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# Conditional Expectation and Bicontractive Projections on Jordan $C^*$-algebras and Their Generalizations*

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On a Banach space $X$ a projection $P \in \mathscr{L}(X)$ is called *bicontractive* if $\|P\| \leqq 1$ and $\|I-P\| \leqq 1$. Such projections may be constructed as follows. Let $\theta$ be an isometry of $X$ onto $X$ of order 2. Then

$$(0.1) \qquad P = \tfrac{1}{2}(I+\theta)$$

is a bicontractive projection. It would be of interest to characterize the class $\mathscr{S}$ of those Banach spaces for which every bicontractive projection is of the above form. Evidently $X \in \mathscr{S}$ if $X'$, its dual, belongs to $\mathscr{S}$.

The purpose of this paper is to show that if $X$ is a $JB^*$-triple, then $X \in \mathscr{S}$. It follows that all $JB^*$-algebras (= Jordan $C^*$-algebras) and their duals and preduals (when existing) belong to $\mathscr{S}$.

By a theorem of Kaup [14, 18], $JB^*$-triples are precisely those Banach spaces (within isometric isomorphism) for which the open unit ball is a bounded symmetric domain (within biholomorphic equivalence). For the precise definition and basic properties of $JB^*$-triples we refer to [11, 14, 18].

In Bernau-Lacey [3] it is shown that $X = L_p(\mu)$, $1 \leqq p < \infty$ belongs to $\mathscr{S}$. Earlier Byrne-Sullivan [4] proved the special cases $1 < p < \infty$ and $\mu$ a probability measure. As a bi-product of the classification of all contractive projections on the space $C_1$ of all trace class operators on a separable Hilbert space, Arazy-Friedman show in [1] that $C_1$, and therefore its predual (the space of all compact operators), belongs to $\mathscr{S}$.

A study of unit preserving bicontractive projections on unital Jordan (operator) algebras was begun in Robertson-Youngson [16]. Using some ideas from [16] and the classification of type $I$ Jordan factors, Stormer [17] proved that every unital bicontractive projection on an arbitrary $C^*$-algebra is of the form (0.1) with $\theta$ a Jordan automorphism of order 2.

In [9] the authors showed that every $C^*$-algebra belongs to $\mathscr{S}$. This result generalized the above mentioned results of Arazy-Friedman and of Stormer.

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The proof was based on the authors' detailed study of contractive projections in a setting of operator algebras without order [7, 8, 10]. It was actually proved for the class of $J^*$-algebras and therefore also showed that all $JC^*$-algebras belong to $\mathcal{S}$.

The proof of our main result below depends only on global properties of $JB^*$-triples and contractive projections thereon, as developed in [11, 15] and Sects. 1 and 2 below. As such it includes as true corollaries all of the above mentioned results of Arazy-Friedman, Robertson-Youngson, Stormer, and the authors on bicontractive projections.

Recall that a $J^*$-algebra is a concrete example of a $JB^*$-triple. Other examples of $JB^*$-triples which are not $J^*$-algebras are the two exceptional $JB^*$-triples of dimensions 16 and 27, which we denote by $C^5$ and $C^6$. The structure of an arbitrary $JB^*$-triple has been studied by the authors in [11] and [12]. As a result of this study some problems for $JB^*$-triples can be reduced to the corresponding problem for $J^*$-algebras and the two exceptional $JB^*$-triples. The former act on Hilbert space and the latter are finite dimensional.

It is natural to attempt a direct proof that $C^5$ and $C^6$ belong to $\mathcal{S}$ and to use this, together with the structure theorem mentioned above, in order to show that an arbitrary $JB^*$-triple belongs to $\mathcal{S}$.

We found that this approach seems to require showing the invariance of the special and exceptional summands of a $JBW^*$-triple under the projection. To prove this would require much of the fine structure of a contractive projection, as developed in [7] and [10] for $J^*$-algebras. Hence it is more efficient to follow the outline that solved the bicontractive projection problem for $J^*$-algebras [9].

This paper is organized as follows. In Sect. 1 we prove some commutativity formulas involving a contractive projection and the Peirce projections associated with an element in the range of the dual projection. The corresponding formulas for the $J^*$-algebra case played important roles in the study of a contractive projection, and the same is true here. The main tools used in the proofs of these formulas are Propositions 1, 2 and 3 of [11]. In Sect. 2 we exploit two results of Kaup [15] to develop the fine structure of a contractive projection $P$ on a $JB^*$-triple $U$. The main results, Theorems 2 and 3, give a concrete realization of $P(U)$ in the second dual $U''$, and a new conditional expectation formula for $P$. Our final result, that $JB^*$-triples belong to the class $\mathcal{S}$, is proved in Sect. 3.

The following are some of the notational conventions used in this paper. If $X$ is a Banach space, $X'$ denotes its normed dual and $X_*$ denotes a predual of $X$, i.e., $(X_*)' \approxeq X$ (isometric). We use the same notation, namely $P_k(v)$, $k=0,1,2$, for the Peirce projections associated with a tripotent $v$, and their adjoints. For a normal functional $f$ on a $JBW^*$-triple $U$, $e(f)$ denotes the tripotent occuring in the polar decomposition of $f$. We write $P_k(f)$ for $P_k(e(f))$. Two functionals $f, g$ are orthogonal, denoted by $f \perp g$, if $e(f) \in P_0(g)\, U$. The symbol $\perp$ will also denote orthogonality of elements of $U$.

The following consequence of [7: Lemma 2.4] and [11: Cor. 1.6] will be used several times:

(0.2) If $P$ is a contractive projection on a $JB^*$-triple $U$ and $f \in P'(U')$, then with $v = e(f)$ we have $P''v = v + P_0(f) P''v$.

More generally:

(0.3) If $x \in U''$, $\|x\| = 1$, and $f(x) = \|f\|$, for some $f \in U'$, then $x = v + P_0(f) x$.

## 1. Commutativity Formulas for Contractive Projections

In this section we use freely the notation and results of Friedman-Russo [11: Sect. 1]. By Dineen [5] and Barton-Timoney [2], if $U$ is a $JB^*$-triple, we may regard $U'$ as the predual of the $JBW^*$-triple $U''$.

**Lemma 1.1.** *Let $U$ be a $JB^*$-triple and let $e$ be a tripotent of $U''$. For each $g \in P_2(e) U'$, let $T(g)$ denote the restriction of $g$ to $P_2(e) U''$. Then $T$ is an isometric isomorphism of $P_2(e) U'$ onto $(P_2(e) U'')_*$.*

*Proof.* The map $T$ is clearly linear, norm decreasing and takes $P_2(e) U'$ into $(P_2(e) U'')_*$. By [11: Prop. 1] $T$ is onto and isometric. □

The proof of the following proposition is the same as [7: Prop. 3.3], with Lemma 1.1 replacing [7: Remark 3.2].

**Proposition 1.2.** *Let $P$ be a contractive projection on a $JB^*$-triple $U$ and let $f \in P'(U')$. Then $P' P_2(f) = P_2(f) P' P_2(f)$.*

*Proof.* Let $B := (P_2(f) U'')_*$, $T$: $P_2(f) U' \to B$ as in Lemma 1.1 and let $V_f$ be the face generated by $T(f)$ in $B^+$. By Emch-King [6, 13] $V_f$ is norm dense in $B^+$. Since $B^+$ linearly spans $B$, it will suffice, by Lemma 1.1, to prove that $P'(T^{-1}(V_f)) \subseteq P_2(f) U'$.

For $\tau \in V_f$, write $T(f) = \alpha\tau + \sigma$ for some $\alpha > 0$ and $\sigma \in V_f$. Then $f = \alpha\tilde{\tau} + \tilde{\sigma}$ where $\tilde{\tau} = T^{-1}(\tau)$ and $\tilde{\sigma} = T^{-1}(\sigma)$, and

$$\begin{aligned}\|f\| &= \|T(f)\| = \alpha\|\tau\| + \|\sigma\| = \alpha\|\tilde{\tau}\| + \|\tilde{\sigma}\| \\ &\geqq \alpha\|P'\tilde{\tau}\| + \|P'\tilde{\sigma}\| \geqq \alpha\|P_2(f) P'\tilde{\tau}\| + \|P_2(f) P'\tilde{\sigma}\| \geqq \|f\|\end{aligned}$$

since $f = P_2(f) P' f = P_2(f) P'(\alpha\tilde{\tau} + \tilde{\sigma})$. Therefore $\|P'\tilde{\tau}\| = \|P_2(f) P'\tilde{\tau}\|$ so by [11: Prop. 1] $P'\tilde{\tau} = P_2(f) P'\tilde{\tau}$. □

In order to prove our second commutativity formula we need a lemma which generalizes [7: Lemma 3.4].

**Lemma 1.3.** *Let $P$ be a contractive projection on a $JB^*$-triple $U$ and let $f \in P'(U')$. Then $P_0(f) P' P_1(f) = 0$.*

*Proof.* With $v = e(f)$ consider the map $\pi$: $P_1(f) U'' \to U'$ defined by $\pi(y) = D(v, y) f$. We shall show that $\pi$ is a linear bijection of $P_1(f) U''$ onto a norm dense subspace $S$ of $P_1(f) U'$. Then we shall show that $P_0(f) P' g = 0$ for all $g \in S$, completing the proof.

We show first that $S \subseteq P_1(f)\,U'$. Let $x \in U''$. By the Peirce rules [11: (1.7)], and the fact that $f = P_2(f)\,f$,

$$\langle D(v,y)\,f, x\rangle = \langle f, P_2(f)\,\{v\,y\,x\}\rangle = \langle f, \{v\,y\,P_1(f)\,x\}\rangle$$
$$= \langle P_1(f)\,D(v,y)\,f, x\rangle.$$

Thus $S \subseteq P_1(f)\,U'$.

Now let $z \in U''$ satisfy $\langle z, S\rangle = 0$, so that $\langle z_1, S\rangle = 0$ where $z_1 = P_1(f)\,z$. Then $f\{v\,y\,z_1\} = 0$ for all $y \in P_1(f)\,U''$ and in particular $f\{v\,z_1\,z_1\} = 0$. Since, by [11: Sect. 1], $\{v\,z_1\,z_1\}$ is positive in $P_2(f)\,U''$ and $f$ is faithful there, $\{v\,z_1\,z_1\} = 0$ and so $z_1 = 0$. Thus $\langle z, P_1(f)\,U'\rangle = 0$, proving that $S$ is norm dense in $P_1(f)\,U'$.

Next, let $g = D(v,y)\,f \in S$ for some $y \in P_1(f)\,U''$. We shall show that $P_0(f)\,P'g = 0$. Since $D(y,v) + D(v,y)$ is hermitian

$$|\langle f, (\exp i\,t(D(y,v) + D(v,y)))\,x\rangle| \leqq 1$$

for all $x \in U''$ with $\|x\| \leqq 1$ and all $t \in R$. Therefore

$$|f(x) + i\,t\,f(\{y\,v\,x\} + \{v\,y\,x\}) + O(t^2)| \leqq 1,$$

and since by the Peirce rules

$$f\{y\,v\,x\} = 0, \qquad |f(x) + i\,t\,f\{v\,y\,x\}| \leqq 1 + O(t^2).$$

Thus

$$\|f + i\,t\,g\| \leqq 1 + O(t^2).$$

Set $w_t = P'(f + i\,t\,g)$. Since $P''v = v + P_0(f)\,P''v$,

$$w_t(v) = \langle P'f, v\rangle + i\,t\langle P'g, v\rangle = 1.$$

Therefore $1 \leqq \|P_2(f)\,w_t\| \leqq \|w_t\| \leqq 1 + O(t^2)$.

Finally, since by Proposition 1.2 $P_0(f)\,P'f = 0$, we have

$$1 + t\,\|P_0(f)\,P'g\| \leqq \|P_2(f)\,w_t\| + \|P_0(f)\,P'(f + i\,t\,g)\|$$
$$= \|(P_2(f) + P_0(f))\,w_t\| \leqq \|w_t\| = 1 + O(t^2),$$

which forces $P_0(f)\,P'g = 0$. □

By using this lemma and Proposition 1.2, the following proposition can be proved exactly as [7: Prop. 3.5].

**Proposition 1.4.** *Let $P$ be a contractive projection on a JB*-triple $U$ and let $f \in P'(U')$. Then $P_0(f)\,P' = P_0(f)\,P'\,P_0(f) = P'\,P_0(f)\,P'$ and $P_0(f)\,P'$ and $P_1(f)\,P'$ are projections.*

*Proof.* Writing $P_k$ for $P_k(f)$ we have

$$P_0\,P' = P_0\,P'(P_2 + P_1 + P_0) = P_0\,P'\,P_2 + P_0\,P'\,P_1 + P_0\,P'\,P_0 = P_0\,P'\,P_0$$

by Proposition 1.2 and Lemma 1.3. Also $P_0\,P'\,P_0\,P' = P_0\,P'\,P' = P_0\,P'$ is a projection.

Let $g \in U'$. Then $\|P_0 P' g\| = \|P_0 P' P_0 P' g\| \leqq \|P' P_0 P' g\| \leqq \|P_0 P' g\|$. By [11: Prop. 1] $P' P_0 P' g = P_0 P' P_0 P' g = P_0 P' g$. Finally,

$$P_1 P' = P_1 P'(P_2 + P_1 + P_0) P' = P_1(P' P_2) P' + P_1(P' P_0 P') + P_1 P' P_1 P' = P_1 P' P_1 P'. \quad \square$$

To establish our final commutativity formula we need a lemma which generalizes and considerably simplifies [7: Cor. 4.2].

**Lemma 1.5.** *Let $v$ be a tripotent in $U''$ where $U$ is a $JB^*$-triple and suppose $f \in P_2(v) U'$ and $g \in P_1(v) U'$. If $f \neq 0$, then $\|f+g\| > \|g\|$.*

*Proof.* Suppose that $\|f+g\| \leqq \|g\|$. Let $u = e(g)$. Then by [11: Prop. 3], $u \in P_1(v) U''$ and $[P_k(v), P_j(u)] = 0$ for all $j, k$. Thus $\langle f+g, u\rangle = \langle g, u\rangle = \|g\| \geqq \|f+g\|$. By [11: Prop. 1], $f+g$ and $g$ belong to $P_2(u) U'$, implying $f \in P_2(u) U'$. Let $x \in U''$ be such that $f(x) \neq 0$ and set $y = P_2(v) P_2(u) x$. Denote by $\#$ the involution on the $JBW^*$-algebra $P_2(u) U''$. Then by Peirce rules with respect to $v$, $y^\# = \{u y u\} \in P_0(v) U''$. Since $g(y) = g(P_1(v) y) = 0$ and $f+g$ is positive, hence self-adjoint on $P_2(u) U''$, we have

$$\overline{f(x)} = \overline{f(y)} = \overline{\langle f+g, y\rangle} = \langle f+g, y^\#\rangle = 0, \quad \text{a contradiction.} \quad \square$$

The proof of our final commutativity formula is the same as [7: Prop. 4.3] with Lemma 1.5 in place of [7: Cor. 4.2].

**Proposition 1.6.** *Let $P$ be a contractive projection on a $JB^*$-triple $U$ and let $f \in P'(U')$. Then $P_1(f) P' = P' P_1(f) P'$, $P_2(f) P' = P' P_2(f) P'$, and $P_2(f) P'$ is a projection.*

*Proof.* We have

$$P' P_1 P' = (P_2 + P_1 + P_0) P' P_1 P' = P_2 P' P_1 P' + P_1 P' P_1 P' + P_0 P' P_1 P' = P_2 P' P_1 P' + P_1 P'.$$

Therefore for arbitrary $g \in U'$,

$$\|P_2(P' P_1 P' g) + P_1(P' g)\| = \|P' P_1 P' g\| \leqq \|P_1(P' g)\|,$$

and by Lemma 1.5, $P_2 P' P_1 P' g = 0$, so $P' P_1 P' = P_1 P'$.

Furthermore,

$$\begin{aligned} P_2 P' &= (1 - P_1 - P_0) P' = P' - P_1 P' - P_0 P' \\ &= P' - P' P_1 P' - P' P_0 P' = P'(1 - P_1 - P_0) P' = P' P_2 P'. \end{aligned}$$

Finally $P_2(P' P_2 P') = P_2 P_2 P' = P_2 P'$. $\square$

We summarize the results of this section in the following theorem.

**Theorem 1.** *Let $P$ be a contractive projection on a $JB^*$-triple $U$ and let $f \in P'(U')$. With $P_k = P_k(f)$, $k = 0\,1, 2$ we have*

(a) *On $U'$, $P' P_2 = P_2 P' P_2$, $P_0 P' = P_0 P' P_0 = P' P_0 P'$,*

$$P_1 P' = P' P_1 P', \quad P_2 P' = P' P_2 P';$$

(b) *On $U''$, $P_2 P'' = P_2 P'' P_2$, $P'' P_0 = P_0 P'' P_0 = P'' P_0 P''$,*

$$P'' P_1 = P'' P_1 P'', \quad P'' P_2 = P'' P_2 P''.$$

## 2. Conditional Expectation Property of a Contractive Projection

In this section we shall use the following two results of Kaup [15] concerning a contractive projection $P$ on a $JB^*$-triple $U$.

(2.1) $P\{PabPc\}=P\{PaPbPc\}$, for $a,b,c\in U$;

(2.2) $P(U)$ is a $JB^*$-triple in the triple product $\{xyz\}_{P(U)}:=P\{xyz\}$, for $x,y,z\in P(U)$.

As noted earlier in (0.2) if $f\in P'(U')$, then $P''v=v+b$ where $v=e(f)$ and $b$ is orthogonal to $v$. The next lemma, which will be needed in Lemma 2.6, shows that in fact $b$ is orthogonal to $e(g)$ where $g$ is an *arbitrary* element of $P'(U')$.

**Lemma 2.1.** *Let $P$ be a contractive projection on a $JB^*$-triple $U$, let $f\in P'(U')$ and let $v=e(f)$ and $b=P''v-v$. Then for any $g\in P'(U')$ we have $b\in P_0(g)U''$.*

*Proof.* Let $u=e(g)$. Set $c=P''u-u$. Then $b\perp v$ and $c\perp u$. We calculate $\lambda:=g\{P''v,P''v,P''u\}$ in two ways. First, by (2.1) applied to $P''$,

$$\begin{aligned}\lambda&=g(P''\{P''v,P''v,P''u\})=g(P''\{P''v,v,P''u\})\\&=g(\{v+b,v,P''u\})=g(\{vvP''u\}).\end{aligned}$$

Second, $\lambda=g(\{v+b,v+b,P''u\})=g\{vvP''u\}+g\{bbP''u\}$. Therefore $g\{bbP''u\}=0$, i.e., $g\{bbu\}+g\{bbc\}=0$.

Let $b=b_2+b_1+b_0$ be the Peirce decomposition of $b$ with respect to $u$. Then by the "Peirce rules for multiplication" and the fact that $g=P_2(u)g$, we have $g\{bbc\}=0$ (so that $g(\{bbu\})=0$), and $g\{bbu\}=g\{b_2b_2u\}+g\{b_1b_1u\}$. Since $\{b_2b_2u\}$ and $\{b_1b_1u\}$ belong to $(P_2(u)U'')^+$ and $g$ is positive we have $g\{b_1b_1u\}=g\{b_2b_2u\}=0$. Moreover $g$ is faithful on $P_2(u)U''$, so that $\{b_1b_1u\}=\{b_2b_2u\}=0$. Finally by [11: p. 73] we have $b_1=b_2=0$. $\square$

It follows from (2.2) that $M:=P''(U'')$ is a $JBW^*$-triple with predual $M_*=P'(U')$. The next proposition gives the connection between the polar decompositions of an element $f\in P'(U')$ with respect to $M$ and $U''$. Together with Lemma 2.1 it will clarify the relationship between orthogonality in $M$ and in $U''$.

**Proposition 2.2.** *Let $P$ be a contractive projection on a $JB^*$-triple $U$, let $f\in P'(U')$, and let $v=e(f)\in U''$. Then $\tilde{v}:=P''v$ is the tripotent occurring in the polar decomposition of $f$ with respect to the $JBW^*$-triple $M:=P''(U'')$.*

*Proof.* We show first that $\tilde{v}$ is a tripotent in $P''(U'')$. By (0.2), we have $\tilde{v}=v+b$ where $b\in P_0(v)U''$. By (2.1) applied to $P''$,

$$\{\tilde{v}\tilde{v}\tilde{v}\}_M=P''\{\tilde{v}\tilde{v}\tilde{v}\}=P''\{\tilde{v},v,\tilde{v}\}=P''\{v+b,v,v+b\}=P''\{vvv\}=P''v=\tilde{v}.$$

Therefore $\tilde{v}$ is a tripotent.

We show next that if $u$ is a tripotent in $P''(U'')$ with $f(u)=\|f\|$, then $u\geqq\tilde{v}$ in $M$. Since $f(u)=\|f\|$, we have $u=v+c$ with $c\in P_0(v)U''$ (by (0.3)). Thus $u=P''u=\tilde{v}+P''c$ and $\tilde{c}:=P''c\in P_0(v)U''$ by Theorem 1. It remains to show that $\tilde{v}$ is orthogonal to $\tilde{c}$ in $M$. By (2.1), we have

$$\{\tilde{v}\,\tilde{v}\,\tilde{c}\}_M = P''\{\tilde{v}\,\tilde{v}\,\tilde{c}\} = P''\{\tilde{v}\,v\,\tilde{c}\} = P''\{v+b, v, \tilde{c}\} = P''\{v, v, \tilde{c}\} = P''0 = 0. \quad \square$$

To obtain the connection between orthogonality in $M$ and $U''$ we need the following

**Lemma 2.3.** *Let $f$ and $g$ be normal functionals on a $JBW^*$-triple $U$. Then $f \perp g$ if and only if $\|f \pm g\| = \|f\| + \|g\|$.*

*Proof.* We may assume $\|f\| = \|g\| = 1$. Let $u = e(g)$, $v = e(f)$.

Suppose first that $f \perp g$. Then $e_{\pm} := u \pm v$ has norm one and $\langle f \pm g, e_{\pm}\rangle = 2$. Therefore $2 \leqq \|f \pm g\| \leqq \|f\| + \|g\| = 2$.

Suppose now that $\|f \pm g\| = \|f\| + \|g\|$, and set $w := e(f+g)$. Then

$$2 = \langle f+g, w\rangle = \langle f, w\rangle + \langle g, w\rangle \leqq |\langle f, w\rangle| + |\langle g, w\rangle| \leqq \|f\| + \|g\| = 2.$$

Thus $\langle f, w\rangle = \langle g, w\rangle = 1$ and so $f, g \in P_2(w)\, U_*$ by [11: Prop. 1]. By the Jordan decomposition in $JBW$-algebras [13: Appendix] $f$ and $g$ are orthogonal states on the $JBW^*$-algebra $U_2(w)$, and hence $f \perp g$. $\square$

**Corollary 2.4.** *For a contractive projection $P$ on a $JB^*$-triple $U$ and $f, g \in P'(U')$, let $v = e(f)$, $u = e(g)$, $\tilde{v} = P''v$, $\tilde{u} = P''u$. Then $u \perp v$ in $U''$ if and only if $\tilde{u} \perp \tilde{v}$ in $M = P''(U'')$.*

*Proof.* For any $h \in P'(U') \subseteq U'$, $\|h\|_{M_*} = \|h\|_{U'}$. By Proposition 2.2 and Lemma 2.3,

$$\begin{aligned} u \perp v \text{ in } U'' &\Leftrightarrow \|f \pm g\|_{U'} = \|f\|_{U'} + \|g\|_{U'} \Leftrightarrow \|f \pm g\|_{M_*} \\ &= \|f\|_{M_*} + \|g\|_{M_*} \Leftrightarrow \tilde{u} \perp \tilde{v} \text{ in } M. \quad \square \end{aligned}$$

To extend this corollary to arbitrary tripotents we need:

**Lemma 2.5.** *Let $v$ be a tripotent in a $JBW^*$-triple $U$. Then there is a family of mutually orthogonal functionals $(f_\alpha) \subseteq U_*$ with $v = \sum_\alpha e(f_\alpha)$ ($w^*$-convergence).*

Proof. Let $A = U_2(v)$ which is a $JBW^*$-algebra with unit $v$. Since the normal states of $A$ are separating [13], $v = \sup\{e_\phi\colon \phi \in A_*^+\}$ where $e_\phi$ is the support projection of $\phi$ in $A$. By Zorn there is an orthogonal family $\{e_{\phi_\alpha}\}$ such that $v = \sum e_{\phi_\alpha}$ in $A$. Then $(e_{\phi_\alpha})$ are pairwise orthogonal tripotents in $U$ and $e_{\phi_\alpha} = e(f_\alpha)$ where $f_\alpha = \phi_\alpha P_2(v)$. $\square$

The following lemma gives a correspondence between tripotents in $M$ and in $U''$ which is needed in order to describe the fine structure of the range of a contractive projection. We first establish some notation.

If $P$ is a contractive projection on a $JB^*$-triple $U$, we let

$$\mathscr{C} := \text{the } w^*\text{-closure of } \operatorname{span}\{e(f)\colon f \in P'(U')\} \subseteq U'',$$

$$\mathscr{O} := \bigcap_{g \in P'(U')} P_0(g)\, U''.$$

It is obvious that $\mathscr{O}$ is a $JB^*$-subtriple of $U''$ and that $\mathscr{C} \perp \mathscr{O}$. Therefore the sum $\mathscr{C} + \mathscr{O}$ is direct and the projection $Q$ of $\mathscr{C} + \mathscr{O}$ onto $\mathscr{C}$ is contractive.

**Lemma 2.6.** *Let $P$ be a contractive projection on a $JB^*$-triple $U$ and let $w$ be a tripotent of the $JBW^*$-triple $M=P''(U'')$. Then $Qw$ is a tripotent of $U''$. Moreover, if $w_1$ and $w_2$ and orthogonal tripotents of $M$, then $Qw_1$ and $Qw_2$ are orthogonal in $U''$.*

*Proof.* Let us apply Lemma 2.5 to a tripotent $w$ of the $JBW^*$-triple $M:=P''(U'')$. We obtain orthogonal elements $(f_\alpha)\subseteq P'(U')$ and by Proposition 2.2 $w=\sum_\alpha (e(f_\alpha)+b_\alpha)$ ($w^*$-convergence in $M$) where, by Lemma 2.1, $b_\alpha\in\mathcal{O}$. By Corollary 2.4 the $e(f_\alpha)$ are orthogonal in $U''$ so $w=\sum_\alpha e(f_\alpha)+\sum_\alpha b_\alpha$, $\sum_\alpha e(f_\alpha)$ $w^*$-converges to a tripotent $Qw\in\mathscr{C}$ of $U''$ and $b:=\sum_\alpha b_\alpha\in\mathcal{O}$ exists in $U''$ as a $w^*$-limit. The second statement follows from Corollary 2.4 and the formula $Qw=\sum_\alpha e(f_\alpha)$. $\square$

By Lemma 2.6 each tripotent of $M=P''(U'')$ lies in $\mathscr{C}+\mathcal{O}$. By the spectral theorem [11: Rk. 1.9] $M\subset\mathscr{C}+\mathcal{O}$, and $Q$ is a homomorphism of $M$ into $U''$, i.e., $Q(\{abc\}_M)=\{QaQbQc\}$. On the other hand, for $x\in P(U)\subseteq M$, and $f\in P'(U')$, $f(x)=f(Qx)$, and

$$\|x\|=\sup\{|\langle f,Qx\rangle|:\ f\in P'(U'),\ \|f\|\leqq 1\}\leqq\|Qx\|\leqq\|x\|.$$

Thus, by restricting $Q$ to $P(U)$, we have:

**Theorem 2.** *Let $P$ be a contractive projection on a $JB^*$-triple $U$. Then the $JB^*$-triple $P(U)$ is isometrically isomorphic to a closed subtriple of $U''$.*

Since it is elementary that the second dual of a $J^*$-algebra is a $J^*$-algebra, we obtain as a consequence of (2.2) and Theorem 2, the main result of [10]:

**Corollary 2.7.** *Let $P$ be a contractive projection on a $J^*$-algebra. Then the range of $P$ is a $JB^*$-triple which is isometrically isomorphic to a $J^*$-algebra.*

Our next result is a conditional expectation formula analogous to (2.1). Recall that the triple product $\{abc\}$ is symmetric and linear in $a$ and $c$ and conjugate linear in $b$. The formula (2.1) was proved in [15] using holomorphic methods. Holomorphic methods are unavailable for the proof of Theorem 3 because of the conjugate linearity in $b$. Both formulas had been proved in [9] for $J^*$-algebras.

**Theorem 3.** *Let $P$ be a contractive projection on a $JB^*$-triple $U$. Then*

$$P\{PaPbc\}=P\{PaPbPc\} \quad \text{for } a,b,c\in U. \tag{2.3}$$

*Proof.* There is no loss of generality in assuming that $a=b$. By approximating $Pa$ by finite linear combinations of orthogonal tripotents of $M=P''(U'')$, it suffices to prove that

$$P''\{w_1w_2x\}=P''\{w_1w_2P''x\} \tag{2.4}$$

whenever $w_1, w_2$ are tripotents of $M$ and $x\in U''$. We only need to consider two cases, namely $w_1=w_2$ and $w_1\perp w_2$.

Note first that by the Peirce rules, for any $a_1, a_2 \in \mathcal{O}$,

$$z \in U'' \quad \text{and} \quad g \in P'(U'), \quad \text{we have } \langle g, \{a_1 a_2 z\}\rangle = 0.$$

Therefore $P''\{a_1 a_2 z\} = 0$.

If $w_1 \perp w_2$, then writing $w_i = Q w_i + b_i = v_i + b_i$ as in Lemma 2.6 implies

$$\{w_1 w_2 x\} = \{v_1 v_2 x\} + \{b_1 b_2 x\} = \{b_1 b_2 x\}.$$

Therefore $P''\{w_1 w_2 x\} = 0 = P''\{w_1 w_2 P'' x\}$.

If $w_1 = w_2 = w$ say, write $w = Q w + b = v + b$ as in Lemma 2.6. Then $P''\{w w x\} = P''\{v v x\} + P''\{b b x\} = P'' D(v, v) x$ and since $D(v, v) = P_2(v) + \frac{1}{2} P_1(v)$, Theorem 1 implies

$$P''\{w w x\} = P''(P_2(v) + \tfrac{1}{2} P_1(v)) x = P''(P_2(v) + \tfrac{1}{2} P_1(v)) P'' x = P''\{w w P'' x\}. \quad \square$$

## 3. Structure of a Bicontractive Projection

In the previous section, we showed that the $JB^*$-triple $P(U)$ is isomorphic to a subtriple of $U''$. Here of course $P$ is any contractive projection on a $JB^*$-triple $U$. In this section we make the assumption that $P$ is bicontractive, i.e., $I - P$ is also contractive.

It turns out that in this case, the isomorphism $Q$ of $P(U)$ into $U''$ is the identity, which results in:

**Proposition 3.1.** *Let $P$ be a bicontractive projection on a $JB^*$-triple $U$. Then $P(U)$ is a $JB^*$-subtriple of $U$.*

*Proof.* Since $M := P''(U'')$ is a $JBW^*$-triple, it suffices to prove that $Q w = w$ for each tripotent $w$ in $M$. By Lemma 2.5 and Lemma 2.6 it suffices to prove that $P'' v = v$ whenever $v = e(f)$ for some $f \in P'(U')$.

Let $b = P'' v - v$. Then, by (2.1),

$$P''\{b b b\} = P''\{v + b, b, v + b\} = P''\{P'' v, b, P'' v\} = P''\{P'' v, P'' b, P'' v\} = 0.$$

By induction $P''(B) = 0$ where $B$ is the $JB^*$-triple generated by $b$. It follows that $I - P''$ restricts to a bicontractive projection on $C v \oplus B$ with range $B$. Then, for any $z \in B$ with $\|z\| \leqq 1$, we have $\|v + z\| \leqq 1$ so that $\|(I - P'')(v + z)\| \leqq 1$, i.e., $\|-b + z\| \leqq 1$. This forces $b = 0$. $\square$

By Proposition 3.1, if $P$ is bicontractive, the conditional expectation formulas (2.1) and (2.3) take on a neater form:

$$P\{a b x\} = \{a b P x\} \quad \text{and} \quad P\{a x b\} = \{a P x b\}$$

for $a, b \in P(U)$ and $x \in U$.

Finally, let $P$ be a bicontractive projection on a $JB^*$-triple $U$, and set $\theta = 2\mathrm{P} - \mathrm{I}$. By the argument in [9: p. 355], the two conditional expectation formulae (2.1), (2.3) and Proposition 3.1 imply that $\theta$ is a homomorphism of $U$.

Since a homomorphism is always contractive (as follows from $\|\{zzz\}\| = \|z\|^3$) we obtain

**Theorem 4.** *Let $P$ be a bicontractive projection on a $JB^*$-triple $U$. Then there is a surjective isometry $\theta$ on $U$ of order 2 such that $P=\frac{1}{2}(I+\theta)$. Thus all $JB^*$-triples, their duals, and all pre-duals of $JBW^*$-triples belong to the class $\mathscr{S}$.*

## References


1. Arazy, J., Friedman, Y.: Contractive projections in $C_1$ and $C_\infty$. Mem. Am. Math. Soc. **13**, No. 200 (1978)
2. Barton, T., Timoney, R.: Weak $*$-continuity of Jordan triple products and its applications. Math. Scand. To appear
3. Bernau, S.J., Lacey, H.E.: Bicontractive projections and reordering of $L_p$ spaces. Pac. J. Math. **69**, 291–302 (1977)
4. Byrne, C., Sullivan, F.E.: Contractive projections with contractive complement in $L_p$-space. J. Multivariate Anal. **2**, 1–13 (1972)
5. Dineen, S.: The second dual of a $JB^*$-triple system. In: Complex Analysis, Functional Analysis and Approximation Theory, pp. 67–69, J. Mujica (ed.). Amsterdam: Elsevier (North Holland) 1986
6. Emch, G.G., King, W.P.C.: Faithful normal states on $JBW$-algebras. Proc. Symp. Pure Math. **38**, 305–307 (1982)
7. Friedman, Y., Russo, B.: Contractive projections on operator triple systems. Math. Scand. **52**, 279–311 (1983)
8. Friedman, Y., Russo, B.: Algebres d'operateurs sans ordre: solution du probleme du projecteur contractif. C.R. Acad. Sci. Paris **296**, 393–396 (1983)
9. Friedman, Y., Russo, B.: Conditional expectation without order. Pac. J. Math. **115**, 351–360 (1984)
10. Friedman, Y., Russo, B.: Solution of the contractive projection problem. J. Funct. Anal. **60**, 56–79 (1985)
11. Friedman, Y., Russo, B.: Structure of the predual of a $JBW^*$-triple. J. Reine u. Angew. Math. **356**, 67–89 (1985)
12. Friedman, Y., Russo, B.: The Gelfand-Naimark theorem for $JB^*$-triples. Duke Math. J. **53**, 139–148 (1986)
13. Iochum, B.: Cones autopolaires et algebres de Jordan. Lecture Notes in Math. **1049**. Berlin-Heidelberg-New York: Springer 1984
14. Kaup, W.: A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces. Math. Z. **183**, 503–529 (1983)
15. Kaup, W.: Contractive projections on Jordan $C^*$-algebras and generalizations. Math. Scand. **54**, 95–100 (1984)
16. Robertson, A.G., Youngson, M.A.: Positive projections with contractive complements on Jordan algebras. J. Lond. Math. Soc. (2) **25**, 365–374 (1982)
17. Størmer, E.: Positive projections with contractive complements on $C^*$-algebra. J. Lond. Math. Soc. (2) **26**, 132–142 (1982)
18. Upmeier, H.: Symmetric Banach manifolds and Jordan $C^*$-algebras. Amsterdam: Holland 1985


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