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Axiom Selection by Maximization: V = Ultimate L vs Forcing Axioms

DISSERTATION

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DOCTOR OF PHILOSOPHY

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by

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DEDICATION

To Mike and Lori Schatz for always encouraging me to pursue my goals
AND
In memory of Jackie, a true and honest friend.

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ABSTRACT OF THE DISSERTATION

Axiom Selection by Maximization: V = Ultimate L vs Forcing Axioms

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This dissertation explores the justification of strong theories of sets extending Zeremelo-Fraenkel set theory with choice and large cardinal axioms. In particular, there are two noted program providing axioms extending this theory: the inner model program and the forcing axiom program. While these programs historically developed to serve different mathematical goals and ends, proponents of each have attempted to justify their preferred axiom candidate on the basis of its supposed maximization potential. Since the maxim of 'maximize' proves central to the justification of ZFC+LCs itself, and shows up centrally in the current debate over how to best extend this theory, any attempt to resolve this debate will need to investigate the relationship between maximization notions and the candidates for a strong theory of sets. This dissertation takes up just this project.

The first chapter of this dissertation describes the history of axiom selection in set theory, focusing on developments since 1980 which have led to the two standard axiom candidates for extending ZFC+LCs: $V = Ult(L)$ and Martin's Maximum. The second chapter explains the justification of the methodological maxim of 'maximize' as an informal principle, and presents two formal explications of the notion: one due to John Steel, the other to Penelope Maddy. The third chapter directly examines whether either approach to axioms can be truly said to maximize over the other. It is shown that the axiom candidates are equivalent in Steel's sense of 'maximize', while in Maddy's sense of 'maximize', Martin's Maximum is found to maximize over $V = Ult(L)$. Given the strong justification of Maddy's explication in terms of the goals of set theory as a foundational discipline, it is argued that this result raises a serious justificatory challenge for advocates of the inner model program. The fourth chapter considers future directions of research, focusing on possible responses to the justificatory challenge, and highlighting issues that must be overcome before a full justificatory story of forcing axioms can be developed.

Chapter 1

A Fork in the Road: Two Contemporary Candidates for A Strong Theory of Sets

In this chapter, we will survey both the historical development of the use of axioms in set theory and the main instances of dispute regarding which axioms should be accepted. We then will lay out the current such dispute facing the set theoretic community, regarding the axioms developed by the inner model and forcing axiom programs. The main goal in doing so will be to lay out an accessible presentation of the mathematical content that will be used throughout this dissertation; excluding footnotes, nothing more than a grasp of introductory metalogic should be required. We then will turn to the key question of this dissertation: given the conflict between these well-developed and well-defended axiom programs, how can the set theoretic community justifiably decide between them? This question will then be dealt with in the remaining three chapters of the dissertation.

1.1 The Axiomatization of Set Theory and The Question of Axiom Selection

While it may be surprising, given its contemporary status as a foundational discipline in mathematics, the mathematical study of sets originally arose as a method for proving general results in diverse areas of mathematics.[1] Cantor initially developed the notion of a set in order to prove theorems about trigonometric series; while this treatment at first only required the bare notion of a collection of mathematical objects, Cantor soon found it productive to consider various operations on these collections, thus turning sets into genuinely mathematical objects of study in their own right. As soon as Cantor began directly studying the properties of sets of natural numbers, certain crucial questions naturally arose, including, especially, the question of how many such sets there were. In 1873, Cantor proved that there were strictly more sets of natural numbers than there were natural numbers themselves,[2] leading to the surprising revelation that there could be distinct sizes of infinite collections! On the other hand, Cantor was able to show that there was a one-to-one correspondence between sets of natural numbers and the real numbers of calculus, and so these two infinite collections were of the same size; as a result, set theorists have to this day used sets of natural numbers as stand-ins for the real numbers. To codify the ordering on these infinite sizes, Cantor introduced the cardinal numbers, each corresponding to a different size of collection: particularly important for our interests will be the first three such numbers, $\aleph_0$–the smallest infinite number, and the size of the natural numbers–and the next two smallest infinite numbers, $\aleph_1$ and $\aleph_2$.[3] Two sets then share a cardinal number just in case they are of the same infinite size. Much as with the natural numbers, it proved fruitful to introduce notions of addition, multiplication, and exponentitation on these numbers, with the discovery of the properties governing these operations

[1]Maddy (1997) is used as a guide for much of this section. Please see this (especially pp. 15-20 and 63-82) or Maddy (1988a) and Maddy (1988b) for more details and analysis.

[2]See Cantor (1874) for this work. For a detailed analysis of Cantor's early work in set theory, see Dauben (1990) Ch. 2 and Ch. 4.

[3]See Cantor (1895) for the first structured development of this theory, and see Dauben (1990) Ch. 7 and 8 for analysis.

on the cardinal numbers occupying much of the work of the early set theorists.

Given Cantor's theorem that there were more sets of natural numbers than there were natural numbers, it was known that the size of the collection of real numbers must be greater than $\aleph_0$; nonetheless, it was not immediately clear whether this set would be the next infinite size, or whether there would be some infinite collections larger than the natural numbers and yet smaller than the reals. Cantor famously conjectured that the former would prove true, and the set of real numbers would have size $\aleph_1$. This conjecture became widely known as the continuum hypothesis (henceforth CH), with the continuum referring to the collection of real numbers.[4] Cantor would go on to expend much effort in trying to prove CH to be true, with little success or lasting progress towards resolving this question,[5] though Cantor's attempts led to the development of many of the central tools and techniques of early set theory. Additionally, the question of whether CH was true attracted great attention throughout the mathematical community, leading to it being named the very first of Hilbert's problems for the mathematical community in the 20th century. Many supposed proofs and refutations were proposed around the turn of the century, with each attempt being quickly refuted; despite these myriad failures, enthusiasm remained undaunted, and the question of CH remained central in pure mathematics.

This flurry of activity came to a point in 1904, through the resolution of another of Cantor's central concerns: the status of the well-ordering principle. The well-ordering principle states that any set can be linearly ordered in such a way that any (non-empty) subset has a unique smallest member. While the natural numbers ordered in the usual way provide a paradigmatic example of such an ordering, for many important collections, no such ordering was known; even for the real numbers no such ordering could be supplied. Nonetheless, Cantor considered the principle to be a basic concept underlying mathematical reasoning that could not sensibly be rejected. In fact, a restricted version of the well-ordering principle was a prerequisite of Cantor's attempts to prove CH: in order

[4] See Cantor (1878) for the original statement of this conjecture. See Dauben (1990) Ch. 6 for the origins of this conjecture. See Hallett (1986) Section 2.3 for an alternative analysis of the origins of this conjecture.

[5] See Dauben (1990) Ch. 12 for a moving account of the struggles his inability to prove the continuum hypothesis caused for Cantor's mental health.

to have the specific cardinality $\aleph_1$, there would need to be a bijection between the continuum and some ordinal, that is the continuum would need to be well-orderable. While Cantor treated the principle as properly basic to set theoretic activity, others sought to prove it, thus securing its place in mathematical reasoning. Despite the lack of a successful proof, many prominent mathematicians including Hilbert and his colleagues expected such a result to be soon forthcoming.

Thus, it shocked the mathematical community when, at the third international congress of mathematics, König announced that he had proven the falsity of the well-ordering principle by showing that the continuum could not be well-ordered! Beyond being a meaningful and important result in its own right, this proof also would have implied that the continuum had no cardinality, and so CH would have to be false! This announcement attracted the attention of a wide range of practicing mathematicians, leading to the conference canceling the concurrent sessions so that all in attendance could hear the proofs of these results; one of the members of the congress who heard this talk was Hilbert's student, Zermelo. Zermelo had expected the well-ordering principle to be true, and spent the night after the talk digging into the proposed disproof of the theorem; the very next day, he was able to announce to the congress that König had misapplied Bernstein's theorem in his result, and therefore had not successfully refuted the principle.[6] Once this mistake was realized, König withdrew his announcement, but noted that he would seek fill in this gap and disprove the principle another way. With this, Zermelo turned his attention to proving the well-ordering principle once and for all.

Later that year, Zermelo made his own announcement to the set theoretic community: he had successfully proven the well-ordering theorem to be true in full generality.[7] While some, including Cantor and Hilbert, accepted the proof as valid, there proved to be much doubt over its veracity throughout Europe.[8] The sources of these concerns were fairly wide-ranging, but mainly took two distinct forms. Firstly, a number of German and British mathematicians expressed concern

[6]See Moore (1982) Ch. 2.1 for an account of König's mistake and the immediate response to it.
[7]See Zermelo (1904).
[8]See Moore (1982) Ch. 2.2 for an account of this original proof.

with the basic proof methods that Zermelo made use of, fearing that these methods could lead to paradox, and thus that Zermelo's result could not be trusted.[9] Given the myriad paradoxes that had been discovered by Russell, Hilbert, and Zermelo himself, such concerns were far from baseless, and thus Zermelo faced the challenge of showing that his methods were safe. The second basic type of concern relied on a new basic principle that Zermelo had formulated and relied on in his result: the claim that for any collection of (non-empty) sets, there would be some function that would choose a unique member of each set. This principle would come to be known as the axiom of choice (henceforth, AC); given that there may be situations were such a function could not be directly defined, however, a number of primarily French mathematicians worried that AC was a dubious assumption at best, and a direct perversion of the fundamentals of mathematical rigor at worst.[10] Thus, Zermelo was forced to defend his proof and his assumptions, thereby coming directly into contact with a major development in mathematical practice in the late 19th and early 20th centuries: the rise of pure mathematics, and the concurrent concerns it raised for the value of mathematical rigor.

The traditional conception of mathematics throughout much of the ancient and early modern world had been intimately tied to physical applications.[11] Galileo famously captured this view of math's nature with his famous claim that math was simply the language of nature. With this in mind, the properness of mathematical concepts or methods would be shown through their successful application to the physical world, with these applications functioning as the certification of any new bit of mathematics. Over the course of the late 18th and 19th centuries, however, mathematical practice became increasingly unmoored from needing direct applications to the world: particularly prominent examples of unapplied developments in math include the development and study of alternative, non-Euclidean geometries, the use of increasingly general concepts in algebra, and the separation of the notions of continuity and limit underlying classical analysis (the traditional name

[9] See Moore (1982) Ch. 2.5 and 2.7 for examples of such responses.

[10] See Moore (1982) Ch. 2.3 and 2.4.

[11] See Maddy (2011) Ch. 1, especially pp. 3-27 for an in depth explanation of the historical shift from this ancient conception of mathematics to the state of pure mathematics today; the material in this paragraph and the following paragraph is largely based on this account.

for the study of calculus). While this work in this new conception of pure mathematics led to many interesting results and a significant amount of excited activity, it also led to new issues: without the physical world underlying the trustworthiness of our mathematical projects, how could we be sure that these activities were proper, and would not lead into outright absurdity and contradiction? Russell's discovery of a paradox in Frege's logical system revealed that such concerns were not mere speculation, and so the many domains of pure mathematics needed to find a new way to guarantee rigor.

A traditional account of rigor separate from physical application can be found back in the ancient world, as exemplified in the geometrical texts of Euclid and the use of axiomatic method. Here, certain basic definitions and axioms are laid out at the beginning of the work and left unproven, with the later results derived from these first principles through various accepted methods. In this way, all of Euclid's results were in theory precisely as trustworthy as the initial axioms and the methods of derivation utilized. As has been noted by historians, Euclid in fact fails to live up to this standard, going beyond his accepted principles and importing geometric and mechanical intuitions in certain key proofs, but nonetheless an ideal of rigor was established that reigned high in the minds of many scientists and mathematicians in the 19th century as pure mathematics developed and split off from a reliance on applications.[12]

One particularly prominent example of the renewed interest in the use of the axiomatic method was Hilbert's axiomatization of geometry.[13] Much like the ancient developers of the axiomatic method, Hilbert began by setting forth a collection of basic concepts and definitions, and then introduced axioms governing the relationships between these concepts. In a novel contemporary use of this method, Hilbert then used this axiomatization to prove the consistency of his variant of Euclidean geometry relative to the consistency of the real numbers and classical analysis: that

[12]See Burgess (2015) pp. 33-38 for an account of the successes and failures of this standard of rigor in Euclid, and its reception in later mathematical thought.

[13]See Hilbert (1903). This approach to geometry is often seen as providing a worked-out example of the goal of Hilbert's program in axiomatics: see Detlefsen (1986) for a standard account of this program and its challenges, and see Zach (2007) especially sections 2.1 and 2.2 for a recent analysis of the role of Hilbert's early relative consistency proofs in his later progam.

is, by interpreting the basic terms of geometry as objects in analysis in a way that made all of the geometric axioms true, Hilbert showed that Euclidean geometry's results would be no less trustworthy than those of the calculus. Hilbert and many of his contemporaries then turned their attention to proving the consistency of classical analysis outright and absolutely, which would conclusively establish the rigor and trustworthiness of geometry beyond any doubt.

In such an environment, with the axiomatic method increasingly used to provide a vouchsafe for methods in pure mathematics that could not be supplied by any physical application, Zermelo naturally looked to use this method to defend his proof of the well-ordering theorem from the many objections it faced. Zermelo laid out a system of explicit axioms that explained the extent of the domain of sets; with a few notable exceptions, these axioms took the form of existence claims, either stating that a particular set theoretic object existed–for example, the axiom of infinity asserts the existence of ω, the set of natural numbers–or that the universe of sets was closed under some particular operation–for example, the axiom of pairing asserts that for any two sets, there exists a set with those as its only members. Amongst these axioms, Zermelo explicitly stated AC, taking the form of just another existence claim.[14]

In order to defend AC and the other axioms, Zermelo asserted that they were utilized throughout mathematical activity, and inseparable to the practice of mathematics at the time. He even went so far as to provide an explicit list of the many uses the axiom and its equivalents had been put in a wide range of mathematical consequences. In fact, it later became clear that many of theses critics had implicitly relied on AC in their own research![15] Curiously, this defense did not take the form of claims that the axioms were immediately obvious, or beyond rational doubt, but instead that mathematics could not make do without them. In this way, Zermelo's attitude towards the axioms resembled that of Russell, for whom an axiom was less obvious than its consequences, and believed only because it was capable of implying the more readily apparent truths of mathematics; nonetheless, these axioms performed an important role, as they were more logically basic, and therefore

[14] See Zermelo (1908b) section 1 for his statement of the full list of axioms.
[15] See Moore (1982)

enabled a proper organization of mathematical knowledge.[16] Using Gödel's later terminology, an axiom can justified through *intrinsic* or *extrinsic* arguments.[17] By the former, an axiom is claimed to follow from the basic concepts of set theory, and so its truth can be seen by anyone with a sufficient understanding of sethood and set membership. By the latter, on the other hand, an axiom is justified on the basis of its consequences, and so its truth can only be discovered by articulating the fruitfulness and desirability of these consequences for specific mathematical purposes. In these terms, Zermelo boldly sought to justify choice on the basis of extrinsic justifications, leaving any recourse to the basic concepts of set theory to the side.

Zermelo was then able to use this new axiomatization to provide an explicit alternate proof of the well-ordering principle.[18] Though Zermelo's arguments for his axioms, and especially for AC, were not able to convince all mathematicians of the propriety of the methods in his proof, this work attracted much attention from practicing set theorists. As more mathematicians began to work in his axiom system, and in particular to make use of AC in various areas of set theory and analysis, more and more traditional results were found to depend on uses of various forms of AC, thus validating Zermelo's claims that mathematics was highly dependent on its acceptance.[19] While his system of axioms was slightly amended by Fraenkel and Skolem in the 1920's, it quickly came to form the standard account of the set theoretic universe: the Zermelo-Fraenkel set theory with choice (henceforth ZFC)[20]:

1. Extensionality: If X and Y have the same elements, then $X = Y$.

2. Pairing: For any sets a and b there is a set $\{a, b\}$ that contains exactly a and b.

3. Separation Schema: If P is a property (with parameter p), then for any X and p there exists

[16]See Russell (1973) for his outline of this "regressive method" for finding and utilizing axioms. See Irvine (1989) for a traditional analysis of the role of this method in Russell's broader epistemic program.

[17]See Gödel (1947).

[18]See Zermelo (1908a).

[19]See Moore (1982) for a through accounting of the various dependencies of mathematical theorems on AC. Many of these results in fact only require dependent choice, or even countable choice; nonetheless, this proved sufficient for Zermelo's purposes, as the critics of AC would not even accept these weaker principles.

[20]This fairly standard presentation of the axioms is taken from Jech (2003), p. 3.

a set $Y = \{u \in X | P(u, p)\}$ that contains all those $u \in X$ that have property P.

4. Union: For any X there exists a set $Y = \bigcup X$, the union of all elements of X.

5. Power Set: For any X there exists a set $Y = \mathcal{P}(X)$, the set of all subsets of X.

6. Infinity: There exists an infinite set.

7. Replacement Schema: If a class F is a function, then for any X there exists a set $Y = F(X) = \{F(x) | x \in X\}$.

8. Foundation: Every non-empty set has a $\in$-minimal element.

9. Choice: Every family of nonempty sets has a choice function.

In spite of the increased and fruitful use of ZFC in set theoretic research, significant doubts remained regarding the acceptance of AC and the associated proof of the well-ordering principle. While some schools rejected the axiom outright on broadly philosophical grounds, as it directly implied the existence of non-definable objects such as a well-ordering of the reals, other concerns were based in questions of rigor: was AC even consistent with the other axioms? The famed incompleteness theorems of Kurt Gödel had shown that there was no hope of proving the consistency of the axiom system ZFC outright, so the best that could be hoped for was a proof of the relative consistency of the axiom system with choice relative to some less-dubitable collection of axioms. In 1938, Gödel himself found just such a result: if ZF (ZFC without the axiom of choice) was consistent, then so is ZFC, so the addition of choice doesn't add any risk to the theory![21]

While this result was quite significant in its own right, more important for our present purposes are the methods Gödel used in his proof. Just as Hilbert had years before found an interpretation of the axioms of geometry working in the background theory of classical analysis, Gödel defined an interpretation of the axioms of ZFC from the background theory ZF. To understand the idea behind

[21] See Gödel (1940).

this interpretation, note that we can think of the set theoretic universe as generated in an iterative hierarchy: we first start with the emptyset, and then take all subsets of the emptyset, and then continue on in this process of taking all possible subsets at each level indefinitely.[22] The structure resulting from this completed process is known as V. Gödel considered a structure generated by a similar process, but instead of taking absolutely any possible subset at each level, only taking the first-order definable subsets. This restricted hierarchy increases in size more slowly than the iterative hierarchy, and Gödel was able to show as a direct result of this slow, methodological construction that the generated structure would be a model of the axioms of ZF, as well as the axiom of choice: this resulting model became known as Gödel's L, or the constructible universe.[23] Since this interpretation was seemingly generating by considering only some sets at each level, it came to be known as an *inner model* interpretation of ZFC.

In addition to being a model of AC, Gödel was able to prove that the constructible universe would also be a model of CH. Thus, Gödel's result served to show that if ZF is consistent, then so is ZFC+CH: in other words, CH can not possibly be disproven from the axioms of ZFC unless those assumptions are themselves inconsistent! While it remained open at this time that ZFC could prove the truth of CH outright, as Cantor had initially hoped, Gödel expressed some skepticism towards this possibility, and no such proof seemed readily forthcoming. Thus, Gödel's proof of the relative consistency of ZFC+CH with respect to ZFC constituted only half a solution to the question of CH: it remained unknown whether CH followed from the axioms of ZFC throughout the 1940's and 1950's.

Then, in the early 1960's, Paul Cohen set out to settle the question of CH completely once and for

[22] See Zermelo (2010a) and Zermelo (2010b) for the original presentation of this view of the set theoretic universe. Martin (1970) served as an important presentation of this conception, leading to its more wide-spread adoption in set theory and philosophy. See also Boolos (1971) and Parsons (1977). For an interesting contemporary treatment of the iterative hierarchy with concepts from modal and plural logics, see Linnebo (2013).

[23] Note that the notion of a definable class modeling the axioms of ZFC can in fact be expressed directly in the language of ZFC. In particular, all that it means to say that L models the axioms of ZFC is that the restriction to L of each axiom–or instance of an axiom schema in the case of separation and replacement–is proved by ZF, where we restrict a sentence to L by relativizing all of its quantifiers to L. As an example, we could express that L models pairing through the claim that $ZF \vdash \forall x, y \in L(\exists z \in L[z = \{x, y\}])$.

all. Suspecting that the many failures to prove ZFC were best explained by its independence from these axioms, Cohen considered alternative methods for developing interpretations withing ZFC. In 1963, Cohen introduced an intricate technique for developing such interpretations: forcing. Using this new method, Cohen was able to develop an interpretation of ZFC+¬CH from the background theory ZFC.[24] Parallel to Gödel's earlier result, this interpretation showed that CH could not be proven working with the theory ZFC unless those axioms were inconsistent. In set theoretic terms, Gödel and Cohen's results jointly showed that CH was independent of standard axioms of ZFC. Thus, any attempt to settle the question of CH from the standard theory of sets was inevitably doomed to failure, as the theory simple wasn't strong enough to take a stand on the important hypothesis.

So, what was this new forcing method?[25] In effect, forcing simulates adding new objects to a model to generate a larger model of ZFC. One particularly intuitive way of thinking about the method relies on countable transitive models (henceforth, ctms)[26]. Through the forcing method, we could add a particularly well-chosen *generic object* to this tiny model in such a way that we preserve all of the axioms of ZFC; we thereby generate a new, larger ctm that contains some particular object. By our choice of the particular generic object, we can guarantee that particular claims are true in the extended model. While thinking about the method in terms of ctms might be illuminating, it proves unnecessary: working in ZF, we can simulate what it would be like to add such a generic object to create a new, *outer model* of ZFC. In doing so, instead of dealing directly with a generic object, we study various partial constuctions of such an object that actually exist; the collection of all such partial constructions is known as a *forcing poset*. The central fact about forcing, discovered by Cohen, is that any theory true in an outer model corresponding to a forcing poset has no greater consistency strength than the initial theory: thus, if we can force a theory T'

[24]See Cohen (1963) and Cohen (1964) for the original presentation of these results. See Cohen (1966) for a slightly more developed exposition of these results. See Moore (1988) for an in-depth historical account of the Cohen's work up to the discovery of the forcing method.

[25]See Kunen (1980) for the standard treatment of this material. See Weaver (2014) for a more recent, significantly more accessible presentation.

[26]A countable transitive model is a model which is of the smallest infinite size ($\aleph_0$) and which is well-behaved in the sense that any members of a set within the model must also be members of the model.

to hold through a poset in T, we know that T' is no riskier than the theory T. In this way, forcing provides a highly general method of proving relative consistency results.

Given the results of Gödel and Cohen, any attempt to settle the central question regarding the size of the collection of real numbers would have to rely on the acceptance of new axioms. While some set theorists gave up on any attempts to decide CH following these discoveries, many more turned their attention to finding new axioms capable of determining the size of the continuum concretely. One potential source of such axioms that was well-studied throughout the middle of the century were the so-called large cardinal axioms.[27] Recall that we can think of the iterative hierarchy as generated by a repeated process of taking every possible subset in consecutive stages. But how many such stages are there? By itself, ZFC proves the existence of many stages, in particular any stages that can be reached through iterations of taking powersets of previous stages and using the axiom of replacement. Further stages are possible, but their existence can not be guaranteed by the axioms of ZFC. The first cardinal beyond the guarenteed stages is called an inaccessible, since it cannot be reached by ZFC's usual methods. If there is an inaccessible cardinal–the smallest of the large cardinals–then that stage in fact forms a model of ZFC. By Gödel's famed incompleteness theorems, if ZFC is consistent, then it cannot prove that there is a model of ZFC; thus the existence of an inaccessible cardinal must go beyond this theory, provided it is consistent. As a result, to assert that such stages existed, new axioms would need to be added to ZFC. From this initial basis, stronger and stronger large cardinal axioms were developed. Each of these large cardinal notions implied the consistency of ZFC, none could be proven to exist from that theory, and additional axioms were required to ensure their existence.

One of the most interesting facts that came out of this initial study into the large cardinal axioms was that there was a linear ordering on the relative power of these notions; that is, for any two large

[27] See Gödel (1947) section 3 for an early account of the large cardinal axioms (there called "strong axioms of infinity"). See Maddy (1988a) sections 3 and 4 and Maddy (1988b) section 6 for a thorough account of the justifcatory arguments proposed for various large cardinal notions. See Jensen (1995) for a "birds-eye view" account of large cardinals and their role in set theory. See Kanamori and Magidor (1978) for a detailed historical account of the development of large cardinal axioms.

cardinal notions, the consistency of one of the large cardinal axioms would imply consistency of the other. As a result, there was a clear hierarchy of the inherent riskiness of the large cardinal axioms. Remarkably, it was found that every known natural theory extending ZFC can both interpret and be interpreted by–in other words, is equiconsistent with–ZFC together with some large cardinal: as a result, the riskiness of any theory under consideration could be measured through the large cardinal hierarchy.[28] In this way, the large cardinals served as a measuring stick for the consistency strength of mathematical theories, serving a central role in attempting to assess the risk of new axioms and new theories.[29] Given the central importance of avoiding undue risk of paradox in pure mathematics, the large cardinals therefore served an indispensible role for practicing set theorists. This fact was widely seen as providing very strong extrinsic evidence for the acceptance of the large cardinal axioms, leading to ZFC+LCs (where LCs stands for whatever particular large cardinal axiom sufficed for present purposes) in effect becoming the standard theory of sets.

While some–including Gödel himself–had initially hoped that these large cardinal axioms might settle the question of CH, the full generality of Cohen's method soon squashed this hope, as forcing was able to show the independence of CH from ZFC+LC for any known large cardinal axiom.[30] Thus, a wholly new source of axiom candidates would be required to settle the question of CH. Two distinct programs arose for finding such axioms, each corresponding to one of the two main methods for generating models of ZFC and proving relative consistency results. The first, the *inner model program*, was inspired by Gödel's constructible universe. Noting many positive features of the constructible universe, the axioms proposed by this program asserted that Gödel's L–or some structure similar to it–in fact encompasses all of V. The second, the *forcing axiom prgoram*, was instead inspired by Cohen's method of forcing. Forcing axioms assert that for well-behaved classes of forcing posets, a generic set already exists in V: as a result, any sentences that can be forced to be true through forcing with that class of posets is already true. We shall consider the origin and

[28]See Steel (2014) section 2 for an account an analysis of this phenomena.

[29]See Maddy (2016) on the foundational role of **Risk Assessment** for more on this use of the large cardinal hierarchy.

[30]See Lévy and Solovay (1967).

development of each of these programs in more detail in turn.

1.2 The Inner Model Program

Though it was initially introduced as a tool for proving a relative consistency result, Gödel's L soon became an object of serious study in its own right. In addition to being a model of CH and AC, the constructible sets were found to have a wide variety of interesting combinatorial properties. As a direct result of these features, Gödel's interpretation provided a very desirable place for set theorists to work, in the sense that many proofs and classification results became remarkably more tractable when it was assumed that all of the sets under consideration were constructible. In this way, the constructible sets provided a fruitful arena to conduct a wide-range of combinatorial studies, with many powerful results readily forthcoming.

Given the desirable features of working in L, in his 1938 paper Gödel formulated and briefly considered the axiom candidate V=L, which simply states that every set is constructible. Given Gödel's relative consistency proof, it was known that ZFC+V=L was no more risky than ZFC itself. Additionally, Gödel saw that V=L was an axiom candidate with significant power, which was capable of settling all of the important open questions of set theory at the time: in its defense, he noted that "[V = L] seems to give a natural completion of the axioms of set theory, in so far as it determines the vague notion of an arbitrary infinite set in a determinate way,".[31] Nonetheless, Gödel's consideration of V=L as a serious candidate for extending ZFC was fairly short-lived. In the 1940's, Gödel came to believe that CH was false, and that instead $2^{\aleph_0} = \aleph_2$,[32] though his arguments for this claim failed to convince the mathematical community beyond a few isolated set theorists.

Thus, despite Gödel's concerns, V=L continued to be considered as an axiom candidate through-

[31] Gödel (1940) p. 179.
[32] Gödel (1947).

out the 1950's, though doubts remained as to its viability. After all, V=L implied that V was an extremely simple and well-behaved structure, well beyond what most practicing set theorists expected. In 1961, Scott discovered a much more conclusive reason for rejecting V=L than Gödel's prior argument: the existence of a particular sort of large cardinal, a measurable cardinal, implied the falsity of V=L.[33] Thus, the orderliness of the theory ZFC+V=L was in direct tension with another axiom candidate, which was soon found to entail significant mathematically beneficial consequences itself. Thus, a clear tension had been found between the desirable features of L and the extrinsic justification of measurable cardinals, which began to be widely recognized as the large cardinal concept was further studied.

This tension was further developed by Silver in 1967, who showed that the existence of a measurable cardinal did not just imply that L did not capture the entirety of the set theoretic universe: it in fact would be wildly and repeatedly wrong, with L not even able to distinguish any two uncountable cardinal numbers.[34] Additionally, under the assumption of a measurable cardinal, there would be a particular set of natural numbers–named $0^\sharp$–which encodes the many ways Silver had found the set theoretic universe goes wrong, and as a result could not exist amongst the constructible sets.[35] This revealed that L is inadequate well before the upper reaches of the iterative hierarchy where a measurable cardinal could be found. Instead, it went wrong at the earliest possible stages, with the forming of sets of natural numbers. This proved to be a serious challenge for defenders of V=L as an axiom candidate, and much of the set theoretic community came to reject the axiom in favor of the study of large cardinals. Nonetheless, this did not mean that the benefits of working in L were completely lost, as L remained as a particular structure within V that could be studied and worked in.

Just what were these desirable features that made working in the constructible universe so productive? Roughly, they can be seen as falling into six main categories:

[33] Scott (1961). See also Jech (2003) Theorem 17.1.

[34] Silver (1971). See also Jech (2003) Theorem 18.1.

[35] $0^\sharp$ was originally introduced in Solovay (1967), with its theory further developed in Silver (1971). See also Jech (2003) pp. 318–328.

1. CH and GCH: As mentioned above, Gödel found that CH is true when evaluated in the constructible universe. In addition, he was able to show that an important strengthening of the principle would also be true: the generalized continuum hypothesis (henceforth, GCH), which says that for any cardinal κ whatsoever, 2^{κ} would be the very next cardinal number. In effect, GCH settles all questions of cardinal arithmetic left open by ZFC, and does so in the simplest possible way. This makes working with the operations on infinite numbers much less technically demanding.

2. Definable Well-Ordering of the Reals: While AC implies that there will be some well-ordering on the set of real numbers, it by itself tells nothing about how complicated that ordering will be. Due to the slow and methodical constructive process of the L hierarchy, however, Gödel found that there is a remarkably simple–in fact projectively definable–well-ordering on all of the constructible reals. Since many interesting objects of study in descriptive set theory can be defined from a well-ordering of the reals, this implies that there is a wide variety of fairly simple so-called pathological sets of real numbers within L.[36]

3. Absoluteness: While typically the (proper class) collection of objects satisfying some property can vary significantly depending on one's background assumptions, the constructible sets show no such relativity: that is, L is the same definable class regardless of what background theory it is studied in, provided the theory includes at least the axioms of ZFC. Due to this fact, Gödel's definition of the constructible universe can be seen as a robust and substantial mathematical class, and not merely an artificial reflection of one's background theory.

4. Anti-Uniformity Principles: In 1972, as part of a broad study of the structure L, Jensen was able to isolate a few combinatorial principles that play essential roles in the proof that L was a model of CH.[37] Primary amongst these principles is Jensen's $\Diamond$, which then inspired the

[36] For example, this implies that there is a projectively definable set of reals which is not Lesbegue measurable, and, similarly, that there is a projectively definable set of reals without the Baire property.

[37] See Jensen (1972), section 5. See also Kunen (2011) pp. 225-233 and Jech (2003) pp. 441-444.

discovery of $\square$ and the $\square_\kappa$'s. While the precise nature and meaning of these terms need not concern us here, they were found to directly imply many of L's important properties. As a result, they are often treated as easily studied proxies for the other typical properties of L.

5. Covering: In 1975, Jensen's studies culminated in his proof of the covering lemma for L: if $0^\sharp$ does not exist, then any set of ordinals is contained within a constructible set of ordinals of the same size.[38] This leads to an important dichotomy, where either a particular large cardinal axiom[39] is true or the constructible universe can approximate V extremely well, even if V=L is false. Jensen noted that if the latter horn of the dichotomy is true, then there is minimal cost to accepting the axiom V=L, and so set theorists might as well accept the benefits of that theory.

6. Fine-Structure: Underlying all of Jensen's work on the constructible universe was his discovery and elaboration of an alternative hierarchy generating the structure L: the so-called J hierarchy. While most stages of Gödel's hierarchy would fail to be models of almost all of the axioms of ZFC, the J hierarchy instead made each stage a well-behaved model of most of this theory. As a result, this hierarchy enabled a detailed analysis of which sets get into the structure, known as the *fine structure* of L.[40] Through the fine structure, Jensen was able to get extraordinary insight into the properties of the constructible universe, enabling far greater success in discovering truths about L than had been previously thought possible.

While all six of these features can be seen as contributing to the desirability of the axiom V=L, Jensen and his colleagues were primarily focused on the last two. In his understanding of L, the important work was done by the covering lemma and the presence of fine structure. Since the fine structure of L could be seen as underlying–though not literally implying–the other properties, however, the rest primarily served to indicate the presence of these more important features. It will

[38]Devlin and Jensen (1975). See also Jech (2003) Theorem 18.30 and Mitchell (2010b).

[39]Though $0^\sharp$ is a set of natural numbers, and so not literally a cardinal, it is seen as a large cardinal axiom since it represents an important stage in the hierarchy of consistency strengths.

[40]See Jensen (1972) for the original development of the fine structure of *L*. See Schindler and Zeman (2010) and Zeman (2002) for standard contemporary accounts of fine structure.

therefore be helpful to have clear language for these properties going forward. We will say that any inner model with a covering theorem–stating that there is a dichotomy between the existence of a large cardinal and having the inner model approximate V reasonably well–is a *core* model. The existence of a fine structure approach for L was seen as imbuing it with a strong sense of *canonicity*, where an inner model is canonical if there is no arbitrary information (especially in the sense of real numbers) that is artificially introduced to the model.[41] As fine structure directly implies canonicity, and there are no known ways of generating a canonical model without utilizing fine structure, we will treat the two concepts as interchangeable: fine structure will be assumed to be the precise mathematical property responsible for the informal idea of canonicity. Note that fine structure tends to directly enable a proof of a covering theorem, and so a canonical inner model will tend to be a core inner model (though not necessarily the other way around). Given the significant benefits of working in such models, but hoping to reconcile this with the strong extrinsic support for large cardinal axioms, Jensen was thus naturally lead to consider whether there could be a canonical, core inner model with sufficient large cardinal strength for set theoretic purposes.

Given that it was measurable cardinals that were found to be incompatible with L by Scott, the first step in finding a satisfactory canonical inner model was to find an inner model capable of containing a measurable.[42] The first attempt to find such a model was to take the constructive process generating L and simply code in all the information required to construct a measure. This is precisely the idea motivating Kunen's model $L[U]$, where U is a measure on some cardinal κ[43]: $L[U]$ is constructed exactly like L , but with a predicate U added to the language of set theory which is evaluated as the measure itself–so $U(X)$ is evaluated as true if and only if X is a set

[41] For these notions of canonicity and coreness, see Sargsyan (2011), especially slides 7-28, and the introduction of Sargsyan (2013). While these ideas have floated around the inner model program since the time of Jensen, Sargsyan is unique in explicitly presenting definitions of the concepts.

[42] See Steel (2010) for a thorough and revealing presentation of the history of canonical inner models from L to inner models of Woodin cardinals.

[43] A measure is a set of subsets of κ that forms a κ-complete non-trivial ultrafiler; for our purposes, all that is needed is that a measure is the standard object witnessing the property of being a measurable cardinal. Note that L will contain all of the ordinals, and therefore κ is itself a constructible set. It is the measure itself that is non-constructible. An inner model will contain a measurable cardinal only if it contains *both* the ordinal and the measure on it.

in the measure.[44] This constructive process then exactly resembles that of L until it reaches the κth stage, at which point a portion of the measure U–restricted to only the subsets of κ contained within L–can be added to the structure, providing a measure on $L[U] \cap \mathcal{P}(\kappa)$: that is, a measure on κ within $L[U]$. With this measure added to the class, $L[U]$ continues constructing definable sets in the usual way. Since it ensures that a measure will be added at a particular stage, the model $L[U]$ will contain a measurable cardinal–in fact, it is the minimal model of ZFC containing that particular measurable. Since it is defined through explicitly hard-coding a relevant witness to a large cardinal property–in this case, a measure–into the constructive process, we will refer to $L[U]$ as an example of a *from above* approach.

Does $L[U]$ provide a canonical, or even a core, inner model of a measurable cardinal? While it was seen to have many of the nice properties of L–in particularly, Kunen was able to show that $L[U]$ is a model of GCH, has a projectively definable well-ordering of the rules, is absolute relative to the choice of U, and modeled the standard anti-uniformity principles–these were seen Jensen and his contemporaries as having little value if not combined with the central coreness and canonicity properties. Regarding the former, Silver discovered that there could be a set of natural numbers coding up substantial information about the structure $L[U]$, much as $0^\sharp$ encodes information about L: due to this similarity, the set was named $0^\dagger$. Working with $0^\dagger$, Dodd and Jensen were able to jointly prove a somewhat restricted covering theorem for $L[U]$: either $0^\dagger$ exists, $L[U]$ covers V, or a forcing extension of $L[U]$ covers V[45]. In this way, $L[U]$ could be thought of as a core model of a measurable. With respect to canonicity, however, $L[U]$ is woefully inadequate. Due to its from above definition, and its inclusion of the wholly arbitrary information regarding how to form the measure U at the κth stage, there simply is no hope for a fine structured approach to $L[U]$. Additionally, the increased large cardinal strength of $L[U]$ over L itself was extremely limited, as the model could not even contain a second measurable cardinal, let alone the full scope of measurable cardinals justified through extrinsic considerations. As a result, Jensen found there to

[44] Kunen (1970). See also Jech (2003) ch. 19 for a contemporary account of the theory of $L[U]$.
[45] Dodd and Jensen (1982a). See also Jech (2003) Theorem 35.16.

be little interest in axiom candidates generated from $L[U]$, instead seeking to find a true canonical inner model of measurable cardinals.

In order to find such a model, Dodd and Jensen began to iteratively apply the model-theoretic technique of taking ultrapowers to the model $L[U]$. By taking the intersection of a transfinite sequence of these ultrapowers, Dodd and Jensen in 1981 discovered the core model K, also known as the core model up to a measurable.[46] Remarkably, however, there also was a fine-structured method for generating this model, by constructing a series of partial approximations–known as mice–of the completed model, where the definition of this process does not require any use be made of any large cardinals. Since this construction refrains from hard-coding any such information into its generation, we will refer to such approaches as *from below*. Since there was a fine-structured approach to K, it is in fact a canonical inner model. Jensen was able to use this fine structure to prove a covering theorem–showing K to be a core model–as well as slightly weakened versions of the other desirable properties of L: GCH is true of K, there is a projectively definable well-ordering of its reals (though this ordering is slightly more complicated than the one on L), it is generically absolute (meaning K cannot be altered by any known forcing methods), and many of the anti-uniformity properties are true of it (though $\diamondsuit$ and $\square$ only hold in restricted contexts).[47] Furthermore, K could contain more large cardinals than L, with $0^\sharp$ being an element of K, provided that V$\neq$L. Nonetheless, K fell short of Jensen's original goal, as it could not contain even a single measurable cardinal.

The discovery of K soon proved to be an instrumental breakthrough in the development of the inner model program for revealing the method of mice iteration. This method gave a wholly new way of generating inner models with fine structure, and was immediately seen to be applicable more widely then Jensen's original J hierarchy. By further generalizing the notion of mice, in 1984 Mitchell was able to define the core model for sequences of measures K^m, which was shown to

[46] Dodd and Jensen (1981). See also Jech (2003) pp. 659–664.
[47] Dodd and Jensen (1982b).

be able to contain many measurable cardinals.[48] As it was generated through the mice iteration method, K^m has fine structure and so is in our terms a canonical inner model of measurable cardinals. Much like K itself, Mitchell was able to use this fine structure to prove weakened versions of the desirable properties of L. Unlike the case of K, however, Mitchell found that there was no corresponding covering theorem of the standard form. Instead, the covering theorem for K^m took a weaker form, namely that either K^m was correct about the cardinal arithmetic of singular cardinals, or it fundamentally misunderstood the properties of uncountable cardinals. This more restricted dichotomy, which is implied by the standard covering theorem for L, is often known as *weak covering*. Since weak covering suffices for many of the uses to which the original covering theorems had been put, K^m is considered to be a core model of multiple measurable cardinals, though it must be noted that this is a non-trivial dilution of the original conception of coreness. It is worth noting, however, that the use to which Jensen had initially put strong covering in defending the axiom V=L–namely, showing that nothing much is lost by "living in" the inner model–is not obviously provided by weak covering alone, so one might naturally wonder whether the intuitive justification for preferring core models is lost in the shift to weak covering properties. We will therefore call this weak coreness, and conclude that K^m is a canonical, weakly core inner model of a measurable.

While K^m had finally achieved Jensen's goal of finding a canonical model of a measurable, it was soon seen that K^m did not suffice for all the extrinsically justified large cardinals, and so a larger inner model was required. Thus began a recurring pattern for canonical and weakly core models of large cardinals. First, from above approaches would lead to the development of a weakly core inner model of a new, stronger large cardinal axiom. Then, through much difficult and painstaking work, a from below approach would lead to a canonical inner model of that axiom. At each step in the process, as the strength of the large cardinals under consideration increased–from an inner model up to a measurable, to one up to a strong cardinal,[49] to progressively more Woodin cardinals[50]–a

[48]Mitchell (1984). See also Jech (2003) pp. 664–665 and Mitchell (2010a).

[49]See Dodd (1982) Ch. 23 for the introduction of the "from above" inner model of a strong cardinal. See Koepke (1989) for the definition of the core model up to a strong cardinal.

[50]See Martin and Steel (1994) and Mitchell and Steel (1994).

further generalization of the mouse method would be required–from mice to extenders to iteration trees of extenders–and the beneficial properties of L would be successively weakened at each step. As the challenges to getting inner models of more and more Woodin cardinals multiplied, and as the positive features of the resulting inner models suffered from increasingly diminishing returns, some in the inner model program–including Jensen himself–began to speculate that there was little hope to finding a satisfactory canonical inner model with sufficient large cardinal strength for all of the purposes of the set theoretic community.

The next important step for the inner model program seemed to be that of supercompact cardinals, a large cardinal notion with consistency strength just above that of Woodin cardinals. Soon after their discovery by Reinhardt and Solovay in 1978, supercompacts quickly proved extremely important for a number of different set theoretic activities.[51] For example, the existence of a supercompact cardinal implies projective determinacy, giving a strong and natural completed theory of the properties of well-behaved sets of real numbers.[52] Additionally, the existence of a supercompact implies determinacy for all sets in the inner model of ZF known as $L(\mathbb{R})$, and therefore the existence of a complete and well-behaved structure theory for $L(\mathbb{R})$.[53] The study of this structure theory has in recent years became a remarkable source of progress for important results in descriptive set theory. Additionally, supercompacts are useful objects for various forcing constructions, enabling relative consistency proofs of a wide variety of strong hypothesis in set theory; while many of these results were later altered to only require the assumption of infinitely many Woodins with a measurable cardinal on top, supercompacts continue to be used to discover new consistency results. Given their centrality for such a wide scope of contemporary research, it was natural to hope for a canonical inner model of a supercompact as an important next step in the inner model program.

[51] Though the theory of supercompacts was developed throughout the 1960's, it only appeared in print in Solovay et al. (1978). See Kanamori (2003) section 22 for an account of the history and basic mathematics of supercompact cardinals.

[52] See Woodin (1988) and Martin and Steel (1989).

[53] See Welch (2015) and Koellner and Woodin (2010) for the mathematics behind this result. See Maddy (2011) pp. 47–51 and Koellner (2014) for analysis of the importance of determinacy in $L(\mathbb{R})$ for the philosophy of set theory.

Unfortunately, there seemed to be a number of serious obstacles to finding such a canonical inner model of a supercompact cardinal. In particular, a number of the beneficial properties of L simply could not possible hold true in the presence of a supercompact. First of all, as a direct implication of projective determinacy, the existence of a supercompact cardinal meant that there could no projectively definable well-ordering of the reals whatsoever, no matter how high the complexity. Similarly, the $\Diamond$ and $\Box$ anti-uniformity principles had been shown to fail given a supercompact, and even the weaker $\Box_\kappa$ principles would fail class many times: in particular, $\Box_\kappa$ would fail whenever κ was above some supercompact cardinal.[54] As a result, it was generally expected that finding any inner model of a single supercompact would require wholly new methods–if it even proved possible!–and would then begin a new difficult process of expanding to inner models of more and more supercompacts.

It was therefore extraordinarily surprising when W. Hugh Woodin discovered that, once the inner model program had achieved the level of a single supercompact cardinal, there were would be no need for a further development of inner models. To understand this result, we first need introduce the notion of a weak extender model: a weak extender model for a supercompact cardinal δ not only agrees with V that δ is supercompact, but requires it is exactly the same collection of measures which witness the supercompactness of δ in both V and the weak extender model.[55] Since being a weak extender model consists in containing a collection of measures, with no necessary iterative process to generate the objects, a weak extender model is an example of the "from above approach" to inner models. Woodin discovered that any weak extender model of a supercomapct would in fact already be an inner model of any known large cardinal notion whatsoever, provided those large cardinals existed in V. In other words, once the inner model program solved the problem of developing a weakly core model of a supercompact cardinal, there would be no need for any further inner models to be developed. If this weakly core model could then inspire a canonical inner model

[54] Solovay (1974).

[55] See Woodin (2017) and especially section 3.1 of this dissertation for more on the current state-of-the-art of the theory of weak extender models and $Ult(L)$. Here, we give a brief, technically undemanding presentation of this material.

of a supercompact, then the inner model program would have found a genuine stopping point, with no further development of inner models required. With this stunning result, the attention of many practicing set theorists–including Woodin himself–turned to the prospect of developing just such an inner model of a supercompact cardinal. Such an inner model–if it exists!–came to be known as the ultimate version of L (henceforth $Ult(L)$).

As might well be expected given the history of inner models of weaker large cardinal assumptions, the first step was to find a from above approach to an inner model of a supercompact. Woodin began by developing an extremely rich and intricate theory of weak extender models of a supercompact cardinal, showing that these models would have a number of highly tractable features; since $Ult(L)$ would eventually need to take the form of some type of weak extender model, these features would also hold true of the eventual candidate for $Ult(L)$. In searching for such a model, Woodin turned his attention to an important notion in set theoretic research: ordinal definability. A set is ordinal definable if it is definable with all of the ordinals permitted as parameters; it is then hereditarily ordinal definable if it is itself ordinal definable, and each of its members is ordinal definable, and each of the members of each of its members is ordinal definable, all the way down. Woodin then proved that under certain conditions, the collection of all hereditarily ordinal definable sets (henceforth HOD) would in fact be a weak extender model of a supercompact.[56] As HOD is in a certain sense the largest definable inner model, this seemed to Woodin to be an intuitive and natural completion of the inner model program: starting with the smallest possible class sized inner model, L, the need to permit study of large cardinals eventually pushed set theorists towards the largest such model, HOD.

But in what ways is HOD an L-like inner model, assuming that it is in fact a weak extender model for a supercomapct? Let us consider each of the six L-like properties in turn. For the first of these, it is unclear–even under the assumption that HOD is a weak extender model–that HOD must be a

[56]In particular, this result requires the existence of an extendible cardinal and of a non-ω strongly measurable regular cardinal in HOD. Under the assumption of the HOD hypothesis, only the former assumption is required. Again, see Section 3.1 for more details.

model of CH, let alone a model of the stronger GCH, and so HOD falls somewhat short of being fully L-like on this count. Additionally, as mentioned above, the existence of a definable well-ordering of the reals and the truth of many anti-uniformity properties must fail in the presence of a supercompact, so the second and fourth L-like properties fail categorically to hold for HOD. Additionally, HOD lacks even the generic absoluteness properties of the earlier canonical inner models, with its make-up being wildly dependent on one's particular background theory, and so the third L-like property also completely fails to be true of HOD. Thus, each of the less important L-like properties either fail outright for HOD or hold in only extremely restricted forms. But what of the properties of coreness and canonicity that became the central goals of the inner model program? To supply a weakly core inner model, there would need to be a weak covering theorem provable for HOD: recent work on the so-called HOD dichotomy has shown that there is much promise of such a result. As such, there seem to be good grounds for expecting HOD to supply a weakly core model of a supercompact, though one lacking in many of the usual consequences of core models.

With respect to the question of whether HOD can be considered a canonical inner model, unfortunately there is much that is not yet known. Unlike L and the previous core models, HOD is entirely defined from above, with no iterative procedure known for generating it in arbitrary circumstances. Given that nothing resembling a fine structured approach for HOD is currently known, there simply is not a clear path to finding a canonical inner model of a supercompact. Additionally, given the lack of typically L-like properties for HOD even under the assumption that it is a weak extender model, it is unlikely that HOD itself could supply a canonical inner model of a supercompact cardinal. Instead, it is conjectured that there will be a canonical inner model contained closely within HOD that is itself presented with a fine-structural approach: this would be $Ult(L)$ itself.[57] In this way, HOD under strong large cardinal assumptions will represent a "from above" approach that serves to lead the set-theoretic community to a more narrow "from below" definition of an inner model, in much the same way that the discovery of $L[U]$ led to the discovery of K. Nonetheless,

[57]This is the $Ult(L)$ conjecture. Again, see section 3.1 for more details.

there are serious barriers to developing a fine structured approach to $Ult(L)$, as Woodin has shown that the method of iteration trees of extenders–used by Mitchell and Steel to develop the canonical inner models of Woodin cardinals–cannot work in the context of a supercompact cardinal. One must take care in how much significance is attributed to the current lack of a fine-structured approach, however: throughout the history of the inner model program, the previous methods for developing fine structure for a particular level of large cardinals were found to fall short for the next level, and a new method needed to be developed in each case, from the J hierarchy to mice to current notions of extenders. For our current purposes, however, we must keep in mind that the prospect of a canonical version of $Ult(L)$ remains merely a promissary note, and a full evaluation of the merits of this program will be conditional on the success or failure of this project.

Assuming for the moment, though, that this project develops in the most promising possible way, we should note that even this would not in itself represent a conclusion to the inner model program: given Jensen and his contemporaries had begun the development of canonical inner models in hopes of finding an axiom candidate capable of reconciling the virtues of V=L with the extrinsic need for large cardinal axioms, there would also need to be an axiom candidate arising from the $Ult(L)$ project. While one might expect that one could simply state V=$Ult(L)$ directly, there no obvious way of directly specifying this informal claim in the language of set theory.[58] As a result, a great deal of ingenuity will be required to find such an axiom candidate, if one is expressible at all. Currently, there is an axiom proposed by Woodin and his colleagues. The proposed axiom can be thought of as simply stating that there is a series of extender-like models of a new form that closely approximate V.[59] It is currently unknown whether this axiom candidate will suffice for capturing the current "from above" approach to $Ult(L)$, let alone the future projected "from below" approach, and so remains open to significant potential future revision. Nonetheless, should these efforts prove successful, the resulting axiom would supply the best known path for extending

[58]Recall that the axiom V=L is expressed through the claim that every set is constructible, which is itself expressible in the language of ZFC as "every set is formed at some stage in the L hierarchy". Lacking a similar "from below" iterative procedure for Ult(L), the usual way of stating an inner model axiom is prevented, and no alternative method has yet been conclusively found.

[59]This current best candidate is introduced and defined explicitly in section 3.1 below as "$V = Ult(L)$.

ZFC+LCs in a way to validate CH.

As a result, throughout this dissertation we will aim to give the inner model program as much benefit of the doubt as possible. In particular, we will assume throughout that at the very least some axiom is forthcoming, and we will accept Woodin's various claims for what such an axiom would imply. Using these assumptions about the eventual results of the $Ult(L)$ program, we will then seek to determine how this best-possible-case axiom compares to the alternatives. To see these alternatives, we now turn to the development of a second program for finding new axioms extending ZFC+LCs. While the inner model program developed out of considerations on Gödel's inner model method for proving the consistency of ZFC+CH, this program developed from considerations of Cohen's method for proving the consistency of ZFC+¬CH.

1.3 The Forcing Axiom Program

In Cohen's original proof of the relative consistency of ¬CH for ZFC, he had utilized a particular forcing poset, corresponding to a particular kind of object to be constructed: a collection of reals. As his methods in the proof came to be further studied, and the true strength and scope of applicability of the method came to be seen, it was realized that this was just one among many possible types of objects that could be constructed through the forcing method. For example, Prikry forcing was developed to allow the construction of cofinal sequences within a particular ordinal,[60] which permitted Silver to show that the generalized continuum hypothesis–and the associated singular cardinal hypothesis–can first fail at a measurable cardinal.[61] There thus developed a lively research program in finding new methods of forcing and using them to prove more and more relative consistency results, thus making the possible consistent extensions of ZFC significantly more

[60] See Kanamori (2003) section 18 for a brief account of the Prikry forcing method, and the surprising ways it enabled a fruitful extension of the theory of measurable cardinals: the below result is just one example of this.

[61] See Kanamori and Magidor (1978) section 25 for a detailed account of this result. Silver's result was originally presented in a draft entitled "G.C.H. and Large Cardinals", which remained unpublished. Related results appear in the much celebrated Silver (1975).

clear.

One of these new methods of forcing was discovered by Solovay and Tennenbaum in 1971: the method of iterated forcing, which in effect allows the simulation of an infinite series of forcings at once. This method arose out of study of Suslin's Hypothesis (henceforth SH), which posited that no trees with a particular combinatorial collection of properties–so-called Suslin trees–exist.[62] It had already been shown through earlier applications of forcing that ¬SH was relatively consistent with ZFC; one simply needed to use forcing to construct a Suslin tree. Attempts to force SH to be true, though, faced a serious obstacle: while it was known that a single tree could be forced to be non-Suslin, how could every possible tree be affected at once? Solovay and Tennenbaum first sought to solve this problem by carrying out each of these forcings, one after the other. Unfortunately, in addition to being an extremely tedious endeavor, it was unclear that such a method would work, as it seemed quite possible that a later forcing in this chain would "undo" the effects of an earlier forcing. By considering the entire chain of forcings as a single, massively complicated forcing that made each possible tree fail to be Suslin at once, however, Solovay and Tennenbaum were able to prove that none of the later forcings in the chain could make an earlier tree be Suslin; the important property of the individual Suslin forcings that enabled this result was that they satisfied many of the so-called κ-chain conditions. Thus, using this new method of iteration, they proved that ¬SH was consistent with ZFC, showing the independence of SH and solving a long-standing open question in combinatorial set theory.

Reflecting on the new methods used in this proof, Martin realized that a single principle, if assumed, could do much of the work done through the iteration of forcings in Solovay and Tennebaum's result. This key principle was the assumption that for any forcing poset that satisfied the countable chain condition–the weakest of the κ chain conditions–would have an already existing completed generic object satisfying any choice of $\aleph_1$ precise mathematical properties–henceforth, we will refer to properties as dense sets, with this being their mathematical guise in the context

[62] Solovay and Tennenbaum (1971). See also Jech (2003) Theorem 15.23 and Theorem 16.16.

of forcing.[63] This principle, now known as $MA(\aleph_1)$, was found by Martin to be very strong, implying directly, for example, that CH is false; by assuming this powerful principle, however, one could avoid the need to engage in any of the difficult technical minutiae associated with the iterated forcing method, and simply prove the relative consistency of SH outright. Furthermore, Solovay and Tennebaum's original proof already constituted a proof of the consistency of $MA(\aleph_1)$ relative to ZFC, so there was no additional amount of risk in assuming the principle. In the interest of separating out the useful technical applications of the principle from the strong consequences such as ¬CH, Martin and Solovay began studying the consequences of a slightly modified principle: namely, that any forcing poset that satisfied the countable chain condition would have an already existing completed generic object satisfying any choice of κ dense sets, where κ is any cardinal below $2^{\aleph_0}$. This principle had already been shown to be consistent relative to ZFC+¬CH, and was immediately seen to be directly implied by ZFC+CH, and so was seen as a safe addition to most base theories then under consideration.[64] This new principle became known as Martin's Axiom (henceforth, MA).

Calling MA an axiom at this time was, however, a bit of a misnomer: even as they introduced it, Martin and Solovay explicitly did not consider it a serious candidate for extending any theory of sets, referring to it as an axiom only within quotation marks![65] Instead, it was used mainly as a convenient technical device to simplify the proofs of a wide variety of results arising from the method of iterated forcing. In practice, one would often first prove a result from the assumptions of ZFC+MA–requiring no actual applications of the forcing method whatsoever–followed by a more difficult and technically tedious proof of the result from the assumption of ZFC+¬MA. In this way, one could prove an independence result for some set theoretic statement from ZFC by first developing a proof for the simpler case, and then using this proof as a guide for the more complicated forcing result required for the remaining case. Through this method of dividing a

[63] See Martin and Solovay (1970) for the classical presentation of this principle and its paradigmatic consequences. See Jech (2003) Ch. 16 and Kunen (2011) sections III.3 and III.4 for more contemporary presentations.

[64] In fact, ZFC+¬MA was found to be consistent relative to ZFC+$2^{\aleph_0} = \kappa$ for any possible consistent choice of κ.

[65] Martin and Solovay (1970) p. 144: "We are then very much in need of an alternative to *CH*. The aim of this paper is to consider one such alternative. We introduce an "axiom" A..."

result into a forcing-free component and only then a more difficult subsequent result, MA served primarily as an organizing principle for directing research into the independent questions of set theory throughout the 1970's. As this work progressed, it was noted that much of the power of MA as a technical tool came from its ability to add some structure to the relatively open-ended theory ZFC+¬CH;[66] nonetheless, it remained a mere technical device throughout the decade instead of a genuine axiom candidate.

Given the significant technical benefits of working from ZFC+MA, the question naturally arose of whether there could be further, similar principles to permit even wider applications of the simplifying proof strategy. The obvious place to look towards when searching for a generalization of MA was the restriction to only considering forcing posets satisfying the countable chain condition (henceforth, ccc). It was already known in 1971 that this requirement could not consistently be dropped outright, but perhaps it could be replaced with a less-strict property. In effect, the way the limitation to ccc posets avoided this contradiction was due to the iterability of the property: if every forcing poset in an iterated chain is ccc, then so will be the iterated forcing considered in itself. This iterability property implied, amongst other beneficial features, that $\aleph_1$ would remain untouched by the long chain of forcings, and thus that much of the basic structure of the small cardinals in the base model would be preserved in the forcing extension. Since this iterability property is what directly enabled the simplifying of iterated forcing in the case of MA, much attention then turned to the development of broader collections of forcing posets that could nonetheless be shown to iterate suitably.

In 1982, Shelah was able to find just such a broader class of posets: drawing on work in the study of the combinatorial properties of stationary sets of ordinals by Jech and Kueker, Shelah introduced the class of proper forcing posets.[67] Roughly speaking, a forcing poset is proper if its forcing

[66]This approach, of using *MA* as a work-around for the weakness of the theory $ZFC + \neg CH$, is explicit even in Martin and Solovay's original work on the axiom. See Martin and Solovay (1970) p. 143: "if we reject *CH* we admit ourselves to be in a state of ignorance about a great many questions which *CH* resolves. While *CH* is a powerful assumption, its negation is in many ways quite weak".

[67]The original presentation of the theory of proper forcing is in Shelah (1982), with a more mature presentation found in Shelah (1998). See Kunen (2011) section V.7 and Abraham (2010) for contemporary presentations.

extensions must preserve stationary collections of countable subsets of ordinals: that is, the forcing must keep big collections of subsets of ordinals big. By requiring such a preservation condition, Shelah was able to enforce that the structure of the small cardinals in the forcing extension would reasonably resemble that of the ground model. As a direct result of this fact, Shelah was able to prove a iteration theorem for proper forcings in direct parallel to the one that Martin and Solovay had developed for ccc forcings.[68] Furthermore, it was shown that every ccc forcing was already a proper forcing, and so Shelah's property was a generalization of the ccc property. While Shelah briefly considers a generalization of MA to all proper posets, the idea of such an axiom is quickly put aside in Shelah's own work; instead, his focus in primarily on using the new iteration theorem to prove a bevy of new results in infinitary combinatorics.

It is only later, in 1984, that the prospect of such an extension of the axiom MA is seriously considered by Baumgartner.[69] Recall that MA asserts the existence of generics for collections of dense sets of any size below that of the continuum. In contrast, Baumgartner noted that the only non-trivial and consistent extension of MA to proper forcing posets was when one considered collections of exactly $\aleph_1$ many precise dense sets that would be required to hold for the generic. With this in mind, Baumgartner introduced the axiom PFA: for any proper poset and any collection of $\aleph_1$ many dense sets, there exists a generic set on the poset hitting each of the dense sets. Baumgartner then proceeded to study its consequences, hoping to evaluate the feasibility of using PFA as a technical device in a similar manner to the original MA.

Baumgartner's first discovery was that PFA had a massive extent of consequences, well beyond those following from MA, or even MA($\aleph_1$). While it was well expected that PFA would have meaningful applications beyond MA–after all, the former was a direct extension of the latter to a wider span of cases–the sheer breadth of these applications took the set theoretic community by surprise. Baumgartner soon discovered the underlying reason for PFA's surprising amount of power: while MA was consistent relative to ZFC, and therefore contained no hidden large cardinal

[68]See Abraham (2010) Section 2 for the now standard presentation of this result.
[69]Baumgartner (1984).

strength, PFA could only be shown the be consistent relative to a supercompact cardinal! While the precise amount of large cardinal strength implied by PFA was then–as it remains today[70]–unknown, it was widely conjectured that PFA would in fact end up implying the consistency of a supercompact. Thus, PFA represented a far riskier and more substantial potential assumption than its inspiration, MA.

Most of the consequences of PFA elaborated by Baumgartner related to rather obscure questions in combinatorial set theory or the application of fairly intricate iterations of complex forcings, so little attention was immediately turned to PFA as an axiom candidate. This lack of serious consideration remained true until the very end of the decade, when Todorčević began studying the open coloring axiom (henceforth, OCA).[71] The OCA was originally developed as a natural extension of notions of colorings from graph theory to continuum sized graphs, stating that any such graph either has a particular sort of coloring or a uncountable clique. Todorčević was able to make use of this axiom to derive a number of important consequences for the structure of the infinite cardinals–in particular, the relative orderliness of regular uncountable cardinals. In addition, Todorčević found that OCA implied that CH failed in a particular way: under OCA, $2^{\aleph_0} = \aleph_2$, and so the continuum is as small as it could possibly be without CH being true. Such a size for the continuum was not widely seen as very plausible at this time; if CH was false, it was often said, it would be wildly false, allowing for a wide scope of interesting behavior of the many uncountable cardinals below the size of the continuum. A major stopping-block for the large continuum program, however, was the lack of a clear axiom that would imply a large continuum in a non-artificial way. In 1989, however, Todorčević proved that PFA implied OCA, and therefore implied $\neg$CH; with this discovery, a naturally arising axiom candidate for $\neg$CH had been discovered. Since this axiom provided for a "small" continuum, with Todorčević's proof the focus of research on $\neg$CH began to

[70]It has been shown that the consistency strength of PFA is somewhere between a supercompact and the existence of an inner model of a single Woodin cardinal, but the precise strength has not been calibrated. While there is much speculation as to how this will turn out, in general it is expected to be on the higher end of this spectrum, at least implying the consistency of infinitely many Woodins with a measurable above.

[71]See Todorčević (1989) for the original presentation and development of his *OCA* principle. Note that this principle is distinct from Abraham and Shelah's earlier *OCA* principle: the exact relationship between these two principles is still somewhat of an open question. Throughout this dissertation, we use *OCA* to refer to Todorčević's version of *OCA*.

shift away from the supposed large values for the continuum.

The importance of this event is quite hard to overstate: while the inner model program had provided axiom candidates implying the truth of CH, it is only with the discovery of PFA's consequences for the size of the continuum that proponents of $\neg$CH had a parallel axiom to defend. It is only at this point that forcing axioms start to become considered as serious axiom candidates, and not mere technical conveniences to be used as part of a proof strategy for deriving independence results. Todorčević then continued to develop throughout the 1990's a full picture of the consequences of ZFC+PFA, often using *OCA* as a useful intermediate principle in proofs.[72] In many cases, including that of $\neg$CH itself, the consequences of PFA are seen to be precisely the opposite of those associated with the inner model program. Of particular relevance for our concerns, Todorčević head earlier proved that PFA implies $\neg\Box_\kappa$ for all uncountable κ, as well as the negations of each of the other anti-uniformity properties associated with L.[73] Despite these developments, work with PFA proves quite difficult, and progress in developing a complete picture of the resulting theory remains slow-going.

As Todorčević's research program progressed, a concurrent development in the study of forcing axioms occurs through Shelah's study of iteration theorems that would have a marked effect on the consideration of forcing axioms as genuine axiom candidates. Due to the great difficulties of working with forcing posets that could not be iterated without collapsing $\aleph_1$, Shelah sought to extend his iteration theorem from proper forcing posets to even wider classes of forcings; by isolating the safely iterable classes of forcing posets, Shelah sought to focus attention on the types of forcing that would prove more tractable. In 1988, working towards this goal, Foreman, Magidor, and Shelah isolated the notion of a stationary-set preserving (henceforth, ssp) notion of forcing: roughly, a forcing poset is ssp if every large subset of $\aleph_1$ in the ground model must remain large in any forcing extension through the poset.[74] Since every proper forcing poset must be ssp, the

[72]See Todorcevic (2014) for Todorčević's own course notes summarizing the key developments during this period.
[73]Todorčević (1984)
[74]Foreman et al. (1988). See Jech (2003) Ch. 37 for a contemporary presentation of the main results from this work.

trio noted that this represented a further generalization of the class of ccc posets. By replacing the concept of a proper forcing in the statement of PFA with that of a ssp forcing, a new axiom candidate was discovered; since Foreman, Magidor, and Shelah proved that no further extension of the class of forcings under consideration could be consistent–in particular, even extending the class of ssp forcings by a single additional forcing poset would generate a contradiction–this axiom became known as Martin's Maximum (henceforth, MM). With the formulation of MM, a strict upper limit to the program of developing generalizations of MA had been found.

In the 1988 paper first introducing the axiom, Foreman, Magidor, and Shelah also go on to develop a rich portrait of the consequences of MM. Since MM directly implies PFA, it was immediately known that it implied all of Baumgartner's consequences for applications of forcing and infinity combinatorics; additionally, Foreman, Magidor, and Shelah were able to show that MM implies $2^{\aleph_0} = \aleph_2$, using different methods from those used in Todorčević's earlier proof. Beyond the features of PFA, the trio proved that MM had even stronger consequences for the structure of regular cardinals than PFA, and that a bevy of reflection properties would be true of large cardinals under the assumption of MM. In this way, the consequences of MM were found to go beyond those of PFA, but to a much less significant extent than had been seen for PFA and MA; this fact was soon explained by the relative consistency proof of MM from the existence of a supercompact cardinal, showing that MM did not contain much–if any–large cardinal strength beyond that of PFA. The closeness of these two axioms was further elaborated by Todorčević in the early 1990's, as he was able to successfully weaken the assumption in many of Foreman, Magidor, and Shelah's proofs from MM to just PFA. As a result, Todorčević's studies into the potential justifications of PFA as an axiom candidate and the further study of the consequences of MM began to coalesce into a single unified research program of the applications of forcing axioms.[75]

Besides these three standard forcing axioms, a number of ways for developing additional so-called

[75] Of the initial group of mathematicians working on strong forcing axioms, it is not entirely clear who became proponents of their adoption as extensions of ZFC+LCs. While Todorčević and Magidor explicitly argue for their justification, the remaining three–Baumgartner, Foreman, and Shelah–stay focused on their applications as technical tools to simplify proofs of independence results, primarily through applications of the associated iteration theorem.

forcing axioms have been developed. These efforts broadly fall into three main types. First, a wide variety of set theorists have considered weakenings of PFA and MM, such as the bounded versions of these axioms; the goal in studying these axioms is to more precisely calibrate the strength of assumption required to prove each of the canonical consequences of the standard forcing axioms.[76] As a result, this program amounts to more of a technical analyses of the standard forcing axioms, and not a genuine alternative. Secondly, there are the direct extensions of the standard forcing axioms, such as MM^{+} or MM^{++}, which increase the strength of either PFA or MM by adding additional requirements for the generic objects that the standard forcing axioms guarantee will exist; as straight-forward strengthenings of the standard forcing axioms, any account of their justification will rely on a prior account of the justification of PFA and MM.[77] Finally, there is the axiom $(\star)$ proposed by Woodin before his work in the inner model program; though this axiom is sometimes called a forcing axiom, its statement and history is wholly separate from the research tradition described above, and relies on a separate justificatory story.[78] Additionally, it is not yet clear whether $(\star)$ is implied by or even equivalent to some suitably defined strengthening of MM.[79] In the absence of clarity on these mathematical questions, and noting the strong differences between the arguments for $(\star)$ and for traditional forcing axioms, for the purposes of this dissertation we will consider $(\star)$ to be a separate entity from the broader family of standard forcing axioms. For these reasons, this dissertation will focus primarily on the three standard forcing axioms: MA, PFA, and MM. Unless explicitly noted otherwise, the term 'forcing axiom' will henceforth refer to just these three axiom candidates.

[76] The study of these weakened versions of forcing axioms begins in earnest with Goldstern and Shelah (1995). See also Moore (2005) and Caicedo and Veličković (2006) for important recent examples of this field of research.

[77] These strengthenings are originally introduced in Foreman et al. (1988). See Viale (2016) for a standard example of the use of these stronger axioms in the study of forcing axioms. It is worth noting that these stronger axioms are in fact quite close to the original counterparts: for example, the standard forcing construction used in Foreman et al. (1988) to prove the consistency of MM in fact generates a model of MM^{++}. It is not yet entirely clear to what extent these alternative axioms are genuinely more powerful than MM.

[78] See Woodin (2010a) for the standard presentation of this material.

[79] See the axiom MM^{+++} introduced in Schindler (2017) for a recent attempt to find a strengthening of MM capable of encapsulating $(\star)$. See Magidor (2012) Conjecture 6.8 and surrounding discussion for reasons for believing $(\star)$ may already be implied by a MM^{++}.

1.4 How to Settle This Dispute?

Out of these two distinct research programs have developed two starkly contrasting approaches for extending the theory ZFC+LCs. On the one hand, implying the truth of CH, there is the inner model program, which arose out of a purposeful search for an axiom capable of reconciling the ability of $V = L$ to serve as a compete answer to the traditional questions of set theory with the mathematical desirability of the large cardinal axioms. On the other hand, implying the falsity of CH, there is the forcing axiom program, which was initially developed as a mere technical convenience for the use of certain forcing methods, and to organize proof efforts for a wide variety of results, only comparatively lately being considered as a candidate extension of ZFC+LCs. Since the two axiom programs differ in their resolution of the question of CH, they cannot be jointly accepted. Thus, the methodological question immediately arises: how can the set theoretic community properly decide between them? Stated more generally: on what basis can one axiom candidate be justifiably preferred over another?

One place one might look for help in settling this dispute would be the methodological notion of *maximize*. Roughly put, the maxim of 'maximize' urges that, since a fundamental mathematical goal of set theory is to serve as a suitable foundation for classical and contemporary mathematics, one should prefer axioms which lead to as wide-ranging and well-populated of domain of sets as possible. The notion of 'maximize' has motivated past cases of successful axiom selection by the mathematical community, as argued by Maddy in her *Naturalism in Mathematics* regarding the choice between $V = L$ and the existence of measurable cardinals, and so we might naturally hope that it would also serve to adjudicate the current dispute between axiom candidates. Furthermore, the informal notion of 'maximize' has already played a significant role in the dispute between forcing axioms and the inner model program: ironically, advocates of both approaches seek to defend their preferred axiom candidates through maximization considerations.[80] Given this track

[80]For the former, see Magidor's EFI talk, especially Magidor (2012) pp. 15-16. For the latter, see Koellner (2017) section 3.3 and the forthcoming paper on this material by Koellner and Woodin. Interestingly, in his PhD thesis, Koellner advocates for Woodin's $(\star)$ axiom on maximization grounds (see Koellner (2003), p. 94); with the elaboration

record of past success, as well as its importance to key figures in the current debate, we will seek to apply the notion of 'maximize' to illuminate, and potentially help settle, the current dispute between forcing axioms and the inner model program.

In order to sort out the use of maximization considerations by advocates of incompatible methodological programs, however, we will first need to become more clear regarding what 'maximize' entails, and how to apply it in concrete circumstances. For this reason, in the next chapter we will consider the motivations for 'maximize'–both intrinsic and extrinsic–, as well as considering how to best formally explicate the informal notion. By so doing, we will aim to develop a precise methodological tool capable of being put to work in comparing particular axiom candidates. Then, in the third chapter, we will return to the two axioms programs, seeking to apply our formal explication to the current dispute directly.

of the Ult(L) project, however, Koellner came to believe that these arguments were in fact fundamentally misguided.

Chapter 2

Axiom Selection and the Maxim of ‘Maximize’

In this chapter, we will introduce and examine one particular methodological maxim particularly relevant to past cases of axiom selection: maximize. Among the goals of set theory as a mathematical discipline Maddy identifies MAXIMIZE and UNIFY. Given these two important goals, one should seek axiom systems that avoid any unnecessary restrictions on what kinds of objects can exist.[1] One *very* informal gloss on this idea is that if a useful mathematical object can exist, we should prefer axiom systems that imply its existence. Unfortunately, this gloss fails to be applicable in practice, as we frequently face trade-offs between what sets can exist in different axiom systems: for example, is it maximizing to have an axiom system with many different non-isomorphic types of unbounded, dense, complete orderings (and so where Suslin’s hypothesis is false) or one with as many bijections between such orderings as possible (and so where Suslin’s hypothesis is true). To settle concrete cases of axiom selection with ‘maximize’, we will find it necessary to shift to a more formal approach; the key question of this chapter will be which formal explication is best able to capture the motivations behind the informal maxim of ‘maximize’.

[1] See Maddy (1997) pp. 208–212.

To begin, we introduce the informal motivation for maximization as serving the distinct mathematical goals of set theory, relying heavily on Maddy's *Naturalism in Mathematics*.[2] With this background in place, we will then introduce her formal explication of maximization from that same work, comparing it to another formal account of maximization first presented by Steel in 2004 and later further elaborated in Steel's "Gödel's Program".[3] I will consider how well each formal explication ties into the methodological motivations behind "maximize", noting a few ways that Maddy's definition seems to more directly capture these motivations. Finally, I will consider a note a hitherto unnoticed feature of Maddy's formal notion when applied to strong theories extending *ZFC* which will make it somewhat easier to later apply this tool to the theories introduced in section 1.

2.1 'Maximize' as a Methodological Maxim

As detailed in the previous chapter, one essential element of set-theoretic practice since the discovery of the independence of CH has been *axiom selection*: finding candidates for new axioms to extend ZFC and determining ways to justify these candidates, thereby deciding between them. Given that historic and contemporary set theorists have given serious effort to justifying preferred axiom candidates, any successful methodology of set theory would need to come to terms with and evaluate the efficacy of these efforts. This is an especially salient question for current philosophical attempts to study set theory given the presence of two well-developed but mutually exclusive programs for developing new axioms: the inner model program and the forcing axioms program. So the questions arises: how do set theorists attempt to adjudicate between axiom candidates, and how should they do so?

In her *Naturalism in Mathematics*, Maddy begins by considering one of the goals of set theory

[2] Maddy (1997)

[3] See Steel (2004) for an early presentation of this approach to 'maximize'. See Steel (2014) for a later development of this material, esp. pp. 165.

as a mathematical discipline, namely providing a foundation for classical mathematics. She notes that achieving this goal requires following two fundamental maxims: maximize and unify.[4] 'Maximize' ensures that the domain of mathematical objects is as generous possible: "if set theory is to play the hoped-for foundational role, then set theory should not impose any limitations of its own: the set theoretic arena in which mathematics is to be modeled should be as generous as possible; the set theoretic axioms from which mathematical theorems are to be proved should be as powerful and fruitful as possible. Thus, the goal of founding mathematics without encumbering it generates the methodological admonition to MAXIMIZE".[5] In addition, UNIFY ensures that there is a single comprehensive theory of sets, enabling all the objects and methods of classical mathematics to be studied together.[6] Combining these two maxims, we find that the aim is to find a theory which provides "a single arena where all the various structures in all the various branches [of math] can co-exist side-by-side, where their interrelations can be studied" and a standard for "what counts as proof".[7] In playing these roles, set theory aims to provide a domain of objects that includes everything that could fall under serious mathematical study, and avoids curtailing any area of study in pure mathematics. For example, Maddy argues that the axiom of a measurable cardinal implies the existence of $0^\sharp$, a well-motivated object of mathematical inquiry, while $V = L$ blocks the existence of this set, and so curtails interesting mathematical work.[8] For this reason, maximize counts against the acceptance of V=L.

It is worthwhile to be clear on the precise nature of the justification of 'maximize' from these foundational goals. The foundational goals are justified by Maddy on firmly extrinsic grounds,

[4]Maddy (1997), p. 208

[5]Maddy (1997) pp. 210–211.

[6]While unify is an important methodological principle in its own right, this dissertation will focus primarily on maximize. For this reason, we will mostly set unify to the side, and instead focus on the effects of maximize on the current debate in axiom selection.

[7]Maddy (2016) p. 16, 15. As Maddy notes, it is not obvious that maximize and unify can both be satisfied: for example, it might turn out that "ZFC can be extended in a number of incompatible ways... and that no mathematically defensible considerations allow us to choose between them" (Maddy (1997) pp. 211–212). In such a case, we might instead turn to a multiverse approach to set theory. For the purposes of this work, we will focus on possible ways we might reconcile maximize and unify, and therefore will not consider multiverse approaches; see Maddy and Meadows (Frth) for more on how multiverse approaches fit into this view of set theory.

[8]See footnote 8 below.

through their ability to enable fruitful mathematical research by providing a single mathematical domain where one can study and compare the multitude of mathematical objects under consideration by the varied disciplines of contemporary mathematics. Considering these goals, Maddy notes that a notion of 'maximize' provides the most effective means of achieving the foundational goal: but this is simply a case of means/ends reasoning as found throughout scientific inquiry. 'Maximize' is thereby able to attain the extrinsic justification of the foundational goals, representing the methodological means for attaining them in practice.

In *Naturalism in Mathematics*, Maddy puts this methodological principle to work in considering the case of V=L and the existence of measurable cardinals, aiming to assess the rationality of the arguments leading set theorists to reject V=L as an axiom candidate on the basis of the mathematical goals of set theory.[9] As a rough (and very simplified) outline of this argument, Maddy notes that 'maximize' encourages that an axiom candidate leading to the existence of some interesting mathematical objects should be preferred over an alternative candidate, provided that the former doesn't require giving up interesting mathematical content of the latter. The existence of a measurable cardinal implies the existence of non-constructible sets, and in particular the large cardinal $0^\sharp$, which conflicts with the axiom V=L. In fact, Maddy shows that V=L cannot even contain any object with the same isomorphism type as $0^\sharp$. Since this would curtail the use of the large cardinal hierarchy as a measure of consistency strength, as well as blocking the fruitful study of the higher large cardinal axioms, Maddy concludes that maximize counts against the acceptance of V=L. Finally, she notes there the axiom of a measurable cardinal does not require forgoing any of the mathematical benefits of working in ZFC+V=L, as L itself provides an interpretation of the theory that even a V=L proponent would have to accept as fair. Thus, Maddy concludes that 'maximize'–properly understood–is able to justify the set theoretic community's rejection of V=L.

Throughout this chapter and the next we will focus solely on the maxim of 'maximize' as a tool for

[9]See Maddy (1997) pp. 212-15 for an abbreviated outline of this argument; see Maddy (1997) pp. 216-32 for the argument in full technical detail. We will introduce much of Maddy's formal tools used in this argument in the subsequent section of this chapter.

settling disputes in axiom selection. This is not to ignore that there are potentially other maxims that might be put to work in settling the current dispute between the inner model and forcing axiom programs. Nonetheless, 'maximize' is both rationally supported by the aims of set theory and justifies previous historically successful instances of extending ZFC. Thus, it is the obvious first method to be used in future instances of incompatible but desirable axioms. As a result, for the remainder of this chapter we will simply explore the best approach to 'maximize' in general, and in the next we will seek to apply it directly to the current axiom candidates; we will postpone any discussion of alternative methods of adjudicating this dispute to the final chapter of this work.

Applied to the axiom candidates from the previous chapter, the maxim of 'maximize' tells us we should ask whether strong forcing axioms or the presumed axiom candidate for $V = Ult(L)$ provide for the existence of objects or proof methods that can't exist under the alternative axiom. Given Maddy's case for the measurable cardinal axiom over the inner model axiom $V = L$, and the fact that this case can be extended for any inner model axiom below the level of a supercompact cardinal, one might initially expect maximize to immediately support forcing axioms on the same grounds. Note, however, that these previous arguments rely on the existence of a particular type of object, namely a sharp that cannot be contained in the relevant inner model. The possible existence of such a sharp-like-notion is referred to as a *anti-inner model theorem*. By Woodin's result regarding inner models for supercompacts, however, there is no anti-inner model theorem at this level, and therefore there cannot be the equivalent of a sharp for $Ult(L)$. Thus, any case against $Ult(L)$ on the grounds of the 'maximize' maxim will have to be distinct from these previous cases of axiom selection; the question of which axiom is to be preferred on these grounds is therefore genuinely open at the outset.

2.2 Two Notions of Maximize

In looking to use a formal explication of 'maximize' in the current axiom selection debate,[10] we note that there are already two well-developed formal accounts in the extant literature.[11] On the one hand, we have Steel's version of maximization, which claims that a maximizing theory is one which is able to interpret any more restrictive theories, so that no content of the restrictive theories are lost.[12] On the other hand, we have Maddy's original explication of 'maximize', which claims that this is merely a necessary, but not sufficient, condition for maximizing; beyond failing to lose any content, a maximizing theory must offer some genuinely new mathematical content. In particular, Maddy's account also requires that there is some object in the maximizing theory that

[10]Some will find the task of finding a formal explication for such a central methodological notion to a hopeless task, fearing that any such precise counterpart will be doomed to supporting obviously unacceptable theories (false positives) and will miss other cases where a theory does intuitively maximize over another (false negatives). Maddy directly considers such problematic "dud" theories which meet the formal criteria for maximizing over a desirable theory can be artificially generated for a wide family of desirable theories (Maddy (1997), pp. 229–31). In order to deal with these issues, we will follow Maddy's lead and informally restrict the scope of the notions of 'maximize' under consideration to theories which are genuine contenders for a strong theory of sets, considering only "natural theories". Note that both formal accounts we will consider below require this move, so any concerns with the notion of "natural theories" apply to all parties in the debate over the correct formal understanding of 'maximize'. For this reason, we will put this issue to the side, accepting the standard restriction of theories under considerations: for our purposes, we simply note that both ZFC+LCs+V=Ult(L) ZFC+LCs+MM are natural theories. Lacking a more firm criteria for "naturalness", we will use the informal guideline proposed by Steel: a natural theory should be a theory that is seriously proposed and considered by practicing set-theorists for its mathematical merits. See Steel (2014) Sections 2 and 3 for his treatment of "naturalness".

[11]Beyond the two accounts that we will consider in this chapter, there are also more overtly syntactic approaches to maximize. A particular syntactic approach will outline a particular class of sentences that are seen as particularly important for set theoretic purposes, and evaluate the maximizing potential by the extent to which they can make sentences in that class true. For example, Woodin's defense of the axiom $(\star)$ argues that it is maximizing because it makes every Π_2 sentence in the language $\mathcal{L} = \{\in, I_{NS}, A\}$–for a non-stationary ideal I_{NS} and $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$–applied to the structure $H(\omega_2)$ true, provided that sentence can be made true through forcing methods (see Woodin (2010a), especially ch. 5, for an in-depth articulation of this material). Nonetheless, a fully general account of such a syntactic version of 'maximize' has not been developed, and so it is unclear how to apply such account to the current or future axiom selection debates. Additionally, it is unclear that such an account would tie into the justifications for 'maximize' discussed above, as the ability of a theory to ensure a wide variety of interesting set theoretic objects does not seem to be directly furthered by focusing on making sentences in some intricately-defined syntactic class true; thus, there would seemingly need to be a separate defense of any syntactic version of maximize. For these reasons, we will not give further consideration to such syntactic accounts in this dissertation.

[12]We will refer to this account as Steel Maximization, as it's basis is presented and defended in Steel (2004), and then further developed in Steel (2014). Similar accounts seem to be suggested less clearly in other places in the literature, however: of particular relevance, a similar approach to maximization seems to play a role in Koellner's "envelope perspective" defense of CH against Todorčević's compactness style objections: see Koellner (2017). As the clearest account of such a consistency strength style approach to maximization, however, we will focus on Steel's particular account throughout the next two chapters. As these alternative accounts appear to be fairly similar to Steel's version, however, we can use that account as a proxy for all such approaches to 'maximize'.

cannot be properly represented in any of the more restrictive theories.

The central idea behind Steel's approach is that many different, even incompatible theories are all of genuine mathematical interest, and can be fruitfully studied for various mathematical purposes. Perhaps the clearest example of this is with the trade-off between the axiom of choice and determinacy axiom; since Zermelo, choice has been recognized as essential for a broad swathe of work in core mathematics, and so plainly necessary for any good foundational theory,[13] and yet there have been many mathematical benefits of studying the consequences of determinacy axioms in restricted inner models of V.[14] Given the utility of studying rejected theories, a maximizing theory should avoid curtailing any of this well-motivated math, and so must provide domains for the theories which are more restrictive but still worthy of sustained development. In more plain terms, this means that a maximizing theory should prove the existence of some interpretation of a more restrictive theory. In this case, we do not face a genuine trade-off between the two theories, as the maximizing theory permits the exploration of the other within this interpretation, and so nothing is lost in shifting to the maximizing theory. Given that one theory proves the existence of an interpretation of another just in case it is of (weakly) greater consistency strength, we will refer to Steel's approach to 'maximize' as a *consistency strength* approach.[15]

As Steel explains it, the central guideline underlying axiom selection is "to maximize interpretive power, to provide a language and theory in which all mathematics, of today, and of the future so far as we can anticipate it today, can be developed".[16] Motivating the restriction of interpretations under consideration is the idea that it is not enough to interpret a more-restrictive theory in some roundabout way; instead, the interpretation and the candidate for V must "agree" on the meanings of the relevant set theoretic vocabularly, so that the interpretation can "preserve their meaning".[17]

[13]See Moore (1982) and Section 1.1 above.

[14]See Maddy (1988b) Section 5.

[15]Note that this is not to imply that all there is to this approach is a guideline to choose the theory with the greatest consistency strength. As will be seen, Steel's restriction to "meaning-preserving interpretations" goes beyond this minimal requirement.

[16]Steel (2014) p. 165.

[17]Steel (2014) p. 165.

While it is not entirely clear what makes an interpretation "meaning preserving"–and in particular how such interpretations might tie into a broader theory of meaning–Steel suggests as a rough sketch that it is those models which are "inner models of generic extensions of models satisfying some large cardinal hypothesis": on these grounds, we will define the *S-fair* interpretations as those which can be generated through iterations of the processes of taking definable inner models and generating forcing extensions.[18] We will call this methodological approach Steel-Maximization (S-Max for short).

On this basis, we define a theory T' weakly S-maximizing over a theory T[19] as follows:

$$T \trianglelefteq_S T' \text{ iff there is some } \varphi(x) \text{ st i). for all } \sigma \in T\ T' \vdash \sigma^{\varphi},$$
$$\text{and ii). } T' \text{ proves that } \varphi \text{ is S-fair.}^{20}$$

We will define a theory T' strongly S-maximizing over a theory T just in case T' weakly S-maximizes over T, but T does not weakly S-maximize over T'.

Maddy's approach as articulated in *Naturalism in Mathematics* can be understood as rationally starting from such a consistency type approach, with the existence of a "fair" interpretation being a necessary but not sufficient condition for a theory to be maximal. For Maddy, an interpretation is "fair" just in case it is a definable inner model of the base theory;[21] in contrast to the S-fair interpretations, we will use the term *M-fair* interpretations to describe this class. Beyond merely

[18] Steel (2014) p. 165.

[19] Given that the actual theories we wish to apply the notion of 'maximize' to are all extensions of ZFC, throughout our formal account of maximize, any theories T and T' will be assumed to be extensions of ZFC. Unless explicitly specified, assume any reference to general theories T and T' include at least these axioms.

[20] For the our purposes we will leave the notions of an "forcing extension model" included in the conception of an S-fair interpretation vague; Steel does not attempt to formally specify this notion. It suffices to note that any set-forcing poset which provably exists in T' and which is capable of forcing a theory T will count as providing an S-fair interpretation of T within T'.

[21] See Maddy (1997) pp. 220–221. Note that Maddy understands "inner model" in a slightly nonstandard way. In particular, an inner model includes both class sized transitive models of ZFC and the various V_κ's for κ an inaccessible cardinal. Throughout our discussion of maximization, "inner model" will instead be understood in the more typical way, including only class sized transitive models. We will suppress the inclusion of set sized models as "inner models" since nothing hangs on this difference in the context of considering the case of the inner model program and forcing axioms, and the suppression will enable a more direct comparison of the two formal approaches to 'maximize'.

having such an interpretation, however, Maddy's version of maximize demands that there also is a genuine mathematical benefit to accepting the maximizing theory. That is, there must be something of mathematical interest that is supplied by the maximizing theory that cannot be found within the original theory. In particular, there should be a pair (X,R) such that no object in the interpretation of the restrictive theory can have any pair (Y,S) that is isomorphic to (X,R).[22] Given that not all isomorphism types are equally of mathematical interest, we also intend that this isomorphism type be mathematically relevant: this is to exclude such objects as odd Gödel codings that have no appeal to practicing mathematicians, and so their existence does not present a genuine advantage for a theory. Unfortunately, there is no clear way to formalize the notion of "genuine mathematical interest", and given that a suitable foundation should aim to provide all the necessary objects for the future development of mathematics, it is desirable to refrain from hewing too closely to current understandings of mathematics. As a result, we will not officially include the demand that the isormophism type (X,R) be particularly useful. Given the focus on the existence of useful isomorphism types in the maximizing theory, we will refer to Maddy's approach to 'maximize' as a *isomorphism type* approach. We will call this approach to maximization Maddy-Maximization (M-Max for short).

In direct comparison to the definition of S-Max, we define a theory T' weakly M-maximizing over a theory T as follows:

$T \trianglelefteq_M T'$ iff there is some $\varphi(x)$ st i). for all $\sigma \in T$ $T' \vdash \sigma^{\varphi}$,

ii). T' proves that φ is an M-fair interpretation,

iii). $T' \vdash \exists x \exists R \subseteq x^2 \forall y \forall S \subseteq y^2 (\varphi(y) \wedge \varphi(S) \rightarrow (x,R) \not\cong (y,S))$.

As before, we will define a theory T' strongly M-maximizing over a theory T just in case T' weakly M-maximizes over T, but T does not weakly M-maximize over T'.

Note that there are two crucial differences between S-Max and M-Max: first, the former permits

[22] See Maddy (1997) pp. 221–222.

the wider scope of S-fair interpretations instead of only M-fair interpretations,[23] and secondly the latter adds an additional requirement of a new isomorphism type in the maximizing model. How then do S-Max and M-Max compare? Note that S-Max solely requires that T' be able to provide a S-fair interpretation of T, with no further requirements. Given that M-Max requires a M-fair interpretation–and all M-fair interpretations are already S-fair–as well as the additional criteria of providing a unique isomorphism type, it is immediately clear that M-Max is a more fine grained notion of maximization: that is, $T \trianglelefteq_M T' \rightarrow T \trianglelefteq_S T'$. The natural question then becomes whether they are in fact equivalent notions of maximization, or, in other words, whether the additional requirement of M-Max ever in fact serves to separate two theories which are equivalent in terms of S-Max. A bit of reflection reveals that these are in fact distinct notions of maximization; while $ZFC + V = L$ and $ZFC + V \neq L$ each have a S-fair interpretation of the other[24] and are therefore S-equivalent, $ZFC + V = L$ cannot prove that there is a M-fair interpretation of $ZFC + V \neq L$, and so $ZFC + V \neq L$ strictly M-maximizes over $ZFC + V = L$. This case however is fairly insignificant: $ZFC + V \neq L$ has never been considered a serious candidate for our best theory of sets. Nonetheless, because of this toy example, we find that M-Max is in fact a strictly more fine grained notion of maximization than S-Max: given that they are extensionally distinct, the question arises of which notion better captures the informal notion of maximize that they attempt to explicate. In other words, which formalization should we use as our formal notion of maximize?

Before considering this question directly, it is worth noting that both formal notions justify the set theoretic community's rejection of V=L in favor of large cardinal axioms, and each does so in much the same way. With regards to the question of $T = ZFC + V = L$ versus $T' = ZFC + \exists\kappa Meas(\kappa)$, Maddy shows that the latter can provide a M-fair interpretation of the former (namely L itself), while the former cannot provide even a S-fair interpretation of the latter under pain of inconsistency; given that the consistency strength of T is strictly less than that of T', this cannot

[23]We here focus on the latter condition, and why it seems to lead to M-Max being a better justified explication of the informal 'maximize' maxim. We will return to the effects of this first difference–between S-Fair and M-Fair interpretations–in Section 4.2 below.

[24]$ZFC + V \neq L$ can interpret $ZFC + V = L$ in the inner model L; $ZFC + V = L$ can interpret $ZFC + V \neq L$ in a plethora of simple forcing extensions.

even be an artifact of the consideration of only M-fair interpretations, as no series of inner or outer model interpretations within T can result in an interpretation of a measurable cardinal unless ZFC itself is inconsistent. Furthermore, Maddy is able to show that there is an isomorphism type in T' that cannot be replicated within T (namely the large cardinal $0^\sharp$).[25] As Maddy notes, this case can be extended to other cases of large cardinal versus inner model axioms: for example, a similar argument shows that $ZFC + \exists\kappa,\lambda(Meas(\kappa) \wedge Meas(\lambda) \wedge \kappa \neq \lambda)$ maximizes over $ZFC + V = L[\mu]$ where μ is the measure on a measurable cardinal, in both the S and M senses. Similarly, $ZFC + \exists\kappa(SC(\kappa))$ both (strongly) S- and M-maximizes over $ZF + AD$.[26] Thus, any way of distinguishing the two formal explications of maximize will need to occur beyond the level of this earlier axiom dispute.

With this in mind, let us return to the question of how well-motivated the two explications are by the foundational goals that underlie the methodological merits of 'maximize' in the first place. After all, it is not enough that a formal explication of 'maximize' gets the "right" answer in past cases of axiom selection: it needs to be that the formalization decides these cases on the basis of the "right reasons". On these grounds, we find an important difference between S- and M-maximization. Recall that, beyond the relatively weak restriction to S-fair interpretations, S-Max amounts to a simply admonition to "choose the theory with greater consistency strength", regardless of the content of the two theories.[27] But consider what this admonition amounts to: given that higher consistency strength directly accords with a greater risk of inconsistency, S-Max amounts to the suggestion that the riskier theory should be chosen no matter what. Such a principle pushes for the mathematical community to maximize the risk of inconsistency regardless of whether there is any corresponding benefit gained by incurring the greater risk; that is, there is no off-setting cost-benefit analysis to be performed between the theories. On the other hand, M-Max requires that a riskier theory also provide some value to offset this risk: namely, some new isomorphism type

[25] See Maddy (1997) Part III, Chapter 6 for this argument in full detail.

[26] This result follows immediately from the result that, given ω many Woodins with a measurable on top, $L(\mathbb{R})$ is provably a model of AD. See Woodin (1988) and Martin and Steel (1989).

[27] As before, this is under the implicit assumption that both theories extend ZFC and are natural theories.

that could not be studied in the less-risky theory. Combined with the informal guideline to focus on the isomorphism types that are genuinely mathematically interesting, M-Max amounts to a methodological imposition to choose the riskier theory when and only when there is mathematical benefits to be gained by doing so. Given the centrality of cost-benefit analysis to mathematical practice, and to set theoretic activity and axiom selection in particular, this seems to be a more sustainable approach to increasing the consistency strength of our best theories of sets:[28] only incur risk when there is a corresponding mathematical benefit.

In addition, there is one more reason to very slightly favor M-Max over S-Max as our formal explication of 'maximize': the more fine-grainedness of the former. While there has been a remarkable degree of orderliness in the consistency strength ordering of most seriously proposed theories of sets, this appears to no longer be something that can be assumed for future theories. In particular, we noted in the first chapter that for any large cardinal notion above that of a supercompact, it appears that there will be equiconsistent theories at that consistency strength, with one endorsing inner model axioms and the other endorsing forcing axioms.[29] The key to this result is the development of more sophisticated methods of constructing inner and outer models; as the subject progresses further, it stands likely that even more powerful methods will be devised, and will generate more and more distinct theories at each level of the large cardinal hierarchy. Given the possibility of an increasingly intricate array of distinct theories at any given consistency strength level, it may prove that the greater power to distinguish theories provided by M-Max is necessary for putting the notion of 'maximize' to work in real debates of axiom selection. This weak reason for preferring M-Max, however, is at this point entirely preliminary: the only pairs of theories that M-Max is capable of distinguishing that S-Max cannot are not both serious candidates for acceptance by the set theoretic community, and so represent only toy examples, such as that of $ZFC + V \neq L$. If this were to persist for current and future debates, it would turn out that the fine grained view provided by M-Max is of no actual mathematical benefit. At this point, we merely

[28]See Maddy (1997) for more on the important role of cost/benefit analyses in the methodology of set theory.

[29]This is given the various conjectures of the proponents of $Ult(L)$. See Section 3.1 below for more on these assumptions.

note that this greater ability to distinguish theories may potentially provide a strong reason to prefer M-Max, if there are genuine axiom selection debates that can only be settled with this more fine-grained perspective.

Thus, while there are some preliminary reasons to find M-Max to be better in keeping with the philosophical motivations underlying the maxim of 'maximize' than S-Max, it is not clear that there is much to be gained in real debates of axiom selection by adopting the more fine-grained notion of M-Max. In the next chapter we will therefore seek to apply both formal explications to the current dispute between the inner model program and forcing axioms, seeing whether either or both is capable of distinguishing the axiom candidates. Before we attempt to put the formalizations to work, however, in the remainder of this chapter we will note a important lemma regarding applications of M-Max to theories T, T' that extend ZFC. In particular, we will see that the assumption that both theories include ZF–guaranteeing the absoluteness of key set-theoretic concepts–and of *AC*–enabling the coding of sets with collections of ordinals–greatly simplifies the search for particular isomorphism types witnessing M-Max. We will explain this lemma, and its effects on our use of M-Max, in the following section, while its proof can be found in Appendix A.

2.3 Re-characterizing M-Max for Extensions of ZFC

Note that a theory T' M-Maximizing over a theory T consists of two distinct conditions: in particular, conditions i) and ii) above together require that T' proves the existence of an M-Fair interpretation of T, while condition iii) requires that T' prove that there is some particular isomorphism type that exists but cannot be captured by the interpretation of T. It is natural to wonder whether the isomorphism type condition, which is usually the difficult condition to verify in practice, might be simplified to a more tractable form. In particular, we ask whether the existence of a non-trivial M-Fair interpretation[30] suffices for the satisfaction of the isomorphism type condition?

[30] Here we just mean an M-Fair interpretation $\varphi(x)$ where $T' \vdash \exists X \neg\varphi(X)$.

We should note that, in complete generality, the answer to this question is a clear negative. To see this, consider the theory $ZFC^- + AFA$, where ZFC^- is ZFC without the axiom of foundation, and AFA is Aczel's anti-foundation axiom.[31] Since ZFC^- suffices to prove that the class of well-founded sets WF is a model of the full theory ZF, we find that $ZFC^- + AFA$ proves that there is an M-Fair interpretation of ZFC. One might naturally expect that $ZFC^- + AFA$ would witness many new isomorphism types, as it permits a development of an entirely new domain of non-well-founded sets; nonetheless, Maddy notes that any isomorphism type in $ZFC^- + AFA$ will in fact already be witnessed within the class WF.[32] In fact, it is precisely examples like these that motivate the necessity of Maddy's formulation of the isomorphism type condition in the first place.

Note, however, that in this example one of the theories–$ZFC^- + AFA$–is fairly weak, in the sense of not including all of the standard axiomatization ZFC. In fact, this relative weakness is what makes the full isomorphism type condition necessary: if both T' and T are extensions of the full theory ZFC, then the existence of a non-trivial M-Fair interpretation of T within T' suffices to establish that there is an isomorphism type that fails to be witnessed in the interpretation of T.[33] That is, for theories extending ZFC, any set outside of a proper inner model interpretation provides a novel isomorphism type.[34] As a result of this fact, we find that we can state an equivalent formulation of the conditions of M-Max for strong theories of sets extending ZFC:[35]

[31]An accessible pointgraph is a directed graph with a distinguished vertex such that for any vertex in the graph, there is a path from the distinguished element to it. We say that an accessible pointgraph A with vertexes V decorates a set X if there is some map $f : X \mapsto V$ st $\forall y,z \in X$, $y \in z$ iff there is some path between $f(y)$ and $f(z)$ in A. The axiom AFA says that every accessible pointgraph decorates some set. This immediately suffices to imply the failure of the axiom of Foundation: to see this, note that a graph with a single point and self-directed edge is an accessible pointgraph; thus, any model of $ZF^- + AFA$ must include some x st $x \in x$. See Aczel (1988) for a development of this theory.

[32]See Maddy (1997) pp. 216–217, especially footnote 5, for this argument.

[33]See Lemma A.1.1 in Appendix A for the proof of this claim.

[34]This is just a bit too loose of a way of stating the result; a bit more precisely, any set outside of the proper inner model interpretation is coded by a collection of sets of ordinals, and the isomorphism type of the union of this singleton of this set of ordinals with its transitive closure cannot exist in the interpretation. See the proof below.

[35]See Theorem A.1.2 in Appendix A for the proof of this claim.

Re-Characterization Theorem:

Let $T, T' \supseteq ZFC$. Then $T \trianglelefteq_M T'$ iff there is some $\varphi(x)$ st

i). for all $\sigma \in T$ $T' \vdash \sigma^{\varphi}$,

ii). T' proves that φ is an M-fair interpretation,

iii). $T' \vdash \exists X(\neg\varphi(X))$.

Thus, when considering theories extending $ZFC + LCs$ we only need to be concerned with showing the existence of non-trivial M-Fair interpretations: the existence of a particular isomorphism type escaping the interpretation automatically follows from the existence of *any* set outside of the interpretation. For our present purposes, this will make it somewhat easier to apply M-Max to our theories, as there has already been much work on studying the existence of various interpretations between theories of forcing axioms and that of $Ult(L)$. Now, with this formal machinery in place, as well as that of S-Max, the question arises of whether these formal tools can have any bearing on the debate between V=Ult(L) and forcing axioms. In the next chapter, we will tackle this question directly.

Chapter 3

Applying ‘Maximize’ to Contemporary Axiom Candidates

In this chapter, we will use the two formal approaches to ‘maximize’ introduced in the previous chapter–S-Max and M-Max–to examine whether either of the contemporary strong theories of sets presented in the first chapter–a theory capturing $V = Ult(L)$ and the theory given by forcing axioms–can be said to maximize over each other. But first, in order to permit the necessary level of precision for applying these tools, we must make some assumptions regarding the eventual shape of an acceptable theory of $V = Ult(L)$: we will consider these in the first section. Then, in the following two sections, we will separately appraise the central question of maximization between these theories using S-Max and M-Max. Finally, we conclude by reflecting on what this study reveals about the current dilemma in contemporary axiom selection.

3.1 Towards a Theory for Ultimate L

In this section we will outline how adherents of the inner model program, in particular the Harvard school,[1] have sought to find a theory capturing $V = Ult(L)$.[2] It is worth noting at the outset that there is some ambiguity in how $Ult(L)$ is used in the literature: it is sometimes used more broadly to mean the core or canonical inner model of a supercompact cardinal, but is sometimes used more strictly to refer to the unique structure satisfying some axiom candidate, especially Woodin's current candidate for $V = Ult(L)$. Throughout this section, we will use $Ult(L)$ in the looser sense, referring just to a uniquely specifiable (in some particular sense) weak extender model for the supercompactness of an extendible cardinal. While this ambiguity might be somewhat unsettling, by the end of this section, with some charitable assumptions made on behalf of the advocates of $Ult(L)$, a more sturdy understanding of the meaning of $Ult(L)$ will be possible.

As mentioned in the first chapter, the minimal requirement for a candidate for $Ult(L)$ is that of a weak extender model for the supercompactness of an extendible cardinal δ, for some cardinal δ.[3] A weak extender model, within some larger V, not only agrees with V on the supercompactness of δ, but does so for precisely the same reason: that is, the proper class of measures witnessing the supercompactness of δ in the weak extender is just the restriction of the proper class of measures witnessing this property in V. In this sense, a weak extender model is an inner model that provides a fair interpretation for the existence of a supercompact cardinal within V. Note that a weak-extender

[1] By the Harvard school we mean the group of mathematicians and philosophers working on the $Ult(L)$ program at Harvard from roughly 2010 through the present. In particular, this group includes Hugh Woodin, Peter Koellner, and Gabriel Goldberg.

[2] It is also worth noting that this section will of necessity require a direct accounting of the technical details of the $Ult(L)$ project. As a result, the mathematical prerequisites of this section are somewhat higher than the remainder of the chapter. With that said, the section can be skipped without losing the narrative of the chapter: however, if this track is taken, then the justification for the assumption that there is a fair interpretation of $ZFC + LCs + V = Ult(L)$ in $ZFC + LCs$ for LCs of at least an extendible cardinal will have to be granted without direct argument.

[3] The current standard source for much of the material from this section is Woodin (2017), especially sections 3 and 7. While this provides a comprehensive account of much of the theory of weak extender models and of the axiom candidate for $V = Ult(L)$, the former appears to be in the process of being superseded by an alternative approach: it seems that much of the proofs in section 3 can instead be proven in a somewhat simpler way directly from the δ-covering, -approximation, and -genericity properties. This alternative approach is mentioned briefly in Woodin (2019). Since the main resulting theorems themselves are unchanged, and this alternative approach has not been fully worked out in print, for this section we will follow Woodin (2017).

model need not provide any of the insight into supercompact cardinals typically associated with core or canonical inner models: in this way, the notion of a weak extender model is, in itself, a very loose "from above" notion of an inner model.

While providing a good inner model of a supercompact cardinal is an interesting feature of weak extender models, well worth studying in its own right, this proved to be just the first of a large number of surprising properties. The initial key discovery regarding weak extender models for the supercompactness of an extendible cardinal is that such a model must also provide a similarly fair interpretation of all known large cardinal assumptions above that of a supercompact: that is, any embedding witnessing a known large-cardinal property in V will also witness that property in the weak extender (through its restriction to that model).[4] Thus, unlike previous inner models of large cardinals, any inner model that closely agrees with V on the nature of a single supercompact cardinal will not face an anti-inner model theorem ruling out the existence of some even larger large cardinal notion. While a weak extender model for a supercompact cardinal need not be a core or canonical inner model, Woodin pointed out that any core or canonical inner model containing a supercompact would seemingly be required to be at least a weak extender model for a supercompact: in this way, the theory of weak extender models represents a minimal requirement for any future theory of a fine-structured inner model at the level of a supercompact. Given the

[4] One might naturally wonder how to evaluate the limitation to "known large cardinal assumptions" in Woodin's common statement of this result: how seriously should we take the possibility of a future large cardinal assumption that escapes this result? Such a large cardinal would necessarily fail to have a presentation in the form of the existence of an elementary embedding from V into some inner model M. Given the assumption of AC, there is only a single serious candidate for such a large cardinal axiom, namely the so-called HOD analogues of choiceless large cardinals. If these choiceless large cardinals prove to be consistent with ZF, then, there will be large cardinals that are unable to be captured in a weak extender model, and so there will in fact be an anti-inner model theorem for weak extender models. Such a theorem would, however, be only the tip of the iceberg regarding problems for the $Ult(L)$ program: this would imply that there is no weak extender model contained within HOD (Woodin (2017) p. 24), and that all of the friendly assumptions made on the behalf of proponents of $Ult(L)$ are false.

Thus, it seems that either there are no consistent candidates for large cardinal assumptions that cannot be captured in a weak extender model, or that the $Ult(L)$ program is thoroughly shattered. Much work remains to be done examining the plausibility of the choiceless cardinals: in particular, their consistency implies the consistency of a proper class of ω-strongly measurable cardinals in HOD, while it is currently unknown if even small finite numbers of these cardinals can consistenly exist in HOD (See Woodin (2017) p. 25, remark 3.43 for the current state-of-the-art on ω-strongly measurable cardinals in HOD.) For our purposes, we will set aside the possibility of choiceless large cardinals in order to present the $Ult(L)$ program in its intended context; if these cardinal prove consistent in the end, then it is quite likely that this would prove the end of the $Ult(L)$ approach to the inner model program as we know it.

surprising discovery, this implied that there would be no limits on the large cardinal strength of a fine-structured inner model of a supercompact, if such a model were discovered.[5]

In this way, a weak extender model for the supercompactness of an extendible cardinal δ must be close to V in the sense of agreeing with V on all known large cardinal notions. There are a wide variety of other ways in which such an inner model must be "close to V", however. First of all, the weak extender will agree with V on all singular cardinals, and the identity of their successors, above δ; as a result of this fact, the weak extender will agree with V on cardinal arithmetic involving singular cardinals above δ more broadly.[6] Secondly, a weak extender model will have strong covering properties in relation to V.[7] In fact, given the assumption of a proper class of supercompact cardinals in V, Woodin has reportedly shown that any inner model with such covering properties must be a weak-extender model for the supercompactness of some δ: that is, only weak extender models can be similarly close to V.[8] Finally, any elementary embedding from a weak extender model into itself is either trivial or pathological, in the sense of having a small critical point relative to δ.[9] In each of these ways, we find that a weak extender model–if such a model exists–is uniquely similar to V in a wide variety of mathematically useful ways.

Given the important and useful properties of weak extender models, then, a natural question arises of whether any such model must exist, given the assumption of sufficient large cardinal strength. At present, this question remains open. Nonetheless, work on the choiceless large cardinals has led to a somewhat plausible assumption (the "HOD hypothesis") capable of resolving this question:

[5] An important distinction must be made here: while such a hypothetical fine-structured approach to a supercompact would capture larger large cardinal assumptions, it would not necessarily provide a fine-structured, "from below" approach to these cardinals. In this way, the proposed $Ult(L)$ structure might provide only a coarse understanding of large cardinals above a supercompact, leaving the task of developing notions of strategic extender models capable of providing a fine-structured understanding of these very large cardinal notions. Whether this possibility holds or not cannot be known until the fine-structure of weak extender models is more fully developed; but it is worth noting that in this way a proof of the assumptions outlined in this section may not be a complete end to the inner model program.

[6] See Woodin (2017) p. 10, Them 3.10.

[7] In particular, the model will have γ-covering for all regular $\gamma > \delta$, where a model M has γ covering if for all $X \subseteq M$ where $|X| < \gamma$, there is some $Y \in M$ st $X \subseteq Y$ and $|Y| < \gamma$.

[8] This theorem was presented in a series of seminar talks given by Peter Koellner at UCI in the Spring of 2019. A published version of this material is forthcoming.

[9] See Woodin (2017) p. 14, Thm 3.19.

that there is a regular cardinal λ above an extendible cardinal which is not ω-strongly measurable in HOD. For present purposes, we need not be too concerned with the detailed machinery surrounding this assumption, but may instead focus on a particular implication. If the HOD hypothesis is true and an extendible cardinal κ exists, then there is a weak extender model for the supercompactness of κ: equivalently, HOD itself is such a model.[10] Importantly for our purposes, this means that the HOD hypothesis implies that there is a definable inner model that provably must be a weak extender model for the supercompactness of an extendible cardinal. Furthermore, if $V \neq HOD$, then there is a definable *proper* inner model that must be a weak extender model.

Note, however, two key problems preventing HOD under these assumptions from truly representing the hoped-for ultimate approach to L: first, there is no clear axiom or theory behind the informal conception of $Ult(L)$ as a definable weak extender model, and, secondly, this approach fails to properly characterize $Ult(L)$ as a canonical inner model. Thus, HOD alone does not suffice for providing a theory of $Ult(L)$ robust enough for the analysis of maximization notions later in this chapter–even assuming HOD is a weak extender model. Towards this end, we must look more directly to the notion of canonicity of an inner model itself. Unfortunately, it does not seem that much can likely be known about the properties of a canonical inner model of a supercompact cardinal until the precise nature of the extender-like models capable of generating a model of a supercompact cardinal is discovered.

Nonetheless, some progress had made been into these questions by reflecting on and abstracting properties from past canonical inner models. Central to past successes of the inner model program is the comparison lemma. The comparison lemma was inspired as an extension of Kunen's comparison lemma, which states that for any two iterable structures of a particular form, there eventually is a pair of their iterates with one an initial segment of the other.[11] Since it is then said that the structure with an iterate initial segment is no stronger than the other to which it is compared, this

[10]See Koellner (2017) p. 3222–3223 and the surrounding material for a list of several key equivalencies to the HOD hypothesis, and a discussion of their importance.

[11]More precisely: if $\mathcal{M}_0 = \langle L_\xi[U], \in, U\rangle$ and $\mathcal{N}_0 = \langle L_\eta[U'], \in, U'\rangle$ are iterable structures, then there is some α and some filter F st $\mathcal{M}_\alpha \equiv \langle L_{\xi'}[F], \in, F\rangle$ and $\mathcal{N}_\alpha \equiv \langle L_{\eta'}[F], \in, F\rangle$. See Steel (2010) p. 1611 or Jech (2003) pp. 348–352.

lemma shows that the strength of any two iterable structures of the proper form can be compared. The comparison lemma proper similarly states that any two mice[12] with sufficient iterability conditions permit a comparison through their eventual iterates, though the necessary conditions are much more involved due to the heightened complexity of mice and iteration strategies compared to Kunen's iterated ultrapowers.[13]

While each version of a comparison lemma for a particular type of canonical inner model makes reference to the precise machinery of that iterable structure, Woodin noted that every known canonical inner model had a corresponding comparison lemma; furthermore, reflecting on the nature of these lemmas, Woodin was able to abstract the essential content of these lemmas as the weak comparison lemma, which allowed the notion of comparibility to be stated without involving particular fine-structural concepts.[14] Weak comparison follows from each form of a comparison lemma, and so Woodin notes that any acceptable fine-structure for a canonical inner model of a supercompact cardinal should also be expected to imply weak comparison. As a result, weak comparison can be used as a proxy for the future fine-structure that will result from further development of the $Ult(L)$ program.[15] Thus, weak comparison–together with a few necessary supporting assumptions needed to permit this principle to do much real mathematical work–presents an initial option for a theory of a canonical inner model of a supercompact.

Noting the difficulty of working with weak comparison and the supporting assumptions, however,

[12]A mouse is a particular type of well-behaved transitive, iterable structure which plays a key role in core model theory: in particular, the core model up to a measurable cardinal K is equivalent to L relatized to the collection of all mice. For present purposes, we simply note that mice represent the"from below" building blocks of the core model K. See Jech (2003) pp. 660–661.

[13]The details of this lemma are quite involved and go beyond the purposes of this section. See Steel (2010) section 3.2, especially Thm 3.11, for a precise statement of this theorem.

[14]Woodin's weak comparison axiom states that for any two finitely generated transitive models of ZFC $\mathcal{M}_0$ and $\mathcal{M}_1$ that are Σ_2-embeddable in V with $\mathbb{R}^{\mathcal{M}_0} = \mathbb{R}^{\mathcal{M}1}$, there must be a transitive N with embeddings $i_0 : \mathcal{M}_0 \mapsto N, i_1 : \mathcal{M}_1 \mapsto N$ with i_0 close to $\mathcal{M}_0$ and i_1 close to $\mathcal{M}_1$. See Woodin (2017) section 6.5 or Goldberg (Frth) for more on the weak comparison principle.

[15]More precisely, the combination of ZFC with the existence of a supercompact, $V = HOD$, and the weak comparison principle can be studied as a proxy for a theory of a canonical model of a supercompact cardinal. It is an open question whether this theory is consistent, which is unlikely to be resolved until the theory of fine-structure for $Ult(L)$ is further developed: Woodin notes that this is a "natural test question for the existence of a generalization of L at the level of supercompact cardinals based on anything like the current methodology for the construction of such inner models" (Woodin (2017) p. 88).

in his dissertation Goldberg takes the process of abstracting from the particular details of the fine-structured versions of the comparison lemma one step further. Seeking a simpler principle more easily used in mathematical proofs, Golberg isolated the Ultrapower Axiom (henceforth, UA): the claim that every pair of ultrapower embeddings admits a comparison.[16] Given the supplemental assumptions used for working with weak comparison, Goldberg was able to show that weak comparison implies UA, and thereby the fact that UA holds in all previously discovered canonical inner models.[17] Thus, due to the centraility of the comparison lemmas ("the central feature of modern inner model theory" (Goldberg (2018) p. 3)), Goldberg notes that ZFC with the existence of a supercompact and UA serves as a falsifiable test case for the eventual fine structured model of a supercompact: "if one could rule out the Ultrapower Axiom from a supercompact cardinal, one would in fact rule out any sort of inner model theory from supercompact cardinals" (Goldberg (2018) p. 3). Furthermore, in stark contrast to weak covering, UA has proven extremely fruitful to work with, enabling a large body of theorems to be developed from this assumption.[18] Of particular relevance for our present purposes, Goldberg was able to use UA to prove a number of the properties typically associated with coreness and canonicity, including that GCH must hold eventually.[19] Thus, $ZFC + \exists\kappa SC(\kappa) + UA$ provides our first serious candidate for a theory approximating the eventual theory of $Ult(L)$:[20] though $ZFC + \exists\kappa SC(\kappa) + UA$ will likely not be the eventual theory used in the $Ult(L)$ program, it provides a stable subtheory of the eventual resulting theory that can be used in analysis as this true theory continues to be developed.

[16]For transitive models of ZFC P,Q and a cofinal elementary embedding $j : P \mapsto Q$ is an ultrapower embedding if there is some $a \in Q$ that generates Q from functions in P, meaning $Q = \{j(f)(a) | f \in P\}$. Note that any embedding generated from an ultafilter in the usual way can be shown to be an ultrapower embedding in this sense. For ultrapower embeddings $j_0 : V \mapsto M_0$, $j_1 : V \mapsto M_1$, a comparison of (j_0, j_1) is a model N and a pair of embeddings (i_0, i_1) st $i_0 : M_0 \mapsto N$ and $i_1 : M_1 \mapsto N$ are themselves ultrapower embeddings, and $i_0 \circ j_0 = i_1 \circ j_1$. See Goldberg (Frth) or Goldberg (2018) Section 2.3 for more details.

[17]A proof of this result is forthcoming in Goldberg (Frth); an early summary of the main results of the dissertation, including this proof, was shared with the author in the Fall of 2018.

[18]See Goldberg's dissertation Goldberg (Frth) for the vast majority of this literature.

[19]That is, for any λ st $\lambda > \kappa$ for some supercompact κ, $2^\lambda = \lambda^+$. See Goldberg (2018) section 6.2 for more on this result.

[20]The assumption of a supercompact is necessary here: as UA is implied by previous versions of covering lemmas, its is true in each of the canonical inner models below a supercompact. Only with the additional assumption of a supercompact, ruling out these earlier models of UA, can Goldberg's axiom be seen as capturing the informal $Ult(L)$ idea.

We have therefore found our first candidate for a theory of $Ult(L)$ articulated with sufficient precision to be capable of being utilized in our analysis of formal maximization notions. With this goal in mind, the question naturally arises of the extent to which interpretations of this theory are known or suspected to exist. Given Goldberg's above comments on the necessity of the consistency of *UA* and the existence of a supercompact cardinal for any hope of an extension of the inner model program to this level of the large cardinal hierarchy, it seems fair to conclude that any satisfactory canonical inner model of a supercompact cardinal must validate the *UA* axiom. Additionally, a canonical inner model must be uniquely identifiable–hence the reference to canonicity–usually by making reference to the fine-structural notions used to build the model from below. With these two facts in mind, there seems to be a clear conjecture underlying Golberg's work with *UA*: that there must eventually be some definable inner model of $ZFC + UA + \exists\kappa SC(\kappa)$. Let us refer to this implicit claim as the *UA* conjecture.

Note, though, that *UA* only aims to capture an abstract and somewhat removed picture of the consequences of canonicity, without dealing with the details of what an extender-based approach to an inner model of a supercompact might entail. Another approach to finding a theory capturing the $Ult(L)$ notion instead tries to engage with this problem more directly. Much of the difficulty of articulating a fine-structural approach for this level comes from the seeming inability of standard, (nonstrategic) extender models[21] of reaching the level of a strongly compact cardinal: Woodin has shown that the least cardinal that is $\kappa^{+\omega}$ strongly-compact must fail to be $\kappa^{+\omega}$ supercompact in a nonstrategic extender model, while Goldberg showed that $UA + GCH$ implies just the opposite for the the least $\kappa^{+\omega}$ strongly-compact cardinal. Gven that $UA + GCH$ is seen as indicating the presence of fine structure, these results show that nonstrategic extenders are *not* the correct

[21] Here we mean any notion of extender models which uses a construction process defining the next model only from a single parameter (the previous model). This is in contrast to strategic extender models, which use a process defining the next strategic extender model from two parameters (the previous model and an iteration strategy which in effect codes the correct way to carry out this process while preserving iterability). Note that partial extender models, first developed by Baldwin and Mitchell and used by Mitchell and Steel to develop inner model theory to the level of Woodin cardinals in the 1990's, are of the former sort. See Steel (2010) footnote 3, Mitchell and Steel (1994), and Woodin (2017) section 5.1 for more on partial extender models; see footnote 25 below for more on strategic extenders.

notion of fine structure at the level of a strongly compact cardinal.[22] Thus, there is a seeming incompatibility between fine-structure as underwritten by *UA* and fine-structure as generated from non-strategic extender models once as the level of a supercompact cardinal is approached. To explain this incompatibility, Woodin and Goldberg note the difficulties in proving the iterability[23] of non-strategic extender models past the finite levels of supercompactness, and suggest that this is due to a failure of iterability at this point.[24] Thus, a new notion of an extender-like model was needed which would be capable of preserving iterability to the level of a supercompact.

To this end, Woodin turned to the notion of strategic extender models, which supplement a series of extender-models with iteration strategies in effect coding the proper way to proceed in these constructions without violating iterability.[25] Crucially, the use of iteration strategies on particular infinitary games in these definitions allows Woodin to apply the fruits of the successful theory of AD^+ in $L(A,\mathbb{R})$ for amenable sets A under the assumption of a supercompact cardinal to the work of developing the theory of strategic extender models at this level.[26] While the state of this project is very much in flux as Woodin continues to work out the details of strategic extender models at higher infinitary levels, for our purposes we need only note that Woodin conjectures that applications of the theory of AD^+ in this way will permit a proof of the iterability of strategic extender models past the finite levels of a supercompact.[27] Let us refer to the conjecture that

[22]See Goldberg (2018) for a brief discussion of this result and the challenges it poses for nonstrategic extender models reaching very large cardinals. While Goldberg's result requires the additional assumption of *GCH*, there is some hope that the assumptions behind this and other related results will eventually be weakened to just *UA*.

[23]A model permitting iterated ultrapowers is said to be iterable if each of its iterates is itself well-founded. See Steel (2010) section 3 for an approachable discussion of iterability, and see Schimmerling (2010) and Neeman (2010) for proofs of iterability for earlier notions of fine-structure.

[24]See Woodin (2017) Remark 7.1 and surrounding material for more on this failure of iterability.

[25]The bulk of Woodin (2017) is concerned with laying out the role and theory of strategic extender models: Sections 4, 5, and 6 lay out the complications that prevent nonstrategic extender models from capturing a supercompact cardinal, and section 7 develops the initial theory of strategic extenders. Note that until the construction process reaches the finite levels of a supercompact, beyond the level of a single Woodin cardinal, the addition of iteration strategies has no effect on the defined models, and so the nonstrategic and strategic hierarchies are identical; this is due to the unique branch theorem, which states that there is only one unique way of carrying this process out (See Woodin (2017) Thm 6.35). After the fist Woodin cardinal, these hierarchies diverge. It is currently conjectured that at some point iterability fails for the non-strategic hierarchy, while iterability continues for the strategic hierarchy up to and including the level of a full supercompact cardinal. See Woodin (2017) section 7 for more on these conjectures.

[26]See Woodin (2017) pp. 90-92 for a brief outline of this work.

[27]See Woodin (2017) pp. 89-90 for a clear presentation of the recent state of the project of proving iterability. Note in particular the claim that "the only credible possibility that remains is that iterability is proved by induction and *not*

iterability will eventually be proved for the full strategic extender hierarchy as Woodin's Iteration Conjecture (henceforth, WIC).

Given the WIC, the eventual model constructed through the strategic extender hierarchy would provide a fully fine-structural, from below approach to a unique weak extender model of a supercompact cardinal, thereby fulfilling the promise of the informal $Ult(L)$ notion. But what would the theory of this canonical structure be? While one might naturally expect any answer to this question to depend on the precise details of the proof of iteration, and thereby be inaccessible for the time being, in another surprising development Woodin found that the connections between strategic extender models and the theory of AD^+ might already provide the answer: under the assumption of a proper class of Woodin cardinals, good approximations to the eventually generated structure exist and can be defined without any reference to fine-structural notions.[28] These determinacy based approximations to $Ult(L)$ form a hierarchy, and it has already been shown that the initial members of the hierarchy are also members of the strategic extender hierarchy. Furthermore, it is expected that there will eventually be a proof that all of the determinacy based approximations must be strategic extender models.[29] Given this expectation, the completion of each hierarchy will be extensionally equivalent, and so the structure ($Ult(L)$ itself) could be theorized through reference to determinacy notions instead of the more difficult fine-structural approach.[30]

It is precisely this possibility that motivated the current formulation of the axiom $V = Ult(L)$

on the basis of some general iteration hypothesis for V. Verifying that this is in fact what happens is the main task ahead." (Woodin (2017) p. 90).

[28]See Woodin (2017) p. 90, especially "in the context of a proper class of Woodin cardinals, there are naturally defined approximations to Ultimate-L and the collection is rich enough to make a definition of the axiom, $V = Ultimate(L)$, possible without specifying the detailed level-by-level definition of Ultimate-L".

[29]Woodin (2017) p. 90: "The conjecture is of course that all the approximations are strategic-extender models and there is quite a bit of evidence for this conjecture... The key issue is whether the axiom $V = Ultimate(L)$ formulated in terms of these approximations must hold in *some* weak extender model for supercompactness assuming that there is an extendible cardinal. Presumably any proof of this must yield as a corollary that these approximations are all strategic-extender models".

[30]While it might seem initially strange that a canonical inner model could be axiomatized through a claim about close approximations to V, without reference to any fine-structural notions, it is worth noting that the same is true for earlier examples of canonical inner models. As a particularly salient example, Woodin notes that there is a very similar presentation of the axiom $V = L$ as the claim that any true Σ_2-sentence in V is true in an inner model of a particular form. Thus, it should not be too surprising that such an axiom could be found for $Ult(L)$. See Woodin (2017) Lemma 7.2.

by Woodin. This axiom states two distinct claims: first, that there is a proper class of Woodin cardinals, and, second, that for any true Σ_2 sentence φ there is a universally Baire set of reals A such that HOD in $L(A,\mathbb{R})$ models φ.[31] It is hoped that any model of this axiom would be so well-approximated by the determinacy based approximations to $Ult(L)$ that it would have to simply be the completion of the hierarchy of these approximations; furthermore, given the assumption that this determinacy hierarchy is a cofinal proper subset of the strategic extender hierarchy, any model of the axiom would also thereby have to be the completion of the strategic extender hierarchy. While there is a flurry of current activity by the Harvard school in developing the consequences of this axiom, some initial results have been quite promising. In particular, it has been shown that $V = Ult(L)$ implies CH[32], and there is much hope that other restricted versions of paridigmatically L-like properties will also be shown to follow from it.[33] Thus, with the candidate axiom for $V = Ult(L)$, we have a second possible theory aiming to capture the informal $Ult(L)$ idea: $ZFC + V = Ult(L)$.

We should note, however, that though there has been significant progress regarding Woodin's current preferred candidate axiom for $V = Ult(L)$ in recent years, and while proponents of the $Ult(L)$ program have great confidence that this will be the eventual axiom capturing $Ult(L)$, it may perhaps be a bit wise to refrain from a complete endorsement of $V = Ult(L)$ as the correct axiom.

[31] The inner models $HOD^{L(A,\mathbb{R})}$ are the determinacy based approximations to $Ult(L)$ described in the above paragraph. So the candidate for $V = Ult(L)$ in effect states that the theory of AD^+ is sufficient to permit the development of these inner models, and that for any sufficiently simple sentence φ there is an approximation of this form that comes close to V, in the sense of agreeing regarding φ. While it has been shown for particular universally Baire sets of reals A that $HOD^{L(A,\mathbb{R})}$ is a strategic extender model, it is currently only conjectured that this is the case for any universally Baire A. A proof of this conjecture would likely be closely tied to a proof of the WIC. See Woodin (2017) Definition 7.14 for the official definition and surrounding discussion.

[32] See Woodin (2017) Theorem 7.26 part a). It is currently unknown whether $V = Ult(L)$ implies the full GCH, though Goldberg's proof of GCH above a strongly compact cardinal from UA is seen as providing some evidence that this implication does in fact hold.

[33] As mentioned in chapter 1, the existence of a supercompact cardinal poses a serious limitation for the extent to which the typical L-like properties can hold in $Ult(L)$. In particular, there is no hope for a "simple" well-ordering of the reals. For the other properties, however, these is some hope. Woodin has shown that $V = Ult(L)$ implies that $V = HOD$ and that V has no proper set-generic grounds (Woodin (2017) Theorem 7.26 parts b) and c)). Together with the HOD hypothesis, the former may lead to a (weak) covering theorem; the latter may underlie a notion of restricted absoluteness, together with further developments in set theoretic geology. As there is much that remains to be proven regarding these properties, however, we will leave the matter here, noting that it is an open question of how L-like the structure characterized by the $V = Ult(L)$ axiom is.

This is especially true given the extremely fast pace at which developments in the $Ult(L)$ program have happened. In particular, there has been a bit of shift regarding the axiom candidate for $V = Ult(L)$ in the past, which past proposed axioms having been found to be too strong to capture the informal $Ult(L)$ notion: for example, while the various axiom candidates attempt to specify a canonical example of a weak extender model of a supercompact, past proposed axioms have been shown to be incompatible with weak extender models.[34] At the time of writing it remains possible that either the current candidate axiom prove to be too strong–by proving incompatible with the notion of a weak extender model–or that it prove to be too weak–by failing to entail UA and thus falling short of a fine-structural theory of $Ult(L)$. For these reasons, we will be somewhat cautious with equating the informal $Ult(L)$ notion and models of this axiom. Nonetheless, given the strong confidence on the part of its proponents, as well as the non-trivial theory of its consequences that has already been developed, we will henceforth treat this axiom as the axiom of $Ult(L)$, and refer to it just as $V = Ult(L)$.

As with the theory $ZFC + \exists\kappa SC(\kappa) + UA$, the question similarly arises of whether a definable interpretation of the theory $ZFC + V = Ult(L)$ is known to exist. As with the earlier question, much remains to be seen through the development of the $Ult(L)$ program, but Woodin makes the expectation quite explicit through the $Ult(L)$ conjecture: $ZFC + LCs$ proves that for any extendible cardinal δ, there is an inner model N such that N is contained in HOD, N is definable from δ, and $N \models V = Ult(L)$.[35] The truth of the $Ult(L)$ conjecture would suffice to show the truth of the WIC,

[34] For examples of this, see Axiom 1 and Axiom 2 in Woodin (2017) pp. 92-93. Note that–though they appear to have somewhat different forms–the current candidate for $V = Ult(L)$ is equivalent to a straightforward restriction of Axiom 2 to only Σ_2-sentences. In the end, Axioms 1 and 2 in fact insist on too much "closeness" between the determinacy inspired approximations and V. A similar incompatibility between the current candidate axiom and the notion of a weak extender model for a supercompact has not been found, but a proof of the consistency of $V = Ult(L)$ holding in a weak extender likely will only be possible after a further development of the associated fine-structural notions, if it is in fact true.

[35] There are many different formulations of the $Ult(L)$ conjecture. For standard treatments, see Woodin (2010b) (in its original form) or Woodin (2017). The most recent presentation in Woodin (2019) is based in the δ-approximation, -covering, and -genericity properties, but can be shown to be equivalent to earlier formulations. See also Koellner (2017) p. 3224 for a more explicitly arithmetical statement of the conjecture and ensuing discussion. Woodin sometimes distinguishes between weak, standard, and strong versions of this conjecture, regarding the large cardinal strength that must be assumed to prove the conjecture: the distinction between these versions of the conjectures need not make a difference for present purposes. If in fact only the weak $Ult(L)$ conjecture is true, the LCs assumption in the theories in sections 3.1 and 3.2 will need to be supplemented with the precise large cardinal assumptions required

and thus that there is a fine-structural approach to a model of $V = Ult(L)$.[36] It seems fair to say that a proof of the $Ult(L)$ conjecture would constitute a completion and a validation of the entire $Ult(L)$ program up to this point.

In summary, in this section we have found three approaches to attempting to capture the informal notion of the Ultimate version of L: the notion of a weak extender model of a supercompact, the theory $ZFC + \exists\kappa SC(\kappa) + UA$, and they theory $ZFC + V = Ult(L)$. Additionally, each has a corresponding conjecture which implies the existence of an inner model representing that approach: the HOD hypothesis, the UA conjecture, and the $Ult(L)$ conjecture. Given these multiple options to use in our analysis of the maximality relations between the $Ult(L)$ program and forcing axioms, a choice must be made of which to pursue. Fortunately, it turns out that not much results from such a choice, due to the deep relationships between the approaches. In particular, both the UA conjecture and the $Ult(L)$ conjecture imply the HOD hypothesis, and can only be satisfied in a weak extender model for a supercompact.[37] Additionally, given the WIC, $V = Ult(L)$ will entail that $Ult(L)$ is a fine-structural model of a supercompact in the traditional sense, and therefore that $UA + \exists\kappa SC(\kappa)$ holds.[38] In fact, the ability of $V = Ult(L)$ to imply $UA + \exists\kappa SC(\kappa)$ serves as a sort of verification that $V = Ult(L)$ is in fact the correct axiom candidate. With this in mind, not much hangs on whether $V = Ult(L)$ or $UA + \exists\kappa SC(\kappa)$ is used as our axiom of $Ult(L)$: given the focus of the Harvard school on $V = Ult(L)$, we will use this as our axiom in the remainder of the chapter.

In conclusion, for our analysis of the maximality relations between the $Ult(L)$ notion and forcing axioms, we will use $V = Ult(L)$ as our axiom. Likewise, we will assume the $Ult(L)$ conjecture

to prove the $Ult(L)$ conjecture.

[36] See Woodin (2017) p. 101: "Proving [the $Ult(L)$ conjecture] would show in a decisive fashion the transcendence of the strategic-extender hierarchy."

[37] To see the implication, note that the HOD hypothesis is equivalent to the provable existence of any weak extender model for a supercompact contained within HOD (Woodin (2017) Theorem 3.39).

[38] In fact, the connection between $V = Ult(L)$ and $UA + \exists\kappa SC(\kappa)$ might be even closer than this. Goldberg conjectures that $UA + GA + \exists\kappa Extendible(\kappa)$ in fact implies $V = Ult(L)$ (where GA is the claim that there are no proper set-generic grounds of V), thereby "recapturing" $V = Ult(L)$ from a collection of its most central consequences (See Goldberg (2017)). This is regarded as a highly speculative conjecture, with far more room for failure than the other conjectures and hypotheses articulated in this section. The truth of it, however, would represent the best possible situation for the $Ult(L)$ program.

(and therefore also the HOD hypothesis and the UA conjecture) on the behalf of adherants of the $Ult(L)$ program, putting this program in its strongest possible position as a compelling and justified way of extending $ZFC + LCs$. We note, however, that not much would change if we were to instead assume only the UA conjecture and use $ZFC + UA + \exists\kappa SC(\kappa)$ as our theory of $Ult(L)$: in fact, all of the results in the following two sections similarly hold with $UA + \exists\kappa SC(\kappa)$ replacing every mention of $V = Ult(L)$, and the UA conjecture replacing every mention of the $Ult(L)$ conjecture. In fact, all that is required for this analysis is that there is some axiom capturing the $Ult(L)$ notion and that there is provably some interpretation of this axiom in $ZFC + LCs$ for some large cardinal strength assumption. But the existence of an axiom and theorem of this sort has been noted above as being a minimal prerequisite for a successful completion of the inner model program: if the assumptions of the $Ult(L)$ conjecture and the UA conjecture are eventually shown to be provably false[39], this would constitute the end of the $Ult(L)$ program and of the inner model project as we know it. Thus, we note that the following results hold not just for the current axiom candidate $V = Ult(L)$, but for any reasonable alteration this axiom undergoes as the $Ult(L)$ program continues to evolve and develop.

3.2 S-Max: Equivalence

With a clear and precise theory for $Ult(L)$ in hand, we can now shift our focus to the main question of this chapter: does either the theory of $Ult(L)$ or the theory of forcing axioms maximize over the other in terms of the formal notions of maximize introduced in chapter 2? Given the justification of the 'maximize' maxim in terms of the mathematical goals of set theory, such a result would provide some defeasible support for the maximizing theory as the better extension of the theory $ZFC + LCs$. On the other hand, finding that the two theories were equivalent according to both formal notions

[39] Woodin stresses that the $Ult(L)$ conjecture is in fact a simple arithmetical statement about the provability of a finite collection of sentences from another finite collection of sentences. As a result, it cannot be independent of ZFC, but instead must be either provably true or false. See Koellner (2017) pp. 3223–3224.

would cast some doubt on the ability of ‘maximize’ to settle disputes in contemporary axiom selection, pointing to the need for find and defending other, hitherto unidentified methodological maxims. Since S-Max is a properly more coarse-grained notion of maximization, we will first tackle the question for this formal explication in the current section; we will then turn to applying M-Max to the question in the following section.

Before S-Max can be properly applied to this question, however, we must deal with a non-trivial asymmetry between the theories $ZFC+V=Ult(L)$ and $ZFC+MM$: namely, the significantly higher large cardinal strength built into the former. As noted in section 3.1, $V=Ult(L)$ explicitly includes the assumption that there is a proper class of Woodin cardinals, giving great large cardinal strength to this axiom. Additionally, $V=Ult(L)$ is expected to imply the iterability of a full hierarchy of fine-structural models eventually culminating in an inner model of the supercompactness of an extendible cardinal: it follows that the assumption of the existence of an extendible cardinal is also implicit in the $V=Ult(L)$ axiom. Thus, it is not entirely correct to think of both of these theories as possible extensions of $ZFC+LCs$, as only the former actually contains the full force of contemporary large cardinal assumptions. Thus, to compare these candidate theories, we first must balance the strength of their large cardinal assumptions. So we will henceforth consider the theories $ZFC+V=Ult(L)(\equiv ZFC+\exists\kappa Ext(\kappa)+PCW+V=Ult(L))$ and $ZFC+\exists\kappa Ext(\kappa)+PCW+MM$, where PCW denotes the claim that there is a proper class of Woodin cardinals. For simplicity, we will denote these theories as $ZFC+LCs+V=Ult(L)$ and $ZFC+LCs+MM$, respectively.[40]

Recall that a theory T' weakly S-Maxes over a theory T if it proves that there is some S-Fair interpretation φ, where both set-generic forcing extensions and definable inner models are explicitly included in the class of S-Fair interpretations. So, to show that $ZFC+LCs+MM$ weakly

[40]As noted, the introduction of an explicit LCs assumption is redundant in the context of $ZFC+V=Ult(L)$. Nonetheless, we will continue to state the theory with the assumption made explicit as a visual reminder that the theories are calibrated to have equivalent large cardinal strength. Additionally, we note that for any LC assumption stronger than an extendible and a proper class of Woodin’s, the same results found in the following two sections will still hold.

S-Maxes over $ZFC + LCs + V = Ult(L)$, it would suffice to prove in the former that a definable inner model interpretation of the latter. But the $Ult(L)$ conjecture states that $ZFC + LCs$ proves for any extendible cardinal δ, there is an inner model interpretation, definable from δ, of $ZFC + LCs + V = Ult(L)$. Letting δ be the smallest extendible cardinal, $ZFC + LCs$ therefore proves that there is an inner model interpretation of $ZFC + LCs + V = Ult(L)$. As an extension of $ZFC + LCs$, we thus find that $ZFC + LCs + MM$ proves that there is an S-Fair interpretation of $ZFC + LCs + V = Ult(L)$ under the assumption that the $Ult(L)$ conjecture is true: that is, it is immediate from our $Ult(L)$-friendly assumptions that $ZFC + LCs + MM$ weakly S-Maxes over $ZFC + LCs + V = Ult(L)$.

Next, we note that showing that $ZFC + LCs + V = Ult(L)$ proves that there is a set-forcing poset which forces $ZFC + LCs + MM$ would suffice to show that the former also weakly S-Maxes over the latter. But this is directly implied by the standard consistency proof of $ZFC + MM$: in particular, Foreman, Magidor, and Shelah showed in 1988 that $ZFC + \exists\kappa SC(\kappa)$ proves that there is a set-generic model of $ZFC + MM$.[41] Since this forcing in known to preserve large cardinals above that of the supercompact used to define the forcing poset, we find that this result also shows that $ZFC + LCs + V = Ult(L)$ proves that there is a set-forcing interpretation of $ZFC + LCs + MM$. We thereby find fairly immediately that the extant theory of forcing axioms suffices to establish that $ZFC + LCs + V = Ult(L)$ weakly S-Maxes over $ZFC + LCs + MM$.

Putting these two previous facts together, we find that the theories of $Ult(L)$ and of forcing axioms are S-Max equivalent: S-Max simply is not fine-grained enough to distinguish between the maximizing potential of the two theories. As a result, if S-Max were indeed the proper formal explication of the informal maxim of 'maximize', then we would find that there is no maximization-based reason to prefer either of our candidate theories over the other. But it was noted at the end of Chapter 2 that there are intuitive reasons to be skeptical that S-Max successfully captures the entirety of the justificatory force of the 'maximize' idea. Particularly salient in regards to this result is the lim-

[41] See Foreman et al. (1988) for the original proof of this result. See Jech (2003) Theorem 37.9 and surrounding discussion for a contemporary presentation of this result.

itation of S-Max to a simple admonishment to prefer theories with greater large cardinal strength. Since the present debate on how to extend $ZFC + LCs$ inherently requires both theories to have the same large cardinal assumptions, it is not particularly surprising that S-Max is unable to do any work in distinguishing the current theory candidates. It therefore seems prudent to refrain from putting too much stock into this initial result. Instead, in the following section, we will investigate whether the more fine-grained notion of M-Max is better able to separate the two theories in terms of their maximizing potential.

3.3 M-Max: Possible Separation

Recall from Chapter 2 that M-Max differs from S-Max in two ways: first, it only permits M-Fair interpretations, and, second, it adds a third isomorphism type requirement. By Theorem A.1.2, for theories extending ZFC it turns out that this new isomorphism type condition is equivalent to the inner model interpretation being provably proper. As a result, for the two theories under present consideration, T' M-Maxes over T if it proves that there is some non-trivial M-Fair interpretation φ. Recalling that, in contrast to the class of S-Fair interpretations, M-Fair interpretations only include definable inner models, the key difference in this analysis is that forcing extension interpretations no longer suffice for weak maximization. With these facts in mind, we now turn to the question of how to evaluate our two theories in terms of M-Max.

The first direction–regarding whether $ZFC + LCs + MM$ weakly M-Maxes over $ZFC + LCs + V = Ult(L)$–is extremely straight-forward, given what has already been established. In section 3.2, it was noted that under the assumption of the $Ult(L)$ conjecture, there is a definable inner model interpretation of $ZFC + LCs + V = Ult(L)$. Since interpretations of this sort are M-Fair, in addition to being S-Fair, we find that the very same interpretation which witnessed the weak S-Maximzation of forcing axioms over $Ult(L)$ also witnesses its (weak) M-Maximization as well. We therefore note that $ZFC + LCs + MM$ weakly M-Maxes over $ZFC + LCs + V = Ult(L)$ without further comment.

The question of the other direction–regarding whether $ZFC + LCs + V = Ult(L)$ also weakly M-Maxes over $ZFC + LCs + MM$–is much more intricately involved. Unlike with the earlier case, the interpretation witnessing the weak S-Maximization of $ZFC + LCs + V = Ult(L)$ over forcing axioms in section 3.2 was a set-forcing interpretation, and therefore fails to provide an M-Fair interpretation. As a result, an additional method of defining interpretations of forcing axioms would be required for the two current theories to be M-Equivalent. So we turn to the question of whether there are any known inner model interpretations of forcing axioms, given sufficient large cardinal strength.

One natural possibility to consider when searching for such an inner model interpretation comes from the theory of the $\mathbb{P}_{max}$ forcing extensions. Originally developed by Woodin[42] in the early 1990's as a further development of the forcing axiom program, it was discovered that forcing with this poset on models of strong determinacy could provide natural interpretations of restricted versions of forcing axioms, assuming the existence of a supercompact in V.[43] Woodin was able to show that under the assumption of a supercompact, the generic extension of $L(\mathbb{R})$ by this forcing poset exists as an inner model of V, and many of the standard consequences of forcing axioms at the level of $H(\omega_2)$ hold here.[44] Furthermore, for certain well-behaved pointclasses of reals Γ, the $\mathbb{P}_{max}$ extension of $L(\Gamma, \mathbb{R})$ can be seen to model restrictions of strong forcing axioms to small cardinals, including $MM(\mathfrak{c})$.[45] Additionally, these forcing extensions of inner models of

[42]Prior to the discovery of the machinery of $Ult(L)$, Woodin's work focused prominently on the development of forcing axioms. Though he advocated for those axioms, and especially his axiom $(\star)$, his later proofs regarding weak extender models for supercompactness lead him to later reject this earlier line of argument. See Woodin (2010a) Definition 5.1 for the definition of $(\star)$ in terms of $\mathbb{P}_{max}$. Interestingly, these early arguments directly invoked maximality considerations, noting that $(\star)$ implied that any Π_2 over $H(\omega_2)$ which can be made true with set-forcing is already true in V. Though Woodin has rejected these arguments, other proponents of this line of reasoning persist: see Schindler (2017) for the current state-of-the art regarding these arguments.

[43]The $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ was first introduced in Woodin (2010a) ch. 4-6. A substantial theory of this forcing poset is developed and articulated throughout this textbook. In effect, forcing with $\mathbb{P}_{max}$ is designed to directly "lift" the structure theory of $\mathcal{P}(\omega)$ under $AD^{L(\mathbb{R})}$ directly to $H(\omega_2)$, preserving as much of these consequences as possible. See Koellner (2017) pp. 3212–3215 for an accessible presentation of this material.

[44]See Woodin (2010a) Theorem 4.54 for the existence of the generic extension, and that is models ZFC. Most of the rest of the chapter proves particular paradigmatic consequences of forcing axioms in this model.

[45]See Woodin (2010a) Section 9.2. Note that the results are slightly stronger than stated above: in fact,

determinacy are generically absolute, and therefore represent informally "canonical" inner models of restricted versions of forcing axioms.[46] Given this rich and structured theory regarding inner models of restricted versions of forcing axioms, one might naturally hope that a full definable inner model of $ZFC+LCs+MM$ could be found through further developments of this literature.

Alas, however, the prospects of a $\mathbb{P}_{max}$ extension providing an M-Fair interpretation of the full theory of $ZFC+LCs+MM$ seem highly implausible, at best. There are three distinct complications for this prospect. First, while the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ itself provides a definable inner model, the more intricate $\mathbb{P}_{max}$ extensions of $L(\Gamma,\mathbb{R})$ for pointclasses Γ are only definable in terms of a parameter. The $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ only models a very limited selection of localized consequences of forcing axioms–in particular, the structure theory of $\mathbb{P}(\omega_1)$ under MM–and not the full axiom MM.[47] On the other hand, the $\mathbb{P}_{max}$ extensions of models of the form $L(\Gamma,\mathbb{R})$ are unable to provide M-Fair interpretations of forcing axioms, requiring the use of the parameter Γ in their definition. Thus, any possible interpretation arising from the theory of $\mathbb{P}_{max}$ extensions will seemingly either be too weak or fail to provide a genuinely M-Fair interpretation.[48] Secondly, even if an interpretation only definable from a parameter were accepted, it currently seems unlikely that $ZFC+LCs+V=Ult(L)$ will prove that any $\mathbb{P}_{max}$ extension of an inner model of determinacy will model all of MM: the process of extending the strength of the forcing axioms true in these models

$L(\Gamma,\mathbb{R})^{\mathbb{P}_{max}} \models MM^{++}(\mathfrak{c})$. Work on extending these results is ongoing: see Larson (2014) (and footnote 49 below) for a discussion of one such extension, to a restricted version of $MM(\mathfrak{c}^+)$.

[46]The term "canonicity" is frequently used in the literature regarding the $\mathbb{P}_{max}$ extensions. Note that this is a wholly separate usage of the term than is found in the literature regarding canonical inner models. To avoid confusion, we will not refer to the $\mathbb{P}_{max}$ extensions in this way throughout the remainder of the dissertation, instead directly stating the properties seen as providing a degree of "canonicity".

[47]It should be noted that these local consequences are particularly important to some advocates of forcing axioms. In particular, Todorčević (2012) presents these consequences for the structure theory of $\mathcal{P}(\omega_1)$ as the most important sources of justification for PFA and MM. In his envelope perspective argument, Koellner argues that since this structure theory concerns only the cumulative hierarchy up to $H(\omega_2)$, and so can be wholly captured in a proper inner model of V, these consequences can be fruitfully studied and developed regardless of whether forcing axioms hold globally in V or not. See Koellner (2017) section 3.3 for a concise and focused treatment of this argument in print. A full-length treatment of the envelope perspective argument is also forthcoming. For present purpose, we note that regardless of whether the envelope perspective successfully nullifies Todorčević's particular argument for forcing axioms, it does not attempt to provide a response to the arguments for forcing axioms on the grounds of 'maximize' under consideration here.

[48]See section 4.2 for more on the restriction of M-Fair interpretations to those which are definable without parameters.

to even $MM(\mathfrak{c}^+)$ has proved painstaking, requiring myriad assumptions and restrictions to be added to the axiom's formulation.[49] Given how artifical and strict the restrictions on MM must be to hold even for the smallest infinite cardinals in a $\mathbb{P}_{max}$ extension, it would be extremely surprising if such an extension were proved to model the full, unrestricted MM.[50] Lastly, even if a $\mathbb{P}_{max}$ extension of an inner model of determinacy modeling MM were to be found, this would still not suffice to provide an interpretation of $ZFC+LCs+MM$: as $\mathbb{P}_{max}$ is a homogenous forcing poset, the large cardinal strength of these extensions is limited to that of the original inner model $L(\Gamma,\mathbb{R})$, which fall well below that of a proper class of Woodins and an extendible. As a result, even if there were hiterto unprecedented developments in the theory of $\mathbb{P}_{max}$ extensions of inner models of determinacy, this would nonetheless fail to provide an interpretation of $ZFC+LCs+MM$. For these three reasons, the method of $\mathbb{P}_{max}$ extensions appears to be unable to provide an M-Fair interpretation of $ZFC+LCs+MM$ in $ZFC+LCs+V=Ult(L)$.

But what about other possible avenues for finding an inner model interpretation of $ZFC+LCs+MM$: might there be a method unrelated to the theory of $\mathbb{P}_{max}$ forcing extensions that could reveal the existence of such an inner model? Unfortunately, this remains an open question: there are no known examples of inner model interpretations of MM, but there also is no general proof showing that such an interpretation is impossible. We note, however, that any such interpretation would have to be generated in a very novel fashion, as the usual methods of defining inner models cannot be used for interpreting $ZFC+LCs+MM$: neither inner models of determinacy[51], nor set-generic grounds[52], nor fine-structural inner models[53] can interpret $ZFC+LCs+MM$ within $ZFC+LCs+$

[49] The current strongest forcing axiom known to hold in a $\mathbb{P}_{max}$ extension is $MM(\mathfrak{c}^+)$ restricted to posets which a) do not reduce the cofinality of ω_3 to ω_1 and b) are stationary set preserving for posets which do not add ω_1 sequences of ordinals. See Larson (2014).

[50] Note that, unlike the case of MA, MM is not equivalent to any of its restrictions to particular infinite cardinals.

[51] As noted above, these models and their forcing extensions which provably are contained in V inherently lack the necessary large cardinal strength

[52] Woodin has shown that $V=Ult(L)$ implies that V is the minimum member of the generic-multiverse: see Woodin (2017) Theorem 7.26 3). For present purposes, we note that this implies that there are no proper set generic grounds of V.

[53] All known fine-structural inner models are models of CH, and so cannot satisfy strong forcing axioms: the same is true for all known coarse-structural inner models. It is believed that any future fine-structural inner model will also model CH, as well as a variety of combinatorial properties inconsistent with MM.

$V = Ult(L)$. As a result, it is not even clear where one might start to look for such an interpretation of forcing axioms!

Given the inability of any of the established methods of defining inner model interpretations to properly interpret $ZFC + LCs + MM$ with an M-Fair interpretation, the prospects for the existence of such an interpretation might seem dubious. We further argue that this possibility is extremely unlikely. Towards this end, note that for all prior stages of the inner model program there has been a clear pattern: a canonical inner model will be found, representing the "smallest" possible class-sized inner model interpretation for a given large cardinal strength, and any other theories with that consistency strength can be interpreted in it through outer model interpretations (or patently unfair interpretations).[54] If this pattern broke at the level of an inner model of a supercompact cardinal, this would represent yet another fashion in which the typical features of the inner model program are incompatible with the level of a supercompact cardinal. But, while earlier instances of failures of paradigmatic properties of the inner model program at the level of a supercompact–such as the lack of a projectively definable well-ordering of the reals–can be explained and understood as the direct result of central consequences of the existence of a supercompact, the existence of non-trivial class-sized inner model interpretations in $Ult(L)$ would seem to be entirely unrelated to the unique properties of supercompact cardinals. At the very least, this would represent one of the most surprising and seemingly inexplicable asymmetries between inner model theory below and above a supercompact cardinal.

Beyond merely being surprising, however, the existence of such an interpretation of forcing axioms within $Ult(L)$ would pose a serious challenge for the extent to which $Ult(L)$ could be seen as L-like in any meaningful way. If there were a definable inner model interpretation of $ZFC + LCs + MM$ then, given the $Ult(L)$ conjecture, the definition of $Ult(L)$ could be carried out in this proper inner model; as a result, there would be a definable proper inner model of the Ultimate version of L within the Ultimate version of L. Furthermore, this process could then be carried out in

[54]See Chapter 1 for more on the history of the inner model program.

the interpretation of $V = Ult(L)$, leading to an infinite descending chain of Ultimate versions of L. This would be in stark contrast to the case of $V = L$, where there are no proper class-sized inner model interpretations whatsoever, and even to the case of canonical inner models of large cardinals below a supercompact, where any such proper class-sized inner model must lose large cardinal strength. Without knowing the exact nature of the definable inner model interpretations of $ZFC + LCs + MM$, it is unclear the extent to which these distinct inner models would necessarily resemble each other. Advocates of $V = Ult(L)$ explicitly hope that $ZFC + V = Ult(L)$ would suffice to settle all important undecidable questions modulo large cardinals[55], so any instances of disagreement over an independently motivated mathematical question would seriously diminish the prospects of $Ult(L)$ as a completion to the challenges of incompleteness. Without knowing more about the eventual details of these inner models, however, at this point we can only note that they may reveal a fundamental lack of even a minor degree of absoluteness.

Regardless of whether this possible descending infinite chain of Ultimate L's would prevent $Ult(L)$ from having the necessary degree of absoluteness expected of an L-like inner model, we argue that it reveals a more significant challenge for the justification of $V = Ult(L)$: a deep lack of canonicity for the $Ult(L)$ structure. Recall that an inner model is considered canonical if there are no arbitrary sets contained in it, and no arbitrary information introduced to it: Sargsyan eloquently explains the canonicity of L as meaning "*no random or artificial information is coded into the model.* Every set in L has a reason for being in it." (Sargsyan (2011), slide 9). In stark contrast to this, if there is a descending infinite chain of models of $V = Ult(L)$, then *any* model of $Ult(L)$ will contain arbitrary sets that fall out of the next proper inner model of $V = Ult(L)$. These sets will not be required by the fine-structural notions–the strategic extender models–used to construct a particular model of $V = Ult(L)$. In this way, $ZFC + LCs + V = Ult(L)$ would seem to clearly fail to identify a principled and robust collection of sets, instead varying significantly depending on the precise

[55]See Goldberg (Frth): "Woodin has proposed the axiom $V = UltimateL$... which, supplemented by large cardinal axioms, is expected to axiomatize the structure of the canonical inner model with a supercompact cardinal, *if it exists*. Therefore, supplemented with large cardinal axioms, the axiom $V = UltimateL$ should decide all questions of set theory."

specifics of where it were constructed. In this way, the existence of a definable interpretation of $ZFC+LCs+MM$ within $ZFC+LCs+V=Ult(L)$ would appear to pose a serious challenge to treating the "Ultimate version of L" as a genuinely canonical inner model. In fact, given the myriad ways that such an interpretation would seem to prevent a significant degree of L-likeness for $Ult(L)$, its existence would raise many questions regarding the justification of $V=Ult(L)$ through the historical goals and methods of the inner model program.

Beyond its effects on the $Ult(L)$ program, the existence of an interpretation of $ZFC+LCs+V=Ult(L)$ within $ZFC+LCs+V=Ult(L)$ would also raise questions about the usefulness of M-Max as a methodological tool for evaluating extensions of $ZFC+LCs$. Recall from Chapter 2 that M-Max can be thought of as formally explicating a concept of having all of the mathematical content of a less-maximizing theory, and containing additional mathematical content that the other theory misses. But if there is an interpretation of $ZFC+LCs+MM$ in $ZFC+LCs+V=Ult(L)$, then we would find that some natural mathematical theories M-maximize over themselves: in terms of the informal gloss on M-Max, this would seemingly mean that some good mathematical theories "think" that they themselves miss out on genuine mathematical content! Such a state would be bizzare to say the least, and may well require a a rethinking of how to apply 'maximize' to theories extending $ZFC+LCs$. It is likely that further study of the specifics of these M-Fair interpretations would grant insight into how inner model interpretations broke with established patterns at the level of a supercompact cardinal, perhaps revealing how to better explicate 'maximize' for dealing with theories at such high levels of consistency strength.

But this all remains deeply speculative, given the complete lack of an obvious way to interpret $ZFC+LCs+MM$ with an inner model interpretation. Since such an interpretation would require novel and unprecedented methods to be developed and utilized, and since its existence would imply a wide bevy of surprising results regarding $Ult(L)$ which seem at odds with the significant theory as a fine-structured inner model so-far developed by Woodin, Koellner, and Goldberg, we here register strong skepticism towards the possibility that any such interpretation exists. For these reasons, we propose the following Conjecture:

Interpretation Non-Existence Conjecture ($INEC$): There is no M-Fair interpretation of $ZFC + LCs + MM$ within $ZFC + LCs + V = Ult(L)$.

With this conjecture in place, we are now ready to evaluate the maximization conditions between the theory of forcing axioms and of $Ult(L)$ in terms of M-Max. Given the $Ult(L)$ conjecture of the Harvard school, we find that $ZFC + LCs + MM$ weakly M-maximizes over $ZFC + LCs + V = Ult(L)$. Additionally, given our INEC, we find that $ZFC + LCs + V = Ult(L)$ cannot weakly M-maximize over $ZFC + LCs + MM$. We therefore find that the following implications hold:

If the $Ult(L)$ conjecture holds, then $ZFC + LCs + V = Ult(L) \trianglelefteq_M ZFC + LCs + MM$

and

If the $INEC$ holds, then $ZFC + LCs + MM \ntrianglelefteq_M ZFC + LCs + V = Ult(L)$

Combining these, we find:

Maximality result: If the $Ult(L)$ conjecture and the $INEC$ are both true, then $ZFC + LCs + V = Ult(L) \triangleleft_M ZFC + LCs + MM$

In conclusion, we note that the Harvard school strongly endorses the truth of the $Ult(L)$ conjecture, and the currently available evidence seems to suggest that the $INEC$ is true; and so, in the best possible justificatory situation for the $Ult(L)$ program, M-Max finds that forcing axioms are strictly more maximizing than $V = Ult(L)$.

3.4 'Maximize' and Contemporary Axiom Selection

In the previous two sections, we studied the maximization relationships between forcing axioms and the $Ult(L)$ program according to S-Max and M-Max. While S-Max found their maximization potential to be equivalent, due to the identical large cardinal strength in the specifications of the two theories, we showed that M-Max finds separation between the two theories, given our two conjectures. In particular, we found that $ZFC + LCs + MM$ strictly M-Maxes over $ZFC + LCs + V = Ult(L)$. In this final section of the chapter, we will evaluate the consequences of this result for the justification of the $Ult(L)$ program.

In Chapter 2, it was noted that the 'maximize' maxim was closely related to the mathematical goals of set theory as a discipline, and so justified on the basis of means-ends analysis. Additionally, it was noted that 'maximize' was shown by Maddy to underlie the justification of $ZFC + LCs$ in the first place, and so seemed to be a natural source of justification for any acceptable extension of this theory. Finally, it was argued that M-Max is the best justified formal explication of the informal 'maximize' notion, and therefore best positioned to be used in future justificatory endeavors. As a result, the likely strict maximization of another natural theory extending $ZFC + LCs$ over $ZFC + LCs + V = Ult(L)$ seems to pose a clear de facto challenge to the justification of the latter as the uniquely correct theory of the set-theoretic universe. But how should an advocate of $V = Ult(L)$ respond to this?

One response, given the dependence of our maximization result on two unproven conjectures, would be to reject one or both of these assumptions, thereby nullifying the result. Unfortunately, this does not seem to be a live option for an advocate of the $Ult(L)$ program. A rejection of the $Ult(L)$ conjecture would likely dash the hopes of the Harvard school to prove the WIC, leaving strategic extender models unable to provide a fine-structural presentation of a supercompact cardinal. Since Woodin notes that strategic extender models seem to be the only possible notion of fine-structure at this level,[56] this would leave the inner model program without any clear path forward. Additionally, we note that the analysis in section 3.3 could instead be carried out with UA and the UA conjecture in place of $V = Ult(L)$ and the $Ult(L)$ conjecture: thus, this response would in fact require an even stronger rejection of the UA conjecture. But the lack of an inner model of UA with a supercompact cardinal has been described by the Harvard school as a potential falsification of the prospects of the inner model program at levels above a supercompact: as a result, its rejection would pose a much more signficant challenge for the justification of the $Ult(L)$ program than our maximization result! For these reasons, a rejection of the $Ult(L)$ conjecture (and also the UA conjecture) does not seem to be a live option for proponents of $V = Ult(L)$.

Another response would be to instead reject the *INEC*, arguing instead that there will eventually be a definable inner model interpretation of $ZFC + LCs + MM$ within $ZFC + LCs + V = Ult(L)$. By itself, a rejection of the *INEC* would seem somewhat implausible: this type of response would naturally also require some sort of account of what novel methods could possibly generate this M-Fair interpretation. Additionally, as noted above, the existence of such an interpretation would raise serious question regarding the extent to which $Ult(L)$ would represent a canonical inner model of a supercompact cardinal, thereby severing the supposedly close connection between $V = Ult(L)$ and previous axioms generated from the inner model program. Given the combination of the implausibility of the falsity of *INEC* and the extent to which its falsity would raise new challenges for the justification of $V = Ult(L)$, rejecting the *INEC* also does not seem to be a live option for supporters of the $Ult(L)$ program.

Thus, any satisfactory response should instead accept the maximality result, but instead question its significance for the current debate in axiom selection. In taking this tact, there are two distinct options for a response. First, one could argue for the importance of 'maximize' as a methodological principle in this dispute, but question whether M-Max provides the correct formal explication of the informal principle. One such approach would be to argue that S-Max instead provides the

[56] See Woodin (2017) p. 89: "the hierarchy cannot consist of models constructed from *just* a sequence of partial extenders. The only alternative at present is an inner model theory based on *strategic* premice."

correct explication of 'maximize', and therefore that $Ult(L)$ and forcing axioms have equivalent maximization strength. This does not seem like a very satisfactory response, however: as noted in Chapter 2, S-Max amounts to a simple admonition to choose a theory with higher large cardinal strength, when possible. Though large cardinals are particularly important examples of set theoretic objects, they by no means are the only objects of study in contemporary set theoretic practice. Thus, explicating 'maximize' in a way that only considers a particular subcollection of set theoretic objects seems arbitrary at best. Additionally, while in Chapter 2 it was only suggested that the coarse-grained nature of S-Max may prevent it from being a useful methodological tool in axiom selection debates, it is only with our maximization result that an actual example of this limitation has been found. Since it is both less well-justified on the basis of the means/ends analysis underlying 'maximize' and does not seem fit for the work of the current axiom selection debate, we note that a restriction to S-Max instead of M-Max would be unwise. On the other hand, an advocate of the $Ult(L)$ program could argue that an alternative formalization of 'maximize' should be preferred to both S-Max and M-Max; we will consider the prospects for such a response further in Section 4.2.

Secondly, a proponent of the $Ult(L)$ program could concede that other theories are better justified on the grounds of the 'maximize' maxim, but argue that $ZFC+LCs+V=Ult(L)$ is still the best choice for extending $ZFC+LCs$ on the basis of other methodological principles. Such a response may well be in keeping with the traditional arguments for $V=L$, which focused on the ease of working in such a well-behaved and regulated model of set theory. An argument of this form for $V=Ult(L)$ would seemingly focus closely on the extent to which $Ult(L)$ can be shown to be L-like in any of the ways described in Section 1.2, but without forcing an adherent to give up the use of large cardinals as a tool for measuring and comparing the consistency strength of theories. Without further development of the $Ult(L)$ program, and especially without a proof of the WIC and $Ult(L)$ conjecture yet in place, it is difficult to see how to evaluate the force that such an argument might have. Additionally, there has been relatively little work in the literature trying to identify what general methodological principles might underlie these lines of reasoning. As such, the burden at this time is on the $Ult(L)$ program to further articulate and defend such an argument.

Beyond arguments from L-likeness, there is one other argumentative route that can be taken for arguing that $V = Ult(L)$ is better justified than alternative theories in spite of the maximization result: that $Ult(L)$ is provably so "close" to V under any theory extending $ZFC + LCs$ to make the fruits of maximization negligible for set theoretic purposes. A particular instance of this "closeness" argument schema would then include a proof, for some particular explication of the notion of an inner model being close to V, that the interpretation of $V = Ult(L)$ must be close in that way, and a methodological argument that the particular sense of closeness allows all the mathematical fruits of the alternative theory to be achieved in the interpretation of $V = Ult(L)$. For some notions of closeness, a proof already exists: for example, the theory of weak extender models suffices to establish that the interpretation provided by the $Ult(L)$ conjecture–if it exists–will be close to any V in the sense of agreeing on large cardinal notions. Other proofs of closeness properties are promised by work on the HOD dichotomy and the HOD conjecture, which would show that the interpretation of $V = Ult(L)$ is close in the sense of being a class-generic ground of V under any theory extending $ZFC + LCs$. But, while there has been much work on proving that particular closeness properties hold between $Ult(L)$ and alternative models of $ZFC + LCs$, there has so-far been relatively little work on providing arguments for why such a notion of closeness might allow $Ult(L)$ to retain all of the important mathematical fruits of the alternative theory. While we note that such "closeness" arguments seem potentially quite promising as a source of justification for $Ult(L)$, it is difficult to evaluate them further without some particular instances being put forward and defended. Thus, as with the L-likeness arguments, the burden at this time is on the $Ult(L)$ program to explicitly provide such an argument.

In conclusion, in this chapter we used the two formal tools presented in Chapter 2 to evaluate the justification of the two contemporary candidates for extending $ZFC + LCs$ described in Chapter 1. In section 3.1, we provided an accessible presentation of the current $Ult(L)$ program, eventually isolating a particular formal theory as a precise formulation of the program. In addition, we highlighted a few crucial assumptions underlying the program that must be granted for argument's sake, especially the $Ult(L)$ conjecture. Then, in section 3.2, we found that using the coarse grained S-Max notion, the theories of forcing axioms and of $Ult(L)$ were found to be equivalent, with no maximization based reason to prefer one to the other. On the other hand, in section 3.3. we used the fine-grained M-max notion to show that, given the truth of a plausible conjecture, the theory of forcing axioms maximizes over that of $Ult(L)$. Finally, in the current section we noted some possible responses to this maximality result available to advocates of the $Ult(L)$ program. In the following, final chapter of this dissertation, we will identify and outline two future projects to continue the study of axiom selection beyond $ZFC + LCs$.

Chapter 4

Future Directions and Conclusion

In this final chapter of the dissertation, we outline two future projects that are intended to continue the author's work on questions of methodology related to the justification of strong theories of sets extending $ZFC+LCs$. The first project, outlined in Section 4.1, arises from the question of the extent to which the maximality result of Ch. 3 should be seen as directly justifying the theory of $ZFC+LCs+MM$, instead of merely providing a justificatory challenge to the $Ult(L)$ program. In reflecting on this question, we suggest that a new, third alternative theory extending $ZFC+LCs$ is suggested; further study of this new theory will help to clarify the justificatory force of 'maximize' for forcing axioms, in addition to the theory representing a mathematically and philosophically interesting object of study in its own right. The second project, outlined in Section 4.2, arises from one of the main live responses for advocates of $Ult(L)$ to the maximality result: namely, offering an alternative formal account of 'maximize' that is both well-justified and does not find $ZFC+LCs+V=Ult(L)$ unduly restrictive. We propose two possible alterations to M-Max which may potentially provide just such an account, and seem worthy of further study. We also raise an important methodological question regarding 'maximize' that appears essential to any possible justification of these alternative accounts. Finally, in Section 4.3, we offer a brief summary of the dissertation, and conclude with a suggestion of how philosophy and mathematics may continue to be fruitfully intertwined in studies of contemporary axiom selection going forward.

4.1 Future Project 1: A Third Alternative to Extending $ZFC + LCs$

In Section 3.4 above, we noted the justificatory challenge that the maximality result poses for the justification of the $Ult(L)$ program. But what of the effects of this result on the justification of the forcing axioms program? At first glance, it would seem that the maximality result provides strong primae facie support for $ZFC + LCs + MM$ as an extension of $ZFC + LCs$: after all, the 'maximize' maxim implicitly underlying the case for $ZFC + LCs$ also provides support for the forcing axiom program over the only other live candidate for extending $ZFC + LCs$. Reflecting on the argument for the maximality result, however, may well give some pause from regarding it as directly providing support for forcing axioms. To see this, note that neither the actual axiom MM nor any of its paradigmatic consequences play any direct role in establishing the maximality result. Instead, the $Ult(L)$ conjecture suffices to show that *any* extension of $ZFC + LCs$ that contradicts $V = Ult(L)$ weakly M-maxes over $ZFC + LCs + V = Ult(L)$; similarly, the nature of $Ult(L)$ as a canonical inner model appears to prevent $ZFC + LCs + V = Ult(L)$ from proving the existence of an M-Fair interpretation of *any* extension of $ZFC + LCs$ that contradicts $V = Ult(L)$.[1] Combining these two facts, it seems that the maximality result in fact reveals that $ZFC + LCs + V \neq Ult(L)$ strictly M-maxes over $ZFC + LCs + V = Ult(L)$. Since the particular choice of an alternative theory to that of $Ult(L)$ seems to be irrelevant for its maximization over $ZFC + LCs + V = Ult(L)$, one might naturally feel some queasiness in regarding the maximality result as providing justification *for* forcing axioms, and not just representing a justificatory challenge to the $Ult(L)$ program.

So why should the maximality result be seen as providing justification specifically for the forc-

[1]Special thanks to Guillaume Massas for stressing this point in private conversation.

ing axiom program? In the current context of the axiom selection debate, it seems that forcing axioms are justified by 'maximize' in large measure because there is no other alternative candidate to $ZFC + LCs + V = Ult(L)$ actually on-offer which similarly M-maxes over $V = Ult(L)$; while the same may well be true for $ZFC + LCs + \neg CH$, this theory is not a natural candidate that bears serious consideration as an extension of $ZFC + LCs$.[2] As a result, our maximality result provides a non-trivial degree of justification for the forcing axiom program, and, at least until another alternative theory with similar maximizing potential is actually put forward by the set-theoretic community, this therefore provides the basis for an argument for accepting $ZFC + LCs + MM$ as the correct extension of $ZFC + LCs$. This support provided by this argument remains highly tentative, however, unless one were sufficiently confident that there would be no other alternative theory ever put forward to extend $ZFC + LCs$ in a different, third way.

So, is there any plausible option for a third candidate theory that might arise in the future? We suggest that there is. To understand this possibility, note that it is somewhat of a historical oddity that the current candidates for possible sizes of the continuum are either $\aleph_1$ (making CH true) or $\aleph_2$ (making CH false, but in the smallest possible way). Historically, there were instead many advocates for an extremely large continuum, well beyond $\aleph_2$ or any of the first infinite cardinals.[3] The main idea behind the large continuum approach seemed to be that the continuum should be as wild and chaotic as possible, permitting a wide variety of different types of sets of reals to exist. While explicit calls for a large continuum were somewhat superseded by the flurry of activity in

[2]In the terminology of Maddy (1997), what the result in Ch. 3 purports to show is that $ZFC + LCs + V = Ult(L)$ is a restrictive theory. Maddy notes that the maxim of 'Maximize' provides a clear admonition to avoid restrictive theories, all other things being equal. It is for the precise reason that a theory T' maximizing over some theory T may be restrictive relative to some other, unrelated theory $T^\star$ that Maddy refrains from suggesting that the M-Maximization of a theory over another theory by itself provides support for accepting the maximizing theory, but instead only for preferring it to its restrictive alternative. See Maddy (1997) pp. 84–85 and 231–232.

[3]See Cohen (1966) for a particularly salient version of this view: "A point of view which the author feels may eventually come to be accepted is that CH is *obviously* false. The main reason that one accepts the Axiom of Infinity is probably that we feel it absurd to think that the process of adding only one set at a time can exhaust the entire universe... Now $\aleph_1$ is the set of countable ordinals and this is merely a special and the simplest way of generating a higher cardinal. The set C is, in contrast, generated by a totally new and more powerful principle, namely the Power Set Axiom. It is unreasonable to expect that any description of a larger cardinal which attempts to build up that cardinal from ideas deriving from the Replacement Axiom can ever reach C. Thus C is greater than $\aleph_n$, $\aleph_\omega$, $\aleph_{\aleph_\omega}$. This point of view regards C as an incredibly rich set given to us by a bold new axiom, which can never be approached by any piecemeal process of construction."

the 1980's and 1990's after the discovery of strong forcing axioms as natural axiom candidates for a small continuum alternative to CH, support for a large value of the continuum quietly continued.[4] Given the historical support for large continuum approaches to resolving the question of CH, as well as the fact that never explicitly rejected by the set-theoretic community–instead becoming somewhat dormant in the face of a clear alternative in the forcing axioms program–it seems that a theory implying that the continuum is above $\aleph_2$ is a natural possible third alternative.

Additionally, as a result of recent developments in set-theoretic practice, there is an obvious source for axiom candidates capturing a large continuum approach: the study of cardinal characteristics of the continuum (henceforth cc's). To understand the notion of a cc, note that there are many mathematical properties which ZFC proves are true of the continuum, but which cannot be true of ω.[5] For a wide variety of such properties, it has been shown to be consistent that they are first true at any κ with $\aleph_0 < \kappa \leq \mathfrak{c}$. A cardinal characteristic is then the first cardinal number at which the property is true, for some particular choice of such a property.[6] Through study of the paradigmatic properties of the continuum, a zoo of such notions has been discovered.[7] While much of the early work on these principles focusing on finding forcing constructions which showed the consistency of various orderings on collections of cc's,[8] recent work has begun to focus more directly on the effects of particular strict orderings of cc's for resolving open questions of ZFC. Interestingly, principles asserting strict inequalities between particular collections of cc's are known to imply $\neg CH$; furthermore, if the collection is larger than three particular cc's, then these principles imply that the continuum is greater than $\aleph_2$. In fact, Blass notes that there are only two sorts of natural

[4]See Jensen (1995) p. 401 for a recent, somewhat poetic statement of this point-of-view: "The Newtonian directs his gaze to the real instead of to the natural numbers. He is less impressed by their clarity than by their boundless multiplicity. The real numbers constitute a gigantic, unfathomable sea. For every principle that generates real numbers, there must be a number not attainable by that principle".

[5]For example, consider the property of there being an unbounded collection of functions on ω: ZFC proves that any countable collection of functions $f : \omega \mapsto \omega$ is bounded almost everywhere by some function $g : \omega \mapsto \omega$, but that a continuum sized collection of such functions is (trivially) unbounded by any particular g.

[6]From the above example, we get the cardinal characteristic of the bounding number $\mathfrak{b}$, defined as the smallest κ st there is a set F of functions $f : \omega \mapsto \omega$ st $|F| = \kappa$ and there is no $g : \omega \mapsto \omega$ that bounds each member of F almost everywhere.

[7]See Blass (2010) for a thorough account of the current state of this literature. See Blass (1996) for a more accessible presentation of this material.

[8]See Kellner et al. (2019) for an example of the current state of the art of this work.

set-theoretic principles that are known to imply the failure of CH: strong forcing axioms, and strict cardinal characteristic inequalities.

Thus, there is a notable tradition in the history of responses to CH that is not represented in the current candidate theories for extending $ZFC+LCs$, and there are possible axioms–the strict cardinal characteristic inequalities–that are able to axiomatize a theory of this unrepresented approach. We suggest, then, that theories including cc inequalities should be studied as a possible third alternative to the inner model and forcing axiom programs. Let us specify this further. Letting *char* be a particular collection of cc's, we propose the axiom schema $CCN(char)$ which states, for each pair a,b in *char*, that $a \neq b$. One particularly interesting instance of this schema would then be $CCN(cich)$, where *cich* is a collection of ten cardinal characteristics which are particularly important for the study of cc's.[9] We note that $ZFC+LCs+CCN(cich) \vdash \mathfrak{c} \geq \aleph_9$, and so provides a candidate theory of a large continuum (relative to $\aleph_2$). It is hoped that this theory could serve as an proxy for any future theory of a large continuum, allowing this third alternative to be better studied and understood.

We conclude this section by suggesting three important projects regarding $ZFC+LCs+CCN(cich)$ that we hope to carry out in the future:

1. Study and develop the mathematical theory of $ZFC+LCs+CCN(cich)$, to better understand its features and consequences.

2. Evaluate the justification of $ZFC+LCs+CCN(cich)$ in terms of 'maximize' and other potential methodological maxims, becoming clearer on whether a large continuum approach can provide an extrinsically motivated extension of $ZFC+LCs$.

3. Compare the maximization potential of $ZFC+LCs+MM$ with that of $ZFC+LCs+CCN(cich)$, aiming to determine the extent to which either candidate is justified in its own right by 'max-

[9] In particular, *cich* is the collection of the ten independent cc's found in Chichon's diagram. See Kellner et al. (2019) for a description of these cc's, and of their importance.

imize'.

4.2 Future Project 2: Expanding the Notion of M-Fair Interpretations

Recall that in Section 3.4 above, it was noted that the maximality result seems to pose a direct challenge to the justification of $ZFC + LCs + V = Ult(L)$. While a number of possible responses to this challenge were noted, perhaps the best available option would be to challenge M-Max as the proper formal explication of the informal maxim of 'maximize'. Since S-Max was found to be somewhat undermotivated as an account of 'maximize' in Ch. 2, it seems that any response of this type would require a new formal approach to 'maximize', beyond those already found in the extant literature. Where might such an alternative explication be found? Recall from Ch. 2 that there were two distinct differences between S-Max and M-Max: first, the latter has an additional isomorphism type condition, and, secondly, that the latter uses the more narrow notion of M-Fair interpretations, instead of the broader notion of S-Fair interpretations. Given the importance of the isomorphism type condition to the usefulness of M-Max as a methodological tool, it therefore seems that the most likely source of a well-justified new explication of 'maximize' would be to alter the scope of permissible fair interpretations in the definition of M-Max.

The first step in providing an alternative explication of 'maximize' in this way would be providing the particular formal account of fair interpretations. We see two particularly interesting options for extending the notion of M-Fair interpretations. First, one could use the notion of S-Fair interpretations in place of M-Fair the definition of M-Max. Unfortunately, this would not be without its technical challenges: it is unclear how to define the isomorphism type condition in the presence of the broader class of S-Fair interpretations. It follows that any attempt to amend M-Max to permit S-Fair interpretations would need to pay close attention to precisely how sets "existing" only in the extension should be treated, and how their isomorphism types could be studied in the ground model. These deep questions have been central to the philosophical understanding of forcing since its discovery by Cohen, and it would be wise not to expect any simple answers. In passing, we simply note that there are likely conceptual challenges ahead in attempting to formulate a version of M-Max that makes sense in the context of outer models.

Another, perhaps more intriguing option for altering the notion of M-Fair arises from reflection on the $\mathbb{P}_{max}$ extensions of models of the form $L(\Gamma, \mathbb{R})$. As we noted in Section 3.3, these models are interesting objects of current set-theoretic study, and have generated a rich theory with abundant consequences for the understanding of forcing axioms, large cardinals, and determinacy.[10] But such models are only definable with a parameter: the set of reals Γ. Given their importance for set-theoretic practice, one might naturally hope to include such models as "good" interpretations. With this in mind, we might seek to expand the class of M-Fair interpretations to include formulas $\varphi(x, P)$ with a parameter. M-Max amended with this broader class of interpretations seems to be a ripe topic of further study, as this addition would permit the inclusion of a wide scope of important set-theoretic models.[11] Of the two possible alterations to the notion of M-Fair interpretations, the author notes that this latter seems more mathematically and philosophically compelling, and more likely to lead to an interesting approach to 'maximize' in its own right.

The second step in providing an alternative explication of 'maximize', after particular formal explications were developed and provided, would be to evaluate the extent to which the formal notion properly captures the justificatory motivations behind the informal maxim of 'maximize' in the first place. While the exact nature of what this task would entail will likely depend heavily on the specifics of the worked-out alternative accounts, we note one particularly important methodolog-

[10] These abundant consequences are the subject of Woodin (2010a).

[11] Of particular importance to applications of this alternative of M-Max would be models related to large cardinal embeddings. Recall that a measurable cardinal κ implies that there is some measure U with particular properties; one of the central discoveries in the history of large cardinals was that such a measure was equivalent to the existence of an elementary embedding $j : V \mapsto M$ for some inner model M. Using the parameter U, the model M would be definable. Using definable methods within such a model M may well open up a fascinating range of inner model interpretations: we are very interested to study the consequences such interperations would have for a formal notion of 'maximize'. Special thanks to Toby Meadows for discussing this possibility and stressing its importance in private conversation.

ical question that these alternatives would likely raise. Maddy notes that inner model interpretations are particularly useful in set-theoretic practice as they allow the import and export of results from the interpreted theories: for example, one would be able to freely apply any theorems from the theory $ZFC+V=L$ if one could explicitly assume that all sets under examination were constructible.[12] Through such applications of the fruits of an interpreted theory, M-Fair interpretations allow the (constrained) use of any mathematical useful results of a more restrictive theory. In fact, this ability to retain the mathematical content of an interpreted theory in some form directly underlies the idea that a M-Maximizing theory need not lose any of the "good math" from the restrictive theory. As a result, we would hope that any future version of M-Max also permit satisfactory import/export conditions. But it is currently unclear the extent to which outer model interpretations or inner models definable from parameters would permit any similar form of import/export of important results. As such, we note that a possible defense of either of the alternatives to M-Max noted above would likely require a deeper analysis of the exact nature of import/export conditions for definable inner models, as well as the extent to which similar conditions can hold for other sorts of interpretations. In addition to leading to a better understanding of the desirability of alternatives to M-Max, it is our hope that such a study of import/export in set theory might better reveal the nature of the 'maximize' maxim more broadly.

We conclude this section by suggesting three important projects regarding alternative to the notion of M-Max that we hope to carry out in the future:

1. Articulate an account of the isomorphism type condition that functions properly in the context of S-Fair interpretations, permitting an intermediate formal approach to maximize properly between S-Max and M-Max.

2. Study the effects of permitting inner models definable only from parameters into the notion of M-Fair interpretations, examining whether such an extension provides a possible response to the maximality result for advocates of $V=Ult(L)$.

[12] See Maddy (2016) especially pp. 15–17.

3. Develop a precise account of nature of import/export between theories and their definable inner model interpretations, allowing an examination of extent to which these benefits are available for other sorts of interpretations.

4.3 Summary and Conclusion

We conclude this dissertation with a brief summary of its contents. In the first chapter, we laid out the history of axiom selection in set theory, focusing considerably on developments since 1980 which have led to the two standard axiom candidates for extending Zermelo-Fraenkel set theory with choice and large cardinals: $V = Ult(L)$ and strong forcing axioms. Throughout this historical presentation, we sought to examine the epistemological motivations for each program, tracing how attempts to expand the programs to higher levels of consistency strength have required shifts and loosenings of these motivations. In the second chapter, we turned to the methodological maxim of 'maximize', presenting the justification of this principle as well as two formal explications of the notion, one owing to John Steel, the other to Penelope Maddy. We concluded that, on the basis of the methodological reasons for preferring 'maximize' in the first place, Maddy's notion is better justified, and therefore should be preferred in axiom selection debates. In the third chapter, we put the material of the first two chapters together, asking whether either approach to axioms can be truly said to maximize over the other, finding that forcing axioms strictly maximize over $V = Ult(L)$, given the assumption of an important conjecture of the $Ult(L)$ program and a plausible conjecture of our own. Finally, in this fourth chapter we presented two future directions for this research: in particular, we examined the possibility for future axioms distinct from either forcing axioms or the inner model program, and considered the prospects of finding an alternative formal approach to 'maximize' by altering the notion of fair interpretations used in Maddy's approach.

Lastly, it is the sincere hope of the author that this dissertation serves as an example of the possible benefits of combining philosophical and mathematical inquiry in studying contemporary axiom selection. Throughout this work, we have aimed to connect genuinely epistemological analysis with techniques from mathematical logic in order to contribute towards settling live questions in the methodology of set theory. We believe that it is only by combining these methods from across disciplines that a complete and satisfactory understanding of the proper methods of deciding between axiom candidates is possible. In our future work, including the projects highlighted above, we hope to continue combining tools from epistemology and from mathematics to generate results of interest to both philosophers and working set theorists.

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Appendix A

Proofs for Chapter 2

A.1 Simplifying the Isomorphism Condition in M-Max for Theories Extending ZFC

Lemma A.1.1. *Let $T, T' \supseteq ZFC$. Assume that $T' \vdash$ "φ is an M-fair interpretation of T" and $\exists X \neg\varphi(X)$. Then $T' \vdash \exists A \forall y \forall S \subseteq y^2 (\varphi(y) \wedge \varphi(S) \rightarrow (A, \in) \ncong (Y, S))$.*

Proof. We work in the theory T'. Since $\exists X$ st $\neg\varphi(X)$, there must be some set of ordinals O such that $\neg\varphi(O)$; otherwise, the φ-sets would be an inner model of ZFC with the same sets of ordinals, and so would be equivalent to V. Since φ is a class sized inner model, note that O cannot itself be an ordinal. Let $A = \{O\} \cup trcl(O)$; note that, since O is a set of ordinals, $trcl(O)$ is itself some ordinal, and that $\forall \alpha \in O (\alpha \in trcl(O))$.

Now, assume for purposes of reductio that there is some Y and S st $\varphi(Y)$, $\varphi(S)$, and $(A, \in) \cong (Y, S)$: in particular, let g be the bijection witnessing $(A, \in) \cong (Y, S)$. Working in φ, we now define a few sets. Let $Y^- = \{y \in Y | \exists z \in Y (ySz)\}$ and let $Y^+ = \{y \in Y | \neg\exists z \in Y (ySz)\}$. Note that Y^+ must be a singleton, as $\exists! c \in A \neg\exists d \in A (c \in d)$. For each $y \in Y^-$, let $y^\star = \{z \in Y^- | zSy\}$. Recursively define a function $f : Y^- \mapsto Ord$ st for all $y \in Y^-$, $f(y) = \alpha$ iff $(y^\star, S) \cong (\alpha, \in)$. Finally, let $B = \{f(y) | y \in Y^- \wedge \exists z \in Y^+(ySz)\}$.

Since $(A, \in) \cong (Y, S)$, we have that $f(Y^-) = trcl(O)$. Since $\{x \in A | \neg\exists y \in O(x \in y)\} = \{O\}$, we find that $g(\{O\}) = Y^+$. Additionally, note that since g^- and f are both S-order preserving, for all $y \in Y^-$ we find that $f(y) = \alpha$ iff $g(\alpha) = y$. Thus, $b \in B$ iff $\exists y \in Y^-(f(y) = b \wedge \exists z \in Y^+(ySz))$ iff $\exists y \in Y^-(f(y) = b \wedge g^-(y) \in O)$ iff $b \in O$. As a result, we find that $B = O$, and so $\varphi(O)$, contradicting our assumption. $\rightarrow\leftarrow$. Thus, our reductio assumption must be false, and there cannot be any Y, S st $\varphi(Y)$, $\varphi(S)$, and $(A, \in) \cong (Y, S)$. This completes our result. □

Theorem A.1.2. *Let $T, T' \supseteq ZFC$. Then $T \leq_M T'$ iff there is some $\varphi(x)$ st*
a). for all $\sigma \in T$ $T' \vdash \sigma^\varphi$,
b). T' proves that φ is an M-fair interpretation,
c). $T' \vdash \exists X(\neg\varphi(X))$.

Proof. $\rightarrow$: Let $T \leq_M T'$. It follows immediately from the definition of M-Max that there is some $\varphi(x)$ st a) and b) hold. Additionally, $T' \vdash \exists x \exists R \subseteq x^2 \forall y \forall S \subseteq y^2(\varphi(y) \wedge \varphi(S) \rightarrow (x, R) \ncong (y, S))$; but then $T' \vdash (\neg\varphi(x)) \vee \neg\varphi(R)$, and so $T' \vdash \exists X \neg(\varphi(X))$.

$\leftarrow$: Assume that there is some $\varphi(x)$ st a), b), and c) above hold. Then i) and ii) from the definition of M-Max follow immediately, and it only remains to show that $T' \vdash \exists x \exists R \subseteq x^2 \forall y \forall S \subseteq y^2(\varphi(y) \wedge \varphi(S) \rightarrow (x, R) \ncong (y, S))$; but this follows immediately from the assumption of c) and Lemma A.1.1 above. □