the elliptic operator (cf. [8]). By the local existence Proposition 3.15 and local estimate Proposition 3.14, Ψ can be extended uniquely to an open neighborhood of the slice

Σ_τ as long as Σ_τ has finite constant expansion and $\lambda_1(\mathcal{L}_{\Sigma_\tau}) > 0$. Thus, there is a maximal interval (T_-, T_+) such that Ψ remains smooth and satisfies $\theta[\Sigma_\tau] \equiv \tau$ and $\lambda_1(\mathcal{L}_{\Sigma_\tau}) > 0$ for all $\tau \in (T_-, T_+)$. In particular, if $|T_+| < \infty$ (resp. $|T_-| < \infty$), then Σ_τ converges smoothly to a CES Σ_{T_+} (resp. Σ_{T_-}) as $\tau \to T_+$ (resp. $\tau \to T_-$). In either case, Σ_{T_+} or Σ_{T_-} is marginally stable; otherwise, the foliation Ψ continues by the local construction, which contradicts the maximality of the interval (T_-, T_+). □

Proposition 3.17 (Local smooth solution)**.** *Suppose (Σ, ν) is a closed smooth strictly stable CES with $\theta \equiv \tau_0$ in (M, g, k). Let Ψ be the maximal stable foliation constructed in Corollary 3.16. Define*

$$v\big(\Psi(\tau, y)\big) = \tau \tag{3.3.20}$$

for all $\tau \in (T_-, T_+)$, $y \in \Sigma$. Then v is a smooth solution to equation (3.3.2) in the region $\Psi\big((T_-, T_+) \times \Sigma\big)$ such that ∇v is nowhere vanishing. Moreover, for all $\tau \in (T_-, T_+)$ there exists $0 < C(\tau) < \infty$ depending continuously on local geometry of Σ_τ and k such that

$$C(\tau)^{-1}\lambda_1(\mathcal{L}_{\Sigma_\tau}) \leq |\nabla v|_{\Sigma_\tau} \leq C(\tau)\lambda_1(\mathcal{L}_{\Sigma_\tau}). \tag{3.3.21}$$

In particular, if $|T_+| < \infty$ (respectively $|T_-| < \infty$), then $\nabla v(x)$ converges to zero uniformly as x on approach to Σ_{T_+} (respectively Σ_{T_-}).

Proof. By definition, v is a smooth function since Ψ is a smooth foliation. Let $\tau \in (T_-, T_+)$. In view of (3.3.18) and (3.3.19) we have

$$1 = \frac{d}{d\tau} v = \langle \nabla v, \psi_\tau \nu_{\Sigma_\tau} \rangle = |\nabla v| \cdot \psi_\tau \quad \text{on } \Sigma_\tau.$$

From the proof of Proposition 3.15, we find $0 < \psi_\tau < \infty$. Thus,

$$0 < |\nabla v| = 1/\psi_\tau < \infty \quad \text{on } \Sigma_\tau. \tag{3.3.22}$$

It follows that the level set $v^{-1}(\tau) = \Sigma_\tau$ is regular and has constant expansion τ. Therefore, v is a classical solution to (3.3.3) in $\Psi\big((T_-, T_+) \times \Sigma\big)$.

Let $\beta_\tau > 0$ denote the (unique up to scaling) eigenfunction of $\mathcal{L}_{\Sigma_\tau}$ associated with the principal eigenvalue $\lambda_1(\mathcal{L}_{\Sigma_\tau})$. Remark that the following argument is independent of the choice of scaling of β_τ. By Harnack inequality, there exists $C(\tau)$ such that

$$\max_{\Sigma_\tau} \beta_\tau \leq C(\tau) \min_{\Sigma_\tau} \beta_\tau \tag{3.3.23}$$

for all $T_- < \tau < T_+$. Here $C(\tau)$ depends on the coefficients of $\mathcal{L}_{\Sigma_\tau}$ and intrinsic diameter of Σ_τ and therefore depends on local geometry of Σ_τ and k. Since both ψ_τ and β_τ are positive and Σ_τ is compact, there exists a constant $b_\tau > 0$ and a point $x_\tau \in \Sigma_\tau$ such that

$$\max_{\Sigma_\tau}(\psi_\tau - b_\tau\beta_\tau) = \psi_\tau(x_\tau) - b_\tau\beta_\tau(x_\tau) = 0.$$

It follows from (3.3.19) that

$$0 \leq \mathcal{L}_{\Sigma_\tau}(\psi_\tau - b_\tau\beta_\tau)(x_\tau) = 1 - b_\tau\lambda_1(\mathcal{L}_{\Sigma_\tau})\beta_\tau(x_\tau).$$

Thus,

$$b_\tau\beta_\tau(x_\tau) \leq \lambda_1(\mathcal{L}_{\Sigma_\tau}).$$

Then the maximum of $\psi_\tau - b_\tau\beta_\tau$ at x_τ and the Harnack inequality (3.3.23) imply that for any $x \in \Sigma_\tau$

$$\psi_\tau(x) \leq b_\tau\beta(x) \leq b_\tau C(\tau)\beta(x_\tau) \leq C(\tau)\lambda_1(\mathcal{L}_{\Sigma_\tau}). \tag{3.3.24}$$

By considering $\min_{\Sigma_\tau}(\psi_\tau - a_\tau\beta_\tau) = 0$ for suitable constant $a_\tau > 0$, we can analogously show that

$$C(\tau)^{-1}\lambda_1(\mathcal{L}_{\Sigma_\tau}) \leq \psi_\tau. \tag{3.3.25}$$

Putting (3.3.22), (3.3.24) and (3.3.25) together, we conclude (3.3.21).

If $|T_\pm| < \infty$, then by Corollary 3.16 Σ_τ converges smoothly to $\Sigma_{T_\pm}$ smoothly as $\tau \to T_\pm$ and therefore $C(\tau)$ can extend continuously to $\tau = T_\pm$. In view of (3.3.21) and Corollary 3.16: $\lambda_1(\mathcal{L}_{\Sigma_{T_\pm}}) = 0$, we find that $|\nabla v|(x)$ converges to 0 uniformly as x goes to $\Sigma_{T_\pm}$. $\square$

When (Σ, ν) is a compact, smooth, marginally stable CES, we are not able to construct a local foliation of CESs around Σ with the operator in Proposition 3.15. Nevertheless, Galloway [19] constructed a local foliation of CESs around Σ by considering the operator

$$\mathcal{T}_0 : C^\infty(\Sigma) \times \mathbb{R} \to C^\infty(\Sigma) \times \mathbb{R}, \quad \mathcal{T}_0(w, \ell) = \Big(\theta[w] - \ell, \int_\Sigma w\Big).$$

The fact that the principal eigenvalue is simple allows him to apply the inverse function theorem with this operator. The drawbacks of the foliation are that the expansion function θ is implicit and that the sheets are not necessarily stable so that we can further extend the foliation.

Proposition 3.18 (cf. [19] the proof of Theorem 3.1). *Suppose (Σ, ν) is a compact smooth marginally stable CES with $\theta \equiv \tau_0$ and $\lambda_1(\mathcal{L}_\Sigma) = 0$ in (M, g, k). Then there exists $\varepsilon > 0$ and a smooth CES foliation $\Psi_0 : (-\varepsilon, \varepsilon) \to M$ satisfying the following properties. Denote $\Psi_0(\tau, \Sigma)$ by Σ_τ. We have*

(1) $\Psi_0(0, \cdot) = \mathrm{Id}_\Sigma(\cdot)$ on Σ.

(2) If $\tilde\Sigma$ is a compact smooth CES in $\Psi\big((-\varepsilon, \varepsilon) \times \Sigma\big)$ and $\tilde\Sigma$ can be expressed as a graph of $w \in C^\infty(\Sigma)$ in normal coordinates around Σ, then $\tilde\Sigma = \Sigma_{\tilde\tau}$ where

$$\tilde\tau = \int_\Sigma w. \tag{3.3.26}$$

3.3.4 Comparison Theorem

Theorem 3.19. *Suppose $\Sigma_0 \subset \partial\Omega$ is a compact, smooth, strictly stable MOTS. Let u be a capillary blowdown limit and let v be the local smooth solution constructed on the annular region $\Psi\big([0, T_+], \Sigma_0\big)$ in Proposition 3.17. Then $u \le v$ in $\Psi\big([0, T_+], \Sigma_0\big)$.*

Proof. Let $\mathcal{A}$ denote the annular region $\Psi\big((0, T_+), \Sigma_0\big)$. Suppose the statement is not true, that is, $u - v > 0$ at some point in $\mathcal{A}$. Since $u - v$ is continuous and $\overline{\mathcal{A}}$ is compact, there exists $x_0 \in \overline{\mathcal{A}}$ such that $(u - v)(x_0) = \max_{\overline{\mathcal{A}}}(u - v) > 0$.

Suppose $x_0 \in \mathcal{A}$ is an interior point. Let $\rho(x) = d(x, x_0)$. Note that $u - v - \rho^4$ has a strict maximum in $\mathcal{A}$ at x_0. Since u_{s_j} converges to u uniformly on $\overline{\mathcal{A}}$, there exists $x_j \in \overline{\mathcal{A}}$ such that $u_{s_j} - v - \rho^4$ has a local maximum at x_j and $x_j \to x_0$. For all sufficiently large j, we have $x_j \in \mathcal{A}$. Since u_{s_j} and v are twice differentiable, the derivative test at x_j shows that

$$\nabla u_{s_j}(x_j) = \nabla v(x_j) + \nabla \rho^4(x_j), \quad \nabla^2 u_{s_j}(x_j) \le \nabla^2 v(x_j) + \nabla^2 \rho^4(x_j).$$

In view of regularized equation (3.3.1), at x_j we have

$$\begin{aligned}
0 = &-\nabla^2 u_{s_j}\big(I - \frac{\nabla u_{s_j}}{\sqrt{s_j^2 + |\nabla u_{s_j}|^2}} \otimes \frac{\nabla u_{s_j}}{\sqrt{s_j^2 + |\nabla u_{s_j}|^2}}\big) \\
&+ \sqrt{s^2 + |\nabla u_{s_j}|^2}\Big\{u_{s_j} + k\big(I - \frac{\nabla u_{s_j}}{\sqrt{s_j^2 + |\nabla u_{s_j}|^2}} \otimes \frac{\nabla u_{s_j}}{\sqrt{s_j^2 + |\nabla u_{s_j}|^2}}\big)\Big\} \\
\ge &-\nabla^2 v_j\big(I - \frac{\nabla u_{s_j}}{\sqrt{s_j^2 + |\nabla u_{s_j}|^2}} \otimes \frac{\nabla u_{s_j}}{\sqrt{s_j^2 + |\nabla u_{s_j}|^2}}\big) \\
&- \nabla^2 \rho^4\big(I - \frac{\nabla u_{s_j}}{\sqrt{s_j^2 + |\nabla u_{s_j}|^2}} \otimes \frac{\nabla u_{s_j}}{\sqrt{s_j^2 + |\nabla u_{s_j}|^2}}\big) \\
&+ \sqrt{s^2 + |\nabla u_{s_j}|^2}\Big\{u_{s_j} + k\big(I - \frac{\nabla u_{s_j}}{\sqrt{s_j^2 + |\nabla u_{s_j}|^2}} \otimes \frac{\nabla u_{s_j}}{\sqrt{s_j^2 + |\nabla u_{s_j}|^2}}\big)\Big\}.
\end{aligned}$$

We remark that

$$\lim_{j\to\infty} \nabla\rho^4(x_j) = \nabla\rho^4(x_0) = 0,$$

$$\lim_{j\to\infty} \nabla^2\rho^4(x_j) = \nabla^2\rho^4(x_0) = 0,$$

$$\lim_{j\to\infty} u_{s_j}(x_j) = u(x_0).$$

Moreover, by Proposition 3.17 we know $v(x_0) \neq 0$ for $x_0 \in \mathcal{A}$ and hence

$$\lim_{j\to\infty} \frac{\nabla u_{s_j}(x_j)}{\sqrt{s_j^2 + |\nabla u_{s_j}(x_j)|^2}} = \frac{\nabla v(x_0)}{|\nabla v(x_0)|}.$$

Therefore, by letting $j \to \infty$ we obtain at x_0

$$\begin{aligned} 0 &\geq -\nabla^2 v(x_0)\big(I - \frac{\nabla v(x_0)}{|\nabla v(x_0)|} \otimes \frac{\nabla v(x_0)}{|\nabla v(x_0)|}\big) \\ &\quad + |\nabla v(x_0)|\{u(x_0) + k\big(I - \frac{\nabla v(x_0)}{|\nabla v(x_0)|} \otimes \frac{\nabla v(x_0)}{|\nabla v(x_0)|}\big)\} \\ &= |\nabla v|\{u(x_0) - v(x_0)\} \end{aligned}$$

where the last equality follows from the fact that v satisfies equation (3.3.3). We then conclude that

$$0 < |\nabla v(x_0)|\{u(x_0) - v(x_0)\} \leq 0$$

which is a contradiction.

Next suppose $x_0 \in \partial\mathcal{A}$. Note that $u = v = 0$ on $\partial\Omega$, so $x_0 \in v^{-1}(T_+)$. We claim that $\mathcal{A} \subset E^-_{u(x_0)}(u)$ and therefore $x_0 \in \partial E^-_{u(x_0)}(u)$. To confirm this, we observe that if $x \in \mathcal{A}$,

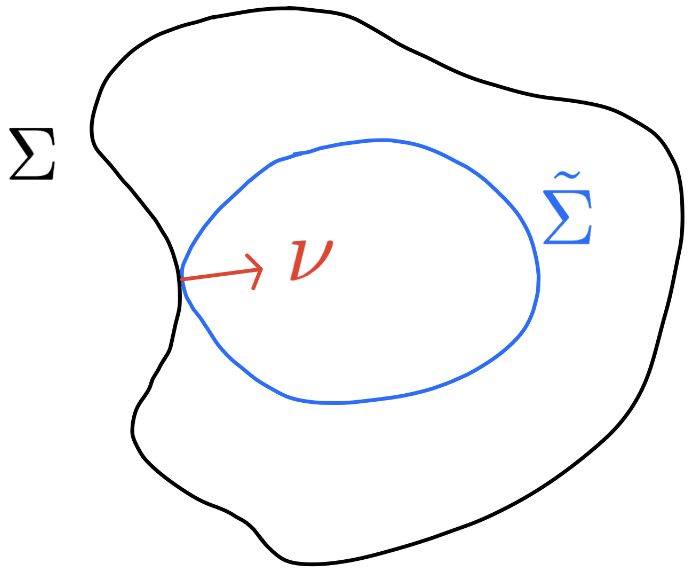

Figure 3.2: The situation excluded by maximum principle where Σ and $\tilde{\Sigma}$ are CESs such that $\theta[\tilde{\Sigma}] > \theta[\Sigma]$ with respect to the common normal vector ν at the contact point.

then $v(x) < T_+$ and by maximality of $u - v$

$$u(x) = u(x) - v(x) + v(x) \leq u(x_0) - v(x_0) + v(x) < u(x_0).$$

By Theorem 3.7, there exists a closed smooth properly embedded surface $\tilde{\Sigma}$ in $u^{-1}(u(x_0))$ passing through x_0 having constant expansion $u(x_0)$ with respect to the unit normal vector pointing outside of $E^-_{u(x_0)}(u)$. In addition, the above clam implies that $\tilde{\Sigma} \subset u^{-1}(u(x_0)) \subset \Omega\backslash\mathcal{A}$. Therefore, $\tilde{\Sigma}$ is enclosed by Σ_{T_+} and two surfaces contact each other at x_0. Since the chosen unit normal vectors $\nu_{\Sigma_{T_+}}$ and $\nu_{\tilde{\Sigma}}$ of these two surfaces agree at x_0 and $\tilde{\Sigma}$ lies on the $+\nu_{\Sigma_{T_+}}$-side of Σ_{T_+}, the maximum principle implies that $\mathrm{H}[\tilde{\Sigma}](x_0) \leq \mathrm{H}[\Sigma_{T_+}](x_0)$. Consequently, we conclude that

$$u(x_0) = \theta[\tilde{\Sigma}](x_0) \leq \theta[\Sigma_{T_+}](x_0) = v(x_0)$$

which contradicts the assumption that $u(x_0) > v(x_0)$.

□

3.4 Structure of Black Hole Regions and Capillary Blow-down Limit

3.4.1 Thin Maximal Domains

For any open subset S of M, we can define the **thickness** of S by

$$\tau(S) = \sup\{\mathrm{diam} B : B \text{ is an open geodesic ball in } S\}.$$

Proposition 3.20. *There exists a constant $R_0 = R_0(M, g, k) > 0$ satisfying the following property. Let $\Theta \in [-\mu_1, \mu_1]$ and let f be a smooth solution to constant expansion equation $\theta[f] = \Theta$ on maximal domain U in (M, g, k). If U is* ***thin*** *in the sense that $\tau(U) < R$, then f has no critical point. Moreover, U is homeomorphic to $f^{-1}(0) \times \mathbb{R}$ with exactly two boundary components $\partial_- U =$ "$f^{-1}(-\infty)$" and $\partial_+ U =$ "$f^{-1}(\infty)$" which are closed smooth CES with expansion Θ.*

Proof. Suppose $x_0 \in U$ is a critical point of f. For $x \in U$, we define

$$\beta(x) := \langle \nu_f, \partial t \rangle|_{(x, f(x))} = (1 + |\nabla f(x)|^2)^{-1/2}.$$

Thus, $\beta(x_0) = 1$. By Harnack-type inequality in Proposition 1.10 (3),

$$c_4 \geq |d \log \beta|^2_{\tilde{g}} = |d \log \beta|^2_g - \frac{\langle df, d \log \beta \rangle^2_g}{1 + |\nabla f|^2_g} \geq \beta^2 |d \log \beta|^2_g = |d\beta|^2_g,$$

where $\tilde{g} = g + df \otimes df$ is the induced metric on the graph of f. Let $\gamma : [0, \ell] \to U$ be a geodesic emitting from $\gamma(0) = x_0$. Then for $s \in [0, \ell]$,

$$\begin{aligned}|\beta(\gamma(s)) - \beta(\gamma(0))| &= \Big| \int_0^s \frac{d}{d\tau}\beta(\gamma(\tau))\, d\tau \Big| \\ &\leq \int_0^s \big|\langle \nabla\beta(\gamma(\tau)), \gamma'(\tau)\rangle_g\big|\, d\tau \\ &\leq c_4 \int_0^s d\tau = c_4 s.\end{aligned}$$

If $\ell < c_4^{-1}$, then $|\beta(\gamma(s)) - 1| < 1$ for any $s \in [0, \ell]$. It follows that $\beta(x) > 0$ for any point x in the closed geodesic ball $\overline{B}_\ell(x_0)$, and hence ∇f is bounded in the geodesic ball $\overline{B}_\ell(x_0)$. The domain U is maximal, so U contains the closed geodesic ball $\overline{B}_\ell(x_0)$ for any $\ell < c_4^{-1}$. Therefore, $\tau(U) \geq 2c_4^{-1}$. Take $R_0 = 2c_4^{-1}$, then we will get a contradiction.

The second statement follows immediately from Morse theory and Theorem 3.6. □

The following proposition states that any thin maximal domain contains a (part of) marginally stable CES.

Proposition 3.21. *Let f, U with $\tau(U) < R_0$ be assumed as in in Proposition 3.20. Let ν and ν' be unit normal vector field on $\partial_- U$ and $\partial_+ U$ respectively chosen as in Theorem 3.6. Suppose $\partial_- U$ and $\partial_+ U$ are stable. There exists $R > 0$ depending on the geometry of ∂U in (M, g, k) such that if $\tau(U) \leq R$, then there exist closed smooth marginally stable CESs $\tilde{\Sigma}_1$ on the $+\nu$-side of $\partial_- U$ and $\tilde{\Sigma}_2$ on the $-\nu'$-side of $\partial_+ U$ such that $\Sigma_i \cap U \neq \emptyset$ for both $i = 1, 2$.*

Proof. By Proposition 3.20, ∂U has exactly two component $\partial_- U =$ "$f^{-1}(-\infty)$" and $\partial_+ U =$ "$f^{-1}(\infty)$" with the same constant expansion Θ. Let ν and ν' be unit normal vector field on $\partial_- U$ and $\partial_+ U$ respectively chosen as in Theorem 3.6. Thus, $\partial_+ U$ is on the $+\nu$-side of $\partial_- U$. We simply denote $\partial_- U$ by Σ. There exists ρ_0 depending on the geometry of Σ in (M, g) such that the normal coordinates $\Upsilon : \Sigma \times (-\rho_0, \rho_0) \to M : (x, \sigma) \to \exp_x(\sigma\nu(x))$ is bijective. For $w \in C^{2,\alpha}(\Sigma)$ with $\|w\|_0 < \rho_0$, let $\mathfrak{Graph}(w)$ denote the graph of w in normal coordinates adapted to Σ and let $\theta[w]$ denote the expansion of $\mathfrak{Graph}(w)$ with respect to $\partial_\sigma^\perp/|\partial_\sigma^\perp|$ where $\partial_\sigma^\perp$ is the projection of ∂_σ onto the normal bundle of $\mathfrak{Graph}(w)$. By the nature of linearization operator $\mathcal{L}_\Sigma$, we define the deviation Q of $\theta[w]$ from its linear approximation around Σ:

$$Q[w] := \big(\theta[w] - \Theta\big) - \mathcal{L}_\Sigma w. \tag{3.4.1}$$

The quadratic term Q depends also on ∇w and $\nabla^2 w$. Here the notation $Q[w]$ is treated as a functional. There exist constants R_0, A depending on geometry of Σ in (M, g) and k such that $0 < \rho_1 < \rho_0$ and for $\|w\|_{2,\alpha} \leq \rho_1$

$$\|Q[w]\|_{0,\alpha} \leq A\|w\|_{2,\alpha}^2. \tag{3.4.2}$$

The standard Schauder estimates applied to (3.4.1) implies that there exists a constant C depending on geometry of Σ in (M, g) and k such that

$$\|w\|_{2,\alpha} \leq C\big(\|w\|_0 + \|\theta[w] - \Theta\|_{0,\alpha} + \|Q[w]\|_{0,\alpha}\big). \tag{3.4.3}$$

Put (3.4.2) into (3.4.3), we obtain

$$\|w\|_{2,\alpha} \leq C\big(\|w\|_0 + \|\theta[w] - \Theta\|_{0,\alpha}\big) + AC\|w\|_{2,\alpha}^2.$$

Let $\delta \in (0, 1)$. If $\|w\|_{2,\alpha} \leq \delta A^{-1}C^{-1}$, then we have estimates

$$\|w\|_{2,\alpha} \leq (1 - \delta)^{-1}C\big(\|w\|_0 + \|\theta[w] - \Theta\|_{0,\alpha}\big), \tag{3.4.4}$$

and

$$\|Q[w]\|_{0,\alpha} \leq \eta\big(\|w\|_0 + \|\theta[w] - \Theta\|_{0,\alpha}\big) \tag{3.4.5}$$

where $\eta := \frac{\delta C}{1-\delta} \in (0,1)$ if δ is chosen small enough. Take $0 < \rho_2 < \rho_1$ such that

$$\rho_2 < \delta(1-\delta)A^{-1}C^{-2}. \tag{3.4.6}$$

Then (3.4.4) and (3.4.5) hold true as long as $\|w\|_0 + \|\theta[w] - \Theta\|_{0,\alpha} \le \rho_2$ by using continuity argument along the family $\{sw\}_{0\le s\le 1}$. In particular, suppose $\partial_+ U$ can be expressed as the graph of $v > 0$ with $\|v\|_0 \le \rho_2$, then $\theta[v] = \Theta$ implies

$$\mathcal{L}_\Sigma v = -Q[v]. \tag{3.4.7}$$

In this case, (3.4.4) and (3.4.5) can reduced to

$$\|v\|_{2,\alpha} \le (1-\delta)^{-1} C\|v\|_0, \tag{3.4.8}$$

and

$$\|Q[v]\|_{0,\alpha} \le \eta\|v\|_0. \tag{3.4.9}$$

Now since Σ is strictly stable, by Corollary 3.16 there exists a maximal foliation Ψ of CES initiated from Σ toward the $+\nu$-side. Let τ be a small positive number, and let $w_\tau > 0$ represent the sheet $\Psi(\Theta+\tau, \Sigma)$ in the maximal foliation Ψ satisfying $\theta[w_\tau] = \Theta + \tau$. Then w_τ satisfies

$$\mathcal{L}_\Sigma w_\tau = \tau - Q[w_\tau]. \tag{3.4.10}$$

Since w_τ and v are both positive, there exists a number $a > 0$ and a point $z \in \Sigma$ such that

$$av \le w_\tau \quad \text{and the equality holds at } z. \tag{3.4.11}$$

By derivative tests,

$$
\begin{aligned}
0 \geq \mathcal{L}_\Sigma(w - av)(z) &= \tau - Q[w_\tau](z) + aQ[v](z) \\
&\geq \tau - \eta(\|w_\tau\|_0 + \tau) - a\eta\|v\|_0 \quad \text{by (3.4.5) and (3.4.9)} \\
&\geq (1-\eta)\tau - 2\eta\|w_\tau\|_0 \quad \text{by (3.4.11)}
\end{aligned}
$$

This implies that

$$\tau \leq 2\eta(1-\eta)^{-1}\|w_\tau\|_0. \tag{3.4.12}$$

Again by continuity argument, (3.4.12) holds true so long as $\|w_\tau\|_0 \leq \{1+2\eta(1-\eta)^{-1}\}^{-1}\rho_2 := \rho_3$. In this case, the Schauder estimates (3.4.4) can be further reduced to

$$\|w_\tau\|_{2,\alpha} \leq (1-\delta)^{-1}\{1 + 2\eta(1-\eta)^{-1}\}C \cdot \|w_\tau\|_0. \tag{3.4.13}$$

Combining (3.4.5) and (3.4.12), we have

$$\|\mathcal{L}_\Sigma w_\tau\|_0 \leq 3\eta\|w\|_0. \tag{3.4.14}$$

Then the Harnack inequality applied to (3.4.10) implies that there exists a constant Λ depending on geometry of Σ in (M, g) and k such that

$$\|w_\tau\|_0 \leq \Lambda\big(\min w_\tau + \|\mathcal{L}_\Sigma w_\tau\|_0\big) \leq \Lambda \min w_\tau + 3\eta\Lambda\|w_\tau\|_0. \tag{3.4.15}$$

If δ is chosen small enough (and so is η) such that $3\eta\Lambda < 1$, then

$$\min w_\tau \geq (1 - 3\eta\Lambda)\Lambda^{-1}\|w_\tau\|_0. \tag{3.4.16}$$

Set $\rho_4 := \frac{1}{2}(1-3\eta\Lambda)\Lambda^{-1}\rho_3$. From (3.4.13), we find the sheet $\Psi(\Theta+\tau, \Sigma)$ is $C^{2,\alpha}$ if $\|w_\tau\|_0 \leq \rho_3$. If the sheets remain stable, then the foliation Ψ would continue and by (3.4.16) sweep the

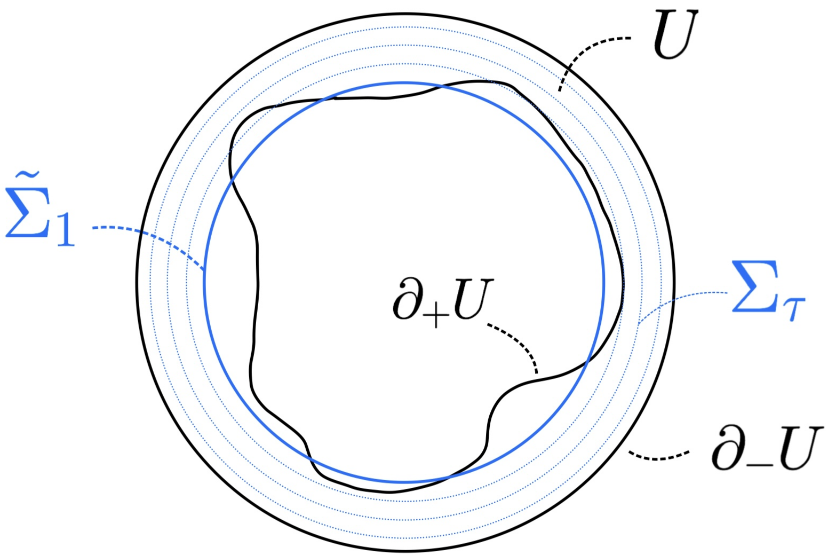

Figure 3.3: Thin maximal domain U containing parts of marginally stable CES $\tilde{\Sigma}_1$. If $\partial_- U$ is stable, then the foliation of stable CESs, Σ_τ, coming out of $\partial_- U$ must terminate before fully immerging in $\partial_+ U$.

region $[0, \rho_4] \times \Sigma$ in normal coordinates. On the other hand, the Harnack inequality applied to (3.4.7) implies that

$$\|v\|_0 \leq \Lambda\big(\min v + \|\mathcal{L}_\Sigma v\|_0\big) \leq \Lambda \min v + \eta\Lambda\|v\|_0. \tag{3.4.17}$$

Thus,

$$\|v\|_0 \leq (1 - \eta\Lambda)^{-1}\Lambda \min v. \tag{3.4.18}$$

We set $R := (1 - \eta\Lambda)\Lambda^{-1}\rho_4$. If $\tau(U) \leq R$, then $\min v \leq \tau(U)$ and (3.4.18) implies that $\partial_+ U \subset (0, \rho_4] \times \Sigma$. It follows that there exists a sheet $\Sigma_\tau := \Psi(\Theta + \tau, \Sigma)$ in the foliation Ψ for some positive number τ such that Σ_τ lies on the $+\nu'$-side of $\partial_+ U$ and contacts $\partial_+ U$ at a point, say p. By maximal principle,

$$\mathrm{H}[\Sigma_\tau](p) \geq \mathrm{H}[\partial_+ U](p). \tag{3.4.19}$$

But this contradicts the fact that $\theta[\partial_+ U] = \Theta < \Theta + \tau = \theta[\Sigma_\tau]$. This means that the foliation Ψ towards the $+\nu$-side of Σ must terminate at a marginally stable CES $\tilde{\Sigma}_1$ which has nonempty intersection with U. See Figure 3.3. Analogously, the maximal foliation Ψ' of CES initiated from $\partial_+ U$ towards the $-\nu'$-side must terminate at a marginally stable CES $\tilde{\Sigma}_2$ which has nonempty intersection with U. □

3.4.2 Structure Theorem

In the subsection, we will investigate the structure of component Ω of Ω_+. We begin by considering $D = \{r_k\}_{k=1}^{\infty} \subset \overline{\Omega}$ a dense countable subset. We will apply Theorem 3.6 multiple times without explicitly mentioning it throughout this subsection. Use diagonal argument and relabeling index j, we may assume for all $r_k \in D$ the sequence $\mathrm{Graph}(\tilde{f}_{s_j}^{(r_k)})$ converges in C^∞_{loc} to either a maximal graph or a cylinder over a closed smooth surface. We then decompose $\mathbb{N}$ as $A \sqcup B$ where

$$k \in A : \tilde{f}_{s_j}^{(r_k)} \text{ converges in } C^\infty_{loc} \text{ to } \tilde{f}_0^{(r_k)} \text{ on maximal domain } U_{r_k} \subset u^{-1}(u(r_k)) \cap \Omega,$$
$$k \in B : \mathrm{Graph}(\tilde{f}_{s_j}^{(r_k)}) \text{ converges in } C^\infty_{loc} \text{ to a cylinder over } \Sigma_{r_k} \subset u^{-1}(u(r_k)) \cap \Omega.$$

Lemma 3.22. *$\{U_{r_k}\}_{k\in A}$ and $\{\Sigma_{r_\ell}\}_{\ell \in B}$ satisfy avoidance property. More precisely,*

(1) For $k \in A$ and $\ell \in B$, $U_{r_k} \cap \Sigma_{r_\ell} = \emptyset$;

(2) If $U_{r_k} \cap U_{r_\ell} \neq \emptyset$ for $k, \ell \in A$, then $U_{r_k} = U_{r_\ell}$;

(3) If $\Sigma_{r_k} \cap \Sigma_{r_\ell} \neq \emptyset$ for $k, \ell \in B$, then $\Sigma_{r_k} = \Sigma_{r_\ell}$.

Proof. To prove (1), suppose $p \in U_{r_k} \cap \Sigma_{r_\ell}$. By Theorem 3.6, as $p \in U_{r_k}$

$$\lim_{j\to\infty} |\nabla f_{s_j}(p)| < +\infty$$

and as $p \in \Sigma_{r_\ell}$

$$\lim_{j \to \infty} |\nabla f_{s_j}(p)| = +\infty.$$

It leads to a contradiction.

To prove (2), suppose $p \in U_{r_k} \cap U_{r_\ell}$. By definition, we have the conversion identity

$$\tilde{f}_{s_j}^{(r_k)}(x) - \tilde{f}_{s_j}^{(r_\ell)}(x) = \tilde{f}_{s_j}^{(r_k)}(r_\ell) \tag{3.4.20}$$

for all $j, k, \ell \in \mathbb{N}$ and $x \in M$. It follows that by letting $s_j \to 0^+$

$$\tilde{f}_0^{(r_k)}(r_\ell) = \tilde{f}_0^{(r_k)}(p) - \tilde{f}_0^{(r_\ell)}(p),$$

and thus

$$\tilde{f}_0^{(r_k)}(x) - \tilde{f}_0^{(r_\ell)}(x) = \tilde{f}_0^{(r_k)}(r_\ell)$$

for all $x \in U_{r_k} \cup U_{r_\ell}$. This means that $\tilde{f}_0^{(r_k)}, \tilde{f}_0^{(r_\ell)}$ only differ by a constant. By Theorem 3.6, $u \equiv u(p)$ in $U_k \cup U_{r_\ell}$ and $\tilde{f}_0^{(r_k)}, \tilde{f}_0^{(r_\ell)}$ are both solutions to $\theta[f] = u(p)$. Since U_{r_k} and U_{r_ℓ} are maximal domains in the sense that solutions blow up on the boundary, we immediately have $U_{r_k} = U_{r_\ell}$.

To show (3), suppose $p \in \Sigma_{r_k} \cap \Sigma_{r_\ell}$. We first claim that Σ_{r_k} and Σ_{r_ℓ} contact at p but can not cross each other. By Theorem 3.6, $\lim_{j\to\infty} \frac{D_{s_j}(p)}{\sqrt{1+|D_{s_j}(p)|^2}}$ is the common unit normal to Σ_{r_k} and Σ_{r_ℓ} along which the expansions are both $u(p)$. Suppose Σ_{r_k} crosses Σ_{r_ℓ}, then there are points $q_\pm$ in $\mathcal{N}^{\pm}_{\delta_{r_k}}(\Sigma_{r_k}, \nu_{\Sigma_{r_k}}) \cap \mathcal{N}^{\mp}_{\delta_{r_\ell}}(\Sigma_{r_\ell}, \nu_{\Sigma_{r_\ell}})$ respectively. It follows from Theorem 3.6 and the conversion identity (3.4.20) at $q_\pm$ that $\lim_{j\to\infty} \tilde{f}_{s_j}^{(r_\ell)}(r_k)$ is both $+\infty$ and $-\infty$, which is a contradiction. Therefore, Σ_{r_k} and Σ_{r_ℓ} contact each other from one side and both have constant expansion $u(p)$. By strong maximum principle, we find $\Sigma_{r_k} = \Sigma_{r_\ell}$. □

Theorem 3.23 (Structure Theorem)**.** *Assume that any compact subset of M contains only*

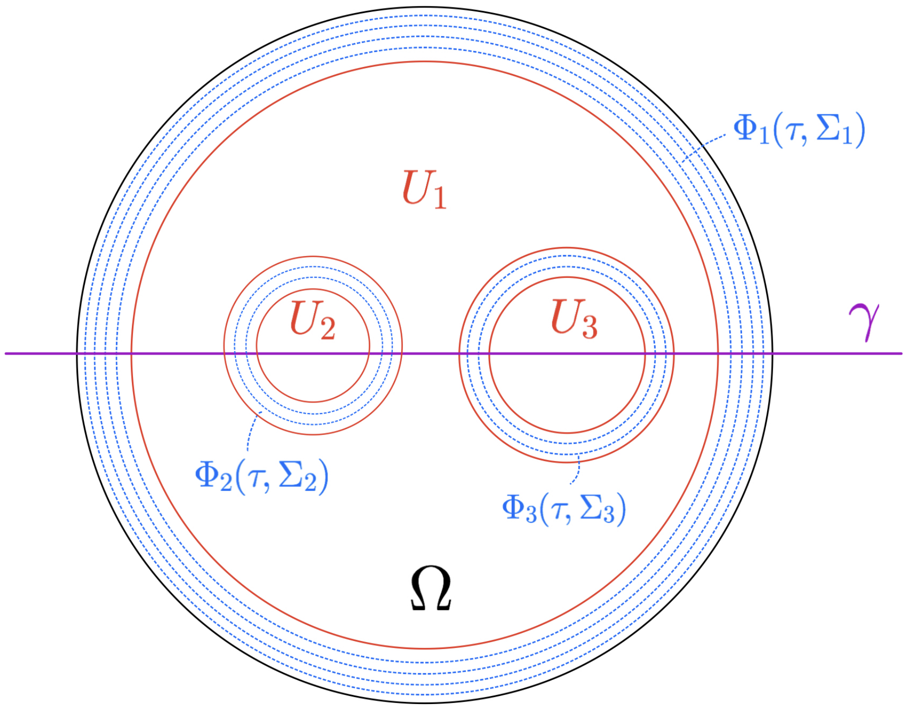

Figure 3.4: Structure of black hole region. The region Ω enclosed by the outermost black circle represents a component of black hole regions Ω_+; The regions U_i's enclosed by red curves represent maximal domains; The blue dotted curves $\Phi_i(\tau,\Sigma_i)$'s represent foliations of CESs; The purple line γ represents a curve crossing Ω. The nontrivial topology occurs in the maximal domain U_1.

a finite number of marginally stable CESs. Let u be a capillary blowndown limit of f_s and let $\Omega\subset\Omega_+$ be a component of black hole region, say $f_{s_j}\to+\infty$ on Ω and $s_jf_{s_j}\to u$ uniformly on Ω. Then there exists a partition

$$\overline{\Omega}=\left(\bigcup_{m=1}^{N_1}U_m\right)\cup\left(\bigcup_{n=1}^{N_2}\Phi_n([0,b_n]\times\Sigma_n)\right)$$

where $1\le N_1,N_2<\infty$, U_m is a maximal domain of a solution to constant expansion equation $\theta[f]=u(U_m)$, and $\Phi_n:[0,b_n]\times\Sigma_n\to M$ is a smooth foliation of closed CES with $\theta[\Phi(\cdot,\Sigma_n)]=u|_{\Phi(\cdot,\Sigma_n)}$ with $b_n\ge 0$ (if $b_n=0$ the foliation degenerates to one sheet of CES). See Figure 3.4 and Figure 3.5.

Proof of Theorem 3.23. For simplicity, we identify and then relabel the objects in $\{U_{r_k}\}$ as

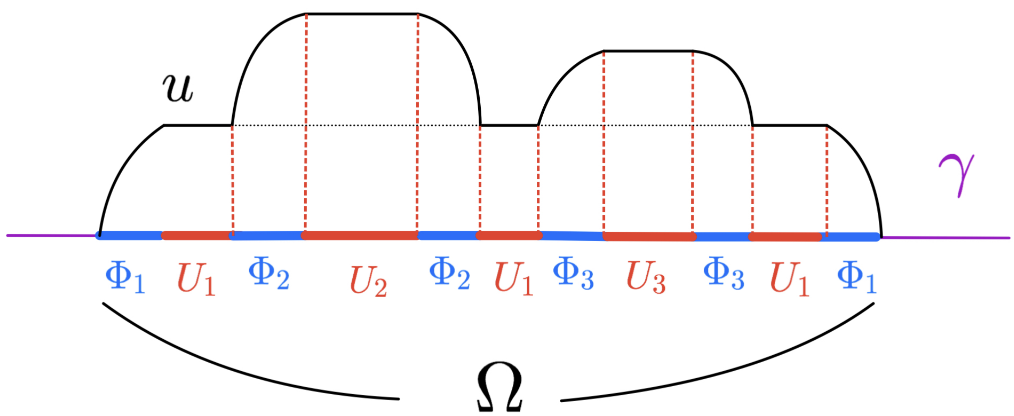

Figure 3.5: Profile of a capillary blowdown limit u over the curve γ in Figure 3.4. The black curve represents the capillary blowdown limit u, which vanishes outside of Ω, changes monotonically along foliations Φ_i's, and stagnates in maximal domains U_i's.

$\{U_m\}_{m=1}^{N_1}$ such that $U_m \cap U_n = \emptyset$ if $m \neq n$.

We prove that $1 \leq N_1 < \infty$. Theorem 3.2 implies that $N_1 \geq 1$. Suppose $N_1 = \infty$. By compactness of $\overline{\Omega}$, avoidance property of U_m and local estimates in Proposition 1.10, there exists a subsequence $\{U_{m'}\}$ and an accumulation CES Σ^* such that $\tau(U_{m'}) < R_0$ as in Proposition 3.20 and boundary components $\partial_\pm U_{m_k}$ converge to Σ from one side smoothly as $m \to \infty$. Proposition 3.21 gives the constant R depending only on the geometry of Σ^* in (M, g) and k. For large enough m', components $\partial_\pm U_{m'}$ can be written as graphs over Σ^* with very small sup-norm, say less than R. This means that $\tau(U_{m'}) \leq R$. By the virtue of estimate (3.4.13), we may assume R is also applicable to $\partial_\pm U_{m'}$ for sufficiently large m'. By finiteness of the number of marginally stable CES in Ω, components of $\partial_\pm U_{m'}$ are strictly stable except finitely many. By Proposition 3.21, for every sufficiently large m' there exists a closed smooth marginally stable CES $\tilde{\Sigma}_{m'}$ which lies between Σ^* and the further boundary component of $U_{m'}$ such that $U_{m'} \cap \tilde{\Sigma}_{m'} \neq \emptyset$. These CES $\Sigma_{m'}$ are distinct because of avoidance property of $\{U_{m'}\}$ and $\tilde{\Sigma}_{m'}$'s relative position to $U_{m'}$ and Σ^*. This means that there are infinitely many distinct closed smooth marginally stable CES in $\overline{\Omega}$, a contradiction to our assumption. Thus, $N_1 < \infty$. As a consequence, we have rather simple topological relations: $\mathrm{Int}(\overline{\Omega} - \bigcup_{m=1}^{N_1} U_m) = \Omega - \bigcup_{m=1}^{N_1} \overline{U}_m$ and $\partial(\overline{\Omega} - \bigcup_{m=1}^{N_1} U_m) = \bigcup_{m=1}^{N_1} \partial U_m \cup \partial\Omega$.

By Proposition 3.8, for every $r_k \in \Omega - \bigcup_{m=1}^{N_1} \overline{U}_m$ where $k \in B$, Σ_{r_k} is a stable with expansion $u(r_k)$. Then we can use Proposition 3.15 for strictly stable CES or 3.18 for marginally stable CES to construct a unique local foliation of CESs around Σ_{r_k}. Since $D \cap (\Omega - \bigcup_{m=1}^{N_1} \overline{U}_m)$ is a dense subset in $\Omega - \bigcup_{m=1}^{N_1} \overline{U}_m$, we conclude that each (open) connected component of $\Omega - \bigcup_{m=1}^{N_1} \overline{U}_m$ is a foliation of CESs. All such foliations can be uniquely and smoothly extended to CES in $\bigcup_{m=1}^{N_1} \partial U_m \cup \partial\Omega$ by connectness of Ω. There may also exist some isolated CESs which are either the common boundaries of two adjacent maximal domains or components of $\partial\Omega$. These isolated CES can be expressed as degenerate foliations. Therefore, the total number of components of foliations of CES is bounded by the number of components of $\bigcup_{m=1}^{N_1} \partial U_m \cup \partial\Omega$. □

Remark 3.24. (1) Without the assumption of finite marginally stable CES in compact sets, there may be infinitely many disjoint maximal domains. This would greatly increase the complexity of the topology of the black hole region and capillary blowdown limit.

(2) Despite the fact that we have $u \in C^{0,1}$ from the construction in Section 3, u is generally not C^1.

Corollary 3.25. *Under the assumption of Theorem 3.23, there exists a sequence $s_j \to 0^+$ such that the sequence of graphs of translated functions $\tilde{f}_{s_j}^{(x_0)}$ converges to a smooth submanifold in an open neighborhood of $(x_0, 0) \in M \times \mathbb{R}$ for* every *$x_0 \in \overline{\Omega}$.*

Proof. We will check that the subsequence s_j obtained by diagonal argument satisfies the claim. Suppose $p \in U_{r_k}$ for $k \in A$, then it follows from the argument in the proof of Lemma 3.22 (ii) that $\tilde{f}_{s_j}^{(p)}$ converges to $\tilde{f}_0^{(r_k)} + C$ in U_{r_k} for some constant C.

Suppose $p \in \overline{\Omega} - \bigcup_{m=1}^{N_1} U_m$. By passing to a further subsequence $s_{j'}$, $\mathrm{Graph}(\tilde{f}_{s_{j'}}^{(p)})$ converges to either a graph on maximal domain U_p or a cylinder over a closed smooth CES Σ_p, either of which satisfies avoidance property together with $\{U_{r_k}, \Sigma_{r_\ell}\}$. Note $(\bigcup_{k\in A} U_{r_k}) \cup (\bigcup_{\ell \in B} \Sigma_{r_\ell})$ is a dense subset of $\overline{\Omega}$. Thus, the limit submanifold of $\mathrm{Graph}(\tilde{f}_{s_{j'}}^{(p)})$ must be a cylinder over a closed smooth CES Σ_p containing p. Then we may encounter two scenarios: $p \in \partial(\overline{\Omega} - \bigcup_{m=1}^{N_1} U_m)$ and $p \in \mathrm{Int}(\overline{\Omega} - \bigcup_{m=1}^{N_1} U_m)$. In the first scenario, either $p \in \partial U_m$ for some m or $p \in \partial\Omega$. By avoidance property, Σ_p lies outside of U_m but inside of $\overline{\Omega}$ and contacts ∂U_m or $\partial\Omega$ respectively at a point p. It follows from Theorem 3.6 that at the point p the unit normal vectors of Σ_p and ∂U_m or $\partial\Omega$ respectively are identical. Since Σ_p and ∂U_m or $\partial\Omega$ respectively both satisfy $\mathrm{H} - \mathrm{K} \equiv u(p)$ with a contact point, strong maximum principle implies that Σ_p is a connected component of ∂U_m or $\partial\Omega$ respectively. In the second scenario, p lies in the interior of one foliation of CESs, $\Phi_n\big((0, b_n) \times \Sigma_n\big)$, in Theorem 3.23. The avoidance property and local uniqueness of foliation around stable CES imply that Σ_p is a sheet of the foliation Φ_n. In both scenarios, we found that Σ_p is uniquely determined. Thus, we can drop the dependence of the choice of subsequence and the convergence holds true for the original sequence s_j. □

The pair (u, η) of capillary blowdown limit and its companion vector field preserve the geometric information of regularized solutions when the blowup occurs.

Corollary 3.26. *Assume that any compact subset of M contains only a finite number of marginally stable CESs. Let u be a capillary blowdown limit of regularized solutions. Then there exists a continuous, piecewise smooth vector field η on M satisfying the following properties.*

(1) $|\eta(x)| \le 1$ *for all* $x \in M$.

(2) If x lies in a maximal domain U of a solution f to constant expansion equation in the Structure Theorem 3.23 or Ω_0 associated with Jang's equation, then η is the horizontal

projection of Gauss map on Graph(f)*:*

$$\eta(x) = \frac{\nabla f(x)}{\sqrt{1 + |\nabla f(x)|^2}}.$$

(3) If x lies in a foliation of CESs in the Structure Theorem 3.23, then $\eta(x)$ is the unit normal to the CES which contains x.

(4) The pair (u, η) satisfies the equation

$$\mathrm{div}_M(\eta) - \mathrm{tr}_g k + k(\eta, \eta) = u \quad \textit{in } M. \tag{3.4.21}$$

Proof. We know from Proposition 1.11 that both Ω_+ and Ω_- have only finitely many connected components (black hole regions). Applying Corollary 3.25 to all black hole regions, there exists a decreasing subsequence $s_{j'} \to 0$ such that

$$\eta(x) := \lim_{j' \to \infty} \frac{\nabla f_{s_{j'}}(x)}{\sqrt{1 + |\nabla f_{s_{j'}}(x)|^2}} \quad \text{exists for all } x \in \overline{\Omega}_+ \cup \overline{\Omega}_-.$$

Proposition 1.11 implies that the above limit $\eta(x)$ also exists for $x \in \Omega_0$ with the same subsequence $s_{j'}$ and

$$\eta(x) = \frac{\nabla f_0(x)}{\sqrt{1 + |\nabla f_0(x)|^2}} \tag{3.4.22}$$

where f_0 is the solution to Jang's equation in Proposition 1.11. Claim (1) is clear. Claim (2) and claim (3) follow from Proposition 3.6. Therefore, η is continuous everywhere, and smooth except across boundaries of maximal domains and $\partial\Omega_0$. Claim (4) records the fact that the solution f in (2) and CES in (3) satisfy constant expansion equation $\theta = u(U)$. □

Corollary 3.27 (Volume estimate for black hole regions)**.** *Let*

$$I = I(M, g) = \inf \frac{A_g(\partial R)^{\frac{3}{2}}}{V_g(R)}$$

be the isoperimetric constant of (M, g) where A and V are area and volume measure with respect to g, and R is any bounded domain whose boundary is nice enough to define area. Suppose Ω is a component of Ω_- or Ω_+. Then we have the volume estimate for Ω:

$$V(\Omega) \geq I^2 \|k\|_{0;\Omega}^{-3} \tag{3.4.23}$$

and the area estimate for $\partial\Omega$:

$$A(\partial\Omega) \geq I^2 \|k\|_{0;\Omega}^{-2}. \tag{3.4.24}$$

Proof. Suppose $\Omega \subset \Omega_+$ is a connected component. Integrating (3.4.21) over Ω and use divergence theorem,

$$A(\partial\Omega) + \int_\Omega u = -\int_{\partial\Omega} \langle \eta, \nu \rangle + \int_\Omega u = -\int_\Omega \left\{ \mathrm{tr}k - k(\eta, \eta) \right\}.$$

Since $u \geq 0$, by isoperimetric inequality we have

$$I^{\frac{2}{3}} A(\Omega)^{\frac{2}{3}} \leq A(\partial\Omega) \leq V(\Omega) \|k\|_{0;\Omega}.$$

Thus,

$$V(\Omega) \geq I^2 \|k\|_{0;\Omega}^{-3}$$

and

$$A(\partial\Omega) \geq I^2 \|k\|_{0;\Omega}^{-2}. \tag{3.4.25}$$

If Ω is a connected component of Ω_-, then we have

$$-A(\partial\Omega) + \int_\Omega u = -\int_\Omega \left\{ \mathrm{tr}k - k(\eta, \eta) \right\}$$

where $u \leq 0$ in Ω. Thus, we conclude the same result as for Ω_+. □

3.5 Trival Capillary Blowdown Limit

This section contributes to the discussion of a very special blowup phenomenon. Given a blowup sequence of regularized solutions, if the speed of caps escaping to infinity is much slower than the contractive rescaling factor s, then it is likely that the rescaled sequence ends up with the trivial capillary blowdown limit, which is identically zero. There is no obvious evidence to exclude this possibility. Nevertheless, the trivial capillary blowdown limit is rigid and gives a topological restriction on black hole regions.

3.5.1 Rigidity of Trivial Capillary Blowdown Limit

In general, the uniqueness of capillary blowdown limits on a given black hole region is not clear. Whereas the trivial capillary blowdown limit has the following rigidity property.

Proposition 3.28. *If there exists a sequence $s_j \to 0^+$ such that*

$$\lim_{j\to\infty} \sup_{x\in M} |u_{s_j}(x)| = 0,$$

then

$$\lim_{s\to 0^+} \sup_{x\in M} |u_s(x)| = 0.$$

The proof is based on the monotonicity property of $\max_M |u_s|$. To show this, we need the following gap estimate:

Lemma 3.29 (Estimate of gap)**.** *Suppose $0 < t < s$ and suppose f_s, f_t are solutions to (1.4.2) and converge to 0 at each infinite end. Denote $u_s = sf_s$ and $u_t = tf_t$.*

(1) If $\min\{\max_M u_s, \max_M u_t\} > 0$, then $\sup_M(f_t - f_s) \le \frac{s-t}{st} \min\{\max_M u_s, \max_M u_t\}$.

(2) If $\max\{\min_M u_s, \min_M u_t\} < 0$, *then* $\frac{s-t}{st}\max\{\min_M u_s, \min_M u_t\} < \inf_M (f_t - f_s)(x)$.

Proof. We may assume $\sup_M(f_t - f_s) > 0$; otherwise, there is nothing to prove. Since $f_t - f_s$ is smooth and decays to zero near infinity, there is $x_0 \in M$ such that $(f_t - f_s)(x_0) = \max_M(f_t - f_s)$. By derivative test, we have

$$\nabla(f_t - f_s)(x_0) = 0, \quad \nabla^2(f_t - f_s) \leq 0.$$

By subtracting regularized equations (1.4.2) associated with s and t, we obtain

$$0 \geq \Big(g^{ij} - \frac{f_s^i f_s^j}{1 + |\nabla f_s|^2}\Big) \frac{\nabla_i \nabla_j (f_t - f_s)}{\sqrt{1 + |\nabla f_s|^2}}(x_0) = t f_t(x_0) - s f_s(x_0). \tag{3.5.1}$$

There are two ways to split the difference. Firstly, we have

$$\begin{aligned} 0 &\geq t f_t(x_0) - s f_s(x_0) \\ &= t f_t(x_0) - t f_s(x_0) + t f_s(x_0) - s f_s(x_0) \\ &= t\big(f_t(x_0) - f_s(x_0)\big) + (t - s) f_s(x_0). \end{aligned}$$

Thus,

$$f_t(x_0) - f_s(x_0) \leq \frac{s-t}{t} f_s(x_0) \leq \frac{s-t}{t} \max f_s = \frac{s-t}{st} \max u_s. \tag{3.5.2}$$

Secondly, we have

$$\begin{aligned} 0 &\geq t f_t(x_0) - s f_s(x_0) \\ &= t f_t(x_0) - s f_t(x_0) + s f_t(x_0) - s f_s(x_0) \\ &= (t - s) f_t(x_0) + s\big(f_t(x_0) - f_s(x_0)\big). \end{aligned}$$

Thus,

$$f_t(x_0) - f_s(x_0) \leq \frac{s-t}{s} f_t(x_0) \leq \frac{s-t}{s} \max f_t = \frac{s-t}{st} \max u_t. \tag{3.5.3}$$

Therefore, $f_t(x) - f_s(x) \leq f_t(x_0) - f_s(x_0) \leq \frac{s-t}{st} \min\{\max u_s, \max u_t\}$.

The result (2) follows analogously. ☐

Corollary 3.30. *Suppose* $0 < t < s$ *and suppose* f_s, f_t *are solutions to (1.4.2) and converge to 0 at each infinite end. Denote* $u_s = sf_s$ *and* $u_t = tf_t$. *Then*

(1) If $\max_M u_s > 0$, *then* $\max_M u_t \leq \max_M u_s$.

(2) If $\min_M u_s < 0$, *then* $\min_M u_s \leq \min_M u_t$.

Proof. Suppose u_t achieves its maximum at $\overline{x}$. We may also assume that $\max_M u_t > 0$; otherwise, there is nothing to prove. Then Lemma 3.29 implies that

$$\begin{aligned} \max_M u_t &= tf_t(\overline{x}) = t(f_t(\overline{x}) - f_s(\overline{x})) + tf_s(\overline{x}) \\ &\leq \frac{s-t}{s} \max u_s + \frac{t}{s} u_s(\overline{x}) \\ &\leq \max_M u_s. \end{aligned}$$

The result (2) follows analogously. ☐

Proof of Proposition 3.28. It follows from Corollary 3.30 that $\sup_M |u_s|$ is increasing in s. Therefore, $\sup_M |u_s|$ converges to zero as $s \to 0^+$ if one sequence does. ☐

3.5.2 Topology of Black Hole Regions with Trivial Capillary Blowdown Limit

The main theorem of this subsection asserts that when the dominant energy condition holds strictly for the initial data set, if a capillary blowdown limit of f_s is trivial in some black hole region Ω, then Ω has rather simple topology.

Theorem 3.31. *Suppose the dominant energy condition holds strictly, i.e.,* $\mu - |J|_g > 0$. *Let* u *be a capillary blowdown limit of* f_s *and* Ω *be a connected component of* Ω_+ *or* Ω_- *with boundary components* $\Sigma_1, \ldots, \Sigma_l$. *Suppose* $u = 0$ *in* Ω. *Then the compactification* $\Omega \cup \{P_1, \ldots, P_l\}$ *by adding a point to each boundary component is homeophorphic to a connected sum of finite number of spherical space forms* S^3/Γ *and* $S^2 \times S^1$.

We begin with the model case where the entire black hole region $\Omega = U$ is one maximal domain of a solution f to Jang's equation.

Proposition 3.32. *Let* U *be a bounded maximal domain of solution* f *to Jang's equation with boundary components* $\{\Sigma_1, \ldots, \Sigma_l\}$. *Suppose the dominant energy condition holds strictly, i.e.,* $\mu - |J|_g \geq \delta$ *for some* $\delta > 0$. *Then every boundary component of* U *is a 2-sphere and the compactification* $U \cup \{P_1, \ldots, P_l\}$ *by adding a point to each boundary component is homeomorphic to a smooth manifold of positive Yamabe type, i.e., the manifold admits a metric such that the scalar curvature is positive (cf. Proposition A.2).*

Remark 3.33. The claim that every boundary component of U is a 2-sphere will follow from the same argument of Proposition 1.13.

To construct a compact smooth manifold out of U, we need the following gluing lemma to cap off the openings ∂U by topological half 3-spheres. Note that the function u in the conformal factor here no longer represents a blowdown limit.

Lemma 3.34. *Suppose (Σ, γ_*) is a 2-dimensional compact manifold with or without boundary. Let $\gamma_s(x) = e^{2w(x,s)}\gamma_*(x)$ for $s \in (a,b)$ be a smooth path in the conformal class of γ_*. Suppose for each $s \in (a,b)$ the first (Neumann if Σ has boundary) eigenvalue of the 2-dimensional conformal Laplacian $\lambda_1(-\Delta_{\gamma_s} + \kappa(\gamma_s)) \geq \lambda_*$ for some $\lambda_* > 0$ where $\kappa(\gamma_s)$ is the Gaussian curvature of Σ with respect to γ_s. Suppose w satisfies the boundary condition*

$$\frac{d}{ds}\Big|_{s=a^+} e^{2w(s,x)} = \frac{d}{ds}\Big|_{s=b^-} e^{2w(x,s)} = 0 \quad \text{for all } x \in \Sigma, \tag{3.5.4}$$

and

$$\sup_{s\in(a,b),x\in\Sigma} \left|\frac{d^2}{ds^2}e^{2w}\right| < \infty.$$

Then the first Neumann eigenvalue of the 3-dimensional conformal Laplacian $\lambda_1(-\Delta_g + \frac{1}{8}\mathrm{R}_g)$ on the cylinder $\mathcal{C} := \Sigma\times(a,b)$ equipped with the warped product $g(x,s) = \gamma_s(x)+ds^2$ is positive ($\geq \frac{\lambda_*}{4} > 0$).

Proof. Let $i_s : \Sigma \hookrightarrow \Sigma \times (a,b)$ denote the inclusion map $i_s(x) = (x,s)$ for $s \in (a,b)$ and $x \in \Sigma$. Then

$$\begin{aligned}\int_{\mathcal{C}} |d\phi|_g^2 dV_g &= \int_a^b \int_\Sigma \Big\{|i_s^* d\phi|^2_{\gamma_s} + |\phi'|^2\Big\} dA_{\gamma_s} ds \\ &= \int_a^b \int_\Sigma |i_s^* d\phi|^2_{\gamma_s} dA_{\gamma_s} ds + \int_a^b \int_\Sigma |\phi'|^2 e^{2w} dA_{\gamma_*} ds\end{aligned}$$

where $\cdot'$ denotes $\frac{d}{ds}\cdot$. By direct computation (see Proposition A.3), the scalar curvature $\mathrm{R}(g)$ of the warped product metric is

$$\mathrm{R}(g) = 2\kappa(\gamma_s) - 4(w'') - 6(w')^2.$$

Let $\phi \in C^1(\mathcal{C})$ be bounded. Then

$$\begin{aligned}
&\frac{1}{8}\int_{\mathcal{C}} \mathrm{R}(g)\phi^2 dV_g \\
&= \frac{1}{4}\int_a^b \int_\Sigma \kappa(\gamma_s)\phi^2 dA_{\gamma_s} ds + \int_a^b \int_\Sigma \Big\{ -\frac{1}{2}w''\phi^2 e^{2w} - \frac{3}{4}(w')^2\phi^2 e^{2w}\Big\} dA_{\gamma_*} ds \\
&= \frac{1}{4}\int_a^b \int_\Sigma \kappa(\gamma_s)\phi^2 dA_{\gamma_s} ds + \int_a^b \int_\Sigma \Big\{ w'\phi\phi' e^{2w} + \frac{1}{4}(w')^2\phi^2 e^{2w}\Big\} dA_{\gamma_*} ds.
\end{aligned}$$

In the last equality, we integrate the second term by parts and use the boundary condition (3.5.4). Putting above computations together gives

$$\begin{aligned}
&\int_{\mathcal{C}} \Big\{ |d\phi|_g^2 + \frac{1}{8}\mathrm{R}(g)\phi^2 \Big\} dV_g \\
&\geq \frac{1}{4}\int_a^b \int_\Sigma \Big\{ |i_s^* d\phi|_{\gamma_s}^2 + \kappa(\gamma_s)\phi^2 \Big\} dA_{\gamma_s} ds + \int_a^b \int_\Sigma e^{-\frac{w}{2}} [(\phi e^{\frac{w}{2}})']^2 dA_{\gamma_s} ds \\
&\geq \frac{1}{4}\int_a^b \lambda_1(-\Delta_{\gamma_s} + \kappa(\gamma_s)) \int_\Sigma \phi^2 dA_{\gamma_s} ds \\
&\geq \frac{\lambda_*}{4}\int_{\mathcal{C}} \phi^2 dV_g.
\end{aligned}$$

Consequently, $\lambda_1(-\Delta_g + \frac{1}{8}\mathrm{R}(g)) \geq \frac{\lambda_*}{4} > 0$. □

Remark 3.35. The boundary condition (3.5.4) is weaker than the condition that $\lim_{s\to a^+} \mathrm{H}[i_s(\Sigma)] = \lim_{s\to b^-} \mathrm{H}[i_s(\Sigma)] = 0$ where $H[i_s(\Sigma)]$ is the mean curvature of $i_s(\Sigma)$ in $\mathcal{C}$ with respect to $\frac{\partial}{\partial s}$. In the following application of this lemma, we will take the cylinder $S^2 \times (0, b)$ to be a flat punctured 3-ball in spherical coordinate near the origin. In this case, $w(x, s) = \log s$ near $s = 0$ and the mean curvature of sphere actually blows up near the origin, whereas $\frac{d}{ds}e^{2w}$ converges to zero near the origin.

Proof of Proposition 3.32. Let $G = \mathrm{Graph}(f, U) \subset (M \times \mathbb{R}, g + dt^2)$ endowed with induced metric $\overline{g} = g + df \otimes df$. Observe that vertical translations generate a Jacobi vector field whose normal component is

$$\beta := \langle -\partial t, \nu_G \rangle = (1 + |\nabla f|^2_g)^{-\frac{1}{2}}.$$

Using identities $\mathcal{L}_G \beta = 0$ and (1.3.11) we find

$$2(\mu - J(\nu)) + |h-k|^2_{\overline{g}} = -2\mathrm{div}_G(\xi + \overline{\nabla}\log\beta) - 2|\xi + \overline{\nabla}\log\beta|^2_{\overline{g}} + \mathrm{R}(\overline{g}) \tag{3.5.5}$$

where $\xi = \big(k(\nu,\cdot)^\sharp\big)^\top$. Choose $t_0 > 0$ sufficiently large to be determined. Let $\phi \in C^1(G)$. Multiplying (3.5.5) by ϕ^2, integrating by parts and using the pointwise Cauchy–Schwartz inequality

$$2\langle X, \overline{\nabla}\phi\rangle_{\overline{g}}\phi - |X|^2_{\overline{g}}\phi^2 \le 2|X|_{\overline{g}}\,|d\phi|_{\overline{g}}|\phi| - |X|^2_{\overline{g}}\phi^2 \le |d\phi|^2_{\overline{g}},$$

we find

$$\begin{aligned}\int_{G\cap\big(M\times(-t_0,t_0)\big)} 2(\mu - J(\nu))\phi^2 dV_{\overline{g}} \le 2\int_{G\cap(\{\pm t_0\}\times M)} &\phi^2|\langle \xi + \overline{\nabla}\log\beta, \eta_\pm\rangle_{\overline{g}}|dA_{\overline{g}} \\ + \int_{G\cap\big(M\times(-t_0,t_0)\big)} &2|d\phi|^2_{\overline{g}} + \mathrm{R}(\overline{g})\phi^2 dV_{\overline{g}}\end{aligned} \tag{3.5.6}$$

where $\eta_\pm = \pm\frac{\nabla f + |\nabla f|^2\partial_t}{|\nabla f|\sqrt{1+|\nabla f|^2}}$ is the conormal on the section $G\cap(M\times\{\pm t_0\})$ pointing out of $G\cap\big(M\times(-t_0,t_0)\big)$ and $dA_{\overline{g}}$ is the area element induced by $\overline{g}|_{G\cap(M\times\{\pm t_0\})}$. Translating G vertically as in Proposition 1.11 (2), G has infinite ends that are $C^{2,\alpha}$-asymptotic to $(\partial U\times\mathbb{R})$. Therefore, $G\cap(M\times\{\pm t_0\})$ converges uniformly to ∂U as $t_0 \to +\infty$. Then the trace theorem implies that there exists constants C, $T > 0$ depending only on geometry of ∂U such that for all $t_0 > T$

$$\int_{G\cap\big(M\times\{\pm t_0\}\big)} \phi^2 dA_{\overline{g}} \le C\int_{G\cap\big(M\times(-t_0,t_0)\big)} |d\phi|^2_{\overline{g}} + \phi^2 dV_{\overline{g}}.$$

Since $\lim_{t_0\to\infty}\eta_\pm = \pm\partial_t$ and $|\overline{\nabla}\log\beta| \le c_4$ in Proposition 1.10, we have

$$\lim_{t_0\to\infty}\langle \xi + \overline{\nabla}\log\beta, \eta_\pm\rangle_{\overline{g}} = k(\nu_{\partial U}, \pm\partial_t) \pm \partial_t \log\beta = 0.$$

We also perturb G to exact cylinders $\partial U \times \mathbb{R}$ with a new metric $\tilde{g} = g|_{\partial U} + dt^2$ for $t_0 - 1 \le |t| \le t_0$ and keep $\tilde{g} = \overline{g}$ for $|t| \le t_0 - 2$. By choosing $t_0 > 0$ large enough, the error term due to perturbation and the boundary integral in (3.5.6) are bounded by ε times $W^{1,2}$-norm of ϕ on $G \cap \big(M \times (-t_0, t_0)\big)$ for a very small $\varepsilon > 0$. By using the strong dominant energy condition $\mu - |J|_g \ge \delta$, the inequality (3.5.6) implies

$$\delta \int_{G\cap\big(M\times(-t_0,t_0)\big)} \phi^2 dV_{\tilde{g}} \le \int_{G\cap\big(M\times(-t_0,t_0)\big)} 3|d\phi|^2_{\tilde{g}} + \mathrm{R}(\tilde{g})\phi^2 dV_{\tilde{g}}. \tag{3.5.7}$$

Let $\Sigma_i \subset \partial U$ be a connected component and let $\gamma^{(i)} = g\big|_{\Sigma_i}$. By Proposition 1.14, we know that Σ_i is a closed stable apparent horizon. Following the same computation for (3.5.6) without the presence of boundary integral (since Σ_i is closed), we have for any $\xi \in C^1(\Sigma_i)$,

$$\delta \int_{\Sigma_i} \xi^2 dA_{\gamma^{(i)}} \le \int_{\Sigma_i} (\mu - J(\nu))\xi^2 dA_{\gamma^{(i)}} \le \int_{\Sigma_i} |d\xi|^2_{\gamma^{(i)}} dA_{\gamma^{(i)}} + \kappa(\gamma^{(i)})\xi^2 dA_{\gamma^{(i)}}. \tag{3.5.8}$$

It follows that the first eigenvalue of the 2-dimensional conformal Laplacian on $(\Sigma_i, \gamma^{(i)})$ is positive. Taking $\xi \equiv 1$, we find

$$0 < \int_{\Sigma_i} \kappa(\gamma^{(i)}) dV_{\gamma^{(i)}}.$$

By Gauss-Bonnet theorem, Σ_i is homeomorphic to S^2.

Next we will fill up the opening of $G \cap \big(M \times (-t_0, t_0)\big)$ by gluing a 3-ball to obtain a closed manifold homeomorphic to $U \cup \{P_1, \dots, P_l\}$ using the trick of path of conformal metrics in [30]. Recall that each Σ_i is homeomorphic to S^2. By abuse of notation, we will identify Σ_i as

S^2 equipped with metric $\gamma^{(i)}$ in the following discussion. By uniformization theorem, there exists $w_i \in C^\infty(S^2)$ such that $\gamma^{(i)} = e^{2w_i}\gamma_*$ where γ_* is the standard round metric on S^2. Let $\eta(s)$ and $a(s)$ be smooth functions on $(0,3)$ such that $0 \leq \eta(s) \leq 1$ for all $s \in (0,3)$,

$$\eta(s) = \begin{cases} 0 & , \quad \text{if } s \in (0,1], \\ 1 & , \quad \text{if } s \in [2,3), \end{cases}$$

and $a(s) \leq 0$ for all $s \in (0,3)$,

$$a(s) = \begin{cases} \log s & , \quad \text{if } s \in (0,\frac{1}{2}], \\ 0 & , \quad \text{if } s \in [2,3). \end{cases}$$

Set

$$\gamma_s^{(i)}(x) = e^{2\eta(s)w_i(x)+2a(s)}\gamma_*(x)$$

so that $\gamma_s^{(i)} = \gamma^{(i)}$ for $s \in [2,3)$. Then the cylinder $\mathcal{C}_i := S^2 \times (0,3)$ equipped with the warped product $\gamma_s^{(i)} + ds^2$ coincides with flat punctured 3-ball in spherical coordinates for $s \in (0,\frac{1}{2})$, and coincides with $(G \cap \big(\Sigma_i \times (-t_0,t_0)\big), \tilde{g})$ for $s \in (2,3)$ with the orientation ∂_s pointing into $G \cap \big(M \times (-t_0,t_0)\big)$. In such a way, we can patch up the opening by gluing a 3-ball $\mathcal{C}_i \cup P_i$ where P_i is the origin in spherical coordinates. We repeat the surgery at other cylindrical ends and then we obtain a new smooth closed manifold $(\hat{M},\hat{g})$ which is homeomorphic to $G \cup \{P_1,\ldots,P_l\} \cong U \cup \{P_1,\ldots,P_l\}$.

To complete the proof, we need to show that $(\hat{M},\hat{g})$ is of the positive Yamabe type. It suffices to show that $\lambda_1(-\Delta_{\hat{g}} + \frac{1}{8}\mathrm{R}(\hat{g}))$ is positive, since this implies that there exists a smooth positive eigenfunction u of $-\Delta_{\hat{g}} + \frac{1}{8}\mathrm{R}(\hat{g})$ on $\hat{M}$ such that

$$\mathrm{R}(u^4\hat{g}) = 8u^{-5}(-\Delta_{\hat{g}}u + \frac{1}{8}\mathrm{R}(\hat{g})u) = 8u^{-4}\lambda_1(-\Delta_{\hat{g}} + \frac{1}{8}\mathrm{R}(\hat{g})) > 0,$$

and therefore $\hat{M}$ admits a metric $u^4\hat{g}$ with positive scalar curvature. Let $\phi \in C^1(\hat{M})$. The relevant bilinear form can be split into the sum of integrals on several portions:

$$\int_{\hat{M}} |d\phi|^2_{\hat{g}} + \frac{1}{8}\mathrm{R}(\hat{g})\phi^2 dV_{\hat{g}} = \sum_{i=1}^{\ell} \int_{\mathcal{C}_i \cup P_i} + \int_{\hat{M} \setminus \bigcup_j (\mathcal{C}_j \cup P_j)} |d\phi|^2_{\hat{g}} + \frac{1}{8}\mathrm{R}(\hat{g})\phi^2 dV_{\hat{g}}.$$

For each integral on $\mathcal{C}_i \cup P_i$, we will use Lemma 3.34 to get a positive lower bound. By definition of $\gamma_s^{(i)}$, it is clear that the conditions for Lemma 3.34, $\frac{d}{ds}\big|_{s=0^+} e^{2(\eta w_i + a)} = \frac{d}{ds}\big|_{s=3^-} e^{2(\eta w_i + a)} = 0$ and $\sup_{C_i} \left|\frac{d^2}{ds^2} e^{2(\eta w_i + a)}\right| < \infty$ hold true. By Proposition A.1, for any $\varphi \in C^\infty(S^2)$ the Gaussian curvature of conformal metric $e^{2\varphi}\gamma_*$ on S^2 is given by

$$\kappa(e^{2\varphi}\gamma_*) = e^{-2\varphi}\Big(\kappa(\gamma_*) - \Delta_{\gamma_*}\varphi\Big). \tag{3.5.9}$$

Let $\xi \in C^\infty(S^2)$. Using (3.5.9), for $s \in (0,3)$ the bilinear form related to the 2-dimensional conformal Laplacian on $(S^2, \gamma_s^{(i)})$ can be rewritten as

$$\begin{aligned}
&\int_{S^2} |d\xi|^2_{\gamma_s^{(i)}} + \kappa(\gamma_s^{(i)})\xi^2 dA_{\gamma_s^{(i)}} \\
&= \int_{S^2} |d\xi|^2_{\gamma_*} + \big\{\kappa(\gamma_*) - \Delta_{\gamma_*}\big(\eta(s)w_i(x) + a(s)\big)\big\}\xi^2 dA_{\gamma_*} \\
&= \int_{S^2} |d\xi|^2_{\gamma_*} + \big(1 - \eta(s)\Delta_{\gamma_*} w_i\big)\xi^2 dA_{\gamma_*} \\
&= \eta(s)\int_{S^2} \Big\{|d\xi|^2_{\gamma_*} + \big(1 - \Delta_{\gamma_*} w_i(x)\big)\xi^2\Big\} dA_{\gamma_*} + (1-\eta(s))\int_{S^2} \Big\{|d\xi|^2_{\gamma_*} + \xi^2\Big\} dA_{\gamma_*} \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

To estimate I, we use (3.5.9) and (3.5.8) to obtain

$$\begin{aligned}
\mathrm{I} &= \eta(s)\int_{S^2} \Big\{|d\xi|^2_{\gamma_*} + \big(1 - \Delta_{\gamma_*} w_i(x)\big)\xi^2\Big\} dA_{\gamma^{(i)}} \\
&= \eta(s)\int_{S^2} \Big\{|d\xi|^2_{\gamma^{(i)}} + \kappa(\gamma^{(i)})\xi^2\Big\} dA_{\gamma^{(i)}} \\
&\geq \eta(s)\delta \int_{S^2} \xi^2 dA_{\gamma^{(i)}} \\
&\geq \eta(s)\delta \inf_{\mathcal{C}_i} e^{2(1-\eta)w_i - 2a} \int_{S^2} \xi^2 dA_{\gamma_s^{(i)}}.
\end{aligned}$$

To estimate II, we use the fact that $\lambda_1(-\Delta_{\gamma_*}) = 2$ to obtain

$$\begin{aligned}
&(1-\eta(s))\int_{S^2}\Big\{|d\xi|^2_{\gamma_*}+\xi^2\Big\}dA_{\gamma_*}\\
&\geq 3(1-\eta(s))\int_{S^2}\xi^2 dA_{\gamma_*}\\
&\geq 3(1-\eta(s))\inf_{\mathcal{C}_i} e^{-2\eta w_i-2a}\int_{S^2}\xi^2 dA_{\gamma_s^{(i)}}.
\end{aligned}$$

Then we can conclude that

$$\int_{S^2}|d\xi|^2_{\gamma_s^{(i)}}+\kappa(\gamma_s^{(i)})\xi^2 dA_{\gamma_s^{(i)}} \geq \big(\eta(s)\delta\inf_{\mathcal{C}_i} e^{2(1-\eta)w_i-2a}+3(1-\eta)\inf_{\mathcal{C}_i} e^{-2\eta w_i-2a}\big)\int_{S^2}\xi^2 dA_{\gamma_s^{(i)}}.$$

Since $a\leq 0$ and $0\leq\eta\leq 1$, the coefficient of the integral on the right is positive for all $s\in(0,3)$. It follows that there exists $\lambda_*>0$ such that $\lambda_1(-\Delta_{\gamma_s^{(i)}}+\kappa(\gamma_s^{(i)}))\geq\lambda_*$ for all $s\in(0,3)$. We repeat the argument on all $\mathcal{C}_i\cup P_i$'s and we may assume λ_* is a lower bound of $\lambda_1(-\Delta_{\gamma_s^{(i)}}+\kappa(\gamma_s^{(i)}))$ for all $\mathcal{C}_i\cup P_i$'s. Using Lemma 3.34, we conclude that

$$\sum_{i=1}^{\ell}\int_{\mathcal{C}_i\cup P_i}|d\phi|^2_{\hat g}+\frac{1}{8}\mathrm{R}(\hat g)\phi^2 dV_{\hat g}\geq\frac{\lambda_*}{4}\sum_{i=1}^{\ell}\int_{\mathcal{C}_i\cup P_i}\phi^2 dV_{\hat g}.$$

From (3.5.7), we find

$$\int_{\hat M\setminus\bigcup_j(\mathcal{C}_j\cup P_j)}|d\phi|^2_{\hat g}+\frac{1}{8}\mathrm{R}(\hat g)\phi^2 dV_{\hat g}\geq\frac{1}{8}\int_{\hat M\setminus\bigcup_j(\mathcal{C}_j\cup P_j)}3|d\phi|^2_{\hat g}+\mathrm{R}(\hat g)\phi^2 dV_{\hat g}\geq\frac{1}{8}\delta\int_{\hat M\setminus\bigcup_j(\mathcal{C}_j\cup P_j)}\phi^2 dV_{\hat g}.$$

Putting all together, we conclude that there exists $\alpha=\alpha(\lambda_*,\delta)>0$ such that

$$\int_{\hat M}|d\phi|^2_{\hat g}+\frac{1}{8}\mathrm{R}(\hat g)\phi^2 dV_{\hat g}\geq\alpha\int_{\hat M}\phi^2 dV_{\hat g}.$$

$\square$

The following topological classification theorem of connected, orientable, closed, Yamabe-positive 3-manifolds is the final component of our proof. This theorem is a byproduct of Perelman's proof of the geometrization theorem, together with the early classification results of Schoen–Yau [38] and Gromov–Lawson [20]. The assertion can be found in the survey paper [11, Theorem 2.1].

Proposition 3.36 (Gromov–Lawson, Schoen–Yau)**.** *Let X^3 be a connected, orientable, compact manifold without boundary with positive Yamabe type. Then X is homeomorphic to a connected sum of finite number of spherical space forms S^3/Γ, where Γ is a finite subgroup of $SO(4)$ acting freely on S^3, and $S^2 \times S^1$.*

Now we are ready to combine Proposition 3.36 for the special case, the Classification Theorem 3.36 together with the Structure Theorem 3.23 of black hole regions to prove Theorem 3.31.

Proof of Theorem 3.31. Without the assumption that there are only finitely many closed smooth marginally stable CES in compact sets in (M, g, k), the Structure Theorem 3.23 implies that

$$\overline{\Omega} = \left(\bigcup_{m=1}^{N_1} U_m \right) \cup \left(\bigcup_{n=1}^{N_2} \Phi_n([0, b_n] \times \Sigma_n) \right)$$

where $1 \leq N_1, N_2 \leq \infty$, U_m is a maximal domain of solution to Jang's equation for all m and Φ_n is a smooth foliation of closed MOTS or MITS for all n. There may be infinitely many maximal domains U_m's. But since the black hole region Ω is bounded, all except finitely many U_m's are *thin* as defined in Proposition 3.20. Proposition 3.20 implies that each *thin* U_m is homeomorphic to a cylinder over its boundary component and Proposition 3.32 implies that the boundary components of *thin* U_m are 2-spheres. Therefore, all *thin* U_m's are homeomorphic to round cylinder $S^2 \times \mathbb{R}$ and contribute nothing to the topological structure of entire connected sum.

Every boundary component of $\Phi_n([0, b_n] \times \Sigma_n)$ is a connected component of ∂U_m or $\partial \Omega$ which is a 2-sphere by Proposition 3.32 and Remark 3.33. Thus, each foliation $\Phi_n([0,b_n]\times\Sigma_n)$ is homeomorphic to a round cylinder $[0,b_n]\times S^2$ (which may degenerate to $\{0\}\times S^2$).

The main contributions to the topological structure of the black hole region come from finitely many *thick* maximal domains. Combining Proposition 3.32 and Proposition 3.36, the compactification of each *thick* maximal domain U_m by adding a point to each boundary component is homeomorhpic to a connected sum of finite number of spherical space forms S^3/Γ and $S^2\times S^1$. On the other hand, we may view thin maximal domains and foliations as cylindrical necks connecting finitely many *thick* maximal domains in the entire connected sum. Consequently, the compactification $\Omega\cup\{P_1,\dots,P_l\}$ is homeomorphic to a connected sum of finite number of spherical space forms S^3/Γ and $S^2\times S^1$. □

Corollary 3.37. *Suppose the dominant energy condition holds strictly, i.e.,* $\mu>|J|$*. Let* u *be a capillary blowdown limit of* f_s *and* $\Omega\subset\Omega_+$ *with boundary components* $\Sigma_1,\dots,\Sigma_l$*. If the compactification* $\Omega\cup\{P_1,\dots,P_l\}$ *by adding a point to each boundary component is not homeophorphic to a connected sum of finite number of spherical space forms* S^3/G *and* $S^2\times S^1$*, then* u *is not trivial in* Ω*.*

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Appendix A

Deformation of Scalar Curvature

Proposition A.1 (Conformal transformation of scalar curvature, [41] Chapter 5)**.** *Let (M^n, g) be a smooth Riemannian manifold with dimension $n \geq 2$. If $n = 2$, then for any smooth function u*

$$\mathrm{R}(e^{2u}g) = e^{-2u}\big(\mathrm{R}(g) - 2\Delta_g u\big),$$

or equivalently, using the Gauss curvature $\kappa = 2\mathrm{R}$,

$$\kappa(e^{2u}g) = e^{-2u}\big(\kappa(g) - \Delta_g u\big);$$

if $n > 2$, then for any positive smooth function u

$$\mathrm{R}(u^{\frac{4}{n-2}}g) = c(n)^{-1}u^{-\frac{n+2}{n-2}}L_g u,$$

*where $L_g := -\Delta_g + c(n)\mathrm{R}(g)$ is call the **conformal Laplacian** and $c(n) = \frac{n-2}{4(n-1)}$.*

Let M^n be a compact manifold with $n \geq 3$ equipped with a background metric g_0. Define the conformal class of g_0 by

$$[g_0] = \left\{ g = e^{2f} g_0 : f \in C^\infty(M) \right\}.$$

The Yamabe invariant of this conformal class is defined to be

$$\mathcal{Y}([g_0]) = \inf \left\{ \int_M \mathrm{R}(g)\, dV_g : g \in [g_0], V(M,g) = 1 \right\}.$$

Using the conformal Laplacian, for a conformal metric $g = u^{\frac{4}{n-2}} g_0$ with $u > 0$

$$\int_M \mathrm{R}_g\, dV_g = c(n)^{-1} \int_M |\nabla_{g_0} u|^2 + c(n)\mathrm{R}(g_0) u^2\, dV_{g_0}.$$

It follows from the variational characterization of the first eigenvalue of the Schrödinger operator, L_{g_0}, that we have the following trichotomy theorem.

Proposition A.2 (Trichotomy theorem, cf. [43]). *Let (M^3, g_0) be a closed, compact, smooth Riemannian manifold with $n \geq 3$. Then the conformal class of g_0 belongs to one of the following three classes:*

(1) $\mathcal{Y}([g_0]) > 0 \Longleftrightarrow \exists g \in [g_0], \mathrm{R}(g) > 0 \Longleftrightarrow \lambda_1(L_{g_0}) > 0.$

(2) $\mathcal{Y}([g_0]) = 0 \Longleftrightarrow \exists g \in [g_0], \mathrm{R}(g) = 0 \Longleftrightarrow \lambda_1(L_{g_0}) = 0.$

(3) $\mathcal{Y}([g_0]) < 0 \Longleftrightarrow \exists g \in [g_0], \mathrm{R}(g) < 0 \Longleftrightarrow \lambda_1(L_{g_0}) < 0.$

Lemma A.3. *Suppose (Σ, γ) is a 2-dimensional smooth manifold. Let $\mathcal{C} := \Sigma \times (a, b)$ and let w be a smooth function on $\mathcal{C}$. Consider the warped product $g(x,s) = e^{2w(x,s)}\gamma(x) + ds^2$ on $\mathcal{C}$. Then*

$$\mathrm{R}(g) = 2\kappa(e^{2w(x,s)}\gamma(x)) - 4(w'') - 6(w')^2,$$

where $'$ means the derivative in s, $\kappa(e^{2w(x,s)}\gamma(x))$ is the Gauss curvature of the slice $\Sigma \times \{s\}$ equipped with the conformal metric $e^{2w(x,s)}\gamma(x)$.

Proof. Let x^1, x^2 be coordinates on Σ such that $\gamma_{ij} = \delta_{ij}$ at a point, and let $x^3 = s$ be the coordinate on (a, b). Take indices $1 \leq i, j, k, \ell \leq 2$ and $1 \leq a, b, c, d \leq 3$. We only need to take extra care of the components involving x^3. We let $\overline{\mathrm{Ric}}$ and $\overline{\mathrm{R}}$ denote the geometric quantities of the slice $\Sigma \times \{s\}$ equipped with the conformal metric $e^{2w}\gamma$. Recall the the Christoffel symbol of g is given by

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd}\big(\partial_d g_{ad} + \partial_a g_{bd} - \partial_d g_{ab}\big).$$

One can compute

$$\Gamma^3_{a3} = 0, \quad \Gamma^3_{ij} = -w' e^{2w}\gamma_{ij}, \quad \Gamma^k_{33} = 0, \quad \Gamma^k_{i3} = w'\delta^k_i.$$

Recall the Riemanian curvature tensor is given by

$$\mathrm{R}^a{}_{bcd} = \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb} + \big(\Gamma^e_{db}\Gamma^a_{ce} - \Gamma^e_{cb}\Gamma^a_{de}\big).$$

Thus, one can compute

$$\begin{aligned}
\mathrm{Ric}_{ij} &= \partial_a \Gamma^a_{ji} - \partial_j \Gamma^a_{ai} + \Gamma^e_{ji}\Gamma^a_{ae} - \Gamma^e_{ai}\Gamma^a_{je} \\
&= \overline{\mathrm{Ric}}_{ij} + \partial_3 \Gamma^3_{ji} - \cancel{\partial_j \Gamma^3_{3i}} + \cancel{\Gamma^e_{ji}\Gamma^3_{3e}} - \Gamma^e_{3i}\Gamma^3_{je} + \Gamma^3_{ji}\Gamma^k_{k3} - \Gamma^3_{ki}\Gamma^k_{j3} \\
&= \overline{\mathrm{Ric}}_{ij} + \big(-w'' - 2(w')^2\big)e^{2w}\delta_{ij} \\
&\quad - (w'\delta^k_i)(-w'e^{2w}\delta_{ij}) + (-w'e^{2w}\delta_{ji})(w'\delta^k_k) - (-w'e^{2w}\delta_{ki})(w'\delta^k_j) \\
&= \overline{\mathrm{Ric}}_{ij} - (w'' + 2(w')^2)e^{2w}\delta_{ij},
\end{aligned}$$

and

$$
\begin{aligned}
\mathrm{Ric}_{33} &= \cancel{\partial_a \Gamma^a_{33}} - \partial_3 \Gamma^a_{a3} + \cancel{\Gamma^e_{33}\Gamma^a_{ae}} - \Gamma^e_{a3}\Gamma^a_{3e} \\
&= -\partial_3(w'\delta^k_k) - (-w'\delta^\ell_k)(-w'\delta^k_\ell) \\
&= -2w'' - 2(w')^2.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\mathrm{R} &= g^{ab}\mathrm{Ric}_{ab} \\
&= \overline{\mathrm{R}} - e^{-2w}\delta^{ij}(w'' + 2(w')^2)e^{2w}\delta_{ij} + \big(-2w'' - 2(w')^2 \big) \\
&= 2\kappa(e^{2w}g) - 4w'' - 6(w')^2.
\end{aligned}
$$

□