Title
Solutions to Monthly Problems 11456 and 11457

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Publication Date
2009-10-10

Peer reviewed

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October 10, 2009

The *Monthly* problem #11456 [1] asks to evaluate

$$\alpha \quad = \quad \lim_{n\to\infty} n \prod_{m=1}^{n} \left(1 - \frac{1}{m} + \frac{5}{4m^2}\right).$$

Numerical computations, using, say, $n = 10^9$, yields the numerical value $3.6898333\ldots$. Using this value as input to the Inverse Symbolic Calculator 2.0 tool (available at `http://glooscap.cs.dal.ca:8087`, one of the output results is the tantalizingly simple expression

$$\alpha \quad \stackrel{?}{=} \quad \frac{e^{\pi} + e^{-\pi}}{2\pi}.$$

Indeed, this result can be established directly by typing the *Maple* command

```
n * product (1 - 1/m + 5/(4*m^2), m = 1..n);
```

which yields the expression

$$\frac{n\Gamma\left(n + 1/2 - i\right)\Gamma\left(n + 1/2 + i\right)}{\Gamma^2(n+1)\Gamma(1/2 - i)\Gamma(1/2 + i)}.$$

After typing `limit(%,n=infinity);` this reduces to

$$\frac{1}{\Gamma(1/2 - i)\Gamma(1/2 + i)}$$

which, after `simplify(%);`, yields the final result:

$$\frac{\cosh \pi}{\pi}.$$

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The *Monthly* problem #11457 [2] asks to evaluate

$$F(a,b) \quad = \quad \int_a^b \arccos\left(\frac{x}{\sqrt{(a+b)x-ab}}\right)\,\mathrm{d}x$$

Here again, computer experimentation (using either *Mathematica* or *Maple*) yields a number of specific results:

$$\begin{aligned} F[0,b] &= b\pi/4 \qquad \text{for } b \geq 0, \\ F[1,b] &= \frac{(b-1)^2\pi}{4(b+1)} \qquad \text{for } b \geq 1, \\ F[2,b] &= \frac{(b-2)^2\pi}{4(b+2)} \qquad \text{for } b \geq 2, \\ F[3,b] &= \frac{(b-3)^2\pi}{4(b+3)} \qquad \text{for } b \geq 3, \end{aligned}$$

which quickly suggest the "obvious" answer:

$$F[a,b] \quad \stackrel{?}{=} \quad \frac{(a-b)^2\pi}{4(a+b)}$$

This result can be established directly by the *Maple* command

```
factor(int(arccos(x/sqrt((a+b)*x - a*b)), x=a..b)) assuming a>0, a<b;
```

References

[1] Raymond Mortini, "Problem 11456," *American Mathematical Monthly*, vol. 116, no. 8 (Oct 2009), pg. 747.

[2] M. L. Glasser, "Problem 11457," *American Mathematical Monthly*, vol. 116, no. 8 (Oct 2009), pg. 747.