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On tangential equivalence of manifolds

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1. Introduction

Let M_1, M_2 be smooth, compact, unbounded manifolds with tangent bundles τ_1, τ_2. The object of this paper is to prove Theorem A below, generalizing the following theorem of Barry Mazur [3][1].

THEOREM OF MAZUR. *Let f: $M_1 \to M_2$ be a homotopy equivalence such that $f^*\tau_2$ is stably equivalent to τ_1. Then $M_1 \times R^k$ is diffeomorphic to $M_2 \times R^k$ for* $k \geqq \dim M + 2$.

The generalization concerns the types of manifolds considered. In § 3 it is shown that M_1 and M_2 need not be compact, provided that f is a *proper* homotopy equivalence. In § 4 the cases where M_1 and M_2 are piecewise linear or topological manifolds are reduced to the smooth case by the use of microbundles. Definitions and the statement of Theorem A occupy § 2, while § 5 consists of remarks.

I am grateful to C.T.C. Wall for suggesting this work.

2. Theorem A

A map f: $X \to Y$ is *proper* if $f^{-1}(C)$ is compact for every compact $C \subset Y$. Two maps f, g: $X \to Y$ are *properly homotopic*, denoted by $f \simeq_{\text{prop}} g$ if there is a proper map F: $X \times I \to Y$ (where I is the closed interval $[0, 1]$) such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. A *proper homotopy equivalence* is a map f: $X \to Y$ such that for some map g: $Y \to X$ it is true that $gf \simeq_{\text{prop}} 1_X$ and $fg \simeq_{\text{prop}} 1_Y$; thus f and g must be proper. (The identity map of X is denoted by 1_X or by 1.)

Define three categories **DIFF**, **PL**, **TOP** as follows:

DIFF: smooth manifolds and smooth maps;

PL : piecewise linear manifolds and piecewise linear maps;

TOP : topological manifolds and continuous maps.

Let **C** be one of these categories. If M is a manifold in **C**, let the *tangent bundle* τ_M of M denote the usual tangent bundle when **C** = **DIFF**, and the *tangent microbundle* of Milnor [6] when **C** = **PL** or **TOP** (If **C** = **PL**, then of course τ_M is the *piecewise linear* tangent microbundle).

Let M, N be manifolds of **C**. A *proper tangential equivalence* f: $M \to N$ is a morphism in **C** such that

[1] Mazur informs me that he has also proved Theorem A; cf. [11].

(i) f is a proper homotopy equivalence;

(ii) $f^*\tau_N$ is stably equivalent to τ_M.

Let $\sim$ denote bundle equivalence in C; then (ii) means that for some k,

$$(f^*\tau_N) \oplus \varepsilon^k \sim \tau_M \oplus \varepsilon^k ,$$

where ε^k is the trivial bundle over M of dimension k.

Let R^k denote euclidean k-space. We always identify M with $M \times \{0\} \subset M \times R^k$.

THEOREM A. *Let* C *be one of the categories* DIFF, PL, TOP. *Let* $f: M_1 \rightarrow M_2$ *be a proper tangential equivalence between unbounded manifolds in* C. *For some integer* $k \geqq 0$, *there exists a* C*-isomorphism*

$$F: M_1 \times R^k \approx M_2 \times R^k ,$$

such that $F \mid M_1 \simeq_{\text{prop}} f$. *If* C = DIFF *then* $k = \dim M + 2$.

3. The non-compact smooth case

In this section, all manifolds etc. are smooth; the boundary bM of the manifold M is empty.

The proof of Theorem A for C = DIFF is, essentially, the same as the proof of Mazur's theorem in [5], using Proposition 3 below in place of Lemma 1 of [5]. The details are left to the reader.

LEMMA 1. *Let* V *be a manifold, and* $f, g: M \rightarrow \operatorname{int} V$ *proper embeddings with trivial normal bundles. Assume* $\dim V \geqq 2 \dim M + 2$. *If* $f \simeq_{\text{prop}} g$, *there is a diffeotopy of* V *carrying* f *into* g, *fixed near* bV. *In particular, there is a diffeomorphism* $h: V \rightarrow V$ *such that*

(a) $hf = g$,

(b) $h = 1$ *in a neighborhood of* bV.

PROOF. First assume $f(M) \cap g(M) = \varnothing$. Since $\dim V \geqq 2 \dim (M \times I)$, there is a proper homotopy $F: M \times I \rightarrow \operatorname{int} V$ from f to g which is also an immersion.

If $\dim V > 2 \dim (M \times I)$, then assume F is an embedding.

If $\dim V = 2 \dim (M \times I) = 2n$, say, proceed as follows, in order to make F into an embedding.

Assume that F has generic double points, which therefore form a closed 0-manifold in $M \times I$, disjoint from $b(M \times I)$. These singularities may be removed by the method of *piping*, due to Penrose-Whitehead-Zeeman [8]. The idea is this: Suppose $F(p) = F(p')$, $p \neq p'$. Let A be an arc in $M \times I$ with one end at p', and meeting $b(M \times I)$ only at its other end $q \in M \times \{0\}$. Assume that p' is the only double point of F lying on A.

Let $B \subset \text{int}(M \times I)$ be an n-disk centered at p, containing no other double points. Because the self intersection (p, p') is generic, we can find an embedding $\varphi_0: B \times [0, 1] \to \text{int } V$ such that $\varphi_0(B \times \{0\}) = B$, and $\varphi_0(\{p\} \times [0, 1]) = F(A)$. Since $\varphi_0(p, 1) = F(q) \in Fb(M \times I)$, one may extend φ_0 to an embedding $\varphi: B \times [0, 2] \to \text{int } V$ such that $\varphi(B \times [1, 2]) \cap F(M \times I) = \varnothing$.

Let $\lambda: B \to [0, 2]$ take the value 2 in a neighborhood of p', and 0 in a neighborhood of bB. Define $F': M \times I \to \text{int } V$ by

$$F'(x) = \begin{cases} F(x)\,, & x \notin B \\ \varphi(x, \lambda(x))\,, & x \in B. \end{cases}$$

This procedure produces a new embedding $F': M \to \text{int } V$ which differs from F only in B, and which has no new double points; moreover, F' has no double points in B.

Let the double points of F be labelled $p_j, p'_j, j = 1, 2, \cdots$, so that $F(p_j) = F(p'_j)$. Since F is a proper immersion, the set of double points has no limit point. The piping process can be applied recursively to each pair (p_j, p'_j), of double points to produce a sequence $F'_1, F'_2, \cdots$, of embeddings $M \times I \to \text{int } V$ such that:

(i) $F'_{j+1} = F'_j$ except in an n-disk B_j, with $p'_j \in B_j \subset \text{int}(M \times I)$:

(ii) the disks B_j are pairwise disjoint, and $\bigcup_j B_j$ is closed in $M \times I$.

(iii) $\bigcup_j F'_j(B_j)$ is closed in V. This will be achieved provided the embeddings φ are carefully chosen. The properness of F is used.

(iv) the double points F'_r are the points p'_j, p_j for $j > r$.

It follows that $\lim F'_j: M \times I \to \text{int } V$ is an embedding which is also a proper homotopy from f to g.

Assume, then, that $F: M \times I \to \text{int } V$ is both an embedding and a proper homotopy from f to g. Extend F to a proper embedding $H: M \times J \to \text{int } V$, $J = [-3, 3]$. Since f has trivial normal bundle, so does H. Therefore, there is an embedding (improper) $G: M \times J \times R^k \to V$ such that $G(x, y, 0) = H(x, y)$, where $k = \dim V - \dim(M \times J)$.

There is clearly a diffeotopy of $M \times J \times R^k$ taking $(x, 0, 0)$ into $(x, 1, 0)$ for all $x \in M$, and which is fixed outside $M \times [-2, 2] \times D^k$ (where $D^k \subset R^k$ is the unit disk). Since G is proper on $M \times J \times D^k$, the image of G is a closed subset of V. Therefore the diffeotopy of $M \times J \times R^k$ can be transferred, *via* G, to a diffeotopy of V as required.

If $f(M) \cap g(M) \neq \varnothing$, choose a proper embedding $f_1: M \to \text{int } V$ such that

(i) $f_1 \simeq_{\text{prop}} f$,

(ii) $f_1(M) \cap f(M) = \varnothing = f_1(M) \cap g(M)$;

such an f_1 exists because $\dim V > 2 \dim M$. The first part of the proof applies to carry f into f_1 and then f_1 into g, by suitable diffeotopies of V, which may be combined to make a diffeotopy of V carrying f into g. Lemma 1 is proved.

REMARK. In Lemma 1, the hypothesis that f and g have trivial normal bundles can easily be dropped.

Let $p: E \to M$ be a smooth orthogonal bundle with fibre D^k. Identify M with the zero section of E. For every smooth map $\lambda: M \to (0, 1)$ of M into the open unit interval, let $\lambda E = \{x \in E: \| x \| \leqq \lambda p(x)\}$. If λ take the constant value a, put $\lambda E = aE$.

LEMMA 2. *Let $p: E \to M$ be as above. If $f: \frac{1}{2}E \to \operatorname{int} E$ is an embedding such that $f \mid M = 1$, then f has an extension to a diffeomorphism of E onto itself.*

PROOF. The conclusion of the lemma is equivalent to the statement

(i) $E - \operatorname{int} f(\frac{1}{2}E) \approx I \times bE$,

since clearly $E - \operatorname{int} \frac{1}{2} E \approx I \times bE$, and any diffeomorphism of bE extends to a diffeomorphism of $I \times bE$. It is easy to see that $E - \operatorname{int} f(\frac{1}{2}E) \approx E - \operatorname{int} f(\lambda E)$ for any $\lambda: M \to (0, \frac{1}{2})$; therefore (i) is equivalent to

(ii) $E - \operatorname{int} f(\lambda E) \approx I \times bE$ for some $\lambda: M \to (0, \frac{1}{2})$.

It is easy to prove that there exists a $\mu: M \to (0, \frac{1}{2})$ such that $b\, f(\mu E)$ is transverse to each radial segment of E; that is, each of the curves $t \to tx$, for $t \in I$ and $x \in bE$, meets $bf(\mu E)$ at a single point, and is not tangent to $bf(\mu E)$. (This is where the assumption that $f \mid M = 1$ is used.) It is clear that $E - \operatorname{int} f(\mu E) \approx I \times bE$, proving (ii), and the lemma.

Combining Lemmas 1 and 2 yields:

PROPOSITION 3. *Let $p: E \to M$ be a smooth orthogonal D^k bundle with $k \geqq \dim M + 2$. If $f: \frac{1}{2}E \to \operatorname{int} E$ is an embedding such that $f \mid M \approx_{\text{prop}} 1$, then f extends to a diffeomorphism of E onto itself.*

4. Reduction to the smooth case

The reduction is effected by observing that the total space $E\nu$ of the normal bundle of a manifold is smoothable. If $f: M_1 \to M_2$ is a proper tangential equivalence, then there is a proper tangential equivalence $\bar{f}: E\nu_1 \to E\nu_2$ covering f, where ν_i is a suitable normal bundle of M_i. Therefore $(E\nu_1) \times R^k \approx (E\nu_2) \times R^k$ for some k, by the results of § 3, or equivalently, $E(\nu_1 \oplus \varepsilon_1^k) \approx E(\nu_2 \oplus \varepsilon_2^k)$, where ε_i^k is the trivial k-dimensional bundle over $M_i (i = 1, 2)$. It follows easily that $E(\nu_1 \oplus \tau_1 \oplus \varepsilon_1^k) \approx E(\nu_2 \oplus \tau_2 \oplus \varepsilon_2^k)$, where τ_i is the tangent microbundle of M_i. If $\nu_1 \oplus \tau_i \sim \varepsilon_i^j$, then this means that $M_1 \times R^{k+j} \approx M_2 \times R^{k+j}$. The details follow.

Let $\hat{\mathrm{C}}$ be one of the categories PL, TOP. The following theorem, due to

J. Kister [2] and also B. Mazur, is true for either value of $\hat{C}$; see also [10].

THEOREM OF KISTER AND MAZUR. *Let* α: $B \xrightarrow{i} E \xrightarrow{j} B$ *be a microbundle in* $\hat{C}$. *There is a neighborhood* E_0 *of* B *in* E *such that* $(j \mid E_0, E_0, B)$ *is a* $\hat{C}$ *locally trivial fibre space* ξ_0 *with fibre* $\hat{C}$ *isomorphic to a euclidean space. Moreover, if* $E_1 \subset E$ *is another neighborhood of* B *making* $(j \mid E_1, E_1, B)$ *into a similar fibre space* ξ_1, *then there is a* $\hat{C}$ *isomorphism* h: $E_0 \to E_1$ *such that the diagram*

$$\begin{tikzcd} & E_0 \arrow[dr, "j"] \arrow[dd, "h"] & \\ B \arrow[ur, "i"] \arrow[dr, "i"'] & & B \\ & E_1 \arrow[ur, "j"'] & \end{tikzcd}$$

commutes.

By virtue of this result, it is assumed hereafter that all microbundles are locally trivial fibre spaces, as in Lemma 1.

LEMMA 4. *If Theorem* A *is true for the category* DIFF, *it is also true for manifolds* M_1, $M_2 \in \hat{C}$ *which are open submanifolds of* R^s.

PROOF. M_1 and M_2 inherit parallelizable differential structures from R^s; these are compatible with the piecewise linear structures if $\hat{C}$ = PL. The smooth case of Theorem A provides a diffeomorphism h: $M_1 \times R^k \to M_2 \times R^k$. If $\hat{C}$ = PL, approximate h by a piecewise linear homeomorphism, using the uniqueness of smooth triangulations.

The next lemma takes place in $\hat{C}$.

LEMMA 5. *Let* h: $N_1 \to N_2$ *be an isomorphism. Let* α_i *be a microbundle over* $N_i (i = 1, 2)$ *such that* $h^*\alpha_2 \sim \alpha_1$. *Then* h *is covered by an isomorphism* $E\alpha_1 \to E\alpha_2$.

PROOF. In view of the assumption concerning microbundles, preceding Lemma 4, Lemma 5 is tautological.

PROPOSITION 6. *If Theorem* A *is true for* C = DIFF, *it is also true for* C=PL *and* C=TOP.

PROOF. Let f: $M_1 \to M_2$ be a proper tangential equivalence in the category $\hat{C}$. We may assume that $f^*\tau_2 \sim \tau_1$ (not merely stably equivalent), since, if not, we can replace M_i by $M_i \times R^k$, where $f^*(\tau_2 \oplus \varepsilon_2^k) \sim \tau_1 \oplus \varepsilon_1^k$. According to Milnor [6] there exist open subsets $N_1, N_2 \subset R^s$ and $\hat{C}$ microbundles

$$\nu_i: M_i \longrightarrow N_i \xrightarrow{p_i} M_i \ .$$

If s is sufficiently large, the stable uniqueness theorem for inverses of microbundles and the assumption that $f^*\tau_2 \sim \tau_1$ together imply that $f^*\nu_2 \sim \nu_1$. It is easy to see that $f: M_1 \to M_2$ is covered by a bundle map $g: \nu_1 \to \nu_2$ which is a proper $\hat{\mathrm{C}}$ tangential equivalence $g: N_1 \to N_2$. By Lemma 4, there exists a $\hat{\mathrm{C}}$ isomorphism $h: N_1 \times R^t \approx N_2 \times R^t$, for some integer $t \geqq 0$. Since $N_i \times R^t = E(\nu_i \oplus \varepsilon_i^t)$, we may assume $t = 0$. Therefore assume the existence of a $\hat{\mathrm{C}}$ isomorphism $h: N_1 \approx N_2$ such that $h \mid M_1 \simeq_{\mathrm{prop}} f$. Observe now that

$$M_i \times R^s \approx E(\nu_i \oplus \tau_i) = E(p_i^* \tau_i) .$$

By Lemma 5 (with $\alpha_i = p_i^*\tau_i$) there exists a $\hat{\mathrm{C}}$ isomorphism $F: E(p_1^*\tau_1) \approx E(p_1^*\tau_2)$ covering $h: N_1 \approx N_2$. Identifying $M_i \times R^s$ with $E(\nu_i \oplus \tau_i)$ proves the lemma and, with it, Theorem A.

5. Remarks

(1) The proof of Theorem A can be refined so as to provide the following upper bound for k in the cases C = PL, TOP:

$$k \leqq p + \dim M_i + 2 ,$$

where M_i can be embedded in R^p with a normal microbundle ν_i such that $f^*\nu_2 \sim \nu_1$.

(2) In the cases C=DIFF, PL of Theorem A, consider this extra hypothesis: M_i is compact, and $f: M_1 \to M_2$ is a *simple* homotopy equivalence. In the smooth case it is known that $M_1 \times D^k \approx M_2 \times D^k$ for sufficiently large k, and the proofs generalize to the piecewise linear case [1], [4]. It can be shown by means of the s-cobordism theorem that if C=DIFF, then $k > n$ suffices, or $k \geqq n$ if M_i is simply connected. One cannot reduce k much further, however, as is shown by the example [9] of a smooth homotopy sphere M^{16} which does not embed smoothly in R^{29} with trivial normal bundle. Thus M^{16} is tangentially equivalent to S^{16}, but $M^{16} \times R^{13} \not\approx S^{16} \times R^{13}$.

(3) Can the dimensional restriction of Lemma 1 be weakened? If M is compact, Palais [7] shows that no restriction is needed. That some restrictions are needed for non-compact manifolds is shown by the following example. Let $K^1 \subset S^3$ be a smooth non-trivial knot, and let $S^1 \subset S^3$ be a trivial knot. Let $R^3 = S^3 - \{\infty\}$, with $\infty \in K^1$. Then no homeomorphism of R^3 carries $X^1 = K^1 - \{\infty\}$ onto a straight line $L^1 = S^1 - \{\infty\}$, since $R^3 - L^1$ and $R^3 - X^1$ have different fundamental groups. There does exist, however, a proper smooth isotopy of the embedding $X^1 \to R^3$ carrying X^1 onto L^1. Such an isotopy is by definition a smooth proper embedding $h: X^1 \times I \to R^3 \times I$ of the form $h(x, t) = (h_t(x), t)$ such that $h_0 \mid X^1 = 1$, is the inclusion, and $h_1(X^1) = L^1$. To construct h_t, let $g: K^1 \times I \to S^3 \times I$ be an isotopy such that

(i) $g_0 \mid K^1 = 1$;
(ii) $g_t(\infty) = \infty$, for all $t \in I$;
(iii) $g_1(K^1) = S^1$;
(iv) $g \mid K^1 \times I - \{(\infty, 1)\}$ is a smooth embedding.

Such a g can be constructed by the well known trick (due to Seifert) of "pulling K^1 tight", in such a way that the "knotted part" of K^1 shrinks down to ∞. Then $g: X^1 \times I \to R^3 \times I$ is the required proper smooth isotopy.

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