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On the Stability of Self-Similar Blow-Up in Nonlinear Wave Equations

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On the Stability of Self-Similar Blow-Up in Nonlinear Wave Equations

A Dissertation submitted in partial satisfaction
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by

Michael Liam McNulty

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To my family

Michael, Adrienne, Patrick, and Kasey McNulty.

ABSTRACT OF THE DISSERTATION

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by

Michael Liam McNulty

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2022
Professor Po-Ning Chen, Co-Chairperson
Professor Birgit Schörkhuber, Co-Chairperson

In this dissertation, we study the stability of self-similar blow-up for two nonlinear wave equations: the equation of motion of the strong-field Skyrme model and the quadratic wave equation.

We begin by studying the stability of an explicitly known, self-similar solution of the equation of motion of the strong-field Skyrme model in the lowest energy supercritical dimension. This model is a particular limiting case of the Skyrme model which is itself a quasilinear modification of wave maps into a sphere. The strong-field Skyrme model restores the scaling invariance not present in the Skyrme model which allows for the existence of self-similar solutions. This equation is a semilinear wave equation that is, in particular, nonlinear in the derivatives of the unknown. As a consequence, standard techniques for a linear stability analysis of this solution do not apply. Via application to a toy model, we present techniques that will be used to study the linear stability of this solution in a forthcoming paper.

Next, we study the stability of an explicitly known, self-similar solution of the quadratic wave equation in the lowest energy supercritical dimension. For radial data close to this self-similar blow-up solution with blow-up time $T = 1$ adjusted along a one-dimensional subspace, we are able to prove convergence of the corresponding solution to the same self-similar solution with a potentially different blow-up time. This result holds true in a region of spacetime that extends beyond the time of blow-up and can be made arbitrarily close to the Cauchy horizon of the singularity. This is achieved via hyperboloidal similarity coordinates.

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Chapter 1

Introduction

In this dissertation, we consider equations of the form

$$\Box u(t,x) = \mathcal{N}\big(t, x, u(t,x), \partial u(t,x)\big) \tag{1.1}$$

for real-valued functions u on $\mathbb{R}^{1+d}$ where

$$\Box = \partial_t^2 - \Delta_x$$

is the d'Alembertian on $\mathbb{R}^{1+d}$,

$$\Delta_x = \partial_{x^1}^2 + \cdots + \partial_{x^d}^2$$

is the Laplacian on $\mathbb{R}^d$, $\mathcal{N} : \mathbb{R}^{1+d} \times \mathbb{R} \times \mathbb{R}^{1+d} \to \mathbb{R}$, and ∂ denotes the spacetime gradient on $\mathbb{R}^{1+d}$. One concerns themselves with solving the *Cauchy problem*, that is, finding a solution of Equation (1.1) subject to the condition that

$$u(0,x) = f(x),\ u_t(0,x) = g(x),\ x \in \mathbb{R}^d \tag{1.2}$$

for some specified functions f, g.

As an example, consider the simple case: $d = 1$, $\mathcal{N} = 0$, and $f, g \in C^\infty(\mathbb{R})$. In this case, one finds explicitly that

$$u(t,x) = \frac{1}{2}\big(f(x+t) + f(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds$$

is the unique solution of Equation (1.1) subject to the constraint, Equation (1.2). Of crucial note is that $u(t,\cdot) \in C^\infty(\mathbb{R})$ for all $t \in \mathbb{R}$. As we will see, such a property is in stark contrast with what can happen for nonlinear problems where solutions with smooth, compactly supported initial data can fail to exist after a finite amount of time.

1.1 Finite-Time Blow-Up

Despite what occurs for linear problems, solutions of the Cauchy problem

$$\begin{cases} \Box u(t,x) = \mathcal{N}\big(t,x,u(t,x),\partial u(t,x)\big) \\ u(0,x) = f(x),\ u_t(0,x) = g(x) \end{cases} \tag{1.3}$$

do not necessarily exist for all time. We say that a solution exists *locally in time* if a solution of Equation (1.3) exists for all $x \in \mathbb{R}^d$ and for $|t| \leq T$ for some $T > 0$. Of course, once one knows that at least one such T exists, then one can consider the largest such T, i.e.,

$$T^* := \sup\{T > 0 : \text{Equation (1.3) has a solution in } [-T,T] \times \mathbb{R}^d\}.$$

If $T^* = \infty$, then we say that the solution exists *globally in time.* Otherwise, we say that the solution *blows up in finite time.* For some standard well-posedness results, we refer the reader to [1], [32], and [38].

By now, it is well-known that finite-time blow-up is common for nonlinear wave equations[1]. For instance, consider the focusing nonlinear wave equation

$$\Box u = |u|^{p-1}u \tag{1.4}$$

on $\mathbb{R}^{1+d}$ with $p > 1$. Equation (1.4) has the explicit solution

$$u_T(t,x) := c_p(T-t)^{-\frac{2}{p-1}},\ T > 0$$

where

$$c_p := \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}$$

which clearly becomes singular as $t \to T^-$. By finite speed of propagation, one can obtain smooth, compactly supported initial data for which the associated Cauchy problem with that data has a solution that blows up in finite time. In fact, Donninger and Schörkhuber in [16] and [17] showed that, given suitable restrictions on the relationship between p and d, there is an entire open set of initial data centered around the previously mentioned data that also leads to finite-time blow-up.

Now, consider a second example

$$\Box u = u_t^2. \tag{1.5}$$

In [26], John demonstrated that any non-trivial smooth solution of Equation (1.5) with compactly supported data blows up in finite time. It is important to contrast this with the similar example

$$\Box u = u_t^2 - |\nabla u|^2. \tag{1.6}$$

[1]In fact, finite-time blow-up is common for nonlinear *evolution equations*, though we will only concern ourselves with wave equations here.

In [28], Klainerman transformed the corresponding problem for Equation (1.6) via the so-called 'Nirenberg-trick' by setting $v = 1 - e^{-u}$ into

$$\begin{cases} \Box v(t,x) = 0 \\ v(0,x) = 1 - e^{-u(0,x)}, v_t(0,x) = u_t(0,x)e^{-u(0,x)} \end{cases}.$$

This is now a linear wave equation and thus a solution exists for all time. A global solution of Equation (1.6) can be obtained by inverting this transformation which can always be done so long as $u(0,x)$ and $u_t(0,x)$ are sufficiently small and compactly supported.

1.2 Wave Maps and the Skyrme Model

Another example of such an equation for which blow-up in finite time occurs is the *wave maps* equation. Before introducing this it will be instructive to first consider the more general *Skyrme model*.

Let's now consider maps $u : \mathbb{R}^{1+d} \to \mathbb{S}^d$. Furthermore, we endow $\mathbb{R}^{1+d}$ with the *Minkowski metric* g given by

$$g_{\mu\nu}dx^\mu dx^\nu = dt^2 - \left((dx^1))^2 + \cdots + (dx^d))^2\right)$$

where $\mu, \nu = 0, \ldots, d$ and we make the convention that $(x^0, \ldots, x^d)$ are coordinates on $\mathbb{R}^{1+d}$ with $x^0 = t$. We call $(\mathbb{R}^{1+d}, g)$ the $(1+d)$-dimensional *Minkowski space*. Furthermore, we view $\mathbb{S}^d$ as a submanifold of the $(1+d)$-dimensional Euclidean space $\mathbb{R}^{1+d}$ with spherical coordinates $(\omega^0, ..., \omega^{d-1})$, with $\omega^0, ..., \omega^{d-2}$ being polar angles and ω^{d-1} the azimuthal angle. We can endow $\mathbb{S}^d$ with a Riemannian metric h defined by

$$h_{ab}d\omega^a d\omega^b = (d\omega^0)^2 + \sin^2(\omega^0)d\Omega_{d-1}(\omega^1, ..., \omega^{d-1})^2,$$

with $d\Omega^2_{d-1}$ denoting the standard round metric on $\mathbb{S}^{d-1}$.

Given this data, we can pull h back to $\mathbb{R}^{1+d}$ which, expressed in local coordinates, has components

$$(u^*h)_{\mu\nu} = h_{ab}(u)\partial_\mu u^a \partial_\nu u^b.$$

For convenience, we set $S_{\mu\nu} := (u^*h)_{\mu\nu}$. At each point of Minkowski space, u^*h is a quadratic form on the corresponding tangent space. Thus, one can consider forming elementary symmetric polynomials of its eigenvalues. For example, S^μ_μ is the trace of u^*h and $\left(S^\mu_\mu\right)^2 - S^{\mu\nu}S_{\mu\nu}$ is the sum of pair products of distinct eigenvalues of u^*h. With this, one can consider the following action functional

$$\mathcal{A}[u] := \int_{\mathbb{R}^{1+d}} \left(\frac{\alpha^2}{2} S^\mu_\mu + \frac{\beta^2}{4}\left(\left(S^\mu_\mu\right)^2 - S^{\mu\nu}S_{\mu\nu}\right)\right) dg \tag{1.7}$$

for some fixed $\alpha, \beta \in \mathbb{R}$ and dg denoting the volume measure on $\mathbb{R}^{1+d}$ induced by the Minkowski metric. We call the action, Equation (1.7), the *Skyrme model*. There are two extreme cases of the Skyrme model worth considering:

1. $\alpha \neq 0$ and $\beta = 0$ (*wave maps*), and

2. $\alpha = 0$ and $\beta \neq 0$ (*strong-field Skyrme model*).

Let's begin by examining the first extreme case. In the physics literature, this is commonly referred to as a *nonlinear sigma model* or *wave maps* in the mathematics literature. Nonlinear sigma models were first introduced for $d = 3$ by Gell-Mann and Lévy in [22] in the context of nuclear and particle physics. The codomain of the field u was intended to model three subatomic particles called π mesons, often also referred to as pions, and the interactions between them.

We will restrict our attention to *co-rotational maps*, i.e., maps $u : \mathbb{R}^{1+d} \to \mathbb{S}^d$ with the property that, expressed in spherical coordinates, take the form

$$u(t, r, \omega) = \big(\psi(t, r), \omega\big)$$

for some $\psi : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ and $\omega \in \mathbb{S}^{d-1}$. In this extreme case, formal critical points of Equation (1.7) with respect to compactly supported variations satisfy the following Euler-Lagrange equation

$$\Box^{\text{rad}}_{r,d}\psi + \frac{d-1}{2}\frac{\sin(2\psi)}{r^2} = 0 \tag{1.8}$$

where

$$\Box^{\text{rad}}_{r,d} = \partial_t^2 - \Delta^{\text{rad}}_{r,d}$$

is the radial d'Alembertian on $\mathbb{R}^{1+d}$ and

$$\Delta^{\text{rad}}_{r,d} = \partial_r^2 + \frac{d-1}{r}\partial_r$$

is the radial Laplacian on $\mathbb{R}^d$. Of utmost importance is to notice that solutions of Equation (1.8) conserve the energy

$$E_{WM}[\psi] = \frac{1}{2}\int_0^\infty \left(\psi_t^2 + \psi_r^2 + (d-1)\frac{\sin^2(\psi)}{r^2}\right) r^{d-1} dr,$$

that is, $E_{WM}[\psi](t) = E_{WM}[\psi](0)$ for all $t \in \mathbb{R}$. In addition, Equation (1.8) exhibits *scaling invariance*, i.e., given $\lambda > 0$ and any solution ψ,

$$\psi_\lambda(t, r) := \psi(t/\lambda, r/\lambda) \tag{1.9}$$

defines another solution of Equation (1.8). Under this rescaling, the conserved energy transforms according to the equation

$$E_{WM}[\psi_\lambda] = \lambda^{d-2} E_{WM}[\psi]. \tag{1.10}$$

Concerning the nature of the associated Cauchy problem, one often develops a feeling of what to expect according to the physical heuristic that those solutions with the least allowable energy will be 'preferred' over those with greater energy. According to Equation (1.10), there are three distinct cases to consider concerning how to produce a solution with less energy when already given a solution.

1. If $d < 2$, then one can rescale any solution as in Equation (1.9) and take $\lambda \to \infty$.

2. If $d = 2$, then one cannot rescale any solution and change λ in order to decrease energy.

3. If $d > 2$, then one can rescale any solution and take $\lambda \to 0^+$.

In the first case, one says that the Cauchy problem associated to Equation (1.8) is *energy subcritical* in which case global existence from large initial data is expected. In the second case, one says that the Cauchy problem is *energy critical* in which case one expects that large data leads to a mix of global existence and finite-time blow-up. In the last case, one says that the Cauchy problem is *energy supercritical* in which case generic large initial data is expected to lead to finite-time blow-up.

We will be interested in energy supercritical problems due to their connections to physical equations such as Einstein's equation of general relativity in three spatial dimensions. For an excellent review, we refer the reader to the survey by Bizoń, [3]. To gain intuition for why finite-time blow-up should be expected, observe that shrinking λ shrinks the support of the solution. Thus, shrinking to smaller scales localizes a solution to a point. Consequently, the solution concentrates at a point and becomes singular, see Figure 1.1.

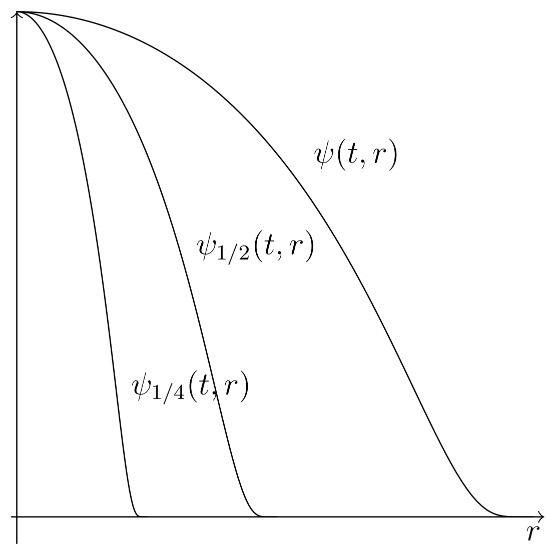

Figure 1.1: A plot of rescaled versions of exemplary, compactly supported solutions of the co-rotational wave maps equation.

In '88, Shatah provided evidence for this heuristic for Equation (1.8). In [34], he proved the existence of smooth, compactly supported data which lead to finite-time blow-up for the corresponding Cauchy problem in $d = 3$. This was achieved by looking for so-called *self-similar solutions*, i.e., solutions of the form

$$\psi(t,r) = f\left(\frac{r}{T-t}\right), T > 0.$$

Self-similar solutions respect the scaling invariance of Equation (1.8). More precisely, they are constant along lines in spacetime of the form $r = c(T-t)$ which all intersect at the spacetime point $(t,r) = (T,0)$, see Figure 1.2. A nontrivial, smooth self-similar solution blows up at this point according to

$$|\partial_r \psi(t,0)| \simeq \frac{1}{T-t} \to_{t\to T^-} \infty.$$

The self-similar ansatz reduces the problem of solving Equation (1.8) to solving

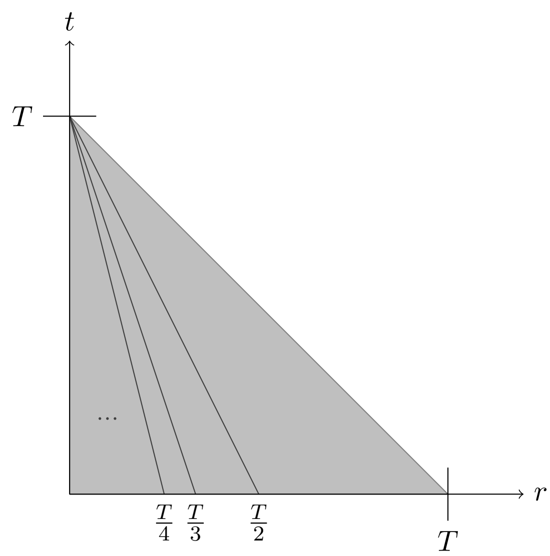

Figure 1.2: A spacetime diagram depicting the backwards lightcone of the spacetime point $(T, 0)$ and rays emanating from the vertex.

the ODE

$$f''(\rho) + \frac{d-1-2\rho^2}{\rho(1-\rho^2)} f'(\rho) - \frac{(d-1)\sin\big(2f(\rho)\big)}{\rho^2(1-\rho^2)^2} = 0 \tag{1.11}$$

where $\rho := \frac{r}{T-t}$ and $' := \frac{d}{d\rho}$. Shatah proved that at least one self-similar solution had to exist by searching for minima of a particular functional whose critical points solved Equation (1.11). A year later, Turok and Spergel in [39] were able to find an explicit self-similar solution in $d = 3$. Twenty-five years after that, Bizoń and Biernat in [4], found the same solution in all dimensions $d \geq 3$ given explicitly by

$$\psi_T^*(t, r) = 2\arctan\left(\frac{r}{\sqrt{d-2}(T-t)}\right), \ T > 0.$$

The stability of this solution has a rather interesting history and, in a sense, started the rigorous study of self-similar blow-up for energy supercritical nonlinear wave equations at large. In '99, Bizoń [2] initiated a numerical study of the stability of this solution for $d = 3$ which, in fact, represented the ground state of a countable family of self-similar solutions of Equation (1.11). Though not rigorously proven, these numerics provided strong evidence for universality of this blow-up profile. In '09, Donninger and Aichelburg [12] provided the first rigorous proof for the nonexistence of unstable solutions of the linearized equation which grow faster than the so-called *gauge instability*. This unstable solution is due to the fact that ψ_T^* represents a one-parameter family of solutions of Equation (1.8) and, consequently, $\partial_T \psi_T^*$ is a solution of the linearized equation which grows like $(T-t)^{-1}$ within backwards lightcones. Clearly, this left open the possibility that there could be other solutions of the linearized equation which could grow like $(T-t)^{-p}$ for $p \in [0,1)$. In '12, Aichelburg, Donninger, and Schörkhuber [18] pushed this range of growth down to $p \in [0, \frac{1}{2}]$ and provided numerical evidence for the nonexistence of solutions in the remaining range. Using this[2], in '11, Donninger [10] showed that if ψ_T^* were *mode stable*, then it was stable. The condition of mode stability referred to $\partial_T \psi_T^*$ being the only solution of the linearized equation which became unbounded with the backwards lightcone. Mode stability could be formulated as a condition for a second-order ordinary differential operator though, at the time, no rigorous proof of this condition being satisfied could be established. In '16, Costin, Donninger, and Xia provided the first rigorous proof for mode stability of ψ_T^* which, along with [10], concluded the first stability result for self-similar blow-up in wave maps for $d = 3$. Then with knowledge of the solution ψ_T^* in higher dimensions, Costin, Donninger, and Glogić [6] improved these methods for proving mode stability in '17 and, consequently, established mode stability of ψ_T^* for $d \geq 4$. Then, in '17, Chatzikaleas, Donninger, and Glogić [5] established stabiltiy of ψ_T^* for $d \geq 5$ and odd. More recently, in '19, Biernat, Donninger, and Schörkhuber [1] introduced a new system of coordinates which

[2] The work in [18] and [10] were done in conjunction and the confusing timeline is due to publication date.

allowed for stability analysis of ψ_T^* outside of backwards lightcones. Stability of ψ_T^* outside of co-rotational symmetry remains an open problem.

Now, let's examine the second extreme case of Equation (1.7): $\alpha = 0$ and $\beta \neq 0$. We refer to this case as the *strong-field Skyrme model*. Restricting our attention again to co-rotational maps, one finds that formal critical points of Equation (1.7) in this extreme case solve[3]

$$\Box^{\text{rad}}_{d-2,r}\psi + \cot(\psi)\big(\psi_t^2 - \psi_r^2\big) + \frac{d-2}{2}\frac{\sin(2\psi)}{r^2} = 0. \tag{1.12}$$

Solutions of this equation conserve the energy

$$E_{SF}[\psi] := \frac{1}{2}\int_0^\infty \frac{\sin^2(\psi)}{r^2}\left(\psi_t^2 + \psi_r^2 + \frac{d-2}{2}\frac{\sin^2(\psi)}{r^2}\right) r^{d-1} dr.$$

Furthermore, Equation (1.12) is invariant under the rescaling (1.9) and the energy transforms according to the equation

$$E_{SF}[\psi_\lambda] = \lambda^{d-4} E_{SF}[\psi].$$

In this case, the associated Cauchy problem is energy supercritical for $d \geq 5$. Again, looking for self-similar solutions reduces the problem of solving Equation (1.12) to the problem of solving an ODE. In fact, if one looks for solutions of the more special form

$$\psi(t,r) = \arccos\left(f\left(\frac{r}{T-t}\right)\right), T > 0,$$

Equation (1.12) reduces to the ODE

$$f''(\rho) + \frac{d-3-2\rho^2}{\rho(1-\rho^2)} f'(\rho) + \frac{(d-2)f(\rho)\big(1-f(\rho)^2\big)}{\rho^2(1-\rho^2)} = 0. \tag{1.13}$$

[3] In fact, this is only true for functions $\psi(t,r)$ which do not vanish for $r > 0$. For our purposes, this will be satisfied.

In [29], it was shown in $d = 5$ that Equation (1.13) has a nontrivial, smooth solution which gives rise to a solution of Equation (1.12) that blows up in finite time. Much like the argument of Shatah, the existence proof in [29] is abstract and does not say much about the nature of a self-similar solution. We, in fact, now have an explicit self-similar solution valid for any $d \geq 5$ and $r \leq \rho_d^*(T-t)$ given by

$$\psi_T^*(t,r) = \arccos\left(f_0\left(\frac{r}{T-t}\right)\right), \; T > 0 \tag{1.14}$$

where

$$f_0(\rho) = \frac{b_1 - b_2\rho^2}{b_1 + \rho^2}$$

with

$$b_1 := \frac{1}{3}\left(2(d-4) + \sqrt{3(d-4)(d-2)}\right),$$
$$b_2 := 2\sqrt{\frac{d-4}{3(d-2)}} + 1$$

and

$$\rho_d^* := \sqrt{\frac{2b_1}{b_2 - 1}}.$$

Observe that $f_0(\rho) < -1$ for $\rho > \rho_d^*$ which implies that ψ_T^* fails to be a real-valued solution of Equation (1.12) for $r > \rho_d^*(T-t)$. A direct calculation shows that this solution blows up like the wave maps solution, i.e.,

$$|\partial_r \psi_T^*(t,0)| \simeq \frac{1}{T-t} \to_{t\to T^-} \infty.$$

Finally, let's return to the general case of Equation (1.7). In the co-rotational setting, formal critical points of this action solve the quasilinear wave equation

$$\begin{aligned}\left(\alpha^2+\beta^2(d-1)\frac{\sin^2\psi}{r^2}\right)(\psi_{tt}-\psi_{rr}) - \left(\alpha^2 + \beta^2(d-3)\frac{\sin^2\psi}{r^2}\right)\frac{d-1}{r}\psi_r \\ + \frac{d-1}{2}\frac{\sin 2\psi}{r^2}\left(\alpha^2 + \beta^2(\psi_t^2 - \psi_r^2) + \frac{\beta^2(d-2)\sin^2\psi}{r^2}\right) = 0.\end{aligned} \tag{1.15}$$

Solutions of Equation (1.15) conserve the energy

$$E_{Sk}[\psi] := \alpha^2 E_{WM}[\psi] + \beta^2 E_{SF}[\psi].$$

Despite the fact that large data solutions, in general, do not exist globally for Equation (1.8) with $d = 3$, Geba and Grillakis [21] established a large data global existence result for the associated Cauchy problem of Equation (1.15) with $d = 3$. One might interpret this as saying that the strong-field Skyrme terms are strong enough to control the finite-time blow-up exhibited by the wave maps terms. Based on the physical heuristic, this should be expected since the strong-field Skyrme model is not energy supercritical for $d = 3$. In fact, this is related to the original motivation of Tony Skyrme's proposal of his model in his series of papers [35, 36, 37].

This should leave one wondering about the nature of the Cauchy problem associated to Equation (1.15) for $d \geq 5$ where the strong-field Skyrme model is energy supercritical. A bit of rearranging, after setting $\alpha = \beta = 1$ for simplicity, yields the equivalent equation

$$\begin{aligned}\Box^{\text{rad}}_{d-2,r}\psi + \cot(\psi)\big(\psi_t^2 - \psi_r^2\big) + \frac{d-2}{2}\frac{\sin(2\psi)}{r^2} \\ + \frac{r^2}{(d-1)\sin^2(\psi)}\left(\Box^{\text{rad}}_{r,d}\psi + \frac{d-1}{2}\frac{\sin(2\psi)}{r^2}\right) = 0.\end{aligned}$$

Though this equation is clearly not scaling invariant, if one rescales any solution ψ to obtain ψ_λ, then ψ_λ solves the λ-dependent equation

$$\begin{aligned}\Box^{\text{rad}}_{d-2,r}\psi_\lambda + \cot(\psi_\lambda)\Big(\big(\psi_\lambda\big)_t^2 - \big(\psi_\lambda\big)_r^2\Big) + \frac{d-2}{2}\frac{\sin(2\psi_\lambda)}{r^2} \\ + \lambda^2\frac{r^2}{(d-1)\sin^2(\psi_\lambda)}\left(\Box^{\text{rad}}_{r,d}\psi_\lambda + \frac{d-1}{2}\frac{\sin(2\psi_\lambda)}{r^2}\right) = 0.\end{aligned}$$

Consequently, as $\lambda \to 0^+$, the wave maps terms become negligible and one might expect that the dynamics of the Skyrme model are dominated by the strong-field Skyrme model. Moreover, we have the relation

$$E_{Sk}[\psi_\lambda] = \lambda^{d-4}\big(\lambda^2 E_{WM}[\psi] + E_{SF}[\psi]\big)$$

which suggests that shrinking to smaller scales is energetically favorable for $d \geq 5$. With this in mind, we make the following conjecture:

Conjecture 1 *The Cauchy problem associated to Equation* (1.15) *exhibits finite-time blow-up for $d \geq 5$ which can be described locally in terms of solutions of Equation* (1.12) *which blow-up in finite time.*

As a first step toward investigating this conjecture, we study the blow-up solution defined in Equation (1.14) for $d = 5$ in Chapter 2. In particular, we investigate its stability, i.e., the evolution of data close to ψ_T^*. Though we will not present a complete proof, we will present techniques which can and will be applied to provide one in a forthcoming paper. With confidence, we state the following conjecture

Conjecture 2 *Data close to ψ_1^*, defined in Equation* (1.14)*, can be evolved according to Equation* (1.12) *within backwards lightcones and their evolution converges to that of ψ_T^* for some T sufficiently close to* 1*.*

The main obstacle complicating a proof of this conjecture can be understood easily by comparing Equations (1.8) and (1.12). Equation (1.12) contains derivative nonlinearities whereas Equation (1.8) does not. Such a nonlinearity in Equation (1.12), when linearized around the solution ψ_T^*, gives rise to a *relatively compact perturbation* of some free operator.

Though this free operator can be understood well, perturbing it by a relatively compact operator prevents the application of previously well-understood techniques which apply to equations such as Equation (1.8) where the perturbation is instead only compact. Thus, we are forced to take a much closer look at the linear evolution of such data which we carry out in Section 2.4. Furthermore, we apply techniques to a toy model which can be used to overcome this obstacle.

1.3 The Quadratic Wave Equation

In establishing Conjecture 2, we track the evolution of perturbations of ψ_1^* within backwards lightcones. This leaves completely open the possibility of finite-time blow-up within the portion of the domain of influence of the perturbation which lies outside of these backwards lightcones. In particular, this includes a region of spacetime beyond the time of blow-up. However, we can use a special system of coordinates to investigate the stability of self-similar blow-up in regions of spacetime strictly larger than the corresponding backwards lightcones.

We will investigate the stability of self-similar blow-up outside of backwards lightcones by means of the radial *quadratic wave equation*

$$\Box_{r,d}^{\text{rad}} u = u^2. \tag{1.16}$$

Observe that solutions of Equation (1.16) conserve the energy

$$E[u] = \frac{1}{2}\int_0^\infty \left(u_t^2 + u_r^2 - \frac{1}{3}u^3\right) r^{d-1} dr.$$

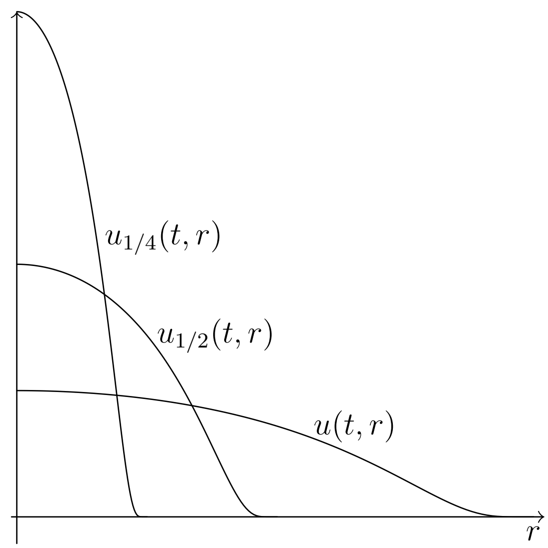

Figure 1.3: A plot of rescaled versions of exemplary, compactly supported solutions of the radial quadratic wave equation.

Furthermore, the rescaling $u \mapsto u_\lambda$, $\lambda > 0$, given by

$$u_\lambda(t,r) = \lambda^{-2} u(t/\lambda, r/\lambda) \tag{1.17}$$

leaves invariant Equation (1.16) and transforms the energy of a given solution u according to

$$E[u_\lambda] = \lambda^{d-6} E[u].$$

Thus, the associated Cauchy problem is energy supercritical for $d \geq 7$. For compactly supported solutions, the rescaling, Equation (1.17), has the effect of shrinking the support of the solution while increasing its amplitude. Thus, as $\lambda \to 0^+$, the solution becomes infinite in size and changes infinitely fast at the origin, see Figure 1.3.

This equation was recently studied by Csobo, Glogić, and Schörkhuber in [9] where they presented for the first time an explicit, spatially nontrivial, blow-up solution which

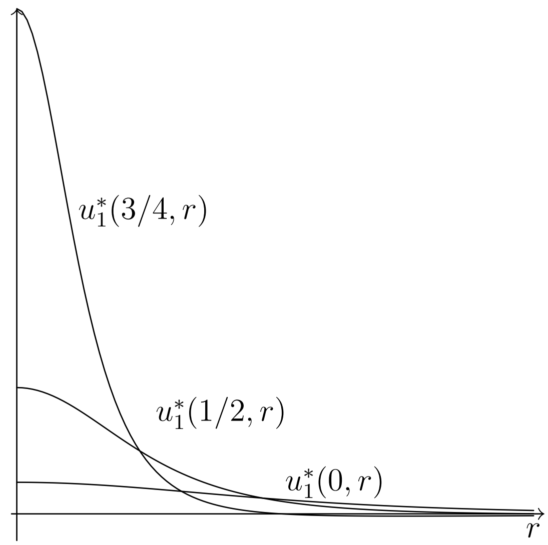

Figure 1.4: A plot of u_1^* at various times throughout its evolution.

takes the form

$$u_T^*(t,r) = \frac{1}{(T-t)^2} U\Big(\frac{r}{T-t}\Big),\ T > 0$$

with

$$U(\rho) = \frac{c_1 - c_2\rho^2}{(\rho^2 + c_3)^2}$$

where

$$c_1 = \frac{4}{25}\big((3d-8)\sqrt{6(d-1)(d-6)} + 8d^2 - 56d + 48\big),$$

$$c_2 = \frac{4}{5}\sqrt{6(d-1)(d-6)},$$

and

$$c_3 = \frac{1}{15}\big(3d - 18 + \sqrt{6(d-1)(d-6)}\big).$$

See Figure 1.4 for a plot of the solution for $d = 7$ and $T = 1$ at various different fixed times.

They were able to show that for $d = 9$, the corresponding solution is *conditionally stable* under non-radial perturbations within backwards lightcones. The term conditionally stable here refers to the need to remove a genuine instability present in the evolution of data close to u_T^*.

To investigate the nature of blow-up outside of backwards lightcones, we employ a coordinate system called *hyperboloidal similarity coordinates*. These coordinates were first introduced in [1] by Biernat, Donninger, and Schörkhuber in order to study the stability of the previously mentioned self-similar solution of the co-rotational wave maps equation outside of backwards lightcones for $d = 3$. Knowledge of the linear wave equation in these coordinates was then extensively developed by Donninger and Ostermann in [14] in order to study the stability of a well-known self-similar blow-up solution of the co-rotational, hyperbolic Yang-Mills equation in all odd space dimensions $d \geq 5$.

We will concern ourselves with studying the evolution of data close to that of u_T^* under Equation (1.16) for $d = 7$, the lowest energy supercritical dimension. The main obstacle posed in this problem compared to that of the analogous problem for Yang-Mills is the genuine instability investigated by Csobo, Schörkhuber, and Glogić. In hyperboloidal similarity coordinates, proving a conditional stability result becomes a subtly difficult problem and we present a new technique for solving it. We have the following soft statement of our result:

Theorem 3 (Soft Statement) *Suitably adjusted, compactly supported perturbations of* u_1^* *can be evolved according to Equation* (1.16) *in a region of spacetime strictly containing the backward lightcone of the point* $(1, 0)$ *and their evolution converges to that of* u_T^* *for some* T *sufficiently close to* 1.

1.4 Outline

Before beginning, we present a brief outline of this dissertation. As there is a significant amount of overlap in the strategy of proof, we simultaneously describe the approaches taken in Chapters 2 and 3. When these strategies diverge from each other, we will begin to describe them separately.

In Chapters 2 and 3, we study the equation of motion of the strong-field Skyrme model and the quadratic wave equation respectively. Using coordinates well-adapted to self-similar blow-up, we are able to rewrite these equations in the abstract form

$$\dot{\Phi} = \mathbf{L}\Phi + \mathbf{N}(\Phi) \tag{1.18}$$

where Φ represents a perturbation of the blow-up solution in these variables, $\dot{}$ represents differentiation with respect to some parameter of evolution, $\mathbf{L}$ is a particular non self-adjoint linear operator associated to linearization around the blow-up solution, and $\mathbf{N}$ is the corresponding nonlinear remainder. First, we establish linear stability of the blow-up solution, i.e., well-posedness and decay for the linearized equation in a particular Sobolev space. This consists of establishing that the linear operator $\mathbf{L}$ generates a strongly continuous semigroup on this Sobolev space and showing that this semigroup decays with the evolution. To show decay, we turn to characterizing the spectrum of $\mathbf{L}$. We achieve this by studying a particular ODE whose solutions can be connected to eigenfunctions of $\mathbf{L}$. In both Chapters 2 and 3, we will see that the evolution of generic data does not decay due to unstable eigenvalues of the operators $\mathbf{L}$. In both situations, the evolution of data away from the corresponding eigenspaces of these unstable eigenvalues will, in fact, decay. It is worth remarking that characterizing the spectrum of $\mathbf{L}$, in both cases, is an extremely difficult problem due to the non self-adjoint nature of these operators. In particular, characterizing the spectrum in Chapter 3 is exceptionally difficult. This will be elaborated on in Section 3.3.2.

At this point, Chapters 2 and 3 diverge. In Chapter 2, proving decay of the semigroup after characterizing the spectrum of $\mathbf{L}$ is an unusually difficult problem. Due to the derivative nonlinearities in Equation (1.12), linearization around ψ_T^* yields a relatively compact perturbation of some free evolution. In all previously studied problems, the analogous linearization yielded a compact perturbation of some free evolution for which a general spectral mapping theorem is now known, see [24]. From the point of view of perturbation theory, a relatively compact perturbation can be thought of as the worst type of small perturbation. A priori, there is no reason to believe that linear stability should hold for our solution. Indeed, in '94, Renardy [31] showed that, in a physically realistic setting, linear stability need not hold true for a hyperbolic PDE. To overcome this difficulty, we initiate an explicit construction of the resolvent $\mathbf{R}_{\mathbf{L}}(\lambda)$. In a forthcoming paper, we will show that using this explicit construction, the resolvent is uniformly bounded on the right-half plane in order to improve the growth bound on the semigroup and obtain the desired decay. We then present techniques for showing uniform boundedness of a toy model resolvent which will also be applied in that same forthcoming paper.

In Chapter 3, obtaining an improved growth bound on the semigroup is simpler due to compactness of the perturbation. After improving this growth bound, we proceed to establish nonlinear stability. That is, we prove global well-posedness and decay for small data for the nonlinear equation (1.18). We reformulate Equation (1.18) as a fixed-point problem via Duhamel's formula,

$$\Phi(s) = \mathbf{S}(s)\Phi(0) + \int_0^s \mathbf{S}(s-s')\mathbf{N}\big(\Phi(s')\big)ds'$$

where $(\mathbf{S}(s))_{s\geq 0}$ here denotes the semigroup generated by $\mathbf{L}$. In general, solutions with small data will not decay due to a genuine unstable eigenfunction of $\mathbf{L}$. Regardless, the data can be adjusted along a one-dimensional subspace in order to remove this instability, ultimately allowing for solutions to decay. This one-dimensional subspace will be spanned by a particular solution of the quadratic wave equation linearized around the blow-up solution which also has a very crucial property in relation to hyperboloidal similarity coordinates. We will elaborate on this in Section 3.5.2.

1.5 Notation and Conventions

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ the sets of natural numbers, integers, real numbers, and complex numbers respectively. By $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ we denote the nonnegative integers. Furthermore, we will denote by $\mathbb{R}_+ := \{a \in \mathbb{R} : a > 0\}$ the set of positive real numbers. Given a complex number $z \in \mathbb{C}$, we denote by $\Re z$ and $\Im z$ its real and imaginary parts respectively. We denote by $\mathbb{H} := \{z \in \mathbb{C} : \Re z > 0\}$ the open right-half complex plane. Given $R > 0$ and $d \in \mathbb{N}$, we denote by $\mathbb{B}^d_R := \{x \in \mathbb{R}^d : |x| < R\}$ the open ball in $\mathbb{R}^d$ of radius R centered at the origin.

Given $x, y \in \mathbb{R}_+$, we say $x \lesssim y$ if there exists a constant $C > 0$ such that $x \leq Cy$. Furthermore, we say that $x \simeq y$ if $x \lesssim y$ and $y \lesssim x$. For a one-parameter family of positive numbers x_λ, y_λ, we say that $x_\lambda \lesssim y_\lambda$ if there exists a constant $C > 0$, independent of the parameter λ, such that $x_\lambda \leq Cy_\lambda$ for all λ.

On a domain $\Omega \subset \mathbb{R}$ and given $f, g \in C^1(\Omega)$, we denote their Wronskian by $W(f,g)(x) = f(x)g'(x) - f'(x)g(x)$. When solving initial value problems for nonlinear wave equations for an unknown u, we will often denote by $u[t_0] := \big(u(t_0,\cdot), u_t(t_0,\cdot)\big)$ the initial data with initial time $t_0 \in \mathbb{R}$. Throughout, we will adopt the Einstein summation convention, i.e., repeated upper and lower indices indicate an implied sum over those indices.

The Schwartz space of functions on $\mathbb{R}^d$ will be denoted by $\mathcal{S}(\mathbb{R}^d)$. We define the Fourier transform by

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(x)dx$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$. For $k \in \mathbb{N}_0$ and $U \subset \mathbb{R}^d$ open and bounded, we define the Sobolev norms and homogeneous Sobolev norms by

$$\|f\|^2_{H^k(U)} := \sum_{|\kappa|\leq k} \|\partial^\kappa f\|^2_{L^2(U)},\ \|f\|^2_{\dot{H}^k(\Omega)} := \sum_{|\kappa|=k} \|\partial^\kappa f\|^2_{L^2(U)}$$

for all $f \in C^\infty(\overline{U})$ where $\kappa = (\kappa_1, \dots, \kappa_d) \in \mathbb{N}^d$ is a multi-index with $|\kappa| := \sum_{i=1}^d \kappa_i$. The Sobolev spaces $H^k(U)$ are then defined as the completion of $C^\infty(\overline{U})$ with respect to the norm $\|\cdot\|_{H^k(U)}$. For $s \in \mathbb{R}$, the Sobolev norms and homogeneous Sobolev norms on $\mathbb{R}^d$ can be defined via the Fourier transform as

$$\|f\|^2_{H^s(\mathbb{R}^d)} := \|\langle\cdot\rangle^s \mathcal{F}f\|^2_{L^2(\mathbb{R}^d)},\ \|f\|^2_{\dot{H}^s(\mathbb{R}^d)} := \||\cdot|^s \mathcal{F}f\|^2_{L^2(\mathbb{R}^d)}$$

where $\langle x\rangle := \sqrt{1+|x|^2}$ denotes the Japanese bracket.

On a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators. For a closed operator $\big(\mathbf{L}, \mathcal{D}(\mathbf{L})\big)$ on the Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(\mathbf{L})$, we denote the resolvent set by $\rho(\mathbf{L}) := \{\lambda \in \mathbb{C} \mid \lambda\mathbf{I} - \mathbf{L} : \mathcal{D}(\mathbf{L}) \subset \mathcal{H} \to \mathcal{H} \text{ is bounded and invertible}\}$. Given $\lambda \in \rho(\mathbf{L})$, we denote by $\mathbf{R}_\mathbf{L}(\lambda) := (\lambda\mathbf{I} - \mathbf{L})^{-1}$ the resolvent operator. Furthermore, we denote by $\sigma(\mathbf{L}) := \mathbb{C} \setminus \rho(\mathbf{L})$ the spectrum of $\big(\mathbf{L}, \mathcal{D}(\mathbf{L})\big)$. In particular we denote by $\sigma_p(\mathbf{L}) := \{\lambda \in \sigma(\mathbf{L}) : \exists \mathbf{u} \in \mathcal{D}(\mathbf{L}) \setminus \{\mathbf{0}\} \text{ such that } \mathbf{u} \in \ker(\lambda \mathbf{I} - \mathbf{L})\}$ the point spectrum of $\big(\mathbf{L}, \mathcal{D}(\mathbf{L})\big)$.

1.6 Functional Setting

Given $R > 0$, $k \in \mathbb{N}_0$, and $d \in \mathbb{N}$, we define the following space of functions

$$H^k_{\text{rad}}(\mathbb{B}^d_R) := \{f : (0, R) \to \mathbb{C} : f(|\cdot|) \in H^k(\mathbb{B}^d_R)\}.$$

We set

$$\mathcal{H}^k_R := H^k_{\text{rad}}(\mathbb{B}^d_R) \times H^{k-1}_{\text{rad}}(\mathbb{B}^d_R)$$

which is a Hilbert space with inner product

$$(\mathbf{f}|\mathbf{g})_{\mathcal{H}^k_R} := (f_1|g_1)_{H^k_{\text{rad}}(\mathbb{B}^d_R)} + (f_2|g_2)_{H^{k-1}_{\text{rad}}(\mathbb{B}^d_R)}$$

and norm

$$\|\mathbf{f}\|^2_{\mathcal{H}^k_R} := \|f_1\|^2_{H^k_{\text{rad}}(\mathbb{B}^d_R)} + \|f_2\|^2_{H^{k-1}_{\text{rad}}(\mathbb{B}^d_R)}$$

where $\mathbf{f} = (f_1, f_2)$ and $\mathbf{g} = (g_1, g_2)$. In the following chapters, we will often omit the dependence on k or R in our notation though this will be made clear with context.

We also consider the space of radial functions which are smooth up to the boundary of $\overline{\mathbb{B}^d_R}$

$$C^\infty_{\text{rad}}(\overline{\mathbb{B}^d_R}) := \{f \in C^\infty(\overline{\mathbb{B}^d_R}) : f \text{ is radial}\}$$

and the space of smooth even functions

$$C^\infty_e[0, R] := \{f \in C^\infty[0, R] : f^{(2j+1)}(0) = 0 \text{ for } j \in \mathbb{N}_0\}.$$

By Lemma 2.1 of [24], there is a one-to-one correspondence between $C^\infty_{\text{rad}}(\overline{\mathbb{B}^d_R})$ and $C^\infty_e[0,R]$. For ease of reading, we attempt to avoid switching between $C^\infty_{\text{rad}}(\overline{\mathbb{B}^d_R})$ and $C^\infty_e[0,R]$ and stick only with $C^\infty_e[0,R]$ whenever possible. We remark that $C^\infty_e[0,R]$ is dense in $H^k_{\text{rad}}(\mathbb{B}^d_R)$ which implies $C^\infty_e[0,R]^2$ is dense in $\mathcal{H}^k_R$.

In Chapter 2, we will take advantage of equivalent norms on $\mathcal{H}^5_1$ which significantly simplify the analysis. First is the the Σ-norm defined by

$$\|u\|^2_{\Sigma_0} := \|u\|^2_{H^1(0,1)} + \sum_{n=2}^{5} \|(\cdot)^{n-2}u^{(n)}\|^2_{L^2(0,1)}$$
$$\|u\|^2_{\Sigma_1} := \|u\|^2_{L^2(0,1)} + \sum_{n=1}^{4} \|(\cdot)^{n-1}u^{(n)}\|^2_{L^2(0,1)}$$
$$\|\mathbf{u}\|^2_{\Sigma} := \|u_1\|^2_{\Sigma_0} + \|u_2\|^2_{\Sigma_1},\ \forall \mathbf{u} \in \mathcal{H}^5_1.$$

This norm is most useful for proving compactness properties of particular operators and also for resolvent estimates. Second is the D_7-Norm and inner product defined by

$$\begin{aligned}(\mathbf{u}|\mathbf{v})_{D_7} :=&(D_7u_1|D_7v_1)_{\dot{H}^3(0,1)} + (D_7u_1|D_7v_1)_{\dot{H}^2(0,1)}\\ &+ (D_7u_1|D_7v_1)_{\dot{H}^1(0,1)} + (D_7u_2|D_7v_2)_{H^2(0,1)}\end{aligned}$$
$$\|\mathbf{u}\|_{D_7} := \sqrt{(\mathbf{u}|\mathbf{u})_{D_7}}$$

for $\mathbf{u} \in \mathcal{H}^5_1$ where

$$D_7u(\rho) := \Big(\frac{1}{\rho}\frac{d}{d\rho}\Big)^2 \big(\rho^5 u(\rho)\big)$$

with inverse on $\mathcal{H}^5_1$ defined by

$$K_7u(\rho) := \rho^{-5}\mathcal{K}^2u(\rho),\ \ \mathcal{K}u(\rho) := \int_0^\rho su(s)ds.$$

This norm is most useful for proving decay of solutions to Equation (2.7). We have the following proposition.

Proposition 4 *We have* $\|\cdot\|_\Sigma \simeq \|\cdot\|_{D_7} \simeq \|\cdot\|_{\mathcal{H}_1^5}$ *on* $\mathcal{H}_1^5$.

Proof. We leave the proof to Section 2.6.1. ■

Chapter 2

On the Stability of Blow-Up for the Strong-Field Skyrme Model

2.1 Introduction

This chapter concerns the equation of motion of the strong-field Skyrme model, i.e.,

$$\psi_{tt} - \psi_{rr} - \frac{d-3}{r}\psi_r + \cot(\psi)\big(\psi_t^2 - \psi_r^2\big) + \frac{d-2}{2}\frac{\sin(2\psi)}{r^2} = 0 \tag{2.1}$$

for $\psi : I \times [0, \infty) \to \mathbb{R}$, $I \subset \mathbb{R}$ an interval containing zero, and $r = |x|$ for $x \in \mathbb{R}^d$. Equation (2.1) exhibits the scaling symmetry $\psi \mapsto \psi_\lambda$,

$$\psi_\lambda(t, r) := \psi(t/\lambda, r/\lambda)$$

for any $\lambda > 0$. Solutions of Equation (2.1) conserve the energy

$$E[\psi] := \frac{1}{2}\int_0^\infty \frac{\sin^2(\psi)}{r^2}\left(\psi_t^2 + \psi_r^2 + \frac{d-2}{2}\frac{\sin^2(\psi)}{r^2}\right)r^{d-1}dr.$$

Under this rescaling, the energy transforms according to the equation

$$E[\psi_\lambda] = \lambda^{d-4} E[\psi].$$

Thus, the associated Cauchy problem is energy supercritical for $d \geq 5$. Restricting our attention to $d = 5$, the lowest energy supercritical dimension, we find the explicit radial, self-similar solution given by

$$\psi_T^*(t, r) = f_0\Big(\frac{r}{T-t}\Big), \; T > 0.$$

where

$$f_0(\rho) = \arccos\Big(\frac{5(1-\rho^2)}{5+3\rho^2}\Big).$$

Note that $\psi_T^*(0, r)$ and $\partial_t \psi_T^*(0, r)$ are smooth for $r \leq \sqrt{5}$. Furthermore, ψ_T^* becomes singular forward in time, i.e.

$$|\partial_r \psi_T^*(t, 0)| \to_{t \to T^-} \infty.$$

As a consequence of finite speed of propagation, one finds that the Cauchy problem

$$\begin{cases} \psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \cot(\psi)\big(\psi_t^2 - \psi_r^2\big) + \frac{3}{2}\frac{\sin(2\psi)}{r^2} = 0 & \text{in } C_T \\ \psi(0, r) = \psi_T^*(0, r), \; \psi_t(0, r) = \partial_t \psi_T^*(0, r) & r \leq T \end{cases},$$

exhibits finite-time blow-up where

$$C_T := \{(t, r) : 0 < t < T, \; 0 \leq r \leq T - t\}$$

is the backward lightcone of the spacetime point $(T, 0)$.

In this chapter, we make significant progress toward establishing the stability of the solution ψ_1^* locally around the blow-up point. That is, we make progress toward showing that for small perturbations f, g, there is a unique choice of T close to 1 so that the Cauchy problem

$$\begin{cases} \psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \cot(\psi)\big(\psi_t^2 - \psi_r^2\big) + \frac{3}{2}\frac{\sin(2\psi)}{r^2} = 0 & \text{in } C_T \\ \psi(0,r) = \psi_1^*(0,r) + f(r),\ \psi_t(0,r) = \partial_t \psi_1^*(0,r) + g(r) & r \leq T \end{cases} \tag{2.2}$$

has a unique solution ψ in C_T and converges to ψ_T^* as $t \to T^-$ within C_T in a suitable sense.

2.1.1 Similarity Coordinates

In this section, we introduce similarity coordinates. This system of coordinates is well-adapted to the self-similar nature of the solution ψ_T^*. In these coordinates, we rewrite the Cauchy problem (2.2) as an abstract initial value problem on some suitably chosen Hilbert space where ψ_T^* becomes an equilibrium of the system.

Given $T > 0$, we define *similarity coordinates* by the map

$$\eta_T : (0,\infty) \times [0,1] \to C_T$$
$$(\tau, \rho) \mapsto \big(T - Te^{-\tau}, e^{-\tau}\rho\big)$$

with inverse given by

$$\eta_T^{-1} : C_T \to (0,\infty) \times [0,1]$$
$$(t,r) \mapsto \left(\log\left(\frac{T}{T-t}\right), \frac{r}{T-t}\right),$$

see Figure 2.1. The derivatives transform according to the equations

$$\partial_t = \frac{e^\tau}{T}(\partial_\tau + \rho\partial_\rho)$$

and

$$\partial_r = \frac{e^\tau}{T}\partial_\rho.$$

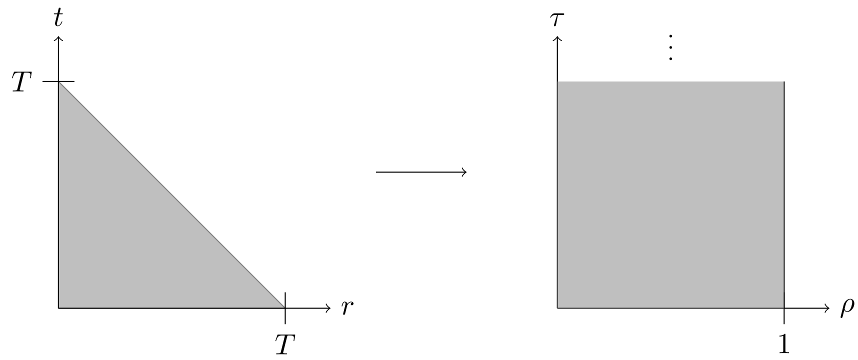

Figure 2.1: A diagram depicting how the map η_T^{-1} transforms the backwards lightcone C_T.

A direct calculation shows that, in these coordinates, ψ_T^* transforms according to the equation

$$(\psi_T^* \circ \eta_T)(\tau, \rho) = \arccos\big(f_0(\rho)\big).$$

Observe that $\psi_T^* \circ \eta_T$ is static and independent of the blow-up time T.

2.1.2 Statement of the Main Result

In this section, we state the intended main result of this chapter concerning the stability of ψ_T^* under small, radial perturbations.

Theorem 5 (*In progress*) *There exist constants $\delta, c, \omega_0 > 0$ such that for any radial initial data (f, g) satisfying*

$$\big\| |\cdot|^{-1}(f,g) \big\|_{H^5(\mathbb{B}^7_{1+\delta}) \times H^4(\mathbb{B}^7_{1+\delta})} \leq \frac{\delta}{c}$$

the following hold:

1. *There exists a unique $T \in [1-\delta, 1+\delta]$ and unique function $\psi : C_T \to \mathbb{R}$ solving Equation (2.1) in the sense of Definition 11 with $\psi[0] = \psi_1^*[0] + (f, g)$.*

2. The solution blows up at $t = T$ and converges to ψ_T^* according to

$$(T-t)^{k-\frac{d}{2}} \big\| |\cdot|^{-1} \big(\psi(t,\cdot) - \psi_T^*(t,\cdot)\big) \big\|_{\dot{H}^k(\mathbb{B}^7_{T-t})} \leq \delta (T-t)^{\omega_0}$$

$$(T-t)^{\ell+1-\frac{d}{2}} \big\| |\cdot|^{-1} \big(\partial_t\psi(t,\cdot) - \partial_t\psi_T^*(t,\cdot)\big) \big\|_{\dot{H}^\ell(\mathbb{B}^7_{T-t})} \leq \delta (T-t)^{\omega_0}$$

for all $k = 0, \ldots, 5$ and $\ell = 0, \ldots, 4$.

Remark 6 *Observe that the normalizing factors $(T-t)^{k-\frac{d}{2}}$ and $(T-t)^{\ell+1-\frac{d}{2}}$ reflect the behavior of the self-similar solution measured in the homogeneous Sobolev norms.*

Remark 7 *Equation* (2.1) *is in the same scaling class as a focusing cubic nonlinear wave equation, i.e. $\Box u = u^3$. With this in mind, one might expect that, within integer Sobolev spaces, that the regularity could be lowered to at least $H^3 \times H^2$ in light of [17]. However, we need to control ψ and its derivatives in L^∞ to make sense of the nonlinearity around ψ_T^* which forces a regularity of $H^5 \times H^4$. Reducing this regularity by even one derivative would require some deeper knowledge of how to control derivatives in this self-similar formulation using the nonlinear structures clearly present.*

2.2 The Wave Equation in Similarity Coordinates

In order to ensure regularity of the solutions, one must impose the boundary condition $\psi(\cdot, 0) = 0$. Thus, it is natural to work with $\hat{\psi}(t,r) := r^{-1}\psi(t,r)$. Consequently, Equation (2.1) becomes

$$\hat{\psi}_{tt} - \hat{\psi}_{rr} - \frac{4}{r}\hat{\psi}_r + r\cot(r\hat{\psi})\big(\hat{\psi}_t^2 - \hat{\psi}_r^2\big) - \frac{2}{r} r\hat{\psi}\cot(r\hat{\psi})\hat{\psi}_r + \frac{\frac{3}{2}\sin(2\hat{\psi}) - 2r\hat{\psi} - (r\hat{\psi})^2\cot(r\hat{\psi})}{r^3} = 0.$$

To ensure that we have a smooth nonlinearity upon linearizing around ψ_T^*, we study the equivalent equation

$$\hat{\psi}_{tt} - \hat{\psi}_{rr} - \frac{6}{r}\hat{\psi}_r + \frac{2}{T-t}\hat{\psi}_t - F_1\big(r\hat{\psi}, \hat{\psi}_r, \hat{\psi}_t, t, r; T\big) - r^{-3}F_2(\hat{\psi}) = 0 \tag{2.3}$$

where

$$F_1(x, y, z, t, r; T) := \frac{2}{T-t}z - r\cot(x)(z^2 - y^2) - \frac{2}{r}(1 - x\cot(x))y$$

and

$$F_2(x) := 2x + x^2\cot(x) - \frac{3}{2}\sin(2x).$$

2.2.1 Free Wave Evolution in Similarity Coordinates

To begin, we first analyze the following damped wave equation

$$\hat{\psi}_{tt} - \hat{\psi}_{rr} - \frac{6}{r}\hat{\psi}_r + \frac{2}{T-t}\hat{\psi}_t = 0$$

in C_T. To that end, we introduce the new variables

$$\chi_1(t, r) := (T-t)\hat{\psi}(t, r), \quad \chi_2(t, r) := (T-t)^2\hat{\psi}_t(t, r).$$

A direct calculation shows that

$$\partial_t\chi_1(t, r) = -\frac{1}{T-t}\chi_1(t, r) + \frac{1}{T-t}\chi_2(t, r)$$

and

$$\partial_t\chi_2(t, r) = -\frac{4}{T-t}\chi_2(t, r) + (T-t)\Delta_{r,7}^{\text{rad}}\chi_1(t, r).$$

We introduce the new variables

$$\psi_j(\tau, \rho) := \chi_j\big(t(\tau, \rho), r(\tau, \rho)\big)$$

to obtain the system

$$\begin{pmatrix} \partial_\tau \psi_1(\tau,\rho) \\ \\ \partial_\tau \psi_2(\tau,\rho) \end{pmatrix} = \begin{pmatrix} -\psi_1(\tau,\rho) + \psi_2(\tau,\rho) - \rho\partial_\rho\psi_1(\tau,\rho) \\ \\ \Delta^{\text{rad}}_{\rho,7}\psi_1(\tau,\rho) - \rho\partial_\rho\psi_2(\tau,\rho) - 4\psi_2(\tau,\rho) \end{pmatrix}. \tag{2.4}$$

In summary, we have the equation

$$\partial_\tau \Psi(\tau) = \tilde{\mathbf{L}}_0 \Psi(\tau) \tag{2.5}$$

where

$$\Psi(\tau)(\rho) := \begin{pmatrix} \psi_1(\tau,\rho) \\ \\ \psi_2(\tau,\rho) \end{pmatrix}$$

and $\tilde{\mathbf{L}}_0$ is defined as

$$\tilde{\mathbf{L}}_0 \mathbf{u}(\rho) := \begin{pmatrix} -\rho u_1'(\rho) - u_1(\rho) + u_2(\rho) \\ \\ \Delta^{\text{rad}}_{\rho,7} u_1(\rho) - \rho u_2'(\rho) - 4u_2(\rho) \end{pmatrix}$$

which we call the *free operator*. We view $\big(\tilde{\mathbf{L}}_0, \mathcal{D}(\tilde{\mathbf{L}}_0)\big)$ as an unbounded operator on the space $\mathcal{H} := \mathcal{H}^5_1$ with domain $\mathcal{D}(\tilde{\mathbf{L}}_0)$ defined by

$$\mathcal{D}(\tilde{\mathbf{L}}_0) := \big\{\mathbf{u} \in C^\infty(0,1)^2 \cap \mathcal{H} : D_7 u_2 \in C^3([0,1]),\ D_7 u_1 \in C^4([0,1]),\ [D_7 u_1]''(0) = 0\big\}$$

where D_7 is defined in Section 1.6. This domain is chosen following the work in [17] though it may be possible to take a simpler domain. As $C^\infty_e[0,1]^2$ is contained in $\mathcal{D}(\tilde{\mathbf{L}}_0)$, we have that $\big(\tilde{\mathbf{L}}_0, \mathcal{D}(\tilde{\mathbf{L}}_0)\big)$ is densely defined on $\mathcal{H}$. Furthermore, we have the following crucial properties of $\big(\tilde{\mathbf{L}}_0, \mathcal{D}(\tilde{\mathbf{L}}_0)\big)$.

Lemma 8 *Let* $\mathbf{u} \in \mathcal{D}(\tilde{\mathbf{L}}_0)$. *Then*

$$\Re(\tilde{\mathbf{L}}_0 \mathbf{u}|\mathbf{u})_{D_7} \leq -\frac{1}{2}\|\mathbf{u}\|^2_{D_7}.$$

Proof. Set $w_j := D_7 u_j$. We begin by computing

$$[D_7(\tilde{\mathbf{L}}_0\mathbf{u})_1]'(\rho) = w_2'(\rho) - \rho w_1''(\rho) - w_1'(\rho)$$

$$[D_7(\tilde{\mathbf{L}}_0\mathbf{u})_1]''(\rho) = w_2''(\rho) - \rho w_1'''(\rho) - 2w_1''(\rho)$$

$$[D_7(\tilde{\mathbf{L}}_0\mathbf{u})_2]'(\rho) = w_1'''(\rho) - \rho w_2''(\rho) - 4w_2'(\rho).$$

Integration by parts yields

$$\Re([D_7(\tilde{\mathbf{L}}_0\mathbf{u})_1]'', [D_7u_1]'')_{L^2(0,1)} = \Re(w_2''|w_1'')_{L^2(0,1)} - \frac{1}{2}|w_1''(1)|^2 - \frac{3}{2}\|w_1''\|^2_{L^2(0,1)}$$
$$\Re([D_7(\tilde{\mathbf{L}}_0\mathbf{u})_1]', [D_7u_1]')_{L^2(0,1)} = \Re(w_2'|w_1')_{L^2(0,1)} - \frac{1}{2}|w_1''(1)|^2 - \frac{1}{2}\|w_1'\|^2_{L^2(0,1)}$$
$$\Re([D_7(\tilde{\mathbf{L}}_0\mathbf{u})_2]', [D_7u_2]')_{L^2(0,1)} = \Re(w_1'''|w_2')_{L^2(0,1)} - \frac{1}{2}|w_2'(1)|^2 - \frac{7}{2}\|w_2'\|^2_{L^2(0,1)}$$
$$\Re(D_7(\tilde{\mathbf{L}}_0\mathbf{u})_2, D_7u_2)_{L^2(0,1)} = \Re(w_1''|w_2)_{L^2(0,1)} - \frac{1}{2}|w_2(1)|^2 - \frac{5}{2}\|w_2\|^2_{L^2(0,1)}.$$

Adding the above four lines, integrating by parts twice, and applying the Cauchy-Schwartz inequality yields the desired estimate. ■

Lemma 9 *For every* $\mathbf{f} \in C_e^\infty[0,1]^2$*, there exists* $\mathbf{u} \in \mathcal{D}(\tilde{\mathbf{L}}_0)$ *such that*

$$-\tilde{\mathbf{L}}_0\mathbf{u} = \mathbf{f}.$$

Proof. Trivially, for $\mathbf{f} = 0$ we have $\mathbf{u} = 0$. Assume $\mathbf{f}$ does not vanish identically. Set

$$F(\rho) := D_7 f_2(\rho) + 3D_7 f_1(\rho) + \rho[D_7 f_1]'(\rho)$$

and define

$$w_1(\rho) := \int_0^\rho \frac{1}{(1-s^2)^2}\int_s^1 (1-t^2)F(t)dtds$$
$$w_2(\rho) := \frac{\rho}{(1-\rho^2)^2}\int_\rho^1 (1-s^2)F(s)ds - D_7 f_1(\rho).$$

As $\mathbf{f} \in C_e^\infty[0,1]^2$, we have that $w_1 \in C^\infty(0,1) \cap C^4[0,1]$, $w_2 \in C^\infty(0,1) \cap C^3[0,1]$, and $w_1''(0) = 0$. Direct calculation shows that $\mathbf{w} := (w_1, w_2)$ solves the system of equations

$$\rho w_1'(\rho) - w_2(\rho) = D_7 f_1(\rho)$$

$$3w_2(\rho) - w_1''(\rho) + \rho w_2'(\rho) = D_7 f_2(\rho).$$

Recall the inverse of D_7, namely K_7 defined in Section 1.6. Applying K_7 to both sides of the equation yields

$$K_7 w_1(\rho) - K_7 w_2(\rho) + \rho[K_7 w_1]'(\rho) = f_1(\rho)$$

$$4K_7 w_2(\rho) - \Delta_{\rho,7}^{\text{rad}} K_7 w_1(\rho) + \rho[K_7 w_1]'(\rho) = f_2(\rho),$$

see [17] Lemma 4.5. Thus, $\mathbf{u} := K_7 \mathbf{w}$ is a solution of $-\tilde{\mathbf{L}}_0 \mathbf{u} = \mathbf{f}$. The properties of $\mathbf{w}$ imply $\mathbf{u} \in \mathcal{D}(\tilde{\mathbf{L}}_0)$ which finishes the proof. ■

The preceding two properties allow us to conclude the following proposition.

Proposition 10 *The operator* $\big(\tilde{\mathbf{L}}_0, \mathcal{D}(\tilde{\mathbf{L}}_0)\big)$ *on* $\mathcal{H}$ *is closable and its closure,* $\big(\mathbf{L}_0, \mathcal{D}(\mathbf{L}_0)\big)$*, is the generator of a strongly continuous one-parameter semigroup of bounded operators on* $\mathcal{H}$*,* $(\mathbf{S}_0(\tau))_{\tau \geq 0}$*, satisfying the estimate*

$$\|\mathbf{S}_0(\tau)\|_{\mathcal{H}} \leq M e^{-\frac{1}{2}\tau}$$

for all $\tau \geq 0$ *and some constant* $M \geq 1$*.*

Proof. Lemmas 8, 9, and the equivalence of the D_7 and $\mathcal{H}$ norms along with the Lumer-Phillips theorem (see [20], p.83 Theorem 3.15) imply the claim. ■

2.2.2 The Strong-Field Skyrme Model Equation of Motion in Similarity Coordinates

We continue by writing Equation (2.3) as a first-order system on $\mathcal{H}$. Using the new variables from Section 2.2.1, we obtain the system

$$\begin{pmatrix} \partial_\tau \psi_1(\tau,\rho) \\ \partial_\tau \psi_2(\tau,\rho) \end{pmatrix} = \begin{pmatrix} -\psi_1(\tau,\rho) + \psi_2(\tau,\rho) - \rho\partial_\rho\psi_1(\tau,\rho) \\ \Delta^{\mathrm{rad}}_{\rho,7}\psi_1(\tau,\rho) - \rho\partial_\rho\psi_2(\tau,\rho) - 4\psi_2(\tau,\rho) \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{F}_1\big(\rho\psi_1(\tau,\rho), \partial_\rho\psi_1(\tau,\rho), \psi_2(\tau,\rho), \rho\big) + \rho^{-3}F_2\big(\psi_1(\tau,\rho)\big) \end{pmatrix} \tag{2.6}$$

where

$$\tilde{F}_1(x,y,z,\rho) := 2z - \rho\cot(x)(z^2 - y^2) + \frac{2}{\rho}\big(x\cot(x) - 1\big)y.$$

The blowup solution becomes

$$\Psi^*_T(\tau)(\rho) := \begin{pmatrix} \frac{T-t}{r}\psi^*_T(t,r) \\ \frac{(T-t)^2}{r}\partial_t\psi^*_T(t,r) \end{pmatrix}\Bigg|_{(t=t(\tau,\rho), r=r(\tau,\rho))} = \begin{pmatrix} \frac{1}{\rho}f_0(\rho) \\ f_0'(\rho) \end{pmatrix}.$$

Since we expect solutions of (2.6) which are initially close to Ψ^*_1 to converge to Ψ^*_T for some T sufficiently close to 1, we insert the ansatz $\Psi = \Psi^*_T + \Phi$ where

$$\Phi(\tau)(\rho) := \begin{pmatrix} \varphi_1(\tau,\rho) \\ \varphi_2(\tau,\rho) \end{pmatrix}.$$

Furthermore, we expand the nonlinearity around $\Big(\frac{1}{\rho}f_0(\rho), \partial_\rho\big(\frac{1}{\rho}f_0(\rho)\big), f_0'(\rho)\Big)$ as follows:

$$\begin{aligned} \tilde{F}_1(\rho\psi_1, \partial_\rho\psi_1, \rho\psi_2, \rho) = {}& \tilde{F}_1\Big(f_0, \partial_\rho\Big(\frac{1}{\rho}f_0\Big), f_0', \rho\Big) + \partial_1\tilde{F}_1\big(f_0, \partial_\rho(\frac{1}{\rho}f_0), f_0', \rho\big)\rho\varphi_1 \\ & + \partial_2\tilde{F}_1\big(f_0, \partial_\rho(\frac{1}{\rho}f_0), f_0', \rho\big)\partial_\rho\varphi_1 + \partial_3\tilde{F}_1\big(f_0, \partial_\rho(\frac{1}{\rho}f_0), f_0', \rho\big)\varphi_2 \\ & + N_1(\rho\varphi_1, \partial_\rho\varphi_1, \varphi_2, \rho) \end{aligned}$$

where

$$N_1(\rho\varphi_1, \partial_\rho\varphi_1, \varphi_2, \rho) := \tilde{F}_1\Big(f_0 + \rho\varphi_1, \partial_\rho\Big(\frac{1}{\rho}f_0, \rho, \rho\Big) + \partial_\rho\varphi_1, f_0' + \varphi_2\Big)$$
$$- \tilde{F}_1\Big(f_0, \partial_\rho\Big(\frac{1}{\rho}f_0\Big), f_0', \rho\Big) - \partial_1\tilde{F}_1\big(f_0, \partial_\rho(\frac{1}{\rho}f_0), f_0', \rho\big)\rho\varphi_1$$
$$- \partial_2\tilde{F}_1\big(f_0, \partial_\rho(\frac{1}{\rho}f_0), f_0', \rho\big)\partial_\rho\varphi_1 - \partial_3\tilde{F}_1\big(f_0, \partial_\rho(\frac{1}{\rho}f_0), f_0', \rho\big)\varphi_2$$

and

$$F_2(\rho\psi_1) = F_2(f_0) + F_2'(f_0)\rho\varphi_1 + N_2(\rho\varphi_1)$$

where

$$N_2(\rho\varphi_1) := F_2(f_0 + \rho\varphi_1) - F_2(f_0) - F_2'(f_0)\rho\varphi_1.$$

In summary, we obtain the equation

$$\partial_\tau\Phi(\tau) = \tilde{\mathbf{L}}\Phi(\tau) + \mathbf{N}\big(\Phi(\tau)\big) \tag{2.7}$$

where

$$\tilde{\mathbf{L}} := \tilde{\mathbf{L}}_0 + \mathbf{L}'$$

with $\mathbf{L}'$, which we refer to as the *potential*[1], defined as

$$\mathbf{L}'\mathbf{u}(\rho) := \begin{pmatrix} 0 \\ V(\rho)u_1(\rho) + \tilde{V}(\rho)\rho\big(\rho u_2(\rho) - u_1'(\rho)\big) \end{pmatrix}$$

where $V, \tilde{V} \in C^\infty[0,1]$ are given explicitly by

$$V(\rho) := -\frac{5\big(21\rho^6 - 375\rho^4 + 1455\rho^2 - 2125\big)}{\big(5+3\rho^2\big)^2\big(5-\rho^2\big)^2}$$

[1] The term potential here is, potentially, not the best choice of terminology due to the presence of derivative terms. However, in the spirit of convention, we will use this anyway.

and

$$\tilde{V}(\rho) := -\frac{2\big(3\rho^2 - 35\big)}{\big(5+3\rho^2\big)\big(5-\rho^2\big)}$$

and $\mathbf{N}$ will be called the *nonlinear remainder* and is defined by

$$\mathbf{N}(\mathbf{u})(\rho) := \begin{pmatrix} 0 \\ N_1\big(\rho u_1(\rho), u_1'(\rho), u_2(\rho), \rho\big) + \rho^{-3} N_2\big(\rho u_1(\rho)\big) \end{pmatrix}.$$

In general, this nonlinearity is not smooth. However, if $u_1 > 0$, then the nonlinearity is smooth. This is easily achieved noting that $\psi_T^* > 0$ within C_T and the Sobolev embedding $L^\infty(\mathbb{B}^7) \hookrightarrow H^5(\mathbb{B}^7)$. Furthermore, $\mathbf{L}'$ is uniformly bounded on the dense set $C_e^\infty[0,1]^2$ and so extends uniquely to a bounded operator on $\mathcal{H}$ which we will, by an abuse of notation, refer to as $\mathbf{L}'$. Thus, we view $\big(\tilde{\mathbf{L}}, \mathcal{D}(\tilde{\mathbf{L}})\big)$ as a bounded perturbation of $\big(\tilde{\mathbf{L}}_0, \mathcal{D}(\tilde{\mathbf{L}}_0)\big)$ by setting $\mathcal{D}(\tilde{\mathbf{L}}) := \mathcal{D}(\tilde{\mathbf{L}}_0)$ from which it follows that $\big(\tilde{\mathbf{L}}, \mathcal{D}(\tilde{\mathbf{L}})\big)$ is densely defined on $\mathcal{H}$.

In the following section, we will show that $\tilde{\mathbf{L}}, \mathcal{D}(\tilde{\mathbf{L}})\big)$ is closable and its closure, denoted by $\big(\mathbf{L}, \mathcal{D}(\mathbf{L})\big)$, generates a strongly continuous one-parameter semigroup of bounded operators on $\mathcal{H}$ denoted by $(\mathbf{S}(\tau))_{\tau \geq 0}$. From this point on, the domains of our operators are fixed and we refrain from referring to them unless necessary. With this semigroup, we formulate the corresponding initial value problem of Equation (2.7) as an abstract integral equation on $\mathcal{H}$ via Duhamel's formula,

$$\Phi(\tau) = \mathbf{S}(\tau)\Phi(0) + \int_0^\tau \mathbf{S}(\tau - s)\mathbf{N}\big(\Phi(s)\big)ds. \tag{2.8}$$

This leads us to the following notion of strong solution for Equation (2.2).

Definition 11 *We say that $\psi : C_T \to \mathbb{R}$ is a* ***strong lightcone solution*** *of Equation* (2.2) *if the corresponding $\Phi : [0,\infty) \to \mathcal{H}$ belongs to $C\big([0,\infty), \mathcal{H}\big)$ and satisfies* (2.8) *for all* $\tau \geq 0$.

The notion of strong lightcone solution allows us to reformulate solving Equation (2.2) into a fixed point problem in terms of similarity coordinates. In fact, given enough regularity of the data, one can show that a strong lightcone solutions yields a classical solution of Equation (2.2), see Proposition 2.2 of [9] for instance.

2.3 Progress Toward Linear Stability

In this section, we demonstrate significant progress toward linear stability of Ψ_T^*. More precisely, we begin by analyzing solutions of the linearized equation, i.e., the equation

$$\partial_\tau \Phi = \tilde{\mathbf{L}}\Phi.$$

First, we show that $\tilde{\mathbf{L}}$ is closable and its closure, denoted by $\mathbf{L}$, is the generator of a strongly continuous one-parameter semigroup of bounded operators on $\mathcal{H}$. In other words, the Cauchy problem for the linearized equation is well-posed on $\mathcal{H}$. In order to show that solutions of Equation (2.7) for sufficiently small initial data exist for all $\tau \geq 0$, we need to obtain sufficient growth bounds on the semigroup generated by $\mathbf{L}$. Due to the non self-adjoint nature of $\mathbf{L}$ and the structure of the potential, this is, in fact, very difficult to obtain. We demonstrate techniques which, in the future, can be easily adapted in order to obtain the improved growth bound on the semigroup.

2.3.1 Well-Posedness of the Linearized Evolution

In this section, we show that $\tilde{\mathbf{L}}$ is closable and its closure, denoted by $\mathbf{L}$, is the generator of a strongly continuous one-parameter semigroup of bounded operators on $\mathcal{H}$.

Proposition 12 *The operator $\tilde{\mathbf{L}}$ is closable and its closure, denoted by $\mathbf{L}$, is the generator of a strongly continuous one-parameter semigroup of bounded operators, $(\mathbf{S}(\tau))_{\tau\geq 0}$, on $\mathcal{H}$.*

Proof. This follows immediately from the bounded perturbation theorem (see p.159, Theorem 1.3 of [20]) and Proposition 10. ■

The bounded perturbation theorem guarantees that the semigroup $(\mathbf{S}(\tau))_{\tau\geq 0}$ satisfies the bound

$$\|\mathbf{S}(\tau)\|_{\mathcal{H}} \leq Me^{(-\frac{1}{2}+M\|\mathbf{L}'\|_{\mathcal{H}})\tau} \tag{2.9}$$

for $M \geq 1$ as in Proposition 10 and all $\tau \geq 0$. As M and $\|\mathbf{L}'\|_{\mathcal{H}}$ can be very large, this bound is useless for showing decay. To improve this bound, we intend to use the Gearhart-Prüss-Greiner theorem (see p.302, Theorem 1.11 of [20]) which necessitates a characterization of $\sigma(\mathbf{L})$. From the decay estimate on the semigroup $(\mathbf{S}_0(\tau))_{\tau\geq 0}$, we infer

$$\sigma(\mathbf{L}_0) \subseteq \{\lambda \in \mathbb{C} : \Re\lambda \leq -\frac{1}{2}\} \tag{2.10}$$

which follows from [20], p. 55, Theorem 1.10. As a first step, we prove the following compactness property of the potential which will allow us to characterize $\sigma(\mathbf{L})$ by checking how $\mathbf{L}'$ affects $\sigma_p(\mathbf{L}_0)$.

Proposition 13 *The operator $\mathbf{L}'$ is compact relative to $\mathbf{L}_0$.*

Proof. We show that this property holds on a dense set which, by the fact that $\tilde{\mathbf{L}}_0$ is densely defined, implies the desired result. Assume that $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subset C_e^\infty[0,1]^2$ is uniformly bounded in the graph norm of $\mathbf{L}_0$, i.e.,

$$\sup_{n\in\mathbb{N}} \|\mathbf{u_n}\|^2_{\mathbf{L}_0} = \sup_{n\in\mathbb{N}} \left(\|\mathbf{u_n}\|^2_{\mathcal{H}} + \|\mathbf{L}_0\mathbf{u_n}\|^2_{\mathcal{H}}\right) < \infty.$$

As $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(\tilde{\mathbf{L}}_0)$, $\mathbf{L}_0$ acts as a classical differential operator. From this assumption and the definition of $\tilde{\mathbf{L}}_0$, we have the following:

$$\sup_{n\in\mathbb{N}} \|u_{1,n}\|_{\Sigma_0} < \infty,$$

$$\sup_{n\in\mathbb{N}} \|u_{2,n}\|_{\Sigma_1} < \infty,$$

$$\sup_{n\in\mathbb{N}} \|u_{2,n} - u_{1,n} - (\cdot)u'_{1,n}\|_{\Sigma_0} < \infty,$$

and

$$\sup_{n\in\mathbb{N}} \left\|\Delta^{\text{rad}}_{\rho,7} u_{1,n} - 2u_{2,n} - (\cdot)u'_{2,n}\right\|_{\Sigma_1} < \infty.$$

Multiplying the fourth quantity by ρ yields

$$\sup_{n\in\mathbb{N}} \big\|(\cdot)\big((\cdot)u'_{2,n} - u''_{1,n}\big)\big\|_{\Sigma_1} < \infty$$

which implies

$$\sup_{n\in\mathbb{N}} \big\|(\cdot)\big((\cdot)u_{2,n} - u'_{1,n}\big)\big\|_{\Sigma_0} < \infty$$

Thus, $\Big((\cdot)\big((\cdot)u_{2,n} - u'_{1,n}\big)\Big)_{n\in\mathbb{N}}$ is uniformly bounded in $H^5_{\text{rad}}(\mathbb{B}^7)$. Lemma 4.2 of [17] implies that $\left(D_7\Big((\cdot)\big((\cdot)u_{2,n} - u'_{1,n}\big)\Big)\right)_{n\in\mathbb{N}}$ is uniformly bounded in $H^3(0,1)$. By the compactness of the Sobolev embedding $H^3(0,1) \hookrightarrow H^2(0,1)$, there exists a subsequence, again denoted by $\left(D_7\Big((\cdot)\big((\cdot)u_{2,n} - u'_{1,n}\big)\Big)\right)_{n\in\mathbb{N}}$, which is Cauchy in $H^2(0,1)$. Applying the same argument to $(u_{1,n})_{n\in\mathbb{N}}$ yields

$$\begin{aligned}\|\mathbf{L}'\mathbf{u}_n - \mathbf{L}'\mathbf{u}_m\|_\Sigma \leq & \Big\|V(\cdot)\big(u_{1,n}-u_{1,m}\big)\Big\|_{\Sigma_1} \\ & + \Big\|\tilde{V}(\cdot)\Big((\cdot)\big((\cdot)u_{2,n}-u_{1,n}'\big)-(\cdot)\big((\cdot)u_{2,m}-u_{1,m}'\big)\Big)\Big\|_{\Sigma_1} \\ \lesssim & \big\|u_{1,n}-u_{1,m}\big\|_{\Sigma_1} + \Big\|(\cdot)\big((\cdot)u_{2,n}-u_{1,n}'\big)-(\cdot)\big((\cdot)u_{2,m}-u_{1,m}'\big)\Big\|_{\Sigma_1} \\ \lesssim & \|D_7\big(u_{1,n}-u_{1,m}\big)\|_{H^2(0,1)} \\ & + \Big\|D_7\Big((\cdot)\big((\cdot)u_{2,n}-u_{1,n}'\big)-(\cdot)\big((\cdot)u_{2,m}-u_{1,m}'\big)\Big)\Big\|_{H^2(0,1)}\end{aligned}$$

where the last inequality follows from Proposition 4. Using the equivalence of the Σ and $\mathcal{H}$ norms from, namely Proposition 4, this shows that $(\mathbf{L}'\mathbf{u}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and must converge. ■

As a consequence, it will suffice to characterize $\sigma_p(\mathbf{L})$ in order to invoke the Gearhart-Prüss-Greiner theorem. Even more, it will turn out that studying $\sigma_p(\mathbf{L})$ amounts to studying $C^\infty[0,1]$ solutions of a certain ODE.

2.3.2 Spectral ODE Analysis

In this section, we begin to make the connection between $\sigma_p(\mathbf{L})$ and a particular ODE. In fact, this connection will make it possible to sufficiently characterize $\sigma_p(\mathbf{L})$. First, observe that for $\mathbf{f} = (f_1, f_2) \in \mathcal{D}(\mathbf{L})$, the equation $(\lambda\mathbf{I} - \mathbf{L})\mathbf{f} = 0$ implies that $f := f_1$ solves the ODE

$$\begin{aligned}-(1-\rho^2)f''(\rho) + \Big(-\frac{6}{\rho} + 2(\lambda+3)\rho + \rho(1-\rho^2)\tilde{V}(\rho)\Big)f'(\rho) \\ +\Big((\lambda+1)(\lambda+4) - V(\rho) - (\lambda+1)\rho^2\tilde{V}(\rho)\Big)f(\rho) = 0\end{aligned} \tag{2.11}$$

on the interval $(0,1)$. We will refer to this as the *spectral ODE*. In fact, the regularity imposed by $\mathbf{f} \in \mathcal{D}(\mathbf{L})$ implies f is a classical solution of Equation (2.11). Regularity of the coefficients implies that $f \in C^\infty(0,1)$. Now, a Frobenius analysis shows that $f \in C^\infty[0,1]$. To that end, we define the following set

$$\Sigma := \{\lambda \in \mathbb{C} : \Re\lambda \geq 0 \text{ and } \exists f \in C^\infty[0,1] \text{ solving Equation (2.11) on } (0,1)\}.$$

We have the following proposition.

Proposition 14 $\Sigma = \{1\}$ *with unique, up to a constant multiple, solution*

$$f(\rho;1) = \frac{1}{\sqrt{5-\rho^2}(5+3\rho^2)}. \tag{2.12}$$

Proof. By direct computation, one sees that $f(\rho;1)$ solves Equation (2.11) with $\lambda = 1$. Let $\lambda \in \mathbb{C} \setminus \{1\}$ with $\Re\lambda \geq 0$. We introduce a new independent variable

$$x = \frac{8\rho^2}{5+3\rho^2}$$

and new dependent variable by the equation

$$f(\rho) = (8-3x)^{\frac{\lambda+2}{2}}(2-x)^{-\frac{1}{2}}y(x)$$

which transforms Equation (2.11) into one of Heun type, namely

$$\ddot{y} + \frac{3(\lambda+5)x^2 - (8\lambda+43)x + 28}{x(1-x)(8-3x)}\dot{y} + \frac{(\lambda-1)\big(3(\lambda+9)x - 5\lambda - 51\big)}{4x(1-x)(8-3x)}y = 0 \tag{2.13}$$

where $\dot{}$ denotes $\dfrac{d}{dx}$. The solution $f(\cdot;1)$ transforms into

$$y(x;1) = 1$$

up to a multiplicative constant.

Frobenius theory implies that any $y \in C^\infty[0,1]$ solving (2.13) is analytic on $[0,1]$. Thus, if y fails to be analytic at $x = 1$, it also fails to be smooth at $x = 1$. Any analytic solution of Equation (2.13) yields an analytic solution of Equation (2.11) as well as the converse. Thus, excluding the existence of an analytic solution of Equation (2.13) for those desired λ also excludes the existence of a $C^\infty[0,1]$ solution of Equation (2.11) for those same λ.

The Frobenius indices at $x = 0$ are $\{0, -\frac{5}{2}\}$. Without loss of generality, such a solution y must have the expansion

$$y(x;\lambda) = \sum_{n=0}^{\infty} a_n(\lambda)x^n, \quad a_0(\lambda) = 1 \tag{2.14}$$

near $x = 0$. Since the finite regular singular points of Equation (2.13) are $x = 0, 1, \dfrac{8}{3}$, y fails to be analytic at $x = 1$ precisely when the radius of convergence of (2.14) is one.

We first derive a recurrence relation for the coefficients $a_n(\lambda)$ given by

$$a_{n+2}(\lambda) = A_n(\lambda)a_{n+1}(\lambda) + B_n(\lambda)a_n(\lambda) \tag{2.15}$$

where

$$A_n(\lambda) = \frac{44n^2 + 8n(4\lambda + 27) + \lambda(5\lambda + 78) + 121}{16(n+2)(2n+9)}$$

$$B_n(\lambda) = -\frac{3(\lambda + 2n - 1)(\lambda + 2n + 9)}{16(n+2)(2n+9)}$$

and $a_{-1}(\lambda) = 0$, and $a_0(\lambda) = 1$. Furthermore, we define

$$r_n(\lambda) := \frac{a_{n+1}(\lambda)}{a_n(\lambda)}.$$

Since $\lim_{n\to\infty} A_n(\lambda) = \frac{11}{8}$, $\lim_{n\to\infty} B_n(\lambda) = -\frac{3}{8}$, the so-called characteristic equation of Equation (2.15) is

$$t^2 - \frac{11}{8}t + \frac{3}{8} = 0.$$

Solutions of this equation are given by $t_1 = \dfrac{3}{8}$ and $t_2 = 1$. By Poincaré's theorem on difference equations, see [19] or [25] Appendix A, we conclude that either $a_n(\lambda) = 0$ eventually in n,

$$\lim_{n\to\infty} r_n(\lambda) = 1 \tag{2.16}$$

or

$$\lim_{n\to\infty} r_n(\lambda) = \frac{3}{8}. \tag{2.17}$$

In fact, $a_n(\lambda)$ cannot go to zero eventually in n since backwards substitution would imply $a_0(\lambda) = 0$ which is a clear contradiction. We show that Equation (2.17) cannot hold.

By plugging Equation (2.15) into the definition of $r_n(\lambda)$, we derive a recurrence relation for $r_n(\lambda)$ given by

$$r_{n+1}(\lambda) = A_n(\lambda) + \frac{B_n(\lambda)}{r_n(\lambda)}$$

with initial condition

$$r_0(\lambda) = \frac{(\lambda - 1)(51 + 5\lambda)}{112}.$$

Furthermore, we define an approximation to $r_n(\lambda)$ given by

$$\tilde{r}_n(\lambda) := \frac{5\lambda^2}{16(n+1)(2n+7)} + \frac{(16n+23)\lambda}{8(n+1)(2n+7)} + \frac{n+3}{n+1}.$$

The quadratic and linear in λ terms are obtained by studying the large $|\lambda|$ behavior of $A_n(\lambda)$ while the constant in λ term is put in by hand in order to mimic the small $|\lambda|$ behavior of the first few iterates of $r_n(\lambda)$. Observe that $\lim_{n\to\infty} \tilde{r}_n(\lambda) = 1$. The approximation $\tilde{r}_n(\lambda)$ is intended to behave like $r_n(\lambda)$ for sufficiently large n. If indeed this is true, then there is hope to exclude Equation (2.17). To show this, we define

$$\delta_n(\lambda) := \frac{r_n(\lambda)}{\tilde{r}_n(\lambda)} - 1$$

and derive a recurrence relation for it given by

$$\delta_{n+1}(\lambda) = \varepsilon_n(\lambda) - C_n(\lambda)\frac{\delta_n(\lambda)}{1+\delta_n(\lambda)}$$

where

$$\varepsilon_n(\lambda) := \frac{A_n(\lambda)\tilde{r}_n(\lambda) + B_n(\lambda)}{\tilde{r}_n(\lambda)\tilde{r}_{n+1}(\lambda)} - 1$$

and

$$C_n(\lambda) := \frac{B_n(\lambda)}{\tilde{r}_n(\lambda)\tilde{r}_{n+1}(\lambda)},$$

by again plugging the recurrence relation for $r_n(\lambda)$ into the definition of $\delta_n(\lambda)$. Direct calculation shows that we have the following explicit expressions for $\varepsilon_n(\lambda)$ and $C_n(\lambda)$ given by

$$C_n(\lambda) = \frac{P_1(n;\lambda)}{P_2(n;\lambda)}$$

and

$$\varepsilon_n(\lambda) = \frac{P_3(n;\lambda)}{P_2(n;\lambda)}$$

where

$$P_1(n;\lambda) := -48(1+n)(7+2n)(-1+2n+\lambda)(9+2n+\lambda),$$

$$\begin{aligned} P_2(n;\lambda) :=& \big(336 + 32n^2 + 16n(13+2\lambda) + \lambda(46+5\lambda)\big) \\ & \times \big(576 + 32n^2 + 16n(17+2\lambda) + \lambda(78+5\lambda)\big), \end{aligned}$$

and

$$\begin{aligned} P_3(n;\lambda) :=& -32(7+2n)(669+322n+40n^2) \\ & + 2\big(11809 + 4n(2742+467n)\big)\lambda \\ & + \big(2611 + 4n(178+9n)\big)\lambda^2. \end{aligned}$$

This shows that given $\lambda \in \mathbb{C} \setminus \{1\}$ with $\Re\lambda \geq 0$, we have that $\varepsilon_n(\lambda) \to 0$ and $C_n(\lambda) \to -\dfrac{3}{8}$ as $n \to \infty$.

For $\lambda \in \mathbb{C} \setminus \{1\}$ with $\Re\lambda \geq 0$, we have the following estimates

$$\begin{aligned} |\delta_{20}(\lambda)| &\leq \frac{1}{4} \\ |C_n(\lambda)| &\leq \frac{3}{8} \\ |\varepsilon_n(\lambda)| &\leq \frac{1}{8} \end{aligned} \tag{2.18}$$

for $n \geq 20$. We discuss the proof of the second estimate since the other two are obtained via the same argument. First, we establish the desired estimate of the imaginary line. Then we can extend the estiamte to $\overline{\mathbb{H}}$ via the Phragmén-Lindelöf principle so long as $C_n(\lambda)$ is analytic and polynomially bounded there.

Observe that for $t \in \mathbb{R}$, The inequality $|C_n(it)| \leq \frac{3}{8}$ is equivalent to the inequality $64|P_1(n,it)|^2 - 9|P_2(n,it)|^2 \leq 0$. For $t \in \mathbb{R}$ and $n \geq 20$, a direct calculation shows that the coefficients of $64|P_1(n,it)|^2 - 9|P_2(n,it)|^2$ are manifestly negative which establishes the desired estimate on the imaginary line. Now, we aim to extend the estimate to all of $\overline{\mathbb{H}}$. As $C_n(\lambda)$ is a rational function of polynomials in $\mathbb{Z}[n,\lambda]$, it is polynomially bounded. Furthermore, a direct calculation of the zeros of $P_2(n,\lambda)$ shows that they are contained in $\mathbb{C} \setminus \overline{\mathbb{H}}$ implying the analyticity of $C_n(\lambda)$ in $\overline{\mathbb{H}}$. Thus, the Phragmén-Lindelöf principle extends the estimate to all of $\overline{\mathbb{H}}$.

With these bounds in hand, we can prove the same bound for δ_n, $n > 20$ by induction. Suppose the estimate holds for some $k > 20$. Then

$$|\delta_{k+1}(\lambda)| \leq \frac{1}{8} + \frac{3}{8}\frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{4}$$

by the triangle inequality, Equation (2.18), and the induction hypothesis. This bound on

$\delta_n(\lambda)$ is now sufficient to exclude Equation (2.17). Suppose Equation (2.17) holds. Then

$$\frac{1}{4} \geq |\delta_n(\lambda)| = \left|1 - \frac{r_n(\lambda)}{\tilde{r}_n(\lambda)}\right| \to_{n\to\infty} \frac{5}{8}$$

which is clearly a contradiction. Thus, Equation (2.16) must hold and so y fails to be analytic at $x = 1$. Therefore, we conclude $\lambda \notin \Sigma$. ■

2.3.3 Spectral Analysis of the Generator

In this section, we analyze $\sigma(\mathbf{L})$. As $\mathbf{L}'$ is compact relative to $\mathbf{L}_0$, it will be sufficient to characterize $\sigma_p(\mathbf{L})$ in order to obtain a characterization of $\sigma(\mathbf{L})$. First, we define the function

$$\mathbf{f}_1^*(\rho) = \begin{pmatrix} f_{1,1}^*(\rho) \\ \\ f_{1,2}^*(\rho) \end{pmatrix} := \begin{pmatrix} f(\rho; 1) \\ \\ \rho f'(\rho; 1) + 2f(\rho; 1) \end{pmatrix}.$$

According to the following proposition, $\mathbf{f}_1^*$ is an eigenfunction of $\mathbf{L}$ with eigenvalue 1.

Proposition 15 *We have*

$$\sigma_p(\mathbf{L}) \subseteq \{\lambda \in \mathbb{C} : \Re\lambda < 0\} \cup \{1\}.$$

Proof. By direct calculation, we see that $\mathbf{f}_1^*$ solves $(1 - \mathbf{L})\mathbf{f}_1^* = 0$, $D_7 f_{1,1}^* \in C^4[0,1]$, $D_7 f_{1,2}^* \in C^3[0,1]$, and $[D_7 f_{1,1}^*]''(0) = 0$ which implies that $1 \in \sigma_p(\mathbf{L})$.

Now, suppose the claim does not hold, i.e., there exists $\lambda \in \sigma_p(\mathbf{L}) \setminus \{1\}$ with $\Re\lambda \geq 0$. Thus, there exists $\mathbf{u} = (u_1, u_2) \in \mathcal{D}(\mathbf{L}) \setminus \{0\}$ such that $\mathbf{u} \in \ker(\lambda - \mathbf{L})$. Unpacking the definition of $\mathbf{L}$, one sees that $(\lambda - \mathbf{L})\mathbf{u} = 0$ implies that u_1 is a solution of Equation (2.11) on $(0, 1)$. Since all coefficients are smooth in $(0, 1)$, we must have $u_1 \in C^\infty(0, 1)$. The

Frobenius indices at the regular singular point $\rho = 0$ are $s_1 = 0$ and $s_2 = -5$. By Frobenius theory, there exists a solution of the form

$$u_{1,1}(\rho) = \sum_{i=0}^{\infty} a_i \rho^i$$

which is analytic at $\rho = 0$. Since $s_1 - s_2 = 6$ there exists a second linearly independent solution of the form

$$u_{1,2}(\rho) = C \log(\rho) u_{1,1}(\rho) + \rho^{-5} \sum_{i=0}^{\infty} b_i \rho^i$$

for some $C \in \mathbb{C}$ and $b_0 = 1$. Observe that $u_{1,2}$ does not belong to the Sobolev space $H^5_{\text{rad}}(B^7)$ due to the second term. Thus, near $\rho = 0$, u_1 must be a multiple of $u_{1,1}$ and we infer $u_1 \in C^\infty[0, 1)$.

Similarly, the Frobenius indices at $\rho = 1$ are $s_1 = 0$ and $s_2 = 1 - \lambda$. If $1 - \lambda \notin \mathbb{Z}$, then we have two linearly independent solutions

$$u_{1,1}(\rho) = \sum_{i=0}^{\infty} a_i (1 - \rho)^i$$

and

$$u_{1,2}(\rho) = (1 - \rho)^{1-\lambda} \sum_{i=0}^{\infty} b_i (1 - \rho)^i.$$

The solution $u_{1,2}$ clearly does not belong to the Sobolev space $H^5_{\text{rad}}(\mathbb{B}^7)$. Thus, near $\rho = 1$, u_1 must be a multiple of $u_{1,1}$ and we infer $u_1 \in C^\infty[0, 1]$.

If $1 - \lambda = k \in \mathbb{N}_0$, we have two fundamental solutions of the form

$$u_{1,1}(\rho) = (1 - \rho)^k \sum_{i=0}^{\infty} a_i (1 - \rho)^i$$

and

$$u_{1,2}(\rho) = \sum_{i=0}^{\infty} b_i (1 - \rho)^i + C \log(1 - \rho) u_{1,1}(\rho).$$

Since $\Re\lambda \geq 0$, we must have that $k = 0$ or 1. Hence, $u_{1,2}$ does not belong to the Sobolev space $H^5_{\text{rad}}(\mathbb{B}^7)$ unless $C = 0$ in which case we conclude $u_1 \in C^\infty[0,1]$.

If $1 - \lambda = -k$, with $k \in \mathbb{N}$, we have two fundamental solutions of the form

$$u_{1,1}(\rho) = \sum_{i=0}^{\infty} a_i (1-\rho)^i$$

and

$$u_{1,2}(\rho) = (1-\rho)^{-k} \sum_{i=0}^{\infty} b_i (1-\rho)^i + C \log(1-\rho) u_{1,1}(\rho).$$

Again, $u_{1,2}$ does not belong to the Sobolev space $H^5_{\text{rad}}(\mathbb{B}^7)$ and we conclude that $u_1 \in C^\infty[0,1]$.

Thus, given $\lambda \in \mathbb{C} \setminus \{1\}$, we have found a nontrivial solution $u_1 \in C^\infty[0,1]$ to (2.11). This clearly contradicts Proposition 14 and so we obtain the desired result. ■

In fact, the eigenspace of 1 is one-dimensional as can be seen by the following proposition.

Proposition 16 *We have* $\ker(1 - \mathbf{L}) = \langle \mathbf{f}_1^* \rangle$.

Proof. The inclusion $\langle \mathbf{f}_1^* \rangle \subseteq \ker(1 - \mathbf{L})$ follows from $1 \in \sigma_p(\mathbf{L})$.

Direct calculation shows that $(1 - \mathbf{L})\mathbf{u} = 0$ is equivalent to the ODEs

$$-(1-\rho^2)u_1''(\rho) + \Big(-\frac{6}{\rho} + 8\rho + \rho(1-\rho^2)\tilde{V}(\rho)\Big)u_1'(\rho) + \Big(10 - V(\rho) - 2\rho^2\tilde{V}(\rho)\Big)u_1(\rho) = 0 \quad (2.19)$$

and

$$u_2(\rho) = \rho u_1'(\rho) + 2u_1(\rho)$$

for $\rho \in (0,1)$ where $\mathbf{u} = (u_1, u_2)$. By direct calculation, one can verify that

$$Q_1(\rho) := \frac{375 + 2125\rho^2 + 10425\rho^4 + 243\rho^6 + 6144\rho^5 \log(1-\rho) - 6144\rho^5 \log(1+\rho)}{3\rho^5\sqrt{5-\rho^2}(5+3\rho^2)}$$

is a solution of Equation (2.19). Furthermore, a direct calculation shows that $\{f^*_{1,1}, Q_1\}$ forms a fundamental system of Equation (2.19). Thus, the general solution of Equation (2.19) is given by

$$u(\rho) = C_1 f^*_{1,1}(\rho) + C_2 Q_1(\rho)$$

for constants $C_1, C_2, \in \mathbb{C}$. The general solution fails to be in the Sobolev space $H^5_{\text{rad}}(\mathbb{B}^7)$ unless $C_2 = 0$. Thus, $\ker(1 - \mathbf{L}) \subset \langle \mathbf{f}^*_1 \rangle$. ■

Remark 17 *The existence of the eigenvalue $\lambda = 1$ is precisely due to the fact that ψ^*_T represents a one-parameter family of solutions of Equation* (2.1)*. Indeed, a direct calculation shows that $\partial_T \psi^*_T$, properly rescaled and expressed in similarity coordinates, is precisely $f(\rho; 1)$ up to a constant multiple.*

Corollary 18 *We have*

$$\sigma(\mathbf{L}) \subseteq \{\lambda \in \mathbb{C} : \Re\lambda < 0\} \cup \{1\}.$$

Proof. Suppose there exists a $\lambda \in \sigma(\mathbf{L}) \setminus \{1\}$ with $\Re\lambda \geq 0$. From Equation (2.10), we see that $\lambda \notin \sigma(\mathbf{L}_0)$ and that $\mathbf{R}_{\mathbf{L}_0}(\lambda)$ exists. From the identity $\lambda - \mathbf{L} = [1 - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)](\lambda - \mathbf{L}_0)$, we see that $1 \in \sigma\big(\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)\big)$. Recall from Proposition 13 that $\mathbf{L}'$ is compact relative to $\mathbf{L}_0$. Thus, $\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)$ is compact from which it follows that $1 \in \sigma_p\big(\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)\big)$. Thus, there exists a nontrivial $\mathbf{f} \in \mathcal{H}$ such that $[1 - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)]\mathbf{f} = 0$. Consequently, $\mathbf{u} := \mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f} \neq 0$ satisfies $(\lambda - \mathbf{L})\mathbf{u} = 0$. Therefore, $\lambda \in \sigma_p(\mathbf{L})$ which contradicts Proposition 15. ■

Corollary 18 shows that 1 is an isolated eigenvalue. As a consequence, we can define the following Riesz projection.

Definition 19 *Let* $\gamma : [0, 2\pi] \to \mathbb{C}$ *be defined by* $\gamma(t) = 1 + \frac{1}{2}e^{it}$*. Then we set*

$$\mathbf{P} := \frac{1}{2\pi i} \int_\gamma \mathbf{R_L}(\lambda) d\lambda.$$

We have the following crucial properties of this Riesz projection.

Proposition 20 *The projection* $\mathbf{P}$ *commutes with the semigroup* $\big(\mathbf{S}(\tau)\big)_{\tau \geq 0}$ *and we have*

$$\operatorname{rg} \mathbf{P} = \langle \mathbf{f}_1^* \rangle.$$

In addition, we have

$$\mathbf{S}(\tau)\mathbf{P}\mathbf{f} = e^\tau \mathbf{P}\mathbf{f}, \quad \tau \geq 0. \tag{2.20}$$

Proof. By definition, $\mathbf{P}$ commutes with $\mathbf{L}$ and thus commutes with the semigroup $\mathbf{S}(\tau)$ (see [27]).

Next, we show that $\langle \mathbf{f}_1^* \rangle = \operatorname{rg} \mathbf{P}$. In fact, it suffices to show $\operatorname{rg} \mathbf{P} \subseteq \langle \mathbf{f}_1^* \rangle$ since the reverse inclusion follows from abstract theory. To see this, first observe that $\mathbf{P}$ decomposes the Hilbert space as $\mathcal{H} = \operatorname{rg} \mathbf{P} \oplus \ker \mathbf{P}$. The operator $\mathbf{L}$ is decomposed into the parts $\mathbf{L}_{\operatorname{rg} \mathbf{P}}$ and $\mathbf{L}_{\ker \mathbf{P}}$ respectively. The spectra of these operators are given by

$$\sigma(\mathbf{L}_{\ker \mathbf{P}}) = \sigma(\mathbf{L}) \setminus \{1\}, \quad \sigma(\mathbf{L}_{\operatorname{rg} \mathbf{P}}) = \{1\}.$$

As a consequence, we claim that $\operatorname{rank} \mathbf{P} := \dim \operatorname{rg} \mathbf{P} < \infty$. If this were not true, then by [27] p. 239 Theorem 5.28 we have that $1 \in \sigma_e(\mathbf{L})$. Furthermore, [27] p. 244 Theorem 5.35 shows that that $\sigma_e(\mathbf{L}) = \sigma_e(\mathbf{L} - \mathbf{L}')$ since $\mathbf{L}'$ is compact relative to $\mathbf{L}_0$. Clearly, $\sigma_e(\mathbf{L} - \mathbf{L}') = \sigma_e(\mathbf{L}_0) \subseteq \sigma(\mathbf{L}_0)$. Thus, we conclude that $1 \in \sigma(\mathbf{L}_0)$ which is clearly a contradiction.

Now, notice that the operator $1-\mathbf{L}_{\operatorname{rg}\mathbf{P}}$ acts on the finite-dimensional Hilbert space $\operatorname{rg}\mathbf{P}$ and, since $\sigma(\mathbf{L}_{\operatorname{rg}\mathbf{P}}) = \{1\}$, 0 is the only spectral point of $1-\mathbf{L}_{\operatorname{rg}\mathbf{P}}$. Thus, $1-\mathbf{L}_{\operatorname{rg}\mathbf{P}}$ is nilpotent, i.e., there exists $k \in \mathbb{N}$ such that

$$(1-\mathbf{L}_{\operatorname{rg}\mathbf{P}})^k \mathbf{u} = 0$$

for all $\mathbf{u} \in \operatorname{rg}\mathbf{P}$ where k is minimal. If $k=1$, then $\langle \mathbf{f}_1^* \rangle = \operatorname{rg}\mathbf{P}$ by Proposition 16.

Suppose $k \geq 2$. Then there exists $\mathbf{u} \in \operatorname{rg}\mathbf{P} \subseteq \mathcal{D}(\mathbf{L})$ such that $(1-\mathbf{L}_{\operatorname{rg}\mathbf{P}})\mathbf{u} \neq 0$ but $(1-\mathbf{L}_{\operatorname{rg}\mathbf{P}})^2\mathbf{u} = 0$. Thus $(1-\mathbf{L})\mathbf{u} = \alpha \mathbf{f}_1^*$ for some $\alpha \in \mathbb{C} \setminus \{0\}$. Without loss of generality, we set $\alpha = -1$. Then the first component of $\mathbf{u}$ solves the ODE

$$-(1-\rho^2)u_1''(\rho) + \Big(-\frac{6}{\rho} + 8\rho + \rho(1-\rho^2)\tilde{V}(\rho)\Big)u_1'(\rho) + \Big(10 - V(\rho) - 2\rho^2\tilde{V}(\rho)\Big)u_1(\rho) = G(\rho)$$

for $\rho \in (0,1)$ where

$$G(\rho) := \frac{35-3\rho^2}{\sqrt{5-\rho^2}\,\big(5+3\rho^2\big)^2}.$$

Recall that we have a fundamental system $\{f_{1,1}^*, Q_1\}$ of the homogeneous equation. Their Wronskian is given explicitly by

$$W(\rho) = \rho^{-6}(1-\rho^2)^{-1}(5-\rho^2)^{-1}(5+3\rho^2)^2.$$

By variation of parameters, the general solution of (2.3.3) can be expressed as

$$\begin{aligned} u_1(\rho) =& C_1 f_{1,1}^*(\rho) + C_2 Q_1(\rho) \\ & - Q_1(\rho)\int_0^\rho \frac{f_{1,1}^*(s)}{W(s)}\frac{G(s)}{1-s^2}ds + f_{1,1}^*(\rho)\int_0^\rho \frac{Q_1(s)}{W(s)}\frac{G(s)}{1-s^2}ds \end{aligned}$$

for some $C_1, C_2 \in \mathbb{C}$ and all $\rho \in (0,1)$. Explicitly, we find

$$\int_0^\rho \frac{f_{1,1}^*(s)}{W(s)}\frac{G(s)}{1-s^2}ds = \frac{\rho^7}{(5+3\rho^2)^4}.$$

Consequently, demanding $u_1 \in H^5_{\text{rad}}(\mathbb{B}^7)$ implies $C_2 = 0$. Thus, we are left with

$$u_1(\rho) = C_1 f^*_{1,1}(\rho) - Q_1(\rho)\frac{\rho^7}{(5+3\rho^2)^4} + f^*_{1,1}(\rho)\int_0^\rho \frac{Q_1(s)}{W(s)}\frac{G(s)}{1-s^2}ds$$

Upon further inspection, we observe that u_1 fails to be in $H^5_{\text{rad}}(\mathbb{B}^7)$ due to the logarithmic behavior of Q_1 near $\rho = 1$. We conclude that there is no such solution in $H^5_{\text{rad}}(\mathbb{B}^7)$. Thus, we must have $k = 1$.

Lastly, observe that Equation (2.20) follows from the facts that $\lambda = 1$ is an eigenvalue of $\mathbf{L}$ with eigenfunction $\mathbf{f}^*_1$ and $\operatorname{rg}\mathbf{P} = \langle \mathbf{f}^*_1 \rangle$. ■

2.4 Explicit Construction of the Resolvent

At this point, we are almost ready to improve the growth bound on the semigroup, $\big(\mathbf{S}(\tau)\big)_{\tau \geq 0}$. In light of the Gearhart-Prüss-Greiner theorem (p.302, Theorem 1.11 of [20]), this requires showing that the resolvent, $\mathbf{R_L}(\lambda)$, is uniformly bounded on $\mathbb{H}$. In fact, due to the eigenvalue $\lambda = 1$, this cannot be done in general. Instead, one can show that the reduced resolvent, $\mathbf{R_L}(\lambda)(\mathbf{I} - \mathbf{P})$, is uniformly bounded. From this, we can infer an improved growth bound on the reduced semigroup, namely $\big(\mathbf{S}(\tau)(\mathbf{I} - \mathbf{P})\big)_{\tau \geq 0}$. For compact potentials, one can appeal to the spectral mapping theorem of [24]. Clearly, our potential is not compact and so this is not available to us. Nevertheless, we aim to establish uniform boundedness by appealing to an explicit construction of the resolvent. In this section, we initiate this construction.

For the remainder of this chapter, we will always set $\varepsilon := \Re\lambda$ and $\omega := \Im\lambda$. First, observe that as a consequence of (2.9), standard semigroup theory allows us to conclude

$$\|\mathbf{R_L}(\lambda)\|_{\mathcal{H}} \leq \frac{M}{\varepsilon + \frac{1}{2} - M\|\mathbf{L}'\|_{\mathcal{H}}} \tag{2.21}$$

for $\varepsilon > -\frac{1}{2} + M\|\mathbf{L}'\|_{\mathcal{H}}$. Furthermore, for $\mathbf{f} \in C^\infty(0,1)^2 \cap \mathcal{H}$, $\mathbf{R_L}(\lambda)(\mathbf{I} - \mathbf{P})\mathbf{f}$ is the solution of a particular ODE. In light of this, we can represent $\mathbf{R_L}(\lambda)(\mathbf{I}-\mathbf{P})\mathbf{f}$ as an integral against some Green's function. For $|\omega| \gg 1$, this Green's function can be understood to leading order in ω as the Green's function one would obtain by representing $\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}$ in the same way. These observations motivate us to decompose $\mathbb{H}$ into three regions: $\mathbb{H} = S_1 \cup S_2 \cup S_3$ where

$$S_1 = \{\lambda \in \mathbb{C} : \varepsilon \geq M\|\mathbf{L}'\|_{\mathcal{H}}\},$$

$$S_2 = \{\lambda \in \mathbb{C} : 0 < \varepsilon < M\|\mathbf{L}'\|_{\mathcal{H}}, |\omega| < R\},$$

and

$$S_3 = \{\lambda \in \mathbb{C} : 0 < \varepsilon < M\|\mathbf{L}'\|_{\mathcal{H}}, |\omega| \geq R\},$$

for some $R > 0$ sufficiently large. On S_1, we immediately have uniform boundedness of the resolvent from Inequality (2.21). On S_2, we can use the fact that the reduced resolvent is continuous on $\overline{S_2}$ to conclude it is uniformly bounded there. Lastly, we can appeal to an explicit construction of the resolvent to show that it is uniformly bounded on S_3. The majority of the construction is devoted to proving that we can understand the Green's function associated to $\mathbf{R_L}(\lambda)(\mathbf{I}-\mathbf{P})\mathbf{f}$ as described in S_3. In fact, since $1 \notin S_3$, it suffices to show that $\mathbf{R_L}(\lambda)\mathbf{f}$ is uniformly bounded on S_3

Let $\mathbf{f} \in C^\infty(0,1)^2 \cap \mathcal{H}$ and $\lambda = \varepsilon + i\omega \in S_3$ for $R > 0$ to be determined. For convenience, set $\mathbf{u} = \mathbf{R_L}(\lambda)\mathbf{f}$. Equivalently, we have that the equation $(\lambda - \mathbf{L})\mathbf{u} = \mathbf{f}$ is satisfied and, using the explicit expressions, reads

$$\begin{cases} \rho u_1'(\rho) + (\lambda+1)u_1(\rho) - u_2(\rho) = f_1(\rho) \\ -u_1''(\rho) - \frac{6}{\rho}u_1'(\rho) + \rho u_2'(\rho) + (\lambda+4)u_2(\rho) - V(\rho)u_1(\rho) - \rho\tilde{V}(\rho)(\rho u_2(\rho) - u_1'(\rho)) = f_2(\rho). \end{cases}$$

The first equation allows to us express u_2 in terms of u_1 and f_1. Consequently, the second equation is equivalent to

$$\begin{aligned} -(1-\rho^2)u''(\rho) + \Big(-\frac{6}{\rho} + 2(\lambda+3)\rho + \rho(1-\rho^2)\tilde{V}(\rho)\Big)u'(\rho) \\ + \big((\lambda+1)(\lambda+4) - V(\rho) - (\lambda+1)\rho^2\tilde{V}(\rho)\big)u(\rho) = F_\lambda(\rho) - \rho^2\tilde{V}(\rho)f_1(\rho) \end{aligned} \tag{2.22}$$

where $u = u_1$ and

$$F_\lambda(\rho) := f_2(\rho) + \rho f_1'(\rho) + (\lambda+4)f_1(\rho). \tag{2.23}$$

Thus,

$$[\mathbf{R_L}(\lambda)\mathbf{f}]_1(\rho) = \int_0^1 G(\rho, s; \lambda)\big[F_\lambda(s) - s^2\tilde{V}(s)f_1(s)\big]ds$$

where G is a Green's function of (2.22).

2.4.1 Constructing the Green's Function

Now, we turn our attention toward constructing a Green's function G in terms of a fundamental system one could obtain for Equation (2.22) with $V = \tilde{V} = F_\lambda = 0$.

When studying the resolvent, we make extensive use of functions of symbol type. Let $I \subseteq \mathbb{R}$, $x_0 \in I$, and $\alpha \in \mathbb{R}$. We say $f \in C^\infty(I)$ is of symbol type and write $f(x) = \mathcal{O}((x-x_0)^\alpha)$ if $|f^{(n)}(x)| \lesssim_n |x-x_0|^{\alpha-n}$ for all $x \in I$ and all $n \in \mathbb{N}_0$. We refer the reader to [11] and [15] for the proofs of various properties of functions of symbol type that we will use throughout.

As a first step, we remove the first-order term in Equation (2.22) using the change of variables

$$v(\rho) = (5-\rho^2)^{\frac{1}{2}}(5+3\rho^2)^{-1}\rho^3(1-\rho^2)^{\frac{\lambda}{2}}u(\rho).$$

This yields the equation

$$v''(\rho) - \frac{6(1-\rho^2) - 2\rho^2(1-\rho^2) + \lambda(\lambda-2)\rho^2}{\rho^2(1-\rho^2)^2}v(\rho) = \frac{\hat{V}(\rho)}{1-\rho^2}v(\rho) \tag{2.24}$$

where

$$\begin{aligned}\hat{V}(\rho) &= \frac{1}{4}\big(-2\rho(1-\rho^2)\tilde{V}'(\rho) + \rho^2(1-\rho^2)\tilde{V}(\rho)^2 + 2(5\rho^2-7)\tilde{V}(\rho) - 4V(\rho)\big)\\ &= \frac{2(9\rho^4+102\rho^2-335)}{(5+3\rho^2)^2}.\end{aligned}$$

Observe that if $v(\cdot;\lambda)$ is a solution of Equation (2.24), then so is $v(\cdot;2-\lambda)$.

Now, we construct a fundamental system for Equation (2.24). An explicit fundamental system for Equation (2.24) with $\hat{V}=0$ is given by

$$\psi_1(\rho;\lambda) = \rho^{-2}(1+\rho)^{1-\frac{\lambda}{2}}(1-\rho)^{\frac{\lambda}{2}}c_+(\rho;\lambda)$$

$$\tilde{\psi}_1(\rho;\lambda) = \psi_1(\rho;2-\lambda) = \rho^{-2}(1-\rho)^{1-\frac{\lambda}{2}}(1+\rho)^{\frac{\lambda}{2}}c_-(\rho;\lambda)$$

where

$$c_+(\rho;\lambda) := 3 + 3(\lambda-1)\rho + (\lambda-1)^2\rho^2 + (\lambda-1)\rho^3 \tag{2.25}$$

$$c_-(\rho;\lambda) := 3 - 3(\lambda-1)\rho + (\lambda-1)^2\rho^2 - (\lambda-1)\rho^3. \tag{2.26}$$

A direct calculation shows that their Wronskian is precisely

$$\begin{aligned}W\big(\psi_1(\cdot;\lambda),\tilde{\psi}_1(\cdot;\lambda)\big)(\rho) =&2(\lambda+2)\lambda(\lambda-1)(\lambda-2)(\lambda-4)\\ =&:W(\lambda).\end{aligned} \tag{2.27}$$

Note that neither solution is well-behaved near $\rho = 0$. A third solution which is well-behaved near $\rho = 0$ is given by

$$\psi_0(\rho;\lambda) := \psi_1(\rho;\lambda) - \tilde{\psi}_1(\rho;\lambda).$$

In what follows, we always impose $0 < \varepsilon < M\|\mathbf{L}'\|_{\mathcal{H}}$. As we proceed with this construction of a fundamental system for Equation (2.24), we will impose conditions on ω to be stated later.

By reduction of order, we obtain a fourth solution according to the following lemma.

Lemma 21 *Let $\lambda = \varepsilon + i\omega$ with $\omega \in \mathbb{R}$. There exists $\delta_0 > 0$ such that*

$$\tilde{\psi}_0(\rho;\lambda) = \psi_0(\rho;\lambda)\int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2} ds$$

is a solution of Equation (2.24) *with $\hat{V} = 0$ for $\rho \in (0, \delta_0\langle\omega\rangle^{-1}]$. Moreover, it satisfies*

$$\tilde{\psi}_0(\rho;\lambda) = -3W(\lambda)^{-1}\rho^{-2}[1 + \mathcal{O}(\rho^2\langle\omega\rangle^2)]$$

for all $\rho \in (0, \delta_0\langle\omega\rangle^{-1}]$. We also have

$$W(\psi_0(\cdot;\lambda), \tilde{\psi}_0(\cdot;\lambda)) = -1.$$

Proof. Taylor expansion yields

$$\psi_0(\rho;\lambda) = -\frac{1}{15}W(\lambda)\rho^3[1 + \mathcal{O}(\rho^2\langle\omega\rangle^2)].$$

Thus, there is a $\delta_0 > 0$ sufficiently small so that $\psi_0(\cdot;\lambda) \neq 0$ on $(0, \delta_0\langle\omega\rangle^{-1}]$. Lemma A.3 of [15] implies that

$$\psi_0(\rho;\lambda)^{-1} = -15W(\lambda)^{-1}\rho^{-3}[1 + \mathcal{O}(\rho^2\langle\omega\rangle^2)],$$

see [15]. A straightforward calculation shows that $\tilde{\psi}_0(\cdot;\lambda)$ is indeed a solution of (2.24) with $\hat{V}=0$. Furthermore,

$$\begin{aligned}\frac{\int_\rho^{\delta_0\langle\omega\rangle^{-1}}\psi_0(s;\lambda)^{-2}ds}{45\rho^{-5}W(\lambda)^{-2}}=&\frac{1}{45}\rho^5W(\lambda)^2\int_\rho^{\delta_0\langle\omega\rangle^{-1}}15^2W(\lambda)^{-2}s^{-6}[1+\mathcal{O}(s^2\langle\omega\rangle^2)]ds\\=&\rho^5\left[s^{-5}\right]_{s=\rho}^{s=\delta_0\langle\omega\rangle^{-1}}+\mathcal{O}(\rho^2\langle\omega\rangle^2)\\=&1+\mathcal{O}(\rho^2\langle\omega\rangle^2).\end{aligned}$$

Lastly, the value of the Wronskian is also a straightforward calculation. ■

Now, we obtain a fundamental system for Equation (2.24) near $\rho=0$ by perturbing the fundamental system $\{\psi_0(\cdot;\lambda),\tilde{\psi}_0(\cdot;\lambda)\}$ of Equation (2.24) with $\hat{V}=0$. This will be achieved in two steps. First, we construct a perturbation of $\psi_0(\cdot;\lambda)$, namely $v_0(\cdot;\lambda)$, by Volterra iterations, i.e., reinterpreting Equation (2.24) as an integral equation of Volterra type. Once we have this solution, we again use reduction of order to produce a second solution $\tilde{v}_0(\cdot;\lambda)$ and prove that it is a perturbation of $\tilde{\psi}_0(\cdot;\lambda)$. For the convenience of the reader, we recall here the standard existence and uniqueness theorem for Volterra integral equations stated in the form we will use. A proof can be found in [33]:

Lemma 22 ([33], Lemma 2.4) *Let $g\in L^\infty(0,1)$ and $K:(0,1)^2\to\mathbb{C}$ satisfy*

$$\mu:=\int_0^1\sup_{0<x<s}|K(x,s)|ds<\infty.$$

Then there exists a unique $f\in L^\infty(0,1)$ solving the equation

$$f(x)=g(x)+\int_0^x K(x,s)f(s)ds.$$

Furthermore, one has the bound

$$\|f\|_{L^\infty(0,1)}\le e^\mu\|g\|_{L^\infty(0,1)}.$$

With this lemma in hand, we obtain a solution of Equation (2.24) via Volterra iterations according to the following lemma.

Lemma 23 *Let $\lambda = \varepsilon + i\omega$ with $|\omega| \geq 1$. There exists $\delta_0 > 0$ such that Equation* (2.24) *has a solution of the form*

$$v_0(\rho; \lambda) = \psi_0(\rho; \lambda) h_0(\rho; \lambda)$$

for all $\rho \in (0, \delta_0 \langle \omega \rangle^{-1}]$ where $h_0(\rho; \lambda) = 1 + \mathcal{O}(\rho^2 \langle \omega \rangle^0)$.

Proof. Using the fundamental system $\{\psi_0(\cdot; \lambda), \psi_1(\cdot; \lambda)\}$, variation of parameters suggests that a solution v_0 should satisfy the integral equation

$$\begin{aligned} v_0(\rho; \lambda) =& \psi_0(\rho; \lambda) - \frac{\psi_0(\rho; \lambda)}{W(\lambda)} \int_0^\rho \psi_1(s; \lambda) \frac{\hat{V}(s)}{1 - s^2} v_0(s; \lambda) ds \\ &+ \frac{\psi_1(\rho; \lambda)}{W(\lambda)} \int_0^\rho \psi_0(s; \lambda) \frac{\hat{V}(s)}{1 - s^2} v_0(s; \lambda) ds. \end{aligned}$$

With $\delta_0 > 0$ as in the proof of Lemma 21, $\psi_0(\cdot; \lambda) \neq 0$ on $(0, \delta_0 \langle \omega \rangle^{-1}]$. Consequently, we set $h_0(\cdot; \lambda) = \frac{v_0(\cdot; \lambda)}{\psi_0(\cdot; \lambda)}$ to obtain the equation

$$h_0(\rho; \lambda) = 1 + \int_0^\rho K(\rho, s \, \lambda) h_0(s; \lambda) ds \tag{2.28}$$

where

$$K(\rho, s; \lambda) = \frac{1}{W(\lambda)} \left(\frac{\psi_1(\rho; \lambda)}{\psi_0(\rho; \lambda)} \psi_0(s; \lambda)^2 - \psi_0(s; \lambda) \psi_1(s; \lambda) \right) \frac{\hat{V}(s)}{1 - s^2}.$$

Using the explicit expression for $\psi_1(\cdot; \lambda)$, the symbol estimates for $\psi_0(\cdot; \lambda)$ and $\psi_0(\cdot; \lambda)^{-1}$, and the fact that $s \leq \rho$, we obtain the estimate

$$|K(\rho, s; \lambda)| \lesssim (\rho^{-2} + s^{-2}) s^3 \leq s$$

for all $0 \leq s \leq \rho \leq \delta_0 \langle \omega \rangle^{-1}$. Thus,

$$\int_0^{\delta_0 \langle \omega \rangle^{-1}} \sup_{\rho \in (0, \delta_0 \langle \omega \rangle^{-1})} |K(\rho, s; \lambda)| ds \lesssim \langle \omega \rangle^{-2}.$$

By Lemma 22, Equation (2.28) has a unique solution and satisfies the estimate

$$|h_0(\rho;\lambda) - 1| \lesssim \int_0^\rho |K(\rho, s;\lambda)| ds \lesssim \rho^2.$$

Consequently, we obtain the estimate

$$h_0(\rho;\lambda) = 1 + O(\rho^2).$$

Since all terms in (2.28) behave like symbols under differentiation, we obtain the symbol estimate

$$h_0(\rho;\lambda) = 1 + \mathcal{O}(\rho^2\langle\omega\rangle^0).$$

■

With this solution, we can obtain a second solution on an interval $(0, \delta_0\langle\omega\rangle^{-1}]$ via reduction of order.

Lemma 24 *Let $\lambda = \varepsilon + i\omega$ with $|\omega| \geq 1$. There exists $\delta_0 > 0$*

$$\tilde{v}_0(\rho;\lambda) = v_0(\rho;\lambda) \int_\rho^{\delta_0\langle\omega\rangle^{-1}} v_0(s;\lambda)^{-2} ds$$

is a solution of Equation (2.24) *for $\rho \in (0, \delta_0\langle\omega\rangle^{-1}]$. Moreover, $\tilde{v}_0$ is of the form*

$$\tilde{v}_0(\rho;\lambda) = \tilde{\psi}_0(\rho;\lambda)\tilde{h}_0(\rho;\lambda)$$

for all $\rho \in (0, \delta_0\langle\omega\rangle^{-1}]$ where $\tilde{h}_0(\rho;\lambda) = 1 + \mathcal{O}(\rho^2\langle\omega\rangle^0)$. We also have

$$W(v_0(\cdot;\lambda), \tilde{v}_0(\cdot;\lambda)) = -1.$$

Proof. Based on the previous symbol estimates, we can choose $\delta_0 > 0$ small enough so that $v_0(\cdot;\lambda)$, $\psi_0(\cdot;\lambda)$, and $\tilde{\psi}_0(\cdot;\lambda)$ have no zeros in $(0, \delta_0\langle\omega\rangle^{-1}]$. Consequently, $\tilde{v}_0(\cdot;\lambda)$ is well-defined on the interval $(0, \delta_0\langle\omega\rangle^{-1}]$. A direct calculation shows that $\tilde{v}_0(\cdot;\lambda)$ is indeed a solution of Equation (2.24).

Using Lemmas 21 and 23, we calculate

$$\begin{aligned}
\frac{\tilde{v}_0(\rho;\lambda)}{\tilde{\psi}_0(\rho;\lambda)} - 1 =& \frac{1}{\tilde{\psi}_0(\rho;\lambda)}\Big(\tilde{v}_0(\rho;\lambda) - \tilde{\psi}_0(\rho;\lambda)\Big) \\
=& \frac{1}{\tilde{\psi}_0(\rho;\lambda)}\Big(v_0(\rho;\lambda)\int_\rho^{\delta_0\langle\omega\rangle^{-1}} v_0(s;\lambda)^{-2}ds - \psi_0(\rho;\lambda)\int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2}ds\Big) \\
=& \frac{\psi_0(\rho;\lambda)}{\tilde{\psi}_0(\rho;\lambda)}\Big([1+\mathcal{O}(\rho^2\langle\omega\rangle^0)]\int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2}[1+\mathcal{O}(s^2\langle\omega\rangle^0)]ds \\
& - \int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2}ds\Big) \\
=& \frac{\psi_0(\rho;\lambda)}{\tilde{\psi}_0(\rho;\lambda)}\Big(\mathcal{O}(\rho^2\langle\omega\rangle^0)\int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2}[1+\mathcal{O}(s^2\langle\omega\rangle^0)]ds \\
& + \int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2}\mathcal{O}(s^2\langle\omega\rangle^0)ds\Big).
\end{aligned}$$

Now, based on the symbol estimates for $\psi_0(\cdot;\lambda)$ and $\tilde{\psi}_0(\cdot;\lambda)$, we find

$$\frac{\psi_0(\rho;\lambda)}{\tilde{\psi}_0(\rho;\lambda)} = \mathcal{O}(\rho^3\langle\omega\rangle^5)\mathcal{O}(\rho^2\langle\omega\rangle^5) = \mathcal{O}(\rho^5\langle\omega\rangle^{10}).$$

Furthermore,

$$\begin{aligned}
\mathcal{O}(\rho^2\langle\omega\rangle^0)\int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2}[1+\mathcal{O}(s^2\langle\omega\rangle^0)]ds &= \mathcal{O}(\rho^2\langle\omega\rangle^0)\int_\rho^{\delta_0\langle\omega\rangle^{-1}} \mathcal{O}(s^{-6}\langle\omega\rangle^{-10})ds \\
&= \mathcal{O}(\rho^{-3}\langle\omega\rangle^{-10})
\end{aligned}$$

and, by a similar calculation, we have

$$\int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2}\mathcal{O}(s^2\langle\omega\rangle^0)ds = \mathcal{O}(\rho^{-3}\langle\omega\rangle^{-10}).$$

Thus, we obtain the symbol estimate

$$\frac{\tilde{v}_0(\rho;\lambda)}{\tilde{\psi}_0(\rho;\lambda)} - 1 = \mathcal{O}(\rho^2\langle\omega\rangle^0).$$

Lastly, the Wronskian follows from a direct calculation. ■

At this point, we have a fundamental system, $\{v_0(\cdot;\lambda), \tilde{v}_0(\cdot;\lambda)\}$, of Equation (2.24) which is defined on the interval $(0, \delta_0\langle\omega\rangle^{-1}]$ and is a perturbation of the fundamental system, $\{\psi_0(\cdot;\lambda), \tilde{\psi}_0(\cdot;\lambda)\}$, of Equation (2.24) with $\hat{V} = 0$. Next, we begin an argument which allows us to perturb the fundamental system $\{\psi_1(\cdot;\lambda), \tilde{\psi}_1(\cdot;\lambda)\}$ to a fundamental system of Equation (2.24) defined on an interval away from $\rho = 0$. We must handle the cases $0 < \varepsilon \leq \frac{3}{2}$ and $\frac{3}{2} < \varepsilon < M\|\mathbf{L}'\|$ separately due to the zeros of $\psi_1(\cdot;\lambda)$ and $\tilde{\psi}_1(\cdot;\lambda)$. A complete argument will appear in a forthcoming paper where we also obtain uniform bounds on the resolvent.

Lemma 25 *Let $\lambda = \varepsilon + i\omega$ with $|\omega| \geq 1$ and $\varepsilon \in (0, \frac{3}{2}]$. There exists $\delta_1 > 0$ such that Equation* (2.24) *has a solution of the form*

$$\tilde{v}_1(\rho;\lambda) = \tilde{\psi}_1(\rho;\lambda)\tilde{h}_1(\rho;\lambda),$$

with

$$\tilde{h}_1(\rho;\lambda) = 1 + O\big((1-\rho)\langle\omega\rangle^{-1}\big),\ |\tilde{h}_1^{(j)}(\rho;\lambda)| \lesssim \langle\omega\rangle^{-1},\ j = 1,2,3,4,5$$

for all $\rho \in [\delta_1\langle\omega\rangle^{-1}, 1)$.

Proof. Again, variation of parameters suggests that a solution of Equation (2.24) should satisfy

$$\begin{aligned}\tilde{v}_1(\rho;\lambda) =& \tilde{\psi}_1(\rho;\lambda) + \frac{\tilde{\psi}_1(\rho;\lambda)}{W(\lambda)}\int_\rho^{\rho_1} \psi_1(s;\lambda)\frac{\hat{V}(s)}{1-s^2}\tilde{v}_1(s;\lambda)ds \\ &- \frac{\psi_1(\rho;\lambda)}{W(\lambda)}\int_\rho^{\rho_1} \tilde{\psi}_1(s;\lambda)\frac{\hat{V}(s)}{1-s^2}\tilde{v}_1(s;\lambda)ds\end{aligned}$$

for some ρ_1 to be determined. A direct calculation shows that for $\varepsilon \in (0, \frac{3}{2}]$, $\tilde{\psi}_1(\cdot;\lambda) \neq 0$ on the interval $(0,1)$. Thus, we set $\tilde{h}_1(\cdot;\lambda) = \frac{\tilde{v}_1(\cdot;\lambda)}{\tilde{\psi}_1(\cdot;\lambda)}$ and obtain the Volterra integral equation

$$\tilde{h}_1(\rho;\lambda) = 1 + \int_\rho^{\rho_1} K(\rho,s;\lambda)\tilde{h}_1(s;\lambda)ds$$

where

$$K(\rho,s;\lambda) = \frac{1}{W(\lambda)}\left(\tilde{\psi}_1(s;\lambda)\psi_1(s;\lambda) - \frac{\psi_1(\rho;\lambda)}{\tilde{\psi}_1(\rho;\lambda)}\tilde{\psi}_1(s;\lambda)^2\right)\frac{\hat{V}(s)}{1-s^2}.$$

Using the explicit formulas, we immediately obtain the estimate

$$|\tilde{\psi}_1(s;\lambda)\psi_1(s;\lambda)| \lesssim (1-\rho)^{\frac{1}{2}}(1-s)^{\frac{1}{2}}(s^{-4} + \langle\omega\rangle^2 s^{-2} + \langle\omega\rangle^4)$$

for $\rho \leq s < 1$. Similarly, we have the estimate

$$\begin{aligned}\left|\frac{\psi_1(\rho;\lambda)}{\tilde{\psi}_1(\rho;\lambda)}\tilde{\psi}_1(s;\lambda)^2\right| &\lesssim \left|\frac{c_+(\rho;\lambda)}{c_-(\rho;\lambda)}\right|(1-\rho)^{-1+\varepsilon}(1-s)^{2-\varepsilon}|c_-(s;\lambda)|^2 \\ &\lesssim (1-\rho)^{\frac{1}{2}}(1-s)^{\frac{1}{2}}(s^{-4} + \langle\omega\rangle^2 s^{-2} + \langle\omega\rangle^4)\end{aligned}$$

for $\varepsilon \in (0, \frac{3}{2}]$ and $\rho \leq s < 1$. Thus, we have the estimate

$$|K(\rho,s;\lambda)| \lesssim (1-\rho)^{\frac{1}{2}}(1-s)^{-\frac{1}{2}}\langle\omega\rangle^{-1}$$

for $\varepsilon \in (0, \frac{3}{2}]$ and $\rho \leq s < 1$. We can now set $\rho_1 = 1$ and obtain

$$\int_{\delta_1\langle\omega\rangle^{-1}}^1 \sup_{\rho\in(\delta_1\langle\omega\rangle^{-1},1)} |K(\rho,s;\lambda)|ds \lesssim \langle\omega\rangle^{-1}$$

for some sufficiently small $\delta_1 > 0$. By Lemma 22, Equation (2.24) has a unique solution $\tilde{h}_1(\cdot;\lambda)$ satisfying

$$|\tilde{h}_1(\rho;\lambda) - 1| \lesssim \int_\rho^1 |K(\rho,s;\lambda)|ds \lesssim (1-\rho)\langle\omega\rangle^{-1}.$$

Differentiating Equation (2.4.1) yields

$$\tilde{h}_1'(\rho;\lambda) = \int_\rho^1 \partial_\rho K(\rho,s;\lambda)\tilde{h}_1(s;\lambda)ds$$

noting the fact that $K(\rho,\rho;\lambda)=0$. Explicitly, we have

$$\partial_\rho K(\rho,s;\lambda) = -\frac{1}{W(\lambda)}\partial_\rho\left(\frac{\psi_1(\rho;\lambda)}{\tilde\psi_1(\rho;\lambda)}\right)\tilde\psi_1(s;\lambda)^2\frac{\hat V(s)}{1-s^2}.$$

Now, note the identity

$$\frac{\tilde\psi_1(s;\lambda)^2}{1-s^2} = -sc_-(s;\lambda)\partial_s\left(\frac{(1-s)^{2-\lambda}(1+s)^\lambda\big(3-2(\lambda-1)s+3s^2\big)}{4s^4}\right).$$

An integration by parts yields

$$\tilde h_1'(\rho;\lambda) = g_1(\rho;\lambda) + \int_\rho^1 K_1(\rho,s;\lambda)\tilde h_1'(\rho,s;\lambda)ds$$

where

$$\begin{aligned}
g_1(\rho;\lambda) :=& -\frac{1}{W(\lambda)}\partial_\rho\left(\frac{\psi_1(\rho;\lambda)}{\tilde\psi_1(\rho;\lambda)}\right)\Bigg(\frac{(1-\rho)^{2-\lambda}(1+\rho)^\lambda\big(3-2(\lambda-1)\rho+3\rho^2\big)}{4\rho^4} \\
&\times \rho c_-(\rho;\lambda)\hat V(\rho)\tilde h_1(\rho;\lambda) \\
&+ \int_\rho^1 \frac{(1-s)^{2-\lambda}(1+s)^\lambda\big(3-2(\lambda-1)s+3s^2\big)}{4s^3}c_-(s;\lambda)\hat V'(s)\tilde h_1(s;\lambda)ds \\
&+ \frac{3(\lambda-1)(\lambda^2-2\lambda+6)}{4(\lambda-3)}\left(\frac{1-\rho}{1+\rho}\right)^{3-\lambda}\rho^{-1}(1+\rho)^4\hat V(\rho)\tilde h_1(\rho;\lambda) \\
&+ \frac{3(\lambda-1)(\lambda^2-2\lambda+6)}{4(\lambda-3)}\int_\rho^1\left(\frac{1-s}{1+s}\right)^{3-\lambda}\partial_s\Big(s^{-1}(1+s)^4\hat V(s)\Big)\tilde h_1(s;\lambda)ds \\
&+ \frac{1}{4}\int_\rho^1\left(\frac{1-s}{1+s}\right)^{2-\lambda}s^{-4}(1+s)^2 \\
&\times\Big(9-24(\lambda-1)s+3(7\lambda^2-14\lambda+10)s^2+17(\lambda-1)^2s^4-12(\lambda-1)s^5\Big) \\
&\times\hat V(s)\tilde h_1(s;\lambda)ds\Bigg)
\end{aligned}$$

and

$$\begin{aligned}
K_1(\rho,s;\lambda) =& -\frac{1}{W(\lambda)}\partial_\rho\left(\frac{\psi_1(\rho;\lambda)}{\tilde\psi_1(\rho;\lambda)}\right)\frac{(1-s)^{2-\lambda}(1+s)^\lambda\big(3-2(\lambda-1)s+3s^2\big)}{4s^3}c_-(s;\lambda)\hat V(s) \\
&- \frac{3(\lambda-1)(\lambda^2-2\lambda+6)}{4(\lambda-3)W(\lambda)}\partial_\rho\left(\frac{\psi_1(\rho;\lambda)}{\tilde\psi_1(\rho;\lambda)}\right)\left(\frac{1-s}{1+s}\right)^{3-\lambda}s^{-1}(1+s)^4\hat V(s).
\end{aligned}$$

For $|\omega| \geq 1$ and $\rho \in [\delta_1 \langle \omega \rangle^{-1}, 1)$, we obtain the estimates

$$|g_1(\rho; \lambda)| \lesssim \langle \omega \rangle^{-1}$$

and

$$|K_1(\rho, s; \lambda)| \lesssim \langle \omega \rangle^{-1}.$$

By Lemma 22, we obtain the estimate

$$|\tilde{h}_1'(\rho; \lambda)| \lesssim \langle \omega \rangle^{-1}$$

for $\varepsilon \in (0, \frac{3}{2}]$, $|\omega| \geq 1$, and $\rho \in [\delta_1 \langle \omega \rangle^{-1}, 1)$. We can perform similar calculations four more times to obtain

$$|\tilde{h}_1^{(j)}(\rho; \lambda)| \lesssim \langle \omega \rangle^{-1}$$

for $j = 2, 3, 4, 5$, $\varepsilon \in (0, \frac{3}{2}]$, $|\omega| \geq 1$, and $\rho \in [\delta_1 \langle \omega \rangle^{-1}, 1)$. ■

For completeness, we state the properties of the solution for $\varepsilon \in [\frac{3}{2}, M\|\mathbf{L}'\|_{\mathcal{H}})$ but do not prove it. The proof is nearly identical to the previous.

Lemma 26 *Let $\lambda = \varepsilon + i\omega$ with $|\omega| \geq 1$ and $\varepsilon \in [\frac{3}{2}, M\|\mathbf{L}'\|_{\mathcal{H}})$. There exists $\delta_1 > 0$ such that Equation* (2.24) *has a solution of the form*

$$v_1(\rho; \lambda) = \psi_1(\rho; \lambda) h_1(\rho; \lambda),$$

with

$$h_1(\rho; \lambda) = 1 + O\big((1-\rho)\langle \omega \rangle^{-1}\big),\ |h_1^{(j)}(\rho; \lambda)| \lesssim \langle \omega \rangle^{-1},\ j = 1, 2, 3, 4, 5$$

for all $\rho \in [\delta_1 \langle \omega \rangle^{-1}, 1)$.

Now, we take $\delta_1 \leq \delta_0$ so that the interval on which the fundamental system $\{v_0(\cdot;\lambda), \tilde{v}_0(\cdot;\lambda)\}$ is defined overlaps with the interval on which the solutions $v_1(\cdot;\lambda)$ and $\tilde{v}_1(\cdot;\lambda)$ are defined. As a consequence, we can extend the solutions $v_1(\cdot;\lambda)$ and $\tilde{v}_1(\cdot;\lambda)$ to the interval $(0, \delta_0\langle\omega\rangle^{-1}]$ as follows.

Lemma 27 *The solutions v_1, $\tilde{v}_1$ can be extended to $\rho \in (0, \delta_0\langle\omega\rangle^{-1}]$, and are of the form*

$$v_1(\rho;\lambda) = \psi_1(\rho;\lambda)h_2(\rho;\lambda),$$

$$\tilde{v}_1(\rho;\lambda) = \tilde{\psi}_1(\rho;\lambda)\tilde{h}_2(\rho;\lambda),$$

where

$$h_2(\rho;\lambda) = [1 + \mathcal{O}(\langle\omega\rangle^{-1}) + \mathcal{O}(\rho^2\langle\omega\rangle^0)]$$

and

$$\tilde{h}_2(\rho;\lambda) = [1 + \mathcal{O}(\langle\omega\rangle^{-1}) + \mathcal{O}(\rho^2\langle\omega\rangle^0)]$$

for all $\rho \in (0, \delta_0\langle\omega\rangle^{-1}]$, $|\omega| \geq 1$.

Proof. For $\lambda \in \mathbb{C}$ as above and $\rho \in (0, \delta_0\langle\omega\rangle^{-1}]$, there exist $a(\lambda), b(\lambda) \in \mathbb{C}$ such that

$$v_1(\rho;\lambda) = a(\lambda)v_0(\rho;\lambda) + b(\lambda)\tilde{v}_0(\rho;\lambda),$$

where

$$a(\lambda) = \frac{W\big(v_1(\cdot;\lambda), \tilde{v}_0(\cdot;\lambda)\big)}{W\big(v_0(\cdot;\lambda), \tilde{v}_0(\cdot;\lambda)\big)} = -W\big(v_1(\cdot;\lambda), \tilde{v}_0(\cdot;\lambda)\big)$$

and

$$b(\lambda) = -\frac{W\big(v_1(\cdot;\lambda), v_0(\cdot;\lambda)\big)}{W\big(v_0(\cdot;\lambda), \tilde{v}_0(\cdot;\lambda)\big)} = W\big(v_1(\cdot;\lambda), v_0(\cdot;\lambda)\big).$$

Note that $\delta_1 \leq \delta_0$ and so the expression for $\tilde{v}_1$ obtained in Lemma 25 is valid at the point $\rho = \delta_0\langle\omega\rangle^{-1}$. Furthermore, the Wronskian of any two solutions must be constant and so we can compute $W\big(v_1(\cdot;\lambda),\tilde{v}_0(\cdot;\lambda)\big)$ and $W\big(v_1(\cdot;\lambda),v_0(\cdot;\lambda)\big)$ by evaluating them at $\rho=\delta_0\langle\omega\rangle^{-1}$. From Lemmas 25 and 24, we have

$$\begin{aligned}
W\big(v_1(\cdot;\lambda),\tilde{v}_0(\cdot;\lambda)\big) =&W\big(\psi_1(\cdot;\lambda)h_1(\cdot;\lambda),\tilde{\psi}_0(\cdot;\lambda)[1+\mathcal{O}(\rho^2\langle\omega\rangle^0)]\big)\\
=&W\big(\psi_1(\cdot;\lambda),\tilde{\psi}_0(\cdot;\lambda)\big)h_1(\delta_0\langle\omega\rangle^{-1};\lambda)[1+\mathcal{O}(\langle\omega\rangle^{-2})]\\
&+\psi_1(\delta_0\langle\omega\rangle^{-1};\lambda)\tilde{\psi}_0(\delta_0\langle\omega\rangle^{-1};\lambda)\big)O(\langle\omega\rangle^{-1})\\
=&W\big(\psi_1(\cdot;\lambda),\tilde{\psi}_0(\cdot;\lambda)\big)[1+O(\langle\omega\rangle^{-1})]+O(\langle\omega\rangle^{-2})
\end{aligned}$$

where we used the fact that $\psi_1(\delta_0\langle\omega\rangle^{-1};\lambda)=\mathcal{O}(\langle\omega\rangle^2)$ and $\tilde{\psi}_0(\delta_0\langle\omega\rangle^{-1};\lambda)\big)=\mathcal{O}(\langle\omega\rangle^{-3})$ which both follow from the explicit expressions and Lemma 21. Explicitly, we have

$$W\big(\psi_1(\cdot;\lambda),\tilde{\psi}_0(\cdot;\lambda)\big)=-W(\lambda)\int_\rho^{\delta_0\langle\omega\rangle^{-1}}\psi_0(s;\lambda)^{-2}ds-\psi_1(\rho;\lambda)\psi_0(\rho;\lambda)^{-1}.$$

Again, since the Wronskian of any two solutions is constant, evaluation at $\rho=\delta_0\langle\omega\rangle^{-1}$ yields

$$W\big(\psi_1(\cdot;\lambda),\tilde{\psi}_0(\cdot;\lambda)\big)=-\psi_1(\delta_0\langle\omega\rangle^{-1};\lambda)\psi_0(\delta_0\langle\omega\rangle^{-1};\lambda)^{-1}=\mathcal{O}(\langle\omega\rangle^0)$$

where we have used $\psi_0(\delta_0\langle\omega\rangle^{-1};\lambda)^{-1}=\mathcal{O}(\langle\omega\rangle^{-2})$. Thus, we have

$$\begin{aligned}
a(\lambda)=&-W\big(v_1(\cdot;\lambda),\tilde{v}_0(\cdot;\lambda)\big)\\
=&\left(W(\lambda)\int_\rho^{\delta_0\langle\omega\rangle^{-1}}\psi_0(s;\lambda)^{-2}ds+\psi_1(\rho;\lambda)\psi_0(\rho;\lambda)^{-1}\right)[1+O(\langle\omega\rangle^{-1})]
\end{aligned}$$

for $\rho \in (0, \delta_0\langle\omega\rangle^{-1}]$. Similarly,

$$
\begin{aligned}
b(\lambda) =& W\big(v_1(\cdot;\lambda), \tilde{v}_0(\cdot;\lambda)\big) \\
=& W\big(\psi_1(\cdot;\lambda)h_1(\cdot;\lambda), \psi_0(\cdot;\lambda)[1+\mathcal{O}(\rho^2\langle\omega\rangle^0)]\big) \\
=& W\big(\psi_1(\cdot;\lambda), \psi_0(\cdot;\lambda)\big)\gamma_1(\delta_0\langle\omega\rangle^{-1};\lambda)[1+\mathcal{O}(\langle\omega\rangle^{-2})] \\
& + \psi_1(\delta_0\langle\omega\rangle^{-1};\lambda)\psi_0(\delta_0\langle\omega\rangle^{-1};\lambda)\big)O(\langle\omega\rangle^{-1}) \\
=& -W(\lambda)[1+O(\langle\omega\rangle^{-1})]
\end{aligned}
$$

where we have used $\psi_0(\delta_0\langle\omega\rangle^{-1};\lambda) = \mathcal{O}(\langle\omega\rangle^2)$ and $W(\lambda) = \mathcal{O}(\langle\omega\rangle^5)$. Thus, we have

$$
\begin{aligned}
v_1(\rho;\lambda) =& v_0(\rho;\lambda)\bigg(W(\lambda)\int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2}ds + \psi_1(\rho;\lambda)\psi_0(\rho;\lambda)^{-1}\bigg)[1+O(\langle\omega\rangle^{-1})] \\
& - \tilde{v}_0(\rho;\lambda)W(\lambda)[1+O(\langle\omega\rangle^{-1})] \\
=& \psi_0(\rho;\lambda)\bigg(W(\lambda)\int_\rho^{\delta_0\langle\omega\rangle^{-1}} \psi_0(s;\lambda)^{-2}ds + \psi_1(\rho;\lambda)\psi_0(\rho;\lambda)^{-1}\bigg) \\
& \times [1+\mathcal{O}(\rho^2\langle\omega\rangle^0)][1+O(\langle\omega\rangle^{-1})] \\
& - \tilde{\psi}_0(\rho;\lambda)W(\lambda)[1+\mathcal{O}(\rho^2\langle\omega\rangle^0)][1+O(\langle\omega\rangle^{-1})] \\
=& W(\lambda)\tilde{\psi}_0(\rho;\lambda)O(\langle\omega\rangle^{-1}) + \psi_1(\rho;\lambda)[1+\mathcal{O}(\rho^2\langle\omega\rangle^0)][1+O(\langle\omega\rangle^{-1})].
\end{aligned}
$$

Now, we use $\tilde{\psi}_0(\rho;\lambda) = \mathcal{O}(\rho^{-2}\langle\omega\rangle^{-5})$ and $\psi_1(\rho;\lambda)^{-1} = \mathcal{O}(\rho^2\langle\omega\rangle^0)$ to obtain

$$
\begin{aligned}
\frac{v_1(\rho;\lambda)}{\psi_1(\rho;\lambda)} =& \frac{\tilde{\psi}_0(\rho;\lambda)}{\psi_1(\rho;\lambda)}\mathcal{O}(\langle\omega\rangle^5)O(\langle\omega\rangle^{-1}) + [1+O(\langle\omega\rangle^{-1})] \\
=& 1+O(\langle\omega\rangle^{-1}).
\end{aligned}
$$

The second solution is obtained via the transformation $\lambda \mapsto 2-\lambda$. ■

Now that we can make sense of the solutions $v_1(\cdot;\lambda)$ and $\tilde{v}_1(\cdot;\lambda)$ on the whole interval $(0,1)$, our last step is to make sense of the solution $v_0(\cdot;\lambda)$ in the region $[\delta_1\langle\omega\rangle^{-1}, 1)$.

Lemma 28 *The solution $v_0(\cdot;\lambda)$ has the representation*

$$v_0(\rho;\lambda) = v_1(\rho;\lambda)h_3(\lambda) - \tilde{v}_1(\rho;\lambda)\tilde{h}_3(\lambda)$$

where

$$h_3(\lambda) = 1 + O(\langle\omega\rangle^{-1})$$

and

$$\tilde{h}_3(\lambda) = 1 + O(\langle\omega\rangle^{-1})$$

for all $\rho \in [\delta_1\langle\omega\rangle^{-1}, 1)$, $|\omega| \geq 1$.

Proof. Again for $\lambda \in \mathbb{C}$ as above, there exist $a(\lambda), b(\lambda) \in \mathbb{C}$ so that

$$v_0(\rho;\lambda) = a(\lambda)v_1(\rho;\lambda) + b(\lambda)\tilde{v}_1(\rho;\lambda)$$

where

$$a(\lambda) = \frac{W(v_0(\cdot;\lambda), \tilde{v}_1(\cdot;\lambda))}{W(v_1(\cdot;\lambda), \tilde{v}_1(\cdot;\lambda))}$$

and

$$b(\lambda) = -\frac{W(v_0(\cdot;\lambda), v_1(\cdot;\lambda))}{W(v_1(\cdot;\lambda), \tilde{v}_1(\cdot;\lambda))}.$$

Direct calculation yields the following expression valid for all $\rho \in [\delta_1\langle\omega\rangle^{-1}, 1)$

$$\begin{aligned} W\big(v_1(\cdot;\lambda), \tilde{v}_1(\cdot;\lambda)\big) =& W\big(\psi_1(\cdot;\lambda), \tilde{\psi}_1(\cdot;\lambda)\big)h_1(\rho;\lambda)\tilde{h}_1(\rho;\lambda) \\ &+ \psi_1(\rho;\lambda)\tilde{\psi}_1(\rho;\lambda)W\big(h_1(\cdot;\lambda), \tilde{h}_1(\cdot;\lambda)\big) \end{aligned}$$

Since the Wronskian must be constant, we can take the limit $\rho \to 1^-$ to calculate it. A direct calculation using the explicit expressions shows that $\lim_{\rho\to 1^-} \psi_1(\rho;\lambda)\tilde{\psi}_1(\rho;\lambda) = 0$. Thus, taking the limit as $\rho \to 1^-$ yields

$$W(v_1(\cdot;\lambda), \tilde{v}_1(\cdot;\lambda)) = W(\lambda).$$

Thus,

$$a(\lambda) = \frac{W(v_0(\cdot;\lambda), \tilde{v}_1(\cdot;\lambda))}{W(\lambda)}$$

and

$$b(\lambda) = -\frac{W(v_0(\cdot;\lambda), v_1(\cdot;\lambda))}{W(\lambda)}.$$

As in the proof of the previous lemma, we evaluate at $\rho = \delta_0\langle\omega\rangle^{-1}$ to obtain

$$W(v_0(\cdot;\lambda), \tilde{v}_1(\cdot;\lambda)) = W(\lambda)[1 + O(\langle\omega\rangle^{-1})].$$

Thus, we have

$$a(\lambda) = 1 + O(\langle\omega\rangle^{-1})$$

Similarly,

$$W(v_0(\cdot;\lambda), v_1(\cdot;\lambda)) = W(\lambda)[1 + O(\langle\omega\rangle^{-1})] \tag{2.29}$$

which yields

$$b(\lambda) = -[1 + O(\langle\omega\rangle^{-1})].$$

Setting $h_3(\lambda) := a(\lambda)$ and $\tilde{h}_3(\lambda) := -b(\lambda)$ yields the desired result. ■

At this point, we have solutions $v_1(\rho;\lambda)$, $\tilde{v}_1(\rho;\lambda)$, and $v_0(\rho;\lambda)$ of Equation (2.24) which are defined for $\rho \in (0,1)$. Upon making the transformation

$$u_j(\rho;\lambda) = (5 + 3\rho^2)(5 - \rho^2)^{-\frac{1}{2}}\rho^{-3}(1 - \rho^2)^{-\frac{\lambda}{2}} v_j(\rho;\lambda),\ j = 0, 1$$

and

$$\tilde{u}_1(\rho;\lambda) = (5 + 3\rho^2)(5 - \rho^2)^{-\frac{1}{2}}\rho^{-3}(1 - \rho^2)^{-\frac{\lambda}{2}} \tilde{v}_1(\rho;\lambda),$$

we obtain solutions of the homogenous version of Equation (2.22). Observe that neither $u_0(\cdot;\lambda)$ nor $u_1(\cdot;\lambda)$ live in the space $H^5_{\text{rad}}(\mathbb{B}^7)$. From the following calculation, we see that $\{u_0(\cdot;\lambda), u_1(\cdot;\lambda)\}$ forms a fundamental system of Equation (2.22).

Lemma 29 *We have*

$$W(u_0(\cdot;\lambda),u_1(\cdot;\lambda))(\rho) = W(\lambda)w_0(\lambda)(5+3\rho^2)^2(5-\rho^2)^{-1}\rho^{-6}(1-\rho^2)^{-\lambda}$$

where $w_0(\lambda) = 1 + O(\langle\omega\rangle^{-1})$ *for* $\rho \in (0,1)$, $|\omega| \gg 1$. *Moreover* $|w_0(\lambda)| \gtrsim 1$ *for all such* λ.

Proof. We compute directly using the transformation

$$W(u_0(\cdot;\lambda),u_1(\cdot;\lambda)) = W(v_0(\cdot;\lambda),v_1(\cdot;\lambda))(5+3\rho^2)^2(5-\rho^2)^{-1}\rho^{-6}(1-\rho^2)^{-\lambda}.$$

From Eqn. (2.29), we found that

$$W(v_0(\cdot;\lambda),v_1(\cdot;\lambda)) = W(\lambda)w_0(\lambda).$$

with $w_0(\lambda) = 1 + O(\langle\omega\rangle^{-1})$. Upon taking $|\omega| \gg 1$, we can guarantee $|w_0(\lambda)| \gtrsim 1$. ∎

Before proceeding, we prove one last useful lemma.

Lemma 30 *We have*

$$\frac{1}{w_0(\lambda)} = 1 + O(\langle\omega\rangle^{-1})$$

for the function w_0 *from Lemma 29 where* $\lambda = \varepsilon + i\omega$ *and* $|\omega| \gg 1$.

Proof. This is an immediate consequence of Lemma 29 if we write

$$\frac{1}{w_0(\lambda)} = \frac{\overline{w_0(\lambda)}}{|w_0(\lambda)|^2}.$$

∎

2.4.2 Decomposing the Resolvent and the Free Resolvent

In this section, we decompose the resolvent using the previously constructed fundamental system $\{u_0(\cdot;\lambda),u_1(\cdot;\lambda)\}$. In parallel, we decompose the free resolvent in order to highlight the similarities. We begin by explicitly constructing the free resolvent. To that end, given $\lambda \in \mathbb{C}$ with $0 < \varepsilon < M\|\mathbf{L}'\|_{\mathcal{H}}$, we note that $\mathbf{R}_{\mathbf{L}_0}(\lambda)$ exists. Thus, for $\mathbf{f} \in C^\infty(0,1)^2 \cap \mathcal{H}$, setting $\mathbf{u} = (u_1, u_2) = \mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}$ implies that u_1 solves

$$-(1-\rho^2)u''(\rho) + \Big(-\frac{6}{\rho} + 2(\lambda+3)\rho\Big)u'(\rho) + (\lambda+1)(\lambda+4)u(\rho) = F_\lambda(\rho) \tag{2.30}$$

with F_λ defined in Equation (2.23). An explicit fundamental system of Equation (2.30) is given by

$$\varphi_1(\rho;\lambda) := \rho^{-5}(1+\rho)^{1-\lambda}c_+(\rho;\lambda)$$

$$\tilde{\varphi}_1(\rho;\lambda) := \rho^{-5}(1-\rho)^{1-\lambda}c_-(\rho;\lambda)$$

where $c_\pm(\rho;\lambda)$ are defined in Equations (2.25) and (2.26). Their Wronskian is given by

$$W(\varphi_1(\cdot;\lambda), \tilde{\varphi}_1(\cdot;\lambda))(\rho) = W(\lambda)\rho^{-6}(1-\rho^2)^{-\lambda}.$$

Note that neither solution is well-behaved near $\rho = 0$. Thus, we define a third solution which is well-behaved near $\rho = 0$ by

$$\varphi_0(\rho;\lambda) := \varphi_1(\rho;\lambda) - \tilde{\varphi}_1(\rho;\lambda).$$

For convenience, we set

$$\Gamma(\rho) := (5+3\rho^2)(5-\rho^2)^{-\frac{1}{2}}.$$

Now, compared to the fundamental system of Equation (2.22) that we just constructed, observe that we have the relations

$$u_1(\rho;\lambda) = \varphi_1(\rho;\lambda)\Gamma(\rho)h_2(\rho;\lambda),$$

$$u_0(\rho;\lambda) = \varphi_0(\rho;\lambda)\Gamma(\rho)h_0(\rho;\lambda),$$

valid for $\rho\langle\omega\rangle \leq \delta_0$, and

$$u_1(\rho;\lambda) = \varphi_1(\rho;\lambda)\Gamma(\rho)h_1(\rho;\lambda),$$

$$\tilde{u}_1(\rho;\lambda) = \tilde{\varphi}_1(\rho;\lambda)\Gamma(\rho)\tilde{h}_1(\rho;\lambda),$$

$$u_0(\rho;\lambda) = \varphi_1(\rho;\lambda)\Gamma(\rho)h_3(\lambda)h_1(\rho;\lambda) - \tilde{\varphi}_1(\rho;\lambda)\Gamma(\rho)\tilde{h}_3(\lambda)\tilde{h}_1(\rho;\lambda)$$

valid for $\rho\langle\omega\rangle \geq \delta_1$ with Wronskian given by

$$W(u_0(\cdot;\lambda), u_1(\cdot;\lambda))(\rho) = W(\lambda)w_0(\lambda)\Gamma(\rho)^2\rho^{-6}(1-\rho^2)^{-\lambda}.$$

Observe that $\varphi_0(\cdot;\lambda), \varphi_1(\cdot;\lambda) \notin H^5_{\text{rad}}(\mathbb{B}^7)$ and so variation of parameters implies

$$\begin{aligned}[\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}]_1(\rho) = &-\varphi_1(\rho;\lambda)\int_0^\rho \frac{\varphi_0(s;\lambda)}{W(\varphi_0(\cdot;\lambda),\varphi_1(\cdot;\lambda))(s)}\frac{F_\lambda(s)}{1-s^2}ds \\ &-\varphi_0(\rho;\lambda)\int_\rho^1 \frac{\varphi_1(s;\lambda)}{W(\varphi_0(\cdot;\lambda),\varphi_1(\cdot;\lambda))(s)}\frac{F_\lambda(s)}{1-s^2}ds.\end{aligned}$$

Upon setting

$$s^6(1-s^2)^{-1+\lambda}\varphi_1(s;\lambda) = (1-s)^{-1+\lambda}c_0(s;\lambda)$$

and

$$s^6(1-s^2)^{-1+\lambda}\tilde{\varphi}_1(s;\lambda) = (1+s)^{-1+\lambda}\tilde{c}_0(s;\lambda)$$

where

$$c_0(s;\lambda) := sc_+(s;\lambda)$$

and

$$\tilde{c}_0(s;\lambda) := sc_-(s;\lambda),$$

we write the more convenient expression for the first component of the free resolvent

$$\begin{aligned}[\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}]_1(\rho) = -\frac{1}{W(\lambda)}\Big(&\varphi_1(\rho;\lambda)\int_0^\rho \big((1-s)^{-1+\lambda}c_0(s;\lambda) - (1+s)^{-1+\lambda}\tilde{c}_0(s;\lambda)\big)F_\lambda(s)ds \\ &+\varphi_0(\rho;\lambda)\int_\rho^1 (1-s)^{-1+\lambda}c_0(s;\lambda)F_\lambda(s)ds\Big).\end{aligned}$$

Now, using the fundamental system obtained in Section 2.4.1, we can write the full resolvent as

$$[\mathbf{R_L}(\lambda)\tilde{\mathbf{f}}]_1(\rho) = -\,u_1(\rho;\lambda)\int_0^\rho \frac{u_0(s;\lambda)}{W(u_0(\cdot;\lambda),u_1(\cdot;\lambda))(s)}\frac{F_\lambda(s)-s^2\tilde{V}(s)\tilde{f}_1(s)}{1-s^2}ds$$
$$-\,u_0(\rho;\lambda)\int_\rho^1 \frac{u_1(s;\lambda)}{W(u_0(\cdot;\lambda),u_1(\cdot;\lambda))(s)}\frac{F_\lambda(s)-s^2\tilde{V}(s)\tilde{f}_1(s)}{1-s^2}ds$$

It is convenient to first introduce a smooth cutoff function to distinguish the regions ρ close to 0 and ρ away from 0. Let $\chi:\mathbb{R}\to[0,1]$ be any smooth function with $\chi(x)=1$ for $|x|\leq\delta_1$, $\chi(x)=0$ for $|x|\geq\delta_0$, with χ decreasing smoothly from 1 to 0 on the interval $\delta_1<x<\delta_0$. Using such a cutoff, we decompose the full resolvent as

$$[\mathbf{R_L}(\lambda)\mathbf{f}]_1(\rho) = \chi(\rho\langle\omega\rangle)[\mathbf{R_L}(\lambda)\mathbf{f}]_1(\rho) + [1-\chi(\rho\langle\omega\rangle)][\mathbf{R_L}(\lambda)\mathbf{f}]_1(\rho).$$

and similarly for the free resolvent. We further decompose by introducing cutoffs inside of the integral as follows:

$$\begin{aligned}
&\chi(\rho\langle\omega\rangle)[\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}]_1(\rho)\\
&\quad= -\frac{\chi(\rho\langle\omega\rangle)}{W(\lambda)}\Bigg(\varphi_1(\rho;\lambda)\int_0^\rho\big((1-s)^{-1+\lambda}c_0(s;\lambda)-(1+s)^{-1+\lambda}\tilde{c}_0(s;\lambda)\big)\chi(s\langle\omega\rangle)F_\lambda(s)ds\\
&\qquad+\varphi_1(\rho;\lambda)\int_0^\rho\big((1-s)^{-1+\lambda}c_0(s;\lambda)-(1+s)^{-1+\lambda}\tilde{c}_0(s;\lambda)\big)[1-\chi(s\langle\omega\rangle)]F_\lambda(s)ds\\
&\qquad+\varphi_0(\rho;\lambda)\int_\rho^1(1-s)^{-1+\lambda}c_0(s;\lambda)\chi(s\langle\omega\rangle)F_\lambda(s)ds\\
&\qquad+\varphi_0(\rho;\lambda)\int_\rho^1(1-s)^{-1+\lambda}c_0(s;\lambda)[1-\chi(s\langle\omega\rangle)]F_\lambda(s)ds\Bigg),
\end{aligned}$$

$$
\begin{aligned}
&\chi(\rho\langle\omega\rangle)[\mathbf{R}_{\mathbf{L}}(\lambda)\mathbf{f}]_1(\rho)\\
&\quad= -\frac{\chi(\rho\langle\omega\rangle)}{W(\lambda)w_0(\lambda)}\bigg(\varphi_1(\rho;\lambda)\Gamma(\rho)h_2(\rho;\lambda)\int_0^\rho \big((1-s)^{-1+\lambda}c_0(s;\lambda)-(1+s)^{-1+\lambda}\tilde{c}_0(s;\lambda)\big)\\
&\qquad\times h_0(s;\lambda)\Gamma^{-1}(s)\chi(s\langle\omega\rangle)\big(F_\lambda(s)-s^2\tilde{V}(s)\tilde{f}_1(s)\big)ds\\
&\qquad+\varphi_1(\rho;\lambda)\Gamma(\rho)h_2(\rho;\lambda)\int_0^\rho \big((1-s)^{-1+\lambda}c_0(s;\lambda)-(1+s)^{-1+\lambda}\tilde{c}_0(s;\lambda)\big)\\
&\qquad\times h_0(s;\lambda)\Gamma^{-1}(s)[1-\chi(s\langle\omega\rangle)]\big(F_\lambda(s)-s^2\tilde{V}(s)\tilde{f}_1(s)\big)ds\\
&\qquad+\varphi_0(\rho;\lambda)\Gamma(\rho)h_0(\rho;\lambda)\int_\rho^1 (1-s)^{-1+\lambda}c_0(s;\lambda)h_2(s;\lambda)\Gamma^{-1}(s)\chi(s\langle\omega\rangle)\\
&\qquad\times\big(F_\lambda(s)-s^2\tilde{V}(s)\tilde{f}_1(s)\big)ds\\
&\qquad+\varphi_0(\rho;\lambda)\Gamma(\rho)h_0(\rho;\lambda)\int_\rho^1 (1-s)^{-1+\lambda}c_0(s;\lambda)h_1(s;\lambda)\Gamma^{-1}(s)[1-\chi(s\langle\omega\rangle)]\\
&\qquad\times\big(F_\lambda(s)-s^2\tilde{V}(s)\tilde{f}_1(s)\big)ds\bigg),
\end{aligned}
$$

$$
\begin{aligned}
&[1-\chi(\rho\langle\omega\rangle)][\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}]_1(\rho)\\
&\quad= -\frac{[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\bigg(\varphi_1(\rho;\lambda)\int_0^\rho \big((1-s)^{-1+\lambda}c_0(s;\lambda)-(1+s)^{-1+\lambda}\tilde{c}_0(s;\lambda)\big)\\
&\qquad\times\chi(s\langle\omega\rangle)F_\lambda(s)ds\\
&\qquad+\varphi_0(\rho;\lambda)\int_\rho^1 (1-s)^{-1+\lambda}c_0(s;\lambda)\chi(s\langle\omega\rangle)F_\lambda(s)ds\\
&\qquad+\varphi_1(\rho;\lambda)\int_0^1 (1-s)^{-1+\lambda}c_0(s;\lambda)[1-\chi(s\langle\omega\rangle)]F_\lambda(s)ds\\
&\qquad-\varphi_1(\rho;\lambda)\int_0^\rho (1+s)^{-1+\lambda}\tilde{c}_0(s;\lambda)[1-\chi(s\langle\omega\rangle)]F_\lambda(s)ds\\
&\qquad-\tilde{\varphi}_1(\rho;\lambda)\int_\rho^1 (1-s)^{-1+\lambda}c_0(s;\lambda)[1-\chi(s\langle\omega\rangle)]F_\lambda(s)ds\bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
&[1-\chi(\rho\langle\omega\rangle)][\mathbf{R_L}(\lambda)\mathbf{f}]_1(\rho)\\
&\quad=-\frac{[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)w_0(\lambda)}\Big(\varphi_1(\rho;\lambda)\Gamma(\rho)h_1(\rho;\lambda)\int_0^\rho\big((1-s)^{-1+\lambda}c_0(s;\lambda)-(1+s)^{-1+\lambda}\tilde{c}_0(s;\lambda)\big)\\
&\qquad\times h_0(s;\lambda)\Gamma^{-1}(s)\chi(s\langle\omega\rangle)\big(F_\lambda(s)-s^2\tilde{V}(s)\tilde{f}_1(s)\big)ds\\
&\qquad+\varphi_0(\rho;\lambda)\Gamma(\rho)h_0(\rho;\lambda)\int_\rho^1(1-s)^{-1+\lambda}c_0(s;\lambda)h_2(s;\lambda)\Gamma^{-1}(s)\chi(s\langle\omega\rangle)\\
&\qquad\quad\times\big(F_\lambda(s)-s^2\tilde{V}(s)\tilde{f}_1(s)\big)ds\\
&\qquad+\varphi_1(\rho;\lambda)\Gamma(\rho)h_3(\lambda)h_1(\rho;\lambda)\int_0^1(1-s)^{-1+\lambda}c_0(s;\lambda)h_1(s;\lambda)\Gamma^{-1}(s)[1-\chi(s\langle\omega\rangle)]\\
&\qquad\times\big(F_\lambda(s)-s^2\tilde{V}(s)\tilde{f}_1(s)\big)ds\\
&\qquad-\varphi_1(\rho;\lambda)\Gamma(\rho)h_1(\rho;\lambda)\int_0^\rho(1+s)^{-1+\lambda}\tilde{c}_0(s;\lambda)\tilde{h}_3(\lambda)\tilde{h}_1(s;\lambda)\Gamma^{-1}(s)[1-\chi(s\langle\omega\rangle)]\\
&\qquad\times\big(F_\lambda(s)-s^2\tilde{V}(s)\tilde{f}_1(s)\big)ds\\
&\qquad-\tilde{\varphi}_1(\rho;\lambda)\Gamma(\rho)\tilde{h}_3(\lambda)\tilde{h}_1(\rho;\lambda)\int_\rho^1(1-s)^{-1+\lambda}c_0(s;\lambda)h_1(s;\lambda)\Gamma^{-1}(s)[1-\chi(s\langle\omega\rangle)]\\
&\qquad\times\big(F_\lambda(s)-s^2\tilde{V}(s)\tilde{f}_1(s)\big)ds\Big).
\end{aligned}
$$

Though we will not carry it out here, the following lemma is necessary in order to control the full resolvent.

Lemma 31 *In the intermediate region $\delta_1\langle\omega\rangle^{-1}\leq\rho\leq\delta_0\langle\omega\rangle^{-1}$ we have the estimate*

$$|h_0(\rho;\lambda)-h_3(\lambda)h_1(\rho;\lambda)|\lesssim\langle\omega\rangle^{-2}$$

and

$$|h_0(\rho;\lambda)-\tilde{h}_3(\lambda)\tilde{h}_1(\rho;\lambda)|\lesssim\langle\omega\rangle^{-2}.$$

Furthermore,

$$h_1(\rho;\lambda)=h_2(\rho;\lambda).$$

Proof. Upon equating the two representations of v_0 available in this region, we obtain the equation

$$\big(h_0(\rho;\lambda) - h_3(\lambda)h_1(\rho;\lambda)\big)\psi_1(\rho;\lambda) - \big(h_0(\rho;\lambda) - \tilde{h}_3(\lambda)\tilde{h}_1(\rho;\lambda)\big)\tilde{\psi}_1(\rho;\lambda) = 0$$

Taking the Wronskian with $\big(h_0(\rho;\lambda) - \tilde{h}_3(\lambda)\tilde{h}_1(\rho;\lambda)\big)\tilde{\psi}_1(\rho;\lambda)$ yields the equation

$$\begin{aligned} &W\big(h_0(\rho;\lambda) - h_3(\lambda)h_1(\rho;\lambda)\big), h_0(\rho;\lambda) - \tilde{h}_3(\lambda)\tilde{h}_1(\rho;\lambda)\psi_1(\rho;\lambda)\tilde{\psi}_1(\rho;\lambda) \\ &\quad + \big(h_0(\rho;\lambda) - h_3(\lambda)h_1(\rho;\lambda))\big)\big(h_0(\rho;\lambda) - \tilde{h}_3(\lambda)\tilde{h}_1(\rho;\lambda)\big)W(\lambda) \\ =&0. \end{aligned}$$

Recalling the explicit expressions for $\psi_1(\cdot;\lambda)$ and $\tilde{\psi}_1(\cdot;\lambda)$ and the order estimates for h_1, $\tilde{h}_1$, h_0, h_3, and $\tilde{h}_3$, we see that the first term cannot cancel the highest order term in the second term unless it is of one order lower in $\langle\omega\rangle$. Thus, either $|h_0(\rho;\lambda) - h_3(\lambda)h_1(\rho;\lambda)| \lesssim \langle\omega\rangle^{-2}$ or $|h_0(\rho;\lambda) - \tilde{h}_3(\lambda)\tilde{h}_1(\rho;\lambda)| \lesssim \langle\omega\rangle^{-2}$. So long as one holds, plugging this order estimate into the original equation shows that the second estimate must also hold. Lastly, the third claim follows from matching the two representations of v_1 in this region. ∎

2.5 Free Resolvent Estimates

In this section, we estimate the free resolvent $\mathbf{R}_{\mathbf{L}_0}(\lambda)$ in $H^5_{\text{rad}}(\mathbb{B}^7) \times H^4_{\text{rad}}(\mathbb{B}^7)$ for $0 < \varepsilon < M\|\mathbf{L}'\|_{\mathcal{H}}$ and $|\omega| \gg 1$. The techniques we use can be applied to obtaining uniform bounds on the full resolvent, $\mathbf{R}_{\mathbf{L}}(\lambda)$, constructed in 2.4 which we intend to carry out in a forthcoming paper. This can be summarized in the following lemma.

Lemma 32 *Fix $R > 0$ sufficiently large. We have the uniform bound*

$$\sup_{\lambda \in \mathbb{C}, 0 < \Re\lambda < -\frac{1}{2} + M\|\mathbf{L}'\|_{\mathcal{H}}, |\Im\lambda| \geq R} \|\mathbf{R}_{\mathbf{L}_0}(\lambda)\|_{\mathcal{H}} < \infty.$$

As a consequence, we have the following theorem.

Theorem 33 *We have the uniform bound*

$$\sup_{\lambda \in \mathbb{H}} \|\mathbf{R}_{\mathbf{L}_0}(\lambda)\| < \infty.$$

Proof. We decompose $\mathbb{H}$ as $\mathbb{H} = S_1 \cup S_2 \cup S_3$ where S_1, S_2, S_3 are defined at the beginning of Section 2.4 with $R > 0$ sufficiently large as in the previous lemma. By standard semigroup theory [20], $\lambda \mapsto \mathbf{R}_{\mathbf{L}_0}(\lambda)$ is uniformly bounded on S_1. Since $\lambda \mapsto \mathbf{R}_{\mathbf{L}_0}(\lambda)$ exists for $\lambda \in \overline{S}_2$, it is uniformly bounded there. By Lemma 32, $\lambda \mapsto \mathbf{R}_{\mathbf{L}_0}(\lambda)$ is uniformly bounded on S_3. ■

2.5.1 Preliminary Calculations and Useful Estimates

In this section, we demonstrate and record some preliminary calculations and collect some useful estimates which will help in proving Lemma 32 in the following section.

First, we define $c_{j+1}(\rho;\lambda)$ and $\tilde{c}_{j+1}(\rho;\lambda)$, $j = 0, \ldots, 4$, recursively by the equations

$$(1-\rho)^{-1+j+\lambda} c_j(\rho;\lambda) = \left((1-\rho)^{j+\lambda} c_{j+1}(\rho;\lambda)\right)'$$

and

$$(1+\rho)^{-1+j+\lambda} \tilde{c}_j(\rho;\lambda) = \left((1+\rho)^{j+\lambda} \tilde{c}_{j+1}(\rho;\lambda)\right)'$$

where $'$ denotes derivative with respect to ρ. We leave the explicit expressions and useful estimates they satisfy for the appendix. As a consequence of their definition, c_j and $\tilde{c}_j$ satisfy the equation $\tilde{c}_j(\rho;\lambda) = (-1)^{j+1} c_j(-\rho;\lambda)$.

By Taylor expansion, we infer the existence of $\delta_0 > 0$ so that we have the following estimate

$$|\varphi_0^{(j)}(\rho;\lambda)| \lesssim \langle\omega\rangle^{5+j},\ \rho\langle\omega\rangle \leq \delta_0.$$

For the same values of ρ, we also have the estimates

$$|\varphi_1^{(j)}(\rho;\lambda)| \lesssim \rho^{-5+j},\ \rho\langle\omega\rangle \leq \delta_0$$

and

$$|\tilde{\varphi}_1^{(j)}(\rho;\lambda)| \lesssim \rho^{-5+j},\ \rho\langle\omega\rangle \leq \delta_0.$$

Now, we let $\delta_1 > 0$ be any number such that $\delta_1 \leq \delta_0$. We have the estimates

$$|\varphi_1^{(j)}(\rho;\lambda)| \lesssim \langle\omega\rangle^{2+j}\rho^{-3},\ \rho\langle\omega\rangle \geq \delta_1$$

and

$$|\tilde{\varphi}_1^{(j)}(\rho;\lambda)| \lesssim (1-\rho)^{1-j-\epsilon}\langle\omega\rangle^{2+j}\rho^{-3},\ \rho\langle\omega\rangle \geq \delta_1$$

for $j \in \mathbb{N}_0$. In addition, we define the quantities

$$\alpha_{i,j}(\rho;\lambda) := \varphi_1^{(i)}\big((1-\rho)^{j+\lambda}c_{j+1}(\rho;\lambda) - (1+\rho)^{j+\lambda}\tilde{c}_{j+1}(\rho;\lambda)\big),$$

$$\beta_{i,j}(\rho;\lambda) := \varphi_0^{(i)}(1-\rho)^{j+\lambda}c_{j+1}(\rho;\lambda),$$

and

$$\gamma_{i,j}(\rho;\lambda) := \varphi_1^{(i)}(1+\rho)^{j+\lambda}\tilde{c}_{j+1}(\rho;\lambda) - \tilde{\varphi}_1^{(i)}(1-\rho)^{j+\lambda}c_{j+1}(\rho;\lambda).$$

These quantities will show up frequently in various boundary terms when estimating the resolvent.

For both the free resolvent near zero and the free resolvent away from zero, their zeroth derivatives will be handled in a different way from their remaining derivatives. For this reason, we begin by writing the zeroth derivative of the free resolvent near zero as

$$
\begin{aligned}
&\chi(\rho\langle\omega\rangle)[\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}]_1(\rho)\\
&\quad=\frac{\chi(\rho\langle\omega\rangle)}{W(\lambda)}\Big(\varphi_1(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big)\chi(s\langle\omega\rangle)F_\lambda'(s)ds\\
&\quad-\alpha_{0,0}(\rho;\lambda)\chi(\rho\langle\omega\rangle)F_\lambda(\rho)\\
&\quad+\varphi_1(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big)[1-\chi(s\langle\omega\rangle)]F_\lambda'(s)ds\\
&\quad-\alpha_{0,0}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]F_\lambda(\rho)\\
&\quad+\varphi_0(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)\chi(s\langle\omega\rangle)F_\lambda'(s)ds+\beta_{0,0}(\rho;\lambda)\chi(\rho\langle\omega\rangle)F_\lambda(\rho)\\
&\quad+\varphi_0(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]F_\lambda'(s)ds+\beta_{0,0}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]F_\lambda(\rho)\Big)
\end{aligned}
$$

which follows after one integration by parts. This motivates defining the following formal integral operators which, together, represent the free resolvent, properly weighted, near zero

$$
\begin{aligned}
[Q_1(\lambda)f](\rho):=\frac{\langle\omega\rangle\chi(\rho\langle\omega\rangle)}{W(\lambda)}\Big(&\varphi_1(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big)\chi(s\langle\omega\rangle)f'(s)ds\\
&-\alpha_{0,0}(\rho;\lambda)\chi(\rho\langle\omega\rangle)f(\rho)\Big),
\end{aligned}
$$

$$
\begin{aligned}
[Q_2(\lambda)f](\rho):=\frac{\langle\omega\rangle\chi(\rho\langle\omega\rangle)}{W(\lambda)}\Big(&\varphi_1(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big)\\
&\times[1-\chi(s\langle\omega\rangle)]f'(s)ds\\
&-\alpha_{0,0}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f(\rho)\Big),
\end{aligned}
$$

$$
\begin{aligned}
[Q_3(\lambda)f](\rho):=\frac{\langle\omega\rangle\chi(\rho\langle\omega\rangle)}{W(\lambda)}\Big(&\varphi_0(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)\chi(s\langle\omega\rangle)f'(s)ds\\
&+\beta_{0,0}(\rho;\lambda)\chi(\rho\langle\omega\rangle)f(\rho)\Big),
\end{aligned}
$$

and

$$[Q_4(\lambda)f](\rho) := \frac{\langle\omega\rangle\chi(\rho\langle\omega\rangle)}{W(\lambda)}\Big(\varphi_0(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds$$
$$+\beta_{0,0}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f(\rho)\Big).$$

Similarly, we write the zeroth derivative of the resolvent away from zero as

$$\begin{aligned}
&[1-\chi(\rho\langle\omega\rangle)][\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}]_1(\rho)\\
&=\frac{[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\Big(\varphi_1(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big)\chi(s\langle\omega\rangle)F_\lambda'(s)ds\\
&-\alpha_{0,0}(\rho;\lambda)\chi(\rho\langle\omega\rangle)F_\lambda(\rho)\\
&+\varphi_0(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)\chi(s\langle\omega\rangle)F_\lambda'(s)ds+\beta_{0,0}(\rho;\lambda)\chi(\rho\langle\omega\rangle)F_\lambda(\rho)\\
&+\varphi_1(\rho;\lambda)\int_0^1 (1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]F_\lambda'(s)ds\\
&-\varphi_1(\rho;\lambda)\int_0^\rho (1+s)^\lambda \tilde{c}_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]F_\lambda'(s)ds\\
&-\tilde{\varphi}_1(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]F_\lambda'(s)ds\\
&+\gamma_{0,0}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]F_\lambda(\rho)\Big)
\end{aligned}$$

This motivates defining the following formal integral operators which, together, represent the free resolvent, properly weighted, away from zero

$$\begin{aligned}
[Q_5(\lambda)f](\rho) :=&\frac{\langle\omega\rangle[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\Big(\varphi_1(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big)\\
&\times \chi(s\langle\omega\rangle)f'(s)ds\\
&-\alpha_{0,0}(\rho;\lambda)\chi(\rho\langle\omega\rangle)f(\rho)\Big)
\end{aligned}$$

$$
\begin{aligned}
[Q_6(\lambda)f](\rho) :=&\frac{\langle\omega\rangle[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\Big(\varphi_0(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)\chi(s\langle\omega\rangle)f'(s)ds \\
&+\beta_{0,0}(\rho;\lambda)\chi(\rho\langle\omega\rangle)f(\rho)\Big)
\end{aligned}
$$

$$
[Q_7(\lambda)f](\rho) := \frac{\langle\omega\rangle[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\varphi_1(\rho;\lambda)\int_0^1 (1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds
$$

$$
\begin{aligned}
[Q_8(\lambda)f](\rho) :=&-\frac{\langle\omega\rangle[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\Big(\varphi_1(\rho;\lambda)\int_0^\rho (1+s)^\lambda \tilde{c}_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds \\
&+\tilde{\varphi}_1(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds \\
&-\gamma_{0,0}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f(\rho)\Big).
\end{aligned}
$$

Returning to the resolvent near zero, we write its first derivative as

$$
\begin{aligned}
&\chi(\rho\langle\omega\rangle)[\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}]_1'(\rho) \\
&\quad=\frac{(\lambda+4)\chi(\rho\langle\omega\rangle)}{W(\lambda)}\Big(\varphi_1'(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big)\chi(s\langle\omega\rangle)f_1'(s)ds \\
&\quad+\varphi_1'(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big)[1-\chi(s\langle\omega\rangle)]f_1'(s)ds \\
&\quad-\varphi_0'(\rho;\lambda)\int_\rho^1 (1-s)^{1+\lambda}c_2(s;\lambda)\chi(s\langle\omega\rangle)f_1''(s)ds-\beta_{1,1}(\rho;\lambda)\chi(\rho\langle\omega\rangle)f_1'(\rho) \\
&\quad-\varphi_0'(\rho;\lambda)\int_\rho^1 (1-s)^{1+\lambda}c_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f_1''(s)ds-\beta_{1,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f_1'(\rho)\Big) \\
&\quad+\frac{\chi(\rho\langle\omega\rangle)}{W(\lambda)}\Big(\varphi_1'(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big) \\
&\qquad\times\chi(s\langle\omega\rangle)\big(f_2(s)+sf_1'(s)\big)'ds \\
&\quad+\varphi_1'(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big)[1-\chi(s\langle\omega\rangle)]\big(f_2(s)+sf_1'(s)\big)'ds \\
&\quad+\varphi_0'(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)\chi(s\langle\omega\rangle)\big(f_2(s)+sf_1'(s)\big)')ds \\
&\quad+\varphi_0'(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]\big(f_2(s)+sf_1'(s)\big)'ds\Big)
\end{aligned}
$$

which follows after integrating by parts once the f_1 term of the integral from $s=\rho$ to $s=1$. This motivates defining the following formal integral operators which, together, represent the f_1 terms in the first derivative of the free resolvent, properly weighted, near zero

$$[T_{1,1}(\lambda)f](\rho) := \frac{\langle\omega\rangle(\lambda+4)\chi(\rho\langle\omega\rangle)}{W(\lambda)}\varphi_1'(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big) \times \chi(s\langle\omega\rangle)f'(s)ds,$$

$$[T_{2,1}(\lambda)f](\rho) := \frac{\langle\omega\rangle(\lambda+4)\chi(\rho\langle\omega\rangle)}{W(\lambda)}\varphi_1'(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big) \times [1-\chi(s\langle\omega\rangle)]f'(s)ds,$$

$$[T_{3,1}(\lambda)f](\rho) := -\frac{\langle\omega\rangle(\lambda+4)\chi(\rho\langle\omega\rangle)}{W(\lambda)}\bigg(\varphi_0'(\rho;\lambda)\int_\rho^1 (1-s)^{1+\lambda}c_2(s;\lambda)\chi(s\langle\omega\rangle)f''(s)ds + \beta_{1,1}(\rho;\lambda)\chi(\rho\langle\omega\rangle)f'(\rho)\bigg),$$

$$[T_{4,1}(\lambda)f](\rho) := -\frac{\langle\omega\rangle(\lambda+4)\chi(\rho\langle\omega\rangle)}{W(\lambda)}\bigg(\varphi_0'(\rho;\lambda)\int_\rho^1 (1-s)^{1+\lambda}c_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f''(s)ds + \beta_{1,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho)\bigg)$$

and the $f_2+(\cdot)f_1'$ terms in the first derivative of the free resolvent, properly weighted, near zero

$$[R_{1,1}(\lambda)f](\rho) := \frac{\langle\omega\rangle\chi(\rho\langle\omega\rangle)}{W(\lambda)}\varphi_1'(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big)\chi(s\langle\omega\rangle)f'(s)ds,$$

$$[R_{2,1}(\lambda)f](\rho) := \frac{\langle\omega\rangle\chi(\rho\langle\omega\rangle)}{W(\lambda)}\varphi_1'(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big) \times [1-\chi(s\langle\omega\rangle)]f'(s)ds,$$

$$[R_{3,1}(\lambda)f](\rho) := \frac{\langle\omega\rangle\chi(\rho\langle\omega\rangle)}{W(\lambda)}\varphi_0'(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)\chi(s\langle\omega\rangle)f'(s)ds,$$

$$[R_{4,1}(\lambda)f](\rho) := \frac{\langle\omega\rangle\chi(\rho\langle\omega\rangle)}{W(\lambda)}\varphi_0'(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds.$$

Similarly, for the first derivative of the free resolvent away from zero, we write

$$\begin{aligned}
&[1-\chi(\rho\langle\omega\rangle)][\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}]_1'(\rho) \\
&= -\frac{(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\Big(\varphi_1'(\rho;\lambda)\int_0^\rho \big((1-s)^{1+\lambda}c_2(s;\lambda)-(1+s)^{1+\lambda}\tilde{c}_2(s;\lambda)\big) \\
&\times \chi(s\langle\omega\rangle)f_1''(s)ds - \alpha_{1,1}(\rho;\lambda)\chi(\rho\langle\omega\rangle)f_1'(\rho) \\
&+ \varphi_0'(\rho;\lambda)\int_\rho^1 (1-s)^{1+\lambda}c_2(s;\lambda)\chi(s\langle\omega\rangle)f_1''(s)ds + \beta_{1,1}(\rho;\lambda)\chi(\rho\langle\omega\rangle)f_1'(\rho) \\
&+ \varphi_1'(\rho;\lambda)\int_0^1 (1-s)^{1+\lambda}c_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f_1''(s)ds \\
&- \varphi_1'(\rho;\lambda)\int_0^\rho (1+s)^{1+\lambda}\tilde{c}_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f_1''(s)ds \\
&- \tilde{\varphi}_1(\rho;\lambda)\int_\rho^1 (1-s)^{1+\lambda}c_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f_1''(s)ds \\
&+ \gamma_{1,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f_1'(\rho)\Big) \\
&+ \frac{[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\Big(\varphi_1(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big) \\
&\times \chi(s\langle\omega\rangle)\big(f_2(s)+sf_1'(s)\big)'ds \\
&+ \varphi_0(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)\chi(s\langle\omega\rangle)\big(f_2(s)+sf_1'(s)\big)'ds \\
&+ \varphi_1(\rho;\lambda)\int_0^1 (1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]\big(f_2(s)+sf_1'(s)\big)'ds \\
&- \varphi_1(\rho;\lambda)\int_0^\rho (1+s)^\lambda\tilde{c}_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]\big(f_2(s)+sf_1'(s)\big)'ds \\
&- \tilde{\varphi}_1(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]\big(f_2(s)+sf_1'(s)\big)'ds\Big)
\end{aligned}$$

which follows after integrating by parts once all of the f_1 terms. This motivates defining the following formal integral operators which, together, represent the f_1 terms in the first derivative of the free resolvent, properly weighted, away from zero

$$
\begin{aligned}
[T_{5,1}(\lambda)f](\rho) := &- \frac{\langle\omega\rangle(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)} \\
&\times \bigg(\varphi_1'(\rho;\lambda)\int_0^\rho \big((1-s)^{1+\lambda}c_2(s;\lambda) - \varphi_1'(\rho;\lambda)(1+s)^{1+\lambda}\tilde{c}_2(s;\lambda)\big) \\
&\times \chi(s\langle\omega\rangle)f''(s)ds - \alpha_{1,1}(\rho;\lambda)\chi(\rho\langle\omega\rangle)f'(\rho)\bigg),
\end{aligned}
$$

$$
\begin{aligned}
[T_{6,1}(\lambda)f](\rho) := &- \frac{\langle\omega\rangle(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\bigg(\varphi_0'(\rho;\lambda)\int_\rho^1 (1-s)^{1+\lambda}c_2(s;\lambda)\chi(s\langle\omega\rangle)f''(s)ds \\
&+ \beta_{1,1}(\rho;\lambda)\chi(\rho\langle\omega\rangle)f'(\rho)\bigg),
\end{aligned}
$$

$$
[T_{7,1}(\lambda)f](\rho) := -\frac{\langle\omega\rangle(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\varphi_1'(\rho;\lambda)\int_0^1 (1-s)^{1+\lambda}c_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f''(s)ds,
$$

$$
\begin{aligned}
[T_{8,1}(\lambda)f](\rho) := &- \frac{\langle\omega\rangle(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\bigg(- \varphi_1'(\rho;\lambda)\int_0^\rho (1+s)^{1+\lambda}\tilde{c}_2(s;\lambda) \\
&\times [1-\chi(s\langle\omega\rangle)]f''(s)ds - \tilde{\varphi}_1(\rho;\lambda)\int_\rho^1 (1-s)^{1+\lambda}c_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f''(s)ds \\
&+ \gamma_{1,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho)\bigg)
\end{aligned}
$$

and the $f_2 + (\cdot)f_1'$ terms in the first derivative of the free resolvent, properly weighted, away from zero

$$
\begin{aligned}
[R_{5,1}(\lambda)f](\rho) := &\frac{\langle\omega\rangle[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\varphi_1'(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda) - (1+s)^\lambda \tilde{c}_1(s;\lambda)\big) \\
&\chi(s\langle\omega\rangle)f'(s)ds
\end{aligned}
$$

$$
[R_{6,1}(\lambda)f](\rho) := \frac{\langle\omega\rangle[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\varphi_0'(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)\chi(s\langle\omega\rangle)f'(s)ds
$$

$$
[R_{7,1}(\lambda)f](\rho) := \frac{\langle\omega\rangle[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\varphi_1'(\rho;\lambda)\int_0^1 (1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds
$$

$$[R_{8,1}(\lambda)f](\rho) := -\frac{\langle\omega\rangle[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\Big(\varphi_1'(\rho;\lambda)\int_0^\rho (1+s)^\lambda \tilde{c}_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds$$
$$+\tilde{\varphi}_1'(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds\Big).$$

For $k = 2,\dots,5$, we define

$$[T_{1,k}(\lambda)f](\rho) := \frac{\rho^{k-2}(\lambda+4)\chi(\rho\langle\omega\rangle)}{W(\lambda)}$$
$$\times\frac{d^{k-1}}{d\rho^{k-1}}\Big(\varphi_1'(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big)\chi(s\langle\omega\rangle)f'(s)ds\Big),$$

$$[T_{2,k}(\lambda)f](\rho) := \frac{\rho^{k-2}(\lambda+4)\chi(\rho\langle\omega\rangle)}{W(\lambda)}$$
$$\times\frac{d^{k-1}}{d\rho^{k-1}}\Big(\varphi_1'(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big)$$
$$\times[1-\chi(s\langle\omega\rangle)]f'(s)ds\Big),$$

$$[T_{3,k}(\lambda)f](\rho) := -\frac{\rho^{k-2}(\lambda+4)\chi(\rho\langle\omega\rangle)}{W(\lambda)}\frac{d^{k-1}}{d\rho^{k-1}}\Big(\varphi_0'(\rho;\lambda)\int_\rho^1 (1-s)^{1+\lambda}c_2(s;\lambda)\chi(s\langle\omega\rangle)f''(s)ds$$
$$+\beta_{1,1}(\rho;\lambda)\chi(\rho\langle\omega\rangle)f'(\rho)\Big),$$

$$[T_{4,k}(\lambda)f](\rho) := -\frac{\rho^{k-2}(\lambda+4)\chi(\rho\langle\omega\rangle)}{W(\lambda)}\frac{d^{k-1}}{d\rho^{k-1}}\Big(\varphi_0'(\rho;\lambda)\int_\rho^1 (1-s)^{1+\lambda}c_2(s;\lambda)$$
$$\times[1-\chi(s\langle\omega\rangle)]f''(s)ds+\beta_{1,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho)\Big),$$

and

$$[R_{1,k}(\lambda)f](\rho) := \frac{\rho^{k-2}\chi(\rho\langle\omega\rangle)}{W(\lambda)}\frac{d^{k-1}}{d\rho^{k-1}}\Big(\varphi_1'(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big)$$
$$\chi(s\langle\omega\rangle)f'(s)ds\Big),$$

$$[R_{2,k}(\lambda)f](\rho) := \frac{\rho^{k-2}\chi(\rho\langle\omega\rangle)}{W(\lambda)}\frac{d^{k-1}}{d\rho^{k-1}}\Big(\varphi_1'(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big)$$
$$\times[1-\chi(s\langle\omega\rangle)]f'(s)ds\Big),$$

$$[R_{3,k}(\lambda)f](\rho) := \frac{\rho^{k-2}\chi(\rho\langle\omega\rangle)}{W(\lambda)}\frac{d^{k-1}}{d\rho^{k-1}}\left(\varphi_0'(\rho;\lambda)\int_\rho^1(1-s)^\lambda c_1(s;\lambda)\chi(s\langle\omega\rangle)f'(s)ds\right),$$

$$[R_{4,k}(\lambda)f](\rho) := \frac{\rho^{k-2}\chi(\rho\langle\omega\rangle)}{W(\lambda)}\frac{d^{k-1}}{d\rho^{k-1}}\left(\varphi_0'(\rho;\lambda)\int_\rho^1(1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds\right)$$

which together represent the f_1 and $f_2+(\cdot)f_1'$ terms in the kth derivative of the free resolvent near zero, respectively. For $k=2,\dots,4$ we define

$$\begin{aligned}[T_{5,k}(\lambda)f](\rho) := &-\frac{\langle\omega\rangle\rho^{k-1}(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\\ &\times\frac{d^{k-1}}{d\rho^{k-1}}\bigg(\varphi_1'(\rho;\lambda)\int_0^\rho\big((1-s)^{1+\lambda}c_2(s;\lambda)-\varphi_1'(\rho;\lambda)(1+s)^{1+\lambda}\tilde{c}_2(s;\lambda)\big)\\ &\times\chi(s\langle\omega\rangle)f''(s)ds-\alpha_{1,1}(\rho;\lambda)\chi(\rho\langle\omega\rangle)f'(\rho)\bigg),\end{aligned}$$

$$\begin{aligned}[T_{6,k}(\lambda)f](\rho) := &-\frac{\langle\omega\rangle\rho^{k-1}(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\\ &\times\frac{d^{k-1}}{d\rho^{k-1}}\bigg(\varphi_0'(\rho;\lambda)\int_\rho^1(1-s)^{1+\lambda}c_2(s;\lambda)\chi(s\langle\omega\rangle)f''(s)ds\\ &+\beta_{1,1}(\rho;\lambda)\chi(\rho\langle\omega\rangle)f'(\rho)\bigg),\end{aligned}$$

$$\begin{aligned}[T_{7,k}(\lambda)f](\rho) := &-\frac{\langle\omega\rangle\rho^{k-1}(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\varphi_1^{(k)}(\rho;\lambda)\\ &\times\int_0^1(1-s)^{1+\lambda}c_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f''(s)ds,\end{aligned}$$

$$\begin{aligned}[T_{8,k}(\lambda)f](\rho) := &-\frac{\langle\omega\rangle\rho^{k-1}(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\\ &\times\frac{d^{k-1}}{d\rho^{k-1}}\bigg(-\varphi_1'(\rho;\lambda)\int_0^\rho(1+s)^{1+\lambda}\tilde{c}_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f''(s)ds\\ &-\tilde{\varphi}_1(\rho;\lambda)\int_\rho^1(1-s)^{1+\lambda}c_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f''(s)ds\\ &+\gamma_{1,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho)\bigg),\end{aligned}$$

and

$$[R_{5,k}(\lambda)f](\rho) := \frac{\langle\omega\rangle\rho^{k-1}[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)} \times \frac{d^{k-1}}{d\rho^{k-1}}\left(\varphi_1'(\rho;\lambda)\int_0^\rho \big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big)\chi(s\langle\omega\rangle)f'(s)ds\right)$$

$$[R_{6,k}(\lambda)f](\rho) := \frac{\langle\omega\rangle\rho^{k-1}[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\frac{d^{k-1}}{d\rho^{k-1}}\left(\varphi_0'(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)\chi(s\langle\omega\rangle)f'(s)ds\right)$$

$$[R_{7,k}(\lambda)f](\rho) := \frac{\langle\omega\rangle\rho^{k-1}[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\varphi_1^{(k)}(\rho;\lambda)\int_0^1 (1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds,$$

$$[R_{8,k}(\lambda)f](\rho) := -\frac{\langle\omega\rangle\rho^{k-1}[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\frac{d^{k-1}}{d\rho^{k-1}}\left(\varphi_1'(\rho;\lambda)\int_0^\rho (1+s)^\lambda\tilde{c}_1(s;\lambda) \times [1-\chi(s\langle\omega\rangle)]f'(s)ds + \tilde{\varphi}_1'(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds\right).$$

which together represent the f_1 and $f_2+(\cdot)f_1'$ terms in the kth derivative of the free resolvent away from zero, respectively. Lastly, for the fifth derivative of the free resolvent away from zero we define

$$[T_{5,5}(\lambda)f](\rho) := -\frac{\rho^3(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)} \times \frac{d^4}{d\rho^4}\left(\varphi_1'(\rho;\lambda)\int_0^\rho \big((1-s)^{1+\lambda}c_2(s;\lambda)-(1+s)^{1+\lambda}\tilde{c}_2(s;\lambda)\big)\chi(s\langle\omega\rangle)f''(s)ds - \alpha_{1,1}(\rho;\lambda)\chi(\rho\langle\omega\rangle)f'(\rho)\right),$$

$$[T_{6,5}(\lambda)f](\rho) := -\frac{\rho^3(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\frac{d^4}{d\rho^4}\left(\varphi_0'(\rho;\lambda)\int_\rho^1 (1-s)^{1+\lambda}c_2(s;\lambda)\chi(s\langle\omega\rangle)f''(s)ds + \beta_{1,1}(\rho;\lambda)\chi(\rho\langle\omega\rangle)f'(\rho)\right),$$

$$[T_{7,5}(\lambda)f](\rho) := -\frac{\rho^3(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\varphi_1^{(5)}(\rho;\lambda)\int_0^1 (1-s)^{1+\lambda}c_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f''(s)ds,$$

$$[T_{8,5}(\lambda)f](\rho) := -\frac{\rho^3(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\frac{d^4}{d\rho^4}\bigg(-\varphi_1'(\rho;\lambda)\int_0^\rho(1+s)^{1+\lambda}\tilde{c}_2(s;\lambda)$$

$$\times[1-\chi(s\langle\omega\rangle)]f''(s)ds-\tilde{\varphi}_1(\rho;\lambda)\int_\rho^1(1-s)^{1+\lambda}c_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f''(s)ds$$

$$+\gamma_{1,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho)\bigg)$$

and

$$[R_{5,5}(\lambda)f](\rho) := \frac{\rho^3[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\frac{d^4}{d\rho^4}\bigg(\varphi_1'(\rho;\lambda)\int_0^\rho\big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big)$$

$$\times\chi(s\langle\omega\rangle)f'(s)ds\bigg),$$

$$[R_{6,5}(\lambda)f](\rho) := \frac{\rho^3[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\frac{d^4}{d\rho^4}\bigg(\varphi_0'(\rho;\lambda)\int_\rho^1(1-s)^\lambda c_1(s;\lambda)\chi(s\langle\omega\rangle)f'(s)ds\bigg),$$

$$[R_{7,5}(\lambda)f](\rho) := \frac{\rho^3[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\varphi_1^{(5)}(\rho;\lambda)\int_0^1(1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds,$$

$$[R_{8,5}(\lambda)f](\rho) := -\frac{\rho^3[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\frac{d^4}{d\rho^4}\bigg(\varphi_1'(\rho;\lambda)\int_0^\rho(1+s)^\lambda\tilde{c}_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds$$

$$+\tilde{\varphi}_1'(\rho;\lambda)\int_\rho^1(1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds\bigg).$$

In summary, we have the following equalities

$$\langle\omega\rangle[\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}]_1(\rho) = \sum_{j=1}^8[Q_j(\lambda)F_\lambda](\rho),$$

$$\langle\omega\rangle[\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}]_1'(\rho) = \sum_{j=1}^8\big([T_{j,1}(\lambda)\tilde{f}_1](\rho)+[R_{j,1}(\lambda)(\tilde{f}_2+(\cdot)\tilde{f}_1')](\rho)\big),$$

$$\rho^{k-2}\chi(\rho\langle\omega\rangle)[\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}]_1^{(k)}(\rho) = \sum_{j=1}^4\Big([T_{j,k}(\lambda)\tilde{f}_1](\rho)+\big[R_{j,k}(\lambda)(\tilde{f}_2+(\cdot)\tilde{f}_1')\big](\rho)\Big)$$

$$\langle\omega\rangle\rho^{k-1}[1-\chi(\rho\langle\omega\rangle)][\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}]_1^{(k)}(\rho) = \sum_{j=5}^8\Big([T_{j,k}(\lambda)\tilde{f}_1](\rho)+\big[R_{j,k}(\lambda)(\tilde{f}_2+(\cdot)\tilde{f}_1')\big](\rho)\Big)$$

for $k = 2, 3, 4$, and

$$\rho^3 [\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f}]_1^{(5)}(\rho) = \sum_{j=1}^{8} \Big([T_{j,5}(\lambda)\tilde{f}_1](\rho) + \big[R_{j,5}(\lambda)(\tilde{f}_2 + (\cdot)\tilde{f}_1')\big](\rho) \Big).$$

With these equalities in mind, we are motivated to prove the following lemma.

Lemma 34 *Fix $R > 0$ sufficiently large. Given $\lambda \in \mathbb{C}$ with $\lambda = \epsilon + i\omega$, $0 < \epsilon < M\|\mathbf{L}'\|_{\mathcal{H}}$ and $|\omega| \geq R$, the operators*

$$Q_j(\lambda) : \mathcal{D}(Q_j(\lambda)) \subset H^4_{rad}(\mathbb{B}^7) \to L^2(0,1),$$

$$T_{j,k}(\lambda) : \mathcal{D}(T_{j,k}(\lambda)) \subset H^5_{rad}(\mathbb{B}^7) \to L^2(0,1),$$

and

$$R_{j,k}(\lambda) : \mathcal{D}(R_{j,k}(\lambda)) \subset H^4_{rad}(\mathbb{B}^7) \to L^2(0,1)$$

with

$$\mathcal{D}(Q_j(\lambda)) = \mathcal{D}(T_{j,k}(\lambda)) = \mathcal{D}(R_{j,k}(\lambda)) = C_e^\infty[0,1]$$

satisfy the uniform bound

$$\sup_{\lambda\in\mathbb{C},\ 0<\epsilon<\|\mathbf{L}'\|,\ |\omega|\geq R} \langle\omega\rangle \|Q_j(\lambda)f\|_{L^2(0,1)} \lesssim \|f\|_{\Sigma_1},\ j = 1,\dots,8,$$

$$\sup_{\lambda\in\mathbb{C},\ 0<\epsilon<\|\mathbf{L}'\|,\ |\omega|\geq R} \|T_{j,k}(\lambda)f\|_{L^2(0,1)} \lesssim \|f\|_{\Sigma_0},\ j = 3,\dots,8,\ k = 1,\dots,5,$$

and

$$\sup_{\lambda\in\mathbb{C},\ 0<\epsilon<\|\mathbf{L}'\|,\ |\omega|\geq R} \|R_{j,k}(\lambda)f\|_{L^2(0,1)} \lesssim \|f\|_{\Sigma_1},\ j = 1,\dots,8,\ k = 1,\dots,5.$$

Precisely how we define the operators $T_{j,k}$ for $j = 1, 2$, $k = 1, \dots, 5$ and $R_{8,5}$ is delicate due to their singular nature. In preparation, we need the following crucial lemma.

Lemma 35 *Suppose* $T : \mathcal{D}(T) \subset L^2(0,1) \to L^2(0,1)$ *is given by*

$$\mathcal{D}(T) := C_c^\infty(0,1),\ [Tf](x) := \int_0^1 K(x,y)f(y)dy,$$

where the kernel $K \in L^1_{loc}\big((0,1)\times(0,1)\big)$ *and satisfies the pointwise bounds*

$$|K(x,y)| \lesssim \min\{x^{-1+\delta}y^{-\delta}, x^{-\delta}y^{-1+\delta}\}$$

for all $x, y \in (0,1)$ *and some fixed* $\delta \in [0, \frac{1}{2})$. *Then* T *extends to a bounded operator on* $L^2(0,1)$.

Proof. Set $I_j := [2^{j-1}, 2^{j+1}]$ and $\tilde{I}_j := [2^{j-1}, 2^{j+1}] \cap (0,1)$, $j \in \mathbb{Z}$ and denote by 1_j the characteristic function of $\tilde{I}_j$. With these intervals we decompose the operator T as follows

$$Tf = \frac{1}{2}\sum_{j\in\mathbb{Z}} T(1_j f).$$

As a consequence, we have

$$\begin{aligned}
\|Tf\|^2_{L^2(0,1)} =& \frac{1}{2}\sum_{j\in\mathbb{Z}} \|1_j Tf\|^2_{L^2(0,1)} \\
=& \frac{1}{8}\sum_{j\in\mathbb{Z}} \Big\| \sum_{k\in\mathbb{Z}} 1_j T(1_k f)\Big\|^2_{L^2(0,1)} \\
\le& \sum_{j\in\mathbb{Z}} \Big| \sum_{k\in\mathbb{Z}} \|1_j T(1_k f)\|_{L^2(0,1)}\Big|^2 \\
\le& \sum_{j\in\mathbb{Z}} \Big| \sum_{k\in\mathbb{Z}} \|T_{jk}\|_{L^2(0,1)} \|1_k f\|_{L^2(0,1)}\Big|^2
\end{aligned}$$

where for each $j, k \in \mathbb{Z}$, the bounded operator $T_{jk} : \mathcal{D}(T_{jk}) \subset L^2(0,1) \to L^2(0,1)$ is given by

$$\mathcal{D}(T_{jk}) := C_c^\infty(0,1),\ [T_{jk}f](x) := \int_0^1 1_j(x)K(x,y)1_k(y)f(y)dy.$$

Observe that if either $j \geq 1$ or $k \geq 1$, then $\|T_{jk}\|_{L^2(0,1)} = 0$ since $\tilde{I}_j = \tilde{I}_k = \emptyset$. If $j < 1$, $k < 1$, and $0 > j \geq k$, then we can use $|K(x,y)| \lesssim x^{-1+\delta}y^{-\delta}$ to show

$$\begin{aligned}
\|T_{jk}\|_{L^2(0,1)}^2 \leq & \int_0^1 \int_0^1 1_j(x)|K(x,y)|^2 1_k(y)dydx \\
= & \int_{\tilde{I}_j} \int_{\tilde{I}_k} |K(x,y)|^2 dydx \\
\lesssim & 2^{j+k} 2^{2(-1+\delta)j} 2^{-2\delta k} \\
= & 2^{-(1-2\delta)(j-k)}.
\end{aligned}$$

If $0 = j > k$, then we again use $|K(x,y)| \lesssim x^{-1+\delta}y^{-\delta}$ to show

$$\begin{aligned}
\|T_{0k}\|_{L^2(0,1)}^2 \leq & \int_0^1 \int_0^1 1_0(x)|K(x,y)|^2 1_k(y)dydx \\
= & \int_{\tilde{I}_0} \int_{\tilde{I}_k} |K(x,y)|^2 dydx \\
\leq & \int_{I_0} \int_{\tilde{I}_k} x^{2(-1+\delta)} y^{-2\delta} dydx \\
\lesssim & 2^k 2^{-2\delta k} \\
= & 2^{-(1-2\delta)(-k)}.
\end{aligned}$$

Similarly, if $0 = j = k$, then we again use $|K(x,y)| \lesssim x^{-1+\delta}y^{-\delta}$ to show

$$\begin{aligned}
\|T_{00}\|_{L^2(0,1)}^2 \leq & \int_0^1 \int_0^1 1_0(x)|K(x,y)|^2 1_0(y)dydx \\
= & \int_{\tilde{I}_0} \int_{\tilde{I}_0} |K(x,y)|^2 dydx \\
\leq & \int_{I_0} \int_{I_0} x^{2(-1+\delta)} y^{-2\delta} dydx \\
\lesssim & 1.
\end{aligned}$$

Now, if $j < k < 0$, we use $|K(x,y)| \lesssim x^{-\delta}y^{-1+\delta}$ to show that $\|T_{jk}\|_{L^2(0,1)}^2 \lesssim 2^{-(1-2\delta)(k-j)}$. Similar to the previous cases, if $j < k = 0$, then we again use $|K(x,y)| \lesssim x^{-\delta}y^{-1+\delta}$ to show

$\|T_{jk}\|^2_{L^2(0,1)} \lesssim 2^{-(1-2\delta)(-j)}$. In summary, we have the bound

$$\|Tf\|^2_{L^2(0,1)} \lesssim \sum_{j\in\mathbb{Z}} \Big| \sum_{k\in\mathbb{Z}} 2^{-(\frac{1}{2}-\delta)|j-k|} \|1_k f\|_{L^2(0,1)} \Big|^2.$$

On the right-hand side, we have the $\ell^2(\mathbb{Z})$ norm of the convolution of the sequences $(2^{-(\frac{1}{2}-\delta)|k|})_k$ and $(\|1_k f\|_{L^2(0,1)})_k$. By assumption on δ, we have that $(2^{-(\frac{1}{2}-\delta)|k|})_k$ belongs to $\ell^1(\mathbb{Z})$. Thus, by Young's inequality we conclude

$$\|Tf\|_{L^2(0,1)} \lesssim \left(\sum_{k\in\mathbb{Z}} \|1_k f\|^2_{L^2(0,1)} \right)^{1/2} = \sqrt{2}\|f\|_{L^2(0,1)}.$$

■

Remark 36 *This proof is only a slight modification of the proof of Lemma 5.5 of [13] for such operators on* $\mathbb{R}_+$.

Remark 37 *We are immediately able to conclude the same result for operators with kernels satisfying the bound* $|K(x,y)| \lesssim \min\{(1-x)^{-1+\delta}(1-y)^{-\delta}, (1-x)^{-\delta}(1-y)^{-1+\delta}\}$ *by applying the transformation* $(x,y) \mapsto (1-x, 1-y)$ *and using Lemma 35 on the resulting operator.*

At this point, we are ready to prove Lemma 34. This will be achieved by establishing pointwise bounds on the various operators that can be squared and integrated to obtain the necessary $L^2(0,1)$ bounds.

2.5.2 Proof of Lemma 34

We begin with the operators $Q_j(\lambda)$, $j = 1, \dots, 8$. Straightforward estimates yield

$$\begin{aligned} \big|[Q_1(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-4}\chi(\rho\langle\omega\rangle)\Big(\rho^{-5}\int_0^\rho \langle\omega\rangle^5 s^7 |f'(s)| ds + \langle\omega\rangle^3 |f(\rho)|\Big) \\ \lesssim& \langle\omega\rangle^{-1}\big(\|f'\|_{L^2(0,1)} + |f(\rho)|\big), \end{aligned}$$

$$\big|[Q_2(\lambda)f](\rho)\big| \lesssim \langle\omega\rangle^{-1}\big(\|f'\|_{L^2(0,1)} + |f(\rho)|\big),$$

and

$$\begin{aligned}\big|[Q_3(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-4}\chi(\rho\langle\omega\rangle)\Big(\langle\omega\rangle^5 \int_\rho^1 \langle\omega\rangle^{-2}|f'(s)|ds + \langle\omega\rangle^3|f(\rho)|\Big) \\ \lesssim& \langle\omega\rangle^{-1}\big(\|f'\|_{L^2(0,1)} + |f(\rho)|\big).\end{aligned}$$

Observe that we have the estimate $|c_1(s;\lambda)| \lesssim \langle\omega\rangle^{-2}\big(1 + \langle\omega\rangle s + \langle\omega\rangle^2 s^2 + \langle\omega\rangle^3 s^3\big)$. In order to control the powers of $\langle\omega\rangle$ in the region $s\langle\omega\rangle \gtrsim 1$, we are forced to integrate by parts the term with $\langle\omega\rangle^m$ m-times for $m = 1, 2, 3$. Doing so yields the estimate

$$\begin{aligned}&\left|\chi(\rho\langle\omega\rangle)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds\right| \\ &\qquad \lesssim \langle\omega\rangle^{-2}\Big(\|f\|_{\Sigma_1} + \langle\omega\rangle^{-1}\big(|f'(\rho)| + |\rho f''(\rho)| + |\rho^2 f^{(3)}(\rho)|\big)\Big).\end{aligned}$$

As a consequence, we have the estimate

$$\begin{aligned}\big|[Q_4(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-4}\chi(\rho\langle\omega\rangle)\bigg(\langle\omega\rangle^5\langle\omega\rangle^{-2}\Big(\|f\|_{\Sigma_1} + \langle\omega\rangle^{-1}\big(|f'(\rho)| + |\rho f''(\rho)| + |\rho^2 f^{(3)}(\rho)|\big)\Big) \\ &+ \langle\omega\rangle^3|f(\rho)|\bigg) \\ \lesssim& \langle\omega\rangle^{-1}\Big(\|f\|_{\Sigma_1} + \langle\omega\rangle^{-1}\big(|f(\rho)| + |f'(\rho)| + |\rho f''(\rho)| + |\rho^2 f^{(3)}(\rho)|\big)\Big).\end{aligned}$$

Now, more straightforward estimates yield

$$\begin{aligned}\big|[Q_5(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-4}[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^2\rho^{-3}\int_0^\rho \langle\omega\rangle^5 s^7\chi(s\langle\omega\rangle)|f'(s)|ds + \langle\omega\rangle^3|f(\rho)|\Big) \\ \lesssim& \langle\omega\rangle^{-1}\big(\|f'\|_{L^2(0,1)} + |f(\rho)|\big)\end{aligned}$$

and

$$\begin{aligned}\big|[Q_6(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-4}[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^5\int_0^\rho \langle\omega\rangle^{-2}\chi(s\langle\omega\rangle)|f'(s)|ds + \langle\omega\rangle^3|f(\rho)|\Big) \\ \lesssim& \langle\omega\rangle^{-1}\big(\|f'\|_{L^2(0,1)} + |f(\rho)|\big).\end{aligned}$$

Similar to how we controlled $Q_4(\lambda)$, integrating by parts the term with $\langle\omega\rangle^m$ m-times for $m = 1, 2, 3$ yields the estimate

$$\left| \int_0^1 (1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds \right| \lesssim \langle\omega\rangle^{-2}\|f\|_{\Sigma_1}.$$

This implies the bound

$$\begin{aligned} \big|[Q_7(\lambda)f](\rho)\big| \lesssim & \langle\omega\rangle^{-4}[1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^2\rho^{-3}\langle\omega\rangle^{-2}\|f\|_{\Sigma_1} \\ \lesssim & \langle\omega\rangle^{-1}\|f\|_{\Sigma_1}. \end{aligned}$$

To control the operator $Q_8(\lambda)$, we first observe that integrating by parts three times yields

$$\begin{aligned} \varphi_1(\rho;\lambda) & \int_0^\rho (1+s)^\lambda \tilde{c}_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds \\ & + \tilde{\varphi}_1(\rho;\lambda)\int_\rho^1 (1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds \\ = & -\varphi_1(\rho;\lambda)\int_0^\rho (1+s)^{3+\lambda}\tilde{c}_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds \\ & - \tilde{\varphi}_1(\rho;\lambda)\int_\rho^1 (1-s)^{3+\lambda}c_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds \\ & + \gamma_{0,3}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)'' - \gamma_{0,2}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)' \\ & + \gamma_{0,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho). \end{aligned}$$

For the integrals we have the bound

$$\begin{aligned} [1-\chi(\rho\langle\omega\rangle)]\Big|\varphi_1(\rho;\lambda) & \int_0^\rho (1+s)^{3+\lambda}\tilde{c}_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds \\ & + \tilde{\varphi}_1(\rho;\lambda)\int_\rho^1 (1-s)^{3+\lambda}c_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds\Big| \\ \lesssim & \langle\omega\rangle^2\rho^{-3}\int_0^1 \langle\omega\rangle^{-2}\big|\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}\big|ds \\ \lesssim & \rho^{-3}\|f\|_{\Sigma_1}. \end{aligned}$$

This implies the bound

$$\begin{aligned}\big|[Q_8(\lambda)f](\rho)\big| \lesssim & \langle\omega\rangle^{-4}[1-\chi(\rho\langle\omega\rangle)]\big(\rho^{-3}\|f\|_{\Sigma_1}+|f^{(3)}(\rho)|+\langle\omega\rangle|f''(\rho)|+\langle\omega\rangle^2|f'(\rho)| \\ & +\langle\omega\rangle^3|f(\rho)|\big) \\ \lesssim & \langle\omega\rangle^{-1}\Big(\|f\|_{\Sigma_1}+\langle\omega\rangle^{-1}\big(|f(\rho)|+|f'(\rho)|+|\rho f''(\rho)|+|\rho^2 f^{(3)}(\rho)|\big)\Big).\end{aligned}$$

Taking $L^2(0,1)$ norms yields the desired bound.

We now turn our attention to controlling the operators $T_{1,k}(\lambda)$ for $k=1,\dots,5$. Unfortunately, this operator is singular and cannot be handled like the operators $Q_j(\lambda)$. Instead, we first define the auxiliary operators $\tilde{T}_{1,k}(\lambda):\mathcal{D}(\tilde{T}_{1,k}(\lambda))\subset L^2(0,1)\to L^2(0,1)$ by

$$\mathcal{D}(\tilde{T}_{1,k}(\lambda))=C_c^\infty(0,1),\ [\tilde{T}_{1,k}f](\rho):=\int_0^1 K_{1,k}(\rho,s;\lambda)f(s)ds$$

with kernels

$$K_{1,1}(\rho,s;\lambda):=\frac{\langle\omega\rangle(\lambda+4)\chi(\rho\langle\omega\rangle)}{W(\lambda)}\varphi_1'(\rho;\lambda)\big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big)\chi(s\langle\omega\rangle)1_{\mathbb{R}^+}(\rho-s)$$

and

$$\begin{aligned}K_{1,k}(\rho,s;\lambda):=\frac{\rho^{k-2}(\lambda+4)\chi(\rho\langle\omega\rangle)}{W(\lambda)}\varphi_1^{(k)}(\rho;\lambda)\big((1-s)^\lambda c_1(s;\lambda)-(1+s)^\lambda\tilde{c}_1(s;\lambda)\big) \\ \times\chi(s\langle\omega\rangle)1_{\mathbb{R}^+}(\rho-s).\end{aligned}$$

Then, for $f\in\mathcal{D}(\tilde{T}_{1,k})$, we have the equalities

$$[T_{1,1}(\lambda)f](\rho)=[\tilde{T}_{1,1}(\lambda)f'](\rho)$$

and

$$[T_{1,k}(\lambda)f](\rho)=[\tilde{T}_{1,k}(\lambda)f'](\rho)+\frac{\rho^{k-2}(\lambda+4)\chi(\rho\langle\omega\rangle)}{W(\lambda)}\sum_{i=1}^{k-1}\big(\alpha_{i,0}(\rho;\lambda)\chi(\rho\langle\omega\rangle)f'(\rho)\big)^{(k-1-i)} \tag{2.31}$$

for $k = 2, \dots, 5$. Straightforward estimates show that the kernels satisfy the bound

$$|K_{1,k}(\rho, s; \lambda)| \lesssim \min\{\rho^{-1}, s^{-1}\}.$$

Thus, Lemma 35 implies that the operators $\tilde{T}_{1,k}$ extend to bounded operators on $L^2(0,1)$. Another straightforward estimate shows

$$\left| \frac{\rho^{k-2}(\lambda+4)\chi(\rho\langle\omega\rangle)}{W(\lambda)} \sum_{i=1}^{k-1} \big(\alpha_{i,0}(\rho;\lambda)\chi(\rho\langle\omega\rangle)f'(\rho)\big)^{(k-1-i)} \right| \lesssim |f'(\rho)| + \langle\omega\rangle^{-1} \sum_{i=1}^{k-1} |f^{(i)}(\rho)|.$$

As a consequence, we have the pointwise bound

$$[T_{1,k}(\lambda)f](\rho) \lesssim \|f'\|_{L^2(0,1)} + |f'(\rho)| + \langle\omega\rangle^{-1}\big(|f''(\rho)| + |\rho f^{(3)}(\rho)| + |\rho^2 f^{(4)}(\rho)|\big)$$

for any $f \in C_e^\infty[0,1]$. Taking $L^2(0,1)$ norms yields the desired bound.

The operators $T_{2,k}(\lambda)$ for $k = 1, \dots, 5$ are handled similarly. We first define the auxiliary operators $\tilde{T}_{2,k}(\lambda) : \mathcal{D}(\tilde{T}_{2,k}) \subset L^2(0,1) \to L^2(0,1)$ by

$$\mathcal{D}(\tilde{T}_{2,k}(\lambda)) = C_c^\infty(0,1), \; [\tilde{T}_{2,k}(\lambda)f](\rho) := \int_0^1 K_{2,k}(\rho, s; \lambda) f(s) ds$$

with kernels

$$K_{2,1}(\rho, s; \lambda) := \frac{\langle\omega\rangle(\lambda+4)\chi(\rho\langle\omega\rangle)}{W(\lambda)} \varphi_1'(\rho;\lambda)\big((1-s)^\lambda c_1(s;\lambda) - (1+s)^\lambda \tilde{c}_1(s;\lambda)\big) \times [1-\chi(s\langle\omega\rangle)] 1_{\mathbb{R}^+}(\rho - s)$$

and

$$K_{2,k}(\rho, s; \lambda) := \frac{\rho^{k-2}(\lambda+4)\chi(\rho\langle\omega\rangle)}{W(\lambda)} \varphi_1^{(k)}(\rho;\lambda)\big((1-s)^\lambda c_1(s;\lambda) - (1+s)^\lambda \tilde{c}_1(s;\lambda)\big) \times [1-\chi(s\langle\omega\rangle)] 1_{\mathbb{R}^+}(\rho - s).$$

Then, for $f \in \mathcal{D}(\tilde{T}_{1,k}(\lambda))$, we have the equalities

$$[T_{2,1}(\lambda)f](\rho) = [\tilde{T}_{2,1}(\lambda)f'](\rho)$$

and

$$[T_{2,k}(\lambda)f](\rho) = [\tilde{T}_{1,k}(\lambda)f'](\rho) + \frac{\rho^{k-2}(\lambda+4)\chi(\rho\langle\omega\rangle)}{W(\lambda)} \sum_{i=1}^{k-1} \big(\alpha_{i,0}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)^{(k-1-i)}$$

for $k = 2, \ldots, 5$. Straightforward estimates show that the kernels satisfy the bound

$$|K_{2,k}(\rho, s;\lambda)| \lesssim \min\{\rho^{-1}, s^{-1}\}.$$

Thus, Lemma 35 implies that the operators $\tilde{T}_{2,k}$ extend to bounded operators on $L^2(0,1)$. Another straightforward estimate shows

$$\left|\frac{\rho^{k-2}(\lambda+4)\chi(\rho\langle\omega\rangle)}{W(\lambda)} \sum_{i=1}^{k-1} \big(\alpha_{i,0}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)^{(k-1-i)}\right| \lesssim |f'(\rho)| + \langle\omega\rangle^{-1}\sum_{i=1}^{k-1}|f^{(i)}(\rho)|.$$

As a consequence, we have the pointwise bound

$$[T_{2,k}(\lambda)f](\rho) \lesssim \|f'\|_{L^2(0,1)} + |f'(\rho)| + \langle\omega\rangle^{-1}\big(|f''(\rho)| + |\rho f^{(3)}(\rho)| + |\rho^2 f^{(4)}(\rho)|\big)$$

for any $f \in C_e^\infty[0,1]$. Taking $L^2(0,1)$ norms yields the desired bound.

We now turn our attention to the operators $T_{3,k}(\lambda)$, $k = 1, \ldots 5$. Straightforward estimates yield

$$\begin{aligned}\big|[T_{3,1}(\lambda)f](\rho)\big| &\lesssim \langle\omega\rangle^{-3}\chi(\rho\langle\omega\rangle)\left(\langle\omega\rangle^6 \int_\rho^1 \langle\omega\rangle^{-3}|f''(s)|ds + \langle\omega\rangle^3|f'(\rho)|\right)\\ &\lesssim \|f''\|_{L^2(0,1)} + |f'(\rho)|,\end{aligned}$$

$$\begin{aligned}\big|[T_{3,2}(\lambda)f](\rho)\big| &\lesssim \langle\omega\rangle^{-4}\chi(\rho\langle\omega\rangle)\left(\langle\omega\rangle^7 \int_\rho^1 \langle\omega\rangle^{-3}|f''(s)|ds + \langle\omega\rangle^4|f'(\rho)|\right)\\ &\lesssim \|f''\|_{L^2(0,1)} + |f'(\rho)|,\end{aligned}$$

$$\begin{aligned}\big|[T_{3,3}(\lambda)f](\rho)\big| \lesssim& \rho\langle\omega\rangle^{-4}\chi(\rho\langle\omega\rangle)\Big(\langle\omega\rangle^{8}\int_\rho^1 \langle\omega\rangle^{-3}|f''(s)|ds + \langle\omega\rangle^{5}|f'(\rho)| + \langle\omega\rangle^{4}|f''(\rho)|\Big)\\ \lesssim& \|f''\|_{L^2(0,1)} + |f'(\rho)| + \langle\omega\rangle^{-1}|f''(\rho)|,\end{aligned}$$

$$\begin{aligned}\big|[T_{3,4}(\lambda)f](\rho)\big| \lesssim& \rho^2\langle\omega\rangle^{-4}\chi(\rho\langle\omega\rangle)\Big(\langle\omega\rangle^{9}\int_\rho^1 \langle\omega\rangle^{-3}|f''(s)|ds + \langle\omega\rangle^{6}|f'(\rho)| + \langle\omega\rangle^{5}|f''(\rho)|\\ &+ \langle\omega\rangle^{4}|f^{(3)}(\rho)|\Big)\\ \lesssim& \|f''\|_{L^2(0,1)} + |f'(\rho)| + \langle\omega\rangle^{-1}\big(|f''(\rho)| + |\rho f^{(3)}(\rho)|\big),\end{aligned}$$

and

$$\begin{aligned}\big|[T_{3,5}(\lambda)f](\rho)\big| \lesssim& \rho^3\langle\omega\rangle^{-4}\chi(\rho\langle\omega\rangle)\Big(\langle\omega\rangle^{10}\int_\rho^1 \langle\omega\rangle^{-3}|f''(s)|ds + \langle\omega\rangle^{7}|f'(\rho)| + \langle\omega\rangle^{6}|f''(\rho)|\\ &+ \langle\omega\rangle^{5}|f^{(3)}(\rho)| + \langle\omega\rangle^{4}|f^{(4)}(\rho)|\Big)\\ \lesssim& \|f''\|_{L^2(0,1)} + |f'(\rho)| + \langle\omega\rangle^{-1}\big(|f''(\rho)| + |\rho f^{(3)}(\rho)| + |\rho^2 f^{(4)}(\rho)|\big).\end{aligned}$$

Taking $L^2(0,1)$ norms yields the desired bound.

We now turn our attention to the operators $T_{4,k}(\lambda)$, $k = 1, \dots 5$. Similar to how we controlled the operator $Q_4(\lambda)$, we first observe that

$$|c_2(s;\lambda)| \lesssim \langle\omega\rangle^{-3}\big(1 + \langle\omega\rangle s + \langle\omega\rangle^2 s^2 + \langle\omega\rangle^3 s^3\big).$$

In the region $s\langle\omega\rangle \gtrsim 1$, it is again useful to integrate by parts the term with $\langle\omega\rangle^m$ m-times in order to control those powers of $\langle\omega\rangle$. This yields the estimate

$$\begin{aligned}&\left|\chi(\rho\langle\omega\rangle)\int_\rho^1 (1-s)^{1+\lambda}c_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f''(s)ds\right|\\ &\qquad\lesssim \langle\omega\rangle^{-3}\Big(\|f\|_{\Sigma_0} + \langle\omega\rangle^{-1}\big(|f''(\rho)| + |\rho f^{(3)}(\rho)| + |\rho^2 f^{(4)}(\rho)|\big)\Big)\end{aligned}$$

As a consequence, we have the estimate

$$\begin{aligned}\big|[T_{4,1}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-3}\chi(\rho\langle\omega\rangle)\Big(\langle\omega\rangle^{6}\langle\omega\rangle^{-3}\Big(\|f\|_{\Sigma_0}+\langle\omega\rangle^{-1}\big(|f''(\rho)|+|\rho f^{(3)}(\rho)|\\&+|\rho^2 f^{(4)}(\rho)|\big)\Big)+\langle\omega\rangle^{3}|f'(\rho)|\Big)\\ \lesssim&\|f\|_{\Sigma_0}+|f'(\rho)|+\langle\omega\rangle^{-1}\big(|f''(\rho)|+|\rho f^{(3)}(\rho)|+|\rho^2 f^{(4)}(\rho)|\big).\end{aligned}$$

Similar straightforward estimates then yield

$$\begin{aligned}\big|[T_{4,2}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-4}\chi(\rho\langle\omega\rangle)\Big(\langle\omega\rangle^{7}\langle\omega\rangle^{-3}\Big(\|f\|_{\Sigma_0}+\langle\omega\rangle^{-1}\big(|f''(\rho)|+|\rho f^{(3)}(\rho)|\\&+|\rho^2 f^{(4)}(\rho)|\big)\Big)+\langle\omega\rangle^{4}|f'(\rho)|\Big)\\ \lesssim&\|f\|_{\Sigma_0}+|f'(\rho)|+\langle\omega\rangle^{-1}\big(|f''(\rho)|+|\rho f^{(3)}(\rho)|+|\rho^2 f^{(4)}(\rho)|\big),\end{aligned}$$

$$\begin{aligned}\big|[T_{4,3}(\lambda)f](\rho)\big| \lesssim& \rho\langle\omega\rangle^{-4}\chi(\rho\langle\omega\rangle)\Big(\langle\omega\rangle^{8}\langle\omega\rangle^{-3}\Big(\|f\|_{\Sigma_0}+\langle\omega\rangle^{-1}\big(|f''(\rho)|+|\rho f^{(3)}(\rho)|\\&+|\rho^2 f^{(4)}(\rho)|\big)\Big)+\langle\omega\rangle^{5}|f'(\rho)|+\langle\omega\rangle^{4}|f''(\rho)|\Big)\\ \lesssim&\|f\|_{\Sigma_0}+|f'(\rho)|+\langle\omega\rangle^{-1}\big(|f''(\rho)|+|\rho f^{(3)}(\rho)|+|\rho^2 f^{(4)}(\rho)|\big),\end{aligned}$$

$$\begin{aligned}\big|[T_{4,4}(\lambda)f](\rho)\big| \lesssim& \rho^2\langle\omega\rangle^{-4}\chi(\rho\langle\omega\rangle)\Big(\langle\omega\rangle^{9}\langle\omega\rangle^{-3}\Big(\|f\|_{\Sigma_0}+\langle\omega\rangle^{-1}\big(|f''(\rho)|+|\rho f^{(3)}(\rho)|\\&+|\rho^2 f^{(4)}(\rho)|\big)\Big)+\langle\omega\rangle^{6}|f'(\rho)|+\langle\omega\rangle^{5}|f''(\rho)|+\langle\omega\rangle^{4}|f^{(3)}(\rho)|\Big)\\ \lesssim&\|f\|_{\Sigma_0}+|f'(\rho)|+\langle\omega\rangle^{-1}\big(|f''(\rho)|+|\rho f^{(3)}(\rho)|+|\rho^2 f^{(4)}(\rho)|\big),\end{aligned}$$

and

$$\begin{aligned}\big|[T_{4,5}(\lambda)f](\rho)\big| \lesssim& \rho^3\langle\omega\rangle^{-4}\chi(\rho\langle\omega\rangle)\Big(\langle\omega\rangle^{10}\langle\omega\rangle^{-3}\Big(\|f\|_{\Sigma_0}+\langle\omega\rangle^{-1}\big(|f''(\rho)|+|\rho f^{(3)}(\rho)|\\&+|\rho^2 f^{(4)}(\rho)|\big)\Big)+\langle\omega\rangle^{7}|f'(\rho)|+\langle\omega\rangle^{6}|f''(\rho)|+\langle\omega\rangle^{5}|f^{(3)}(\rho)|+\langle\omega\rangle^{4}|f^{(4)}(\rho)|\Big)\\ \lesssim&\|f\|_{\Sigma_0}+|f'(\rho)|+\langle\omega\rangle^{-1}\big(|f''(\rho)|+|\rho f^{(3)}(\rho)|+|\rho^2 f^{(4)}(\rho)|\big).\end{aligned}$$

Taking $L^2(0,1)$ norms yields the desired bound.

Next, we turn our attention to the operators $T_{5,k}(\lambda)$, $k = 1, \dots 5$. Straightforward estimates yield

$$\begin{aligned}\big|[T_{5,1}(\lambda)f](\rho)\big| \lesssim &\langle\omega\rangle^{-3}[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^{3}\rho^{-3}\int_0^\rho \langle\omega\rangle^{-3}\chi(s\langle\omega\rangle)|f''(s)|ds + \langle\omega\rangle^{-3}\rho^{-6}|f'(\rho)|\Big)\\ \lesssim &\|f''\|_{L^2(0,1)} + |f'(\rho)|,\end{aligned}$$

$$\begin{aligned}\big|[T_{5,2}(\lambda)f](\rho)\big| \lesssim &\langle\omega\rangle^{-3}\rho[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^{4}\rho^{-3}\int_0^\rho \langle\omega\rangle^{-3}\chi(s\langle\omega\rangle)|f''(s)|ds + \langle\omega\rangle^{-3}\rho^{-7}|f'(\rho)|\Big)\\ \lesssim &\|f''\|_{L^2(0,1)} + |f'(\rho)|,\end{aligned}$$

$$\begin{aligned}\big|[T_{5,3}(\lambda)f](\rho)\big| \lesssim &\langle\omega\rangle^{-3}\rho^2[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^{5}\rho^{-3}\int_0^\rho \langle\omega\rangle^{-3}\chi(s\langle\omega\rangle)|f''(s)|ds + \langle\omega\rangle^{-3}\rho^{-7}|f''(\rho)|\\ &+ \langle\omega\rangle^{-3}\rho^{-8}|f'(\rho)|\Big)\\ \lesssim &\|f''\|_{L^2(0,1)} + |f'(\rho)| + \langle\omega\rangle^{-1}|f''(\rho)|,\end{aligned}$$

$$\begin{aligned}\big|[T_{5,4}(\lambda)f](\rho)\big| \lesssim &\langle\omega\rangle^{-3}\rho^3[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^{6}\rho^{-3}\int_0^\rho \langle\omega\rangle^{-3}\chi(s\langle\omega\rangle)|f''(s)|ds\\ &+ \langle\omega\rangle^{-3}\rho^{-7}|f^{(3)}(\rho)| + \langle\omega\rangle^{-3}\rho^{-8}|f''(\rho)| + \langle\omega\rangle^{-3}\rho^{-9}|f'(\rho)|\Big)\\ \lesssim &\|f''\|_{L^2(0,1)} + |f'(\rho)| + \langle\omega\rangle^{-1}\big(|f''(\rho)| + |\rho f^{(3)}(\rho)|\big),\end{aligned}$$

and

$$\begin{aligned}\big|[T_{5,5}(\lambda)f](\rho)\big| \lesssim &\langle\omega\rangle^{-4}\rho^3[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^{7}\rho^{-3}\int_0^\rho \langle\omega\rangle^{-3}\chi(s\langle\omega\rangle)|f''(s)|ds\\ &+ \langle\omega\rangle^{-3}\rho^{-7}|f^{(4)}(\rho)| + \langle\omega\rangle^{-3}\rho^{-8}|f^{(3)}(\rho)| + \langle\omega\rangle^{-3}\rho^{-9}|f''(\rho)|\\ &+ \langle\omega\rangle^{-3}\rho^{-10}|f'(\rho)|\Big)\\ \lesssim &\|f''\|_{L^2(0,1)} + |f'(\rho)| + \langle\omega\rangle^{-1}\big(|f''(\rho)| + |\rho f^{(3)}(\rho)| + |\rho^2 f^{(4)}(\rho)|\big).\end{aligned}$$

Taking $L^2(0,1)$ norms yields the desired bound.

Next, we turn our attention to the operators $T_{6,k}(\lambda)$, $k = 1, \dots 5$. Again, straightforward estimates yield

$$\begin{aligned}\big|[T_{6,1}(\lambda)f](\rho)\big| &\lesssim \langle\omega\rangle^{-3}[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^{6}\int_\rho^1 \langle\omega\rangle^{-3}|f''(s)|ds + \langle\omega\rangle^{3}|f'(\rho)|\Big)\\ &\lesssim \|f''\|_{L^2(0,1)} + |f'(\rho)|,\end{aligned}$$

$$\begin{aligned}\big|[T_{6,2}(\lambda)f](\rho)\big| &\lesssim \langle\omega\rangle^{-4}[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^{7}\int_\rho^1 \langle\omega\rangle^{-3}|f''(s)|ds + \langle\omega\rangle^{4}|f'(\rho)|\Big)\\ &\lesssim \|f''\|_{L^2(0,1)} + |f'(\rho)|.\end{aligned}$$

$$\begin{aligned}\big|[T_{6,3}(\lambda)f](\rho)\big| &\lesssim \langle\omega\rangle^{-5}[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^{8}\int_\rho^1 \langle\omega\rangle^{-3}|f''(s)|ds + \langle\omega\rangle^{4}|f''(\rho)| + \langle\omega\rangle^{5}|f'(\rho)|\Big)\\ &\lesssim \|f''\|_{L^2(0,1)} + |f'(\rho)| + \langle\omega\rangle^{-1}|f''(\rho)|,\end{aligned}$$

$$\begin{aligned}\big|[T_{6,4}(\lambda)f](\rho)\big| &\lesssim \langle\omega\rangle^{-6}[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^{9}\int_\rho^1 \langle\omega\rangle^{-3}|f''(s)|ds + \langle\omega\rangle^{4}|f^{(3)}(\rho)| + \langle\omega\rangle^{5}|f''(\rho)|\\ &\qquad + \langle\omega\rangle^{6}|f'(\rho)|\Big)\\ &\lesssim \|f''\|_{L^2(0,1)} + |f'(\rho)| + \langle\omega\rangle^{-1}\big(|f''(\rho)| + |\rho f^{(3)}(\rho)|\big),\end{aligned}$$

and

$$\begin{aligned}\big|[T_{6,5}(\lambda)f](\rho)\big| &\lesssim \langle\omega\rangle^{-7}[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^{10}\int_\rho^1 \langle\omega\rangle^{-3}|f''(s)|ds + \langle\omega\rangle^{4}|f^{(4)}(\rho)| + \langle\omega\rangle^{5}|f^{(3)}(\rho)|\\ &\qquad + \langle\omega\rangle^{6}|f''(\rho)| + \langle\omega\rangle^{7}|f'(\rho)|\Big)\\ &\lesssim \|f''\|_{L^2(0,1)} + |f'(\rho)| + \langle\omega\rangle^{-1}\big(|f''(\rho)| + |\rho f^{(3)}(\rho)| + |\rho^2 f^{(4)}(\rho)|\big).\end{aligned}$$

Taking $L^2(0,1)$ norms yields the desired bound.

Next, we turn our attention to the operators $T_{7,k}(\lambda)$, $k = 1, \dots 5$. Similar to how we integrated by parts for the operators $T_{4,k}(\lambda)$, we obtain the estimate

$$\left| \int_0^1 (1-s)^{1+\lambda} c_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f''(s)ds \right| \lesssim \langle\omega\rangle^{-3}\|f\|_{\Sigma_0}.$$

This implies the bounds

$$\begin{aligned}\big|[T_{7,1}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-3}[1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^{3}\rho^{-3}\langle\omega\rangle^{-3}\|f\|_{\Sigma_0}\\ \lesssim& \|f\|_{\Sigma_0},\end{aligned}$$

$$\begin{aligned}\big|[T_{7,2}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-3}\rho[1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^{4}\rho^{-3}\langle\omega\rangle^{-3}\|f\|_{\Sigma_0}\\ \lesssim& \|f\|_{\Sigma_0},\end{aligned}$$

$$\begin{aligned}\big|[T_{7,3}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-3}\rho^{2}[1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^{5}\rho^{-3}\langle\omega\rangle^{-3}\|f\|_{\Sigma_0}\\ \lesssim& \|f\|_{\Sigma_0},\end{aligned}$$

$$\begin{aligned}\big|[T_{7,4}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-3}\rho^{3}[1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^{6}\rho^{-3}\langle\omega\rangle^{-3}\|f\|_{\Sigma_0}\\ \lesssim& \|f\|_{\Sigma_0},\end{aligned}$$

and

$$\begin{aligned}\big|[T_{7,5}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-4}\rho^{3}[1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^{7}\rho^{-3}\langle\omega\rangle^{-3}\|f\|_{\Sigma_0}\\ \lesssim& \|f\|_{\Sigma_0}.\end{aligned}$$

Taking $L^2(0,1)$ norms yields the desired bound.

Now, we turn our attention to the operators $T_{8,k}(\lambda)$, $k = 1, \dots, 5$. Though these operators are not singular, controlling them is more delicate than the rest. For convenience, recall

$$[T_{8,1}(\lambda)f](\rho) = \frac{\langle\omega\rangle(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\Big(\varphi_1'(\rho;\lambda)\int_0^\rho (1+s)^{1+\lambda}\tilde{c}_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f''(s)ds$$

$$+\tilde{\varphi}_1'(\rho;\lambda)\int_\rho^1 (1-s)^{1+\lambda}c_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f''(s)ds$$

$$-\gamma_{1,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho)\Big).$$

For the integrals, observe that integrating by parts three times yields

$$\begin{aligned}
&\varphi_1'(\rho;\lambda)\int_0^\rho (1+s)^{1+\lambda}\tilde{c}_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f''(s)ds \\
&+\tilde{\varphi}_1'(\rho;\lambda)\int_\rho^1 (1-s)^{1+\lambda}c_2(s;\lambda)[1-\chi(s\langle\omega\rangle)]f''(s)ds \\
=&-\varphi_1'(\rho;\lambda)\int_0^\rho (1+s)^{4+\lambda}\tilde{c}_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds \\
&-\tilde{\varphi}_1'(\rho;\lambda)\int_\rho^1 (1-s)^{4+\lambda}c_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds \\
&+\gamma_{1,4}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f''(\rho)\big)''-\gamma_{1,3}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f''(\rho)\big)' \\
&+\gamma_{1,2}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f''(\rho).
\end{aligned}$$

As a consequence, we obtain the bound

$$\begin{aligned}
[1-\chi(\rho\langle\omega\rangle)]\Big|&\varphi_1'(\rho;\lambda)\int_0^\rho (1+s)^{4+\lambda}\tilde{c}_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds \\
&+\tilde{\varphi}_1'(\rho;\lambda)\int_\rho^1 (1-s)^{4+\lambda}c_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds\Big| \\
&\lesssim[1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^3\rho^{-3}\int_0^1 \langle\omega\rangle^{-3}s^3\big|\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}\big|ds \\
&\lesssim\langle\omega\rangle^3\|f\|_{\Sigma_0}.
\end{aligned}$$

This implies the bound

$$
\begin{aligned}
\big|[T_{8,1}(\lambda)f](\rho)\big| \lesssim & \langle\omega\rangle^{-3}[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^3\|f\|_{\Sigma_0} + |f^{(4)}(\rho)| + \langle\omega\rangle|f^{(3)}(\rho)| + \langle\omega\rangle^2|f''(\rho)| \\
& + \langle\omega\rangle^3|f'(\rho)|\Big) \\
\lesssim & \|f\|_{\Sigma_0} + |f'(\rho)| + |f''(\rho)| + |\rho f^{(3)}(\rho)| + |\rho^2 f_1^{(4)}(\rho)|.
\end{aligned}
$$

A straightforward calculation involving one integration by parts yields

$$
\begin{aligned}
[T_{8,2}(\lambda)f](\rho) = & -\frac{\langle\omega\rangle\rho(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\Big(\varphi_1''(\rho;\lambda)\int_0^\rho (1+s)^{2+\lambda}\tilde{c}_3(s;\lambda) \\
& \times \big([1-\chi(s\langle\omega\rangle)]f''(s)\big)' ds \\
& + \tilde{\varphi}_1''(\rho;\lambda)\int_\rho^1 (1-s)^{2+\lambda}c_3(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)' ds \\
& - \gamma_{2,2}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f''(\rho) - \gamma_{1,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f''(\rho) \\
& + \big(\gamma_{1,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)'\Big).
\end{aligned}
$$

For the integrals, integrating by parts twice yields

$$
\begin{aligned}
& \varphi_1''(\rho;\lambda)\int_0^\rho (1+s)^{2+\lambda}\tilde{c}_3(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f_1''(s)\big)' ds \\
& + \tilde{\varphi}_1''(\rho;\lambda)\int_\rho^1 (1-s)^{2+\lambda}c_3(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f_1''(s)\big)' ds \\
= & \varphi_1''(\rho;\lambda)\int_0^\rho (1+s)^{4+\lambda}\tilde{c}_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)} ds \\
& + \tilde{\varphi}_1''(\rho;\lambda)\int_\rho^1 (1-s)^{4+\lambda}c_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)} ds \\
& - \gamma_{2,4}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f''(\rho)\big)'' + \gamma_{2,3}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f''(\rho)\big)'.
\end{aligned}
$$

This implies the bound

$$
\begin{aligned}
[1-\chi(\rho\langle\omega\rangle)]\Big|&\varphi_1''(\rho;\lambda)\int_0^\rho (1+s)^{4+\lambda}\tilde{c}_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds \\
&+\tilde{\varphi}_1''(\rho;\lambda)\int_\rho^1 (1-s)^{4+\lambda}c_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds\Big| \\
&\lesssim [1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^4\rho^{-3}\int_0^1 \langle\omega\rangle^{-3}s^3\big|[1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}\big|ds \\
&\lesssim \langle\omega\rangle^3\rho^{-1}\|f\|_{\Sigma_0}.
\end{aligned}
$$

As a consequence, we obtain the bound

$$
\begin{aligned}
\big|[T_{8,2}(\lambda)f](\rho)\big| \lesssim&\langle\omega\rangle^{-3}\rho[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^3\rho^{-1}\|f\|_{\Sigma_0}+\langle\omega\rangle|f^{(4)}(\rho)|+\langle\omega\rangle^2|f^{(3)}(\rho)| \\
&+\langle\omega\rangle^3|f''(\rho)|+\langle\omega\rangle^3\rho^{-1}|f'(\rho)|\Big) \\
\lesssim&\|f\|_{\Sigma_0}+|f'(\rho)|+|f''(\rho)|+|\rho f^{(3)}(\rho)|+|\rho^2 f_1^{(4)}(\rho)|.
\end{aligned}
$$

Another straightforward calculation involving one integration by parts yields

$$
\begin{aligned}
[T_{8,3}(\lambda)f](\rho)=&\frac{\langle\omega\rangle\rho^2(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\Big(\varphi_1^{(3)}(\rho;\lambda)\int_0^\rho (1+s)^{3+\lambda}\tilde{c}_4(s;\lambda) \\
&\times\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)''ds \\
&+\tilde{\varphi}_1^{(3)}(\rho;\lambda)\int_\rho^1 (1-s)^{3+\lambda}c_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)''ds \\
&-\gamma_{3,3}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f''(\rho)\big)'+\big(\gamma_{2,2}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f''(\rho)\big)' \\
&+\big(\gamma_{1,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f''(\rho)\big)'-\big(\gamma_{1,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)''\Big).
\end{aligned}
$$

For the integrals, integrating by parts once yields

$$
\begin{aligned}
&\varphi_1^{(3)}(\rho;\lambda)\int_0^\rho (1+s)^{3+\lambda}\tilde{c}_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)''ds\\
&+\tilde{\varphi}_1^{(3)}(\rho;\lambda)\int_\rho^1 (1-s)^{3+\lambda}c_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)''ds\\
=&-\varphi_1^{(3)}(\rho;\lambda)\int_0^\rho (1+s)^{4+\lambda}\tilde{c}_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds\\
&-\tilde{\varphi}_1^{(3)}(\rho;\lambda)\int_\rho^1 (1-s)^{4+\lambda}c_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds\\
&+\gamma_{3,4}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f''(\rho)\big)''.
\end{aligned}
$$

Thus, we have the estimate

$$
\begin{aligned}
[1-\chi(\rho\langle\omega\rangle)]\bigg|&\varphi_1^{(3)}(\rho;\lambda)\int_0^\rho (1+s)^{4+\lambda}\tilde{c}_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds\\
&+\tilde{\varphi}_1^{(3)}(\rho;\lambda)\int_\rho^1 (1-s)^{4+\lambda}c_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds\bigg|\\
\lesssim&[1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^5\rho^{-3}\int_0^1\langle\omega\rangle^{-3}s^3\big|\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}\big|ds\\
\lesssim&\langle\omega\rangle^3\rho^{-2}\|f\|_{\Sigma_0}.
\end{aligned}
$$

This implies the bound

$$
\begin{aligned}
\big|[T_{8,3}(\lambda)f](\rho)\big|\lesssim&\langle\omega\rangle^{-3}\rho^2[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^3\rho^{-2}\|f\|_{\Sigma_0}+\langle\omega\rangle^2|f^{(4)}(\rho)|+\langle\omega\rangle^3|f^{(3)}(\rho)|\\
&+\langle\omega\rangle^3\rho^{-1}|f''(\rho)|+\langle\omega\rangle^3\rho^{-2}|f'(\rho)|\Big)\\
\lesssim&\|f\|_{\Sigma_0}+|f'(\rho)|+|f''(\rho)|+|\rho f^{(3)}(\rho)|+|\rho^2f^{(4)}(\rho)|.
\end{aligned}
$$

Again, another straightforward calculation involving one integration by parts yields

$$
\begin{aligned}
[T_{8,4}(\lambda)f](\rho) = &- \frac{\langle\omega\rangle\rho^3(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\Big(\varphi_1^{(4)}(\rho;\lambda)\int_0^\rho (1+s)^{4+\lambda}\tilde{c}_5(s;\lambda) \\
&\times \big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds \\
&+ \tilde{\varphi}_1^{(4)}(\rho;\lambda)\int_\rho^1 (1-s)^{4+\lambda}c_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds \\
&- \gamma_{4,4}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f''(\rho)\big)'' + \Big(\gamma_{3,3}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f''(\rho)\big)'\Big)' \\
&- \big(\gamma_{2,2}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f''(\rho)\big)'' - \big(\gamma_{1,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f''(\rho)\big)'' \\
&+ \big(\gamma_{1,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)^{(3)}\Big).
\end{aligned}
$$

For the integrals, we obtain the bound

$$
\begin{aligned}
[1-\chi(\rho\langle\omega\rangle)]\Big|&\varphi_1^{(4)}(\rho;\lambda)\int_0^\rho (1+s)^{4+\lambda}\tilde{c}_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds \\
&+ \tilde{\varphi}_1^{(4)}(\rho;\lambda)\int_\rho^1 (1-s)^{4+\lambda}c_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds\Big| \\
&\lesssim [1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^6\rho^{-3}\int_0^1 \langle\omega\rangle^{-3}s^3\big|[1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}\big|ds \\
&\lesssim \langle\omega\rangle^3\rho^{-3}\|f\|_{\Sigma_0}.
\end{aligned}
$$

This implies the bound

$$
\begin{aligned}
\big|[T_{8,4}(\lambda)f](\rho)\big| \lesssim &\langle\omega\rangle^{-3}\rho^3[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^3\rho^{-3}\|f\|_{\Sigma_0} + \langle\omega\rangle^3|f^{(4)}(\rho)| + \langle\omega\rangle^3\rho^{-1}|f^{(3)}(\rho)| \\
&+ \langle\omega\rangle^3\rho^{-2}|f''(\rho)| + \langle\omega\rangle^3\rho^{-3}|f'(\rho)|\Big) \\
\lesssim &\ \|f\|_{\Sigma_0} + |f'(\rho)| + |f''(\rho)| + |\rho f^{(3)}(\rho)| + |\rho^2 f^{(4)}(\rho)|.
\end{aligned}
$$

One last straightforward calculation shows

$$[T_{8,5}(\lambda)f](\rho) = -\frac{\rho^3(\lambda+4)[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\Big(\varphi_1^{(5)}(\rho;\lambda)\int_0^\rho (1+s)^{4+\lambda}\tilde{c}_5(s;\lambda)$$

$$\times\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds$$

$$+\tilde{\varphi}_1^{(5)}(\rho;\lambda)\int_\rho^1 (1-s)^{4+\lambda}c_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds$$

$$+\gamma_{4,4}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f''(\rho)\big)^{(3)} - \Big(\gamma_{4,4}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f''(\rho)\big)''\Big)'$$

$$+\Big(\gamma_{3,3}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f''(\rho)\big)'\Big)'' - \big(\gamma_{2,2}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f''(\rho)\big)^{(3)}$$

$$-\big(\gamma_{1,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f''(\rho)\big)^{(3)} + \big(\gamma_{1,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)^{(4)}\Big).$$

Observe that the coefficient of $f^{(5)}(\rho)$ vanishes. For the integrals, we obtain the bound

$$[1-\chi(\rho\langle\omega\rangle)]\Big|\varphi_1^{(5)}(\rho;\lambda)\int_0^\rho (1+s)^{4+\lambda}\tilde{c}_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds$$

$$+\tilde{\varphi}_1^{(5)}(\rho;\lambda)\int_\rho^1 (1-s)^{4+\lambda}c_5(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}ds\Big|$$

$$\lesssim [1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^7\rho^{-3}\int_0^1 \langle\omega\rangle^{-3}s^3\big|[1-\chi(s\langle\omega\rangle)]f''(s)\big)^{(3)}\big|ds$$

$$\lesssim \langle\omega\rangle^4\rho^{-3}\|f\|_{\Sigma_0}.$$

This implies the bound

$$\big|[T_{8,5}(\lambda)f](\rho)\big| \lesssim \langle\omega\rangle^{-4}\rho^3[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^4\rho^{-3}\|f\|_{\Sigma_0} + \langle\omega\rangle^3\rho^{-1}|f^{(4)}(\rho)| + \langle\omega\rangle^3\rho^{-2}|f^{(3)}(\rho)|$$

$$+\langle\omega\rangle^3\rho^{-3}|f''(\rho)| + \langle\omega\rangle^3\rho^{-4}|f'(\rho)|\Big)$$

$$\lesssim \|f\|_{\Sigma_0} + |f'(\rho)| + |f''(\rho)| + |\rho f^{(3)}(\rho)| + |\rho^2 f^{(4)}(\rho)|.$$

Taking $L^2(0,1)$ norms yields the desired bound.

We turn our attention to the operators $R_{1,k}(\lambda)$, $k = 1,\dots,5$. Straightforward estimates show

$$\big|[R_{1,1}(\lambda)f](\rho)\big| \lesssim \langle\omega\rangle^{-4}\chi(\rho\langle\omega\rangle)\rho^{-6}\int_0^\rho \langle\omega\rangle^5 s^7|f'(s)|ds$$
$$\lesssim \|f'\|_{L^2(0,1)},$$

$$\big|[R_{1,2}(\lambda)f](\rho)\big| \lesssim \langle\omega\rangle^{-5}\chi(\rho\langle\omega\rangle)\bigg(\rho^{-7}\int_0^\rho \langle\omega\rangle^5 s^7|f'(s)|ds + \langle\omega\rangle^4|f'(\rho)|\bigg)$$
$$\lesssim \|f'\|_{L^2(0,1)} + \langle\omega\rangle^{-1}|f'(\rho)|,$$

$$\big|[R_{1,3}(\lambda)f](\rho)\big| \lesssim \rho\langle\omega\rangle^{-5}\chi(\rho\langle\omega\rangle)\bigg(\rho^{-8}\int_0^\rho \langle\omega\rangle^5 s^7|f'(s)|ds + \langle\omega\rangle^5|f'(\rho)| + \langle\omega\rangle^4|f''(\rho)|\bigg)$$
$$\lesssim \|f'\|_{L^2(0,1)} + \langle\omega\rangle^{-1}\big(|f'(\rho)| + |\rho f''(\rho)|\big),$$

$$\big|[R_{1,4}(\lambda)f](\rho)\big| \lesssim \rho^2\langle\omega\rangle^{-5}\chi(\rho\langle\omega\rangle)\bigg(\rho^{-9}\int_0^\rho \langle\omega\rangle^5 s^7|f'(s)|ds + \langle\omega\rangle^5\rho^{-1}|f'(\rho)| + \langle\omega\rangle^5|f''(\rho)|$$
$$+ \langle\omega\rangle^4|f^{(3)}(\rho)|\bigg)$$
$$\lesssim \|f'\|_{L^2(0,1)} + \langle\omega\rangle^{-1}\big(|f'(\rho)| + |\rho f''(\rho)| + |\rho^2 f^{(3)}(\rho)|\big),$$

$$\big|[R_{1,5}(\lambda)f](\rho)\big| \lesssim \rho^3\langle\omega\rangle^{-5}\chi(\rho\langle\omega\rangle)\bigg(\rho^{-10}\int_0^\rho \langle\omega\rangle^5 s^7|f'(s)|ds + \langle\omega\rangle^5\rho^{-2}|f'(\rho)| + \langle\omega\rangle^5\rho^{-1}|f''(\rho)|$$
$$+ \langle\omega\rangle^5|f^{(3)}(\rho)| + \langle\omega\rangle^4|f^{(4)}(\rho)|\bigg)$$
$$\lesssim \|f'\|_{L^2(0,1)} + \langle\omega\rangle^{-1}\big(|f'(\rho)| + |\rho f''(\rho)| + |\rho^2 f^{(3)}(\rho)| + |\rho^3 f^{(4)}(\rho)|\big).$$

Taking $L^2(0,1)$ norms yields the desired bound.

Now, we turn our attention to the operators $R_{2,k}(\lambda)$, $k = 1, \dots, 5$. Straightforward estimates show

$$\big|[R_{2,1}(\lambda)f](\rho)\big| \lesssim \langle\omega\rangle^{-4}\chi(\rho\langle\omega\rangle)\rho^{-6}\int_0^\rho \langle\omega\rangle^5 s^7|f'(s)|ds$$
$$\lesssim \|f'\|_{L^2(0,1)},$$

$$
\begin{aligned}
\big|[R_{2,2}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-5}\chi(\rho\langle\omega\rangle)\bigg(\rho^{-7}\int_0^\rho \langle\omega\rangle^5 s^7|f'(s)|ds + \langle\omega\rangle^4|f'(\rho)|\bigg) \\
\lesssim& \|f'\|_{L^2(0,1)} + \langle\omega\rangle^{-1}|f'(\rho)|,
\end{aligned}
$$

$$
\begin{aligned}
\big|[R_{2,3}(\lambda)f](\rho)\big| \lesssim& \rho\langle\omega\rangle^{-5}\chi(\rho\langle\omega\rangle)\bigg(\rho^{-8}\int_0^\rho \langle\omega\rangle^5 s^7|f'(s)|ds + \langle\omega\rangle^5|f'(\rho)| + \langle\omega\rangle^4|f''(\rho)|\bigg) \\
\lesssim& \|f'\|_{L^2(0,1)} + \langle\omega\rangle^{-1}\big(|f'(\rho)| + |\rho f''(\rho)|\big),
\end{aligned}
$$

$$
\begin{aligned}
\big|[R_{2,4}(\lambda)f](\rho)\big| \lesssim& \rho^2\langle\omega\rangle^{-5}\chi(\rho\langle\omega\rangle)\bigg(\rho^{-9}\int_0^\rho \langle\omega\rangle^5 s^7|f'(s)|ds + \langle\omega\rangle^5\rho^{-1}|f'(\rho)| + \langle\omega\rangle^5|f''(\rho)| \\
&+ \langle\omega\rangle^4|f^{(3)}(\rho)|\bigg) \\
\lesssim& \|f'\|_{L^2(0,1)} + \langle\omega\rangle^{-1}\big(|f'(\rho)| + |\rho f''(\rho)| + |\rho^2 f^{(3)}(\rho)|\big),
\end{aligned}
$$

and

$$
\begin{aligned}
\big|[R_{2,5}(\lambda)f](\rho)\big| \lesssim& \rho^3\langle\omega\rangle^{-5}\chi(\rho\langle\omega\rangle)\bigg(\rho^{-10}\int_0^\rho \langle\omega\rangle^5 s^7|f'(s)|ds + \langle\omega\rangle^5\rho^{-2}|f'(\rho)| + \langle\omega\rangle^5\rho^{-1}|f''(\rho)| \\
&+ \langle\omega\rangle^5|f^{(3)}(\rho)| + \langle\omega\rangle^4|f^{(4)}(\rho)|\bigg) \\
\lesssim& \|f'\|_{L^2(0,1)} + \langle\omega\rangle^{-1}\big(|f'(\rho)| + |\rho f''(\rho)| + |\rho^2 f^{(3)}(\rho)| + |\rho^3 f^{(4)}(\rho)|\big).
\end{aligned}
$$

Taking $L^2(0,1)$ norms yields the desired bound.

We turn our attention to the operators $R_{3,k}(\lambda)$, $k = 1, \ldots, 5$. Straightforward estimates show

$$
\begin{aligned}
\big|[R_{3,1}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-4}\chi(\rho\langle\omega\rangle)\langle\omega\rangle^6\int_\rho^1 \langle\omega\rangle^{-2}|f'(s)|ds \\
\lesssim& \|f'\|_{L^2(0,1)},
\end{aligned}
$$

$$
\begin{aligned}
\big|[R_{3,2}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-5}\chi(\rho\langle\omega\rangle)\bigg(\langle\omega\rangle^7\int_\rho^1 \langle\omega\rangle^{-2}|f'(s)|ds + \langle\omega\rangle^4|f'(\rho)|\bigg) \\
\lesssim& \|f'\|_{L^2(0,1)} + \langle\omega\rangle^{-1}|f'(\rho)|,
\end{aligned}
$$

$$\begin{aligned}\big|[R_{3,3}(\lambda)f](\rho)\big| \lesssim& \rho\langle\omega\rangle^{-5}\chi(\rho\langle\omega\rangle)\bigg(\langle\omega\rangle^{8}\int_\rho^1\langle\omega\rangle^{-2}|f'(s)|ds+\langle\omega\rangle^4|f''(\rho)|+\langle\omega\rangle^5|f'(\rho)|\bigg)\\ \lesssim& \,\|f'\|_{L^2(0,1)}+\langle\omega\rangle^{-1}\big(|f'(\rho)|+|\rho f''(\rho)|\big),\end{aligned}$$

$$\begin{aligned}\big|[R_{3,4}(\lambda)f](\rho)\big| \lesssim& \rho^2\langle\omega\rangle^{-5}\chi(\rho\langle\omega\rangle)\bigg(\langle\omega\rangle^{9}\int_\rho^1\langle\omega\rangle^{-2}|f'(s)|ds+\langle\omega\rangle^4|f^{(3)}(\rho)|+\langle\omega\rangle^5|f''(\rho)|\\ &+\langle\omega\rangle^6|f'(\rho)|\bigg)\\ \lesssim& \,\|f'\|_{L^2(0,1)}+\langle\omega\rangle^{-1}\big(|f'(\rho)|+|\rho f''(\rho)|+|\rho^2 f^{(3)}(\rho)|\big),\end{aligned}$$

and

$$\begin{aligned}\big|[R_{3,5}(\lambda)f](\rho)\big| \lesssim& \rho^3\langle\omega\rangle^{-5}\chi(\rho\langle\omega\rangle)\bigg(\langle\omega\rangle^{10}\int_\rho^1\langle\omega\rangle^{-2}|f'(s)|ds+\langle\omega\rangle^4|f^{(4)}(\rho)|+\langle\omega\rangle^5|f^{(3)}(\rho)|\\ &+\langle\omega\rangle^6|f''(\rho)|+\langle\omega\rangle^7|f'(\rho)|\bigg)\\ \lesssim& \,\|f'\|_{L^2(0,1)}+\langle\omega\rangle^{-1}\big(|f'(\rho)|+|\rho f''(\rho)|+|\rho^2 f^{(3)}(\rho)|+|\rho^3 f^{(4)}(\rho)|\big).\end{aligned}$$

Taking $L^2(0,1)$ norms yields the desired bound.

We now turn our attention to the operators $R_{4,k}(\lambda)$, $k=1,\dots,5$. Using the same estimate we obtained for $Q_4(\lambda)$, we obtain the bounds

$$\begin{aligned}\big|[R_{4,1}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-4}\chi(\rho\langle\omega\rangle)\langle\omega\rangle^6\langle\omega\rangle^{-2}\Big(\|f\|_{\Sigma_1}+\langle\omega\rangle^{-1}\big(|f'(\rho)|+|\rho f''(\rho)|+|\rho^2 f^{(3)}(\rho)|\big)\Big)\\ \lesssim& \,\|f\|_{\Sigma_1}+\langle\omega\rangle^{-1}\big(|f'(\rho)|+|\rho f''(\rho)|+|\rho^2 f^{(3)}(\rho)|\big),\end{aligned}$$

$$\begin{aligned}\big|[R_{4,2}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-5}\chi(\rho\langle\omega\rangle)\bigg(\langle\omega\rangle^7\langle\omega\rangle^{-2}\Big(\|f\|_{\Sigma_1}+\langle\omega\rangle^{-1}\big(|f'(\rho)|+|\rho f''(\rho)|+|\rho^2 f^{(3)}(\rho)|\big)\Big)\\ &+\langle\omega\rangle^4|f'(\rho)|\bigg)\\ \lesssim& \,\|f\|_{\Sigma_1}+\langle\omega\rangle^{-1}\big(|f'(\rho)|+|\rho f''(\rho)|+|\rho^2 f^{(3)}(\rho)|\big),\end{aligned}$$

$$
\begin{aligned}
\big|[R_{4,3}(\lambda)f](\rho)\big| \lesssim& \rho\langle\omega\rangle^{-5}\chi(\rho\langle\omega\rangle)\Big(\langle\omega\rangle^{8}\langle\omega\rangle^{-2}\Big(\|f\|_{\Sigma_1}+\langle\omega\rangle^{-1}\big(|f'(\rho)|+|\rho f''(\rho)| \\
&+|\rho^2 f^{(3)}(\rho)|\big)\Big)+\langle\omega\rangle^{4}|f''(\rho)|+\langle\omega\rangle^{5}|f'(\rho)|\Big) \\
\lesssim& \|f\|_{\Sigma_1}+\langle\omega\rangle^{-1}\big(|f'(\rho)|+|\rho f''(\rho)|+|\rho^2 f^{(3)}(\rho)|\big),
\end{aligned}
$$

$$
\begin{aligned}
\big|[R_{4,4}(\lambda)f](\rho)\big| \lesssim& \rho^2\langle\omega\rangle^{-5}\chi(\rho\langle\omega\rangle)\Big(\langle\omega\rangle^{9}\langle\omega\rangle^{-2}\Big(\|f\|_{\Sigma_1}+\langle\omega\rangle^{-1}\big(|f'(\rho)|+|\rho f''(\rho)| \\
&+|\rho^2 f^{(3)}(\rho)|\big)\Big)+\langle\omega\rangle^{4}|f^{(3)}(\rho)|+\langle\omega\rangle^{5}|f''(\rho)|+\langle\omega\rangle^{6}|f'(\rho)|\Big) \\
\lesssim& \|f\|_{\Sigma_1}+\langle\omega\rangle^{-1}\big(|f'(\rho)|+|\rho f''(\rho)|+|\rho^2 f^{(3)}(\rho)|\big),
\end{aligned}
$$

and

$$
\begin{aligned}
\big|[R_{4,5}(\lambda)f](\rho)\big| \lesssim& \rho^3\langle\omega\rangle^{-5}\chi(\rho\langle\omega\rangle)\Big(\langle\omega\rangle^{10}\langle\omega\rangle^{-2}\Big(\|f\|_{\Sigma_1}+\langle\omega\rangle^{-1}\big(|f'(\rho)|+|\rho f''(\rho)| \\
&+|\rho^2 f^{(3)}(\rho)|+|\rho^3 f^{(4)}(\rho)|\big)\Big)+\langle\omega\rangle^{4}|f^{(4)}(\rho)|+\langle\omega\rangle^{5}|f^{(3)}(\rho)|+\langle\omega\rangle^{6}|f''(\rho)| \\
&+\langle\omega\rangle^{7}|f'(\rho)|\Big) \\
\lesssim& \|f\|_{\Sigma_1}+\langle\omega\rangle^{-1}\big(|f'(\rho)|+|\rho f''(\rho)|+|\rho^2 f^{(3)}(\rho)|+|\rho^3 f^{(4)}(\rho)|\big).
\end{aligned}
$$

Taking $L^2(0,1)$ norms yields the desired bound.

Now, we turn our attention to the operators $R_{5,k}(\lambda)$, $k=1,\dots,5$. Straightforward estimates yield

$$
\begin{aligned}
\big|[R_{5,1}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-4}[1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^{3}\rho^{-3}\int_0^\rho \langle\omega\rangle^{-2}\chi(s\langle\omega\rangle)|f'(s)|ds \\
\lesssim& \|f'\|_{L^2(0,1)},
\end{aligned}
$$

$$
\begin{aligned}
\big|[R_{5,2}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-4}\rho[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^{4}\rho^{-3}\int_0^\rho \langle\omega\rangle^{-2}\chi(s\langle\omega\rangle)|f'(s)|ds+\langle\omega\rangle^{3}\rho^{-1}|f'(\rho)|\Big) \\
\lesssim& \|f'\|_{L^2(0,1)}+\langle\omega\rangle^{-1}|f'(\rho)|,
\end{aligned}
$$

$$
\begin{aligned}
\big|[R_{5,3}(\lambda)f](\rho)\big| \lesssim & \langle\omega\rangle^{-4}\rho^2[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^5\rho^{-3}\int_0^\rho \langle\omega\rangle^{-2}\chi(s\langle\omega\rangle)|f'(s)|ds + \langle\omega\rangle^3\rho^{-1}|f''(\rho)| \\
& + \langle\omega\rangle^3\rho^{-2}|f'(\rho)|\Big) \\
\lesssim & \|f'\|_{L^2(0,1)} + \langle\omega\rangle^{-1}\big(|f'(\rho)| + |\rho f''(\rho)|\big),
\end{aligned}
$$

$$
\begin{aligned}
\big|[R_{5,4}(\lambda)f](\rho)\big| \lesssim & \langle\omega\rangle^{-4}\rho^3[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^6\rho^{-3}\int_0^\rho \langle\omega\rangle^{-2}\chi(s\langle\omega\rangle)|f'(s)|ds + \langle\omega\rangle^3\rho^{-1}|f^{(3)}(\rho)| \\
& + \langle\omega\rangle^3\rho^{-2}|f''(\rho)| + \langle\omega\rangle^3\rho^{-3}|f'(\rho)|\Big) \\
\lesssim & \|f'\|_{L^2(0,1)} + \langle\omega\rangle^{-1}\big(|f'(\rho)| + |\rho f''(\rho)| + |\rho^2 f^{(3)}(\rho)|\big),
\end{aligned}
$$

and

$$
\begin{aligned}
\big|[R_{5,5}(\lambda)f](\rho)\big| \lesssim & \langle\omega\rangle^{-5}\rho^3[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^7\rho^{-3}\int_0^\rho \langle\omega\rangle^{-2}\chi(s\langle\omega\rangle)|f'(s)|ds + \langle\omega\rangle^3\rho^{-1}|f^{(4)}(\rho)| \\
& + \langle\omega\rangle^3\rho^{-2}|f^{(3)}(\rho)| + \langle\omega\rangle^3\rho^{-3}|f''(\rho)| + \langle\omega\rangle^4\rho^{-3}|f'(\rho)|\Big) \\
\lesssim & \|f'\|_{L^2(0,1)} + \langle\omega\rangle^{-1}\big(|f'(\rho)| + |\rho f''(\rho)| + |\rho^2 f^{(3)}(\rho)| + |\rho^3 f^{(4)}(\rho)|\big).
\end{aligned}
$$

Taking $L^2(0,1)$ norms yields the desired bound.

We turn our attention to the operators $R_{6,k}(\lambda)$, $k = 1, \dots, 5$. Straightforward estimates yield

$$
\begin{aligned}
\big|[R_{6,1}(\lambda)f](\rho)\big| \lesssim & \langle\omega\rangle^{-4}[1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^6\int_\rho^1 \langle\omega\rangle^{-2}|f'(s)|ds \\
\lesssim & \|f'\|_{L^2(0,1)},
\end{aligned}
$$

$$
\begin{aligned}
\big|[R_{6,2}(\lambda)f](\rho)\big| \lesssim & \langle\omega\rangle^{-5}[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^7\int_\rho^1 \langle\omega\rangle^{-2}|f'(s)|ds + \langle\omega\rangle^4|f'(\rho)|\Big) \\
\lesssim & \|f'\|_{L^2(0,1)} + \langle\omega\rangle^{-1}|f'(\rho)|,
\end{aligned}
$$

$$\begin{aligned}\big|[R_{6,3}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-6}[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^{8}\int_\rho^1 \langle\omega\rangle^{-2}|f'(s)|ds + \langle\omega\rangle^{4}|f''(\rho)| + \langle\omega\rangle^{5}|f'(\rho)|\Big)\\ \lesssim& \|f'\|_{L^2(0,1)} + \langle\omega\rangle^{-1}\big(|f'(\rho)| + |\rho f''(\rho)|\big),\end{aligned}$$

$$\begin{aligned}\big|[R_{6,4}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-7}[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^{9}\int_\rho^1 \langle\omega\rangle^{-2}|f'(s)|ds + \langle\omega\rangle^{4}|f^{(3)}(\rho)| + \langle\omega\rangle^{5}|f''(\rho)|\\ &+ \langle\omega\rangle^{6}|f'(\rho)|\Big)\\ \lesssim& \|f'\|_{L^2(0,1)} + \langle\omega\rangle^{-1}\big(|f'(\rho)| + |\rho f''(\rho)| + |\rho^2 f^{(3)}(\rho)|\big),\end{aligned}$$

and

$$\begin{aligned}\big|[R_{6,5}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-8}[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^{10}\int_\rho^1 \langle\omega\rangle^{-2}|f'(s)|ds + \langle\omega\rangle^{4}|f^{(4)}(\rho)| + \langle\omega\rangle^{5}|f^{(3)}(\rho)|\\ &+ \langle\omega\rangle^{6}|f''(\rho)| + \langle\omega\rangle^{7}|f'(\rho)|\Big)\\ \lesssim& \|f'\|_{L^2(0,1)} + \langle\omega\rangle^{-1}\big(|f'(\rho)| + |\rho f''(\rho)| + |\rho^2 f^{(3)}(\rho)| + |\rho^3 f^{(4)}(\rho)|\big).\end{aligned}$$

Taking $L^2(0,1)$ norms yields the desired bound.

We turn our attention to the operators $R_{7,k}(\lambda)$, $k = 1, \ldots, 5$. Using the bound we obtained to control $Q_7(\lambda)$, we obtain the bounds

$$\begin{aligned}\big|[R_{7,1}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-4}[1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^{3}\rho^{-3}\langle\omega\rangle^{-2}\|f\|_{\Sigma_1}\\ \lesssim& \|f\|_{\Sigma_1},\end{aligned}$$

$$\begin{aligned}\big|[R_{7,2}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-4}\rho[1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^{4}\rho^{-3}\langle\omega\rangle^{-2}\|f\|_{\Sigma_1}\\ \lesssim& \|f\|_{\Sigma_1},\end{aligned}$$

$$\begin{aligned}\big|[R_{7,3}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-4}\rho^2[1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^{5}\rho^{-3}\langle\omega\rangle^{-2}\|f\|_{\Sigma_1}\\ \lesssim& \|f\|_{\Sigma_1},\end{aligned}$$

$$\begin{aligned}\big|[R_{7,4}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-4}\rho^3[1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^6\rho^{-3}\langle\omega\rangle^{-2}\|f\|_{\Sigma_1}\\ \lesssim&\|f\|_{\Sigma_1},\end{aligned}$$

and

$$\begin{aligned}\big|[R_{7,5}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-5}\rho^3[1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^7\rho^{-3}\langle\omega\rangle^{-2}\|f\|_{\Sigma_1}\\ \lesssim&\|f\|_{\Sigma_1}.\end{aligned}$$

Taking $L^2(0,1)$ norms yields the desired bound.

Last, we turn our attention to the operators $R_{8,k}(\lambda)$, $k=1,\dots,5$. Similar to the operators $T_{8,k}(\lambda)$, controlling these operators is also very delicate. In fact, $R_{8,5}(\lambda)$ is also singular and a fair amount of care will need to be taken to control it. For convenience, recall

$$\begin{aligned}[R_{8,1}(\lambda)f](\rho) =& -\frac{\langle\omega\rangle[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\bigg(\varphi_1'(\rho;\lambda)\int_0^\rho (1+s)^\lambda\tilde{c}_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds\\ &+\tilde{\varphi}_1'(\rho;\lambda)\int_\rho^1(1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds\bigg).\end{aligned}$$

For the integrals, integrating by parts yields

$$\begin{aligned}&\varphi_1'(\rho;\lambda)\int_0^\rho (1+s)^\lambda\tilde{c}_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds\\ &+\tilde{\varphi}_1'(\rho;\lambda)\int_\rho^1(1-s)^\lambda c_1(s;\lambda)[1-\chi(s\langle\omega\rangle)]f'(s)ds\\ =&-\varphi_1'(\rho;\lambda)\int_0^\rho(1+s)^{3+\lambda}\tilde{c}_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds\\ &-\tilde{\varphi}_1'(\rho;\lambda)\int_\rho^1(1-s)^{3+\lambda}c_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds\\ &+\gamma_{1,3}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)''-\gamma_{1,2}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)'\\ &+\gamma_{1,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho)\end{aligned}$$

For the integrals, we obtain the bound

$$
\begin{aligned}
[1-\chi(\rho\langle\omega\rangle)]\Big|\varphi_1'(\rho;\lambda)\int_0^\rho (1+s)^{3+\lambda}\tilde{c}_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds \\
+\tilde{\varphi}_1'(\rho;\lambda)\int_\rho^1 (1-s)^{3+\lambda}c_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds\Big| \\
\lesssim [1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^3\rho^{-3}\int_0^1 \langle\omega\rangle^{-2}s^3\big|\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}\big|ds \\
\lesssim \langle\omega\rangle\rho^{-3}\|f\|_{\Sigma_1}.
\end{aligned}
$$

This implies the bound

$$
\begin{aligned}
\big|[R_{8,1}(\lambda)f](\rho)\big| \lesssim &\langle\omega\rangle^{-4}[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle\rho^{-3}\|f\|_{\Sigma_1}+\langle\omega\rangle|f^{(3)}(\rho)|+\langle\omega\rangle^2|f''(\rho)| \\
&+\langle\omega\rangle^3|f'(\rho)|\Big) \\
\lesssim &\|f\|_{\Sigma_1}+\langle\omega\rangle^{-1}\big(|f'(\rho)|+|\rho f''(\rho)|+|\rho^2 f^{(3)}(\rho)|\big).
\end{aligned}
$$

A straightforward calculation involving one integration by parts shows

$$
\begin{aligned}
[R_{8,2}(\lambda)f](\rho) = &-\frac{\langle\omega\rangle\rho[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\Big(-\varphi_1''(\rho;\lambda)\int_0^\rho (1+s)^{1+\lambda}\tilde{c}_2(s;\lambda) \\
&\times\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)'ds \\
&-\tilde{\varphi}_1''(\rho;\lambda)\int_\rho^1 (1-s)^{1+\lambda}c_2(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)'ds \\
&+\gamma_{2,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho)\Big)
\end{aligned}
$$

which follows after one integrations by parts and recalling $\gamma_{1,0}(\rho;\lambda)=0$. Two integration by parts yields

$$\begin{aligned}
&\varphi_1''(\rho;\lambda)\int_0^\rho (1+s)^{1+\lambda}\tilde{c}_2(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)'ds \\
&+\tilde{\varphi}_1''(\rho;\lambda)\int_\rho^1 (1-s)^{1+\lambda}c_2(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)'ds \\
=&\varphi_1''(\rho;\lambda)\int_0^\rho (1+s)^{3+\lambda}\tilde{c}_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds \\
&+\tilde{\varphi}_1''(\rho;\lambda)\int_\rho^1 (1-s)^{3+\lambda}c_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds \\
&-\gamma_{2,3}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)''+\gamma_{2,2}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)'
\end{aligned}$$

For the integrals, we obtain the bound

$$\begin{aligned}
[1-\chi(\rho\langle\omega\rangle)]\Big|&\varphi_1''(\rho;\lambda)\int_0^\rho (1+s)^{3+\lambda}\tilde{c}_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds \\
&+\tilde{\varphi}_1''(\rho;\lambda)\int_\rho^1 (1-s)^{3+\lambda}c_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds\Big| \\
\lesssim&[1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^4\rho^{-3}\int_0^1\langle\omega\rangle^{-2}s^3\big|\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}\big|ds \\
\lesssim&\langle\omega\rangle^2\rho^{-3}\|f\|_{\Sigma_1}.
\end{aligned}$$

This implies the bound

$$\begin{aligned}
\big|[R_{8,2}(\lambda)f](\rho)\big|\lesssim&\langle\omega\rangle^{-4}\rho[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^2\rho^{-3}\|f\|_{\Sigma_1}+\langle\omega\rangle^2|f^{(3)}(\rho)|+\langle\omega\rangle^3|f''(\rho)| \\
&+\langle\omega\rangle^3\rho^{-1}|f'(\rho)|\Big) \\
\lesssim&\ \|f\|_{\Sigma_1}+\langle\omega\rangle^{-1}\big(|f'(\rho)|+|\rho f''(\rho)|+|\rho^2 f^{(3)}(\rho)|\big).
\end{aligned}$$

Another straightforward calculation involving one integration by parts shows

$$
\begin{aligned}
[R_{8,3}(\lambda)f](\rho) = & -\frac{\langle\omega\rangle\rho^2[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\bigg(\varphi_1^{(3)}(\rho;\lambda)\int_0^\rho(1+s)^{2+\lambda}\tilde{c}_3(s;\lambda) \\
& \times\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)''ds \\
& + \tilde{\varphi}_1^{(3)}(\rho;\lambda)\int_\rho^1(1-s)^{2+\lambda}c_3(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)''ds \\
& - \gamma_{3,2}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)' - \gamma_{2,1}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)' \\
& + \big(\gamma_{2,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)'\bigg).
\end{aligned}
$$

One integration by parts yields

$$
\begin{aligned}
& \varphi_1^{(3)}(\rho;\lambda)\int_0^\rho(1+s)^{2+\lambda}\tilde{c}_3(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)''ds \\
& + \tilde{\varphi}_1^{(3)}(\rho;\lambda)\int_\rho^1(1-s)^{2+\lambda}c_3(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)''ds \\
= & -\varphi_1^{(3)}(\rho;\lambda)\int_0^\rho(1+s)^{3+\lambda}\tilde{c}_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds \\
& - \tilde{\varphi}_1^{(3)}(\rho;\lambda)\int_\rho^1(1-s)^{3+\lambda}c_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds \\
& + \gamma_{3,3}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)''.
\end{aligned}
$$

For the integrals, we obtain the bound

$$
\begin{aligned}
[1-\chi(\rho\langle\omega\rangle)]\bigg| & \varphi_1^{(3)}(\rho;\lambda)\int_0^\rho(1+s)^{3+\lambda}\tilde{c}_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds \\
& + \tilde{\varphi}_1^{(3)}(\rho;\lambda)\int_\rho^1(1-s)^{3+\lambda}c_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds\bigg| \\
& \lesssim [1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^5\rho^{-3}\int_0^1\langle\omega\rangle^{-2}s^3\big|\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}\big|ds \\
& \lesssim \langle\omega\rangle^3\rho^{-3}\|f\|_{\Sigma_1}.
\end{aligned}
$$

This implies the bound

$$
\begin{aligned}
\big|[R_{8,3}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-4}\rho^2[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^3\rho^{-3}\|f\|_{\Sigma_1}+\langle\omega\rangle^3|f^{(3)}(\rho)|+\langle\omega\rangle^3\rho^{-1}|f''(\rho)| \\
&+\langle\omega\rangle^3\rho^{-2}|f'(\rho)|\Big) \\
&\lesssim \|f\|_{\Sigma_1}+\langle\omega\rangle^{-1}\big(|f'(\rho)|+|\rho f''(\rho)|+|\rho^2 f^{(3)}(\rho)|\big).
\end{aligned}
$$

Another straightforward calculation involving one integration by parts shows

$$
\begin{aligned}
[R_{8,4}(\lambda)f](\rho) =& -\frac{\langle\omega\rangle\rho^3[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\bigg(-\varphi_1^{(4)}(\rho;\lambda)\int_0^\rho(1+s)^{3+\lambda}\tilde{c}_4(s;\lambda) \\
&\times\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds \\
&-\tilde{\varphi}_1^{(4)}(\rho;\lambda)\int_\rho^1(1-s)^{3+\lambda}c_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds \\
&+\gamma_{4,3}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)''+\gamma_{3,2}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)'' \\
&-\Big(\gamma_{3,2}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)'\Big)'-\Big(\gamma_{2,1}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)'\Big)' \\
&+\Big(\gamma_{2,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho)\Big)''\bigg)
\end{aligned}
$$

which follows after one integration by parts. For the integrals, we obtain the bound

$$
\begin{aligned}
[1-\chi(\rho\langle\omega\rangle)]\bigg|&\varphi_1^{(4)}(\rho;\lambda)\int_0^\rho(1+s)^{3+\lambda}\tilde{c}_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds \\
&+\tilde{\varphi}_1^{(4)}(\rho;\lambda)\int_\rho^1(1-s)^{3+\lambda}c_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds\bigg| \\
\lesssim& [1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^6\rho^{-3}\int_0^1\langle\omega\rangle^{-2}s^3\big|\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}\big|ds \\
\lesssim& \langle\omega\rangle^4\rho^{-3}\|f\|_{\Sigma_1}.
\end{aligned}
$$

This implies the bound

$$\begin{aligned}\big|[R_{8,4}(\lambda)f](\rho)\big| \lesssim& \langle\omega\rangle^{-4}\rho^3[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^4\rho^{-3}\|f\|_{\Sigma_1}+\langle\omega\rangle^3\rho^{-1}|f^{(3)}(\rho)+\langle\omega\rangle^3\rho^{-2}|f''(\rho)| \\ &+\langle\omega\rangle^3\rho^{-3}|f'(\rho)|\Big) \\ \lesssim& \|f\|_{\Sigma_1}+\langle\omega\rangle^{-1}\big(|f'(\rho)|+|\rho f''(\rho)|+|\rho^2 f^{(3)}(\rho)|\big).\end{aligned}$$

Lastly, a straightforward calculation shows

$$\begin{aligned}[R_{8,5}(\lambda)f](\rho) =& -\frac{\rho^3[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)}\Bigg(-\varphi_1^{(5)}(\rho;\lambda)\int_0^\rho (1+s)^{3+\lambda}\tilde{c}_4(s;\lambda) \\ &\times\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds \\ &-\tilde{\varphi}_1^{(5)}(\rho;\lambda)\int_\rho^1(1-s)^{3+\lambda}c_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds \\ &-\gamma_{4,3}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)^{(3)}+\Big(\gamma_{4,3}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)''\Big)' \\ &+\Big(\gamma_{3,2}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)''\Big)'-\Big(\gamma_{3,2}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)'\Big)'' \\ &-\Big(\gamma_{2,1}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)]f'(\rho)\big)'\Big)''+\Big(\gamma_{2,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)]f'(\rho)\Big)^{(3)}\Bigg).\end{aligned}$$

For the first integral, we obtain the bound

$$\begin{aligned}&[1-\chi(\rho\langle\omega\rangle)]\left|\varphi_1^{(5)}(\rho;\lambda)\int_0^\rho(1+s)^{3+\lambda}\tilde{c}_4(s;\lambda)\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}ds\right| \\ &\lesssim[1-\chi(\rho\langle\omega\rangle)]\langle\omega\rangle^7\rho^{-3}\int_0^1\langle\omega\rangle^{-2}s^3\big|\big([1-\chi(s\langle\omega\rangle)]f'(s)\big)^{(3)}\big|ds \\ &\lesssim\langle\omega\rangle^5\rho^{-3}\|f\|_{\Sigma_1}.\end{aligned}$$

The second integral, like the operators $T_{1,k}$ and $T_{2,k}$, is too singular at $\rho=1$ to be controlled in a straightforward manner. Instead, we define $\tilde{R}_{8,5}(\lambda):\mathcal{D}(\tilde{R}_{8,5}(\lambda))\subset L^2(0,1)\to L^2(0,1)$ by

$$\mathcal{D}(\tilde{R}_{8,5}(\lambda))=C_c^\infty(0,1),\ [\tilde{R}_{8,5}(\lambda)f](\rho):=\int_0^1 K_{8,5}(\rho,s;\lambda)f(s)ds$$

where the kernel is given by

$$K_{8,5}(\rho, s; \lambda) := \frac{\rho^3[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)} \tilde{\varphi}_1^{(5)}(\rho;\lambda)(1-s)^{3+\lambda} c_4(s;\lambda) s^{-3} 1_{\mathbb{R}^+}(s-\rho).$$

A straightforward estimate shows that the kernel satisfies the pointwise bound

$$|K_{8,5}(\rho, s; \lambda)| \lesssim \min\{(1-\rho)^{-1}, (1-s)^{-1}\}.$$

Thus, Lemma 35 implies that $\tilde{R}_{8,5}(\lambda)$ extends to a bounded operator on $L^2(0,1)$. Thus, we have the equality

$$\begin{aligned}
[R_{8,5}(\lambda)f](\rho) = &-\frac{\rho^3[1-\chi(\rho\langle\omega\rangle)]}{W(\lambda)} \bigg(-\varphi_1^{(5)}(\rho;\lambda) \int_0^\rho (1+s)^{3+\lambda} \tilde{c}_4(s;\lambda) \\
&\times \big([1-\chi(s\langle\omega\rangle)] f'(s)\big)^{(3)} ds \\
&- \gamma_{4,3}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)] f'(\rho)\big)^{(3)} + \Big(\gamma_{4,3}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)] f'(\rho)\big)''\Big)' \\
&+ \Big(\gamma_{3,2}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)] f'(\rho)\big)''\Big)' - \Big(\gamma_{3,2}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)] f'(\rho)\big)'\Big)'' \\
&- \Big(\gamma_{2,1}(\rho;\lambda)\big([1-\chi(\rho\langle\omega\rangle)] f'(\rho)\big)'\Big)'' + \Big(\gamma_{2,1}(\rho;\lambda)[1-\chi(\rho\langle\omega\rangle)] f'(\rho)\Big)^{(3)}\bigg) \\
&+ \bigg[\tilde{R}_{8,5}(\lambda)\bigg(s^3\Big(\big[1-\chi\big((\cdot)\langle\omega\rangle\big)\big] f'\Big)^{(3)}\bigg)\bigg](\rho)
\end{aligned}$$

for $f \in C_e^\infty[0,1]$. Consequently, we obtain the bound

$$\begin{aligned}
\big|[R_{8,5}(\lambda)f](\rho)\big| \lesssim &\langle\omega\rangle^{-5}\rho^3[1-\chi(\rho\langle\omega\rangle)]\Big(\langle\omega\rangle^5\rho^{-3}\|f\|_{\Sigma_1} + \langle\omega\rangle^3\rho^{-1}|f^{(4)}(\rho)| + \langle\omega\rangle^3\rho^{-2}|f^{(3)}(\rho)| \\
&+ \langle\omega\rangle^3\rho^{-3}|f''(\rho)| + \langle\omega\rangle^3\rho^{-4}|f'(\rho)|\Big) \\
\lesssim &\|f\|_{\Sigma_1} + \langle\omega\rangle^{-1}\big(|f'(\rho)| + |\rho f''(\rho)| + |\rho^2 f^{(3)}(\rho)| + |\rho^3 f^{(4)}(\rho)|\big).
\end{aligned}$$

Taking $L^2(0,1)$ norms yields the desired bound.

2.6 Appendix

2.6.1 Proof of Equivalent Norms

In this section, we prove Proposition 4. It suffices to prove the equivalence on $C_e^\infty[0,1]^2$ as the rest of $\mathcal{H}$ follows by density. We begin with the following lemma.

Lemma 38 *We have that* $\|\mathbf{u}\|_\Sigma^2 \lesssim \|D_7 u_1\|_{H^3(0,1)}^2 + \|D_7 u_2\|_{H^2(0,1)}^2$ *for all* $\mathbf{u} \in C_e^\infty[0,1]^2$.

Proof. It suffices to show

$$\|u\|_{\Sigma_0}^2 \lesssim \|D_7 u\|_{H^3(0,1)}^2$$

for all $u \in C_e^\infty[0,1]^2$. A proof of the analogous estimate for $\|\cdot\|_{\Sigma_1}$ can be found in [17] Lemma C.3. Begin by setting $w := D_7 u$. Then $w \in C^\infty[0,1]$ and $w^{(2n)}(0) = 0$ for $n \in \mathbb{N}_0$. Upon applying the fundamental theorem of calculus and using the fact that $w(0) = w''(0) = 0$, we have that

$$w(\rho) = \rho w'(0) + \int_0^\rho \int_0^\sigma \int_0^\tau w'''(r) dr d\tau d\sigma.$$

By setting $\mathcal{V}w(\rho) := \int_0^\rho w(s)ds$, we infer that

$$K_7 w(\rho) = k_7 w'(0) + K_7 \mathcal{V}^3 w'''(\rho)$$

for some constant $k_7 \in \mathbb{R}$. We have

$$\|u\|_{H^1(0,1)} \lesssim \|D_7 u\|_{H^2(0,1)}$$

as a consequence of Lemma C.3 of [17]. Thus, it is sufficient to show

$$\|(\cdot)^{n-2}(K_7 w)^n\|_{L^2(0,1)}^2 \lesssim \|w\|_{H^3(0,1)}^2$$

for $n = 2, 3, 4, 5$. By direct computation, we find

$$(K_7 w)''(\rho) = \sum_{j=0}^{2} \beta_{2,j} \rho^{-3-2j} \mathcal{K}^j \mathcal{V}^3 w'''(\rho)$$

$$\rho (K_7 w)'''(\rho) = \sum_{j=0}^{2} \beta_{3,j} \rho^{-3-2j} \mathcal{K}^j \mathcal{V}^3 w'''(\rho) + \alpha_{3,2} \rho^{-2} \mathcal{V}^2 w'''(\rho)$$

$$\rho^2 (K_7 w)^{(4)}(\rho) = \sum_{j=0}^{2} \beta_{4,j} \rho^{-3-2j} \mathcal{K}^j \mathcal{V}^3 w'''(\rho) + \alpha_{4,2} \rho^{-2} \mathcal{V}^2 w'''(\rho) + \alpha_{4,1} \rho^{-1} \mathcal{V} w'''(\rho)$$

$$\rho^3 (K_7 w)^{(5)}(\rho) = \sum_{j=0}^{2} \beta_{5,j} \rho^{-3-2j} \mathcal{K}^j \mathcal{V}^3 w'''(\rho) + \alpha_{5,2} \rho^{-2} \mathcal{V}^2 w'''(\rho)$$
$$+ \alpha_{5,1} \rho^{-1} \mathcal{V} w'''(\rho) + w'''(\rho)$$

for some constants $\beta_{n,j}, \alpha_{n,j} \in \mathbb{Z}$. Since $(\mathcal{K}^2 \mathcal{V}^3 w''')^{(n)}(0) = 0$ for $n = 0, 1, 2, 3, 4, 5, 6$, $(\mathcal{K} \mathcal{V}^3 w''')^{(n)}(0) = 0$ for $n = 0, 1, 2, 3, 4$, and $(\mathcal{V}^3 w''')^{(n)}(0) = 0$ for $n = 0, 1, 2$, the desired inequality follows by repeated application of Hardy's inequality (see [17] Lemma A.1 for the applicable version of Hardy's inequality). ■

Having established a relationship between the Σ-norm and quantities involving the D_7 operator, we can make a connection to the D_7-norm according to the following lemma.

Lemma 39 *We have that* $\|\mathbf{u}\|_{D_7}^2 \simeq \|D_7 u_1\|_{H^3(0,1)}^2 + \|D_7 u_2\|_{H^2(0,1)}^2$ *for* $\mathbf{u} \in C_e^\infty[0,1]^2$.

Proof. The inequality $\|\mathbf{u}\|_{D_7}^2 \lesssim \|D_7 u_1\|_{H^3(0,1)}^2 + \|D_7 u_2\|_{H^2(0,1)}^2$ follows from the definition of $\| \cdot \|_{D_7}$. For the reverse direction, it suffices to show that

$$\|D_7 u_1\|_{L^2(0,1)}^2 \lesssim \|D_7 u_1\|_{\dot{H}^1(0,1)}^2.$$

This is a simple consequence of the fact that $[D_7 u_1](0) = 0$. Begin by applying the fundamental theorem of calculus

$$D_7 u_1(\rho) = \int_0^\rho [D_7 u_1]'(s) ds.$$

Thus,

$$|D_7 u_1(\rho)| \leq \int_0^1 \left|[D_7 u_1]'(s)\right| ds \leq \|D_7 u'\|_{L^2(0,1)}^{\frac{1}{2}}$$

by the Cauchy-Schwarz inequality. The desired estimate follows by squaring and integrating the above inequality. ■

As a consequence, we have $\|\mathbf{u}\|_\Sigma \lesssim \|\mathbf{u}\|_{D_7}$. The reverse inequality is rather easy though we state it for completeness.

Lemma 40 *We have that* $\|\mathbf{u}\|_{D_7} \lesssim \|\mathbf{u}\|_\Sigma$ *for all* $\mathbf{u} \in C_e^\infty[0,1]^2$.

Proof. This is an immediate consequence of the definition of D_7 and the triangle inequality.

■

Lastly, we must now establish the equivalence with the radial Sobolev norms. We do so by showing equivalence with the Σ-norm.

Lemma 41 *Let* $u \in C_e^\infty[0,1]$. *Then*

$$\|u\|_{\Sigma_0} \simeq \|u(|\cdot|)\|_{H^5(\mathbb{B}^7)}$$

and

$$\|u\|_{\Sigma_1} \simeq \|u(|\cdot|)\|_{H^4(\mathbb{B}^7)}.$$

Proof. The second estimate is proven in [17] Lemma B.1. First, we recall that

$$|u^{(n)}(|x|)|^2 \lesssim \sum_{|\alpha|=n} |\partial_x^\alpha u(|x|)|^2.$$

For a proof, see [24]. From this inequality, it follows

$$\begin{aligned}\|(\cdot)^3 u^{(5)}\|^2_{L^2(0,1)} &\simeq \|u^{(5)}(|\cdot|)\|^2_{L^2(\mathbb{B}^7)} \\ &\lesssim \sum_{|\alpha|=n} \|\partial_x^\alpha u(|x|)\|^2_{L^2(\mathbb{B}^7)} \\ &\lesssim \|u(|\cdot|)\|^2_{H^5(\mathbb{B}^7)}.\end{aligned}$$

We control lower order derivatives by use of Lemma 2.12 of [24] as follows:

$$\begin{aligned}\|(\cdot)^2 u^{(4)}\|_{L^2(0,1)} &\lesssim |u^{(4)}(1)| + \|(\cdot)^3 u^{(5)}\|^2_{L^2(0,1)} \\ &\lesssim \|(\cdot)^3 u^{(4)}\|^2_{L^2(0,1)} + \|(\cdot)^3 u^{(5)}\|^2_{L^2(0,1)}.\end{aligned}$$

The fourth derivative term can be controlled by $\|u(|\cdot|)\|_{H^5(\mathbb{B}^7)}$ by the same argument we used for the fifth derivative term. Repeating this argument for each derivative yields the first direction of the desired result. For the reverse direction, we recall the identity

$$\sum_{|\alpha|=n} \partial_x^\alpha u(|x|) = \sum_{j=1}^{n} u^{(j)}(|x|) \frac{P_j(x)}{|x|^{2n-j}}$$

for some homogeneous polynomials P_j of degree n which satisfy the estimate $|P_j(x)| \lesssim |x|^n$. Consequently, for $2 \leq n \leq 5$, we have

$$\begin{aligned}\|u(|\cdot|)\|^2_{\dot{H}^n(\mathbb{B}^7)} &\lesssim \sum_{j=1}^{n} \|(\cdot)^{3+j-n} u^{(j)}\|^2_{L^2(0,1)} \\ &\leq \|(\cdot)^{-1} u'\|^2_{L^2(0,1)} + \sum_{j=2}^{n} \|(\cdot)^{j-2} u^{(j)}\|^2_{L^2(0,1)} \\ &\lesssim \sum_{j=2}^{n} \|(\cdot)^{j-2} u^{(j)}\|^2_{L^2(0,1)}\end{aligned}$$

where the last line follows from $u'(0) = 0$ and an application of Hardy's inequality. For the first derivative, observe that

$$\|u(|\cdot|)\|^2_{\dot{H}^1(\mathbb{B}^7)} \lesssim \|(\cdot)^3 u'\|^2_{L^2(0,1)} \leq \|u'\|^2_{L^2(0,1)}$$

and similarly for the zeroth derivative. This proves the claim. ∎

2.6.2 Explicit Expressions for Resolvent Estimates

The functions $c_j(s;\lambda)$ and $\tilde{c}_j(s;\lambda)$ are given explicitly by

$$c_0(s;\lambda) = 3s + 3(\lambda-1)s^2 + (\lambda-1)^2 s^3 + (\lambda-1)s^4,$$

$$c_1(s;\lambda) = -\frac{1}{(\lambda+1)(\lambda+4)}\big(15 + 15\lambda s + 6(\lambda^2-1)s^2 + \lambda(\lambda^2-1)s^3 + (\lambda^2-1)s^4\big),$$

$$\begin{aligned} c_2(s;\lambda) = &\frac{1}{(\lambda+1)(\lambda+3)(\lambda+4)(\lambda+5)}\big(48(\lambda+4) + 3(11\lambda^2+40\lambda-11)s \\ &+ (9\lambda^3+33\lambda^2-9\lambda-33)s^2 + (\lambda+1)^2(\lambda^2+2\lambda-3)s^3 \\ &+ (\lambda^3+3\lambda^2-\lambda-3)s^4\big), \end{aligned}$$

$$\begin{aligned} c_3(s;\lambda) = &-\frac{1}{(\lambda+1)(\lambda+3)(\lambda+4)(\lambda+5)(\lambda+6)}\big(105(\lambda+5) + 3(19\lambda^2+85\lambda-34)s \\ &+ 6(2\lambda^3+9\lambda^2-2\lambda-9)s^2 + (\lambda^4+5\lambda^3+5\lambda^2-5\lambda-6)s^3 \\ &+ (\lambda^3+3\lambda^2-\lambda-3)s^4\big), \end{aligned}$$

$$\begin{aligned} c_4(s;\lambda) = &\frac{1}{(\lambda+1)(\lambda+3)(\lambda+4)(\lambda+5)(\lambda+6)(\lambda+7)}\big(192(\lambda+6) \\ &+ 3(29\lambda^2+156\lambda-73)s + 3(5\lambda^3+27\lambda^2-5\lambda-27)s^2 \\ &+ (\lambda+3)^2(\lambda^2-1)s^3 + (\lambda^3+3\lambda^2-\lambda-3)s^4\big), \end{aligned}$$

$$\begin{aligned} c_5(s;\lambda) = &-\frac{1}{(\lambda+1)(\lambda+3)(\lambda+4)(\lambda+5)(\lambda+6)(\lambda+7)(\lambda+8)}\big(315(\lambda+7) \\ &+ 3(41\lambda^2+259\lambda-132)s + 6(3\lambda^3+19\lambda^2-3\lambda-19)s^2 \\ &+ (\lambda^4+7\lambda^3+11\lambda^2-7\lambda-12)s^3 + (\lambda^3+3\lambda^2-\lambda-3)s^4\big), \end{aligned}$$

and $\tilde{c}_j(s;\lambda) = (-1)^{j+1}c_j(-s;\lambda)$.

Chapter 3

Conditionally Stable Blow-Up for the Quadratic Wave Equation on the Whole Space

3.1 Introduction

This chapter concerns the following radial quadratic wave equation

$$u_{tt} - u_{rr} - \frac{d-1}{r}u_r = u^2 \tag{3.1}$$

for $u : I \times [0, \infty) \to \mathbb{R}$, $I \subset \mathbb{R}$ an interval containing zero, and $r = |x|$ for $x \in \mathbb{R}^d$. Equation (3.1) exhibits the scaling symmetry $u \mapsto u_\lambda$,

$$u_\lambda(t, r) := \lambda^{-2}u(t/\lambda, r/\lambda)$$

for any $\lambda > 0$. This rescaling leaves invariant the energy norm $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ precisely when $d = 6$ which defines the energy critical case.

Equation (3.1), in all supercritical dimensions $d \geq 7$, (3.1) has a smooth, radial, self-similar solution for $(t, r) \in [0, T) \times [0, \infty)$ given by

$$u_T^*(t, r) := \frac{1}{(T-t)^2} U\Big(\frac{r}{T-t}\Big), \ T > 0 \tag{3.2}$$

with

$$U(\rho) := \frac{c_1 - c_2 \rho^2}{(\rho^2 + c_3)^2}$$

where

$$c_1 = \frac{4}{25}\big((3d-8)\sqrt{6(d-1)(d-6)} + 8d^2 - 56d + 48\big),$$

$$c_2 = \frac{4}{5}\sqrt{6(d-1)(d-6)},$$

and

$$c_3 = \frac{1}{15}\big(3d - 18 + \sqrt{6(d-1)(d-6)}\big)$$

which becomes singular forward in time as $t \to T^-$. This solution was recently introduced in [9] by Csobo, Glogić, and Schörkhuber where they established a co-dimension one stability result for u_T^* under non-radial perturbations in $d = 9$. Their result tracked the evolution of such data within the backwards lightcone of the blow-up point $(T, 0)$. Of course, even radial, compactly supported perturbations of u_T^* influence the corresponding solution in a region of spacetime strictly larger than these lightcones. One should be left wondering if the solution could blow-up in this larger region of spacetime.

Before addressing this issue, first observe that u_T^*, as stated, is not defined for $t = T$. However, inserting the expression for $U\big(\frac{r}{T-t}\big)$ into the right-hand side of Equation

(3.2) and rearranging shows that it is precisely given by

$$u_T^*(t,r) = \frac{c_1(T-t)^2 - c_2 r^2}{\big(r^2 + c_3(T-t)^2\big)^2}$$

which is well-defined for all $(t,r) \neq (T,0)$. With this in mind, we can properly address the stability of u_T^* outside of lightcones which, in particular, includes regions of spacetime where $t > T$. We address precisely this issue in $d = 7$, the lowest energy supercritical dimension, where u_T^* takes the form

$$u_T^*(t,r) = \frac{24\big(21(T-t)^2 - 5r^2\big)}{\big(3(T-t)^2 + 5r^2\big)^2}.$$

The key ingredient allowing us to access this larger region of spacetime is the coordinate system called *hyperboloidal similarity coordinates*. These coordinates were first introduced in [1] for the study of the stability of self-similar blow-up in wave maps outside of backwards lightcones. They are well-adapted to self-similarity much like standard similarity coordinates typically used in the study of self-similar blow-up. However, hyperboloidal similarity coordinates have the significant advantage that they cover regions of spacetime past the blow-up time.

Surprisingly, the stability of u_T^* is a subtly difficult problem to approach. To begin understanding this, first observe that the linearized equation, $\Box u = 2u_T^* u$, has the following two smooth solutions given explicitly by

$$F_1^*(t,r) = \frac{(T-t)\big(7(T-t)^2 - 15r^2\big)}{\big(5r^2 + 3(T-t)^2\big)^3}$$

and

$$F_4^*(t,r) = \frac{1}{\big(5r^2 + 3(T-t)^2\big)^3}.$$

After we reformulate the problem in hyperboloidal similarity coordinates, we will see that these two solutions correspond to exponentially growing solutions of the linearized equation, i.e., solutions which can destroy any hope for stability. Regardless, these instabilities can be accounted for in a systematic way. As a first step, notice that the existence of the first solution is precisely due to the time translation symmetry of Equation (3.1). More precisely, observe that $\partial_T u_T^* = -432 F_1^*$. Consequently, we can account for this instability by adjusting the blow-up time. On the other hand, F_4^* does not appear to have a connection with any spacetime symmetry. To account for the instability due to F_4^*, one might expect that adjusting perturbations of u_T^* by some multiple of F_4^* could stabilize the evolution. Unfortunately, our techniques rely crucially on the data having compact support and, clearly, F_4^* does not have this property. Thus, one might expect multiplying by a cutoff could fix this issue. This almost works, but doing so carelessly poses major issues in controlling the evolution of data along hyperboloids.

The main novelty of our work is in presenting a technique which stabilizes the evolution of data close to that of a self-similar blow-up solution which has at least two unstable directions, one not coming from any spacetime symmetry, in a region of spacetime which can be made arbitrarily close to the Cauchy horizon of the singularity. Put simply, we achieve this by constructing a particular smooth solution of the linearized equation which has a special property allowing us to control the evolution of such data. Doing so requires a delicate interplay between the standard Cauchy evolution of data along hypersurfaces of constant physical time and that of data along hypersurfaces of constant hyperboloidal time. This will be elaborated on extensively in Section 3.5.2.

3.1.1 Hyperboloidal Similarity Coordinates

In this section, we introduce the hyperboloidal similarity coordinate system and summarize their essential properties. Given $T > 0$, we define *hyperboloidal similarity coordinates* by the map

$$\begin{aligned} \eta_T : \mathbb{R} \times [0, \infty) &\to \mathbb{R} \times [0, \infty) \\ (s, y) &\mapsto \big(T + e^{-s} h(y), e^{-s} y\big) \end{aligned}$$

where

$$h(y) = \sqrt{2 + y^2} - 2$$

which we call the *height function*. We remark that η_T defines a diffeomorphism onto its image. The specific form of h is arbitrary except for the fact that the level-sets

$$\{(s, y) \in \mathbb{R} \times [0, \infty) : s = c\},\ c \in \mathbb{R},$$

are Cauchy surfaces which asymptote to a forward lightcone. Furthermore, $y = \frac{1}{2}$ corresponds to the backwards lightcone of the singularity $(T, 0)$. See Figure 3.1

Observe that taking $h(y) = -1$ returns standard similarity coordinates. It is precisely due to the nontrivial nature of the height function that the coordinates cover a region of spacetime outside of the backwards lightcone of $(T, 0)$.

3.1.2 Statement of the Main Result

We are now ready to state the main result of this chapter. Our theorem concerns the evolution of small, smooth, radial, and suitably adjusted perturbations of u_1^* according to Equation (3.1).

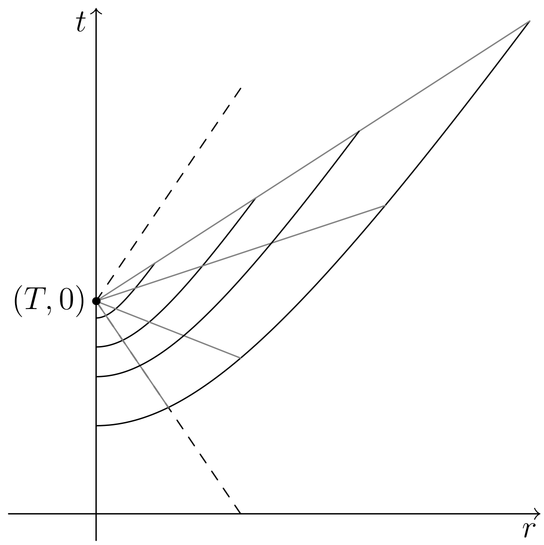

Figure 3.1: A spacetime diagram depicting the hyperboloidal coordinate system. The dashed lines are the boundary of the forward and backward lightcones of the point $(T, 0)$. The hyperboloids correspond to $s = \text{const}$ and the rays emanating from $(T, 0)$ correspond to $y = \text{const}$.

Theorem 42 *Fix $R \geq \frac{1}{2}$ and consider the spacetime region*

$$\Omega_{T,R} = \{(t, r) \in \mathbb{R} \times [0, \infty) : 0 \leq t < T + br\},\ b = \frac{h(R)}{R},$$

see Figure 3.2. There exist positive constants $\delta, r_0, M_0, \omega_0$ such that the following holds.

1. *There exists a pair of functions $\big(Y_1, Y_2\big) \in C_e^\infty[0,\infty)^2$ supported in the interval $[0, r_0)$ such that for any pair of radial functions $(f, g) \in C_e^\infty[0, \infty)^2$ also supported in the interval $[0, r_0)$ and satisfying*

$$\|(f, g)\|_{H^{10}_{rad}(\mathbb{R}^7)\times H^9_{rad}(\mathbb{R}^7)} \leq \frac{\delta}{M_0^2},$$

there exists $\alpha \in [-\frac{\delta}{M_0}, \frac{\delta}{M_0}]$, $T \in [1-\frac{\delta}{M_0}, 1+\frac{\delta}{M_0}]$, and a unique function $u \in C^\infty(\Omega_{T,R})$

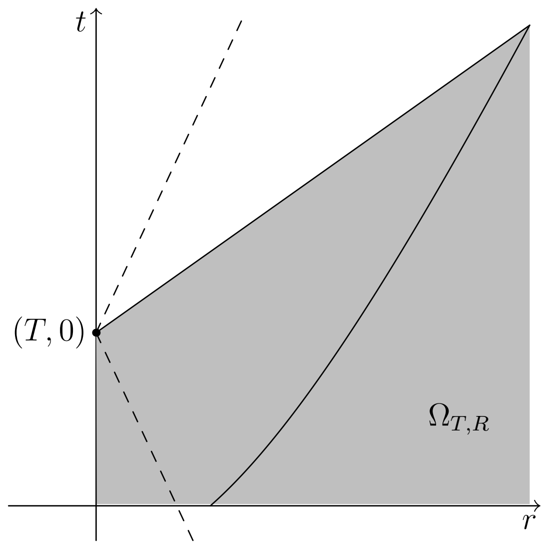

Figure 3.2: A spacetime diagram depicting the region $\Omega_{T,R}$. The dashed lines are the boundary of the forward and backward lightcones of the blow-up point $(T, 0)$. As R increases, the top of the shaded region approaches the forward lightcone.

that satisfies

$$\begin{cases} \left(\partial_t^2 - \partial_r^2 - \frac{d-1}{r}\partial_r\right)u(t,r) = u(t,r)^2 & (t,r) \in \Omega_{T,R} \\ u(0,r) = u_1^*(0,r) + \alpha Y_1(r) + f(r) & r \in [0,\infty) \\ \partial_t u(0,r) = \partial_t u_1^*(0,r) + \alpha Y_2(r) + g(r) & r \in [0,\infty) \end{cases} . \tag{3.3}$$

2. *The solution u converges to u_T^* in the sense that*

$$\begin{aligned} e^{-2s}\|(u \circ \eta_T)(s,\cdot) - (u_T^* \circ \eta_T)(s,\cdot)\|_{H^6_{rad}(\mathbb{B}^7_R)} &\leq \delta e^{-\omega_0 s} \\ e^{-2s}\|\partial_s(u \circ \eta_T)(s,\cdot) - \partial_s(u_T^* \circ \eta_T)(s,\cdot)\|_{H^5_{rad}(\mathbb{B}^7_R)} &\leq \delta e^{-\omega_0 s} \end{aligned} \tag{3.4}$$

for all $s \geq 0$.

3. *In the spacetime region $\Omega_{T,R} \setminus \eta_T\big([s_0,\infty) \times [0,R)\big)$, where $s_0 = \log\left(-\frac{2h(0)}{2+r_0}\right)$, we have $u = u_1^*$.*

Some remarks on Theorem 42 are in order.

1. *The adjustment term.* The functions Y_1 and Y_2 will be constructed in Section 3.5.2. As will be seen, they derive from a particular solution of the quadratic wave equation linearized around u_1^*, i.e. $\Box u = 2u_1^* u$. Their presence in the data allows us to account for the instability due to F_4^* with $T = 1$ in the evolution of small perturbations of u_1^*.

2. *High degree of regularity.* Within integer Sobolev spaces, one might expect to work at the regularity $H^3 \times H^2$ in light of the well-posedness theory developed in [14]. Our techniques necessitate the use of various Sobolev embeddings which impose the high degree of regularity appearing in Theorem 42.

3. *Normalizing factors.* The factors e^{-2s} in Equation (3.4) reflect the convergence of the corresponding blow-up profiles. This can be seen explicitly by observing that the blow-up solution transforms according to

$$(u_T^* \circ \eta_T)(s, y) = -\frac{24e^{2s}\left(5y^2 - 21h(y)^2\right)}{\left(3h(y)^2 + 5y^2\right)^2}.$$

4. *Higher space dimensions.* Theorem 42 is stated only for $d = 7$ though it can certainly be generalized to $d = 9$ using the spectral results in [9]. For odd space dimensions $d \geq 11$, it is unclear whether or not the spectral problem can be solved. If it can, then our result can also be generalized to higher odd space dimensions.

5. *The spectral problem.* As mentioned in the previous remark, a key step in our proof is solving a particular spectral problem. The spectral problem we solve is exceptionally difficult compared to similar problems encountered in the stability of self-similar blow-up. This will be expanded upon in Section 3.3.2.

3.2 The Wave Equation in Hyperboloidal Similarity Coordinates

In this section, we reformulate Equation (3.1) as a first-order system in hyperboloidal similarity coordinates. First, we review the well-posedness theory for the radial wave equation in hyperboloidal similarity coordinates as developed in [14] by Donninger and Ostermann.

3.2.1 Free Wave Evolution in Hyperboloidal Similarity Coordinates

Let $d \in \mathbb{N}$ and $u \in C^\infty(\mathbb{R} \times (0,\infty))$. With $v = u \circ \eta_T$, we infer by the chain rule the following transformation

$$\left(\Big(\partial_t^2 - \partial_r^2 - \frac{d-1}{r}\partial_r\Big)u\right) \circ \eta_T = -g^{00}(\partial_s^2 - c_{11}^d \partial_y - c_{12}\partial_y^2 - c_{20}^d\partial_s - c_{21}\partial_y\partial_s)v$$

where

$$\begin{aligned}
g^{00}(s,y) =& - e^{2s}\frac{1-h'(y)^2}{\big(yh'(y)-h(y)\big)^2} \\
c_{11}^d(y) =& - \frac{d-1}{y}\frac{\big(yh'(y)-h(y)\big)h(y)}{1-h'(y)^2} + \frac{y^2-h(y)^2}{1-h'(y)^2}\frac{yh''(y)}{yh'(y)-h(y)} + 2\frac{h(y)h'(y)-y}{1-h'(y)^2} \\
c_{12}(y) =& \frac{h(y)^2-y^2}{1-h'(y)^2} \\
c_{20}^d(y) =& -1 - \frac{d-1}{y}\frac{\big(yh'(y)-h(y)\big)h'(y)}{1-h'(y)^2} + \frac{y^2-h(y)^2}{1-h'(y)^2}\frac{h''(y)}{yh'(y)-h(y)} \\
c_{21}(y) =& 2\frac{h(y)h'(y)-y}{1-h'(y)^2}.
\end{aligned}$$

Definition 43 *Let $R > 0$ and $d, k \in \mathbb{N}$. We define the* ***free radial wave evolution*** *as* *the unbounded operator* $\left(\tilde{\mathbf{L}}_d, \mathcal{D}(\tilde{\mathbf{L}}_d)\right)$*, with* $\mathcal{D}(\tilde{\mathbf{L}}_d) = C_e^\infty[0,R]^2$*, on* $\mathcal{H}_R^k$ *by*

$$\tilde{\mathbf{L}}_d\mathbf{f}(y) := \begin{pmatrix} f_2(y) \\ c_{11}^d(y)f_1'(y) + c_{12}(y)f_1''(y) + c_{20}^d(y)f_2(y) + c_{21}(y)f_2'(y) \end{pmatrix}.$$

The linear, radial wave equation in hyperboloidal similarity coordinates is equivalent to the first-order system

$$\partial_s \mathbf{v}(s,\cdot) = \tilde{\mathbf{L}}_d \mathbf{v}(s,\cdot)$$

where

$$\mathbf{v}(s,\cdot) = \begin{pmatrix} v(s,\cdot) \\ \partial_s v(s,\cdot) \end{pmatrix}.$$

In [14], it was shown that $\left(\tilde{\mathbf{L}}_d, \mathcal{D}(\tilde{\mathbf{L}}_d)\right)$ is closable and its closure, $\left(\mathbf{L}_d, \mathcal{D}(\mathbf{L}_d)\right)$, generates a strongly continuous semigroup which we recall here.

Lemma 44 ([14], Theorem 2.1) *Let* $R \geq \frac{1}{2}$*,* $d, k, \in \mathbb{N}$*, such that* $d \geq 3$ *is odd and* $k \geq \frac{d-1}{2}$*. The operator* $\left(\tilde{\mathbf{L}}_d, \mathcal{D}(\tilde{\mathbf{L}}_d)\right)$ *is closable and its closure,* $\left(\mathbf{L}_d, \mathcal{D}(\mathbf{L}_d)\right)$*, is the generator of a strongly continuous semigroup* $\left(\mathbf{S}_d(s)\right)_{s\geq 0}$ *of bounded operators on* $\mathcal{H}_R^k$ *with the property that there exists* $M \geq 1$ *such that*

$$\|\mathbf{S}_d(s)\mathbf{f}\|_{\mathcal{H}_R^k} \leq M e^{\frac{s}{2}} \|\mathbf{f}\|_{\mathcal{H}_R^k}$$

for all $s \geq 0$ *and* $\mathbf{f} \in \mathcal{H}_R^k$*.*

3.2.2 The Quadratic Wave Equation in Hyperboloidal Similarity Coordinates

Now, we can reformulate Equation (3.1) as a first-order system in hyperboloidal similarity coordinates. For the remainder of this chapter, we fix $d = 7$. We look for solutions of the form $u = \tilde{u} + u_T^*$, for some T sufficiently close to 1 yet to be determined, where $\tilde{u}$ represents some small perturbation of u_T^*. Equation (3.1) becomes

$$\left(\partial_t^2 - \partial_r^2 - \frac{6}{r}\partial_r + V_T\right)\tilde{u} = \tilde{u}^2$$

where $V_T := -2u_T^*$. Setting $\tilde{v} = \tilde{u} \circ \eta_T$, we obtain the equation

$$\partial_s^2 \tilde{v} = c_{11}\partial_y \tilde{v} + c_{12}\partial_y^2 \tilde{v} + c_{20}\partial_s \tilde{v} + c_{21}\partial_y \partial_s \tilde{v} + \frac{V_T \circ \eta_T}{g^{00}}\tilde{v} - \frac{\tilde{v}^2}{g^{00}}$$

where $c_{11} := c_{11}^7$ and $c_{20} := c_{20}^7$. Observe that the function

$$V(y) := \frac{V_T\big(\eta_T(s,y)\big)}{g^{00}(s,y)}$$

is in $C_e^\infty[0,R]$ for any $R > 0$. Furthermore, we write

$$-\frac{\tilde{v}(s,y)^2}{g^{00}(s,y)} = e^{2s} N(y, e^{-2s}\tilde{v})$$

where

$$N(y,x) := \frac{\big(yh'(y) - h(y)\big)^2}{1 - h'(y)^2} x^2.$$

Upon setting

$$\tilde{\mathbf{v}}(s,\cdot) := \begin{pmatrix} \tilde{v}(s,\cdot) \\ \partial_s \tilde{v}(s,\cdot) \end{pmatrix},$$

the quadratic wave equation, as a first-order system in hyperboloidal similarity coordinates, takes the form

$$\partial_s \tilde{\mathbf{v}}(s,\cdot) = (\tilde{\mathbf{L}}_7 + \mathbf{L}')\tilde{\mathbf{v}}(s,\cdot) + e^{2s}\mathbf{N}\big(e^{-2s}\tilde{\mathbf{v}}(s,\cdot)\big)$$

where $\mathbf{L}'$ is defined by

$$\mathbf{L}'\mathbf{f}(y) := \begin{pmatrix} 0 \\ V(y) f_1(y) \end{pmatrix}$$

and

$$\mathbf{N}\big(\mathbf{f}\big)(y) := \begin{pmatrix} 0 \\ \\ N(y, f_1(y)) \end{pmatrix}.$$

As $V \in C_e^\infty[0, R]$ for any $R > 0$, we see that $\mathbf{L}' \in \mathcal{B}(\mathcal{H}_R^k)$ for any $R > 0$ and $k \in \mathbb{N}$. An autonomous equation is obtained by setting $\Phi(s) := e^{-2s}\tilde{\mathbf{v}}(s, \cdot)$ which yields

$$\partial_s \Phi(s) = (\tilde{\mathbf{L}}_7 - 2\mathbf{I} + \mathbf{L}')\Phi(s) + \mathbf{N}\big(\Phi(s)\big). \tag{3.5}$$

In what follows, we set $\tilde{\mathbf{L}} := \tilde{\mathbf{L}}_7 - 2\mathbf{I} + \mathbf{L}'$ in which case $\big(\tilde{\mathbf{L}}, \mathcal{D}(\tilde{\mathbf{L}})\big)$ is an unbounded, densely defined operator on $\mathcal{H}_R^k$ with $\mathcal{D}(\tilde{\mathbf{L}}) := \mathcal{D}(\tilde{\mathbf{L}}_7)$ for $R \geq \frac{1}{2}$. From this point on, we refrain from referring to the domains of the various operators unless absolutely necessary. Furthermore, we fix the space $\mathcal{H}_R := \mathcal{H}_R^6$ for $R \geq \frac{1}{2}$.

Our analysis of Equation (3.5) proceeds in two steps: linear stability and nonlinear stability. First, given $R \geq \frac{1}{2}$, we show that the operator $\tilde{\mathbf{L}}$ is closable and its closure, $\mathbf{L}$, is the generator of a strongly continuous semigroup $\big(\mathbf{S}(s)\big)_{s \geq 0}$. In other words, the abstract initial value problem

$$\begin{cases} \partial_s \Phi(s) = \mathbf{L}\Phi(s) \\ \\ \Phi(0) = \Phi_0 \end{cases},$$

is well-posed in $\mathcal{H}_R$. Our intent is to show that the solutions Φ decay, i.e., perturbations of the blow-up solution become small. However, a careful analysis of the spectrum of $\mathbf{L}$ will allow us to conclude that this is not true in general. After characterizing the unstable portion of the spectrum of $\mathbf{L}$, we will claim linear stability for a co-dimension two subspace of initial data in $\mathcal{H}_R$.

Using this semigroup, we can reformulate the nonlinear problem as the integral equation

$$\Phi(s) = \mathbf{S}(s)\Phi_0 + \int_0^s \mathbf{S}(s-s')\mathbf{N}\big(\Phi(s')\big)ds'$$

for Φ_0 in a small ball in $\mathcal{H}_R$. We would like to show that for such data, this equation has a unique solution which exists for all $s \geq 0$ and decays exponentially. Similar to the linear stability analysis, this decay will not be true in general again due to the spectrum of $\mathbf{L}$. However, we can consider a modified equation which removes this unstable portion of the data. Ultimately, our goal will be to show that by adjusting the blow-up time and our perturbation, this modification term vanishes. This will be explained and carried out in Section 3.5. At that point, we will claim nonlinear stability.

3.3 Linear Stability

3.3.1 Well-Posedness of the Linearized Evolution

First, we show that $\tilde{\mathbf{L}}$ is closable and its closure, $\mathbf{L}$, is the generator of a strongly continuous semigroup $\big(\mathbf{S}(s)\big)_{s\geq 0}$ of bounded operators on $\mathcal{H}_R$. In fact, this is a very simple consequence of Lemma 44.

Lemma 45 *Let $R \geq \frac{1}{2}$. The operator $\tilde{\mathbf{L}}$ is closable and its closure, denoted by $\mathbf{L}$, is the generator of a strongly continuous semigroup $\big(\mathbf{S}(s)\big)_{s\geq 0}$ of bounded operators on $\mathcal{H}_R$ satisfying the estimate*

$$\|\mathbf{S}(s)\|_{\mathcal{H}_R} \leq M e^{\left(-\frac{3}{2}+M\|\mathbf{L}'\|_{\mathcal{H}_R}\right)s} \tag{3.6}$$

for $M \geq 1$ as in Lemma 44 and all $s \geq 0$.

Proof. From Lemma 44, we infer the existence of a strongly continuous semigroup $\big(\mathbf{S}_7(s)\big)_{s\geq 0}$

of bounded operators on $\mathcal{H}_R$ satisfying the estimate $\|\mathbf{S}_7(s)\|_{\mathcal{H}_R} \leq Me^{\frac{s}{2}}$ for some $M \geq 1$ and all $s \geq 0$ generated by the the operator $\mathbf{L}_7$. As a consequence, the operator $\mathbf{L}_7 - 2\mathbf{I}$, with $\mathcal{D}(\mathbf{L}_7 - 2\mathbf{I}) = \mathcal{D}(\mathbf{L}_7)$, generates the strongly continuous semigroup $\left(\mathbf{S}_0(s)\right)_{s\geq 0}$ of bounded operators on $\mathcal{H}_R$ given by $\mathbf{S}_0(s) = e^{-2s}\mathbf{S}_7(s)$, $s \geq 0$ which satisfies the estimate $\|\mathbf{S}_7(s)\|_{\mathcal{H}_R} \leq Me^{-\frac{3}{2}s}$ for all $s \geq 0$. Since $\mathbf{L}' \in \mathcal{B}(\mathcal{H}_R)$, the bounded perturbation theorem implies that $\mathbf{L}$ generates a strongly continuous semigroup $\left(\mathbf{S}(s)\right)_{s\geq 0}$ of bounded operators on $\mathcal{H}_R$ satisfying the claimed estimate. ■

3.3.2 Spectral Analysis

In order to the growth bound, Inequality (3.6), we first need to characterize the spectrum of $\mathbf{L}$. In fact, for $R \geq \frac{1}{2}$, $V \in C_e^\infty[0, R]$ and compactness of the embedding $H^6_{\text{rad}}(\mathbb{B}^7) \hookrightarrow H^5_{\text{rad}}(\mathbb{B}^7)$ imply that $\mathbf{L}'$ is a compact operator on $\mathcal{H}_R$. According to the following lemma, this allows us to restrict our attention to understanding the point spectrum of $\mathbf{L}$.

Lemma 46 *Let $R \geq \frac{1}{2}$ and $\epsilon > 0$. The set $S_\epsilon := \sigma(\mathbf{L}) \cap \{\lambda \in \mathbb{C} : \Re\lambda \geq -\frac{3}{2} + \epsilon\}$ consists of finitely many eigenvalues of* $\mathbf{L}$*, all of which have finite algebraic multiplicity.*

Proof. Standard semigroup theory, Theorem 1.10.ii of [20] for instance, implies that $\sigma(\mathbf{L}_7 - 2\mathbf{I}) \subseteq \{\lambda \in \mathbb{C} : \Re\lambda \leq -\frac{3}{2}\}$. Since $\mathbf{L}'$ is compact, Theorem B.1.i of [24] implies the claim. ■

To improve the growth bound for the semigroup $\left(\mathbf{S}(s)\right)_{s\geq 0}$, we need to characterize the unstable portion of the spectrum, i.e., $\sigma(\mathbf{L}) \cap \overline{\mathbb{H}}$. Theorem B.1.ii of [24] tells us that in order to achieve this goal, we need only characterize $\sigma_p(\mathbf{L}) \cap \overline{\mathbb{H}}$. First, we define two functions which will be shown to be eigenfunctions of $\mathbf{L}$.

Definition 47 *We set*

$$\mathbf{f}_1^*(y) := \frac{(7h(y)^2 - 15y^2)h(y)}{(5y^2 + 3h(y)^2)^3} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

and

$$\mathbf{f}_4^*(y) := \frac{1}{(5y^2 + 3h(y)^2)^3} \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

for $y \in [0, \infty)$.

Proposition 48 *Let* $R \geq \frac{1}{2}$. *We have that*

$$\sigma_p(\mathbf{L}) \cap \overline{\mathbb{H}} = \{1, 4\}.$$

Furthermore, $\ker(\mathbf{I} - \mathbf{L}) = \langle \mathbf{f}_1^* \rangle$ *and* $\ker(4\mathbf{I} - \mathbf{L}) = \langle \mathbf{f}_4^* \rangle$

Proof. For $\lambda = 1, 4$, direct calculations verify that $\mathbf{f}_\lambda^* \in \mathcal{D}(\tilde{\mathbf{L}})$ and $\langle \mathbf{f}_\lambda^* \rangle \subseteq \ker(\lambda\mathbf{I} - \mathbf{L})$. Thus, $\{1, 4\} \subseteq \sigma_p(\mathbf{L}) \cap \overline{\mathbb{H}}$.

Now, we aim to show $\sigma_p(\mathbf{L}) \cap \overline{\mathbb{H}} \subseteq \{1, 4\}$ after which we will conclude $\ker(\lambda\mathbf{I} - \mathbf{L}) \subseteq \langle \mathbf{f}_\lambda^* \rangle$ for $\lambda = 1, 4$. To that end, suppose $\lambda \in \sigma_p(\mathbf{L}) \cap \overline{\mathbb{H}}$. Thus, there exists $\mathbf{f}_\lambda = (f_{\lambda,1}, f_{\lambda_2}) \in \mathcal{D}(\mathbf{L}) \setminus \{\mathbf{0}\}$ with $(\lambda\mathbf{I} - \mathbf{L})\mathbf{f}_\lambda = \mathbf{0}$. A direct calculation shows that $f_{\lambda,1}$ solves the ODE

$$f''_{\lambda,1} + \frac{c_{11}(y) + (\lambda + 2)c_{21}(y)}{c_{12}(y)} f'_{\lambda,1} + \frac{(\lambda + 2)(c_{20}(y) - \lambda - 2) + V(y)}{c_{12}(y)} f_{\lambda,1} = 0 \tag{3.7}$$

weakly on the interval $(0, R)$ and $f_{\lambda,2} = (\lambda + 2)f_{\lambda,1}$. Furthermore, since $f_{\lambda,1} \in H^6_{\text{rad}}(\mathbb{B}^7)$, Sobolev embedding implies $f_{\lambda,1} \in C^2(0, R)$ and is a classical solution of Equation (3.7) on $(0, R)$. As a consequence, the function $v(s, y) := e^{(\lambda+2)s} f_{\lambda,1}(y)$ is in $C^2(\mathbb{R}_+ \times (0, R))$ and is a classical solution of the equation

$$\partial_s^2 v = c_{11}\partial_y v + c_{12}\partial_y^2 v + c_{20}\partial_s v + c_{21}\partial_y \partial_s v + Vv$$

on $\mathbb{R}_+ \times (0, R)$. Upon setting

$$V_0(y) = -\frac{48(21 - 5y^2)}{(5y^2 + 3)^2},$$

and $v(s, y) =: w\Big(s - \log\big(-h(y)\big), -\frac{y}{h(y)}\Big)$, we find that w is a classical solution of the equation

$$\Big(\partial_\tau^2 + 2\rho\partial_\rho\partial_\tau - \big(1 - \rho^2\big)\partial_\rho^2 - \frac{6}{\rho}\partial_\rho + \partial_\tau + 2\rho\partial_\rho + V_0(\rho)\Big)w(\tau, \rho) = 0$$

on $\mathbb{R}_+ \times (0, 1)$. In terms of $f_{\lambda,1}$, we have

$$\begin{aligned} w(\tau, \rho) &= e^{(\lambda+2)\tau}\Big(\frac{2}{2 + \sqrt{2(1 + \rho^2)}}\Big)^{\lambda+2} f_{\lambda,1}\Big(\frac{2\rho}{2 + \sqrt{2(1 + \rho^2)}}\Big) \\ &=: e^{(\lambda+2)\tau} f(\rho). \end{aligned}$$

Thus, f is a classical solution of the ODE

$$(1 - \rho^2)f''(\rho) + \Big(\frac{6}{\rho} - 2(\lambda + 3)\rho\Big)f'(\rho) - \Big((\lambda + 2)(\lambda + 3) - \frac{48(21 - 5\rho^2)}{(5\rho^2 + 3)^2}\Big)f(\rho) = 0. \quad (3.8)$$

Smoothness of the coefficients implies $f \in C^\infty(0, 1)$. Furthermore, Equation (3.8) has two regular singular points: $\rho = 0$ with Frobenius indices $\{0, -5\}$ and $\rho = 1$ with Frobenius indices $\{0, 1 - \lambda\}$. The Frobenius analysis in the proof of Proposition 3.2 of [24] with $d = 5$ allows us to conclude that $f \in C^\infty[0, 1]$. Now, our goal is to show that Equation (3.8) has solutions in $C^\infty[0, 1]$ only when $\lambda = 1$ or 4, in which case $\mathbf{f}_\lambda \in \langle \mathbf{f}_\lambda^* \rangle$.

First, observe that the functions

$$f(\rho; 1) := \frac{7 - 15\rho^2}{(5\rho^2 + 3)^3}$$

$$f(\rho; 4) := \frac{1}{(5\rho^2 + 3)^3}$$

are indeed solutions in $C^\infty[0,1]$ with $\lambda = 1$ and $\lambda = 4$ respectively. To investigate $C^\infty[0,1]$ solutions of Equation (3.8) with $\lambda \neq 1, 4$, we first 'remove' the eigenvalues $\lambda = 1$ and $\lambda = 4$ by performing so-called *supersymmetric removal*. For an in-depth discussion of this procedure, we refer the reader to [25], Appendix B and [23], Section 2.5. We begin by making the change of variables

$$f(\rho) =: \rho^{-3}(1-\rho^2)^{-\frac{\lambda}{2}} g(\rho)$$

which transforms Equation (3.8) into

$$-g''(\rho) - \frac{2\left(95\rho^6 - 729\rho^4 + 405\rho^2 - 27\right)}{\rho^2\left(1-\rho^2\right)^2\left(5\rho^2+3\right)^2} g(\rho) = -\frac{(\lambda+2)(\lambda-4)}{\left(1-\rho^2\right)^2} g(\rho). \tag{3.9}$$

Consequently, $g(\rho;4) = \rho^3(1-\rho^2)^2 f(\rho;4)$ is a solution of Equation (3.9) with $\lambda = 4$. Our goal is to factor the left-hand side of Equation (3.9) using the solution $g(\rho;4)$. Following the procedure in [25], Appendix B and [23], Section 2.5, the left-hand side can be factored as

$$\begin{aligned} &-\partial_\rho^2 - \frac{2\left(95\rho^6 - 729\rho^4 + 405\rho^2 - 27\right)}{\rho^2\left(1-\rho^2\right)^2\left(5\rho^2+3\right)^2} \\ &\quad = \Big(-\partial_\rho - \frac{-5\rho^4 - 36\rho^2 + 9}{-5\rho^5 + 2\rho^3 + 3\rho} \Big) \Big(\partial_\rho - \frac{-5\rho^4 - 36\rho^2 + 9}{-5\rho^5 + 2\rho^3 + 3\rho} \Big). \end{aligned}$$

Setting $\tilde{g}(\rho) := g'(\rho) - \frac{-5\rho^4-36\rho^2+9}{-5\rho^5+2\rho^3+3\rho} g(\rho)$ and defining $\tilde{g}(\rho) =: \rho^3(1-\rho^2)^{\frac{\lambda}{2}} \tilde{f}(\rho)$ produces the new equation

$$(1-\rho^2)\tilde{f}''(\rho) + \Big(\frac{6}{\rho} - 2(\lambda+3)\rho\Big)\tilde{f}'(\rho) - \Big((\lambda+2)(\lambda+3) - \frac{18\left(5\rho^4 + 30\rho^2 - 3\right)}{\rho^2\left(5\rho^2+3\right)^2}\Big)\tilde{f}(\rho) = 0.$$

Observe that $\tilde{f}(\rho;4) := \rho^{-3}(1-\rho^2)^{-2}\tilde{g}(\rho;4)$, where $\tilde{g}(\rho;4) := g'(\rho;4) - \frac{-5\rho^4-36\rho^2+9}{-5\rho^5+2\rho^3+3\rho} g(\rho;4)$, is identically zero. In this sense, we have 'removed' the eigenvalue $\lambda = 4$ by transforming the corresponding solution, $f(\rho;4)$, into the trivial solution. Under the above transformations

$f(\rho;1)$ transforms into $\tilde{f}(\rho;1) := -\frac{3\rho}{(3+5\rho^2)^2}$. Repeating the same transformations but with the factorization given by the solution $\tilde{f}(\rho;1)$ instead produces the new equation

$$(1-\rho^2)\hat{f}''(\rho)+\Big(\frac{6}{\rho}-2(\lambda+3)\rho\Big)\hat{f}'(\rho)-\Big((\lambda+2)(\lambda+3)-\frac{6\left(35\rho^4+18\rho^2-21\right)}{\rho^2\left(5\rho^2+3\right)^2}\Big)\hat{f}(\rho)=0 \quad (3.10)$$

for the corresponding new dependent variable $\hat{f}$.

Now, we show that Equation (3.10) has no solutions if $\lambda \in \overline{\mathbb{H}}$ other than the zero solution. We achieve this by expanding any nontrivial, analytic solution around the regular singular point $\rho = 0$ and showing that if $\lambda \in \overline{\mathbb{H}}$, then this solution cannot be analytically continued past $\rho = 1$.

Observe that Equation (3.10) has seven regular singular points: $\rho = 0, \pm 1, \pm i\sqrt{\frac{3}{5}}$, and $\pm\infty$. We begin our analysis of Equation (3.10) by first reducing the number of regular singular points to four via the transformation

$$\rho = \sqrt{\frac{3x}{8-5x}},\ \tilde{f}(\rho) = x(8-5x)^{\frac{\lambda+2}{2}}y(x)$$

which transforms Equation (3.10) into its Heun form

$$y''(x)+\Big(\frac{11}{2x}+\frac{\lambda}{x-1}+\frac{1}{2(x-\frac{8}{5})}\Big)y'(x)+\frac{5(\lambda+2)(\lambda+8)x-(\lambda+26)(3\lambda+4)}{20x(x-1)(x-\frac{8}{5})}y(x)=0 \quad (3.11)$$

with the four regular singular points $x = 0, 1, \frac{8}{5}, \infty$. Frobenius theory implies that any $y \in C^\infty[0,1]$ solving Equation (3.11) is analytic on $[0,1]$. In addition, any analytic solution of Equation (3.11) yields an analytic solution of Equation (3.10) as well as the converse. Thus, to exclude the existence of an analytic solution of Equation (3.10), we exclude the existence of an analytic solution of Equation (3.11).

At $x = 0$, the Frobenius indices are $\{0, -\frac{9}{2}\}$. Without loss of generality, we may assume that a solution for a fixed λ, denoted by $y(\cdot;\lambda)$, has the expansion

$$y(x;\lambda) = \sum_{n=0}^{\infty} a_n(\lambda)x^n, \ a_0(\lambda) = 1 \tag{3.12}$$

near $x = 0$. Since the finite regular singular points of Equation (3.11) are $x = 0, 1, \frac{8}{5}$, $y(\cdot;\lambda)$ fails to be analytic at $x = 1$ precisely when the radius of convergence of (3.12) is equal to one. To that end, we derive a recurrence relation for the coefficients given by

$$a_{n+2}(\lambda) = A_n(\lambda)a_{n+1}(\lambda) + B_n(\lambda)a_n(\lambda) \tag{3.13}$$

where

$$A_n(\lambda) = \frac{3\lambda^2 + 114\lambda + 52n^2 + 32\lambda n + 348n + 400}{16(n+2)(2n+13)}$$

and

$$B_n(\lambda) = -\frac{5(\lambda + 2n + 2)(\lambda + 2n + 8)}{16(n+2)(2n+13)}$$

with $a_{-1}(\lambda) = 0$. For $n \in \mathbb{N}_0$, we define

$$r_n(\lambda) := \frac{a_{n+1}(\lambda)}{a_n(\lambda)}.$$

Since $\lim_{n\to\infty} A_n(\lambda) = \frac{13}{8}$ and $\lim_{n\to\infty} B_n(\lambda) = -\frac{5}{8}$, the so-called characteristic equation of Equation (3.13) is

$$t^2 - \frac{13}{8}t + \frac{5}{8} = 0$$

which has solutions $t_1 = \frac{5}{8}$ and $t_2 = 1$. Poincaré's theorem for difference equations, see [19] or [25] Appendix A, implies that either $a_n(\lambda)$ is zero eventually in n or

$$\lim_{n\to\infty} r_n(\lambda) = \frac{5}{8} \tag{3.14}$$

or

$$\lim_{n\to\infty} r_n(\lambda) = 1. \tag{3.15}$$

We aim to prove that Equation (3.15) holds true.

First, observe that $a_n(\lambda)$ cannot eventually be zero since, otherwise, backwards substitution would allow us to conclude that $a_0(\lambda) = 0$ which is in clear contradiction with $a_0(\lambda) = 1$. To rule out Equation (3.14), we first derive a recurrence relation for $r_n(\lambda)$ given by

$$r_{n+1}(\lambda) = A_n(\lambda) + \frac{B_n(\lambda)}{r_n(\lambda)} \tag{3.16}$$

with initial condition

$$r_0(\lambda) = \frac{a_1(\lambda)}{a_0(\lambda)} = A_{-1}(\lambda) = \frac{1}{176}(\lambda + 26)(3\lambda + 4).$$

Furthermore, we define an approximate solution of Equation (3.16) by

$$\tilde{r}_n(\lambda) := \lambda^2 \left(\frac{3}{16(n+1)(2n+11)} + \frac{9}{4000n^2} \right) + \lambda \left(\frac{16n+41}{8(n+1)(2n+11)} - \frac{1}{13n} \right) + \frac{4n+19}{4n+22}$$

for $n \in \mathbb{N}$ which we call a *quasisolution*. This quasisolution is intended to mimic the behavior of the actual solution $r_n(\lambda)$ for large n. Observe that for fixed $\lambda \in \overline{\mathbb{H}}$, $\lim_{n \to \infty} \tilde{r}_n(\lambda) = 1$. If indeed $r_n(\lambda)$ remains close to the quasisolution, then we can exclude Equation (3.14) implying that Equation (3.15) must hold. To prove this, we define

$$\delta_n(\lambda) := \frac{r_n(\lambda)}{\tilde{r}_n(\lambda)} - 1$$

to measure the difference between $r_n(\lambda)$ and the quasisolution and derive a recurrence relation for this difference given by

$$\delta_{n+1}(\lambda) = \varepsilon_n(\lambda) - C_n(\lambda) \frac{\delta_n(\lambda)}{1 + \delta_n(\lambda)}$$

where

$$\varepsilon_n(\lambda) = \frac{A_n(\lambda)\tilde{r}_n(\lambda) + B_n(\lambda)}{\tilde{r}_n(\lambda)\tilde{r}_{n+1}(\lambda)} - 1$$

and

$$C_n(\lambda) = \frac{B_n(\lambda)}{\tilde{r}_n(\lambda)\tilde{r}_{n+1}(\lambda)}. \tag{3.17}$$

For $n \geq 5$, we have the following estimates

$$\begin{aligned} |\delta_5(\lambda)| &\leq \frac{1}{4} \\ |\varepsilon_n(\lambda)| &\leq \frac{64+5n}{120(4+n)} \\ |C_n(\lambda)| &\leq \frac{56+25n}{40(4+n)}. \end{aligned}$$

We will prove the third estimate while the first and second are established analogously. First, we bring $C_n(\lambda)$ into the form of a rational function, namely $C_n(\lambda) = \frac{P_1(n,\lambda)}{P_2(n,\lambda)}$ for polynomials $P_1, P_2 \in \mathbb{Z}[n, \lambda]$. Explicit expressions are provided in the Appendix (3.6.1). We can prove the estimate by first establishing it on the imaginary line and then extending it to all of $\overline{\mathbb{H}}$. This extension can be achieved by showing that $C_n(\lambda)$ is analytic and polynomially bounded on $\overline{\mathbb{H}}$ at which point the Phragmén-Lindelöf principle achieves the desired extension.

Observe that for $t \in \mathbb{R}$, The inequality $|C_n(it)| \leq \frac{56+25n}{40(4+n)}$ is equivalent to the inequality $(40(4+n))^2|P_1(n,it)|^2 - (56+25n)^2|P_2(n,it)|^2 \leq 0$. For $t \in \mathbb{R}$ and $n \geq 5$, a direct calculation shows that the coefficients of $(40(4+n))^2|P_1(n,it)|^2-(56+25n)^2|P_2(n,it)|^2$ are manifestly negative which establishes the desired estimate on the imaginary line. Now, we aim to extend the estimate to all of $\overline{\mathbb{H}}$. As $C_n(\lambda)$ is a rational function of polynomials in $\mathbb{Z}[n, \lambda]$, it is polynomially bounded. Furthermore, a direct calculation of the zeros of $P_2(n, \lambda)$ shows that they are contained in the open left half plane implying the analyticity of $C_n(\lambda)$ in $\overline{\mathbb{H}}$. Thus, the Phragmén-Lindelöf principle extends the estimate to all of $\overline{\mathbb{H}}$.

By induction, we prove

$$|\delta_n(\lambda)| \leq \frac{1}{4}$$

for all $n \geq 5$. The base case $n = 5$ clearly holds. Suppose the estimate holds for some $n > 5$. Using the estimates for $C_n(\lambda)$ and $\varepsilon_n(\lambda)$, we find that

$$\delta_{n+1}(\lambda) \leq \frac{64 + 5n}{120(4 + n)} + \frac{1}{3}\frac{56 + 25n}{40(4 + n)} = \frac{1}{4}.$$

Thus, the estimate holds for all $n \geq 5$.

Now, suppose Equation (3.14) holds true. Then

$$\frac{1}{4} \geq \lim_{n\to\infty} |\delta_n(\lambda)| = \lim_{n\to\infty} \left|\frac{r_n(\lambda)}{\tilde{r}_n(\lambda)} - 1\right| = \frac{3}{8}$$

which is a clear contradiction. Thus, it must be the case that Equation (3.15) holds. By undoing the previous transformations, we conclude that Equation (3.7) has no smooth solution on the interval $(0, R)$ which can be smoothly extended to $[0, R]$ for $\lambda \in \overline{\mathbb{H}}$ other than those which correspond to $\lambda = 1$ and 4, i.e. $f_{\lambda,1} = \alpha f^*_{\lambda,1}$ for some $\alpha \in \mathbb{C} \setminus \{0\}$. Thus, $\ker(\lambda\mathbf{I} - \mathbf{L}) \subseteq \langle \mathbf{f}^*_\lambda \rangle$ and $(\sigma_p(\mathbf{L}) \cap \overline{\mathbb{H}}) \setminus \{1, 4\} = \emptyset$. ■

Remark 49 *A natural first guess for a quasisolution would be*

$$\tilde{r}_n(\lambda) = \lambda^2 \left(\frac{3}{16(n + 1)(2n + 11)}\right) + \lambda \left(\frac{16n + 41}{8(n + 1)(2n + 11)}\right) + \frac{4n + 19}{4n + 22}$$

following the methods in [7], [23], and [25]. The quadratic and linear terms in λ come from studying the large $|\lambda|$ behavior of $A_n(\lambda)$ while the constant in λ term comes from fitting the first few iterates of $r_n(\lambda)$ for small $|\lambda|$. However, it appears that this quasisolution does not work when trying to obtain any reasonable estimates on $\delta_5(\lambda)$, $\varepsilon_n(\lambda)$, and $C_n(\lambda)$. Lower-order corrections to the linear and quadratic terms, to the best of our knowledge, appear to *be essential in obtaining such estimates. This may be important for solving future spectral problems with this method.*

3.3.3 Decay of the Linearized Flow

Lemma 48 shows that $1, 4 \in \sigma_p(\mathbf{L})$ are isolated. This allows us to define the following spectral projections.

Definition 50 *Fix $R \geq \frac{1}{2}$. Let $\gamma_1 : [0, 2\pi] \to \mathbb{C}$ and $\gamma_4 : [0, 2\pi] \to \mathbb{C}$ be defined by $\gamma_1(t) = 1 + \frac{1}{2}e^{it}$ and $\gamma_4(t) = 4 + \frac{1}{2}e^{it}$. Then we set*

$$\mathbf{P}_j := \frac{1}{2\pi i} \int_{\gamma_j} \mathbf{R}_{\mathbf{L}}(\lambda) d\lambda, \; j = 1, 4.$$

Proposition 51 *Let $R \geq \frac{1}{2}$. The operators $\mathbf{P}_j \in \mathcal{B}(\mathcal{H}_R)$, $j = 1, 4$, commute with the semigroup $\left(\mathbf{S}(s)\right)_{s \geq 0}$ and are mutually transversal, i.e.,*

$$\mathbf{P}_1 \mathbf{P}_4 = \mathbf{P}_4 \mathbf{P}_1 = \mathbf{0}.$$

Furthermore, we have

$$\operatorname{rg} \mathbf{P}_j = \langle \mathbf{f}_j^* \rangle$$

and

$$\mathbf{S}(s)\mathbf{P}_j \mathbf{f} = e^{js} \mathbf{P}_j \mathbf{f}, \; s \geq 0, \; \mathbf{f} \in \mathcal{H}_R, \; j = 1, 4.$$

Proof. Boundedness, transversality, and commuting with semigroup follow from abstract theory, see [20, 27]. In the following, we handle both cases $j = 1$ and $j = 4$ simultaneously until the very end at which point the arguments slightly diverge.

Now, we aim to show $\operatorname{rg} \mathbf{P}_j = \langle \mathbf{f}_j^* \rangle$. The inclusion $\langle \mathbf{f}_j^* \rangle \subseteq \operatorname{rg} \mathbf{P}_j$ follows from abstract theory, see [27]. For the reverse inclusion, first observe that $\mathbf{P}_j$ decomposes $\mathcal{H}_R$ as

$\mathcal{H}_R = \operatorname{rg} \mathbf{P}_j \oplus \operatorname{rg}(\mathbf{I} - \mathbf{P}_j)$ and the operator $\mathbf{L}$ decomposes into the parts $\mathbf{L}|_{\operatorname{rg} \mathbf{P}_j}$ and $\mathbf{L}|_{\operatorname{rg}(\mathbf{I}-\mathbf{P}_j)}$ acting on $\operatorname{rg} \mathbf{P}_j$ and $\operatorname{rg}(\mathbf{I} - \mathbf{P}_j)$ respectively. The spectra of these operators are

$$\sigma(\mathbf{L}|_{\operatorname{rg} \mathbf{P}_j}) = \{j\}, \ \sigma(\mathbf{L}|_{\operatorname{rg}(\mathbf{I}-\mathbf{P}_j)}) = \sigma(\mathbf{L}) \setminus \{j\}.$$

We claim that $\operatorname{rg} \mathbf{P}_j$ is finite-dimensional. To see this, suppose that $\dim \operatorname{rg} \mathbf{P}_j = \infty$. Then, Theorem 5.28 of [27] implies that $j \in \sigma_e(\mathbf{L})$. Since $\mathbf{L}'$ is compact and the essential spectrum is stable under compact perturbations, we also have that $j \in \sigma_e(\mathbf{L} - \mathbf{L}')$. This is clearly a contradiction since $\mathbf{L} - \mathbf{L}' = \mathbf{L}_7 - 2\mathbf{I}$ and $\sigma(\mathbf{L}_7 - 2\mathbf{I}) \subseteq \{z \in \mathbb{C} : \Re z \leq -\frac{3}{2}\}$.

Thus, the part $\mathbf{L}|_{\operatorname{rg} \mathbf{P}_j}$ acts on a finite-dimensional Hilbert space with spectrum $\sigma(\mathbf{L}_{\operatorname{rg} \mathbf{P}_j}) = \{j\}$. Consequently, $j\mathbf{I} - \mathbf{L}_{\operatorname{rg} \mathbf{P}}$ is nilpotent since 0 is its only spectral point and is an eigenvalue. So, there exists a minimal $\ell_j \in \mathbb{N}$ with $(j\mathbf{I} - \mathbf{L}_{\operatorname{rg} \mathbf{P}_j})^{\ell_j} \mathbf{f} = \mathbf{0}$ for all $\mathbf{f} \in \operatorname{rg} \mathbf{P}_j$. If $\ell_j = 1$, then the reverse inclusion follows.

Suppose $\ell_j \neq 1$. Then there exists a nonzero $\mathbf{f}_j \in \operatorname{rg} \mathbf{P}_j \subset H^6_{\text{rad}}(\mathbb{B}^7_R) \times H^5_{\text{rad}}(\mathbb{B}^7_R) \subset C^2(0, R) \times C^1(0, R)$ such that $\mathbf{f}_j \in \ker(j\mathbf{I} - \mathbf{L}_{\operatorname{rg} \mathbf{P}_j}) \subseteq \ker(j\mathbf{I} - \mathbf{L})$. By Lemma 48, we have $\ker(j\mathbf{I} - \mathbf{L}) = \langle \mathbf{f}^*_j \rangle$. Thus, $\mathbf{f}_j$ solves the equation

$$\alpha \mathbf{f}^*_j = (j\mathbf{I} - \mathbf{L})\mathbf{f}_j.$$

for some $\alpha \in \mathbb{C} \setminus \{0\}$. Without loss of generality, we take $\alpha = 1$. Consequently, the first component of $\mathbf{f}_j$ solves the ODE

$$f''_{j,1}(y) + p_j(y) f'_{j,1}(y) + q_j(y) f_{j,1}(y) = \frac{G_j(y)}{c_{12}(y)} \tag{3.18}$$

where

$$p_j(y) := \frac{c_{11}(y) + (j+2)c_{21}(y)}{c_{12}(y)},$$

$$q_j(y) := \frac{(j+2)(c_{20}(y) - j - 2) + V(y)}{c_{12}(y)},$$

and

$$G_j(y) = c_{21}(y)\frac{df^*_{j,1}}{dy} + (c_{20}(y) - 4 - 2j)f^*_{j,1}(y).$$

Let $I_j(y)$ be an antiderivative of $p_j(y)$. For instance, the explicit functions

$$I_1(y) = \frac{\left(y^2+1\right)\sqrt{3\sqrt{y^2+2} - y + 4}\sqrt{3\sqrt{y^2+2} + y + 4}}{y^6\left(1-4y^2\right)\sqrt{y^2+2}\sqrt{y^2 + 2\sqrt{y^2+2} + 3}}$$

and

$$I_4(y) = \frac{\left(y^2+1\right)\left(8y^2 + 24\sqrt{y^2+2} + 34\right)^2}{y^6\left(1-4y^2\right)^4\sqrt{y^2+2}\sqrt{y^2 + 2\sqrt{y^2+2} + 3}}$$

suffice. A fundamental system for the homogeneous equation is obtained via reduction of order

$$\phi_j(y) := f^*_{j,1}(y)$$
$$\psi_j(y) := f^*_{j,1}(y)\int_{\frac{1}{4}}^{y} \exp\big(-I_j(y')\big) f^*_{j,1}(y)^{-2} dy'$$

where the lower bound of integration in ψ_j is chosen arbitrarily. Observe that

$$\exp\big(-I_j(y)\big) \simeq y^{-6}\Big(\frac{1}{2} - y\Big)^{-j}$$

which implies that for the second solution we have the asymptotics

$$|\psi_1(y)| \simeq y^{-5}\left|\log\Big(\frac{1}{2} - y\Big)\right|, \quad |\psi_1'(y)| \simeq y^{-6}\Big(\frac{1}{2} - y\Big)^{-1}$$

and

$$|\psi_4(y)| \simeq y^{-5}\Big(\frac{1}{2} - y\Big)^{-3}, \quad |\psi_4'(y)| \simeq y^{-6}\Big(\frac{1}{2} - y\Big)^{-4}.$$

By Abel's identity, the Wronskian is precisely $\exp\big(-I_j(y')\big)$ up to some constant multiple. This implies the asymptotics

$$|W(\phi_j, \psi_j)(y)| \simeq y^{-6}\Big(\frac{1}{2} - y\Big)^{-j}.$$

Variation of parameters yields a solution of Equation (3.27)

$$f_{j,1}(y) = c_1^{(j)}\phi_j(y) + c_2^{(j)}\psi_j(y) - \phi_j(y)\int_0^y \frac{\psi_j(\rho)}{W(\phi_j,\psi_j)(\rho)}\frac{G_j(\rho)}{c_{12}(\rho)}d\rho + \psi_j(y)\int_0^y \frac{\phi_j(\rho)}{W(\phi_j,\psi_j)(\rho)}\frac{G_j(\rho)}{c_{12}(\rho)}d\rho$$

for $y \in (0, \frac{1}{2})$. Taking the limit $y \to 0^+$ yields $c_2^{(j)} = 0$. Thus, we are left with

$$f_{j,1}(y) = c_1^{(j)}\phi_j(y) - \phi_j(y)\int_0^y \frac{\psi_j(\rho)}{W(\phi_j,\psi_j)(\rho)}\frac{G_j(\rho)}{c_{12}(\rho)}d\rho + \psi_j(y)\int_0^y \frac{\phi_j(\rho)}{W(\phi_j,\psi_j)(\rho)}\frac{G_j(\rho)}{c_{12}(\rho)}d\rho$$

Based on the above asymptotics we find

$$\lim_{y\to\frac{1}{2}^-}\int_0^y \frac{\psi_j(\rho)}{W(\phi_j,\psi_j)(\rho)}\frac{G_j(\rho)}{c_{12}(\rho)}d\rho$$

exists. As a consequence, in order to control the third term near $y = \frac{1}{2}$, we must have

$$\int_0^{\frac{1}{2}} \frac{\phi_j(\rho)}{W(\phi_j,\psi_j)(\rho)}\frac{G_j(\rho)}{c_{12}(\rho)}d\rho = 0$$

For $j = 4$, the integrand has a definite sign. Thus, the above integral vanishing yields a contradiction. For $j = 1$, the integral can be computed explicitly and is nonzero which again yields a contradiction. Thus, we must have $(j\mathbf{I} - \mathbf{L})\mathbf{f} = \mathbf{0}$ for all $\mathbf{f} \in \operatorname{rg}\mathbf{P}_j$ which, with Proposition 48, implies $\operatorname{rg}\mathbf{P}_j \subseteq \langle \mathbf{f}_j^* \rangle$.

Lastly, the claim

$$\mathbf{S}(s)\mathbf{P}_j\mathbf{f} = e^{js}\mathbf{P}_j\mathbf{f},\ s \geq 0,\ \mathbf{f} \in \mathcal{H}_R,\ j = 1, 4$$

is a direct consequence of $\operatorname{rg}\mathbf{P}_j = \langle \mathbf{f}_j^* \rangle$ ■

We now state and prove the main result on the linearized equation.

Theorem 52 *Fix $R \geq \frac{1}{2}$. Let $\mathbf{P} := \mathbf{P}_1 + \mathbf{P}_4$. Then there exist $\omega_0 > 0$ and $M \geq 1$ such that*

$$\|\mathbf{S}(s)(\mathbf{I} - \mathbf{P})\mathbf{f}\|_{\mathcal{H}_R} \leq Me^{-\omega_0 s}\|(\mathbf{I} - \mathbf{P})\mathbf{f}\|_{\mathcal{H}_R}$$

for all $s \geq 0$ *and* $\mathbf{f} \in \mathcal{H}_R$.

Proof. This is an immediate consequence of [24], Theorem B.1.iii since Lemma 46 and Proposition 48 imply that there exists $0 < \epsilon < \frac{3}{2}$ such that the set S_ϵ consists of at most finitely many eigenvalues all with finite algebraic multiplicity and negative real part. ■

3.4 Nonlinear Stability

3.4.1 Well-Posedness and Decay of the Nonlinear Evolution

We now turn our attention to the nonlinear problem

$$\begin{cases} \partial_s \Phi(s) = \mathbf{L}\Phi(s) + \mathbf{N}\big(\Phi(s)\big) \\ \Phi(0) = \Phi_0 \end{cases} \tag{3.19}$$

for initial data Φ_0 contained in a small ball in $\mathcal{H}_R$. Equipped with the semigroup $\big(\mathbf{S}(s)\big)_{s\geq 0}$ we appeal to Duhamel's formula and reformulate Equation (3.19) as the integral equation

$$\Phi(s) = \mathbf{S}(s)\Phi(0) + \int_0^s \mathbf{S}(s-s')\mathbf{N}\big(\Phi(s')\big)ds'. \tag{3.20}$$

As a first step, we prove a mapping property and local Lipschitz bound on the nonlinearity.

Lemma 53 *Fix* $R \geq \frac{1}{2}$. *We have* $\mathbf{N} : \mathcal{H}_R \to \mathcal{H}_R$ *and the bound*

$$\|\mathbf{N}(\mathbf{f}) - \mathbf{N}(\mathbf{g})\|_{\mathcal{H}_R} \lesssim \big(\|\mathbf{f}\|_{\mathcal{H}_R} + \|\mathbf{g}\|_{\mathcal{H}_R}\big)\|\mathbf{f} - \mathbf{g}\|_{\mathcal{H}_R}$$

for all $\mathbf{f}, \mathbf{g} \in \mathcal{H}_R$.

Proof. Recalling the definition of $\mathbf{N}$, we find

$$\begin{aligned}
\|\mathbf{N}(\mathbf{f})-\mathbf{N}(\mathbf{g})\|_{\mathcal{H}_R} &= \Big\|N\big(\cdot,f_1(\cdot)\big)-N\big(\cdot,g_1(\cdot)\big)\Big\|_{H^5_{\text{rad}}(\mathbb{B}^7_R)} \\
&\lesssim \|f_1^2-g_1^2\|_{H^5_{\text{rad}}(\mathbb{B}^7_R)} \\
&\lesssim \|f_1+g_1\|_{H^5_{\text{rad}}(\mathbb{B}^7)}\|f_1-g_1\|_{H^5_{\text{rad}}(\mathbb{B}^7_R)} \\
&\lesssim \big(\|\mathbf{f}\|_{\mathcal{H}_R}+\|\mathbf{g}\|_{\mathcal{H}_R}\big)\|\mathbf{f}-\mathbf{g}\|_{\mathcal{H}_R}
\end{aligned}$$

where the second to third line follows from the Banach algebra property of $H^5_{\text{rad}}(\mathbb{B}^7_R)$. The claim $\mathbf{N}:\mathcal{H}_R\to\mathcal{H}_R$ follows from $\mathbf{N}(\mathbf{0})=\mathbf{0}$. ■

Due to the instabilities associated to the eigenvalues $\lambda=1,4$, Equation (3.20) will not, in general, have global solutions that decay. Instead, we consider a modified equation which allows us to correct for these instabilities and achieve global existence and decay. Upon reconnecting to the problem in physical coordinates, we will in fact show that for suitably adjusted perturbations of u_1^*, there is a choice of T close to 1 for which this modification vanishes and the corresponding solution converges to u_T^*.

Definition 54 *For $R\geq\frac{1}{2}$ and ω_0 from Theorem 52, we define the Banach space*

$$\mathcal{X}_R := \{\Phi\in C([0,\infty),\mathcal{H}_R) : \|\Phi\|_{\mathcal{X}_R}<\infty\}$$

where

$$\|\Phi\|_{\mathcal{X}_R} := \sup_{s>0}\Big(e^{\omega_0 s}\|\Phi(s)\|_{\mathcal{H}_R}\Big).$$

Furthermore, we define $\mathbf{C}_j:\mathcal{X}_R\times\mathcal{H}_R\to \operatorname{rg}\mathbf{P}_j$, $j=1,4$ by

$$\mathbf{C}_j(\Phi,\mathbf{f}) := \mathbf{P}_j\bigg(\mathbf{f}+\int_0^\infty e^{-js'}\mathbf{N}\big(\Phi(s')\big)ds'\bigg)$$

and set $\mathbf{C}:=\mathbf{C}_1+\mathbf{C}_4$.

With this, we study the modified equation

$$\Phi(s) = \mathbf{S}(s)\big[\mathbf{f} - \mathbf{C}(\Phi, \mathbf{f})\big] + \int_0^s \mathbf{S}(s-s')\mathbf{N}\big(\Phi(s')\big)ds'. \tag{3.21}$$

For Equation (3.21), we show that for all sufficiently small data $\mathbf{f}$, there exists a unique solution in the space $\mathcal{X}_R$. In other words, the nonlinear problem is globally well-posed for all sufficiently small initial data and the corresponding solutions decay exponentially as $s \to \infty$.

Proposition 55 *Fix* $R \geq \frac{1}{2}$*. For all sufficiently large* $c > 0$ *and sufficiently small* $\delta > 0$ *and any* $\mathbf{f} \in \mathcal{H}_R$ *satisfying* $\|\mathbf{f}\|_{\mathcal{H}_R} \leq \frac{\delta}{c}$*, there exists a unique solution* $\Phi_{\mathbf{f}} \in C([0,\infty), \mathcal{H}_R)$ *of Equation* (3.21) *that satisfies* $\|\Phi_{\mathbf{f}}(s)\|_{\mathcal{H}_R} \leq \delta e^{-\omega_0 s}$ *for all* $s \geq 0$*. Furthermore, the solution map* $\mathbf{f} \mapsto \Phi_{\mathbf{f}}$ *is Lipschitz as a function from a small ball in* $\mathcal{H}_R$ *to* $\mathcal{X}_R$*.*

Proof. Set

$$\mathcal{Y}_\delta := \{\Phi \in \mathcal{X}_R : \|\Phi\|_{\mathcal{X}_R} \leq \delta\}$$

and define the map

$$\mathbf{K}_{\mathbf{f}}(\Phi)(s) := \mathbf{S}(s)\big[\mathbf{f} - \mathbf{C}(\Phi, \mathbf{f})\big] + \int_0^s \mathbf{S}(s-s')\mathbf{N}\big(\Phi(s')\big)ds'.$$

We aim to show that $\mathbf{K}_{\mathbf{f}} : \mathcal{Y}_\delta \to \mathcal{Y}_\delta$ and is a contraction.

First, observe that by Theorem 52 and Proposition 51 we obtain

$$\mathbf{P}_j\mathbf{K}_{\mathbf{f}}(\Phi)(s) = -\int_s^\infty e^{j(s-s')}\mathbf{P}_j\mathbf{N}\big(\Phi(s')\big)ds'.$$

From Lemma 53 and the fact that $\mathbf{N}(\mathbf{0}) = \mathbf{0}$, we have the estimate

$$\begin{aligned}
\|\mathbf{P}_j \mathbf{K_f}(\Phi)(s)\|_{\mathcal{H}_R} &\lesssim e^{js} \int_s^\infty e^{-js'} \|\Phi(s')\|_{\mathcal{H}_R}^2 ds' \\
&\lesssim e^{js} \|\Phi\|_{\mathcal{X}_R}^2 \int_s^\infty e^{-js' - 2\omega_0 s'} ds' \\
&\lesssim \delta^2 e^{-2\omega_0 s}.
\end{aligned}$$

By Proposition 51, we have $(\mathbf{I} - \mathbf{P})\mathbf{C}(\Phi, \mathbf{f}) = \mathbf{0}$ which implies

$$(\mathbf{I} - \mathbf{P})\mathbf{K_f}(\Phi)(s) = \mathbf{S}(s)(\mathbf{I} - \mathbf{P})\mathbf{f} + \int_0^s \mathbf{S}(s - s')(\mathbf{I} - \mathbf{P})\mathbf{N}\big(\Phi(s')\big) ds'.$$

By Theorem 52, we obtain

$$\begin{aligned}
\|(\mathbf{I} - \mathbf{P})\mathbf{K_f}(\Phi)(s)\|_{\mathcal{H}_R} &\lesssim \frac{\delta}{c} \|\mathbf{f}\|_{\mathcal{H}_R} + \int_0^s e^{-\omega_0(s-s')} \|\mathbf{N}\big(\Phi(s')\big)\|_{\mathcal{H}_R} ds' \\
&\lesssim \frac{\delta}{c} e^{-\omega_0 s} + e^{-\omega_0 s} \int_0^s e^{\omega_0 s'} \|\Phi(s')\|_{\mathcal{H}_R}^2 ds' \\
&\lesssim \frac{\delta}{c} e^{-\omega_0 s} + \|\Phi\|_{\mathcal{X}_R}^2 e^{-\omega_0 s} \int_0^s e^{-\omega_0 s'} ds' \\
&\lesssim \frac{\delta}{c} e^{-\omega_0 s} + \delta^2 e^{-\omega_0 s}
\end{aligned}$$

for all $s \geq 0$. Thus, for all sufficiently large c and sufficiently small δ, we can ensure

$$\|\mathbf{K_f}(\Phi)(s)\|_{\mathcal{H}_R} \leq \delta e^{-\omega_0 s}.$$

Consequently, we see that $\mathbf{K_f} : \mathcal{Y}_\delta \to \mathcal{Y}_\delta$.

We claim that $\mathbf{K_f}$ is a contraction map on $\mathcal{Y}_\delta$. Given $\Phi, \Psi \in \mathcal{Y}_\delta$,

$$\mathbf{P}_j \mathbf{K_f}(\Phi)(s) - \mathbf{P}_j \mathbf{K_f}(\Psi)(s) = -\int_s^\infty e^{j(s-s')} \mathbf{P}_j \Big(\mathbf{N}\big(\Phi(s')\big) - \mathbf{N}\big(\Psi(s')\big) \Big) ds'.$$

By Lemma 53

$$\begin{aligned}
&\|\mathbf{P}_j\mathbf{K}_{\mathbf{f}}(\Phi)(s) - \mathbf{P}_j\mathbf{K}_{\mathbf{f}}(\Psi)(s)\|_{\mathcal{H}_R} \\
&\qquad\lesssim e^{js}\int_s^\infty e^{-js'}\big(\|\Phi(s')\|_{\mathcal{H}_R} + \|\Psi(s')\|_{\mathcal{H}_R}\big)\|\Phi(s') - \Psi(s')\|_{\mathcal{H}_R}ds' \\
&\qquad\lesssim \delta\|\Phi - \Psi\|_{\mathcal{X}_R}e^{js}\int_s^\infty e^{-js'-2\omega_0 s'}ds' \\
&\qquad\lesssim \delta e^{-2\omega_0 s}\|\Phi - \Psi\|_{\mathcal{X}_R}.
\end{aligned}$$

Furthermore,

$$(\mathbf{I}-\mathbf{P})\mathbf{K}_{\mathbf{f}}(\Phi)(s) - (\mathbf{I}-\mathbf{P})\mathbf{K}_{\mathbf{f}}(\Psi)(s) = \int_0^s \mathbf{S}(s-s')(\mathbf{I}-\mathbf{P})\Big(\mathbf{N}\big(\Phi(s')\big) - \mathbf{N}\big(\Psi(s')\big)\Big)ds'.$$

By Theorem 52 and Lemma 53

$$\begin{aligned}
&\|(\mathbf{I}-\mathbf{P})\mathbf{K}_{\mathbf{f}}(\Phi)(s) - (\mathbf{I}-\mathbf{P})\mathbf{K}_{\mathbf{f}}(\Psi)(s)\|_{\mathcal{H}_R} \\
&\qquad\lesssim \int_0^s e^{-\omega_0(s-s')}\big(\|\Phi(s')\|_{\mathcal{H}_R} + \|\Psi(s')\|_{\mathcal{H}_R}\big)\|\Phi(s') - \Psi(s')\|_{\mathcal{H}_R}ds' \\
&\qquad\lesssim \delta\|\Phi - \Psi\|_{\mathcal{X}_R}e^{-\omega_0 s}\int_0^s e^{-\omega_0 s'}ds' \\
&\qquad\lesssim \delta e^{-\omega_0 s}\|\Phi - \Psi\|_{\mathcal{X}_R}.
\end{aligned}$$

Thus,

$$\|\mathbf{K}_{\mathbf{f}}(\Phi) - \mathbf{K}_{\mathbf{f}}(\Psi)\|_{\mathcal{X}_R} \lesssim \delta\|\Phi - \Psi\|_{\mathcal{X}_R}$$

and by considering smaller δ if necessary, we see that $\mathbf{K}_{\mathbf{f}}$ is a contraction on $\mathcal{Y}_\delta$. The Banach fixed point theorem implies the existence of a unique fixed point $\Phi_{\mathbf{f}} \in \mathcal{Y}_\delta$ of $\mathbf{K}_{\mathbf{f}}$.

We now claim that the solution map $\mathbf{f} \mapsto \Phi_{\mathbf{f}}$ is Lipschitz. Observe that

$$\begin{aligned}
\|\Phi_{\mathbf{f}} - \Phi_{\mathbf{g}}\|_{\mathcal{X}_R} &= \|\mathbf{K}_{\mathbf{f}}(\Phi_{\mathbf{f}}) - \mathbf{K}_{\mathbf{g}}(\Phi_{\mathbf{g}})\|_{\mathcal{X}_R} \\
&\leq \|\mathbf{K}_{\mathbf{f}}(\Phi_{\mathbf{f}}) - \mathbf{K}_{\mathbf{f}}(\Phi_{\mathbf{g}})\|_{\mathcal{X}_R} + \|\mathbf{K}_{\mathbf{f}}(\Phi_{\mathbf{g}}) - \mathbf{K}_{\mathbf{g}}(\Phi_{\mathbf{g}})\|_{\mathcal{X}_R} \\
&\lesssim \delta\|\Phi_{\mathbf{f}} - \Phi_{\mathbf{g}}\|_{\mathcal{X}_R} + \|\mathbf{K}_{\mathbf{f}}(\Phi_{\mathbf{g}}) - \mathbf{K}_{\mathbf{g}}(\Phi_{\mathbf{g}})\|_{\mathcal{X}_R}.
\end{aligned}$$

A direct calculation shows

$$\mathbf{K}_{\mathbf{f}}(\Phi_{\mathbf{g}})(s) - \mathbf{K}_{\mathbf{g}}(\Phi_{\mathbf{g}})(s) = \mathbf{S}(s)(\mathbf{I} - \mathbf{P})(\mathbf{f} - \mathbf{g}).$$

Theorem 52 yields

$$\|\mathbf{K}_{\mathbf{f}}(\Phi_{\mathbf{g}})(s) - \mathbf{K}_{\mathbf{g}}(\Phi_{\mathbf{g}})(s)\|_{\mathcal{H}_R} \lesssim e^{-\omega_0 s}\|\mathbf{f} - \mathbf{g}\|_{\mathcal{H}_R}.$$

Thus, we have

$$\|\Phi_{\mathbf{f}} - \Phi_{\mathbf{g}}\|_{\mathcal{X}_R} \lesssim \delta\|\Phi_{\mathbf{f}} - \Phi_{\mathbf{g}}\|_{\mathcal{X}_R} + \|\mathbf{f} - \mathbf{g}\|_{\mathcal{H}_R}$$

Again, considering smaller δ if necessary yields the result. ■

At this point, we can, in fact, prove a genuine co-dimension 2 stability result if we allow ourselves to evolve data specified on a fixed hyperboloid in spacetime. However, we aim to instead prove a conditional stability result starting with data specified at $t = 0$.

3.5 Preparation of Hyperboloidal Initial Data

In this section, we evolve data of the form $u_1^*[0] + (f, g)$ for sufficiently small, smooth, compactly supported, radial functions f, g into the region $\Omega_{T,R}$ according to Equation (3.1). As this region is not foliated by surfaces of constant physical time, this evolution must occur in two steps: first along hypersurfaces of constant physical time and then along hypersurfaces of constant hyperboloidal time. Carrying out this two-step evolution is rather nontrivial due to the fact that $\mathbf{L}$ has more than one unstable eigenvalue. In order to explain the nontrivial nature of this problem properly, we will first describe the most natural approach one might naively attempt and then describe how we adapt this approach.

For the moment, let T be some number close to 1 and $R \geq \frac{1}{2}$. The region $\Omega_{T,R}$ can be covered by a mix of hyperboloids and slices of constant physical time. With this in mind, a natural first step would be to solve the Cauchy problem

$$\begin{cases} \Box u = u^2 \\ u[0] = u_1^*[0] + (f, g) \end{cases} \tag{3.22}$$

for some short amount of time. To continue the evolution to the rest of $\Omega_{T,R}$, one might expect to evaluate this solution on some hyperboloid and evolve the solution further using the nonlinear theory developed in Section 3.4. In fact, this is precisely what is done in [1] and [14]. Of course, Equation (3.21) is not the quadratic wave equation due to the correction term. So, one might hope that there exists at least one choice of T for which the correction term, $\mathbf{C}$, vanishes. If this were true, then solutions of Equation (3.21) could in fact yield solutions of the quadratic wave equation in $\Omega_{T,R}$. An obstruction to this is that the correction term is a sum of two terms, one for each unstable eigenvalue. That is, $\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_4$ with $\mathbf{C}_1$ correcting for the eigenvalue $\lambda = 1$ and $\mathbf{C}_4$ correcting for the eigenvalue $\lambda = 4$. Without an additional parameter to vary, one cannot hope to guarantee the vanishing of both correction terms.

Recall the solution F_4^* of the quadratic wave equation linearized around u_T^*, namely

$$F_4^*(t,r) = \frac{1}{\left(5r^2 + 3(T-t)^2\right)^3}.$$

Translating to hyperboloidal similarity coordinates, we have $(F_4^* \circ \eta_T)(s,y) = [(e^{6s}\mathbf{f}_4^*(y)]_1$. The role of the correction term $\mathbf{C}_4$ is to remove the contribution of $\mathbf{f}_4^*$ from hyperboloidal initial data. Thus, it seems plausible to expect that for data of the form $u_1^*[0]+(f,g)+\alpha F_4^*[0]$, the correction term $\mathbf{C}_4$ might vanish for at least one choice of α. As stated, it is not possible to guarantee this and continue the evolution of such data along hyperboloids using techniques as in [1] or [14] where hyperboloidal similarity coordinates were first developed. This is due to the fact that the physical evolution of this data cannot necessarily be contained in a single ball on a hyperboloid and, as a consequence, the nonlinear theory from Section 3.4 cannot be applied in a meaningful way.

As a remedy, one might expect that data of the form $u_1^*[0] + (f,g) + \alpha\chi F_4^*[0]$, for some smooth cutoff function χ and some choice of α and T might work. Though this may be possible, it appears extremely difficult to continue the evolution of such data along hyperboloids in a controllable way. The reason for this difficulty is that one proves that there are parameters α and T for which $\mathbf{C}$ vanishes via a fixed point argument. In order to run this fixed point argument one needs two crucial pieces of information. One crucial bit of information needed is uniform control of the derivatives of the solution of Equation (3.22). If $\alpha\chi F_4^*$ were a solution of the quadratic wave equation or if χF_4^* were a solution of the quadratic wave equation equation linearized around u_1^*, then this uniform control could be obtained. Unfortunately, neither of the two seem to be easy to satisfy in relation to the second bit of information. Running the fixed point argument will eventually necessitate that the spectral projection $\mathbf{P}_4$ applied to a portion of the data not vanish. Proving non-vanishing of the spectral projection applied to this portion of the data appears to be extremely difficult unless it happens to be something rather specific.

Finally, we have enough information to adapt the naive approach. Let's say that any sufficiently smooth perturbation of u_1^* of unit size can be evolved for at least a length of time $t_0 > 0$. For technical reasons, we impose the condition that our perturbation have support contained in the interval $[0, r_0)$ with $r_0 := \frac{t_0}{4}$. With this, we have two conditions which determine the proper replacement, denoted by (Y_1, Y_2), for the term $\chi F_4^*[0]$ in our perturbation:

1. We require that (Y_1, Y_2) when restricted to a particular hyperboloid be precisely $\chi_{t_0}\mathbf{f}_4^*$, up to some constant multiple, where χ_{t_0} is a rather specific smooth cutoff function with support determined by t_0. The support of χ_{t_0} is chosen precisely so that, when viewed in spacetime, its domain of influence at $t = 0$ is contained within the interval $[0, r_0)$. Furthermore, the support is also chosen so that $\mathbf{P}_4$ applied to the previously mentioned portion of the solution does not vanish.

2. We also require that Y_1 be the restriction of a solution of the linearized equation $\Box u = 2u_1^* u$ at $t = 0$ with Y_2 being its time derivative at $t = 0$.

The first property ensures the desired condition involving the spectral projection while the second ensures the necessary uniform control on derivatives of the local solution obtained by solving Equation (3.22). These two properties are met by solving the linearized equation in two different ways; first in hyperboloidal similarity coordinates and second in physical coordinates. The condition on the support of the cutoff ensures that both ways of solving the linearized equation produce the same result in the overlapping region. Before carrying this out, we outline the construction and proof.

Our goal is to construct a smooth solution of the linearized equation $\Box u = 2u_1^* u$ in the region Λ_{t_0} defined by

$$\Lambda_{t_0} := [-t_0, t_0] \times [0, \infty) \cup \{(t, r) \in \mathbb{R} \times [0, \infty) : -r + r_0 \leq t \leq r - r_0\}$$

satisfying the above two properties. To achieve this, one can first solve the abstract initial value problem

$$\begin{cases} \partial_s \Phi(s) = \mathbf{L}\Phi(s) \\ \Phi(s_0) = \chi_{t_0} \mathbf{f}_4^* \end{cases}$$

for $s \geq s_0$ with

$$s_0 := \log\Big(-\frac{2h(0)}{2 + r_0}\Big)$$

on the space $\mathcal{H}^k_{1/2}$ for any $k \in \mathbb{N}$. The cutoff χ_{t_0} is chosen to be non-increasing and to have support contained in the interval $[0, y_0)$ with y_0 to be defined later. This number is chosen precisely so that the domain of influence of $\operatorname{supp}(\chi_{t_0})$, when viewed in spacetime, at $t = 0$ is contained in the interval $[0, r_0)$. As a consequence, one can prove that the solution is smooth and translates it to a smooth solution of $\Box u = 2u_1^* u$ in the spacetime region $\eta_1([s_0, \infty) \times [0, \frac{1}{2}))$ see Figure 3.3. Let's call this solution u_{lin}. Of course, the spacetime region $\eta_1([s_0, \infty) \times [0, \frac{1}{2}))$ does not contain all of Λ_{t_0}. In order to extend the domain of u_{lin}, we first realize that along the initial hyperboloid, $t = 1 + e^{-s_0} h(y_0)$ is the first time at which, beyond that time and along the initial hyperboloid, the cutoff χ_{t_0} is guaranteed to vanish. Thus, the uniqueness of solutions of linear wave equations, see Lemma 12.8 of [32] for instance, guarantees that u_{lin} vanishes in the dark gray portion of $\eta_1([s_0, \infty) \times [0, \frac{1}{2}))$ depicted in Figure 3.3. With this in mind, we smoothly extend $u_{\text{lin}}(0, \cdot)$ by zero, i.e., we

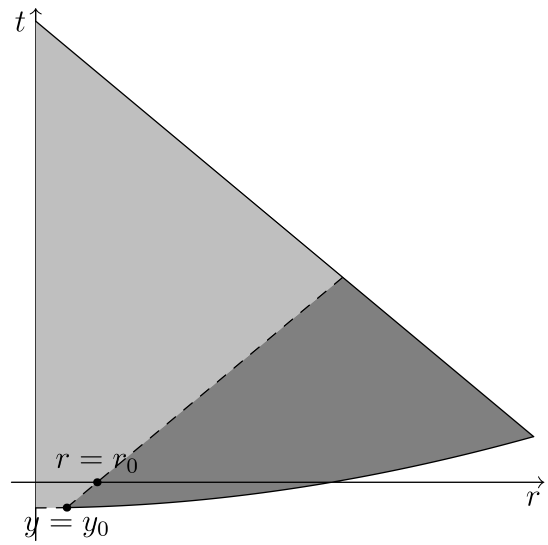

Figure 3.3: A spacetime diagram depicting the region $\eta_1([s_0,\infty)\times[0,\frac{1}{2}))$. The cutoff χ_{t_0} is supported on the dashed portion of the hyperboloid $\eta_1(\{s_0\}\times[0,\frac{1}{2}))$ in the bottom left corner of the picture. The solution produced is guaranteed to vanish in the dark gray region and is potentially nonzero in the light gray region.

define functions

$$U_1(r) := \begin{cases} u_{\text{lin}}(0,r) & r \leq r_0 \\ 0 & r \geq r_0 \end{cases} \tag{3.23}$$

and

$$U_2(r) := \begin{cases} \partial_t u_{\text{lin}}(0,r) & r \leq r_0 \\ 0 & r \geq r_0 \end{cases}. \tag{3.24}$$

We then use these functions as initial data for the Cauchy problem

$$\begin{cases} \Box u = 2u_1^* u & \text{in } \Lambda_{t_0} \\ u[0] = \big(U_1, U_2\big) \end{cases}$$

which is guaranteed to have a unique smooth solution since u_1^* is smooth in Λ_{t_0} and U_1, U_2 are smooth, see for instance Theorem 3.2 of [38]. Since this solution and the original agree on the initial hyperboloid, they must agree wherever Λ_{t_0} and $\eta_1([s_0,\infty)\times[0,\frac{1}{2}))$ overlap.

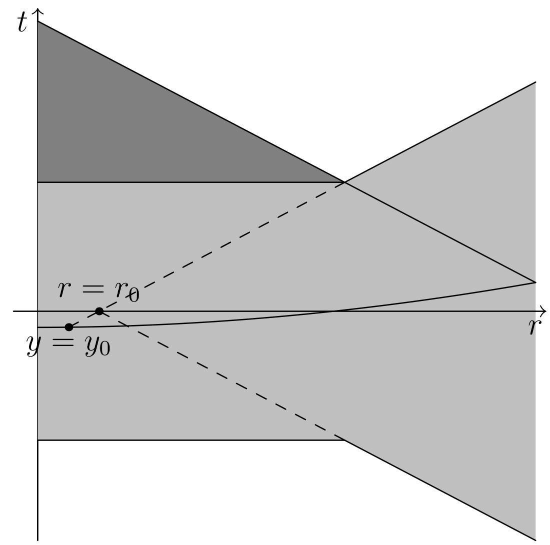

Figure 3.4: A spacetime diagram depicting the domain of the smoothly extended u_{lin}. The darker region depicts a portion of where u_{lin} was originally defined. The lighter region depicts where u_{lin} has been extended by solving Equation (3.29).

Thus, we will have achieved extending u_{lin} to a larger region of spacetime which, in fact, strictly contains Λ_{t_0}, see Figure 3.4. Thus, we have a $C^\infty(\Lambda_{t_0})$ solution of the linearized equation which satisfies our two conditions.

Equipped with the solution u_{lin}, we set $(Y_1, Y_2) = u_{\text{lin}}[0]$ and evolve small, smooth, radial perturbations of u_1^* of the form $(f, g) + \alpha(Y_1, Y_2)$ with $\text{supp}(f, g) \subseteq [0, r_0)$. Adjusting the size of (f, g) and of $|\alpha|$ allows us to ensure $(f, g) + \alpha(Y_1, Y_2)$ is of unit size, i.e., data of the form $u_1^*[0] + (f, g) + \alpha(Y_1, Y_2)$ can be evolved in Λ_{t_0} via the quadratic wave equation. Furthermore, since u_{lin} solves the linearized equation, we are able to obtain the necessary uniform control on the corresponding solution. Then, by allowing α and T to vary, we are indeed able to run the necessary fixed point argument proving the vanishing of the correction term. Thus, we are able to successfully continue the evolution into all of $\Omega_{T,R}$.

3.5.1 Local Existence for Perturbations of u_1^* at $t = 0$

In this section, we prove a local existence and uniqueness result for sufficiently smooth, radial perturbations of u_1^* of unit size. This will allow us to define the spacetime region Λ_{t_0} setting the stage for proving the main result. First, we review the standard Cauchy theory for nonlinear wave equations on balls. For the proofs of the following claims, we refer the reader to [1].

The solution of the Cauchy problem

$$\begin{cases} \Box u = 0 & \text{in } \mathbb{R}^{1+d} \\ u[0] = (f, g) \end{cases}$$

is given by the *wave propagators*

$$u(t, \cdot) = \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|}g$$

for $f, g \in \mathcal{S}(\mathbb{R}^d)$ where $\phi(|\nabla|)f = \mathcal{F}^{-1}(\phi(|\cdot|)\mathcal{F}f)$ for $\phi \in C(\mathbb{R})$. The wave propagators extend to rough data $(f, g) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$ by density. Furthermore, they satisfy the following estimates.

Proposition 56 *Let $d \geq 3$. Then there exists a continuous function $\gamma_d : [0, \infty) \to [1, \infty)$ such that*

$$\left\|\partial_t^\ell \cos(t|\nabla|)f\right\|_{\dot{H}^k(\mathbb{B}^d_{T-t})} \leq \|f\|_{\dot{H}^{k+\ell}(\mathbb{B}^d_T)},$$
$$\left\|\partial_t^\ell \frac{\sin(t|\nabla|)}{|\nabla|}f\right\|_{\dot{H}^k(\mathbb{B}^d_{T-t})} \leq \|f\|_{\dot{H}^{k+\ell-1}(\mathbb{B}^d_T)},$$
$$\left\|\partial_t^\ell \cos(t|\nabla|)f\right\|_{L^2(\mathbb{B}^d_{T-t})} \leq \gamma_d(T)\|f\|_{\dot{H}^{1+\ell}(\mathbb{B}^d_T)},$$

and

$$\left\|\partial_t^\ell \frac{\sin(t|\nabla|)}{|\nabla|}f\right\|_{L^2(\mathbb{B}^d_{T-t})} \leq \gamma_d(T)\|f\|_{\dot{H}^{\ell}(\mathbb{B}^d_T)}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$*,* $T > 0$*,* $t \in [0, T)$*,* $k \in \mathbb{N}$*, and* $\ell \in \mathbb{N}_0$*.*

Again by density, these estimates hold for data in homogeneous Sobolev spaces as well.

Clearly, the wave propagators are only defined for functions on all of $\mathbb{R}^d$. We can extend the definition of the wave propagators to functions defined on balls via Sobolev extensions as follows.

Lemma 57 *Let* $d \in \mathbb{N}$*. For any* $r > 0$ *there exists a linear map* $\mathcal{E}_{r,d} : L^2(\mathbb{B}_r^d) \to L^2(\mathbb{R}^d)$ *such that* $\mathcal{E}_{r,d} f|_{\mathbb{B}_r^d} = f$ *a.e.. If* $f \in H^k(\mathbb{B}_r^d)$ *for* $k \in \mathbb{N}$*, then* $\mathcal{E}_{r,d} f \in H^k(\mathbb{R}^d)$*. Furthermore, there exists a constant* $C_{r,k,d} > 0$ *such that*

$$\|\mathcal{E}_{r,d} f\|_{H^k(\mathbb{R}^d)} \leq C_{r,k,d} \|f\|_{H^k(\mathbb{B}_r^d)}$$

for all $k \in \mathbb{N}_0$ *and* $f \in H^k(\mathbb{B}_r^d)$*.*

Definition 58 *Let* $T > 0$*,* $t \in [0, T)$*, and* $d \in \mathbb{N}$*,* $d \geq 3$*. Then we define*

$$\cos(t|\nabla|), \frac{\sin(t|\nabla|)}{|\nabla|} : L^2(\mathbb{B}_{T-t}^d) \to L^2(\mathbb{B}_{T-t}^d)$$

by

$$\cos(t|\nabla|) f := \big(\cos(t|\nabla|)\mathcal{E}_{T,d} f\big)\big|_{\mathbb{B}_{T-t}^d}$$

and

$$\frac{\sin(t|\nabla|)}{|\nabla|} f := \left(\frac{\sin(t|\nabla|)}{|\nabla|}\mathcal{E}_{T,d} f\right)\Big|_{\mathbb{B}_{T-t}^d}$$

where $\mathcal{E}_{T,d}$ *is a Sobolev extension as in Lemma 57.*

We remark that the wave propagators in Definition 58 are independent of the choice of Sobolev extension. Furthermore, Proposition 56 implies that the wave propagators are bounded linear maps from $H^k(\mathbb{B}_T^d)$ to $H^k(\mathbb{B}_{T-t}^d)$ for all $k \in \mathbb{N}_0$, $T > 0$, and $t \in [0, T)$.

When solving nonlinear problems, we reformulate the corresponding Cauchy problem as a fixed point problem using the wave propagators in the following Banach space.

Definition 59 *Let $k \in \mathbb{N}_0$, $T > 0$, $d \in \mathbb{N}$, and $T' \in (0,T)$. We define a Banach space $X_T^k(T')$ which consists of functions*

$$u : \bigcup_{t \in [0,T']} \{t\} \times \mathbb{B}^d_{T-t} \to \mathbb{R}$$

such that $u(t,\cdot) \in H^k(\mathbb{B}^d_{T-t})$ for each $t \in [0,T']$ and the map $t \to \|u(t,\cdot)\|_{H^k(\mathbb{B}^d_{T-t})}$ is continuous on $[0,T']$. Furthermore, we set

$$\|u\|_{X_T^k(T')} := \max_{t \in [0,T']} \|u(t,\cdot)\|_{H^k(\mathbb{B}^d_{T-t})}$$

In this Banach space, we will look for solutions of the following equation.

Definition 60 *Let $k \in \mathbb{N}$, $T > 0$, $T' \in (0,T)$, and, $d \geq 3$. Let $\mathcal{N}$ be some nonlinear operator. We say that a function $u : \bigcup_{t \in [0,T']} \{t\} \times \mathbb{B}^d_{T-t} \to \mathbb{R}$ is a **strong H^k solution** of the Cauchy problem*

$$\begin{cases} \Box u = \mathcal{N}\big(\cdot, u(\cdot)\big) & \text{in } \bigcup_{t \in [0,T']} \{t\} \times \mathbb{B}^d_{T-t} \\ u[0] = (f,g) \end{cases}$$

if $u \in X_T^k(T')$ and

$$u(t,\cdot) = \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|}g + \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|}\mathcal{N}(s,\cdot,u(s,\cdot))ds$$

for all $t \in [0,T']$.

Remark 61 *Though we do not specify the particular nonlinear operator in our definition of strong solution, when we actually look for strong solutions it will be for a specific nonlinearity which will be clearly stated.*

Using the standard Cauchy theory just reviewed, we can evolve a large class of perturbations of u_1^* via the quadratic wave equation at least for some amount of time $t_0 > 0$. In fact, the estimates that t_0 must satisfy will be very important in the remainder of the argument and so we make them apparent.

Lemma 62 *There exists $t_0 \in (0, \frac{4}{9})$ such that for all $(f,g) \in H^{10}(\mathbb{B}_1^7) \times H^9(\mathbb{B}_1^7)$ satisfying*

$$\|f\|_{H^{10}(\mathbb{B}_1^7)} + \|g\|_{H^9(\mathbb{B}_1^7)} \leq 1,$$

the initial value problem

$$\begin{cases} \Box u = u^2 & \text{in } \bigcup_{t\in[0,t_0]}\{t\} \times \mathbb{B}_{1-t}^7 \\ u[0] = u_1^*[0] + (f,g) \end{cases}$$

has a unique strong H^{10} solution in the truncated lightcone $\bigcup_{t\in[0,t_0]}\{t\} \times \mathbb{B}_{1-t}^7$.

Proof. We seek to solve the Cauchy problem

$$\begin{cases} \Box \varphi = \varphi^2 + 2u_1^*\varphi \quad \text{in } \bigcup_{t\in[0,t_0]}\{t\} \times \mathbb{B}_{1-t}^7 \\ \varphi[0] = (f,g) \end{cases}.$$

A strong H^{10} solution of this Cauchy problem yields a strong H^{10} solution of the original Cauchy problem by setting $u = u_1^* + \varphi$. For $t_0 > 0$, set

$$Y(t_0) := \{\varphi \in X_1^{10}(t_0) : \|\varphi\|_{X_1^{10}(t_0)} \leq 2\gamma\}$$

where $\gamma := \max_{s\in[0,\frac{1}{2}]} \gamma(1-s)$ and $\gamma(\cdot)$ is the continuous function from Proposition 56. Define a map $\mathcal{K}_{f,g}$ on $Y(t_0)$ by

$$\mathcal{K}_{f,g}(\varphi)(t) = \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|}g + \int_0^t \frac{\sin\big((t-s)|\nabla|\big)}{|\nabla|}\mathcal{N}\big(s,\cdot,\varphi(s,\cdot)\big)ds$$

where

$$\mathcal{N}\big(t,x,\varphi(t,x)\big)=\varphi^2(t,x)+2u_1^*(t,|x|)\varphi(t,x).$$

By the Banach algebra property, we infer the existence of a constant $0<C<\infty$ such that

$$\|\tilde{f}\tilde{g}\|_{H^9(\mathbb{B}^7_r)}\leq C\|\tilde{f}\|_{H^9(\mathbb{B}^7_r)}\|\tilde{g}\|_{H^9(\mathbb{B}^7_r)} \tag{3.25}$$

for any $\tilde{f},\tilde{g}\in H^9(\mathbb{B}^7_r)$ and $r\in[\frac{1}{2},1]$. As a consequence of this bound and the standard Cauchy theory, we have

$$\begin{aligned}\|\mathcal{K}_{f,g}(\varphi)(t)\|_{H^{10}(\mathbb{B}^7_{1-t})}\leq&\gamma\|f\|_{H^{10}(\mathbb{B}^7_1)}+\gamma\|g\|_{H^9(\mathbb{B}^7_1)}+\gamma\int_0^t\|\mathcal{N}(s,\cdot,\varphi(s,\cdot))\|_{H^9(\mathbb{B}^7_{1-s})}ds\\ \leq&\gamma+\gamma\int_0^t\big(\|\varphi(s,\cdot)^2\|_{H^9(\mathbb{B}^7_{1-s})}+2\|u_1^*(s,\cdot)\varphi(s,\cdot)\|_{H^9(\mathbb{B}^7_{1-s})}\big)ds\\ \leq&\gamma+\gamma Ct_0\big(\|\varphi\|^2_{X_1^{10}(t_0)}+2\|u_1^*\|_{X_1^{10}(t_0)}\|\varphi\|_{X_1^{10}(t_0)}\big)\\ \leq&\gamma+\gamma Ct_0\big((2\gamma)^2+2\|u_1^*\|_{X_1^{10}(\frac{4}{9})}2\gamma\big).\end{aligned}$$

Thus, $\mathcal{K}_{f,g}:Y(t_0)\to Y(t_0)$ provided t_0 is small enough so that the inequality

$$t_0C\big((2\gamma)^2+2\|u_1^*\|_{X_1^{10}(\frac{4}{9})}2\gamma\big)\leq 1$$

holds. Similarly, given $\varphi,\psi\in Y(t_0)$, we have

$$\begin{aligned}&\|\mathcal{K}_{f,g}(\varphi)(t)-\mathcal{K}_{f,g}(\psi)(t)\|_{H^{10}(\mathbb{B}^7_{1-t})}\\ &\quad\leq\gamma\int_0^t\|\mathcal{N}(s,\cdot,\varphi(s,\cdot))-\mathcal{N}(s,\cdot,\psi(s,\cdot))\|_{H^9(\mathbb{B}^7_{1-s})}ds\\ &\quad\leq\gamma\int_0^t\big(\|(\varphi(s,\cdot)-\psi(s,\cdot))^2\|_{H^9(\mathbb{B}^7_{1-s})}\\ &\qquad+2\|u_1^*(s,\cdot)(\varphi(s,\cdot)-\psi(s,\cdot))\|_{H^9(\mathbb{B}^7_{1-s})}\big)ds\\ &\quad\leq\gamma Ct_0\big(4\gamma+2\|u_1^*\|_{X_1^{10}(\frac{4}{9})}\big)\|\varphi-\psi\|_{X_1^{10}(t_0)}.\end{aligned}$$

Thus $\mathcal{K}_{f,g}$ is Lipschitz on $Y(t_0)$ with Lipschitz constant at most $\frac{1}{2}$ provided t_0 is small enough so that the inequality

$$\gamma t_0 C\big(4\gamma + 2\|u_1^*\|_{X_1^{10}(\frac{4}{9})}\big) \leq \frac{1}{2}$$

holds. Thus, $\mathcal{K}_{f,g}$ is a contraction on the closed subspace $Y(t_0)$ of the Banach space $X_1^{10}(t_0)$. As a consequence, the Banach fixed point theorem implies the existence of a unique fixed point $\varphi_{f,g,\alpha} \in Y(t_0)$ of $\mathcal{K}_{f,g}$. ■

3.5.2 Construction of an Adjustment Term

In this section, we use the number t_0 produced in Lemma 62 to construct a particular solution of the linearized equation $\Box u = 2u_1^* u$ which is smooth in the spacetime region Λ_{t_0}. This function will have the crucial property that on a particular hyperboloid, it will be a suitably cutoff version of the unstable eigenfunction $f_{4,1}^*$. This property will allow us to run a fixed point argument to show that perturbations of u_1^* at $t = 0$, adjusted by this solution of the linearized equation, evolve according to the quadratic wave equation into something which converges to u_T^* for T close to 1. In order to properly motivate this, we first prove one of the crucial properties which allows us to successfully run this fixed point argument and then proceed with the construction.

With $t_0 > 0$ as in Lemma 62 and $r_0 := \frac{t_0}{4}$, we define the number

$$y_0 := e^{s_0} \frac{(4+\sqrt{2})r_0^2 + 4(2+\sqrt{2})r_0}{8(r_0 + 2 + \sqrt{2})}. \tag{3.26}$$

This number is chosen by solving the equation $r - r_0 = 1 + e^{-s_0} h(e^{s_0} r)$ for r and setting

$y_0 = e^{s_0} r$. For reasons to be made clear very soon, consider the function

$$F(y) = \frac{f_{4,1}(y)}{W(y;4)} \frac{\tilde{G}_4(y)}{c_{12}(y)} - \Big(\frac{c_{21}(y) f_{4,1}^*(y)^2}{W(y;4) c_{12}(y)}\Big)'$$

for $y \in [0, \frac{1}{2}]$ where

$$\tilde{G}_4(y) = c_{21}(y) \frac{df_{4,1}^*}{dy} + \big(c_{20}(y) - 12\big) f_{4,1}^*(y)$$

and

$$W(y;4) = \frac{\left(y^2+1\right)\left(1-4y^2\right)^{-4}\left(3\sqrt{y^2+2}-y+4\right)^2\left(3\sqrt{y^2+2}+y+4\right)^2}{y^6\sqrt{y^2+2}\sqrt{y^2+2\sqrt{y^2+2}+3}}.$$

A direct calculation shows that there is $\delta_0 > 0$ so that $F(y) > 0$ for $y \in [0, \delta_0)$. Now, let $\chi_{t_0} : [0, \infty) \to [0, 1]$ be any smooth cutoff function with $\chi_{t_0}(y) = 1$ for $y \leq \frac{1}{2} \min\{y_0, \delta_0\}$ and $\chi_{t_0}(y) = 0$ for $y \geq \frac{3}{4} \min\{y_0, \delta_0\}$. With this, we state and prove the following crucial lemma.

Lemma 63 *Fix $R \geq \frac{1}{2}$. Then $(\mathbf{P}_4(\chi_{t_0} \mathbf{f}_4^*) | \mathbf{f}_4^*)_{\mathcal{H}_R} \neq 0$.*

Proof. Observe that $\chi_{t_0} \mathbf{f}_4^* \in C_e^\infty[0, R]^2$. Given $\lambda \in \rho(\mathbf{L})$, recall that the second component of the resolvent can be re-expressed in terms of the first component by the equation

$$[\mathbf{R_L}(\lambda)\mathbf{f}]_2 = (\lambda + 2)[\mathbf{R_L}(\lambda)\mathbf{f}]_1 - f_1$$

for any $\mathbf{f} = (f_1, f_2) \in \mathcal{H}$. This implies

$$\begin{aligned}
[\mathbf{P}_4 \mathbf{f}]_2 &= \lim_{\lambda \to 4} (\lambda - 4)[\mathbf{R_L}(\lambda)\mathbf{f}]_2 \\
&= \lim_{\lambda \to 4} (\lambda - 4)\Big((\lambda + 2)[\mathbf{R_L}(\lambda)\mathbf{f}]_1 - f_1\Big) \\
&= 6[\mathbf{P}_4 \mathbf{f}]_1.
\end{aligned}$$

Thus, it suffices to study $[\mathbf{P}_4(\chi_{t_0} \mathbf{f}_4^*)]_1$.

For $\lambda \in \rho(\mathbf{L})$ the first component of $\mathbf{u} := \mathbf{R_L}(\lambda)(\chi_{t_0}\mathbf{f}_4^*)$ solves the ODE

$$\begin{aligned} u''(y) + \frac{c_{11}(y) + (\lambda+2)c_{21}(y)}{c_{12}(y)}u'(y) + \frac{(\lambda+2)(c_{20}(y)-\lambda-2)+V(y)}{c_{12}(y)}u(y) \\ = \frac{\chi_{t_0}(y)\tilde{G}_\lambda(y) + c_{21}(y)\chi_{t_0}'(y)f_{4,1}^*(y)}{c_{12}(y)} \end{aligned} \tag{3.27}$$

on the interval $(0, R)$ where

$$\tilde{G}_\lambda(y) = c_{21}(y)\frac{df_{4,1}^*}{dy} + \big(c_{20}(y) - 4 - 2\lambda\big)f_{4,1}^*(y).$$

For the homogeneous equation, the Frobenius indices at the singular point $y = 0$ are $\{0, -5\}$ while at $y = \frac{1}{2}$ they are $\{0, 1-\lambda\}$. Denote by $\phi_1(\cdot;\lambda)$ a solution of the homogeneous version of Equation (3.27) taking the index -5 at $y = 0$ and $\phi_0(\cdot;\lambda)$ a solution of the homogeneous version of Equation (3.27) taking the index 0 at $y = 0$. Observe that $\phi_0(\cdot;\lambda)$ must take the index $1-\lambda$ at $y = \frac{1}{2}$ since, if otherwise, $\phi_0(\cdot;\lambda) \in C^\infty[0,1]$ and this is excluded by Proposition 48 for $\lambda \in \rho(\mathbf{L})$. The Wronskian is $W\big(\phi_1(\cdot;\lambda), \phi_0(\cdot;\lambda)\big)(y) = C(\lambda)W(y;\lambda)$ where $C(\lambda)$ is some constant depending on λ and

$$W(y;\lambda) = \frac{(y^2+1)\left(1-4y^2\right)^{-\lambda}\left(3\sqrt{y^2+2} - y + 4\right)^{\lambda/2}\left(3\sqrt{y^2+2} + y + 4\right)^{\lambda/2}}{y^6\sqrt{y^2+2}\sqrt{y^2+2\sqrt{y^2+2}+3}}.$$

The fact that $\lambda = 4$ is a simple pole of the resolvent implies that $C(\lambda)$ must vanish to order one at $\lambda = 4$.

Since neither of these two fundamental solutions live in $H^6_{\text{rad}}(\mathbb{B}^7_{1/2})$, variation of parameters implies

$$\begin{aligned} &[\mathbf{R_L}(\lambda)(\chi_{t_0}\mathbf{f}_4^*)]_1(y) \\ &= \phi_1(y;\lambda)\int_0^y \frac{\phi_0(\tilde{y};\lambda)}{W\big(\phi_0(\cdot;\lambda),\phi_1(\cdot;\lambda)\big)(\tilde{y})} \frac{\chi_{t_0}(\tilde{y})\tilde{G}_\lambda(\tilde{y}) + c_{21}(\tilde{y})\chi_{t_0}'(\tilde{y})f_{4,1}^*(\tilde{y})}{c_{12}(\tilde{y})}d\tilde{y} \\ &\quad + \phi_0(y;\lambda)\int_y^{\frac{1}{2}} \frac{\phi_1(\tilde{y};\lambda)}{W\big(\phi_0(\cdot;\lambda),\phi_1(\cdot;\lambda)\big)} \frac{\chi_{t_0}(\tilde{y})\tilde{G}_\lambda(\tilde{y}) + c_{21}(\tilde{y})\chi_{t_0}'(\tilde{y})f_{4,1}^*(\tilde{y})}{c_{12}(\tilde{y})}d\tilde{y}. \end{aligned}$$

By repeated integration by parts, one can indeed see that the above expression is in $H^6_{\mathrm{rad}}(\mathbb{B}^7_{1/2})$.

Since $\mathbf{f}_4^*$ is an eigenfunction, we must have that $\phi_0(\cdot;4)$ and $\phi_1(\cdot;4)$ are multiples of $f_{4,1}$. Consequently, we obtain

$$[\mathbf{P}_4(\chi_{t_0}\mathbf{f}_4^*)]_1(y) = Cf_{4,1}^*(y)\int_0^{\frac{1}{2}} \frac{f_{4,1}^*(\tilde{y})}{W(\tilde{y};4)} \frac{\chi_{t_0}(\tilde{y})\tilde{G}_4(\tilde{y}) + c_{21}(\tilde{y})\chi_{t_0}'(\tilde{y})f_{4,1}^*(\tilde{y})}{c_{12}(\tilde{y})}d\tilde{y}$$

for some $C \in \mathbb{C} \setminus \{0\}$. An integration by parts yields

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \frac{f_{4,1}^*(\tilde{y})}{W(\tilde{y};4)} \frac{\chi_{t_0}(\tilde{y})\tilde{G}_4(\tilde{y}) + c_{21}(\tilde{y})\chi_{t_0}'(\tilde{y})f_{4,1}^*(\tilde{y})}{c_{12}(\tilde{y})}d\tilde{y} \\
&\quad = \int_0^{\frac{1}{2}} \chi_{t_0}(\tilde{y})\Big(\frac{f_{4,1}^*(\tilde{y})}{W(\tilde{y};4)}\frac{\tilde{G}_4(\tilde{y})}{c_{12}(\tilde{y})} - \Big(\frac{c_{21}(\tilde{y})f_{4,1}^*(\tilde{y})^2}{W(\tilde{y};4)c_{12}(\tilde{y})}\Big)'\Big)d\tilde{y}
\end{aligned}$$

By definition of χ_{t_0}, the integrand is positive within $\mathrm{supp}(\chi_{t_0})$ which implies

$$[\mathbf{P}_4(\chi_{t_0}\mathbf{f}_4^*)]_1(y) = C_{t_0}f_{4,1}^*(y)$$

for some $C_{t_0} \neq 0$. This implies the desired claim. ■

With the above lemma in mind, we are motivated to construct a function which, on an initial hyperboloid, is precisely $\chi_{t_0}\mathbf{f}_4^*$.

Lemma 64 *Let $t_0 > 0$ be defined as in Lemma 62. There exists a smooth, radial solution of $\Box u = 2u_1^* u$ in the spacetime region Λ_{t_0} with the following two properties:*

1. *$(u \circ \eta_1)(s_0, y) = e^{-2s_0}\chi_{t_0}(y)f_{4,1}^*(y)$ for all $y \in [0,\infty)$ and*

2. *$u|_{t=0}$ has support contained in the interval $[0, r_0)$.*

Proof. The proof proceeds in two steps. First, we evolve the data $\chi_{t_0}\mathbf{f}_4^*$ according to the linearized equation in the hyperboloidal formulation. As this data is smooth, we can in fact show that the solution yields a smooth solution of $\Box u = 2u_1^* u$ in the spacetime region $\eta_1\big([s_0, \infty) \times [0, \frac{1}{2})\big)$. Second, we can extend the domain of this solution to the rest of the claimed spacetime region by evolving it in a standard way. Again, the data will be smooth and so must be the solution.

Let $k \in \mathbb{N}$, $k \geq 6$ and consider the abstract initial value problem

$$\begin{cases} \partial_s \Phi(s) = \mathbf{L}\Phi(s) \\ \Phi(s_0) = \chi_{t_0} \mathbf{f}_4^* \end{cases}$$

on the space $\mathcal{H}^k_{1/2}$. For $k = 6$, we have that the unique solution in $C([s_0, \infty), \mathcal{H}^6_{1/2})$ is given by

$$\Phi(s) := \mathbf{S}(s - s_0)(\chi_{t_0} \mathbf{f}_4^*) \; s \geq s_0$$

where $\big(\mathbf{S}(s)\big)_{s \geq 0}$ is the strongly continuous semigroup from Lemma 45. For $k > 6$, Lemma 44 along with the bounded perturbation theorem imply that $\mathbf{L}$ is also the generator of a strongly continuous semigroup of bounded operators on $\mathcal{H}^k_{1/2}$ which we will denote by $\big(\mathbf{S}_k(s)\big)_{s \geq 0}$. Observe that for any $k \geq 6$, $\mathbf{S}(s)|_{\mathcal{H}^k_{1/2}} = \mathbf{S}_k(s)$ for all $s \geq 0$ following the argument of Lemma 3.5 of [9]. Since $\chi_{t_0} \mathbf{f}_4^* \in C_e^\infty[0, R]^2$, we have as a consequence that Φ is the unique solution in $C([s_0, \infty), \mathcal{H}^k_{1/2})$ for any $k \geq 6$. Consequently, $\Phi(s) \in \mathcal{H}^k_{1/2}$ for any $k \in \mathbb{N}$, $k \geq 6$ and $s \geq s_0$ which, by Sobolev embedding, implies $\Phi(s)(|\cdot|) \in C^\infty(\mathbb{B}^7_{1/2})^2$ for all $s \geq s_0$.

Since $\chi_{t_0} \mathbf{f}_4^* \in C_e^\infty[0, R]^2$, Theorem 6.1.5 of [30] implies that $\Phi \in C^1([s_0, \infty), \mathcal{H}^k_{1/2})$ and

$$\partial_s \Phi(s) = \mathbf{S}(s - s_0)\big(\mathbf{L}(\chi_{t_0} \mathbf{f}_4^*)\big). \tag{3.28}$$

Thus, $\partial_s \Phi(s)(|\cdot|) \in C^\infty(\mathbb{B}^7_{1/2})^2$ for all $s \geq s_0$. From Equation (3.28), we find that $\partial_s \Phi \in$

$C^1([s_0,\infty),\mathcal{H}^k_{1/2})$. Thus, by an inductive argument, we conclude that for all $m\in\mathbb{N}$, $\Phi\in C^m([s_0,\infty),\mathcal{H}^k_{1/2})$ with $\partial_s^m\Phi(s) = \mathbf{S}(s-s_0)\big(\mathbf{L}^m(\chi_{t_0}\mathbf{f}_4^*)\big)$. Again by Sobolev embedding, $\partial_s^m\Phi(s)(|\cdot|)\in C^\infty(\mathbb{B}^7_{1/2})^2$ for all $m\in\mathbb{N}$ and $s\geq s_0$. Since all s and y derivatives exist to arbitrary order and are continuous, we conclude that $\Phi\in C^\infty([s_0,\infty)\times[0,\frac{1}{2}))$.

Upon defining $v(s,y) := e^{2s}\phi_1(s,y)$, we have that $u_{\text{lin}}(t,r) := \big(v\circ\eta_1^{-1}\big)(t,r)$ is a smooth solution of the $\Box u = 2u_1^*u$ in the spacetime region $\eta_1([s_0,\infty)\times[0,\frac{1}{2}))$. To begin extending the domain of u_{lin}, recall the smooth extensions of $u_{\text{lin}}(0,\cdot)$ and $\partial_t u_{\text{lin}}(0,\cdot)$, namely U_1 and U_2 defined as

$$U_1(r) := \begin{cases} u_{\text{lin}}(0,r) & r\leq r_0 \\ 0 & r\geq r_0 \end{cases}$$

and

$$U_2(r) := \begin{cases} \partial_t u_{\text{lin}}(0,r) & r\leq r_0 \\ 0 & r\geq r_0 \end{cases}.$$

Clearly, for $r>r_0$ all derivatives of U_1 and U_2 vanish. For $r<r_0$, all derivatives of U_1 and U_2 exist and are continuous since, for such r, U_1 and U_2 are the composition of smooth functions. All r-derivatives of $u_{\text{lin}}(0,\cdot)$ from the left vanish and $u_{\text{lin}}(0,r_0)$ by the choice of cutoff χ_{t_0} implying smoothness of U_1 and U_2 at $r=r_0$.

Now, consider the Cauchy problem

$$\begin{cases} \Box u = 2u_1^*u & \text{in } \Lambda_{t_0} \\ u[0] = (U_1,U_2) \end{cases}. \tag{3.29}$$

Since $u_1^*\in C^\infty(\Lambda_{t_0})$ and radial along with $U_1(|\cdot|), U_2(|\cdot|)\in C^\infty(\mathbb{R}^7)$, the Cauchy problem (3.29) has a unique radial solution $u\in C^\infty(\Lambda_{t_0})$. Furthermore, since the data satisfies

$\text{supp}(U_1, U_2) \subseteq [0, r_0)$ and is specified at $t = 0$, finite speed of propagation implies that $u = 0$ in the spacetime region $\{(t, r) \in \mathbb{R} \times [0, \infty) : r_0 - r \leq t \leq r_0 + r,\ r \geq r_0\}$. Finally, uniqueness of solutions to linear wave equations allows us to conclude that the solution we have just produced satisfies $u(t, r) = u_{\text{lin}}(t, r)$ for $(t, r) \in \eta_1([s_0, \infty) \times [0, \frac{1}{2}))$, i.e., the solution u smoothly extends u_{lin}. Consequently, we have a function $u_{\text{lin}} \in C^\infty(\Lambda_{t_0})$ which solves the linearized equation $\Box u = 2u_1^* u$ and satisfies $u_{\text{lin}}(1 + e^{-s_0} h(y), y) = e^{-2s_0} \chi_{t_0}(y) f_{4,1}^*(y)$. ■

3.5.3 Local Existence for Adjusted Perturbations of u_1^* at $t = 0$

In this section, we evolve perturbations of u_1^* adjusted by some multiple of u_{lin} in the spacetime region Λ_{t_0} according to the quadratic wave equation. In this region, we will obtain uniform control over enough derivatives of the evolution in order to continue the it via the hyperboloidal formulation. For convenience, given $m \in \mathbb{N}$, $\delta, \epsilon > 0$, we define

$$\mathcal{B}^m_{\delta,\epsilon} := \{(f, g) \in C_e^\infty([0, \infty))^2 : \text{supp}(f, g) \subset [0, \epsilon), \|(f, g)\|_{H^m_{\text{rad}}(\mathbb{R}^7) \times H^{m-1}_{\text{rad}}(\mathbb{R}^7)} \leq \delta\}.$$

Lemma 65 *Let $t_0 > 0$ be as in Lemma 62. For all sufficiently small $\delta > 0$ and sufficiently large $M_0 > 0$, we have that for all $(f, g) \in \mathcal{B}^{10}_{\delta/M_0^2, t_0/4}$ and $|\alpha| \leq \frac{\delta}{M_0}$, the initial value problem*

$$\begin{cases} \Box u = u^2 \\ u[0] = u_1^*[0] + \alpha u_{lin}[0] + (f, g) \end{cases} \tag{3.30}$$

has a unique radial solution $u \in C^\infty(\Lambda_{t_0})$ of the form $u = u_1^* + \alpha u_{lin} + \varphi_{f,g,\alpha}$ with

$$\sup_{(t,r) \in \Lambda_{t_0}} |\partial_t^i \partial_r^j \varphi_{f,g,\alpha}(t, r)| \lesssim \frac{\delta}{M_0^2}$$

for all $i, j \in \mathbb{N}_0$ with $i + j \leq 6$.

Proof. Since $u_{\text{lin}}[0], (f, g) \in C_e^\infty[0, \infty)^2$ and are supported in the interval $[0, r_0)$, we certainly can ensure that for all $\delta > 0$ sufficiently small and $M_0 > 0$ sufficiently large, the inequality $\|\alpha u_{\text{lin}}[0] + (f, g)\|_{H^{10}(\mathbb{R}^7)\times H^9(\mathbb{R}^7)} \leq 1$ holds. Thus, by Lemma 62, Equation (3.30) has a unique strong H^{10} solution in the truncated lightcone up to time t_0. It remains to show that our solution is of the stated form in Λ_{t_0}. To that end, we consider the Cauchy problem

$$\begin{cases} \Box\varphi = \varphi^2 + 2(u_1^* + \alpha u_{\text{lin}})\varphi + \alpha^2 u_{\text{lin}}^2 \\ \varphi[0] = (f, g) \end{cases} . \tag{3.31}$$

A strong H^{10} solution of (3.31) yields a strong H^{10} solution of the original Cauchy problem by setting $u = u_1^* + \alpha u_{\text{lin}} + \varphi$. Set

$$Y'(t_0) := \left\{ \varphi \in X_1^{10}(t_0) : \|\varphi\|_{X_1^{10}(t_0)} \leq 2\gamma \frac{\delta}{M_0^2} \right\}.$$

Define a map $\mathcal{K}_{f,g,\alpha}$ on $Y'(t_0)$ by

$$\mathcal{K}_{f,g,\alpha}(\varphi)(t) := \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|}g + \int_0^t \frac{\sin\big((t-s)|\nabla|\big)}{|\nabla|}\mathcal{N}_\alpha\big(s, \cdot, \varphi(s, \cdot)\big)ds,\ t \in [0, t_0]$$

where

$$\mathcal{N}_\alpha\big(t, x, \varphi(t, x)\big) := \varphi^2(t, x) + 2(u_1^*(t, |x|) + \alpha u_{\text{lin}}(t, |x|))\varphi(t, x) + \alpha^2 u_{\text{lin}}^2(t, |x|).$$

Similar to the proof of Lemma 62, we have

$$
\begin{aligned}
&\|\mathcal{K}_{f,g,\alpha}(\varphi)(t)\|_{H^{10}(\mathbb{B}^7_{1-t})} \\
&\quad\leq \gamma\|f\|_{H^{10}(\mathbb{B}^7_1)} + \gamma\|g\|_{H^9(\mathbb{B}^7_1)} + \gamma\int_0^t \|\mathcal{N}_\alpha(s,\cdot,\varphi(s,\cdot))\|_{H^9(\mathbb{B}^7_{1-s})}ds \\
&\quad\leq \gamma\frac{\delta}{M_0^2} + \gamma C\int_0^t \big(\|\varphi(s,\cdot)\|^2_{H^9(\mathbb{B}^7_{1-s})} \\
&\qquad + 2(\|u_1^*(s,\cdot)\|_{H^9(\mathbb{B}^7_{1-s})} + |\alpha|\|u_{\text{lin}}(s,\cdot)\|_{H^9(\mathbb{B}^7_{1-s})})\|\varphi(s,\cdot)\|_{H^9(\mathbb{B}^7_{1-s})} \\
&\qquad + \alpha^2\|u_{\text{lin}}(s,\cdot)\|^2_{H^9(\mathbb{B}^7_{1-s})}\big)ds \\
&\quad\leq \gamma\frac{\delta}{M_0^2} + \gamma C t_0\big(\|\varphi\|^2_{X_1^{10}(t_0)} + 2(\|u_1^*\|_{X_1^{10}(\frac{4}{9})} + |\alpha|\|u_{\text{lin}}\|_{X_1^{10}(t_0)})\|\varphi\|_{X_1^{10}(t_0)} \\
&\qquad + \alpha^2\|u_{\text{lin}}\|^2_{X_1^{10}(t_0)}\big) \\
&\quad\leq \gamma\frac{\delta}{M_0^2} + \gamma C t_0\Big((2\gamma\frac{\delta}{M_0^2})^2 + 2\big(\|u_1^*\|_{X_1^{10}(\frac{4}{9})} + \frac{\delta}{M_0}\|u_{\text{lin}}\|_{X_1^{10}(t_0)}\big)2\gamma\frac{\delta}{M_0^2} \\
&\qquad + (\frac{\delta}{M_0})^2\|u_{\text{lin}}\|^2_{X_1^{10}(t_0)}\Big)
\end{aligned}
$$

where the constant C is the same constant as in Equation (3.25). Thus, we have that $\mathcal{K}_{f,g,\alpha} : Y'(t_0) \to Y'(t_0)$ provided that the inequality

$$
Ct_0\Big((2\gamma)^2\frac{\delta}{M_0^2} + 2\big(\|u_1^*\|_{X_1^{10}(\frac{4}{9})} + \frac{\delta}{M_0}\|u_{\text{lin}}\|_{X_1^{10}(t_0)}\big)2\gamma + \delta\big(\|u_{\text{lin}}\|_{X_1^{10}(t_0)}\big)^2\Big) \leq 1
$$

holds. Recalling that $t_0 > 0$ was chosen so that the inequality

$$
Ct_0\big((2\gamma)^2 + 2\|u_1^*\|_{X_1^{10}(\frac{4}{9})}2\gamma\big) \leq 1
$$

held, we observe that by considering smaller δ if necessary, we can ensure that the desired inequality is satisfied. Similarly, given $\varphi, \psi \in Y'(t_0)$,

$$
\begin{aligned}
&\|\mathcal{K}_{f,g,\alpha}(\varphi)(t) - \mathcal{K}_{f,g,\alpha}(\psi)(t)\|_{H^k(\mathbb{B}^7_{1-t})} \\
&\quad \leq \gamma \int_0^t \|\mathcal{N}_\alpha(s,\cdot,\varphi(s,\cdot)) - \mathcal{N}_\alpha(s,\cdot,\psi(s,\cdot))\|_{H^9(\mathbb{B}^7_{1-s})} ds \\
&\quad \leq \gamma \int_0^t \big(\|(\varphi(s,\cdot) - \psi(s,\cdot))^2\|_{H^9(\mathbb{B}^7_{1-s})} \\
&\qquad + 2\|(u_1^*(s,\cdot) + \alpha u_{\text{lin}}(s,\cdot))(\varphi(s,\cdot) - \psi(s,\cdot))\|_{H^9(\mathbb{B}^7_{1-s})}\big) ds \\
&\quad \leq \gamma C t_0 \Big(4\gamma \frac{\delta}{M_0^2} + 2\big(\|u_1^*\|_{X_1^{10}(\frac{4}{9})} + \frac{\delta}{M_0}\|u_{\text{lin}}\|_{X_1^{10}(t_0)}\big)\Big)\|\varphi - \psi\|_{X_1^{10}(t_0)}.
\end{aligned}
$$

Thus $\mathcal{K}_{f,g,\alpha}$ is Lipschitz on $Y'(t_0)$ with Lipschitz constant at most $\frac{1}{2}$ provided that the inequality

$$
C\gamma t_0 \Big(4\gamma \frac{\delta}{M_0^2} + 2\big(\|u_1^*\|_{X_1^{10}(\frac{4}{9})} + \frac{\delta}{M_0}\|u_{\text{lin}}\|_{X_1^{10}(t_0)}\big)\Big) \leq \frac{1}{2}
$$

holds. Again, recalling that $t_0 > 0$ was chosen so that, in addition, the inequality

$$
C\gamma t_0 \big(4\gamma + 2\|u_1^*\|_{X_1^{10}(t_0)}\big) \leq \frac{1}{2}
$$

held, we observe that by considering smaller δ if necessary, we can ensure that the desired inequality is satisfied. Thus, $\mathcal{K}_{f,g,\alpha}$ is a contraction on the closed subspace $Y'(t_0)$ of the Banach space $X_1^{10}(t_0)$. The Banach fixed point theorem implies the existence of a unique fixed point, namely $\varphi_{f,g,\alpha} \in Y'(t_0)$, of $\mathcal{K}_{f,g,\alpha}$. Setting $u_{f,g,\alpha} = u_1^* + \alpha u_{\text{lin}} + \varphi_{f,g,\alpha}$ yields the unique solution solution of the Cauchy problem (3.30). Theorems 2.12 and 2.14 of [1] imply $u_{f,g,\alpha} \in C^\infty(\Lambda_{t_0})$. The stated bound on $\varphi_{f,g,\alpha}$ follows from $\varphi_{f,g,\alpha} \in Y'(t_0)$, Sobolev embedding, and finite speed of propagation. ■

3.5.4 The Initial Data Operator

In this section, we use the solution obtained in Lemma 65 to obtain data on a family of hyperboloids in order to continue its evolution into $\Omega_{T,R}$ using the nonlinear theory developed in Section 3.4.1. By studying the properties of this restriction, we will eventually be able to find at least one hyperboloid in this family for which this solution can be continued according to the quadratic wave equation from that hyperboloid. First, we define a map which sends the data in Lemma 65 to the restriction of the solution obtained in Lemma 65 on a family of hyperboloids.

Definition 66 *Let $R \geq \frac{1}{2}$ and $t_0 > 0$ as in Lemma 62. Let $\delta > 0$ be sufficiently small and $M_0 > 0$ be sufficiently large so that, given $(f,g) \in \mathcal{B}^{10}_{\delta/M_0^2, t_0/4}$ and $\alpha \in [-\frac{\delta}{M_0}, \frac{\delta}{M_0}]$, the unique solution of Cauchy problem (3.30), $u_{f,g,\alpha} \in C^\infty(\Lambda_{t_0})$, exists. Then we set*

$$\mathbf{U}((f,g),\alpha,\beta) := e^{-2s}\begin{pmatrix} u_{f,g,\alpha} \circ \eta_{1+\beta} - u^*_{1+\beta} \circ \eta_{1+\beta} \\ \partial_s(u_{f,g,\alpha} \circ \eta_{1+\beta}) - \partial_s(u^*_{1+\beta} \circ \eta_{1+\beta}) \end{pmatrix}\Bigg|_{s=s_0}.$$

*We call $\mathbf{U}$ the **initial data operator**.*

We have the following mapping properties of the initial data operator.

Lemma 67 *Let $R \geq \frac{1}{2}$ and $t_0 > 0$ as in Lemma 62. For all $\delta > 0$ sufficiently small and $M_0 > 0$ sufficiently large, the initial data operator $\mathbf{U} : \mathcal{B}^{10}_{\delta/M_0^2, t_0/4} \times [-\frac{\delta}{M_0}, \frac{\delta}{M_0}]^2 \to \mathcal{H}_R$ is well-defined and for any $(f,g) \in \mathcal{B}^{10}_{\delta/M_0^2, t_0/4}$, the map $\mathbf{U}((f,g),\cdot) : [-\frac{\delta}{M_0}, \frac{\delta}{M_0}]^2 \to \mathcal{H}_R$ is continuous. Furthermore, there exists $\gamma_{t_0} \in \mathbb{R} \setminus \{0\}$ such that*

$$\mathbf{U}((f,g),\alpha,\beta) = \gamma_{t_0}\beta\mathbf{f}^*_1 + \alpha\chi_{t_0}\mathbf{f}^*_4 + \mathbf{V}((f,g),\alpha,\beta), \tag{3.32}$$

and $\mathbf{V}((f,g),\alpha,\beta)$ *satisfies the bound*

$$\|\mathbf{V}((f,g),\alpha,\beta)\|_{\mathcal{H}_R} \lesssim \frac{\delta}{M_0^2} + |\alpha|^2 + |\beta|^2.$$

Proof. The initial data operator is well-defined since the hyperboloids $\eta_{1+\beta}(\{s_0\}\times[0,R))$, $\beta \in [-\frac{\delta}{M_0}, \frac{\delta}{M_0}]$, lie entirely in Λ_{t_0} for sufficiently small δ and sufficiently large M_0. Continuity of $\mathbf{U}((f,g),\cdot)$ follows from $u^*_{1+\beta} \in C^\infty(\Lambda_{t_0})$ and continuous dependence on α of $u_{f,g,\alpha}$. To see the stated expansion of $\mathbf{U}((f,g),\alpha,\beta)$, we insert the form of $u_{f,g,\alpha}$ and group terms as follows

$$\begin{aligned}
\mathbf{U}((f,g),\alpha,\beta) =& e^{-2s}\begin{pmatrix} (u_1^* + \alpha u_{\text{lin}} + \varphi_{f,g,\alpha})\circ\eta_{1+\beta} - u^*_{1+\beta}\circ\eta_{1+\beta} \\ \partial_s((u_1^* + \alpha u_{\text{lin}} + \varphi_{f,g,\alpha})\circ\eta_{1+\beta}) - \partial_s(u^*_{1+\beta}\circ\eta_{1+\beta})\end{pmatrix}\Bigg|_{s=s_0} \\
=& e^{-2s}\begin{pmatrix} u_1^*\circ\eta_{1+\beta} - u^*_{1+\beta}\circ\eta_{1+\beta} \\ \partial_s(u_1^*\circ\eta_{1+\beta}) - \partial_s(u^*_{1+\beta}\circ\eta_{1+\beta})\end{pmatrix}\Bigg|_{s=s_0} \\
&+ \alpha e^{-2s}\begin{pmatrix} u_{\text{lin}}\circ\eta_{1+\beta} - u_{\text{lin}}\circ\eta_1 \\ \partial_s(u_{\text{lin}}\circ\eta_{1+\beta}) - \partial_s(u_{\text{lin}}\circ\eta_1)\end{pmatrix}\Bigg|_{s=s_0} \\
&+ \alpha e^{-2s}\begin{pmatrix} u_{\text{lin}}\circ\eta_1 \\ \partial_s(u_{\text{lin}}\circ\eta_1)\end{pmatrix}\Bigg|_{s=s_0} \\
&+ e^{-2s}\begin{pmatrix} \varphi_{f,g,\alpha}\circ\eta_{1+\beta} \\ \partial_s(\varphi_{f,g,\alpha}\circ\eta_{1+\beta})\end{pmatrix}\Bigg|_{s=s_0}.
\end{aligned}$$

Now, recall

$$(u_T^*\circ\eta_T)(s,y) = -\frac{24e^{2s}\big(5y^2 - 21h(y)^2\big)}{\big(3h(y)^2 + 5y^2\big)^2}.$$

which is clearly independent of T. Thus, for the first term we can write

$$u_1^*(1+\beta+e^{-s}h(y),e^{-s}y)-u_{1+\beta}^*(1+\beta+e^{-s}h(y),e^{-s}y)$$
$$=\partial_T u_1^*(T+e^{-s}h(y),e^{-s}y)|_{T=1}\beta+r_1(\beta,s,y)\beta^2$$

where

$$r_1(\beta,s,y)=\int_0^1\Big(\int_0^1\partial_\beta^2 u_1^*(1+\beta xz+e^{-s}h(y),e^{-s}y)dz\Big)xdx.$$

We have that r_1 is smooth and bounded since $u_1^*\in C^\infty(\Lambda_{t_0})$. Furthermore, recalling $e^{-2s}(\partial_T u_T^*\circ\eta_T)|_{T=1}(s,y)=432e^s f_{1,1}^*(y)$ along with a similar claim for the s derivative yields the first term in the stated expansion. For the second term, we can write

$$u_{\text{lin}}(1+\beta+e^{-s}h(y),e^{-s}y)-u_{\text{lin}}(1+e^{-s}h(y),e^{-s}y)=r_2(\beta,s,y)\beta$$

where

$$r_2(\beta,s,y)=\int_0^1\partial_\beta u_{\text{lin}}(1+\beta x+e^{-s}h(y),e^{-s}y)dx$$

which is smooth and bounded since $u_{\text{lin}}\in C^\infty(\Lambda_{t_0})$. Recalling

$$e^{-2s}\begin{pmatrix}u_{\text{lin}}\circ\eta_1\\ \partial_s(u_{\text{lin}}\circ\eta_1)\end{pmatrix}\Bigg|_{s=s_0}=\chi_{t_0}\mathbf{f}_4^*$$

yields the second claimed term in the expansion. $O(\alpha\beta)$ and $O(\beta^2)$ terms are obtained in $\mathbf{V}$ from r_1 and r_2 along with their s derivatives. Lastly, the $O(\frac{\delta}{M_0^2})$ term in $\mathbf{V}$ follows from Inequality (65). ■

3.5.5 Hyperboloidal Evolution

At this point, we are finally ready to continue the evolution of the data we began to evolve in Section 3.5.3. We achieve this by evolving the data $\mathbf{U}\big((f,g),\alpha,\beta\big)$ according to the nonlinear theory developed in Proposition 55. By a fixed point argument, we will show that there is at least one choice of α and β for which the correction term vanishes. In other words, there is at least one choice of α and β for which evolving $\mathbf{U}\big((f,g),\alpha,\beta\big)$ according to the modified equation is equivalent to that of the quadratic wave equation.

Proposition 68 *Let $R \geq \frac{1}{2}$ and $t_0 > 0$ be as in Lemma 62. Then for all sufficiently large $M_0 > 0$, there exists $\delta > 0$ such that for any pair $(f,g) \in \mathcal{B}^{10}_{\delta/M_0^2, t_0/4}$, there exists $(\alpha_{f,g}, \beta_{f,g}) \in [-\frac{\delta}{M_0}, \frac{\delta}{M_0}]^2$ and a unique function $\Phi_{f,g} \in C([s_0,\infty), \mathcal{H}_R)$ that satisfies*

$$\Phi_{f,g}(s) = \mathbf{S}(s-s_0)\mathbf{U}((f,g),\alpha_{f,g},\beta_{f,g}) + \int_{s_0}^{s} \mathbf{S}(s-s')\mathbf{N}(\Phi_{f,g}(s'))ds',$$

and $\|\Phi_{f,g}(s)\|_{\mathcal{H}_R} \leq \delta e^{-\omega_0 s}$ for all $s \geq s_0$.

Proof. First, let $\delta > 0$ be sufficiently small and $M_0 > 0$ sufficiently large so that the initial data operator is well-defined. Furthermore, the expansion of the initial data operator, Equation (3.32), implies that $\big\|\mathbf{U}\big((f,g),\alpha,\beta\big)\big\|_{\mathcal{H}_R} \lesssim \frac{\delta}{M_0}$ for all $(f,g) \in \mathcal{B}^{10}_{\delta/M_0^2, t_0/4}$ and $(\alpha,\beta) \in [-\frac{\delta}{M_0}, \frac{\delta}{M_0}]^2$. Thus, we require $M_0 \gtrsim c$ so that $\big\|\mathbf{U}\big((f,g),\alpha,\beta\big)\big\|_{\mathcal{H}_R} \leq \frac{\delta}{c}$ for δ and c as in Proposition 55. Thus, for $(f,g) \in \mathcal{B}^{10}_{\delta/M_0^2, t_0/4}$ and $(\alpha,\beta) \in [-\frac{\delta}{M_0}, \frac{\delta}{M_0}]^2$, Proposition 55 implies the existence of a unique $\Phi_{f,g,\alpha,\beta} \in C([s_0,\infty), \mathcal{H}_R)$ that satisfies

$$\begin{aligned}\Phi_{f,g,\alpha,\beta}(s) =& \mathbf{S}(s-s_0)\Big[\mathbf{U}\big((f,g),\alpha,\beta\big) - \mathbf{C}\Big(\Phi_{f,g,\alpha,\beta}, \mathbf{U}\big((f,g),\alpha,\beta\big)\Big)\Big] \\ &+ \int_{s_0}^{s} \mathbf{S}(s-s')\mathbf{N}(\Phi_{f,g,\alpha,\beta}(s'))ds'\end{aligned}$$

with the stated decay. If $\mathbf{C}\Big(\Phi_{f,g,\alpha,\beta}, \mathbf{U}\big((f,g),\alpha,\beta\big)\Big) = \mathbf{0}$, then we are done. To this end, define the function $\Gamma_{f,g} : [-\frac{\delta}{M_0}, \frac{\delta}{M_0}]^2 \to \mathbb{R}^2$ by $\Gamma_{f,g} = \Big(\Gamma^{(1)}_{f,g}, \Gamma^{(4)}_{f,g}\Big)$ where

$$\Gamma^{(1)}_{f,g}(\alpha,\beta) = \Big(\mathbf{C}_1\Big(\Phi_{f,g,\alpha,\beta}, \mathbf{U}\big((f,g),\alpha,\beta\big)\Big)\Big|\mathbf{f}_1^*\Big)_{\mathcal{H}_R}$$

$$\Gamma^{(4)}_{f,g}(\alpha,\beta) = \Big(\mathbf{C}_4\Big(\Phi_{f,g,\alpha,\beta}, \mathbf{U}\big((f,g),\alpha,\beta\big)\Big)\Big|\mathbf{f}^*_4\Big)_{\mathcal{H}_R}$$

Proposition 55 and continuity of the initial data operator implies continuity of $\Gamma_{f,g}$. According to the expansion of the initial data operator and transversality of the spectral projections, there exists a nonzero constant $\tilde{\gamma}_{t_0}$ such that

$$\Gamma^{(1)}_{f,g}(\alpha,\beta) = \tilde{\gamma}_{t_0}\beta + \alpha(\mathbf{P}_1(\chi_{t_0}\mathbf{f}^*_4)|\mathbf{f}^*_1)_{\mathcal{H}_R} + \phi^{(1)}_{f,g}(\alpha,\beta)$$

and

$$\Gamma^{(4)}_{f,g}(\alpha,\beta) = \alpha(\mathbf{P}_4(\chi_{t_0}\mathbf{f}^*_4)|\mathbf{f}^*_4)_{\mathcal{H}_R} + \phi^{(4)}_{f,g}(\alpha,\beta)$$

where $\phi_{f,g} = \Big(\phi^{(1)}_{f,g}(\alpha,\beta), \phi^{(4)}_{f,g}(\alpha,\beta)\Big) : [-\frac{\delta}{M_0}, \frac{\delta}{M_0}]^2 \to \mathbb{R}^2$ is continuous and satisfies the estimate $|\phi_{f,g}| \lesssim \frac{\delta}{M_0^2} + \delta^2$.

The equation $\Gamma_{f,g}(\alpha,\beta) = 0$ is equivalent to the existence of a fixed point of the map $(\alpha,\beta) \mapsto \mathbf{A}^{-1}_{t_0}\phi_{f,g}(\alpha,\beta)$ where

$$\mathbf{A}_{t_0} = -\begin{pmatrix} (\mathbf{P}_1(\chi_{t_0}\mathbf{f}^*_4)|\mathbf{f}^*_1)_{\mathcal{H}_R} & \tilde{\gamma}_{t_0} \\ (\mathbf{P}_4(\chi_{t_0}\mathbf{f}^*_4)|\mathbf{f}^*_4)_{\mathcal{H}_R} & 0 \end{pmatrix}.$$

This matrix is invertible since $\tilde{\gamma}_{t_0}, (\mathbf{P}_4(\chi_{t_0}\mathbf{f}^*_4)|\mathbf{f}^*_4)_{\mathcal{H}_R} \neq 0$, the second following from Lemma 63. Denoting by $\|\cdot\|_{M_2(\mathbb{C})}$ the matrix norm on $M_2(\mathbb{C})$, we have that

$$\begin{aligned} |\mathbf{A}^{-1}_{t_0}\phi_{f,g}(\alpha,\beta)| &\leq \|\mathbf{A}^{-1}_{t_0}\|_{M_2(\mathbb{C})}|\phi_{f,g}(\alpha,\beta)| \\ &\lesssim \Big(\frac{\delta}{M_0^2} + \delta^2\Big). \end{aligned}$$

Thus, for M_0 sufficiently large, one can take $\delta \lesssim \frac{1}{2M_0}$ in order to show that this map sends $[-\frac{\delta}{M_0}, \frac{\delta}{M_0}]^2$ to itself. Thus, the Brouwer fixed point theorem implies the existence of a fixed point. An inductive argument analogous to the proof of Lemma 64 shows that $\Phi_{f,g}$ is indeed smooth. ■

3.5.6 Proof of the Main Result

By Lemma 65 and Proposition 68, there exist positive constants $t_0, \delta, M_0 > 0$ such that for any pair of functions $(f, g) \in \mathcal{B}^{10}_{\delta/M_0^2, t_0/4}$, there exists $(\alpha, \beta) \in [-\frac{\delta}{M_0}, \frac{\delta}{M_0}]^2$ and a unique $u \in C^\infty(\Omega_{T,R})$ solving Equation (3.3) where $T = 1 + \beta$. For $(s, y) \in [s_0, \infty) \times [0, R)$, we have the equality

$$(u \circ \eta_T)(s, y) = (u_T \circ \eta_T)(s, y) + e^{2s}\phi_1(s)(y).$$

By Theorem 2.12 of [1], we have $u = u_1^*$ in $\Omega_{T,R} \setminus \eta_T([s_0, \infty) \times [0, R))$. Lastly, the stated convergence follows from the decay of $\Phi_{f,g}$.

3.6 Appendix

3.6.1 Explicit Expressions for Proposition 48

$C_n(\lambda) = \frac{P_1(n,\lambda)}{P_2(n,\lambda)}$ and $\varepsilon_n(\lambda) = \frac{P_3(n,\lambda)}{P_2(n,\lambda)}$ where

$$P_1(n, \lambda) = -845000000n^2(n + 1)^3(2n + 11)(\lambda + 2n + 2)(\lambda + 2n + 8),$$

$$\begin{aligned} P_2(n, \lambda) =& \big(1287\lambda^2 + 52(192\lambda^2 + 4125\lambda + 9500)n^2 + 2000(48\lambda + 299)n^3 + 104000n^4 \\ &+ \lambda(1521\lambda - 44000)n\big)\big(52(246\lambda^2 + 5125\lambda + 23000) \\ &+ 4(2496\lambda^2 + 125625\lambda + 728000)n^2 \\ &+ 6000(16\lambda + 169)n^3 + 104000n^4 + (21489\lambda^2 + 673000\lambda + 3198000)n\big), \end{aligned}$$

and

$$
\begin{aligned}
P_3(n,\lambda) =& -33462\lambda^2(117\lambda^2-4000\lambda-4000)-4000(74344\lambda^2+1012375\lambda+14196000)n^6 \\
&+1000(57408\lambda^3-1058203\lambda^2-34820500\lambda-237276000)n^5 \\
&-8(292032\lambda^4-66226875\lambda^3-854777500\lambda^2+11226312500\lambda+52728000000)n^4 \\
&-10(2021409\lambda^4-113476350\lambda^3-1831007400\lambda^2 \\
&\quad +9749350000\lambda+33360600000)n^3 \\
&-5(6739551\lambda^4-166553400\lambda^3-1369066800\lambda^2 \\
&\quad +8543600000\lambda+19468800000)n^2 \\
&+78000000(4\lambda-65)n^7-325\lambda(22113\lambda^3-1579680\lambda^2 \\
&\quad +11482600\lambda+14080000)n.
\end{aligned}
$$

Also, $\delta_5(\lambda)=\frac{R_1(\lambda)}{R_2(\lambda)}$ where

$$
\begin{aligned}
R_1(\lambda) =& -597051\lambda^{12}-43222410\lambda^{11}+5068245600\lambda^{10}+633420595440\lambda^9 \\
&+23910688879632\lambda^8+308544639036000\lambda^7-3181221429731200\lambda^6 \\
&-155692128689456640\lambda^5-2167560072357216256\lambda^4-15251720333529661440\lambda^3 \\
&-55976373542617907200\lambda^2-95372978774016000000\lambda-51994908426240000000
\end{aligned}
$$

and

$$
\begin{aligned}
R_2(\lambda) =& 2(64623\lambda^2+4285625\lambda+38025000) \\
&\times \big(81\lambda^{10}+19710\lambda^9+1886400\lambda^8+92781360\lambda^7+2577603408\lambda^6+41940364000\lambda^5 \\
&+401332867200\lambda^4+2206815715840\lambda^3+6537890727936\lambda^2+8994221424640\lambda \\
&+3845731123200\big).
\end{aligned}
$$

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