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Solution Spaces for Linear Equations in Valued D-Fields

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Abstract

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In his 1997 thesis, Thomas Scanlon developed the model theory of a class of valued fields, which allow for the consideration of a difference field and a related differential field in the same structure. In this theory, fields are endowed with a derivative like operator D, interacting strongly with a valuation. The operator specializes to a derivative in the residue field, but in the valued field is interdefinable with a nontrivial automorphism. The theory was shown to have good model theoretic properties, most notably quantifier elimination.

We look at solutions to linear D-equations in these fields, with the goal of using the residue differential field to better understand the behavior of the difference field solutions. First, we show that the dimension of a maximal solution space to such an equation as a vector space over the constants is completely determined by the structure induced on the residue field. We then find reasonable conditions on the base field sufficient to assure uniqueness for the field extension generated by these solutions. Finally, we provide examples of automorphism groups in the theory; in particular, we show that nonlinear relations in the residue field may not lift to the valued field.

Contents

Chapter 1

Introduction

1.1 Motivation

A model complete theory of valued D-fields, called $\widetilde{\mathrm{VDF}}$, was developed by Scanlon in [18]. In this theory, the fields are endowed with a additive operator D, interacting in a natural way with the valuation. The D-operator specializes to a derivation in the residue field, but in the valued field but obeys a twisted Leibniz rule:

$$\mathrm{D}(xy) = x\,\mathrm{D}\,y + y\,\mathrm{D}\,x + \varepsilon\,\mathrm{D}\,x\,\mathrm{D}\,y$$

where ε is a fixed element of positive valuation. Such a D-operator is interdefinable with an automorphism σ of the valued field, defined by $\sigma(x) := x + \varepsilon\,\mathrm{D}\,x$. Additional axioms, notably one demanding that there are D-constants at every valuation and that an analogue of Hensel's lemma holds, assure that the theory has quantifier elimination.

This setting should allow for some information from the well understood differential fields downstairs to be lifted to the more complicated difference fields upstairs. This analysis is feasible thanks to the good model theoretic properties of $\widetilde{\mathrm{VDF}}$, in particular the aforementioned quantifier elimination. However, the theory also presents its own challenges, even in the relatively simple setting of solution spaces to linear equations. This thesis addresses some of these challenges and begins work on such an analysis.

The first difficulty arises from the fact that not all difference field extensions are compatible with the axioms for $\widetilde{\mathrm{VDF}}$. Therefore, it may not always be possible to adjoin as many solutions to a given equation as one would like, and the solution spaces to linear difference equations will often be strictly smaller than those traditionally considered. Furthermore, while it is easy to see that the axioms $\widetilde{\mathrm{VDF}}$ will have an effect on solution space size, it requires some work to see exactly what this effect will be.

In Chapter 4, we show that dimension of the solutions space to a linear D-equation in $\widetilde{\mathrm{VDF}}$ depends in a strong and systematic way on the structure induced on the residue field. With this result, what was once a complication can now be considered a feature of the theory, as it allows for further distinctions about the structure of the solutions to D-equations than are possible purely algebraically.

A second hurdle is the fact that it is not known whether $\widetilde{\mathrm{VDF}}$ has prime models. Thus, it is not immediately obvious that the extension generated by a maximal set of solutions to a linear D-equation will be unique. In Chapter 5, we will show that with reasonable assumptions on the constants of the base field, these extensions are unique up to isomorphism. We will also demonstrate the necessity of these assumptions by exhibiting an example where, without them, the desired uniqueness result fails.

These two elements allow for a meaningful model theoretic Galois theory, outlined at the end of Chapter 5. In Chapter 6, we take a closer look at both sides of this correspondence by working through several examples.

Chapter 2

Preliminaries

In this chapter, we begin with preliminaries on differential algebra, then touch briefly on extensions of these ideas to difference fields. This is followed by a review of some relevant model theory, specifically the relation of internality and the construction of type definable binding groups.

Unless otherwise indicated, all rings considered in this thesis are commutative and contain an identity element. All fields are of characteristic zero.

2.1 Differential Algebra

This section contains definitions of some standard terms from differential algebra, as they appear in [11]. The items in this section are, for the most part, the differential analogues of basic algebraic objects.

Definition 2.1.1. A *derivation* on a ring R is a map $\partial : R \to R$ satisfying, for all a, b in R:
$\partial(a+b) = \partial(a) + \partial(b)$ and
$\partial(ab) = \partial(a)b + a\partial(b)$.

The standard differential identities follow from this definition. In particular, $\partial(1) = 0$, and for $a, b \in R$ with b invertible and $n \in \mathbb{N}$, $\partial(a^n) = na^{n-1}\partial a$, and $\partial(\frac{a}{b}) = \frac{\partial(a)b - a\partial(b)}{b^2}$

Definition 2.1.2. A *differential ring* is a pair (R, ∂) where R is a ring and ∂ is a derivation on R.

Similarly, if R is a field, the pair (R, ∂) is called a *differential field*.

When it will not cause confusion, a differential ring (R, ∂) may sometimes be referred to by its underlying ring R, and for $a \in R$, $\partial(a)$ may be abbreviated by a', $\partial^2(a)$ by a'', etc.

Definition 2.1.3. The *constants* of a differential ring (R, ∂) are the elements of R in the kernel of ∂. They will be denoted C_R, so $C_R = \{c \in R : \partial(c) = 0\}$.

Equivalently, $c \in C_R$ if for all $a \in R$, $\partial(ca) = c\partial(a)$. The constants of a differential ring form a ring, and the constants of a differential field form a field.

Proposition 2.1.4. *If R is a differential ring, and $a \in R$ is algebraic over C_R, then $a \in C_R$.*

Proof. Let $P(x) = \sum_{i=0}^{n} c_i x^i$ be a minimal polynomial for a over C_R. Since $P(a) = 0$,

$$D(P(a)) = \left(\sum_{i=1}^{n} c_i(i)a^{i-1}\right)\partial(a) = 0$$

So either $\partial(a) = 0$ or $(\sum_{i=1}^{n} c_i(i)a^{i-1}) = 0$, contradicting the minimality of P. □

Definition 2.1.5. A differential ring (S, ∂_S) is a *differential extension* of (R, ∂_R) if S is an extension of R as a ring, and for all $a \in R$, $\partial_S(a) = \partial_R(a)$.

Definition 2.1.6. A *differential homomorphism* from (R, ∂_R) to (S, ∂_S) is a ring homomorphism $\phi : R \to S$ making the following diagram commute:

$$\begin{array}{ccc} R & \xrightarrow{\phi} & S \\ \downarrow{\scriptstyle \partial_R} & & \downarrow{\scriptstyle \partial_S} \\ R & \xrightarrow{\phi} & S \end{array}$$

Definition 2.1.7. A *differential ideal* I of (R, ∂) is an ideal of R such that $\partial(I) \subseteq I$.

If I is an ideal of a differential ring (R, ∂), then ∂ induces the structure of a differential ring on R/I exactly when I is a differential ideal.

Definition 2.1.8. A *simple differential ring* is a differential ring whose only differential ideals are (0) and R.

A simple differential ring need not be simple as a ring. For example, $z\mathbb{C}[z]$ is up a nontrivial ring ideal in $\mathbb{C}[z]$, but are not closed under the derivation $\partial = \frac{d}{dz}$. In fact, no nontrivial ideals in $(\mathbb{C}[z], \frac{d}{dz})$ are closed under $\frac{d}{dz}$, making $(\mathbb{C}[z], \frac{d}{dz})$ a simple differential ring.

Definition 2.1.9. A *differential algebra* over (R, ∂_R) is a differential ring (S, ∂_S) given together with a homomorphism of differential rings $(R, \partial_R) \to (S, \partial_S)$.

Definition 2.1.10. The *tensor product* of two differential algebras, (T, ∂_T) and (S, ∂_S), over a differential ring R is the ring $T \otimes_R S$, with $\partial_{T \otimes_R S}$ defined for $t \in T$ and $s \in S$ by,

$$\partial_{T \otimes_R S}(t \otimes_R s) = (\partial_T(t) \otimes_R s) + (t \otimes_R \partial_S(s)).$$

To that the map $\partial_{T\otimes_R S}$ is a well defined derivation, check first the case where the derivation on R is trivial. For general R, observe that $T\otimes_R S$ is the quotient of $T\otimes_C S$ by the ring ideal I generated by elements of the form $\{1\otimes_C 1.r - r.1\otimes_C 1\}$ for $r\in R$, and that this I is a differential ideal.

Definition 2.1.11. For a differential ring R, $R\{X\}$ is the *ring of differential polynomials* over R in the variable X. It is the ring of polynomials in countably many variables

$$R\{X\} := R[X, X', X'', \dots, X^{(n)}, \dots]$$

with the derivation on R extended so that $\partial(X^{(n)}) = X^{(n+1)}$.

A differential polynomial can be considered as a differential operator, via the map $R\{X\}\to \mathrm{End}(R)$ that takes $X^{(n)}$ to ∂_R^n and $a\in R$ to left multiplication by a.

Given a differential ring (R,∂_R) and a multiplicative subset $S\subseteq R$, we can consider the localization $S^{-1}R$. As usual, the elements are classes of ordered pairs (r,s) where $r\in R$ and $s\in S$ modulo the equivalence relation dened by $(r_1,s_1)\sim(r_2,s_2)$ if there is some $t\in S$ with $t(s_2r_1 - s_1r_2) = 0$. This localization $S^{-1}R$ can be given the structure of a differential ring.

Proposition 2.1.12. *Given a differential ring (R,∂_R) and a S a multiplicative subset of R, ∂_R extendes uniquely to $S^{-1}R$ by the quotient rule $\partial(r,s) = (s\partial r - r\partial s, s^2)$.*

Proof. It is clear that if ∂_R extends to $S^{-1}R$, it must obey the quotient rule, so if an extension exists, it is unique.

For this extension to be well defined, it must preserve the equivalence classes that make up $S^{-1}R$. Thus, the derivation will preserve equivalence classes if whenever there is some $t\in R$ such that

$$t(r_1s_2 - r_2s_1) = 0$$

there is also a $t^*\in R$ such that

$$t^*\left((\partial(r_1)s_1 - r_1\partial(s_1))s_2^2 - (\partial(r_2)s_2 - r_2\partial(s_2))s_1^2\right) = 0.$$

A straightforward verification shows that t^2 is such a t^*.

□

Proposition 2.1.13. *If (F,∂_F) is a differential field (of characteristic 0), then ∂_F extends uniquely to the algebraic closure $\widetilde{F}$ of F.*

Proof. To see that the derivation extends uniquely to any finite extension,

Suppose $a\in\widetilde{F}\setminus F$ and let $P(x) = \sum_{i=0}^n f_i x^i$ be its minimal polynomial over F with $f_1 = 1$, and using the primitive element theorem suppose further that a is a primitive root of P, so $P(x)$ splits in $F(a)$.

Now suppose we have extended the derivation ∂_F to a derivation ∂ on all of $F(a)$. By applying this ∂ to the minimal polynomial, we can solve for $\partial(a)$ in terms of a and elements in F. Any element in $F(a)$ is an F linear combination of $\{1, a, a^2, \dots a^{n-1}\}$, and for any such linear combination, $b \coloneqq \sum_{i=0}^{n} f_i a^i$, ∂b can be computed in terms of this ∂a, so if the derivation can be extended, it is unique.

To see that this gives a valid derivation, one checks that for $b_1, b_2 \in K(a)$, the Leibniz rule

$$\partial(b_1 b_2) = b_1 \partial b_2 + b_2 \partial b_1$$

holds, which reduces to to showing that $\partial(a^k) = k a^{k-1} \partial(a)$ for $n \leq k \leq 2n-1$. This can be done by induction, making use of the minimal polynomial for a.

Since $\widetilde{F}$ is the union of $F(a)$ over all such a, we see that ∂_F has a unique extension to $\widetilde{F}$. □

2.2 Differential Galois Theory

Differential Galois theory is the study of the differential field extensions generated to solutions to differential equations and of their automorphism groups. Here, we recall some of the basic ideas and definitions from the differential Galois theory of linear differential equations. An excellent introduction can be found in [11], which we continue to follow here; for a more exhaustive reference, turn to [21].

If L is an order n linear differential operator over a differential field F, then the solutions to the equation $L = 0$ will form a vector space over the constants of dimension at most n. Furthermore, there is some differential extension $K \supseteq F$ in which $L = 0$ has n solutions, linearly independent over the constants of K. If $\{f_1 \dots f_n\}$ is a set of n solutions to $L = 0$ in a differential field K, linearly independent over the constants of K, then $\{f_1 \dots f_n\}$ is a *fundamental system of solutions* of L in K.

If $K \supseteq F$ is a differential field extension and the constants of K are the constants of F, then K is a *no new constants* extension of F. If C_F is algebraically closed, then for any L over F, there is a $K \supseteq F$ containing a fundamental system of solutions to L with $C_K = C_F$. Such extensions are called Picard-Vessiot extensions; they are the smallest differential field extensions containing a full set of solutions to a given equation.

The assumption that C_F be algebraically closed is relatively benign, as the derivation on any differential field extends uniquely to its algebraic closure by Proposition 2.1.13 and the constants of an algebraically closed differential field will themselves be algebraically closed by Proposition 2.1.4.

Definition 2.2.1. Let (F, ∂) be a differential field and L, a linear differential operator of order n over K. A *Picard-Vessiot extension* of F for L is a differential extension $F \subseteq K$ such that:

1. The constants of K are the constants of F,
2. $L = 0$ has n solutions in K linearly independent over these constants
3. K is generated over F as a differential field by the solutions of $L = 0$ in K.

An order n linear differential equation can also be written as $\partial(x) = Ax$, where $A \in Gl_n(K)$, x is a vector, and ∂ acts on x component-wise. We say that M is a *fundamental matrix* for the equation $\partial(x) = Ax$ if M is an invertible matrix satisfying $\partial(M) = AM$. A Picard-Vessiot extension can then be equivalently defined as a no new constants extension of F generated as a field by the entries of M.

If the constants of F are algebraically closed, then any two Picard Vessiot extensions of F for L are isomorphic over F as differential fields.

Picard Vessiot extensions play the role of Galois extensions in differential Galois theory.

Let $\mathrm{Aut}(K/F)$ be the group of field automorphisms of K over F; it is a subgroup of $\mathrm{GL_n(C_K)}$. Let DGal(K/F) be the group of differential automorphisms of K over F; that is, the subgroup of $\mathrm{Aut}(K/F)$ of elements σ such that for all $a \in K$, $\partial(\sigma(a)) = \sigma(\partial(a))$. Then DGal(K/F), considered as a subgroup of $\mathrm{GL_n(C_K)}$, is an algebraic group. Furthermore, it can be shown using Proposition 2.1.13 that any finite Galois extension of $K \supseteq F$ is a Picard Vessiot extension for some equation, and that in this case the differential Galois group will be the same as the ordinary Galois group of the extension.

For a Picard Vessiot extension $K \supseteq F$, there is a Galois correspondence between intermediate differential field extensions and subgroups of $\mathrm{DGal}(K/F)$.

Theorem 2.2.2 (Fundamental Theorem of Differential Galois Theory)**.** *Let $K \supseteq F$ be a Picard Vessiot Extension, and set $G := \mathrm{DGal}(K/F)$. Then there is a lattice inverting bijective correspondence:*

$$\{\textit{differential subfields } F \subseteq E \subseteq K\} \leftrightarrow \{\textit{Zariski closed subgroups } H \textit{ of } G\}$$

given by:

$$E \to \{\textit{elements in } G \textit{ fixing } E\}$$
$$\{\textit{the fixed field of } H\} \leftarrow H$$

with Picard Vessiot extensions corresponding to normal subgroups.

While we will consider only linear equations in this thesis, it is worth noting that Kolchin developed a differential Galois theory for a wider class of equations. The extensions considered are called strongly normal.

Definition 2.2.3. Let K and L be differential fields with $K \subseteq L$, both inside a universal differential field $\mathcal{U}$. The extensions L/K is *strongly normal* if and only if:

1. $C_K = C_L$ is algebraically closed;
2. L is finitely generated over K
3. if $\sigma : \mathcal{U} \to \mathcal{U}$ is a differential automorphism fixing K pointwise, then $\langle L, C_{\mathcal{U}} \rangle = \langle \sigma(L), C_{\mathcal{U}} \rangle$.

Picard Vessiot extensions are strongly normal, but certain nonlinear differential equations also give rise to strongly normal extensions, and can therefore be shown to have a good Galois theory.

2.3 Difference Fields

A difference ring (R, σ) is ring R together with a ring automorphism $\sigma : R \to R$. The constants of R are the elements $c \in R$ with $\sigma(c) = c$, and are denoted C_R. A difference ring that is a field is a difference field. When working in a difference ring, the image of $x \in R$ under the automorphism σ is sometimes denoted $\sigma(x)$, and sometimes x^σ.

A Galois theory of linear difference equations has been developed in analogy with differential Galois theory; a good reference on the subject is [20]. A fundamental system of solutions for an order n linear equation $L(x) = 0$ over a difference field F is again a set of n solutions to the equation in $K \supseteq F$, linearly independent over the constants of K. If K is a no new constants extension of F, generated by such a fundamental system of solutions, K is again called a *Picard Vessiot extension* of F for L. If the constants of F are algebraically closed, such a K exists and is unique up to isomorphism over F, and the expected Galois correspondence holds.

However, in difference fields, the requirement that the constants be algebraically closed is a significant restriction. A difference field (F, σ) might contain nonconstant elements algebraic over C_F, and σ might not have a unique extension to the algebraic closure of F. One consequence of this is that some equations may never have any nonzero solutions in a difference field with algebraically closed constants. The simplest example of such an equation is $\sigma(x) = -x$. Since the square of any solution is a constant, any nonzero solution will be a nonconstant element algebraic over the constants.

The Galois theory of difference equations has therefore been expanded in multiple directions. In order to include equations like $\sigma(x) = -x$, one can consider field extensions whose fields of constants are not algebraically closed, or Picard Vessiot rings that are not necessarily integral (a solution to $\sigma(x) = x$ in such a ring would also satisfy $x^2 = 0$). A survey of these is given in [3], where it is also proved that in the cases they coincide, three reasonable approaches lead to essentially the same Galois theory.

An algorithm for computing the Galois group of an order two difference equation over (K, σ), for $K = k(z)$, where k is a finite algebraic extension of $\mathbb{Q}$ and $\sigma(z) = z + 1$ is given in [6]. We will use this algorithm in the examples, in order to compare model theoretic automorphism groups in valued D-fields to the standard difference Galois groups for the same equations.

To a second order linear difference equation $\sigma^2 x + a\sigma x + bx = 0$, the procedure associates a first order nonlinear difference equation $x\sigma x + ax + b = 0$ called the *Riccati equation*. If the Riccati equation has a solution u in the base field, then the difference operator $\sigma^2 + a\sigma + b$ factors as $(\sigma - \frac{b}{u})(\sigma - u)$. The algorithm draws the following conclusions about the difference Galois group $\mathcal{G}$ based on the number of solutions to the Riccati equation in $\tilde{K} := (\bar{\mathbb{Q}}(t))$.

1. If the Riccati equation has no solutions in $\tilde{K}$, then $\mathcal{G}$ is irreducible.

2. If the Riccati equation has exactly one solution u in $\tilde{K}$, then $\mathcal{G}$ is reducible, but not completely reducible.

3. If the Riccati equation has exactly two solutions u_1 and u_2 in $\tilde{K}$ then $\mathcal{G}$ is completely reducible, but not an algebraic subgroup of $\{c.Id : c \in \bar{\mathbb{Q}}^\times\}$.

4. If the Riccati equation more than two solutions, then $\mathcal{G}$ is an algebraic subgroup of $\{c.Id : c \in \bar{\mathbb{Q}}^\times\}$.

In the cases we encounter, the Riccati equation will have two solutions. It is also shown in [6] that if $\mathcal{G}^0$ is the identity component of $\mathcal{G}$, and $\mathcal{G}$ is a reducible difference Galois group in this setting, then $\mathcal{G}/\mathcal{G}^0$ is finite cyclic. Thus, the problem is reduced to considering such reducible subgroups of $\mathrm{GL}(2, C_K)$. This can be done by inspection, as the subgroup must contain the matrix

$$\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$$

corresponding to equation in its factored form.

2.4 Model Theoretic View

There is also a model theoretic Galois theory, which in its most basic form relates definably closed sets in some model with (quotients of) groups of automorphisms of that model. We introduce this Galois theory here; for basic model theoretic definition and concepts, see [13], [7], or [17].

For two definable sets $\mathbf{Q}$ and $\mathbf{C}$ in some model $\mathcal{U}$, the model theoretic Galois group $\mathrm{MGal} := \mathrm{Aut}(\mathbf{Q}/\mathbf{C})(\mathcal{U})$ of $\mathbf{Q}$ over $\mathbf{C}$ in $\mathcal{U}$ is the group of automorphisms of $\mathbf{Q}$ induced by automorphisms of $\mathcal{U}$ fixing $\mathbf{C}$ pointwise. If $\mathbf{Q}$ is *internal* to $\mathbf{C}$; meaning there is some finite tuple $\bar{b}$ such that in any model, the elements of $\mathbf{Q}$ are definable over $\mathbf{C}\bar{b}$, this automorphism group is type definable. When $\mathbf{Q}$ and $\mathbf{C}$ are definably closed, and $\mathrm{Th}(\mathcal{U})$ eliminates imaginaries, there is a Galois correspondence between type definable subgroups of MGal and definably closed substructures S of $\mathcal{U}$ with $\mathbf{C} \subseteq S \subseteq \mathbf{Q}$.

An excellent introduction to the subject is the recent [14], while a complete technical treatment with minimal assumptions on the theory is given in [8]. A detailed look at the "internality" relation and exhaustive treatment of the Galois correspondence is given in [9].

In the early 1980's, the connection between these model theoretic automorphism groups and differential galois theory was noted in [16]. Later, this connection was more fully developed by Pillay in [15], which uses the model theoretic framework to generalize the Kolchin theory. The setting for this work is DCF_0, the theory of differentially closed fields of characteristic zero. These universal differential fields were described in the 1950's by Kolchin and Robinson, and later shown to have the following finite axiomatization by Lenore Blum.

Definition 2.4.1. A differential field K is *differentially closed* if for all $f, g \in K\{X\}$, with order(f)>order(g), there is some $x \in K$ with $f(x) = 0$ and $g(x) \neq 0$.

Differentially closed fields are the subject of [12]. Any differential field of characteristic zero embeds into a model of DCF_0. The theory has quantifier elimination, is complete and ω-stable, and eliminates imaginaries.

Using these facts, we can uniquely associate to any linear differential equation L over a differential field K the set $\mathbf{Q}_L$ of solutions to L in a prime model $\mathcal{M}$ of DCF_0 over K. If $\mathbf{C}$ is the set of constants, then $\mathbf{Q}_L$ is $\mathbf{C}$-internal; $\mathbf{Q}_L$ is a finite dimensional vector space over $\mathbf{C}$, if $\bar{b}$ is a basis for $\mathbf{Q}_L$ over $\mathbf{C}$ every element of $\mathbf{Q}$ is definable over $\mathbf{C}\bar{b}$. The model theoretic Galois group of $\mathbf{Q}_L$ over $\mathbf{C}$ in $\mathcal{M}$ is therefore type definable.

By ω-stability, any type definable group in DCF_0 is in fact definable. Furthermore, one can identify the model theoretic Galois group in DCF_0 with the standard differential Galois group associated to L. The construction of the group used the fact that L was a linear equation only in establishing that $\mathbf{Q}_L$ was $\mathbf{C}$ internal; everything above makes sense in DCF_0 for any sets $\mathbf{Q}$ and $\mathbf{C}$ satisfying the internality relation. This can be shown to include any case where $\mathbf{C}$ is the set of constants and the extension generated by $\mathbf{Q}$ is a strongly normal extension of the base field.

Chapter 3

Valued D-fields

3.1 $\mathcal{D}_e$-Rings

Everything in this section comes directly from [18]. For ease of reference, throughout this section and the two that follow, we retain the notation of that paper.

Let $\mathcal{L}$ be the language of rings, along with an extra function symbol D and a constant e.

Definition 3.1.1. A $\mathcal{D}_e$**-ring** is an $\mathcal{L}$-structure R such that

- R is a ring,
- e some element in R, and
- D is a additive map from R to R, satisfying $D(1) = 0$ and the twisted Leibniz rule $D(xy) = D(x)y + xD(y) + eD(x)D(y)$.

Given D, we can define an endomorphism σ of R by the formula $\sigma(x) := x + eDx$, and the twisted Leibniz rule can be rewritten as $D(xy) = xDy + \sigma(y)Dx$.

The definition above makes sense for any ring R and any $e \in R$, but there are some special cases. If $e = 0$, there is no twisting, $\sigma = \text{id}$ and D is a derivation. If $e \neq 0$, σ is a nontrivial endomorphism, and this twisting means that D operator can no longer be a derivation. Furthermore, if e is not a zero divisor, then σ and D are interdefinable, and D can be recovered from σ by $eD(x) = (\sigma(x) - x)$, making (R, D) equivalent to the difference ring (R, σ). When e is invertible, the situation is recognizable as that of a σ-derivation, $\delta_\sigma(x) := \gamma(x^\sigma - x)$ with $\gamma = e^{-1}$. Modules over rings with such an operator are deeply explored in [1].

Many facts familiar from differential algebra remain true in the D-ring setting. The constants of a D-ring form a ring, and the constants of a D-field form a field. D-extensions and D-algebras are defined in the expected way. If R is a D-ring and $I \subseteq R$ is an ideal, we say I is a D-ideal if $D(I) \subseteq I$, in which case the structure induced on R/I is also that of a D-ring.

We now check other basic properties of D-rings. The following two rules for applying the D-operator will be useful later.

Proposition 3.1.2. *If R is a D-ring, $x \in R$, and n is a positive integer,*

$$D(x^n) = \sum_{i=1}^{n} \binom{n}{i} e^{i-1} x^{n-i} (Dx)^i$$

Proof. This is proved by induction in [18]. If e is not a zero divisor, it can be seen more directly using the fact that $\sigma(x) = x + eDx$ is an endomorphism. Since $\sigma(x^n) = \sigma(x)^n$, we have that $x^n + eD(x^n) = (x + eDx)^n$, and $eD(x^n) = (x + eDx)^n - x^n$, from which the identity follows. □

Proposition 3.1.3. *If R is a D-ring and x is an invertible element of R, then*

$$D\left(\frac{1}{x}\right) = -\frac{Dx}{x\sigma(x)}$$

Proof.

$$D(1) = D(x^{-1}x) = x^{-1}Dx + xDx^{-1} + eDxDx^{-1} = 0,$$

$$x^{-1}Dx = -(x + eDx)Dx^{-1},$$

$$Dx^{-1} = -\frac{Dx}{x(x + eDx)} = -\frac{Dx}{x\sigma(x)}.$$

□

It is proved in [18] that for a D-ring R and a multiplicative, σ-closed subset $S \subseteq R$ containing the identity, there is a unique D-ring structure on the localization $S^{-1}R$, as was the case with differential fields. The D-operator extends by the rule above, which is shown to be well defined using the universal property of $S^{-1}R$.

It is also shown in [18] that for a D-ring R, there is an extension of D-rings $R \to R\langle X \rangle$ universal with respect to simple extensions of R. As a ring $R\langle X \rangle := R[\{D^n X\}_{n=0}^{\infty}]$, the polynomial ring in countably many indeterminates. The D-operator can be extended to this ring in at least two ways, but for universality it is extended so that $D(D^n X) = D^{n+1} X$. We call $R\langle X \rangle$ with this structure the *ring of D-polynomials over R.*

There are several reasonable notions of complexity for D-polynomials. We start with the most basic.

Definition 3.1.4. The *order* of a nonconstant D-polynomial $P(X)$ over a D ring R is the least n for which $P(X) \in R[X, \dots D^n X]$. The order of a nonzero constant D-polynomial is -1, and the order of the zero D-polynomial is ∞.

Definition 3.1.5. The *degree* of an order n D-polynomial $P(X)$ is the degree of $P(X)$ in the variable $D^n X$ if $n \geq 0$. The degree of a constant D-polynomial is 0. The degree of the zero D-polynomial is ∞.

A D-polynomial of order n and degree d has *order-degree* (n,d). Order-degrees are ordered lexicographically, and a D-polynomial P is said to be *simpler* than a D-polynomial Q, denoted $P \ll Q$ if the order-degree of P is less than the order-degree of Q in that ordering.

In the next example, we apply D to an order one D-polynomial over the constants, to provide some insight into the structure of $R\langle X\rangle$.

Example 3.1.6. Let R be a D-ring and $P(X) = \sum_{i=0}^{n} c_i X^i$ be a polynomial over the constants of R. Then

$$D(P(X)) = \sum_{i=1}^{n} e^{i-1}(DX)^i \left(\sum_{m=i}^{n} \binom{m}{i} c_m X^{m-i} \right)$$

Proof.

$$D\left(\sum_{i=1}^{n} c_i X^i\right) = \sum_{i=1}^{n} c_i D(X^i) = \sum_{i=0}^{n} c_i \left(\sum_{k=0}^{i} \binom{i}{k} e^{k-1} X^{i-k} (DX)^k \right),$$

then regroup around $e^{k-1}(DX)^k$. □

As expected, applying the D-operator increases the order of P. However, unlike in a differential field, the degree of $D(P)$ is the same as the degree of P, as the last term of $D(P)$ will be $c_n e^{n-1}(DX)^n$. When we move to valued D-fields, the "e^{n-1}" in this expression will assure that such terms of high degree will also have high valuation.

In Chapter 5, we will use a lemma from [18] that requires a more refined notion of complexity for an inductive argument.

Definition 3.1.7. The *total degree* of a D polynomial P, denoted $T.\deg(P)$, is $(\deg_{X^{(i)}} P)_{i=0}^{\infty}$.

For any P, $T.\deg(P) \in \mathbb{N}^{<\omega} := \{(n_j)_{j=0}^{\infty} : n_j \in \mathbb{N} \text{ and } n_j = 0 \text{ for } j \gg 0\}$. The set $\mathbb{N}^{<\omega}$ is well ordered by $(n_j)_{j=0}^{\infty} < (m_j)_{j=0}^{\infty}$ if and only if there is some N such that $n_N < m_N$ and for all $j > N$, $n_j \leq m_j$. With this ordering on the total degrees, we write $P \prec Q$ if $T.\deg(P) < T.\deg(Q)$. This means that $P \prec Q$ if there is some N for which $\deg_{D^N} P < \deg_{D^N} Q$ and for all higher order terms, the degree of P never exceeds the degree of Q.

In addition to applying the D operator to D-polynomials, we will also want to be able to take their derivatives. If P is a D-polynomial of the form $P(X) = F(X, DX, ..., D^n X)$, define $\frac{\partial}{\partial X^{(i)}} P$ to be the D-polynomial $\frac{\partial}{\partial X_i} F(X, DX, ..., D^n X)$, the derivative of F with respect to the variable $D^i X$. We will need this primarily to make sense of a version of Hensel's lemma for valued D-fields; however in [18], it is used extensively in inductive arguments on $T.\deg$, since for any i, $\frac{\partial}{\partial X^{(i)}} P \preceq P$.

3.2 Valued Fields

We will look at $\mathcal{D}_e$-rings in the context of valued fields. Recall the definition of a valued field.

Definition 3.2.1. A *valued field* is a pair (K, v) where K is a field and v is a map from K onto $\Gamma \cup \{\infty\}$, where Γ is an ordered abelian group, called the value group and

1. $v(x) = \infty \Leftrightarrow x = 0$,
2. $v(xy) = v(x) + v(y)$ (so v is a group homomorphism $K^\times \to \Gamma$), and
3. $v(x + y) \geq \min\{v(x), v(y)\}$ (the *"ultrametric inequality"*).

From these axioms, we can conclude that $v(x + y) > \min\{v(x), v(y)\}$ then $v(x) = v(y)$. Suppose $x + y = z$ and $v(z) > v(x)$ and $v(z) > v(y)$. Then $v(y) = v(z - x) \geq v(x)$ and $v(x) = v(z - y) \geq v(y)$, which is only possible if $v(x) = v(y)$.

For a valued field let K, we let R_K denote the ring of integers of K, so $R_K := \{x \in K : v(x) \geq 0\}$. This ring has a unique maximal ideal $\mathfrak{m}_K := \{x \in K : v(x) > 0\}$, and the quotient $R/\mathfrak{m}$ is the *residue field* of K, denoted k_K. The subscripts are dropped when doing so will not cause confusion. The quotient map $R \to R/\mathfrak{m}$ is denoted π.

Definition 3.2.2. An extension $L \supseteq K$ of valued fields is *immediate* if L and K have the same value group and residue field.

Definition 3.2.3. An extension $L \supseteq K$ of valued fields is *ramified* if the value group of L is strictly larger than the value group of K, i.e. $\Gamma_L \supsetneq \Gamma_K$.

Definition 3.2.4. A valued field is said to be *maximally complete* if it has no proper immediate extensions.

Kaplansky showed in [10] that valued fields of residue field characterstic zero have unique maximal immediate extensions, and described the structure of such extensions. A key tool in his proof, and in much subsequent work in valued fields, was the existing notion of a pseudo-convergent sequence.

Definition 3.2.5. A *pseudo-convergent sequence* is a limit ordinal indexed sequence $\{x_\alpha\}_{\alpha<\kappa}$ of elements of a valued field K such that $(\forall \alpha < \beta < \gamma < \kappa)\,(v(x_\alpha - x_\beta) < v(x_\beta - x_\gamma))$.

If there is some $c \in K$ such that $(\forall \alpha < \beta < \kappa)\,(v(x_\alpha - c) < v(x_\beta - c))$, then c is a *pseudo-limit* of $\{x_\alpha\}$, and $\{x_\alpha\}$ *pseudo-converges* to c.

For any pseudo-convergent (p.c.) sequence $\{x_\alpha\}$ in K, $\{v(x_\alpha - x_{\alpha+1})\}$ is a strictly increasing sequence in Γ_K. It is unbounded if and only if $\{x_\alpha\}$ is a convergent sequence; otherwise, it isolates a cut in Γ_K.

Define the the *breadth* of a p.c. sequence $\{x_\alpha\}$ to be $\{\rho \in \Gamma \cup \infty : (\forall \alpha) v(x_\alpha - x_{\alpha+1}) < \rho\}$; so a p.c. sequence converges if and only if its breadth is $\{\infty\}$. Let c be a pseudolimit of $\{x_\alpha\}$. Then any element $b \in K$ (or in any extension of K) satisfying $v(b - c) \in \text{breadth}(\{x_\alpha\})$ is also a pseudolimit of $\{x_\alpha\}$ (and conversely).

If $\{x_\alpha\}$ has no pseudo-limits in K, then it is a *strict* pseudo-convergent sequence.

An important attribute of some valued fields is satisfying a condition known as Hensel's Lemma, which can be stated in many equivalent forms. The one we will make the most use of is the following.

Hensel's Lemma: Let K be a valued field and R its valuation ring. Given any $P(x) \in R[x]$ and $a \in R$ satisfying $v(P(a)) > 0$ and $v(P'(a)) = 0$, there is a $b \in R$ with $P(b) = 0$ and $v(a - b) > 0$.

Not all valued fields satisfy Hensel's lemma, but it is a powerful tool in those that do. Valued fields in which Hensel's lemma holds are said to be *henselian.* All maximally complete valued fields are henselian, but the converse is false. Every valued field K has a *henselisation* K^h, an immediate extension universal for extensions of K satisfying Hensel's lemma.

Valued fields have long been objects of interest to model theorists, dating from at least the 1950's. A good survey of the interactions of between model theory and the study of valued fields is given in the introduction to [4]. In particular, because of the ordering on the value group, no theory of valued fields can be stable. However, the theory of algebraically closed valued fields, called ACVF, is an especially tractable example of an unstable theory, because it is largely controlled by a stable part. This is explored very precisely in [5], where a notion called "metastability" is introduced. It is also shown in [5] that algebraically closed valued fields have elimination of imaginaries in a reasonable extension of the standard valued field language, a result which can also be taken as a complete description of the un-eliminable imaginaries in the natural language.

3.3 $\widetilde{\text{VDF}}$: Axioms and Consequences

As noted above, D-rings generalize rings with both difference and differential operators. Valued D-fields allow us to consider these two cases in the same structure.

Definition 3.3.1. A *valued D-field* is a valued field K considered as a $\mathcal{D}_e$-ring, with $v(e) \geq 0$ and $v(Dx) \geq v(x)$.

Since K is a field, e must be 0 or invertible. For the case of interest in this thesis, we take $e \neq 0$ and $v(e) > 0$. The D-operator on K is then interdefinable with an endomorphism of K, while the structure induced on k by D is that of a differential field.

In [18], the model theory of a particular class of these valued D-fields, there called $(\mathbf{k}, \mathbf{G})$-D-henselian fields, is developed. We now present the axioms for these fields, as they appear in that paper.

The sorts are (K, k, Γ) where K is a valued field, k its residue field, and Γ its value group. The fields K and k, both of characteristic zero, are considered in the language of $\mathcal{D}_e$ rings. In the model completion, k will be linearly differentially closed, in the following sense.

Definition 3.3.2. A differential field k is *linearly differentially closed* if any non-zero linear differential operator $L \in k[D]$ is surjective as a map $L : k \to k$.

The group Γ is an ordered abelian group with divisibility predicates; k and Γ may also have additional structure. A symbol ∞ is added to the language in a natural way, and there are maps $v : K \to \Gamma \cup \infty$ for the valuation and $\pi : K \to k \cup \infty$ for the residue map. The ring of integers in K is definable, and denoted $\mathcal{O}_K := \{a \in K : v(a) \geq 0\}$. The constants of K are also definable and for now denoted $K^D := \{a \in K : D(a) = 0\}$.

Definition 3.3.3. Let $\mathbf{k}$ be a linearly differentially closed field, also closed under n^{th} roots. Let $\mathbf{G}$ be an ordered abelian group, and suppose that that $\text{Th}(\mathbf{k})$ and $\text{Th}(\mathbf{G})$ eliminate quantifiers. A $(\mathbf{k}, \mathbf{G})$-*D-valued field* is a multisorted structure with sorts (K, k, Γ) satisfying the following axioms:

1. K and k are $\mathcal{D}_e$ fields of characteristic zero and $k \models Th_\forall(\mathbf{k})$.
2. K is a valued field, whose value group is a subgroup of Γ via the valuation v and whose residue field is a subfield of k via the residue map π, and $v(e) > 0$.
3. $\forall x \in K, v(Dx) \geq v(x)$ and $\pi(Dx) = D\pi(x)$.
4. $\Gamma \models Th_\forall(\mathbf{G})$.

Definition 3.3.4. A $(\mathbf{k}, \mathbf{G})$-*D-henselian field* is a structure satisfying the four axioms above, as well as:

5. $(\forall x \in K)\left[([\exists y \in K] y^n = x) \leftrightarrow n | v(x)\right]$.
6. $\Gamma = v((K^D)^\times)$ (*"K has enough constants"*).
7. $k = \pi(\mathcal{O}_K)$ (*"π is onto"*).
8. (D-Hensel's Lemma): If $P \in \mathcal{O}_K\langle X \rangle$ is a D-polynomial, $a \in \mathcal{O}$, and $v(P(a)) > 0 = v(\frac{\partial}{\partial X^{(i)}} P(a))$ for some i, then there is $b \in K$ with $P(b) = 0$ and $v(a - b) \geq v(P(a))$.
9. $\Gamma \equiv \mathbf{G}$
10. $k \equiv \mathbf{k}$

The theory of $(\mathbf{k}, \mathbf{G})$-D-Henselian fields theory is model complete, and eliminates quantifiers in the field sort, up to the theories of the residue field and value group.

Example 3.3.5. As described in [18], for a fixed $\mathbf{G}$ and $\mathbf{k}$ the generalized power series fields $\mathbf{k}((\varepsilon^{\mathbf{G}}))$ provide canonical models for the theory of $(\mathbf{k},\mathbf{G})$-D-henselian fields. These fields are defined as a set by:

$$\mathbf{k}((\varepsilon^{\mathbf{G}})) \coloneqq \{f : \mathbf{G} \to \mathbf{k} : \operatorname{supp}(f) \coloneqq \{x \in G : f(x) \neq 0\} \text{ is well-ordered in G}\}.$$

Elements in $\mathbf{k}((\varepsilon^{\mathbf{G}}))$ can be considered as formal power series

$$f \leftrightarrow \sum_{\gamma \in \mathbf{G}} f(\gamma)\varepsilon^{\gamma}$$

with addition and multiplication defined in the expected way.

For $f \in \mathbf{k}((\varepsilon^{\mathbf{G}}))$, $v(f) = \min\{\operatorname{supp}(f)\}$. To determine $D(f)$, let ∂ be the derivation on $\mathbf{k}$ and let e be any element with $v(e) > 0$. On $\mathbf{k}$, define

$$\sigma(x) = \sum_{n=0}^{\infty} \frac{\partial^n x}{n!} e^n$$

and extend σ to the rest of $\mathbf{k}((\varepsilon^{\mathbf{G}}))$ by

$$\sigma(f) = \sum_{\gamma \in \mathbf{G}} \sigma(f(\gamma))\varepsilon^{\gamma}.$$

Then $D(f)$ can be recovered by the identity $Df = e^{-1}(\sigma(f) - f)$. With this structure, $\mathbf{k}((\varepsilon^{\mathbf{G}}))$ is a maximally complete as a valued field, and thus henselian.

We will often work in these fields and use model completeness to draw more general conclusions.

Notation 3.3.6. Of special interest in this thesis are $(\mathbf{k},\mathbf{G})$-D-henselian fields for $\mathbf{k}$ differentially closed. We call the theory of $(\mathbf{k},\mathbf{G})$-D-Henselian fields for $\mathbf{k}$ differentially closed $\widetilde{\mathrm{VDF}}$. Any $(\mathbf{k},\mathbf{G})$-D-field K can be embedded into a model M of $\widetilde{\mathrm{VDF}}$; take M to be a $(\mathbf{k}',\mathbf{G})$-D-Henselian field where $\mathbf{k}'$ is a differential closure of $\mathbf{k}$. This will be discussed in more detail in Chapter 5.

From this point on, we will refer to a D-operator on a valued field K as D, and the D-operator induced on the residue field as ∂, as a reminder that the residue operator is in fact a derivation (so the second half of Axiom 3 would now read "$\pi(\mathrm{D}\,x) = \partial\pi(x)$"). To further emphasize the connection with differential fields, we will from now on refer to the constants of K as C_K and the constants of k as C_k. A D-henselian field is any valued D-field tin which the D-Hensel's Lemma holds.

3.4 Algebra in Valued D-Fields

In this section, we explore basic algebra in the valued D-field setting, and establish some facts that will be useful in what follows.

First, note that the requirement that $v(\mathrm{D}\,x) \geq v(x)$ implies that the endomorphism defined by $\sigma(x) = x + e\,\mathrm{D}\,x$ is valuation preserving. In fact, we can say more. In [4] and elsewhere, valued fields are considered with their leading term or "RV" structure. For $x \in K^{\times}$, $\mathrm{rv}(x)$ is the image of x in the quotient $K^{\times}/(1+\mathfrak{m})$. Thus, for $x, y \in K$, $\mathrm{rv}(x) = \mathrm{rv}(y)$ if and only if $v(x-y) > \min\{v(x), v(y)\}$; equivalently, if $v(x-y) > v(x)$ or $v(x-y) > v(y)$.

For any $\delta \in \Gamma$ with $\delta > 0$, we can also consider the ideal $\mathfrak{m}_\delta := \{x \in R : v(x) > \delta\}$, and set $\mathrm{rv}_\delta(x)$ to be the image of x in $K^{\times}/(1+\mathfrak{m}_\delta)$, so $\mathrm{rv}_0(x) = \mathrm{rv}(x)$, and if $\alpha, \beta \in \Gamma$ with $\alpha < \beta$, then for any $x, y \in K$, $\mathrm{rv}_\beta(x) = \mathrm{rv}_\beta(y)$ implies $\mathrm{rv}_\alpha(x) = \mathrm{rv}_\alpha(y)$. The image of the map rv_δ is the uneliminable imaginary sort RV_δ.

Proposition 3.4.1. *If K is a valued* D*-field with $v(e) > 0$, and σ is the endomorphism defined by $\sigma(x) = x + e\,\mathrm{D}\,x$, then $rv(x^\sigma) = rv(x)$.*

Proof. The endomorphism σ takes x and adds to it something of strictly higher valuation, thereby preserving the leading term. By the definition of valued D-field, $v(\mathrm{D}\,x) \geq v(x)$, and by assumption $v(e) > 0$, so $v(e\,\mathrm{D}\,x) = v(\mathrm{D}\,x) + v(e) > v(x)$. Then since

$$x^\sigma - x = e\,\mathrm{D}\,x$$

we have $v(x^\sigma - x) > v(x)$. By the above, this is equivalent to $rv(x^\sigma) = rv(x)$. □

Next, a few remarks about henselizations.

As we will be dealing mostly with linear equations, it is worth noting that if $P(X) = \sum_{i=0}^{n} r_i\,\mathrm{D}^i(X)$ is a linear D-equation with $\min\{v(r_i)\} = 0$, then P satisfies the hypotheses of the D-Hensel's Lemma at any approximate root, since $\frac{\partial}{\partial X^{(j)}} P(a) = r_j$ regardless of the choice of a. Therefore, when working with linear equations in D-henselian fields, we will apply DHL without rechecking this condition.

One major complication in going from the study of differential fields to the study of difference fields is that, in a difference field, elements algebraic over the constants may not be constant themselves. Instead, they might be elements of finite orbit under the difference operator. Difference fields that have algebraically closed fields of constants are in many ways much simpler than those that do not, and many theorems of differential algebra apply only to the restricted case of such fields.

In $\widetilde{\mathrm{VDF}}$, the restriction that the algebraic closure of the constants contains no nonconstant elements is a consequence of the axioms.

Proposition 3.4.2. *If K is a* D*-henselian field with enough constants, then C_K is relatively algebraically closed in K; if $a \in K$ is algebraic over C_K, then* $\mathrm{D}\,a = 0$.

Proof. We must first establish the following weaker claim.

Claim. *If K is a valued* D*-field with enough constants, and $a \in K$ is algebraic over C_K and $a \neq 0$, then $v(\mathrm{D}\, a) > v(a)$.*

Proof. Our definition of valued D-field requires that for all $x \in K$, $v(\mathrm{D}\, x) \geq v(x)$. We show that for a C_K-algebraic, the inequality must be strict.

Since there are constants at every valuation, we may scale by a constant to assume that $v(a) = 0$. Let

$$P := \sum_{i=0}^{n} c_i x^i$$

be a minimal polynomial for a over C_K. We may again scale to assume that P is minimally integral; ie $\min\{v(c_i)\} = 0$. Then $\pi(P)$ is a nonzero polynomial over the constants of the residue field, and $\pi(a)$ is a solution to $\pi(P)(x) = 0$. Since the residue field is a differential field, anything algebraic over the constants is itself a constant, and $\partial(\pi(a)) = \pi(\mathrm{D}\, a) = 0$, so $v(\mathrm{D}\, a) > 0 = v(a)$. □

Now, suppose that there were some $a \in K$, algebraic over C_K, with $\mathrm{D}\, a \neq 0$. Again, we may assume without loss that $v(a) = 0$ and that a has a minimally integral minimal polynomial P over C_K.

From the above, we know that $v(\mathrm{D}\, a) > 0$. Let $v(\mathrm{D}\, a) = \alpha$. If $Q(x) := \mathrm{D}\, x$, then $v(Q(a)) = v(\mathrm{D}\, a) > 0$, and $\frac{\partial}{\partial X^{(1)}} Q(a) = 1$, so $v(\frac{\partial}{\partial X^{(1)}} Q(a)) = 0$, and DHL applies to Q at a. From this, we can find a $b \in K$ such that $\mathrm{D}\, b = 0$ and $v(a-b) \geq v(\mathrm{D}\, a) = \alpha$. Now consider $m := a - b$. The element m is algebraic over the constants, as it satisfies $P(x+b) = 0$, $v(m) = v(a-b)$, which by DHL is at least α, and $v(\mathrm{D}\, m) = v(\mathrm{D}(a-b)) = v(\mathrm{D}\, a - \mathrm{D}\, b) = v(\mathrm{D}\, a) = \alpha$, contradicting the previous claim.

□

Corollary 3.4.3. *If K is a valued* D*-field, then C_K is relatively algebraically closed in K; if $a \in K$ is algebraic over C_K, then $\mathrm{D}\, a = 0$.*

Proof. Any valued D-field K can be extended to a D-henselian field L with enough constants.

□

As we plan to consider linear D-equations, it will be useful to recall the definition of the Wronskian and to define its D-analogue.

Definition 3.4.4. If $y_1, ..., y_s$ are elements of a differential ring R, then their *Wronskian*, denoted $w(y_1, ..., y_s)$, is the determinant of the $s \times s$ matrix whose i^{th} column is $(y_i, \partial y_i, \ldots \partial^{s-1} y_i)^{\top}$

This notion may be extended to any D-ring, where D may not be a derivation.

Definition 3.4.5. If $y_1, ..., y_s$ are elements of a D-ring T, then their D-*Wronskian*, denoted $w_D(y_1, ..., y_s)$, is the determinant of the $s \times s$ matrix whose i^{th} column is $(y_i, \mathrm{D}\, y_i, \ldots \mathrm{D}^{s-1}\, y_i)^{\top}$

When it is clear from the context that we are working in the D-ring setting, this will be referred to simply as the Wronskian and denoted $w(y_1 \dots y_s)$ as in the differential case.

In a differential field F, elements $y_1, \dots, y_n \in F$ are linearly independent over the constants of F if and only if $w(y_1, \dots, y_s) = 0$. This is also true in D-fields. The standard proofs for differential fields work with slight modifications; the key is the following simple observation.

Proposition 3.4.6. *Let K be a field and σ an automorphism of K. Let $y_1, \dots, y_n \in K$. Then $y_1, \dots, y_n$ are linearly dependent over the σ-constants C_K^σ of K if and only if $y_1^\sigma, \dots, y_n^\sigma$ are.*

Proof. Suppose $\sum_{i=1}^n c_i y_i^\sigma = 0$ with $c_i \in C_K^\sigma$, not all zero. Then

$$\sigma^{-1}\left(\sum_{i=1}^n c_i y_i^\sigma\right) = \sum_{i=1}^n \sigma^{-1}\left(c_i y_i^\sigma\right) = \sum_{i=1}^n c_i y_i = \sigma^{-1}(0) = 0.$$

□

From here we follow [11], using the twisted Leibniz rule in the form $D(ab) = aDb + b^\sigma Da$. The above lemma will allow us to work with the twisting.

Lemma 3.4.7. *Let K be a valued* D*-field with field of constants C_K. Then $y_1, \dots, y_n$ are linearly dependent over C_K if and only if $w(y_1 \dots y_n) = 0$.*

Proof. In the first direction, if $(y_1, \dots y_n)$ are linearly dependent over C_K, there are $c_1, \dots, c_n \in C_K$ such that $\sum_{i=1}^n c_i y_i = 0$. Applying D^j to this equation, we see that $\sum_{i=1}^n c_i \, \mathrm{D}^j \, y_i = 0$ for any j, and so the c_i's are a nontrivial solution to the system of linear equations

$$\sum_{i=1}^n (\mathrm{D}^j \, y_i) x_i = 0 \text{ for } 0 \le j \le n-1.$$

The determinant of the matrix of coefficients of this system is $w(y_1, \dots, y_n)$, which must therefore be zero.

On the other hand, if $w(y_1, \dots, y_n) = 0$, then there are $c_1 \dots c_n \in K$ such that for $0 \le j \le n-1$:

$$\sum_{i=1}^n (D^j y_i) c_i = 0$$

To show that all the c_i can be taken to be in C_K, arrange the indices so that $c_1 \neq 0$, then divide through by c_1 to let $c_1 = 1$. Then apply D to obtain

$$\sum_{i=1}^n (\mathrm{D}^{j+1} \, y_i) c_i + \sum_{i=1}^n \sigma(\mathrm{D}^j \, y_i) \, \mathrm{D}(c_i) = 0$$

For $0 \le j \le n-2$, the first sum is zero by the preceding equation. Since $c_1 = 1$, $\mathrm{D}(c_1) = 0$, and the first term in the second sum is also zero. So for $0 \le j \le n-2$, $\mathrm{D}(c_2), \ldots, \mathrm{D}(c_n)$ is a solution to the system of linear equations

$$\sum_{i=2}^{n} \sigma(\mathrm{D}^j\, y_i) x_i = \sum_{i=2}^{n} (\mathrm{D}^j\, \sigma(y_i)) x_i = 0$$

The determinant of the matrix of coefficients for this equation is $w(y_2^\sigma, \ldots, y_n^\sigma)$. If $w(y_2^\sigma, \ldots, y_n^\sigma) \neq 0$, then the solution $\mathrm{D}(c_2), \ldots, \mathrm{D}(c_n)$ trivial, so $y_2^\sigma, \ldots y_n^\sigma$ are linearly dependent over C_K, and so are $y_2 \ldots y_n$. If $w(y_2^\sigma, \ldots, y_n^\sigma) = 0$, proceed by induction until a linear dependence relation between $y_i^{\sigma^m}, \ldots, y_n^{\sigma^m}$ over C_K is found for some i and m. Applying σ^{-m} then gives a relation between $y_i, \ldots, y_m$ without changing the coefficients.

$\square$

Chapter 4

Solutions to Linear $\mathbb{D}$-Equations

4.1 Valuation Compatibility

In [19], it is shown that if (K, v) is a valued field and $\partial : K \to K$ is a derivation preserving the ring of integers, then (K, ∂) is not differentially closed. While our setting differs in both the action of the operator and in its interaction with the valuation, this result suggests a similar question for valued D-fields. Since the requirement that $v(\varepsilon) > 0$ demands a nontrivial valuation, we ask instead:

"Are models of $\widetilde{\mathrm{VDF}}$ D-closed?"

The answer is clearly no, if we expect D-closed to mean satisfying a D-equivalent of the Blum axioms for differential fields. Many simple equations, notably $\varepsilon \,\mathrm{D}\, x - x = 0$, cannot have any nonzero solutions in a model of $\widetilde{\mathrm{VDF}}$, even though they may have solutions in an ordinary difference field. This is because the relation $\mathrm{D}\, x = \varepsilon^{-1} x$ implies $v(\mathrm{D}\, x) = v(x) - v(\varepsilon) < v(x)$, violating the axiom $v(\mathrm{D}\, x) \geq v(x)$. We call such solutions *valuation incompatible*.

On the other hand, the D-Hensel's Lemma and differentially closed residue field guarantee that linear D-equations that induce nontrivial equations on the residue field will always have nontrivial solutions. In fact, C_k linearly independent solutions to the residue equation can be lifted to C_K linearly independent solutions to the original equation, giving a lower bound on the C_K dimension of the solution space.

Combining these two observations, one can produce linear D-equations whose solutions in $\widetilde{\mathrm{VDF}}$ form vector spaces over the constants of dimension greater than zero, but strictly less than the order of the equation. Two examples are worked out in Chapter 6.

The question then becomes:

"Given a linear D-equation $\circledast$ over a valued D-field K, can we systematically determine the dimension of a maximal C_K-vector space of solutions to $\circledast$ in a model $\widetilde{\mathrm{VDF}}$?"

The answer to this question is yes. In fact, the example of $\varepsilon \operatorname{D} x - x = 0$, which specializes to $x = 0$ in the residue field, demonstrates the only hurdle to adjoining solutions to a linear D-equation in $\widetilde{\mathrm{VDF}}$. The C_K dimension of the solution space of a linear D-equation is completely determined by the structure it induces on the residue field.

To arrive at this result, we first need a few definitions.

Definition 4.1.1. A *D*-polynomial $P(X)$ over a valued D-field K is said to be *minimally integral* the minimum of the valuations of its coefficients is zero. Equivalently, P is over R and $\pi(P) \neq 0$.

Since K is a field, any D-polynomial over K is equivalent to one in minimally integral form. Suppose $v(b) = \min\{v(a_S) : a_S$ is a coefficient of $P\}$; then $b^{-1}P$ is a minimally integral polynomial with the same zero set as P.

For a given polynomial, this minimally integral form is not unique, but the valuations of the coefficients are. Furthermore, if P is of order n, we can take the coefficients of P to be indexed by $\mathbb{N}^n$ in the natural way and ordered lexicographically, and can always assume that P has been scaled by the coefficient whose index is greatest among those of minimal valuation. This will ensure that the corresponding polynomial in the residue field is monic.

Definition 4.1.2. The *residual order* of a minimally integral D-polynomial P is the greatest n such that D^n appears in some term of P whose coefficient has valuation zero. Equivalently, $\pi(P)$ has order n as a ∂-polynomial on k.

A linear D-operator $L := \sum a_i \operatorname{D}^{(i)}$ is minimally integral if $\min\{v(a_i)\} = 0$. In that case, its residual order is $\max\{i : v(a_i) = 0\}$, the order of the operator induced on the residue field.

For the remainder of this section, K is a valued D-field with enough constants, satisfying D-Hensel's lemma, with value group Γ, residue field k, and ring of integers R. We let L be a minimally integral linear D-operator over K and $l := \pi(L)$ the ∂-operator induced on k by L.

We will show that the solutions to L in K form a vector space over C_K of dimension equal to the dimension of the C_k vector space of solutions to l in k. If k is differentially closed, or at least closed with respect to linear differential equations, this dimension will be equal to the residual order of L. The first step is to establish a connection between the solutions to L in K and the solutions to l in the k.

Lemma 4.1.3. *Let K, k, L, and l be as above, and suppose that $a_1, \ldots, a_n$ are solutions to l in k, linearly independent over C_k. Then there are $A_1, \ldots, A_n \in R$, such that for all i, $\pi(A_i) = a_i$, $L(A_i) = 0$, and $A_1, \ldots, A_n$ are linearly independent over C_K.*

Proof. Since π is onto, there are elements $B_1, \ldots, B_n \in R$ with $\pi(B_i) = a_i$ for all i. These B_i are approximate solutions to the linear equation $L = 0$, so DHL applies to L at each B_i, giving $A_1, \ldots, A_n \in R$, with $\pi(A_i) = a_i$ and $L(A_i) = 0$.

Suppose that $\sum_{i=1}^{n} c_i A_i = 0$ with $c_i \in C_K$ not all zero. By scaling and rearranging terms we may assume that for all i, $c_i \in R$, and that $v(c_1) = 0$. Then $\sum_{i=1}^{n} \pi(c_i) a_i = 0$, each $\pi(c_i)$ is a constant, and $\pi(c_1) \neq 0$. □

We will also need the following definition.

Definition 4.1.4. For $X \subseteq K$ a definable set and $a \in K$ the *proximity* of a to X is $\rho(a, X) := \sup\{v(a - x) : x \in X\}$.

Since X is definable, so is $\{v(a-x) : x \in X\}$. If the value group is divisible, it is o-minimal, and every definable set will have a supremum, so $\rho(a, X)$ will be a well defined element of the value group. If the value group is not o-minimal, the type of $\rho(a, X)$ may not be realized in Γ. The proximity $\rho(a, X)$ is then the cut described by this type.

When X is the zero set of some D-polynomial $P(x)$, another reasonable measure of proximity would be the distance from $P(a)$ to zero, which will always exist. The next lemma shows that, in many important cases, the two measures will be the same.

Lemma 4.1.5. *Suppose that $P(x) \in R[x]_D$ is a D-polynomial over R, $a \in R$, and DHL applies to P at a. Let $X := \{b \in K : P(b) = 0\}$. Then $\rho(a, X) = v(P(a))$, and there is some $b \in X$ for which this proximity is attained.*

Proof. For any $c \in R$, $P(a) \equiv P(c) \bmod v(a-c)$. If $c \in X$, $P(c) = 0$ and $P(a) \equiv 0 \bmod v(a-c)$, so $v(a - c)$ cannot exceed $v(P(a))$. Therefore $\rho(a, X)$ is bounded above by $v(P(a))$. By DHL, there is some $b \in X$ with $v(a - b) \geq P(a)$, so $\rho(a, X)$ is at least $v(P(a))$, and the two are equal. For the b provided by DHL, $v(a - b) = v(P(a)) = \rho(a, X)$. □

With this in hand, we move onto the main result.

Theorem 4.1.6. *Let L be a minimally integral linear* D*-operator over a* D*-henselian field K with enough constants, and suppose that the solutions to $l := \pi(L)$ form an n-dimensional C_k vector space in the residue field k. Then the solutions to L in K have dimension n as a C_K-vector space.*

Proof. Suppose that $L := \sum_{i=0}^{m} a_i \operatorname{D}^{(i)}$ has order m. It is clear that the solutions to L form a vector space V over C_K of dimension at most m.

Because the residue field k is a differential field, the solutions to $l := \pi(L)$ are a vector space over C_k, the constants of the residue field, which we assume to have dimension $n \leq m$. By Lemma 4.1.3 any basis of this vector space can be lifted to n linearly independent solutions to L in K, so the dimension of V is at least n.

It remains to be shown that the dimension of V is at most n. To do this, we will start with the n-dimensional C_K vector space established above, and demonstrate that any solution a to $L = 0$ in K is in this vector space.

Suppose we have $f_1, \ldots, f_n \in R$, linearly independent over C_K with $Lf_i = 0$, $v(f_i) = 0$, and $\pi(f_1), \ldots, \pi(f_n)$ linearly independent over C_k.

Let U be the C_K vector space generated by $\{f_1,\dots,f_n\}$, and let $Q(x) \coloneqq w(f_1,\dots,f_n,x)$. By Lemma 3.4.7, the set of solutions to $Q(x)=0$ is exactly U.

Let T be the C_k vector space generated by $\{\pi(f_1),\dots,\pi(f_n)\}$. By our assumptions, we know that T can also be viewed as the space of solutions to l in the residue field, i.e. $T \coloneqq \{x \in k : \sum_{i=0}^{n} \pi(a_i)\partial^i x = 0\}$.

Lemma 4.1.7. *If $Q(x) \coloneqq w(f_1,\dots,f_n,x)$, then $\frac{\partial Q}{\partial X_n} = (-1)^n w(f_1,\dots,f_n)$. In particular, its valuation is always zero.*

Proof. By expanding $w(f_1,\dots,f_n,x)$ along the last column, we obtain

$$Q(x) = \sum_{i=0}^{n} (-1)^i \operatorname{D}^i(x)\cdot |M_i|$$

where M_i is the $n\times n$ matrix whose jth column is $(f_j \dots \operatorname{D}^{i-1} f_j, \operatorname{D}^{i+1} f_j, \dots \operatorname{D}^{n+1} f_j)^\mathsf{T}$. The coefficient of $\operatorname{D}^n(x)$ in this expansion is $(-1)^n w(f_1,\dots,f_n)$, so $\frac{\partial Q}{\partial X_n} = (-1)^n w(f_1,\dots,f_n)$.

Since we have taken $\{\pi(f_1),\dots,\pi(f_n)\}$ to be linearly independent over the constants of the residue field, $\pi(w(f_1\dots f_n)) = w(\pi(f_1),\dots,\pi(f_n)) \neq 0$, so $v(w(f_1,\dots,f_n)) = 0$. □

Lemma 4.1.8. *If $La=0$, then there is some $b\in U$ with $v(a-b)=\rho(a,U)$.*

Proof. Since $Q(x)=0$ exactly when $x\in U$, we must find a $b\in K$ with $Q(b)=0$ and $v(a-b)=\rho(a,U)$.

Since $La=0$, $\pi(a)$ is a solution to $l=0$, and can be expressed as a C_k-linear combination of $\{\pi(f_1),\dots,\pi(f_n)\}$. Therefore, $\pi(Q(a)) = w(\pi(a),\pi(f_1),\dots,\pi(f_n)) = 0$, and $Q(a)$ has valuation greater than 0.

By Lemma 4.1.7 $\frac{\partial Q}{\partial X_n}(a) = w(f_1,\dots,f_n)$, which has valuation zero. Therefore, DHL applies to Q at a, and by Lemma 4.1.5 such a b must exist. □

Lemma 4.1.9. *For any $\epsilon\in\Gamma$, there is an injective linear map $\{x\in K : Lx=0 \;\&\; v(x)\geq \epsilon\}/\{x\in K : Lx=0 \;\&\; v(x)>\epsilon\}\to T$*

Proof. Pick $c\in C_K$ with $v(c)=-\epsilon$. For $y\in\{x\in K : Lx=0 \;\&\; v(x)\geq\epsilon\}$, let $F : y\mapsto cy$. Then $G\coloneqq \pi\circ F$ is a linear map from $\{x\in K : Lx=0 \;\&\; v(x)\geq\epsilon\}$ to T. If y_1 and y_2 have the same image under G, then $y_1-y_2\in\{x\in K : Lx=0 \;\&\; v(x)>\epsilon\}$, so the map induced by G from $\{x\in K : Lx=0 \;\&\; v(x)\geq\epsilon\}/\{x\in K : Lx=0 \;\&\; v(x)>\epsilon\}$ to T is injective and linear. □

Lemma 4.1.10. *For any $\epsilon\in\Gamma$ and $y\in R$ with $Ly=0$, there is an injective affine map $\{x\in K : Lx=0 \;\&\; v(x-y)\geq\epsilon\}/E_\epsilon\to T$ where E_ϵ is the equivalence relation $xE_\epsilon y \coloneqq v(x-y)>\epsilon$.*

Proof. If $Lx=0$, then $L(x-y)=0$. Now apply Lemma 4.1.9. □

For a fixed $y \in R$ with $Ly = 0$, set $V_{y,\epsilon} := \{x \in K : Lx = 0 \ \& \ v(x - y) \geq \epsilon\}/E_\epsilon$ as above. Then $V_{y,\epsilon}$ is a vector space over $C_K/E_0 \cong C_k$. Since the map in Lemma 4.1.10 is injective, it must have dimension less than or equal to n.

Similarly, let $U_{y,\epsilon}$ be $\{x \in K : Qx = 0 \ \& \ v(x - y) \geq \epsilon\}/E_\epsilon$. It is also a vector space over $C_K/E_0 \cong C_k$, of dimension exactly n. By a translating and scaling as above, we can construct for any $y \in U$ and $\epsilon \in \Gamma$ an isomorphism $U_{y,\epsilon} \cong T$.

Lemma 4.1.11. *If $La = 0$, then $a \in U$*

Proof. Suppose $La = 0$, but $a \notin U$. Let $\alpha := \rho(a, U)$, and take $b \in U$ with $v(a - b) = \alpha$.

Let **a** be the equivalence class of a in $V_{b,\alpha}$, so **a** consists of elements of $x \in V$ with $v(x - a) > \alpha$ and let z be its image under the above injection $\iota : V_{b,\alpha} \hookrightarrow T$, so z is an element of k. If c is the constant used in Lemma 4.1.9, then $z = \pi(c(a - b))$.

Let **b** be the class of $U_{b,\alpha}$ that maps to z under the same translation and scaling, which must exist since the map from $U_{b,\alpha}$ to T is an isomorphism. Let r be a representative of **b**; since we used the same scaling as above, this means $\pi(c(r - b)) = z$.

Since $U \subseteq V$, $r \in V$. Because $r \in V$ and $v(r-b) \geq \alpha$, $r \in V_{b,\alpha}$. From the fact that $r \in \mathbf{b}$ and $\iota(\mathbf{b}) = z$, it follows that $r \in \mathbf{a}$.

So $r \in U$, and $v(a - r) > \alpha$, a contradiction.

□

Equivalently, since $\pi(c(a - b)) = z$ and $\pi(c(r - b)) = z$, $v(c(a - b) - c(r - b)) > 0$, so $v(c(a - r)) = v(c) + v(a - r) > 0$, and $v(a - r) > -v(c) = \alpha$.

The translation and scaling in Lemmas 4.1.9 and 4.1.10 may also be seen directly in the following calculation:

Let a, b and α be as above, and let $d := a - b$, so $Ld = La - Lb = 0$ and $v(d) = \alpha$. Find a $c \in C_K$ with $v(c) = -\alpha$, which must exist because K has constants at every valuation, and let $g := cd$. Note that $g \in U$ if and only if $a \in U$.

Since g is a solution to $Lx = 0$, by the same argument used above for a, DHL applies to Q at g, providing an $h \in U$ with $v(g - h) > 0$. Let $j = g - h$ and $r = b + c^{-1}h$. Since b and h are both in U, so is r. Then

$$\begin{aligned}
v(a - r) &= v(a - (b + c^{-1}(h)) \\
&= v(a - (b + c^{-1}(g + j)) \\
&= v(a - b - c^{-1}(c(a - b) + j) \\
&= v(a - b - (a - b) - (c^{-1}j)) \\
&= \alpha + v(j) \\
&> \alpha
\end{aligned}$$

So either $a \in U$ and therefore $\rho(a, U) = \infty$, or as above we have found $r \in U$ with $v(a - r) > \rho(a, U)$, a contradiction.

From this, it follows that $V = U$ and $\dim(V) = n$.

□

Corollary 4.1.12. *Let L be a minimally integral linear* D*-operator of residual order n over $K \models \widetilde{VDF}$. Then the solutions to L in K form an n-dimensional vector space over the constants of K.*

Proof. Since $K \models \widetilde{\mathrm{VDF}}$, it is a D-henselian field with enough constants, so the above theorem applies. As k is differentially closed, the solutions to $\pi(L)$ in k will have dimension n as a C_k-vector space. □

4.2 D-fundamental systems of solutions

Given a linear D-equation ⊛ over a valued D-field K, we will be interested in adjoining as many solutions to ⊛ as possible without growing the constants. Therefore, sets of solutions to ⊛ that are maximal in this sense will play an important role. Keeping with the terminology of difference and differential fields, we will call them D-fundamental systems of solutions.

Definition 4.2.1. A D**-fundamental system of solutions** of a linear D-equation ⊛ of residual order n over a valued D-field K is a n-tuple $(f_1 \dots f_n)$ of elements in some extension K' of K such that

- Each f_i is a solution to ⊛.
- The set $\{f_1 \dots f_n\}$ is linearly independent over the constants of K'.
- The set $\{\pi(f_1) \dots \pi(f_n)\}$ is linearly independent over the constants of $\pi(K')$.
- The constants of $K\langle f_1 \dots f_n \rangle$ are the constants of K.

In general, if $L \supseteq K$ is an extension of valued D-fields and $C_L = C_K$, we call L a *no new constants* extension of K.

If K is a valued D-field whose constants C_K form a maximally complete valued subfield with residue field C_k algebraically closed, then any linear D-equation ⊛ over K will have a fundamental system of solutions; a differential fundamental system will exist for the residue equation, and can be lifted by DHL. Lemma 5.1.1 assures that this can be done without adding to the value group. Since C_K is assumed to be maximal, this implies that the extension will add no new constants.

If $\vec{f}$ and $\vec{g}$ are both fundamental systems of solutions to some linear D-equation ⊛ in some extension L of K, then $\vec{f} = A\vec{g}$ where A is some matrix over C_L; otherwise we could construct a solution space C_L dimension greater than n.

Chapter 5

Galois Theory

5.1 PVD-extensions

A vital element in the model theoretic approach to differential Galois theory is the existence of unique prime models over arbitrary sets for DCF_0, which provide a good notion of differential closure in which to work. However, the existence of unique prime models in DCF_0 is consequence of the fact that the theory is totally transcendental. Because of the ordering on the value group, $\widetilde{\mathrm{VDF}}$ is not even stable. It is not known, in general, whether $\widetilde{\mathrm{VDF}}$ such admits prime models. Therefore, it is not immediately clear that the field extensions generated by distinct fundamental systems of solutions to the same equation will be isomorphic. Fortunately, modulo a few reasonable assumptions on the constants of the base field, we will see that they are.

The following lemma (7.12 in [18]) will be useful.

Lemma 5.1.1. *Let K be a valued D-field. Given a type $p \in S_{1,k}(k_K)$ and a* D*-polynomial $P \in \mathcal{O}_K\langle X\rangle$ such that*

- *If $x \models p$ then $\pi(P)$ is of minimal total degree among nonzero $Q(X) \in \pi(\mathcal{O}_K)\langle X\rangle$ with $Q(x) = 0$, and*
- *$T.deg(P) = T.deg(\pi(P))$,*

there is a unique (up to $\mathcal{L}_K$-isomorphism) D*-field $L = K\langle a\rangle$ such that $P(a) = 0$ and $\pi(a) \models p$. The extension $K \subseteq L$ is unramified.*

The version of this lemma appearing in [18] does not include the condition that L is unramified in the statement of its conclusion; however, it is clear from the proof that the value group does not grow in the construction of $K\langle a\rangle$.

Furthermore, the requirement that $\mathrm{T.deg}(P) = \mathrm{T.deg}(\pi(P))$ is slightly stronger than is necessary. In fact, what is needed in the proof is that order-degree of P is equal to the order-degree of $\pi(P)$ and that the highest degree term of any order occurring in P has nonzero residue. We will use this fact in the last example in Chapter 6.

The next necessary ingredient is a slight modification of Theorem 5.8 in [2]. There, Azgin and van den Dries prove uniqueness for certain valued difference field extensions. Because the difference operators on their valued fields must induce nontrivial automorphisms the residue fields, the result does not immediately apply to our valued D-field case, where the residue field automorphism is the identity. However, by judiciously replacing σ with δ in most of their proof, the following can be recovered.

Theorem 5.1.2. *Let K be a valued* D*-field of equicharacteristic zero, whose residue field is linearly differentially closed. All maximal immediate extensions of K are K-isomorphic.*

Together, these two pieces allow us to prove the following extension of Theorem 5.1.2.

Theorem 5.1.3. *Let K be a valued* D*-field with residue field k. Let k' be the differential closure of k. Then there is a maximal unramified extension $K' \supseteq K$ with residue field k', unique up to isomorphism over K.*

Proof. To see that such a field must exist, note that for any valued D-field K and any extension k' of k, we can construct an unramified valued D-field extension L of K with residue field k' by repeatedly applying Lemma 5.1.1.

Now let K' be a maximal immediate extension of L. Such a K' must exist as the class of immediate extensions of L up to isomorphism is a set. This K' is an unramified extension of K having k as its residue field.

If k' is linearly differentially closed, such a K' is unique up to isomorphism over K. Given two candidate fields, we will show that they are isomorphic by first finding isomorphic intermediate subfields, then applying Theorem 5.1.2.

Lemma 5.1.4. *Let K be a valued* D*-field with residue field k, and suppose $k' \supseteq k$ is linearly differentially closed. If K_1 and K_2 are two maximal unramified valued* D*-field extensions of K each with residue field k', then there are subextensions $L_1 \subseteq K_1$ and $L_2 \subseteq K_2$ with $\pi(L_1) = \pi(L_2) = k'$ and $L_1 \cong_K L_2$.*

Proof. By a back and forth argument, it suffices to show that for each $a \in k'$, there is an $a_1 \in K_1$ and an $a_2 \in K_2$ such that $\pi(a_1) = \pi(a_2) = a$ and $K\langle a_1\rangle \cong_K K\langle a_2\rangle$.

Given $a \in k'$, find a minimal D-polynomial Q for a over k, and a lifting P of Q to $\mathcal{O}_K$ such that the hypotheses of Lemma 5.1.1 apply. Since $\pi(K_1) = \pi(K_2) = k'$, there are $b_i \in K_i$ such that $\pi(b_i) = a$. By the minimality of Q, there is some derivative of Q that does not vanish at a, so DHL applies to P at b_i. Since K_1 and K_2 are maximal and k' is linearly differentially closed, DHL holds in K_1 and K_2, giving $a_i \in K_i$ with $\pi(a_i) = a$ and $P(a_i) = 0$. Then $K\langle a_1\rangle \cong_K K\langle a_2\rangle$ by Lemma 5.1.1. □

Corollary 5.1.5. *With notation as in Lemma 5.1.4, K_1 and K_2 are isomorphic over K.*

Proof. Identifying L_1 and L_2 by the isomorphism in Lemma 5.1.4, K_1 and K_2 are both maximal immediate extensions of a valued field L whose residue field is linearly differentially closed. The result then follows, as promised, by Theorem 5.1.2. □

□

Given a linear D-equation ⊛ over a valued D-field K, and a fundamental system of solutions $\vec{f}$ of ⊛ in some extension of K, $K\langle\vec{f}\rangle$ will be our analogue of a Picard-Vessiot extension for the equation ⊛. With some restrictions on the constants of K ond k, this extension is independent of the choice of $\vec{f}$, up to isomorphism over K.

Theorem 5.1.6. *Let K be a valued* D*-field with C_K maximal and $\pi(C_K)$ algebraically closed. Let ⊛ be a linear* D*-equation over K and let $\vec{f}$ and $\vec{g}$ be* D*-fundamental systems of solutions to ⊛. Then $K_f := K\langle\vec{f}\rangle$ and $K_g := K\langle\vec{g}\rangle$ are isomorphic over K.*

Proof. As above, let k be the residue field of K, and let k' be the differential closure of k. Since C_k is algebraically closed, $C_{k'} = C_k$ and the residue fields of K_f and of K_g embed into k' over k.

By Theorem 5.1.3, we may find maximal unramified extensions K_1 of K_f and K_2 of K_g, both with residue field k'. Because K_f and K_g are unramified extensions of K, K_1 and K_2 are also maximal unramified extensions of K with a common differentially closed residue field. By the theorem, they are therefore isomorphic over K. Via this isomorphism K_f and K_g embed over K into a common maximally complete field M with residue field k' and value group Γ_K.

Since $\vec{f}$ and $\vec{g}$ are fundamental systems of solutions for ⊛, we have $\vec{f} = A\vec{g}$ for some matrix of constants in M. By maximality of C_K, $C_K = C_M$ and we conclude that the images of K_f and of K_g in M are equal. □

Note that the requirement that C_K be maximal cannot be dropped, as shown in Example 6.1.1.

5.2 Liaison Groups

We have shown that, given a linear D-equation ⊛ over a valued D-field K, we can associate to ⊛ a unique valued D-field extension L generated by a D-fundamental system of solutions to ⊛. We would now like to consider the automorphisms of this extension.

Let $\mathbf{Q}$ be the definable set of solutions to ⊛ and let $\mathbf{C}$ be the set of solutions to the equation $\mathrm{D}\,x = 0$. Then in any model $\mathcal{M}$ of $\widetilde{\mathrm{VDF}}$, the set $\mathbf{Q}(\mathcal{M})$ of points of $\mathbf{Q}$ in $\mathcal{M}$ is a vector space over $\mathbf{C}(\mathcal{M})$, the constants of $\mathcal{M}$. A basis of this vector space is given by a D-fundamental system $\vec{f}$ of solutions to ⊛, so the points of $\mathbf{Q}(\mathcal{M})$ are all definable over

$\{\mathbf{C}(\mathcal{M}) \cup \vec{f}\}$. This remains true in any extension $\mathcal{N} \supseteq \mathcal{M}$; $\mathbf{Q}(\mathcal{N})$ will be definable over $\{\mathbf{C}(\mathcal{N}) \cup \vec{f}\}$ for the same $\vec{f}$. The definable set $\mathbf{Q}$ of solutions to $\circledast$ is therefore internal to the definable set $\mathbf{C}$ of constants.

The group $\mathrm{MGal}(\mathbf{Q}/\mathbf{C})$ of model theoretic automorphisms of $\mathbf{Q}$ over $\mathbf{C}$ can thus be identified with a type definable group $\mathcal{G}$, and in any $\mathcal{U}$ there is a Galois correspondence between type definable subgroups of $\mathcal{G}$ and definably closed subsets S of $\mathcal{U}^{eq}$ with $\mathrm{dcl}^{eq}(\mathbf{C}) \subseteq S \subseteq \mathrm{dcl}^{eq}(\mathbf{Q})$.

Elements of the automorphism group are determined by their action on the internalizing set $\vec{f}$. Since the image of $\vec{f}$ under any automorphism will be another fundamental system of solutions to $\circledast$, and any two fundamental systems of solutions differ by a constant matrix, there is a map from the model theoretic automorphism group into the $\mathrm{GL}(\mathbf{Q})$, the group of linear transformations of the $\mathbf{C}$ vector space $\mathbf{Q}$, which can be identified with $\mathrm{GL}_n(\mathbf{C})$, using the basis $\vec{f}$.

As the theory $\widetilde{\mathrm{VDF}}$ has quantifier elimination, the structure preserved by this group is exactly that which can be described using quantifier free formulas. However, the presence of the valuation map means that these quantifier free formulas will describe more than the algebraic structure. This means that the model theoretic automorphism group will be a proper subgroup of the group preserving just the algebraic structure. Furthermore, subgroups of $\mathcal{G}$ can be defined using the valuation, leading to points on the group side of the Galois correspondence that do not occur as subgroups in the standard difference Galois correspondence and therefore can not correspond to subfields.

This is related to the failure of $\widetilde{\mathrm{VDF}}$ to eliminate imaginaries. In DCF_0, a consequence of elimination of imaginaries is that in every $\mathcal{M} \models \mathrm{DFC}_0$, every definably closed subset of $\mathcal{M}^{eq}$ is actually a substructure of $\mathcal{M}$, so the "field" side elements of the Galois correspondence are actually fields. In $\widetilde{\mathrm{VDF}}$, the "field" side elements of the correspondence will include at least the imaginaries of [5] coming from the valued field language, and possibly more, as shown in Example 6.2.1.

Finally, in a consequence of the ω-stability of DCF_0 is that every type definable group in that theory is actually definable. This also fails in $\widetilde{\mathrm{VDF}}$; a simple example of an equation with a model theoretic automorphism group that is type definable but not definable is given in Example 6.3.1.

Chapter 6

Examples

In this chapter, we look at several linear D-equations equations and their solution spaces in models of $\widetilde{\mathrm{VDF}}$. These examples are chosen to illustrate issues of definabilty and dimension, and to demonstrate the necessity of the maximality assumption on the constants in Chapter 5. For most of these equations, we compute the model theoretic automorphism group of the solutions space over some base field, usually the constants. We then compare these groups with the traditional difference Galois groups associated to the same equations, and to the differential Galois groups of the equation induced on the residue field. In the difference field case, we rely heavily on the algorithm from [6] outlined in the preliminaries.

To understand the internal structure of models of $\widetilde{\mathrm{VDF}}$, we will need to consider valuation ideals. These will allow us to look at approximations to solutions, and to iteratively construct such approximations. For $\alpha \in \Gamma$, these ideals are $\mathfrak{m}_{>\alpha} =: \{x \in R : v(x) > \alpha\}$ and $\mathfrak{m}_{\geq\alpha} =: \{x \in R : v(x) \geq \alpha\}$. If a and b in R are such that $v(a-b) > \alpha$ or $v(a-b) \geq \alpha$, we will sometimes write $a \equiv_{>\alpha} b$, or $a \equiv_{\geq\alpha} b$.

6.1 Maximality of Constants

The first example demonstrates the necessity of the requirement that the constants be maximal in Theorem 5.1.6, by showing that uniqueness may fail if the constants are assumed only to be algebraically closed.

Example 6.1.1. Let K be a valued D-field with enough constants, such that C_K is algebraically closed but not maximal. Assume that there is some $a \in K$ with $\pi(a) \neq 0$ such that the equation $\mathrm{D}\,x = ax$ has no nonzero solutions in K, but $\partial x = \pi(a)x$ has a full set of solutions in k, the residue field of K.

Let K' be a maximal immediate extension of K. Then $C_{K'}$ is also maximal. Take $c \in C_{K'} \setminus C_K$ with $\pi(c) = 1$. Such a c must exist because if d is any constant in $C_{K'} \setminus C_K$ with $v(d) \geq 0$, then for any $b \in C_K$ with $0 < v(b) < \infty$, we can take $c := 1 + bd$.

Let $\mathcal{U} \models \widetilde{\mathrm{VDF}}$ and take $f \in \mathcal{U}$ such that $K_f := K\langle f \rangle \subseteq \mathcal{U}$ is a no new constants extension of K. Let $g = cf$ and $K_g := K\langle g \rangle \subseteq \mathcal{U}$. Then K_g and K_f are not isomorphic as valued D-fields over K. To see this it suffices to show that $\mathrm{tp}(g/K)$ is not realized in K_f.

Since the residue equation $\partial x = \pi(a)x$ has a full set of solutions in k, we can build a pseudoconvergent sequence $\{x_\alpha\}$ approximating g in K, and therefore also in K_f. Then $\{f^{-1}x_\alpha\}$ is a pseudoconvergent sequence in K_f approximating c.

Now suppose there were some $\widetilde{g} \in K_f$ realizing $\mathrm{tp}(g/K)$. Then $\widetilde{g}$ would be a pseudolimit of $\{x_\alpha\}$ in K_f, and $f^{-1}\widetilde{g}$ would be a pseudolimit of $\{f^{-1}x_\alpha\}$ in C_{K_f}. The element $f^{-1}\widetilde{g}$ would therefore satisfy $tp(c/K)$, which we assumed to be unrealized in K, thereby contradicting the assertion that K_f introduced no new constants.

6.2 Fixed Fields and Imaginaries

Example 6.2.1. In the next two examples, we take as our base field K the generalized power series $\mathbb{C}(t)((\varepsilon^{\mathbb{Q}}))$ in the rational functions over $\mathbb{C}$, with $v(\varepsilon) = 1$, $\partial := \frac{d}{dt}$ in the residue field, and the D-operator defined on K as in Example 3.3.5 with $e = \varepsilon$. Since ε is a constant, $C_K = \mathbb{C}((\varepsilon^{\mathbb{Q}}))$.

We will use order one D-equations over this K to demonstrate some of the interesting structure on both sides of the model theoretic Galois correspondence in $\widetilde{\mathrm{VDF}}$.

First we will show that, because $\widetilde{\mathrm{VDF}}$ does not have elimination of imaginaries, certain subgroups of our automorphism groups will have"fixed fields" that are not fields.

Consider the equation:

$$\mathrm{D}\,x = x$$

1. *In the residue field*

   The equation reduces to $\partial x = x$ in the residue field. The solutions are constant multiples of "e^t", which is transcendental over our base field k. Therefore, the differential Galois group is $G_m(C_k)$, the multiplicative group of constants of the residue field.

2. *In a difference field*

   This equation is equivalent to $\sigma(x) = (1+\varepsilon)x$, the solutions to which can be shown to be transcendental over K. The difference Galois group associated to the equation is therefore $G_m(C_K)$.

3. *In a valued* D*-field*

   Any difference field solution to $Dx = x$ is obviously compatible with the valuation condition $v(\mathrm{D}\,x) \geq v(x)$, so the valued D-field extension L associated to the equation coincides with the difference field extension for the equation. However, D-field automorphisms of L must preserve both the difference field structure and the valuation, so the group of D-field automorphisms will be a subgroup of $G_v(C_K) = \{g \in G_m(C_K) : v(g) = 0\}$.

By Lemma 5.1.1, the type of an element $a \in L$ over K is completely determined by the type of $\pi(a)$ over the residue field k of K and a T.deg preserving choice of lifting of the minimal D-polynomial of $\pi(a)$ over k to K. Since $\mathrm{D}\,x - x = 0$ has the same T.deg as $\partial x - x = 0$, any two solutions to $\mathrm{D}\,x - x = 0$ with the same residue will have the same type over K, and every element in $G_v(C_K)$ preserves both these conditions. Thus, the group of D-field automorphisms of L over K is exactly $G_v(C_K)$.

4. *A definable subgroup*

   Let $H_{>0} := \{g \in G_M(C_K) : v(1-g) > 0\}$. Then $H_{>0}$ is a definable subgroup of $G_v(C_K)$. When it acts on the solutions to $\mathrm{D}\,x = x$, it preserves their leading terms. However, any two solutions with the same leading term are in the same orbit under the action of this group. The "fixed field" of this definable subgroup is therefore not a field, but is instead the imaginary structure of the leading terms of solutions to $\mathrm{D}\,x = x$.

6.3 Definability

The next example shows that the groups we need to consider may be properly type definable.

Example 6.3.1. We again take as our base field K the generalized power series $\mathbb{C}(t)((\varepsilon^{\mathbb{Q}}))$ in the rational functions over $\mathbb{C}$, with $v(\varepsilon) = 1$, $\partial := \frac{d}{dt}$ in the residue field, and the D-operator defined on K as in Example 3.3.5 with $e = \varepsilon$.

We look at the K-automorphisms of the difference, differential, and D-extensions associated to the equation

$$\mathrm{D}\,x = \varepsilon x$$

1. *The Standard Groups*

   Difference field: The equation $\mathrm{D}\,x = \varepsilon x$ is equivalent to $\sigma(x) = (1+\varepsilon^2)x$. Solutions to this equation are transcendental over the rational functions, and the corresponding group of automorphisms is therefore the multiplicative group of the constants: $G_m(C_K)$.

   Residue field: The equation $\mathrm{D}\,x = \varepsilon x$ reduces to $\partial x = 0$, so the solutions to the equation induced on the residue field are exactly the constants. As these are already in the base field, the differential Galois group is trivial.

2. *Valued* D*-field Group*

We expect the group of automorphisms of solutions to $\mathrm{D}\,x = \varepsilon x$ preserving the valued D-structure to be some subgroup of $G_v(C_K) \coloneqq \{g \in G_m(C_K) : v(g) = 0\}$ that reduces to the trivial group in the residue field. The simplest example of such a subgroup is $H_{>0} \coloneqq \{g \in G_m(C_K) : v(1-g) > 0\}$, as above. Any two solutions with the same image in the residue field can be mapped to one another by an element of this group.

However, the higher valuation ideals allow us to identify polynomial approximations to solutions, which must then be fixed by any automorphism that fixes $\mathbb{C}(t)$. The above group $H_{>0}$ does not fix this structure and therefore cannot be the group that we want.

Let f be any solution to $\mathrm{D}\,x = \varepsilon x$ in some no new constants extension L of K. We can assume without loss that $v(f) = 0$.

To approximate f in K, we start with the fact that

$$\mathrm{D}\,f \equiv 0 \mod \mathfrak{m}_{>1}$$

So up to $\mathfrak{m}_{>1}$, f is a constant, and since $C_L = C_K$, there must be some constant $c_0 \in K$ congruent to f mod $\mathfrak{m}_{>1}$. Rewrite f as $c_0 + \varepsilon f_0$.

As c_0 and ε are constants, and f is a solution to the original equation, we have:

$$\mathrm{D}(c_0 + \varepsilon f_0) = \varepsilon\,\mathrm{D}\,f_0 = \varepsilon(c_0 + \varepsilon f_0)$$

Reducing mod $v(\varepsilon^2) = 2$, this gives:

$$\varepsilon\,\mathrm{D}\,f_0 \equiv_{>2} \varepsilon c_0 \implies \mathrm{D}\,f_0 \equiv_{>2} c_0 \implies f_0 \equiv_{<2} c_0 t + c_1$$

for some $c_1 \in C_K$, so $f = (1+t)c_0 + c_1 + \varepsilon^2 f_1$ for some f_1.

This process can be repeated; while the twisted Leibniz rule destroys the familiar rules for integration, it is still always possible to calculate antiderivatives for polynomials; for example, the antiderivative of t^2 is $\frac{1}{6}(2t^3 - 3\varepsilon t^2 + \varepsilon^2 t)$.

By this iterative process, we can see that for any $n \in \omega$, f is congruent modulo n to something already in the base field. However, this is all we know. If c is a constant that is equal to 1 mod all the standard valuations, there is no first order formula with parameters in the base field satisfied by f but not cf. The group of automorphisms must be equal to the subgroup of the multiplicative group of the constants made up of such c's.

It is clear from its definition that this group is type definable, but not definable. This group comes up a lot in our calculations, for the remainder of these examples it will be denoted H_ω in analogy with the valuation restricted subgroup of the previous example.

6.4 Vector Space Dimension

The two examples below demonstrate two different reasons that a linear D-equation may be lacking in solutions: the equation may be hiding a trivial relation, or there can be issues with valuation compatibilty. Which case we are in is not apparent from the unfactored operators, but in both cases the C_K-vector space dimension of the solutions can be easily read off by looking at the residue field.

In both of the following examples, we take our base field K to be the generalized power series field $\mathbb{C}((\varepsilon^{\mathbb{Q}}))$, with the trivial D-operator and the constant e identified with ε.

Example 6.4.1. We first look at solutions to the equation

$$\varepsilon \, \mathrm{D}^2 x + (1-\varepsilon) \, \mathrm{D} \, x - x = 0.$$

1. *In the residue field*

   The equation reduces to $\partial x - x = 0$ in the residue field, so solutions are constant multiples of the exponential function. As these solutions are transcendental over the constants, their automorphism group is $G_m(C_k)$.

2. *In a difference field*

   Since $\varepsilon \, \mathrm{D}^2 x + (1-\varepsilon) \, \mathrm{D} \, x - x = 0$ has an order one reduction, we know from Chapter 4 that the solutions to this equation in a valued D-field will form a one dimensional vector space over the constants. To check what the solutions will be in a difference, field, let us first express the equation in terms of the automorphism σ.

   The identities

   $$\mathrm{D} \, x = \frac{1}{\varepsilon}(\sigma x - x) \text{ and } \mathrm{D}^2 \, x = \frac{1}{\varepsilon^2}(\sigma^2 x - 2\sigma x + x)$$

   allow us to write

   $$\begin{aligned} \varepsilon \, \mathrm{D}^2 \, x + &(1-\varepsilon) \, \mathrm{D} \, x - x = \\ &\frac{1}{\varepsilon}(\sigma^2 x - 2\sigma x + x) + \frac{1-\varepsilon}{\varepsilon}(\sigma x - x) - x = \\ &\frac{1}{\varepsilon}\big((\sigma^2 x - 2\sigma x + x) + (1-\varepsilon)(\sigma x - x) - \varepsilon x\big) = \\ &\frac{1}{\varepsilon}\big(\sigma^2 x - 2\sigma x + x + \sigma x - x - \varepsilon\sigma x + \varepsilon x - \varepsilon x\big) = \\ &\frac{1}{\varepsilon}\big(\sigma^2 x - (1+\varepsilon)(\sigma x)\big). \end{aligned}$$

   Since $\varepsilon \neq 0$, this equation is equivalent to

   $$\sigma^2 x - (1+\varepsilon)\sigma x = 0,$$

and since $\sigma(\varepsilon) = \varepsilon$, we can apply σ^{-1} to get

$$\sigma x - (1 + \varepsilon)x = 0$$

Thus, in this case, the *difference* field solutions have dimension at most one over the constants. Valuation compatibility did not come into play; the equation is simply hiding a trivial relation: $\varepsilon \mathrm{D}^2 + (1-\varepsilon)\mathrm{D} - \mathrm{id}$ factors as $(\mathrm{D} - \mathrm{id})(\varepsilon \mathrm{D} + \mathrm{id})$ and is therefore equivalent to "$\mathrm{D}\, x = x$ or $\sigma x = 0$."

As above, the difference Galois group associated to the equation $\mathrm{D}\, x = x$ is $G_m(C_K)$.

3. *In a valued* D*-field*

   We have already seen that this equation reduces to

$$\mathrm{D}\, x = x \text{ or } \sigma x = 0$$

   Again, as shown above, the group of valued D-field automorphisms of this equation will be $G_v(C_K) := \{g \in G_m(C_K) : v(g) = 0\}$.

By swapping signs, we arrive at an equation where the valuation condition is relevant:

$$(\mathrm{D} + \mathrm{id})(\varepsilon \mathrm{D} - \mathrm{id}) = \varepsilon \mathrm{D}^2 - (1 - \varepsilon)\mathrm{D} - \mathrm{id}$$

Example 6.4.2. The equation $\varepsilon \mathrm{D}^2 x - (1 - \varepsilon)\mathrm{D}\, x - x = 0$.

1. *In the residue field*

   The equation specializes to $\partial x = -x$, so the solutions come from adjoining "e^{-t}" to the base field, and the automorphism group over the constants is again the full multiplicative group of the constants, $G_m(C_k)$.

2. *In a difference field*

   To see how the solution space is affected by the valuation condition, we first look at solutions in a difference field extension.

   Using the identities above, we can see that the equation $\varepsilon \mathrm{D}^2 x - (1 - \varepsilon)\mathrm{D}\, x - x = 0$ is equivalent to

$$\frac{1}{\varepsilon}(\sigma^2 x - 2\sigma x + x) - \frac{1-\varepsilon}{\varepsilon}(\sigma x - x) - x = 0,$$

   which simplifies to

$$\sigma^2 x - (3 - \varepsilon)\sigma x + 2(1 - \varepsilon)x = 0.$$

   This does not reduce to an order one equation, so the solutions will form a two dimensional vector space over the constants in a sufficiently large difference field extension.

To check for relations between these solutions, we apply the Hendriks algorithm to compute the standard difference Galois group.

The associated Ricatti equation is:

$$u\sigma u - (3-\varepsilon)u + 2(1-\varepsilon) = 0.$$

Since we are looking for a solution in the constants, this is equivalent to

$$u^2 - (3-\varepsilon)u + 2(1-\varepsilon) = 0$$

which factors as

$$(u-2)(u-(1-\varepsilon)) = 0.$$

Thus, the Ricatti equation has two distinct solutions in the constants, $u_1 = 2$ and $u_2 = (1-\varepsilon)$, and the standard difference Galois group is "completely reducible, but not an algebraic subgroup of $\{c.Id | c \in C_K\}$." It is therefore a subgroup of the group D of 2×2 diagonal matrices over C_K.

As the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 1-\varepsilon \end{pmatrix}$$

is an element of the group and it lies in no proper subgroup (since for all $i, j \in \mathbb{Z}_{\geq 1}$, $2^i(1-\varepsilon)^j \neq 1$), the group must be all of D.

3. *In the valued* D*-field*

Again, we know from Chapter 4 that this equation can have at most a one dimensional space of solutions over the constants of a valued D-field. However, we have seen above that in the standard difference theory, it is possible to adjoin two solutions that are linearly independent over the constants. Therefore, something in our valued D-field axioms must be preventing us from adjoining the second solution.

To see directly how the valuation condition will lead to this restriction, we show that if h is the D-wronskian of two linearly independent solutions to the equation, we cannot have $v(\mathrm{D}\, h) \geq h$.

Suppose that f and g are both solutions to $\varepsilon\, \mathrm{D}^2 x - (1-\varepsilon)\, \mathrm{D}\, x - x = 0$ in some $L \models \widetilde{\mathrm{VDF}}$. Let $h := f\, \mathrm{D}\, g - g\, \mathrm{D}\, f$. Then

$$\begin{aligned} \mathrm{D}\, h &= \mathrm{D}(f\, \mathrm{D}\, g - g\, \mathrm{D}\, f) \\ &= \mathrm{D}\, f\, \mathrm{D}\, g + f\, \mathrm{D}^2 g + \varepsilon\, \mathrm{D}\, f\, \mathrm{D}^2 g - \mathrm{D}\, g\, \mathrm{D}\, f - g\, \mathrm{D}^2 f - \varepsilon\, \mathrm{D}^2 f\, \mathrm{D}\, g \\ &= f\, \mathrm{D}^2 g - g\, \mathrm{D}^2 f + \varepsilon\, \mathrm{D}\, f\, \mathrm{D}^2 g - \varepsilon\, \mathrm{D}^2 f\, \mathrm{D}\, g \\ &= (f + \varepsilon\, \mathrm{D}\, f)(\mathrm{D}^2 g) - (g + \varepsilon\, \mathrm{D}\, g)(\mathrm{D}^2 f) \end{aligned}$$

Substituting $\mathrm{D}^2 f = \frac{1}{\varepsilon}((1-\varepsilon)\mathrm{D}\, f + f)$ and $\mathrm{D}^2 g = \frac{1}{\varepsilon}((1-\varepsilon)\mathrm{D}\, g + g)$, we get

$$\begin{aligned}(f+\varepsilon\mathrm{D}\, f)(D^2 g) &= \frac{1}{\varepsilon}(f+\varepsilon\mathrm{D}\, f)((1-\varepsilon)\mathrm{D}\, g + g)\\ &= \frac{1}{\varepsilon}\left((1-\varepsilon) f\,\mathrm{D}\, g + fg + (\varepsilon-\varepsilon^2)\mathrm{D}\, f\,\mathrm{D}\, g + \varepsilon g\,\mathrm{D}\, f\right)\end{aligned}$$

and

$$\begin{aligned}(g+\varepsilon\mathrm{D}\, g)(D^2 f) &= \frac{1}{\varepsilon}(g+\varepsilon\mathrm{D}\, g)((1-\varepsilon)\mathrm{D}\, f + f)\\ &= \frac{1}{\varepsilon}\left((1-\varepsilon) g\,\mathrm{D}\, f + fg + (\varepsilon-\varepsilon^2)\mathrm{D}\, f\,\mathrm{D}\, g + \varepsilon f\,\mathrm{D}\, g\right)\end{aligned}$$

Subtracting the two yields

$$\begin{aligned}\mathrm{D}\, h &= \frac{1}{\varepsilon}\left((1-\varepsilon)(f\,\mathrm{D}\, g - g\,\mathrm{D}\, f) - \varepsilon(f\,\mathrm{D}\, g - g\,\mathrm{D}\, f)\right)\\ &= \frac{1}{\varepsilon}(1-2\varepsilon)h = \left(\frac{1}{\varepsilon}+2\right)h.\end{aligned}$$

Since the equation $\varepsilon\mathrm{D}\, x = (1-2\varepsilon)x$ has no nonzero solutions in any valued D-field satisfying $v(\mathrm{D}\, x) \geq v(x)$, $h = 0$.

From this and Lemma 3.4.7, we can conclude that $f = cg$ for some constant c. Therefore, the space of solutions to $\varepsilon\mathrm{D}^2 x - (1-\varepsilon)\mathrm{D}\, x - x = 0$ in $K \models \widetilde{\mathrm{VDF}}$ form a one dimensional vector space over C_K.

A similar calculation will work for any order two linear difference equation of residual order one. For higher order equations, the proof in Chapter 4 appears to be necessary to establish the C_K-vector space dimension of the solutions.

In this particular case, it is clear from the choice of equation that the difference field solutions occurring in the valued D-field are setting are those that satisfy $\mathrm{D}\, x + x = 0$. The equation $\varepsilon\mathrm{D}\, x - x = 0$ has no nonzero solutions compatible with the requirement $v(\mathrm{D}\, x) \geq v(x)$. In fact, because of the vector space structure on the solutions, the solution space for any linear D-equation ⊛ can also be described as the solution space of a linear D-equation or equations having the same order and residual order, although it might not always be so easy to read off.

The group of difference field automorphisms of these solutions over K is again $G_m(C_K)$. Any D-field automorphism must preserve both the difference structure and valuation, so it is clear that the group we want will be a subgroup of $G_v(C_K) := \{g \in G_m(C_K) : v(g) = 0\}$.

To see that there are no further restrictions on the group, we again note that by Lemma 5.1.1, the type of an element $a \in L$ over K is completely determined by the type of $\pi(a)$ over the residue field k of K and a T.deg preserving choice of lifting of the minimal D-polynomial of $\pi(a)$ over k to K. Since $\mathrm{D}\, x + x = 0$ has the same T.deg as $\partial x + x = 0$, any two solutions to $\mathrm{D}\, x + x = 0$ with the same residue will have the same type over K, and every element in $G_v(C_K)$ preserves this type.

6.5 Algebraic Relations

The examples below explore the algebraic relations that may hold inside these vector spaces of solutions. In contrast to Theorem 4.1.6, which showed that the C_K linear structure of the solution spaces lifts from the residue field, we show that nonlinear algebraic relations between solutions in residue field cannot be expected to lift to relations between solutions in the valued field.

As in the first examples, we take as our base field K the generalized power series $\mathbb{C}(t)((\varepsilon^{\mathbb{Q}}))$ in the rational functions over $\mathbb{C}$, and consider solutions in $K' \supseteq K$ with $K' \models \widetilde{\mathrm{VDF}}$.

Example 6.5.1. The equation $\mathrm{D}^2\, x + x = 0$.

1. *In the residue field*

   Solutions to $\mathrm{D}^2\, x + x = 0$ in a valued D-field reduce to solutions to the equation

   $$\partial^2 x + x = 0$$

   in the residue field. These solutions, in turn, are linear combinations of solutions to the equations $\partial x = ix$ and $\partial x = -ix$.

   Let f be any nonzero solution to $\partial x = ix$. Then, by the chain rule, $g = f^{-1}$ is a solution to $\partial x = -ix$. These two solutions are linearly independent over C_k, and therefore generate a full set of solutions to $\partial^2 x + x = 0$. This space of solutions is a two dimensional vector space over C_k. However, because of the algebraic relation $g = f^{-1}$, the Picard Vessiot extension for the equation over C_k is generated as a field by only one element, and the differential Galois group of the equation is thus the multiplicative group of the constants.

2. *In the Valued* D*-field*

   Since $\mathrm{D}^2\, x + x = 0$ has the same complexity as its reduction, any basis for the solutions to $\partial^2 x + x = 0$ in the residue field will lift to a basis of the original equation in K'.

   Let f and g be as above, and let F and G be solutions to $\mathrm{D}\, x = ix$ and $\mathrm{D}\, x = -ix$, respectively, such that $\pi(F) = f$ and $\pi(G) = g$. Then F and G are a basis of solutions for $\mathrm{D}^2\, x + x = 0$, but the relation $F^{-1} = G$ does not hold, since if $\widetilde{F} := F^{-1}$,

$$\mathrm{D}\,\widetilde{F} = \frac{-iF}{F\sigma(F)} = \sigma\left(\frac{-i}{F}\right) \neq -i\widetilde{F}.$$

In fact, $\widetilde{F}$ is not a solution to $\mathrm{D}^2\,x + x = 0$ at all, but rather satisfies $\mathrm{D}^2\,x + \sigma^2 x = 0$, or equivalently $(1+\varepsilon^2)D^2x + 2\varepsilon\,\mathrm{D}\,x + x = 0$.

Furthermore, F and G satisfy *no* algebraic relation over the constants. This is a consequence of the more general calculation worked out in the following example.

However, if a is a solution to $\mathrm{D}\,x = \varepsilon x$, aF^{-1} is a solution to $\mathrm{D}\,x = -ix$, so using the methods of example 6.3.1, we can find an approximation of G in $K\langle F\rangle$.

The group of automorphisms of $K\langle F, G\rangle$ over K is therefore $G_v(C_K) \times H_\omega$. The reflects in a concrete way the idea that the algebraic relation coming from the residue field *almost* holds.

Example 6.5.2. Other liftings of the residue equation $\partial^2 x + x = 0$.

We now consider all order two liftings of the differential equation $\partial^2 x + x = 0$ over C_K, to determine if any lifting will correspond to a transcendence degree one field extension.

After possibly scaling by a constant, any degree two lifting of $\partial^2 x + x = 0$ will be of the form

$$D^2 x + \varepsilon a\,\mathrm{D}\,x + (1+\varepsilon b)x = 0 \tag{6.5.1}$$

with $v(a)$ and $v(b)$ both greater than or equal to zero.

First we compute the group of automorphisms to the difference field solution space of Equation 6.5.1 using the algorithm from [6]. Since the order of these liftings will match their residual order, the valued D-field solutions will coincide with the difference field solutions to the equation, and the valued D-field automorphisms group will be a subgroup of the difference Galois group.

1. *In a difference field*

   We follow the Hendriks Algorithm [6].

   (a) *Express as a difference equation*

   Via the identification $\sigma(x) = x + \varepsilon\,\mathrm{D}\,x$, Equation 6.5.1 becomes

   $$\frac{1}{\varepsilon^2}\left(\sigma^2(x) - 2\sigma(x) + x\right) + a(\sigma(x) - x) + (1+\varepsilon b)x = 0.$$

   Multiplying through by ε^2 gives:

   $$\left(\sigma^2(x) - 2\sigma(x) + x\right) + \varepsilon^2 a(\sigma(x) - x) + \varepsilon^2(1+\varepsilon b)x = 0.$$

And collecting like terms yields:

$$\sigma^2(x) - (2 - \varepsilon^2 a)\sigma(x) + (1 - \varepsilon^2 a + \varepsilon^2 + \varepsilon^3 b)x = 0$$

(b) *The Riccati Equation*

The corresponding Riccati equation is

$$u\sigma(u) - (2 - \varepsilon^2 a)u + (1 - \varepsilon^2 a + \varepsilon^2 + \varepsilon^3 b) = 0$$

(c) *Roots of the Riccati Equation*

As we have taken our base field to be the constants, the Riccati equation is equivalent to

$$u^2 - (2 - \varepsilon^2 a)u + (1 - \varepsilon^2 a + \varepsilon^2 + \varepsilon^3 b) = 0$$

which has for solutions

$$\frac{2 - \varepsilon^2 a \pm \sqrt{(2 - \varepsilon^2 a)^2 - 4(1 - \varepsilon^2 a + \varepsilon^2 + \varepsilon^3 b)}}{2}$$

The discriminant simplifies to

$$\varepsilon^4 a^2 - 4\varepsilon^2 - 4\varepsilon^3 b = -\varepsilon^2(4 + 4\varepsilon b - \varepsilon^2 a^2)$$

Assume for now that C_K is algebraically closed. Since this discriminant will never be zero, the Ricatti equation will always have exactly two solutions in C_K.

(d) *Consequences for group*

By the Hendriks algorithm, the regular difference Galois group G of the equation is then completely reducible, but not an algebraic subgroup of $\{c.Id : c \in G_m(C_K)\}$, regardless of our choices of a and b.

(e) *Subgroup calculation*

Let G be the regular difference Galois group of the equation, and let u_1 and u_2 denote the two roots of the Riccati equation. Either G is the full group of diagonal matrices over C_K, or for some m and n in $\mathbb{Z}_{\neq 0}$, we have $(u_1)^n(u_2)^m = 1$.

Assume such m and n exist. Now, u_1 and u_2 are given by

$$u_1 = 1 - \frac{\varepsilon^2 a}{2} + \varepsilon i\sqrt{1 + \varepsilon b - \frac{\varepsilon^2 a}{4}} \text{ and } u_2 = 1 - \frac{\varepsilon^2 a}{2} - \varepsilon i\sqrt{1 + \varepsilon b - \frac{\varepsilon^2 a}{4}}$$

reduced modulo $\mathfrak{m}_{>v(\varepsilon)}$, this gives

$$(u_1)^n \equiv 1 + ni\varepsilon \text{ and } (u_2)^m \equiv 1 - mi\varepsilon$$

from which it follows that $m = n$, and $u_1 u_2$ is a root of unity. The full reduction map π is injective on roots of unity and $\pi(u_1 u_2) = 1$, but multiplying out the full expressions shows that for no choice of a and b is $u_1 u_2 = 1$. Thus, no such m and n exist, and the standard difference Galois group of *any* degree two lifting of $\partial^2 x + x = 0$ over an algebraically closed C_K is the full group of 2 by 2 diagonal matrices over C_K.

2. *In a valued D-field*

   Fix $f \in k$ with $\partial f = if$. Then by DHL there is some F in K' satisfying Equation 6.5.1 and $\pi(F) = F$, and by Lemma 5.1.1 its type over K is completely determined by this information. Similarly, taking $g \in k$ to be f^{-1} for this f, g satisfies $\partial g = -ig$ and there is some lifting $G \in K'$ of g that satisfies Equation 6.5.1, and this G can be chosen so that $K\langle F, G\rangle$ is an unramified no new constants extension of K.

   By the calculation above, $G \notin K_F := K\langle F\rangle$, but since $\pi(F^{-1}) = \pi(G) = g$, $v(F^{-1} - G) > 0$, and this relationship must be preserved by the valued D-field automorphisms of $K\langle F, G\rangle$. To go even further, let $(F^{-1} - G) = B$, and let $v(A) = \alpha$. There is some constant $c \in C_K$ with $v(A) = \alpha$; let B be cA, so $v(B) = 0$, and let $\pi(B) = b$. There is some $J \in K_F$ with $\pi(J) = b$ as well, so $(F^{-1} + c^{-1}J)$ is a better approximation of G than F^{-1}. This process can be repeated to provide successively better approximations of G in K_F. By this method one cannot rule out the accuracy of these approximations approaching a limit γ in the value group, but it is clear that this is the worst that can happen. Thus, the automorphisms of $K\langle F, G\rangle$ over K_F cannot be any larger than $H_{>\gamma} := \{g \in G_m(C_K) : v(g) > \gamma\}$, which is type definable if $\gamma \notin \Gamma$.

   For such a lifting of $\partial x + x = 0$, the group of automorphisms of $K\langle F, G\rangle$ over K is therefore always strictly smaller than $G_v(C_K) \times G_v(C_K)$ the valuation preserving subgroup of the difference Galois group, but larger than $G_v(C_K)$, the smallest lifting of the residue group.

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