Conceptual Foundations for Orthogonal Gridding in Three Dimensions

By

Arturo A. Palomino Lazo
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Approved:

Dr. Fabian Bombardelli

Dr. Levent Kavvas

Dr. Holly Oldroyd

Committee in Charge

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Abstract

This work shows how differential forms and boundary conditions can be conveniently expressed in an appropriately constructed orthogonal or near-orthogonal coordinate system. As a special case, two coordinate systems are constructed that map a $\mathbb{R}^3$ rectangle to two irregularly shaped open channels and perturbation methods are used to handle a Laplacian equation in different magnitudes.

Additionally, other attempts at constructing such coordinate systems are presented along with their challenges.

Acknowledgments

I want to thank my wife for seven years and love of my life for believing in me and supporting me especially when I could not, my parents for raising me to believe that education is the key to all the freedoms in life, and my sister for being there always. All the teachers who inspired me to greater things and somehow decided that I am capable and worthy of a higher education, and primarily that putting pencil to paper can sometimes produce mathematical art.

I want to additionally thank my advisor Prof. Bombardelli for the years of unending patience and guidance, Prof. Oldroyd for working with my absentmindedness and lending an ear to my strange ideas, and Prof. Kavvas for showing me that a whole Saturday doing math can be fun.

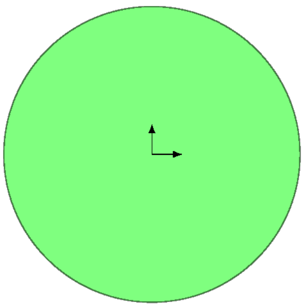
CHAPTER 1

Introduction

Mathematical modeling of physical phenomena relies overwhelmingly on the numeric or analytic solution of differential equations. Differential forms such as the Laplacian, the gradient, the curl, and operators constructed from these are most simply expressed in Cartesian coordinates. However, boundary conditions can only be simply expressed with Cartesian variables when these boundaries are rectangular. Coordinate transforms can be applied whenever a boundary is most simply expressed in a new coordinate system, a testament to this is the usage of polar or spherical coordinates for circular or infinite boundaries.

$$PDE: \quad \frac{\partial^2 \Phi(x,y)}{\partial x^2} + \frac{\partial^2 \Phi(x,y)}{\partial y^2} = 0$$

$$BC: \quad \Phi(x, \pm\sqrt{R^2 - x^2}) = c$$



$$\text{PDE:} \quad \frac{\partial^2 \Phi(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi(r,\theta)}{\partial \theta^2} = 0$$

$$\text{BC:} \quad \Phi(R,\theta) = c$$

FIGURE 1.1. A Dirichlet boundary condition on a circular boundary is better treated with polar coordinates, for some circle of radius R with center at the origin and r the distance from that origin

For certain cases, a change of spatial variables shifts the difficulty of meeting the boundary condition to the expression of the differential equation, this is especially useful when the symmetry of a problem can lead to similarity solutions. In the case depicted in Figure 1.1, the quantity Φ can be assumed to be symmetric in θ and the differential forms in θ can be set to zero, this is not possible in the Cartesian form.

Currently, aside from rectangular and circular boundaries, the increased number of differential terms that come from using non-Cartesian coordinates is untenable, and so analytic methods are generally limited to highly symmetric boundaries. Any other boundaries require differencing schemes and computational grid generation. A way to reduce the number of differential terms that using non-Cartesian coordinate systems comes with the choice of orthogonal over skewed curvilinear coordinates [**4**, **12**], because these eliminate spatial cross-terms (SCT). Such a scheme would introduce the fewest additional differential terms, have the highest possible accuracy for any given numerical differencing scheme, easiest enforcing of boundary conditions, and simplest implementation of turbulence models [**11**, **12**]. An orthogonal coordinate system is optimal either to solve a differential equation analytically or to produce a numerically accurate computational grid, because computational grids are most accurate wherever they are most orthogonal.

In two dimensions, coordinate systems produced through the use of complex functions are always orthogonal because such functions are always conformal mappings[1]. These can be extended to doubly connected and multiply connected [**2**] regions. Unfortunately, there isn't a three-dimensional equivalent of complex variables[2]. The construction of orthogonal coordinates in three dimensions is currently limited to highly symmetric boundaries, such as vertically or axially symmetric boundaries which amount to two-dimensional orthogonal coordinates extruded vertically or rotated axially respectively.

Currently, three-dimensional orthogonal coordinate systems let alone computational grids are impossible: "There apparently is no system, hyperbolic or elliptic, that will give complete orthogonality in 3D in general" (**1**-9) [**12**] and "for a three dimensional complex geometry, a fully orthogonal grid may not exist" (**7**-1) [**12**]. There exist numerous methods that achieve near orthogonality or so-called quasi-conformal maps by use of elliptic and hyperbolic grid generation schemes and the use of control functions; the implementation of this gridding, though applicable to all manner of mechanics and dynamics problems, cannot be used for analytic methods in the same way that an orthogonal coordinate system could. Additionally, differencing schemes treat finitely small spaces as linear and orthogonal and fail to quantify how much this deviation from orthogonality affects their consistency.

This work shows how to construct near-orthogonal three dimensional coordinate systems. These systems do not make cross-terms equal to zero, but instead reduce their magnitude to a quantifiable limit. This makes it possible to solve these differential equations analytically through perturbation methods, because these cross terms become a known order of magnitude smaller than the rest of the equation and the variable or variables of interest can be expressed as asymptotic series [**13**], and the differential equation can be expressed as a hierarchy of differential equations of descending magnitude. Computationally, this allows for the production of differencing schemes that quantify the numerical inconsistencies produced by assuming local orthogonality.

[1]This is only true for functions of a complex variable and not its conjugate.

[2]There are four and eight-dimensional equivalents, but these cannot produce three-dimensional coordinates unfortunately.

CHAPTER 2

Conformal Mapping and Orthogonal Coordinate Systems

2.1. Definition of Terms

2.1.1. Orthogonality. A coordinate system (or a subset of a coordinate system) possesses orthogonality as long as its coordinates meet at right angles everywhere. A sufficient condition for orthogonality is that given a (u, v) coordinate system where u and v can be expressed as functions of (x, y), $\nabla u \cdot \nabla v = 0$ everywhere. I show in Appendix C that this is equivalent to the Laplacian in the coordinate system not containing cross-terms $\left(\text{like } \frac{\partial}{\partial u}\frac{\partial}{\partial v}\right)$.

2.1.2. Conformal Mapping. A conformal map is one such that it transforms one coordinate space to another without affecting its local angles, i.e. one that transforms orthogonal coordinate systems without changing their orthogonality.

At present, for the purposes of physical modeling, conformal mapping is limited to the complex functions of the type

$$x + iy = f(u + iv)$$

where x and y are the coordinates in a 2-dimensional flat space and f is any given function with $u+iv$ as inputs. Liouville's theorem on conformal mapping [**7**] states that there is no three-dimensional equivalent to the above.

2.2. The Utility of Conformal Mapping

Conformal mapping was at one point "central to the practical solution of most physics and engineering problems" [**1**] and there is at the very least one physical question about continuum and field theory that is most appropriately addressed with the theory of conformal mapping: what occurs at points arbitrarily close to a sharp edge or corner. However, conformal mapping fell in disfavor due to three major shortcomings:

- The apparently limited scope of boundary conditions.
- Limitation to singly connected regions.
- Limitation to two-dimensional problems.

There is a rich history of conformal mapping being used to solve groundwater flow, potential flow, electrostatics, and heat flow; in other words, anywhere where the Laplacian operator appears (such as in the Poisson operator). Indeed, if conformal mapping did not have these three major shortcomings, it may well have continued being used in these fields. Perhaps at one point in the future, some analog of the triumphs of conformal mapping may be found again, significantly as its original shortcomings relax:

- In the past decade, conformal mappings of doubly connected [**5**] and multiply connected [**2**] regions have become possible.
- All Dirichlet, Neumann, and oblique boundary conditions can be accommodated in a conformal map [**5**].

- Since near-orthogonal coordinate systems can be produced for geometries that lack azimuthal symmetry (see Chapter 3) a similar methodology may be applied to any given curved surface, leading to something similar to conformal mapping for any given three-dimensional boundary and the easing of differential equations be it in perturbation or fully analytic form.

CHAPTER 3

Constructed Near- and Fully- Orthogonal Coordinate Systems

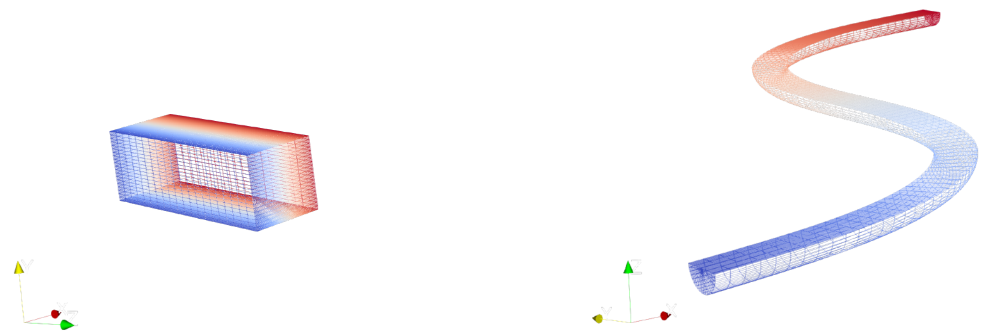
This chapter explores the mathematical implication of one fully-orthogonal and one near-orthogonal coordinate systems that model the shape of a sinuous open channel of different cross-sections. The construction of these coordinate systems is presented in Chapter 4.1 as well as how to create channels of different shapes and cross-sections. Current and future research on extending this procedure to more complex geometries is explored in later chapters.

3.1. Near Orthogonal Sinuous Open Channel with Semi Circular Cross-Section

The parametric equation:

$$\boldsymbol{r}_{sc}(\phi,\rho,\theta)=\begin{bmatrix}x(\phi,\rho,\theta)\\ y(\phi,\rho,\theta)\\ z(\phi,\rho,\theta)\end{bmatrix}=\begin{bmatrix}\mathcal{R}e\left\{\int_0^{2\pi\phi+i\varepsilon\rho\cos\theta}\exp(i\frac{\pi}{4}\left(1-\cos(w)\right))\right\}dw\\ \mathcal{I}m\left\{\int_0^{2\pi\phi+i\varepsilon\rho\cos\theta}\exp(i\frac{\pi}{4}\left(1-\cos(w)\right))\right\}dw\\ \rho\sin\theta\end{bmatrix}$$

where the $\mathcal{R}e$ and $\mathcal{I}m$ operators extract the real and imaginary parts of a complex expression[1]. This expression, known from here on as the Sinuous Semi-Circular Channel Function $\boldsymbol{r}_{sc}$ maps the rectangular domain $(\phi,\rho,\theta)\in[0,1]\times[0,1]\times[\pi,2\pi]$ to the shape of a sinuous and semi-circular open channel of non-dimensional length 1 and non-dimensional channel radius $\varepsilon=\mathcal{O}(0.1)$



(a) Domain $(\phi,\rho,\theta)\in[0,1]\times[0,1]\times[\pi,2\pi]$ (b) Image of sinuous semi-circular channel function

FIGURE 3.1. Domain and image of sinuous semi-circular channel function, domain and image colored blue for $\phi=0$ and red for $\phi=1$

[1]The outputs of $\mathcal{R}e$ and $\mathcal{I}m$ are always real and therefore the expression above is purely real

3.1.1. Orthogonality of the Sinuous Semi-Circular Channel Coordinate System. The coordinate system defined by the variables (ϕ, ρ, θ) is near-orthogonal. The same Fortran program created to plot this channel was used to approximate $\frac{\hat{\partial \boldsymbol{r}}}{\partial \phi}$, $\frac{\hat{\partial \boldsymbol{r}}}{\partial \rho}$, $\frac{\hat{\partial \boldsymbol{r}}}{\partial \theta}$ through the centered method with $\Delta\phi = \Delta\rho = \Delta\theta = 1 \times 10^{-10}$ to find that their dot products are near zero:

Dot Product	Maximum Value over Domain	Average Value over Domain
$\frac{\hat{\partial \boldsymbol{r}}}{\partial \phi} \cdot \frac{\hat{\partial \boldsymbol{r}}}{\partial \rho}$	1.9852160943679541E-004	2.8132505045507806E-005
$\frac{\hat{\partial \boldsymbol{r}}}{\partial \rho} \cdot \frac{\hat{\partial \boldsymbol{r}}}{\partial \theta}$	6.0454666042044691E-002	9.9470386083527866E-003
$\frac{\hat{\partial \boldsymbol{r}}}{\partial \theta} \cdot \frac{\hat{\partial \boldsymbol{r}}}{\partial \phi}$	3.8516866739242844E-003	9.5710515455647583E-005

This is a good numerical measure of orthogonality, but a more rigorous way to identify the orthogonality of this coordinate system is to find the metric tensor [**10**] and determine how close it is to being diagonal. The derivation of the below expression can be found in Appendix B and is repeated in Equation (B.2)

$$g_{sc} = \begin{bmatrix} 4\pi^2 e^{-2S} & 0 & 0 \\ 0 & \varepsilon^2 \left| e^{-2S} \cos\theta + i \sin\theta \right|^2 & \varepsilon \cos\theta \sin\theta (1 - e^{-2S}) \\ 0 & \varepsilon \cos\theta \sin\theta (1 - e^{-2S}) & \left| \cos\theta + i e^{-2S} \sin\theta \right|^2 \end{bmatrix} \tag{3.1}$$

where

$$S = \frac{\pi}{4} \sin(2\pi\phi) \sinh(\varepsilon\rho\cos\theta)$$

This metric shows that the coordinate system is orthogonal insofar as $(1 - e^{-2S}) << 1$; for $\epsilon << 1$, this is certainly the case.

3.1.2. Laplacian of the Sinuous Semi-Circular Channel Coordinate System. Equation (3.1) results in a Laplacian operator of different orders of magnitude, please see Appendix B.1.1 for a derivation:

$$
\begin{aligned}
\mathcal{O}(1) : \nabla^2\{\cdot\} =& 4\pi^2 \frac{\partial^2}{\partial \phi^2}\{\cdot\} + \frac{\partial^2}{\partial \theta^2}\{\cdot\} \\
\mathcal{O}(\varepsilon) : \nabla^2\{\cdot\} =& -2\pi^3 \rho \sin(2\pi\phi)\cos(\theta)\frac{\partial^2}{\partial \phi^2}\{\cdot\} \\
& - 6\pi^3 \cos(\theta)\sin(2\pi\phi)\frac{\partial}{\partial \phi}\{\cdot\} \\
& + \frac{\pi\rho}{8}\sin(2\pi\phi)\left((\cos(3\theta)-\cos(\theta))\frac{\partial^2}{\partial \theta^2}\{\cdot\} - 3(\sin(3\theta) - 3\sin(\theta))\frac{\partial}{\partial \theta}\{\cdot\}\right) \\
\mathcal{O}(\varepsilon^2) : \nabla^2\{\cdot\} =& \pi^4\rho^2\sin^2(2\pi\phi)\cos^2(\theta)\frac{\partial^2}{\partial \phi^2}\{\cdot\} \\
& + \pi\rho\cos(\theta)\sin(2\pi\phi)(3\cos^2(\theta) + 6\pi^2\sin(2\pi\phi)\cos(\theta) - 2)\frac{\partial}{\partial \phi}\{\cdot\} \\
& + 2\frac{\partial^2}{\partial \rho^2}\{\cdot\} \\
& - \pi\rho\sin(2\pi\phi)\cos(\theta)(3\sin^2(\theta) - 1)\frac{\partial}{\partial \rho}\{\cdot\} \\
& + \frac{\pi^2\rho^2}{4}\sin^2(2\pi\phi)\sin^2(2\theta)\frac{\partial^2}{\partial \theta^2}\{\cdot\} \\
& + \frac{\pi}{2}\sin(2\pi\phi)\cos(\theta)(4\pi\rho^2\sin(2\pi\phi)\cos^2(\theta) + 2\cos(\theta) - 3\pi\rho^2\sin(2\pi\phi))\sin(\theta)\frac{\partial}{\partial \theta}\{\cdot\} \\
& + \pi\rho\sin(2\pi\phi)\cos^2(\theta)\sin\theta\frac{\partial^2}{\partial \rho \partial \theta}\{\cdot\}
\end{aligned}
$$

And lower orders of magnitude. As it can be seen above, the cross-term only appears on the second order of magnitude, with $\varepsilon = \mathcal{O}(0.1)$, this means that it's only 1% relevant to the solution of a differential equation where the Laplacian is the only possible source of cross-terms

3.1.3. Diffusion Equation in a Sinuous Semi-Circular Channel. Consider the diffusion of a contaminant on this rectangular sinuous channel expressed in Cartesian coordinates where the diffusion tensor is purely diagonal and isotropic with value equal to 1:

$$
\begin{aligned}
\frac{\partial c}{\partial t} =& \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \\
IC : & \, c\big|_{x>0} = 0 \\
BC : & \, c\big|_{x=0} = 1 \\
BC : & \, c\big|_{(x,y,z)=\text{Channel bottom}} = 1
\end{aligned}
\tag{3.2}
$$

The second boundary condition in the above expression is nearly-impossible to simply express. Alternatively, (ϕ, ρ, θ) coordinates produces this equation instead:

$$
\begin{aligned}
\frac{\partial c}{\partial t} =& \nabla^2 c \\
IC : c|_{\phi>0} &= 0 \\
BC : c|_{\phi=0} &= 1 \\
BC : c|_{\rho=1} &= 0
\end{aligned}
$$

This makes the second boundary condition simply expressible; additionally, the differential expression itself is much easier to solve since the Laplacian can be separated in cascading orders of magnitude, please see Appendix B.1.1 for a derivation. A perturbation method procedure that addresses this is to expand the variable c in orders of magnitude:

$$c = c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + H.O.T$$

The highest order of magnitude equation will be:

$$\mathcal{O}(1): \frac{\partial c_0}{\partial t} = 4\pi^2 \frac{\partial^2 c_0}{\partial \phi^2} + \frac{\partial^2 c_0}{\partial \theta^2} \tag{3.3}$$

Consider how many times easier Equation (3.3) is than Equation (3.2). However, this equation accounts for 90% of the behavior of c in ϕ and θ due to the next equation being $\mathcal{O}(0.1)$ since $\varepsilon = \mathcal{O}(0.1)$. To account for the next 9%, the next order of magnitude of the equation must be accounted for:

$$
\begin{aligned}
\mathcal{O}(0.1): \frac{\partial c_1}{\partial t} =& 4\pi^2 \frac{\partial^2 c_1}{\partial \phi^2} + \frac{\partial^2 c_1}{\partial \theta^2} \\
& - 2\pi^3 \rho \sin(2\pi\phi) \cos(\theta) \frac{\partial^2 c_0}{\partial \phi^2} - 6\pi^3 \cos(\theta) \sin(2\pi\phi) \frac{\partial c_0}{\partial \phi} \\
& + \frac{\pi\rho}{8} \sin(2\pi\phi) \left((\cos(3\theta) - \cos(\theta)) \frac{\partial^2 c_0}{\partial \theta^2} - 3(\sin(3\theta) - 3\sin(\theta)) \frac{\partial c_0}{\partial \theta} \right)
\end{aligned} \tag{3.4}
$$

Whatever c_0 satisfies Equation (3.3) is simply plugged into the lower order of magnitude expression of ∇^2 and acts as a forcing in the otherwise identical equation for c_1. Having solved c_0 and c_1, the solution to the diffusion problem is 99% accurate in ϕ and θ. For an extra 0.9%, the solutions for c_0 and c_1 can be plugged into Equations (B.5) and (B.4), and act as forcings of Equation (B.3) acting on c_2. To obtain the first 90% accurate solution in ρ, the equation will have to be solved until the first differential in ρ occurs which is in $\mathcal{O}(\varepsilon^2)$:

$$\begin{aligned}\frac{\partial c_2}{\partial t} =& B.3\{c_2\} + B.4\{c_1\} + \pi^4\rho^2\sin^2(2\pi\phi)\cos^2(\theta)\frac{\partial^2 c_0}{\partial\phi^2}\\ &+ \pi\rho\cos(\theta)\sin(2\pi\phi)(3\cos^2(\theta) + 6\pi^2\sin(2\pi\phi)\cos(\theta) - 2)\frac{\partial c_0}{\partial\phi}\\ &+ 2\frac{\partial^2 c_0}{\partial\rho^2} - \pi\rho\sin(2\pi\phi)\cos(\theta)(3\sin^2(\theta) - 1)\frac{\partial c_0}{\partial\rho}\\ &+ \frac{\pi^2\rho^2}{4}\sin^2(2\pi\phi)\sin^2(2\theta)\frac{\partial^2 c_0}{\partial\theta^2}\\ &+ \frac{\pi}{2}\sin(2\pi\phi)\cos(\theta)(4\pi\rho^2\sin(2\pi\phi)\cos^2(\theta) + 2\cos(\theta) - 3\pi\rho^2\sin(2\pi\phi))\sin(\theta)\frac{\partial c_0}{\partial\theta}\\ &+ \pi\rho\sin(2\pi\phi)\cos^2(\theta)\sin\theta\frac{\partial^2 c_0}{\partial\rho\partial\theta}\end{aligned}$$

Since the behavior of c_0 and c_1 are known in ϕ and θ, the problem really becomes

$$\begin{aligned}\frac{\partial c_2}{\partial t} =& 4\pi^2\frac{\partial^2 c_2}{\partial\phi^2} + \frac{\partial^2 c_2}{\partial\theta^2} + 2\frac{\partial^2 c_0}{\partial\rho^2} - \pi\rho\sin(2\pi\phi)\cos(\theta)(3\sin^2(\theta) - 1)\frac{\partial c_0}{\partial\rho} + \pi\rho\sin(2\pi\phi)\cos^2(\theta)\sin(\theta)\frac{\partial^2 c_0}{\partial\rho\partial\theta}\\ &+ \text{Known forcings from } c_0 \text{ and } c_1\end{aligned}$$

which can be made even simpler by limiting the expansion of c to be $\mathcal{O}(\varepsilon)$ only:

$$\begin{aligned}0 =& 2\frac{\partial^2 c_0}{\partial\rho^2} - \pi\rho\sin(2\pi\phi)\cos(\theta)(3\sin^2(\theta) - 1)\frac{\partial c_0}{\partial\rho} + \pi\rho\sin(2\pi\phi)\cos^2(\theta)\sin(\theta)\frac{\partial^2 c_0}{\partial\rho\partial\theta}\\ &+ \text{Known forcings from } c_0 \text{ and } c_1\end{aligned}$$

But this is just one of the ways a solution for the problem can be sought in this new coordinate system, if the average behavior of c in ϕ is desired, differentials in θ and/or ρ can be set to zero from the start, or c could also be assumed to be:

$$c = P(\phi)T(\theta)\sum_{i=1}^{\infty} a_n\sin(n(\rho - \varepsilon))$$

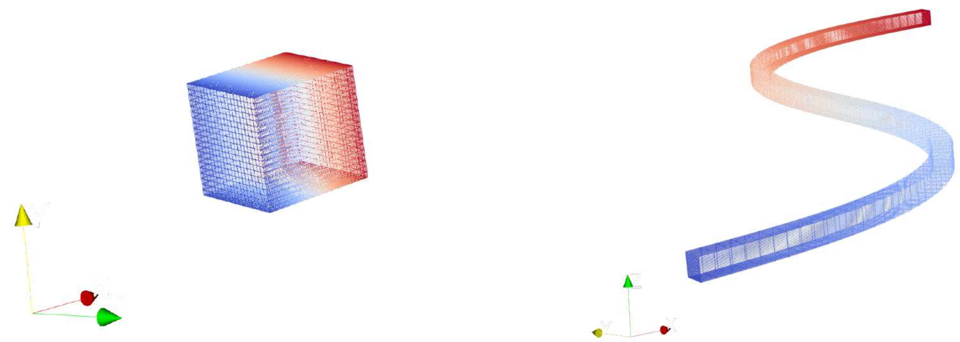
to accomodate the boundary condition from the start. Other differential forms can also be treated this way using tensor calculus [**3**].

3.2. Orthogonal Sinuous Open Channel with Rectangular Cross Section

The parametric equation:

$$\boldsymbol{r}_r(\phi,\rho,\theta) = \begin{bmatrix} x(\phi,\rho,\theta)\\ y(\phi,\rho,\theta)\\ z(\phi,\rho,\theta)\end{bmatrix} = \begin{bmatrix} \mathcal{R}e\left\{\int_0^{2\pi\phi+ib\varepsilon\psi}\exp(i\frac{\pi}{4}(1-\cos(w)))\right\}dw\\ \mathcal{I}m\left\{\int_0^{2\pi\phi+ib\varepsilon\psi}\exp(i\frac{\pi}{4}(1-\cos(w)))\right\}dw\\ h\varepsilon\zeta\end{bmatrix}$$

This expression, known from here on as the Sinuous Rectangular Channel Function $\boldsymbol{r}_r$ maps the rectangular domain $(\phi,\psi,\zeta) \in [0,1]\times[0,1]\times[0,1]$ to the shape of a sinuous rectangular channel of non-dimensional length 1, non-dimensional channel height $h\varepsilon = \mathcal{O}(0.1)$, and non-dimensional channel width $b\varepsilon = \mathcal{O}(0.1)$. It should be noted that h and b are non-dimensional numbers and are physically equal to the ratio of the channel's height over the channel's total length and the ratio of the channel's base over the channel's total length respectively.



(a) Domain $(\phi, \rho, \theta) \in [0,1] \times [0,1] \times [0,1]$ (b) Image of sinuous semi-circular channel function

FIGURE 3.2. Domain and image of sinuous rectangular channel function, domain and image colored blue for $\phi = 0$ and red for $\phi = 1$

3.2.1. Orthogonality of the Sinuous Rectangular Channel Coordinate System. The coordinate system defined by the variables (ϕ, ψ, ζ) should be orthogonal by construction. The same Fortran program created to plot this channel was used to approximate $\frac{\partial \hat{\boldsymbol{r}}}{\partial \phi}$, $\frac{\partial \hat{\boldsymbol{r}}}{\partial \psi}$, $\frac{\partial \hat{\boldsymbol{r}}}{\partial \zeta}$ through the centered method with $\Delta\phi = \Delta\psi = \Delta\zeta = 1 \times 10^{-10}$ to find that their dot products are not all zero:

Dot Product	Maximum Value over Domain	Average Value over Domain
$\frac{\partial \hat{\boldsymbol{r}}}{\partial \phi} \cdot \frac{\partial \hat{\boldsymbol{r}}}{\partial \psi}$	1.7886751019938023E-005	3.0996919513083402E-006
$\frac{\partial \hat{\boldsymbol{r}}}{\partial \psi} \cdot \frac{\partial \hat{\boldsymbol{r}}}{\partial \zeta}$	0	0
$\frac{\partial \hat{\boldsymbol{r}}}{\partial \zeta} \cdot \frac{\partial \hat{\boldsymbol{r}}}{\partial \phi}$	0	0

With the exception of the first dot products, it would appear that the coordinate systems is fully orthogonal, but a more rigorous way to identify the orthogonality of this coordinate system is to find the metric tensor [**10**] and determine how close it is to being diagonal. The derivation of the below expression can be found in Appendix B.2 and is repeated in Equation (B.6)

$$g_r = \begin{bmatrix} 4\pi^2 e^{-2S} & 0 & 0 \\ 0 & b^2\varepsilon^2 e^{-2S} & 0 \\ 0 & 0 & h^2\varepsilon^2 \end{bmatrix} \tag{3.5}$$

where

$$S = \frac{\pi}{4} \sin(2\pi\phi) \sinh(b\epsilon\psi)$$

This metric shows that the coordinate system is unconditionally, fully orthogonal. This is by construction, because $\int_0^{2\pi\phi + ib\varepsilon\psi} \Phi dw$ is a complex function, and having z have no dependency on ϕ or ψ is tantamount to vertically extruding the complex-function/conformal-map. See Chapter 2.2 for more on conformal maps.

3.2.2. Laplacian of the Sinuous Rectangular Channel Coordinate System. Equation (B.6) results in a Laplacian operator of different orders of magnitude:

$$
\begin{aligned}
\mathcal{O}(1) : \nabla^2\{\cdot\} =& 4\pi^2 \frac{\partial^2}{\partial \phi^2} \\
\mathcal{O}(\varepsilon) : \nabla^2\{\cdot\} =& -2\pi^3 b\psi \sin(2\pi\phi) \frac{\partial^2}{\partial \phi^2}\{\cdot\} \\
\mathcal{O}(\varepsilon^2) : \nabla^2\{\cdot\} =& 2\pi^4 b^2 \psi^2 \sin^2(2\pi\phi) \frac{\partial^2}{\partial \phi^2}\{\cdot\} + 2\left(h^2 + b^2\right) \frac{\partial^2}{\partial \psi^2}\{\cdot\} + 2\left(h^2 + b^2\right) \frac{\partial^2}{\partial \zeta^2}\{\cdot\}
\end{aligned}
$$

And lower orders of magnitude, please see Appendix (B.2.1) for a derivation of this Laplacian expression.

3.2.3. Diffusion Equation in a Rectangular Semi-Circular Channel. Consider the diffusion of a contaminant on this rectangular sinuous channel expressed in Cartesian coordinates where the diffusion tensor is purely diagonal and isotropic with value equal to 1:

$$
\begin{aligned}
\frac{\partial c}{\partial t} =& \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \\
IC : c|_{x>0} =& 0 \\
BC : c|_{x=0} =& 1 \\
BC : c|_{(x,y,z)=\text{Channel base}} =& 1
\end{aligned}
\tag{3.6}
$$

The second boundary condition in the above expression is nearly-impossible to simply express, consider instead using the (ϕ, ψ, ζ) coordinates:

$$
\begin{aligned}
\frac{\partial c}{\partial t} =& \nabla^2 c \\
IC : c|_{\phi>0} =& 0 \\
BC : c|_{\phi=0} =& 1 \\
BC : c|_{\zeta=1} =& 0 \\
BC : c|_{\psi=0} =& 0 \\
BC : c|_{\psi=1} =& 0
\end{aligned}
$$

This makes the second boundary condition simply expressible, additionally, the differential expression itself is much easier to solve since the Laplace equation can be separated in cascading orders of magnitude, please see Appendix B.2.1 for a derivation. A perturbation method procedure that addresses this is to expand the variable c in orders of magnitude:

$$c = c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + H.O.T$$

The highest order of magnitude equation will be:

$$\mathcal{O}(1) : \frac{\partial c_0}{\partial t} = 4\pi^2 \frac{\partial^2 c_0}{\partial \phi^2} \tag{3.7}$$

Consider how many times easier Equation (3.7) is than Equation (3.6). However, this equation accounts for 90% of the behavior of c in ϕ due to the next equation being $\mathcal{O}(0.1)$ since $\varepsilon = \mathcal{O}(0.1)$.

To account for the next 9%, the next order of magnitude of the equation must be accounted for:

$$\mathcal{O}(0.1) : \frac{\partial c_1}{\partial t} = -2\pi^3 b\psi \sin(\phi)\frac{\partial^2 c_0}{\partial \phi^2} + \frac{\partial^2 c_1}{\partial \phi^2} \tag{3.8}$$

Whatever c_0 satisfies Equation (3.7) is simply plugged into the lower order of magnitude expression of ∇^2 and acts as a forcing in the otherwise identical equation for c_1. Having solved c_0 and c_1, the solution to the diffusion problem is 99% accurate in ϕ and ψ. To find the solution's behavior in ζ, the next order of magnitude will need to be solved:

$$\frac{\partial c_2}{\partial t} = 4\pi^2\frac{\partial^2 c_2}{\partial \phi^2} - 2\pi^3 b\psi \sin(2\pi\phi)\frac{\partial^2 c_1}{\partial \phi^2} + 2\pi^4 b^2\psi^2 \sin^2(2\pi\phi)\frac{\partial^2 c_0}{\partial \phi^2} + 2\left(h^2+b^2\right)\frac{\partial^2 c_0}{\partial \psi^2} + 2\left(h^2+b^2\right)\frac{\partial^2 c_0}{\partial \zeta^2}$$

Since the behavior of c_0 and c_1 against ϕ and ψ are known, the problem is really

$$\frac{\partial c_2}{\partial t} = 4\pi^2\frac{\partial^2 c_2}{\partial \phi^2} + 2(h^2+b^2)\frac{\partial^2 c_0}{\partial \zeta^2} + \text{Forcings by } c_0 \text{ and } c_1$$

The expression of more involved differential equations can likewise be made simpler through tensor calculus [**3**].

CHAPTER 4

Method to Create Conformally Extruded Coordinate Systems

A number of methods were tried to produce orthogonal or near-orthogonal coordinate systems in three dimensions, the final method used in Chapter 3 began as an attempt of superimposing one conformal map onto each of the transects of another. To produce a channel-like curvature, the Bézier equation was used but with complex variable $w = u + iv$:

$$x + iy = B(u + iv) = P_0(1 - w)^3 + 3P_1(1 - w)^2 w + 3P_2(1 - w)w^2 + P_3 w^3 \tag{4.1}$$

The real form of this equation is explored in [**6**]; replacing the usual t for a w which is a standard complex variable. Observe that the Bézier equation is equal to P_0 for $w = 0$, and P_3 for $w = 1$, the P constants identify the so-called control points, which guide the shape of the Bézier curve. The complex version of this curve plots the same curve for real values of w, and including some small imaginary values $\pm\varepsilon$ produce what can best be described as a "flownet" in the language of ideal fluids.

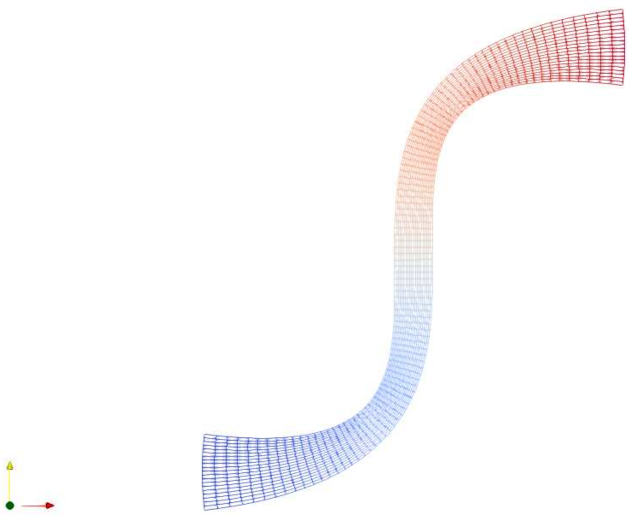


FIGURE 4.1. Image of complex Bézier function for domain $(u, v) \in [0, 1] \times [-\varepsilon, \varepsilon]$ with control points $P_0 = 0$, $P_1 = 1$, $P_2 = i$, $P_3 = 1 + i$. (Hereon referred to as flownet)

The image in Figure 4.1 is by construction fully orthogonal everywhere in two dimensions albeit self-intersecting for a large enough domain by virtue of being a complex function and therefore a conformal map.

To plot the complex Bézier function in $\mathbb{R}^2$ instead of the complex plane, a parametric equation can be made:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{R}e\{B(w)\} \\ \mathcal{I}m\{B(w)\} \end{bmatrix}$$

To extrude this onto $\mathbb{R}^3$, it is sufficient to add a third parameter:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathcal{R}e\{B(w)\} \\ \mathcal{I}m\{B(w)\} \\ z \end{bmatrix}$$

The next step was to project onto it the image of an ellipse:

$$\Upsilon(w) = \cosh(w) = \cosh(u)\cos(v) + i\sinh(u)\sin(v)$$

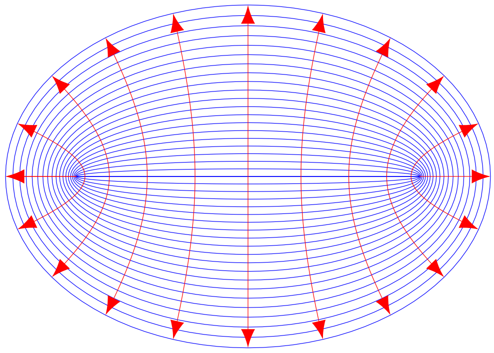


FIGURE 4.2. Image of cosh function in the complex plane

onto each of the transverse planes of the flownet in Figure 4.1. The method of accomplishing this was not obvious, much trial and error produced the following method:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathcal{R}e\{B(\phi + i\mathcal{R}e\{\Upsilon(\rho + i\theta)\})\} \\ \mathcal{I}m\{B(\phi + i\mathcal{R}e\{\Upsilon(\rho + i\theta)\})\} \\ \mathcal{I}m\{\Upsilon(\rho + i\theta)\} \end{bmatrix} \tag{4.2}$$

The orthogonality of this parametrization is small but not negligible, and the transverse shape shrinks and lengthens at the same rate as the flownet's transverse lengths in Figure 4.1. As it turns out, the orthogonality and the uneven lengthening are related in a way that is only obvious after finding the metric of this coordinate transform in Equation (A.1).

Because of this, the equation is better expressed in terms of its derivative, or Hodograph [**6**]:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathcal{R}e\{\int_0^{\phi + i\mathcal{R}e\{\Upsilon(\rho + i\theta)\}} \Phi(w)dw\} \\ \mathcal{I}m\{\int_0^{\phi + i\mathcal{R}e\{\Upsilon(\rho + i\theta)\}} \Phi(w)dw\} \\ \mathcal{I}m\{\Upsilon(\rho + i\theta)\} \end{bmatrix}$$

This way, the Hodograph equation denoted in the rest of this work as Φ, can be monitored to account for how much it stretches the transverse of the flownet in Figure 4.1. Equation (A.1) shows that the transverse stretching produced by the above expression is equal to $|\Phi(\phi+i\mathcal{R}e\{\Upsilon(\rho+i\theta)\})|$. Functions $\lambda(\rho)$ and $\gamma(\theta)$ are then introduced which independently scale ρ and θ without affecting orthogonality for a given (x, y, z) point. Finally, to accomodate for the desired domain of ϕ, a constant can be multiplied by it. This produces the expression for conformally extruded coordinate systems:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathcal{R}e\{\int_0^{b\phi+i\mathcal{R}e\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\}} \Phi(w)dw\} \\ \mathcal{I}m\{\int_0^{b\phi+i\mathcal{R}e\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\}} \Phi(w)dw\} \\ \mathcal{I}m\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\} \end{bmatrix} \tag{4.3}$$

where the image of Υ is extruded through the transects of the complex function $\int_0^w \Phi(w')dw'$.

4.1. Method to Create Near Orthogonal Coordinate Systems for Channel-Like Geometries

It may be possible to get closer to orthogonality by reparametrizing Equation (4.1) so that $|\frac{dB}{du}| = 1$ by scaling w:

$$B(w) \to B(f(w))$$

to make the metric from Equation (A.1) be more orthogonal, this produces a set of complicated differential equations. A way around this is by observing that the hodograph [**6**] only needs to lie very close to the unit complex circle. Restricting hodographs to functions of the form:

$$\Phi(w) = e^{if(w)}$$

where $f(w)$ is a complex function that is purely real for real values of w. The task is then finding $f(w)$ that matches the original channel's center-line, many functions were tried that did not sufficiently match the center-line of the flownet in Figure 4.1, but this one worked subjectively:

$$f(w) = \frac{\pi}{4}(1 - \cos w)$$

which makes:

$$\Phi(w) = e^{i\frac{\pi}{4}(1-\cos w)} \tag{4.4}$$

This choice of Φ in Equation (4.3) along with the circular and rectangular images for Υ produce the semi-circular and rectangular sinuous coordinate systems respectively,

4.2. Possibility of non-Channel-like Coordinate Systems

The coordinate systems produced in Sections 3.1 and 3.2 share a near-unit hodograph as defined in Equation (4.4):

$$\Phi(w) = e^{if(w)} = e^{i\frac{\pi}{4}(1-\cos w)}$$

The construction of this hodograph is not unique, an infinite number of functions can be made through Schwarz-Christoffel transforms [**5**] to meet the required condition that $f(w)$ needs to be near-real for some domain of w.

Additionally, the argument $\cosh(u+iv)$ can be individually scaled with $u \to \xi(u)$ and $v \to \eta(v)$ with functions that go to infinity to produce functions that go from purely real in some domain of u to purely real in some domain of v, and back to purely real in some domain of u. This would accomplish a coordinate system that is orthogonal along the perimeter of a sinuous rectangle, and from these an interlocking set of coordinate systems could be made to produce a grid that is orthogonal everywhere along its edges and only near-orthogonal at its center-faces.

Finally, the hodograph was defined in two dimensions because complex variables were used to produce the first coordinate systems. But this is not a necessity if the system starts in three dimensions. Orthogonal coordinates such as the ones defined in Section 5.3 can easily produce three-dimensional hodographs to integrate and produce new coordinate systems. The difficulty here is that complex integration has no adequate equivalent in three dimensions. In future research, I will start with an expression like

$$\int_0^{\phi} \boldsymbol{r}_{||}(u, v, w) \cdot ds + \int_0^{\psi} \boldsymbol{r}_{\vdash}(\phi, v, w) \cdot ds + \int_0^{\zeta} \boldsymbol{r}_{\top}(u, v, w) \cdot ds$$

where $\boldsymbol{r}_{||}$, $\boldsymbol{r}_{\vdash}$, and $\boldsymbol{r}_{\top}$ are mutually orthogonal unit vectors guided by some three-dimensional hodograph as the ones in Section 5.3, because this retains some of the essential geometry that complex integration accomplishes.

CHAPTER 5

Three Dimensional Unit Hodographs

This chapter expands on the idea of a unit hodograph in three dimensions for the possibility of producing near-orthogonal coordinate systems for non-channel-like geometries.

5.1. Definitions

5.1.1. Conformability. An orthogonal coordinate system possesses conformability as long as two of its coordinates can be transformed via some complex-valued function (e.g., $f(u+iv) = \phi+i\psi$ where (u, v) and (ϕ, ψ) are both orthogonal coordinate systems) without losing orthogonality. It will be shown in this Chapter that a sufficient condition for conformability is that the Laplacian in the coordinate system does not contain any first-order terms.

5.1.1.1. *The Simplest Conformable Coordinate System.* The trivial functions $u(x, y) = x$ and $v(x, y) = y$ that correspond to the Cartesian coordinate system satisfy orthogonality and have the additional property that the real and imaginary parts of a given function of $u(x, y) + iv(x, y)$ will also be orthogonal, for example:

$$
\begin{aligned}
&u(x, y) = x, v(x, y) = y \\
&(u(x, y) + iv(x, y))^2 = (x + iy)^2 = x^2 - y^2 + 2ixy \\
&\phi(x, y) + i\psi(x, y) = (u(x, y) + iv(x, y))^2 = (x + iy)^2 = x^2 - y^2 + 2ixy
\end{aligned}
$$

Observe that $\phi(x, y) = x^2 - y^2$ has a gradient that is orthogonal to the gradient of $\psi(x, y) = 2xy$.

5.1.1.2. *A Non-Conformable Orthogonal Coordinate System.* The polar coordinate system is orthogonal but not conformable:

$$r(x, y) = \sqrt{x^2 + y^2}, \theta(x, y) = \arctan\left(\frac{y}{x}\right)$$

$$
\begin{aligned}
\phi(x, y) + i\psi(x, y) =& \left(\sqrt{x^2 + y^2} + i\arctan\left(\frac{y}{x}\right)\right)^2 \\
=& x^2 + y^2 - \left(\arctan\left(\frac{y}{x}\right)\right)^2 + 2i\sqrt{x^2 + y^2}\arctan\left(\frac{y}{x}\right)
\end{aligned}
$$

$\phi(x, y) = x^2 - y^2 - \left(\arctan\left(\frac{y}{x}\right)\right)^2$ and $\psi(x, y) = 2\sqrt{x^2 + y^2}\arctan\left(\frac{y}{x}\right)$ are not orthogonal, which can be checked by their gradients.

5.1.2. Orthogonal Wrap. A two-dimensional coordinate system must become (or remain) orthogonal after being projected onto said surface to wrap a surface orthogonally.

5.1.3. Conformal Wrap. A two-dimensional coordinate system must become (or remain) conformable after being projected onto said surface to wrap a surface conformably.

5.2. Orthogonally Wrapping a Sphere

A spherical coordinate system with $r = 1$ orthogonally wraps the unit sphere by default:

$$\begin{aligned} x &= \cos\varphi \sin\phi \\ y &= \sin\varphi \sin\theta \\ z &= \cos\theta \end{aligned} \quad \Rightarrow \quad \begin{aligned} \varphi &= \arctan\left(\tfrac{y}{x}\right) \\ \theta &= \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) \end{aligned}$$

This can be seen in how the gradients of φ and θ have a zero dot-product:

$$\nabla\varphi = \frac{1}{x^2+y^2}\begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}, \nabla\theta = \frac{1}{\sqrt{x^2+y^2}(x^2+y^2+z^2)}\begin{bmatrix} xz \\ yz \\ -(x^2+y^2) \end{bmatrix}$$

$$\nabla\varphi \cdot \nabla\theta = \frac{1}{(x^2+y^2)^{3/2}(x^2+y^2+z^2)}(-xyz + xyz + (x^2+y^2)0) = 0$$

The projection of this orthogonal wrapping onto the Gaussian surfaces produced by the unit sphere produces the current spherical system. However, this orthogonal wrapping is not conformable and therefore one and only one three-dimensional orthogonal grid can be obtained from this extrusion. A method for orthogonally wrapping irregular solids is shown in Appendix F.

5.3. Conformably Wrapping a Sphere

The Laplacian of the $\varphi - \theta$ coordinates along the unit sphere is not separable:

$$\nabla^2 \{\} = \left[\frac{1}{\cos^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\theta^2} - \tan\theta\frac{\partial}{\partial\theta}\right]$$

However, there is a way to scale one of the coordinates to make this Laplacian separable. Using Equation (D.2), the proper scaling will yield:

$$\begin{aligned} x &= \cos\psi \operatorname{sech}\zeta \\ y &= \sin\psi \operatorname{sech}\zeta \\ z &= \tanh\zeta \end{aligned} \quad \Rightarrow \quad \begin{aligned} \psi &= \arctan\left(\tfrac{y}{x}\right) \\ \zeta &= \operatorname{arctanh}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \end{aligned}$$

with Laplacian:

$$\nabla^2 = \cosh(\zeta)\left(\frac{\partial^2}{\partial\psi^2} + \frac{\partial^2}{\partial\zeta^2}\right)$$

this Laplacian is perfectly separable:

$$\cosh(\zeta)\left(\frac{\partial^2}{\partial\psi^2} + \frac{\partial^2}{\partial\zeta^2}\right) = \cosh(\zeta)\left(\frac{\partial}{\partial\psi} - i\frac{\partial}{\partial\zeta}\right)\left(\frac{\partial}{\partial\psi} + i\frac{\partial}{\partial\zeta}\right)$$

therefore the (ψ, ζ) orthogonal wrap is perfectly conformable, there exists more ways to wrap the unit sphere, infinitely many in fact, but fewer than the ways in which an infinite flat surface can be orthogonally gridded. As a consequence, orthogonal spherical coordinates can be produced that accommodate any boundary conditions on the surface of the sphere, such as continental coasts or tectonic plates.

5.3.1. A Periodic Conformal Map on the Surface of a Sphere. Not every complex function will transform the conformable grid ($\psi - \zeta$) into a conformal coordinate system for the entire sphere because of self-intersection. Geometrically speaking, once the orthogonal grid makes its way around the sphere, the orthogonal grid will intersect itself and ψ coordinates are not guaranteed to intersect ζ coordinates at right angles; however, this can be avoided in the case of periodic orthogonal grids because the ψ coordinates will meet themselves in parallel and therefore continue to intersect ζ coordinates orthogonally and vice-versa.

A function of periodicity 2π can be obtained by adding an infinite number of source terms (or injection wells in the context of groundwater mechanics) at distance 2π apart:

$$\theta = \theta + i\psi, \qquad X = x + iy$$

$$\begin{aligned}\theta &= \cdots + \log(X + 2\pi) + \log(X) + \log(X - 2\pi) + \cdots \\ &= \sum_{n=-\infty}^{\infty} \log(X + 2n\pi) \\ &= \int \sum_{n=-\infty}^{\infty} \frac{1}{X + 2n\pi}\, dX\end{aligned}$$

the last step comes from the fundamental theorem of calculus. From [**1**], the infinite sums like the one inside the integral converges to a trigonometric function:

$$\begin{aligned}\theta &= \int \frac{\pi}{2\pi} \cot\left(\frac{\pi}{2\pi} X\right) dX \\ &= \int \frac{1}{2} \cot\left(\frac{1}{2} X\right) dX \\ &= \ln\left(\sin\left(\frac{1}{2} X\right)\right) \\ &\Rightarrow X = 2 \arcsin\left(\mathrm{e}^{\theta}\right)\end{aligned}$$

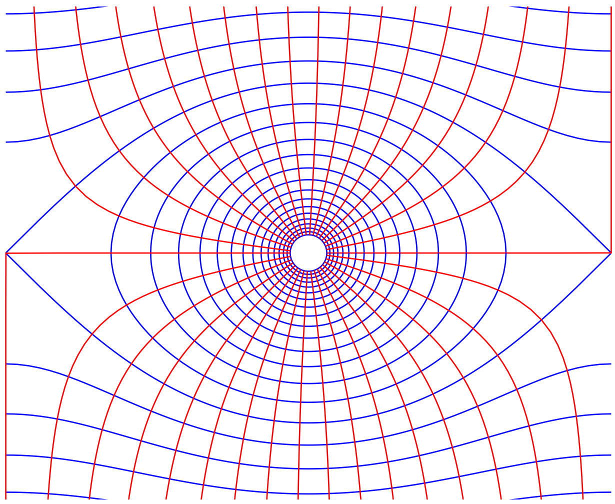


FIGURE 5.1. Conformal map $f(\theta) = 2 \arcsin \mathrm{e}^{\theta}$

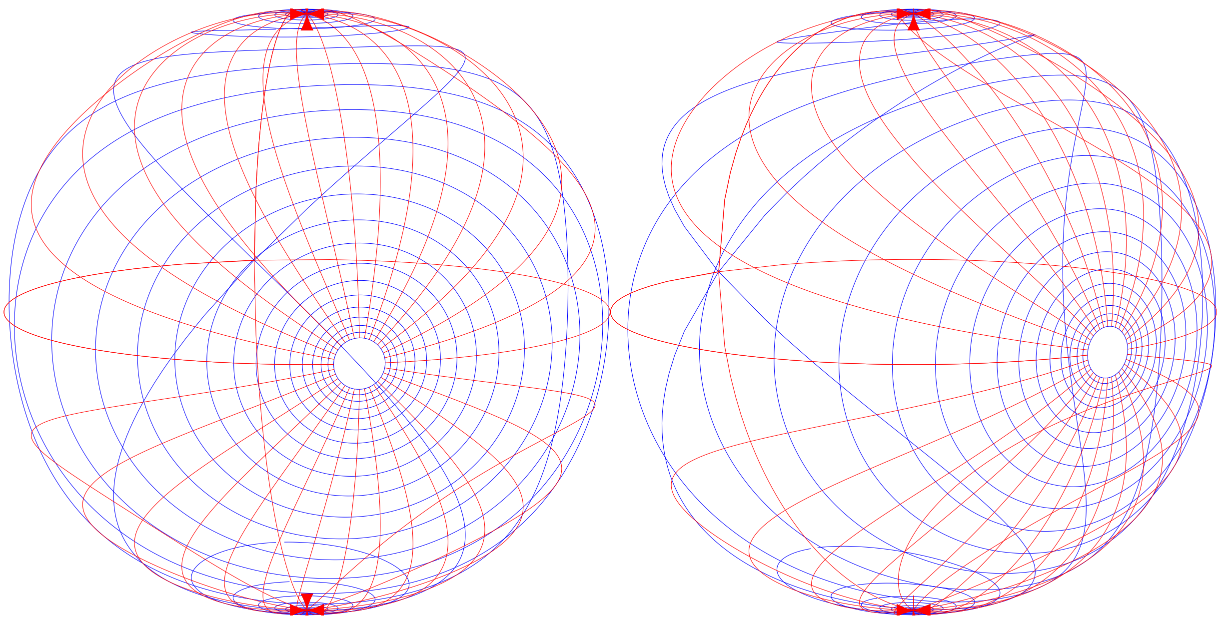
This conformal map can be projected onto the conformable spherical coordinates (ψ, ζ) introduced in Section 5.3 by applying the conformal map:

$$\psi + i\zeta = 2 \arcsin\left(\mathrm{e}^{u+iv}\right)$$

This results in a new spherical coordinate system that is as orthogonal as the original:

$$\begin{aligned}
x &= r\cos\left(\mathcal{R}e\left\{2\arcsin\left(\mathrm{e}^{u+iv}\right)\right\}\right)\operatorname{sech}\left(\mathcal{I}m\left\{2\arcsin\left(\mathrm{e}^{u+iv}\right)\right\}\right)\\
y &= r\sin\left(\mathcal{R}e\left\{2\arcsin\left(\mathrm{e}^{u+iv}\right)\right\}\right)\operatorname{sech}\left(\mathcal{I}m\left\{2\arcsin\left(\mathrm{e}^{u+iv}\right)\right\}\right)\\
z &= r\tanh\left(\mathcal{I}m\left\{2\arcsin\left(\mathrm{e}^{u+iv}\right)\right\}\right)\\
r &= \sqrt{x^2+y^2+z^2}\\
u &= \mathcal{R}e\left(\log\left(\sin\left(\frac{\arctan\left(\frac{y}{x}\right)+i\,\operatorname{arctanh}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)}{2}\right)\right)\right)\\
v &= \mathcal{I}m\left(\log\left(\sin\left(\frac{\arctan\left(\frac{y}{x}\right)+i\,\operatorname{arctanh}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)}{2}\right)\right)\right)
\end{aligned}$$

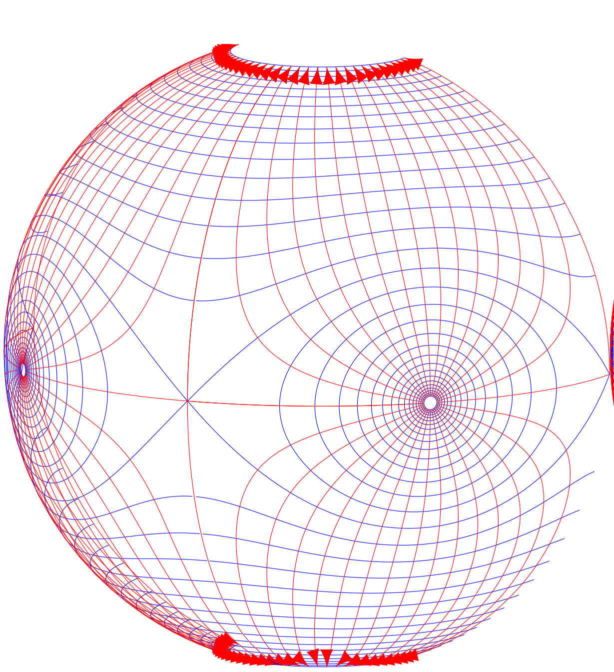
These equations are plotted in Figure 5.2:



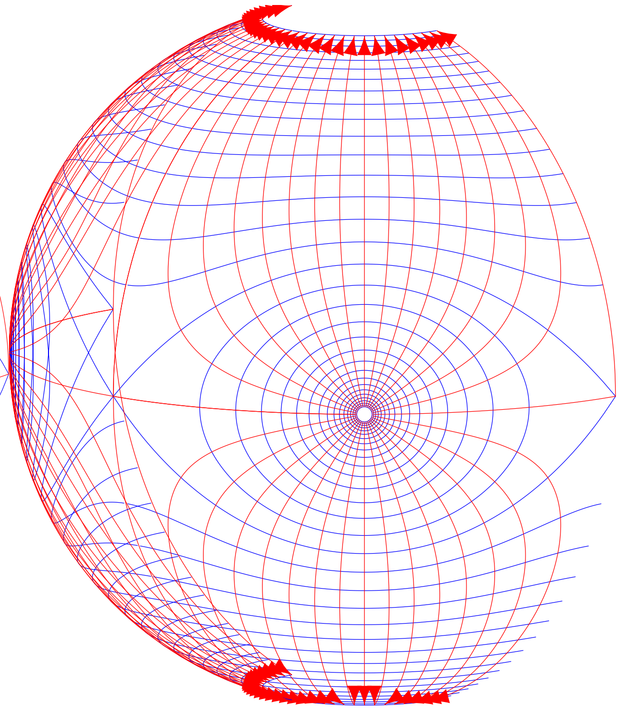
(a) View of alternative conformal wrap

(b) Same view rotated 30° in the z axis

FIGURE 5.2. Conformal map $\psi + i\zeta = 2\arcsin \mathrm{e}^{\psi' + i\zeta'}$ on single layer of alternative coordinate system



(a) View of second alternative conformal wrap



(b) Same view rotated 30° in the z axis

FIGURE 5.3. Conformal map $\psi + i\zeta = \frac{1}{2}\arcsin \mathrm{e}^{\psi' + i\zeta'}$ on single layer of alternative coordinate system

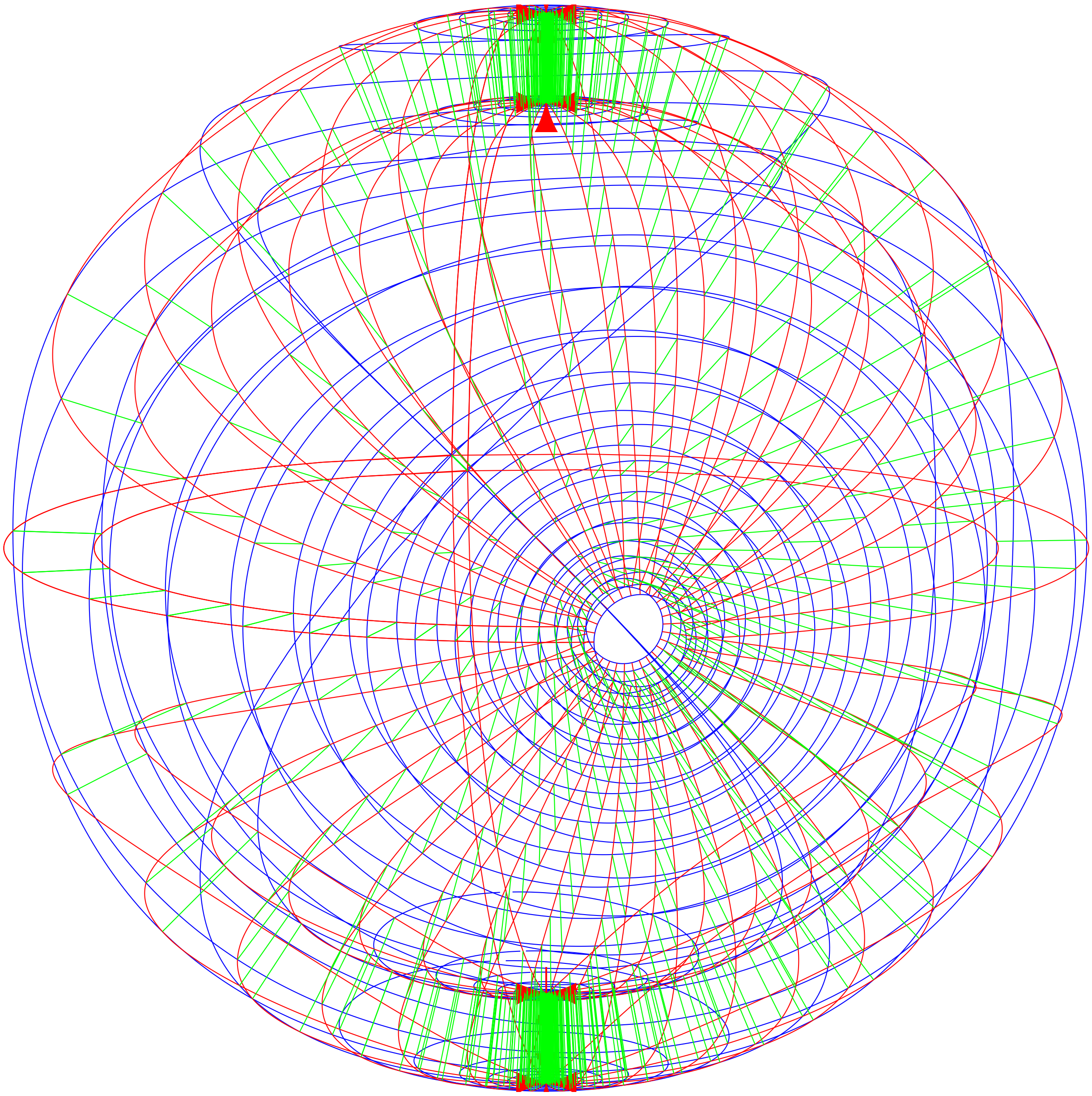


FIGURE 5.4. Projection of conformal map $\psi + i\zeta = 2 \arcsin e^{\psi' + i\zeta'}$ onto the unit sphere and one Gaussian sphere near it

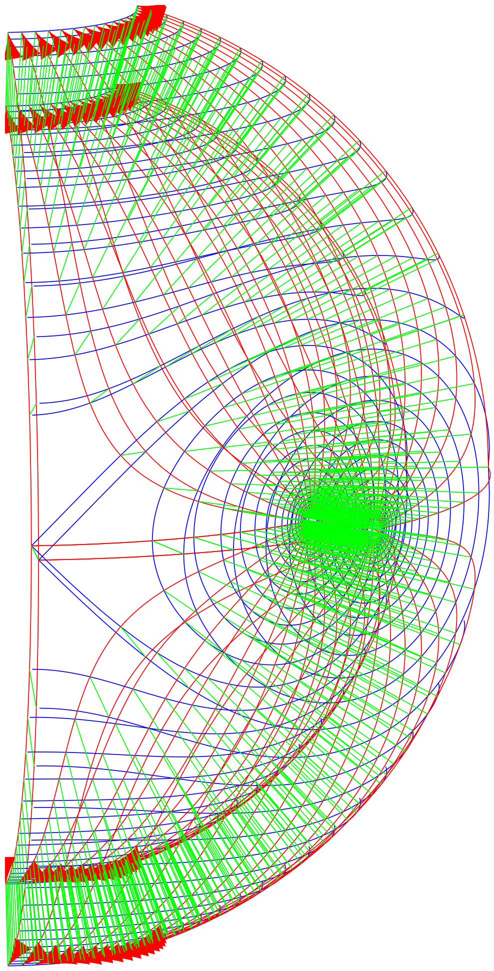
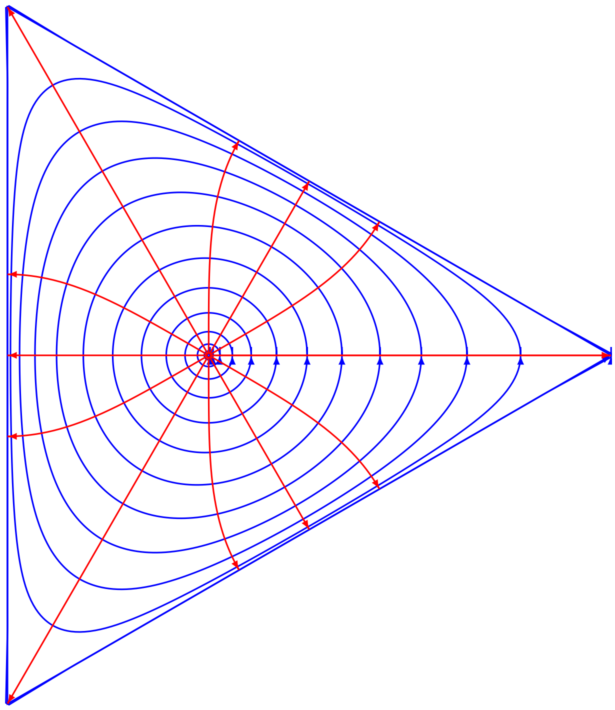


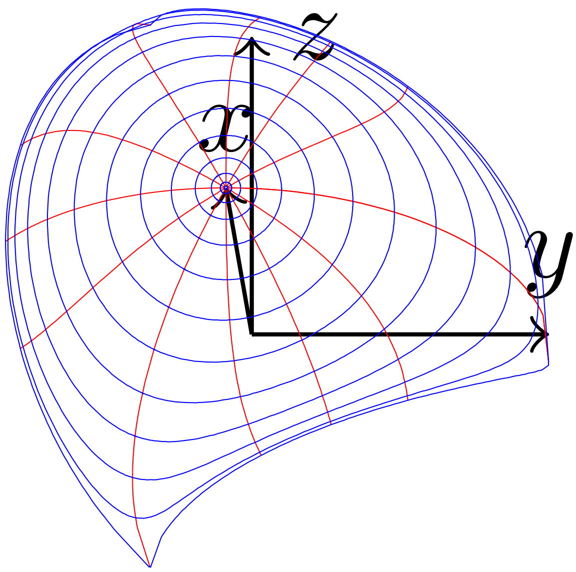
FIGURE 5.5. Conformal map $\psi + i\zeta = \frac{1}{2} \arcsin \mathrm{e}^{\psi' + i\zeta'}$ near the unit sphere

Any periodic conformal map can be used to fully conformally wrap the unit sphere if the (ψ, ζ) coordinate system from Section 5.3 is used.

5.3.2. Some Self Bounded Conformal Maps on the Surface of a Sphere. As I mentioned at the beginning of this Section, another set of conformal maps that can be projected to the surface of a sphere and then extruded into three-dimensional space are those conformal maps that are by construction bounded inside a single period of the azimuth angle. Such is the case of Schwarz-Christoffel transformations of a strip to a polygon [**5**]:

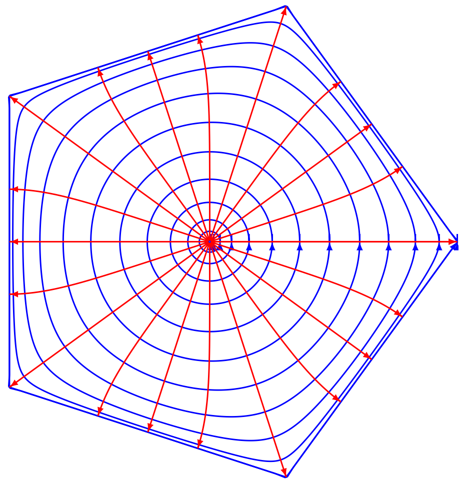


(a) Conformal map $\psi + i\zeta = \int_0^{\zeta'+i\psi'} \zeta'^{-\frac{2}{3}} \left(\zeta' - \mathrm{e}^{\frac{2\pi}{3}}\right)^{-\frac{2}{3}} \left(\zeta' - \mathrm{e}^{\frac{4\pi}{3}}\right)^{-\frac{2}{3}} d\zeta'$ on the plane

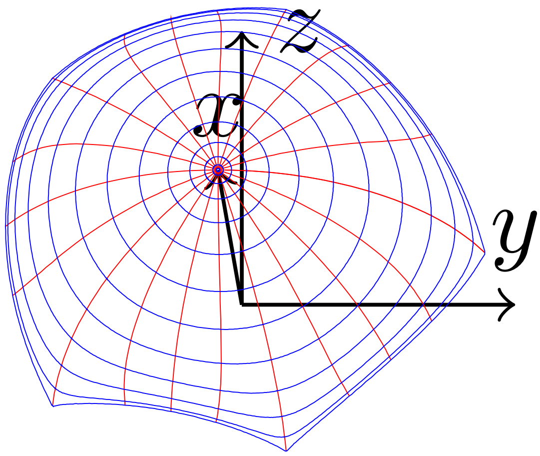


(b) The same conformal map projected onto a sphere continues to be orthogonal

FIGURE 5.6. A self-enclosed conformal map continues to be orthogonal when projected onto a conformably wrapped sphere



(a) Conformal map $\psi + i\zeta = \int_0^{\zeta'+i\psi'} \zeta'^{-\frac{2}{5}} \left(\zeta' - e^{\frac{2\pi}{5}}\right)^{\frac{-2}{5}} \left(\zeta' - e^{\frac{4\pi}{5}}\right)^{-\frac{2}{5}} \left(\zeta' - e^{\frac{6\pi}{5}}\right)^{-\frac{2}{5}} \left(\zeta' - e^{\frac{8\pi}{5}}\right)^{-\frac{2}{5}} d\zeta'$ on the plane



(b) Same conformal map projected onto a sphere

FIGURE 5.7. Another self-enclosed conformal map that remains orthogonal when plotted in a conformably wrapped sphere.

But this can also be accomplished by only including enough of a conformal map in such a way that it does not wrap completely around the sphere. The ζ coordinate can go as far as it needs to without ever reaching the top of the sphere, but the ψ coordinate needs to be observed not to span a domain radius greater than 2π.

CHAPTER 6

Conclusions

A near-orthogonal coordinate system can be made to accommodate a channel that has a curved center-line and is otherwise near-prismatic. The change of variables makes boundary conditions more simply expressible and differential forms more easily solvable, albeit requiring perturbation methods to solve. This work is a first in a series of coordinate system constructions, and while the methods introduced here are applicable to three-dimensional regions quantifiably close to a given curve, methods applicable to regions within a given curved perimeter and a given curved volume are not too far from reach. It may be possible to produce similar differential forms for more complex geometries and the numerical modeling requiring of matrix inversion of arbitrary accuracy could be made much simpler, particularly in that three-dimensional problems could be made two- or one-dimensional albeit requiring twice as many or three times as many time steps. Perhaps some geometries and differential equations could be solved analytically rather than numerically at all. Additionally, the methodologies from this work could also make it possible to grid a given geometry of space in a near- or fully- orthogonal way, giving account to how much detail is lost by assuming orthogonality at the grid scale.

APPENDIX A

Derivation of Metric Tensor for Extruded Coordinate Systems

For any parametrization of (x, y, z) space as variables of (ϕ, ρ, θ), the metric tensor [**10**] can be defined as:

$$g = \begin{bmatrix} \frac{\partial \boldsymbol{r}}{\partial \phi} \cdot \frac{\partial \boldsymbol{r}}{\partial \phi} & \frac{\partial \boldsymbol{r}}{\partial \phi} \cdot \frac{\partial \boldsymbol{r}}{\partial \rho} & \frac{\partial \boldsymbol{r}}{\partial \phi} \cdot \frac{\partial \boldsymbol{r}}{\partial \theta} \\ \frac{\partial \boldsymbol{r}}{\partial \rho} \cdot \frac{\partial \boldsymbol{r}}{\partial \phi} & \frac{\partial \boldsymbol{r}}{\partial \rho} \cdot \frac{\partial \boldsymbol{r}}{\partial \rho} & \frac{\partial \boldsymbol{r}}{\partial \rho} \cdot \frac{\partial \boldsymbol{r}}{\partial \theta} \\ \frac{\partial \boldsymbol{r}}{\partial \theta} \cdot \frac{\partial \boldsymbol{r}}{\partial \phi} & \frac{\partial \boldsymbol{r}}{\partial \theta} \cdot \frac{\partial \boldsymbol{r}}{\partial \rho} & \frac{\partial \boldsymbol{r}}{\partial \theta} \cdot \frac{\partial \boldsymbol{r}}{\partial \theta} \end{bmatrix}$$

Which is identical to the Jacobian matrix left multiplied with its transpose, the Jacobian matrix being denoted as:

$$J = \frac{\partial(x, y, z)}{\partial(\phi, \rho, \theta)} = \begin{bmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} \end{bmatrix}$$

To find the metric tensor then it is first necessary to find the derivative of each of the Cartesian coordinates (x, y, z) against each of the curvilinear coordinates (ϕ, ρ, θ). This is slighlty complicated by the fact that the coordinates were created using $\mathcal{R}e$ and $\mathcal{I}m$ operators. This however can be solved by producing some complex expressions, taking the general expression for Conformally Extruded Coordinate Systems Equation (4.3) in Chapter 4:

$$\boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathcal{R}e\left\{\int_0^{b\phi + i\mathcal{R}e\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\}} \Phi(w)dw\right\} \\ \mathcal{I}m\left\{\int_0^{b\phi + i\mathcal{R}e\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\}} \Phi(w)dw\right\} \\ \mathcal{I}m\{\Upsilon(\lambda(\rho) + i\gamma(\theta))\} \end{bmatrix}$$

$\frac{\partial z}{\partial \phi}$, $\frac{\partial z}{\partial \rho}$, $\frac{\partial z}{\partial \theta}$ can be obtained directly:

$$
\begin{aligned}
\frac{\partial z}{\partial \phi} =& \frac{\partial}{\partial \phi} \left(\mathcal{I}m\{\Upsilon(\lambda(\rho) + i\gamma(\theta))\} \right) \\
=& 0 \\
\frac{\partial z}{\partial \rho} =& \frac{\partial}{\partial \rho} \left(\mathcal{I}m\{\Upsilon(\lambda(\rho) + i\gamma(\theta))\} \right) \\
=& \frac{\partial \lambda}{\partial \rho} \mathcal{I}m \left\{ \frac{\partial \Upsilon}{\partial \lambda} \right\} \\
\frac{\partial z}{\partial \theta} =& \frac{\partial}{\partial \theta} \left(\mathcal{I}m\{\Upsilon(\lambda(\rho) + i\gamma(\theta))\} \right) \\
=& \frac{\partial \gamma}{\partial \theta} \mathcal{I}m \left\{ \frac{\partial \Upsilon}{\partial \gamma} \right\} \\
=& \frac{\partial \gamma}{\partial \theta} \mathcal{I}m \left\{ i \frac{\partial \Upsilon}{\partial \lambda} \right\} \\
=& \frac{\partial \gamma}{\partial \theta} \mathcal{R}e \left\{ \frac{\partial \Upsilon}{\partial \lambda} \right\}
\end{aligned}
$$

The same method would be difficult to apply for x and y, however, a shortcut can be found by forming the complex relation:

$$
x + iy = \int_0^{b\phi + i\mathcal{R}e\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\}} \Phi(w)dw
$$

Which can be differentiated directly by the fundamental theorem of calculus:

$$
\frac{\partial(x + iy)}{\partial(b\phi + i\mathcal{R}e\{\Upsilon(\lambda(\rho) + i\gamma(\theta))\})} = \Phi(b\phi + i\mathcal{R}e\{\Upsilon(\lambda(\rho) + i\gamma(\theta))\})
$$

In the nature of complex differentiation [**1**], $\frac{\partial x}{\partial \phi}$ and $\frac{\partial y}{\partial \phi}$ can be obtained by only allowing ϕ to variate while keeping $\mathcal{R}e\{\Upsilon(\lambda(\rho) + i\gamma(\theta))\}$ constant:

$$
\begin{aligned}
\frac{\partial(x + iy)}{\partial b\phi} =& \Phi(\phi + i\mathcal{R}e\{\Upsilon(\lambda(\rho) + i\gamma(\theta))\}) \\
\frac{\partial(x + iy)}{\partial \phi} =& b\Phi(\phi + i\mathcal{R}e\{\Upsilon(\lambda(\rho) + i\gamma(\theta))\}) \\
\Rightarrow \frac{\partial x}{\partial \phi} =& b\mathcal{R}e \left\{ \Phi(\phi + i\mathcal{R}e\{\Upsilon(\lambda(\rho) + i\gamma(\theta))\}) \right\} \\
\Rightarrow \frac{\partial y}{\partial \phi} =& b\mathcal{I}m \left\{ \Phi(\phi + i\mathcal{R}e\{\Upsilon(\lambda(\rho) + i\gamma(\theta))\}) \right\}
\end{aligned}
$$

In a slighlty more involved manner, $\frac{\partial x}{\partial \rho}$ and $\frac{\partial y}{\partial \rho}$ can be obtained using the chain rule:

$$
\begin{aligned}
\frac{\partial(x+iy)}{i\partial(\mathcal{R}e\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\})} =& \Phi(\phi + i\mathcal{R}e\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\}) \\
\frac{\partial(x+iy)}{\partial(\mathcal{R}e\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\})} =& i\Phi(\phi + i\mathcal{R}e\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\}) \\
\frac{\partial(x+iy)}{\partial\lambda} =& i\frac{\partial}{\partial\lambda}\left(\mathcal{R}e\left\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\right\}\right)\Phi(\phi + i\mathcal{R}e\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\}) \\
=& i\mathcal{R}e\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\Phi \\
\frac{\partial(x+iy)}{\partial\rho} =& i\frac{\partial\lambda}{\partial\rho}\mathcal{R}e\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\Phi \\
\frac{\partial x}{\partial\rho} =& \mathcal{R}e\left\{i\frac{\partial\lambda}{\partial\rho}\mathcal{R}e\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\Phi\right\} \\
=& -\frac{\partial\lambda}{\partial\rho}\mathcal{R}e\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\mathcal{I}m\{\Phi\} \\
\frac{\partial y}{\partial\rho} =& \mathcal{I}m\left\{i\frac{\partial\lambda}{\partial\rho}\mathcal{R}e\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\Phi\right\} \\
=& \frac{\partial\lambda}{\partial\rho}\mathcal{R}e\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\mathcal{R}e\{\Phi\}
\end{aligned}
$$

In an even more involved manner, $\frac{\partial x}{\partial \theta}$ and $\frac{\partial y}{\partial \theta}$ can be found

$$
\begin{aligned}
\frac{\partial(x+iy)}{i\partial(\mathcal{R}e\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\})} =& \Phi(\phi + i\mathcal{R}e\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\}) \\
\frac{\partial(x+iy)}{\partial(\mathcal{R}e\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\})} =& i\Phi(\phi + i\mathcal{R}e\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\}) \\
\frac{\partial(x+iy)}{\partial\gamma} =& i\frac{\partial}{\partial\gamma}\left(\mathcal{R}e\left\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\right\}\right)\Phi(\phi + i\mathcal{R}e\{\Upsilon(\lambda(\rho)+i\gamma(\theta))\}) \\
=& -i\mathcal{I}m\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\Phi \\
\frac{\partial(x+iy)}{\partial\theta} =& -i\frac{\partial\gamma}{\partial\theta}\mathcal{I}m\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\Phi \\
\frac{\partial x}{\partial\theta} =& \mathcal{R}e\left\{-i\frac{\partial\gamma}{\partial\theta}\mathcal{I}m\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\Phi\right\} \\
=& \frac{\partial\gamma}{\partial\theta}\mathcal{I}m\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\mathcal{I}m\{\Phi\} \\
\frac{\partial y}{\partial\theta} =& \mathcal{I}m\left\{-i\frac{\partial\gamma}{\partial\theta}\mathcal{I}m\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\Phi\right\} \\
=& -\frac{\partial\gamma}{\partial\theta}\mathcal{I}m\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\mathcal{R}e\{\Phi\}
\end{aligned}
$$

The Jacobian matrix is therefore:

$$J = \frac{\partial(x,y,z)}{\partial(\phi,\rho,\theta)} = \begin{bmatrix} b\mathcal{R}e\Phi & b\mathcal{I}m\Phi & 0 \\ -\frac{\partial\lambda}{\partial\rho}\mathcal{R}e\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\mathcal{I}m\Phi & \frac{\partial\lambda}{\partial\rho}\mathcal{R}e\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\mathcal{R}e\Phi & \frac{\partial\lambda}{\partial\rho}\mathcal{I}m\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\} \\ \frac{\partial\gamma}{\partial\theta}\mathcal{I}m\left\{\frac{\partial\Upsilon}{\partial\gamma}\right\}\mathcal{I}m\Phi & -\frac{\partial\gamma}{\partial\theta}\mathcal{I}m\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\mathcal{R}e\Phi & \frac{\partial\gamma}{\partial\theta}\mathcal{R}e\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\} \end{bmatrix}$$

which makes the metric tensor:
(A.1)

$$g = JJ^T = \begin{bmatrix} b^2|\Phi|^2 & 0 & 0 \\ 0 & \left(\frac{\partial\lambda}{\partial\rho}\right)^2 \left||\Phi|\mathcal{R}e\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\} + i\mathcal{I}m\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\right|^2 & \frac{\partial\lambda}{\partial\rho}\frac{\partial\gamma}{\partial\theta}\mathcal{R}e\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\mathcal{I}m\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\left(1-|\Phi|^2\right) \\ 0 & \frac{\partial\lambda}{\partial\rho}\frac{\partial\gamma}{\partial\theta}\mathcal{R}e\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\mathcal{I}m\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\left(1-|\Phi|^2\right) & \left(\frac{\partial\gamma}{\partial\theta}\right)^2 \left|\mathcal{R}e\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\} + i|\Phi|\mathcal{I}m\left\{\frac{\partial\Upsilon}{\partial\lambda}\right\}\right|^2 \end{bmatrix}$$

The Laplacian can be expressed using the metric tensor [**3**] from Equation (A.1), the Laplacian in each of the three coordinates (ϕ, ρ, θ) can be expressed as:

(A.2)

$$\begin{aligned}\nabla^2 =& \frac{1}{\sqrt{det(g)}}\frac{\partial}{\partial\phi}\left(\sqrt{det(g)}g_{\phi,\phi}\right)\frac{\partial}{\partial\phi}\{\cdot\} + g_{\phi,\phi}\frac{\partial^2}{\partial\phi^2}\{\cdot\} \\ &+ \frac{1}{\sqrt{det(g)}}\left(\frac{\partial}{\partial\rho}\left(\sqrt{det(g)}g_{\rho,\rho}\right) + \frac{\partial}{\partial\theta}\left(\sqrt{det(g)}g_{\rho,\theta}\right)\right)\frac{\partial}{\partial\rho}\{\cdot\} + g_{\rho,\rho}\frac{\partial^2}{\partial\rho^2}\{\cdot\} \\ &+ \frac{1}{\sqrt{det(g)}}\left(\frac{\partial}{\partial\rho}\left(\sqrt{det(g)}g_{\theta,\rho}\right) + \frac{\partial}{\partial\theta}\left(\sqrt{det(g)}g_{\theta,\theta}\right)\right)\frac{\partial}{\partial\theta}\{\cdot\} + g_{\theta,\theta}\frac{\partial^2}{\partial\theta^2}\{\cdot\} \\ &+ 2g_{\rho,\theta}\frac{\partial^2}{\partial\rho\partial\theta}\end{aligned}$$

APPENDIX B

Metric Tensor for Sinuous Channel-like Coordinate Systems

Sinuous channel-like coordinate systems are a specific case of extruded coordinate systems as defined in Chapter 4 where

$$\Phi(w) = \exp\left(i\frac{\pi}{4}(1-\cos(w))\right) \tag{B.1}$$

This is done on purpose because the metric tensor as defined in Equation (A.1) becomes diagonal for $|\Phi(w)| = 1$. And the above expression has this quality for real inputs w, and deviates from this diagonality by some measureable amount depending on the function $\Upsilon(w)$

B.1. Metric Tensor for Semi-Circular Sinuous Channel Coordinate System

To make a Semi-Circular Sinuous Channel, the transect must be a semi-circle. By construction, the transect is mapped by the Υ function. To resemble the well-known polar coordinates, Υ must be:

$$\Upsilon(\ln\varepsilon\rho + i\theta) = \varepsilon\rho\cos\theta + i\varepsilon\rho\sin\theta$$

Together with the expression for Φ found in Equation (B.1) makes:

$$\begin{aligned}
\lambda(\rho) &= \varepsilon\rho \\
\gamma(\theta) &= \theta \\
\Phi(\phi + i\varepsilon\rho\cos\theta) &= \exp\left(-\frac{\pi}{4}\sin(\phi)\sinh(\varepsilon\rho\cos\theta)\right)\exp\left(i\frac{\pi}{4}(1-\cos\phi\cosh(\varepsilon\rho\cos\theta))\right)
\end{aligned}$$

The imaginary arguments of the first will be so repeated that I will express it as S:

$$S = \frac{\pi}{4}\sin(2\pi\phi)\sinh(\varepsilon\rho\cos\theta)$$

Using the results of Appendix A found in Equation (A.1), the metric tensor becomes:

$$g_{sc} = \begin{bmatrix} e^{-2S} & 0 & 0 \\ 0 & \varepsilon^2\left|e^{-2S}\cos\theta + i\sin\theta\right|^2 & \varepsilon\cos\theta\sin\theta(1-e^{-2S}) \\ 0 & \varepsilon\cos\theta\sin\theta(1-e^{-2S}) & \left|\cos\theta + ie^{-2S}\sin\theta\right|^2 \end{bmatrix} \tag{B.2}$$

B.1.1. Laplacian for Semi-Circular Sinuous Channel Coordinate System. The best way to obtain the Laplacian of this coordinate system is by using a computer algebra system and defining the expression in Equation (A.2), to then feed it the metric in Equation (B.2), I personally used Maxima [**8**][1]. The result is a series of differential equations with different orders of magnitude:

(B.3)

$$\mathcal{O}(1) : \nabla^2\{\cdot\} = 4\pi^2 \frac{\partial^2}{\partial\phi^2}\{\cdot\} + \frac{\partial^2}{\partial\theta^2}\{\cdot\}$$

(B.4)

$$\begin{aligned}\mathcal{O}(\varepsilon) : \nabla^2\{\cdot\} =& -2\pi^3\rho\sin(2\pi\phi)\cos(\theta)\frac{\partial^2}{\partial\phi^2}\{\cdot\}\\ &-6\pi^3\cos(\theta)\sin(\phi)\frac{\partial}{\partial\phi}\{\cdot\}\\ &+\frac{\pi\rho}{8}\sin(2\pi\phi)\left((\cos(3\theta)-\cos(\theta))\frac{\partial^2}{\partial\theta^2}\{\cdot\}-3(\sin(3\theta)-3\sin(\theta))\frac{\partial}{\partial\theta}\{\cdot\}\right)\end{aligned}$$

(B.5)

$$\begin{aligned}\mathcal{O}(\varepsilon^2) : \nabla^2\{\cdot\} =& \pi^4\rho^2\sin^2(2\pi\phi)\cos^2(\theta)\frac{\partial^2}{\partial\phi^2}\{\cdot\}\\ &+\pi\rho\cos(\theta)\sin(2\pi\phi)(3\cos^2(\theta)+6\pi^2\sin(2\pi\phi)\cos(\theta)-2)\frac{\partial}{\partial\phi}\{\cdot\}\\ &+2\frac{\partial^2}{\partial\rho^2}\{\cdot\}\\ &-\pi\rho\sin(2\pi\phi)\cos(\theta)(3\sin^2(\theta)-1)\frac{\partial}{\partial\rho}\{\cdot\}\\ &+\frac{\pi^2\rho^2}{4}\sin^2(2\pi\phi)\sin^2(2\theta)\frac{\partial^2}{\partial\theta^2}\{\cdot\}\\ &+\frac{\pi}{2}\sin(2\pi\phi)\cos(\theta)(4\pi\rho^2\sin(2\pi\phi)\cos^2(\theta)+2\cos(\theta)-3\pi\rho^2\sin(2\pi\phi))\sin(\theta)\frac{\partial}{\partial\theta}\{\cdot\}\\ &+\pi\rho\sin(2\pi\phi)\cos^2(\theta)\sin\theta\frac{\partial^2}{\partial\rho\partial\theta}\{\cdot\}\end{aligned}$$

And lower orders of magnitude.

B.2. Metric Tensor for Rectangular Sinuous Channel Coordinate System

To make a Rectangular Sinuous Channel, the transect must be a rectangle. By construction, the transect is mapped by the Υ function.

$$\Upsilon(\psi,\zeta) = b\varepsilon\psi + ih\varepsilon\zeta$$

Together with the expression for Φ found in Equation (B.1) makes:

$$\begin{aligned}\lambda(\psi) =& b\varepsilon\\ \gamma(\zeta) =& h\varepsilon\\ \Phi(\phi+ib\varepsilon\psi) =& \exp\left(-\frac{\pi}{4}\sin(\phi)\sinh(b\varepsilon\psi)\right)\exp\left(i\frac{\pi}{4}(1-\cos\phi\cosh(b\varepsilon\psi))\right)\end{aligned}$$

[1] I would have used it earlier in the derivations, but Maxima does not integrate in the complex domain.

The arguments of the above exponents will be so repeated that I will express them as S and Θ respectively

$$S = \frac{\pi}{4} \sin(\phi) \sinh(b\varepsilon\psi)$$

Using the results of Appendix (A) found in Equation (A.1), the metric tensor becomes:

$$g_r = \begin{bmatrix} 4\pi^2 e^{-2S} & 0 & 0 \\ 0 & b^2\varepsilon^2 e^{-2S} & 0 \\ 0 & 0 & h^2\varepsilon^2 \end{bmatrix} \tag{B.6}$$

B.2.1. Laplacian for Rectangular Sinuous Channel Coordinate System. The best way to obtain the Laplacian of this coordinate system is by using a computer algebra system and defining the expression in Equation (A.2), to then feed it the metric in Equation (B.6), I personally used Maxima [**8**][2]. The result is a series of differential equations with different orders of magnitude:

$$\mathcal{O}(1) : \nabla^2\{\cdot\} = 4\pi^2 \frac{\partial^2}{\partial\phi^2} \tag{B.7}$$

$$\mathcal{O}(\varepsilon) : \nabla^2\{\cdot\} = -\,2\pi^3 b\psi \sin(2\pi\phi) \frac{\partial^2}{\partial\phi^2}\{\cdot\} \tag{B.8}$$

$$\mathcal{O}(\varepsilon^2) : \nabla^2\{\cdot\} = 2\pi^4 b^2\psi^2 \sin^2(2\pi\phi) \frac{\partial^2}{\partial\phi^2}\{\cdot\} + 2\left(h^2 + b^2\right) \frac{\partial^2}{\partial\psi^2}\{\cdot\} + 2\left(h^2 + b^2\right) \frac{\partial^2}{\partial\zeta^2}\{\cdot\} \tag{B.9}$$

And lower orders of magnitude.

[2]I would have used it earlier in the derivations, but Maxima does not integrate in the complex domain.

APPENDIX C

Laplacian Operator When The Curvilinear Coordinates Are Functions of (x, y, z)

Consider a change of variables $(x, y, z) \to (u, v, w)$ such that each u, v, and w are functions of x, y, z. It is well known that the Laplacian can be expressed using the coefficients of the metric tensor g [**9**]. But it can also be expressed in terms of the gradient and Laplacian of u, v, and w respectively through the use of the chain rule:

$$\begin{aligned}
\frac{\partial}{\partial x_i} =& \frac{\partial u_j}{\partial x_i}\frac{\partial}{\partial u_j} \\
\frac{\partial^2}{\partial x_i^2} =& \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_i}\frac{\partial}{\partial u_j}\right) \\
=& \frac{\partial u_j}{\partial x_i}\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial u_j}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_i}\right)\frac{\partial}{\partial u_j} \\
=& \frac{\partial u_j}{\partial x_i}\frac{\partial u_k}{\partial x_i}\frac{\partial}{\partial u_j}\left(\frac{\partial}{\partial u_k}\right) + \frac{\partial^2 u_j}{\partial x_i^2}\frac{\partial}{\partial u_j} \\
=& \nabla u_j \cdot \nabla u_k \frac{\partial^2}{\partial u_j \partial u_k} + \nabla^2 u_j \frac{\partial}{\partial u_j}
\end{aligned}$$

In the case of orthogonal coordinates:

$$\nabla^2 = \nabla u_j \cdot \nabla u_j \frac{\partial^2}{\partial u_j^2} + \nabla^2 u_j \frac{\partial}{\partial u_j} \tag{C.1}$$

APPENDIX D

Coordinate Scaling

Consider a change of variables $(u,v,w) \to (f(u), g(v), h(w))$. The new Laplacian can be found in terms of derivatives of f, g, and h as well as the dot products of u, v, and w:

$$\begin{aligned}
\frac{\partial}{\partial x_i} =& \frac{\partial u_j}{\partial x_i}\frac{\partial f_k}{\partial u_j}\frac{\partial}{\partial f_k} \\
\frac{\partial^2}{\partial x_i^2} =& \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_i}\frac{\partial f_k}{\partial u_j}\frac{\partial}{\partial f_k}\right) \\
=& \frac{\partial u_j}{\partial x_i}\frac{\partial f_k}{\partial u_j}\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial f_k}\right) + \frac{\partial u_j}{\partial x_i}\frac{\partial}{\partial x_i}\left(\frac{\partial f_k}{\partial u_j}\right)\frac{\partial}{\partial f_k} + \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_i}\right)\frac{\partial f_k}{\partial u_j}\frac{\partial}{\partial f_k} \\
=& \frac{\partial u_j}{\partial x_i}\frac{\partial f_k}{\partial u_j}\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial f_k}\right) + \left[\frac{\partial u_j}{\partial x_i}\frac{\partial}{\partial x_i}\left(\frac{\partial f_k}{\partial u_j}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_i}\right)\frac{\partial f_k}{\partial u_j}\right]\frac{\partial}{\partial f_k} \\
=& \frac{\partial u_j}{\partial x_i}\frac{\partial f_k}{\partial u_j}\frac{\partial u_l}{\partial x_i}\frac{\partial f_m}{\partial u_l}\frac{\partial}{\partial f_m}\left(\frac{\partial}{\partial f_k}\right) + \left[\frac{\partial u_j}{\partial x_i}\frac{\partial u_l}{\partial x_i}\frac{\partial}{\partial u_l}\left(\frac{\partial f_k}{\partial u_j}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_i}\right)\frac{\partial f_k}{\partial u_j}\right]\frac{\partial}{\partial f_k} \\
=& \frac{\partial u_j}{\partial x_i}\frac{\partial f_k}{\partial u_j}\frac{\partial u_l}{\partial x_i}\frac{\partial f_m}{\partial u_l}\left(\frac{\partial^2}{\partial f_k \partial f_m}\right) + \left[\frac{\partial u_j}{\partial x_i}\frac{\partial u_l}{\partial x_i}\left(\frac{\partial^2 f_k}{\partial u_j \partial u_l}\right) + \left(\frac{\partial^2 u_j}{\partial x_i^2}\right)\frac{\partial f_k}{\partial u_j}\right]\frac{\partial}{\partial f_k} \\
=& \frac{\partial u_k}{\partial x_i}\frac{\partial f_k}{\partial u_k}\frac{\partial u_m}{\partial x_i}\frac{\partial f_m}{\partial u_m}\left(\frac{\partial^2}{\partial f_k \partial f_m}\right) + \left[\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_i}\left(\frac{\partial^2 f_k}{\partial u_k^2}\right) + \left(\frac{\partial^2 u_k}{\partial x_i^2}\right)\frac{\partial f_k}{\partial u_k}\right]\frac{\partial}{\partial f_k} \\
=& \nabla u_k \cdot \nabla u_m f_k' f_m' \left(\frac{\partial^2}{\partial f_k \partial f_m}\right) + \left[\nabla u_k \cdot \nabla u_k f_k'' + \nabla^2 u_k f_k'\right]\frac{\partial}{\partial f_k}
\end{aligned}$$

For an orthogonal system, this further reduces to:

$$\nabla^2 = \nabla u_k \cdot \nabla u_k (f_k')^2 \frac{\partial^2}{\partial f_k^2} + \left[\nabla u_k \cdot \nabla u_k f_k'' + \nabla^2 u_k f_k'\right]\frac{\partial}{\partial f_k} \tag{D.1}$$

To eliminate the first-order differential, it is then only necessary to find the f_k that solves the equation

$$\nabla u_k \cdot \nabla u_k f_k'' + \nabla^2 u_k f_k' = 0$$

which can be found to be

$$f_k = \int \exp\left(-\int \frac{\nabla^2 u_k}{\nabla u_k \cdot \nabla u_k}\, du_k\right) du_k \tag{D.2}$$

APPENDIX E

Additional Separability of the Laplace and Helmholtz Operators

The Laplacian operator shows up in a number of physical problems. It can be shown that a separation of variables in the diffusion or wave equations among other equations will result in the Helmholtz operator [**9**]. Which in turn is separable by use of the so-called Stäckel matrix [**9**]. An unexpected consequence of eliminating the first-order differential terms from the Laplacian operator produces an additional way of separating the Laplace and Helmholtz Operators.

E.1. Laplacian of Cylindrical Systems

Starting out from the Cartesian Laplacian:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Any two of the variables can be conformally mapped to produce an orthogonal coordinate system with translational symmetry in the direction of the coordinate that was not conformally mapped. Given a conformal map

$$g(x + iy) = \phi + i\psi$$

of which its inverse:

$$f(\phi + i\psi) = x + iy$$

can provide the exact form of the Laplacian previously derived:

$$\nabla^2 = \frac{1}{f'\overline{f'}} \left(\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial \psi^2} \right) + \frac{\partial^2}{\partial z^2}$$

E.1.1. Separation of Variables in Any Given Cylindrical Coordinate System. For a function $U(\phi, \psi, z)$ that satisfies Helmholtz equation, a separation of variables is possible such that

$$U = \Omega(\phi + i\psi)Z(z)$$

which will always reduce the Helmholtz equation to:

$$\begin{aligned}
\nabla^2 U + k^2 U &= \left(\frac{1}{f'\overline{f'}} \left(\frac{\partial}{\partial \phi} - i\frac{\partial}{\partial \psi} \right) \left(\frac{\partial}{\partial \phi} + i\frac{\partial}{\partial \psi} \right) + \frac{\partial^2}{\partial z^2} + k^2 \right) \Omega(\phi + i\psi)Z(z) \\
&= \left(\mathrm{e}^{-2\phi} \left(\frac{\partial}{\partial \phi} - i\frac{\partial}{\partial \psi} \right) \left(\frac{\partial}{\partial \phi} + i\frac{\partial}{\partial \psi} \right) + \frac{\partial^2}{\partial z^2} + k^2 \right) \Omega(\phi + i\psi)Z(z) \\
&= \Omega(\phi + i\psi)Z''(z) + k^2\Omega(\phi + i\psi)Z(z)
\end{aligned}$$

This is satisfied by any $Z'' + k^2 Z = 0$

E.1.2. First Mention of the Conformable Polar Coordinate System. This coordinate system can be obtained from making the polar system conformable through the coordinate scaling shown in Appendix D

$$((x, y, z) \to (\phi, \psi, z)) \begin{cases} x = \mathrm{e}^{\phi} \cos \psi & \phi = \frac{1}{2} \log \left(x^2 + y^2\right) \\ y = \mathrm{e}^{\phi} \sin \psi & \psi = \arctan \left(\frac{y}{x}\right) \\ z = z & z = z \end{cases}$$

This coordinate system is also mentioned in [**9**]. It is presented alongside the more common

$$((x, y, z) \to (r, \psi, z)) \begin{cases} x = r \cos \psi & r = \sqrt{x^2 + y^2} \\ y = r \sin \psi & \psi = \arctan \left(\frac{y}{x}\right) \\ z = z & z = z \end{cases}$$

Without any prior explanation or further treatment (whereas the handbook explains how to separate Laplace's and Helmholtz equations in the traditional system, this is not done for the new system). However, Section II of the same handbook hints that this new coordinate system was arrived at using a "transformation in the complex plane" [**9**] or a conformal map being extruded in the z direction. The same handbook reveals that the Laplacian will be:

$$\nabla^2 = \mathrm{e}^{-2\phi} \left(\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial \psi^2}\right) + \frac{\partial^2}{\partial z^2}$$

E.1.3. Separation of Laplace's Equation in New Circular-Cylinder Coordinate System. The fact that the second-order differentials of ϕ and ψ have the same coefficient allows for a second way of separation of variables:

$$\begin{aligned} \nabla^2 U &= 0 \\ \nabla^2 U &= \left(\mathrm{e}^{-2\phi} \left(\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial \psi^2}\right) + \frac{\partial^2}{\partial z^2}\right) U \\ &= \left(\mathrm{e}^{-2\phi} \left(\frac{\partial}{\partial \phi} - i\frac{\partial}{\partial \psi}\right)\left(\frac{\partial}{\partial \phi} + i\frac{\partial}{\partial \psi}\right) + \frac{\partial^2}{\partial z^2}\right) U \end{aligned}$$

which leads to a two-fold instead of a three-fold separation of variables:

$$U = \Omega(\phi + i\psi) Z(z)$$

Much in the same way as Wirtinger Derivatives [**14**], it can be seen by inspection that Ω will always be zero:

$$\left(\frac{\partial}{\partial \phi} + i\frac{\partial}{\partial \psi}\right) \Omega(\phi + i\psi) = 0$$

producing:

$$\begin{aligned} \nabla^2 U &= \left(\mathrm{e}^{-2\phi} \left(\frac{\partial}{\partial \phi} - i\frac{\partial}{\partial \psi}\right)\left(\frac{\partial}{\partial \phi} + i\frac{\partial}{\partial \psi}\right) + \frac{\partial^2}{\partial z^2}\right) \Omega(\phi + i\psi) Z(z) \\ &= \left(\mathrm{e}^{-2\phi} \left(\frac{\partial}{\partial \phi} - i\frac{\partial}{\partial \psi}\right)\left(\frac{\partial}{\partial \phi} + i\frac{\partial}{\partial \psi}\right) + \frac{\partial^2}{\partial z^2}\right) \Omega(\phi + i\psi) Z(z) \\ &= \Omega(\phi + i\psi) Z''(z) \end{aligned}$$

Therefore any function of $\phi + i\psi$ will work as long as $Z'' = 0$. Hence Ω can be made to fit any given boundary or initial value problem.

E.2. Separation of Helmholtz Equation in New Circular-Cylinder Coordinate System

The Laplace operator appears in a number of physical equations, such as the diffusion and wave equations. As it is mentioned in [**9**], separation of variables will result in a temporal and spatial component, the spatial component will satisfy Helmholtz equation and may be further separated into three components much in a similar way as Laplace's Equation when using:

$$U = \Omega(\phi + i\psi)Z(z)$$

It can be seen by inspection that Ω will always be zero:

$$\left(\frac{\partial}{\partial\phi} + i\frac{\partial}{\partial\psi}\right)\Omega(\phi + i\psi) = 0$$

which produces:

$$\begin{aligned}\nabla^2 U + k^2 U &= \left(\mathrm{e}^{-2\phi}\left(\frac{\partial}{\partial\phi} - i\frac{\partial}{\partial\psi}\right)\left(\frac{\partial}{\partial\phi} + i\frac{\partial}{\partial\psi}\right) + \frac{\partial^2}{\partial z^2} + k^2\right)\Omega(\phi + i\psi)Z(z)\\ &= \left(\mathrm{e}^{-2\phi}\left(\frac{\partial}{\partial\phi} - i\frac{\partial}{\partial\psi}\right)\left(\frac{\partial}{\partial\phi} + i\frac{\partial}{\partial\psi}\right) + \frac{\partial^2}{\partial z^2} + k^2\right)\Omega(\phi + i\psi)Z(z)\\ &= \Omega(\phi + i\psi)Z''(z) + k^2\Omega(\phi + i\psi)Z(z)\end{aligned}$$

Therefore any function of $\phi + i\psi$ will work as long as $Z'' + k^2 Z = 0 \Rightarrow Z = A\cos(kz) + B\sin(kz)$. Hence Ω can be made to fit any given boundary or initial value problem.

E.3. Discussion

Equation (D.2) shows that for any given orthogonal coordinate system, there must exist one and only one scaling that eliminates the first-order differential term in the Laplacian.
Furthermore, Equations (C.1) and (D.2) imply that whenever a coordinate system is scaled to remove the first-order differential, this scaling will further have the property that:

$$\nabla^2 f(u) = 0$$

E.4. A Spherical Laplacian Without First-Order Differentials

The Laplacian in spherical coordinates is [**9**]:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\psi^2} \tag{E.1}$$

where r is the distance from the origin, ψ is the azimuthal angle, and θ is the zenith angle. The function ϕ required to remove the first-order differential in r will be:

$$\phi(r) = \int \exp\left(-\int \frac{\nabla^2 r}{\nabla r \cdot \nabla r}\, dr\right) dr$$

from Equation (D.2). The terms inside the integral can be read off of Equation (E.1) by observing that it must match Equation (C.1) :

$$\phi(r) = \int \exp\left(-\int \frac{2/r}{1}\, dr\right) dr = \frac{1}{r} \Rightarrow r = \frac{1}{\phi}$$

The function ζ required to remove the first-order differential in θ will be:

$$\zeta(\theta) = \int \exp\left(-\int \frac{\nabla^2\theta}{\nabla\theta \cdot \nabla\theta}\, d\theta\right) d\theta$$

from Equation (D.2). The terms inside the integral can be read off of Equation (E.1) by observing that it must match Equation (C.1) :

$$\zeta(\theta) = \log\left(\tan\left(\frac{\theta}{2}\right)\right) \Rightarrow \theta = 2\arctan\left(\mathrm{e}^{\zeta}\right)$$

These two ϕ and ζ functions turn the old spherical system into:

$$(x, y, z) \to (r, \psi, \theta) = \begin{cases} x = r\sin\theta\cos\psi \\ y = r\sin\theta\sin\psi \\ z = r\cos\theta \end{cases}$$

$$\Downarrow$$

$$(x, y, z) \to (\phi, \psi, \zeta) = \begin{cases} x = \frac{1}{\phi}\operatorname{sech}(\zeta)\cos\psi & \phi = \frac{1}{\sqrt{x^2+y^2+z^2}} \\ y = \frac{1}{\phi}\operatorname{sech}(\zeta)\sin\psi & \psi = \arctan\left(\frac{y}{x}\right) \\ z = \frac{1}{\phi}\tanh(\zeta) & \zeta = \operatorname{arcsinh}\left(\frac{z}{\sqrt{x^2+y^2}}\right) \end{cases} \tag{E.2}$$

E.4.1. Laplacian of New Spherical Coordinate System. Equation (C.1) or Equation (D.1) can be used to calculate:

$$\nabla^2 = \phi^4\frac{\partial^2}{\partial\phi^2} + \phi^2\cosh(\zeta)\left(\frac{\partial^2}{\partial\psi^2} + \frac{\partial^2}{\partial\zeta^2}\right)$$

As it can be seen above, there is no first-order differential term, but more conveniently, it is possible to further separate the second part of the Laplacian:

$$\nabla^2 = \phi^4\frac{\partial^2}{\partial\phi^2} + \phi^2\cosh(\zeta)\left(\frac{\partial}{\partial\psi} - i\frac{\partial}{\partial\zeta}\right)\left(\frac{\partial}{\partial\psi} + i\frac{\partial}{\partial\zeta}\right) \tag{E.3}$$

E.4.2. Separation of Variables in Laplace's Equation of the New Spherical Coordinate System. Laplace's Equation in the traditional spherical system separates into three ordinary differential equations [**9**]:

$$U = R(r)\Theta(\theta)\Psi(\psi)$$

$$\nabla^2 U = 0 \Rightarrow \begin{cases} R'' + \frac{2}{r}R' - \frac{\alpha}{r^2}R = 0 \\ \Theta'' + \cot\theta\Theta' + (\alpha - \beta\csc^2\theta)\Theta = 0 \\ \Psi'' + \beta\Psi = 0 \end{cases}$$

with well-known analytic solutions.

The new spherical coordinate system separates into two equations instead:

$$U = \Phi(\phi)\Omega(\psi + i\zeta)$$

$$\nabla^2 U = 0 \Rightarrow \phi^4\Phi'' = 0$$

with solution

$$\Phi = A\phi + B$$

E.4.3. Separation of Variables in New Spherical Coordinate System. The Helmholtz operator in the traditional Spherical system separates into three ordinary differential equations [**9**]:

$$U = R(r)\Theta(\theta)\Psi(\psi)$$

$$\nabla^2 U + k^2 U = 0 \Rightarrow \begin{cases} R'' + \frac{2}{r}R' + (k^2 - \alpha/r^2)R = 0 \\ \Theta'' + \cot\theta\Theta' + (\alpha - \beta\csc^2\theta)\Theta = 0 \\ \Psi'' + \beta\Psi = 0 \end{cases}$$

With well-known analytic solutions.
The new spherical coordinate system separates into two equations instead:

$$U = \Phi(\phi)\Omega(\psi + i\zeta)$$

$$\nabla^2 U + k^2 U = 0 \Rightarrow \phi^4\Phi'' + k^2\Phi = 0$$

With solution

$$\Phi = A\phi\cos\left(\frac{k}{\phi}\right) + B\phi\sin\left(\frac{k}{\phi}\right)$$

APPENDIX F

Orthogonally Wrapping an Irregular Solid

A given two-dimensional orthogonal grid can be, for lack of a better word, “inflated” to describe a new orthogonal grid on a curved surface.

F.0.1. Inflating a Conformal Map. Consider a conformal map resting on a flat surface:

$$x + iy = f(u+iv) \Rightarrow \begin{array}{l} x = \Re\{f(u+iv)\} \\ y = \Im\{f(u+iv)\} \end{array}$$

$$\Rightarrow \begin{array}{l} \partial x/\partial u = \Re\{f'(u+iv)\} \\ \partial y/\partial u = \Im\{f'(u+iv)\} \end{array} \quad \begin{array}{l} \partial x/\partial v = \Re\{if'(u+iv)\} = -\Im\{f'(u+iv)\} \\ \partial y/\partial v = \Im\{if'(u+iv)\} = \Re\{f'(u+iv)\} \end{array} \quad \Rightarrow \begin{array}{l} \frac{\partial x}{\partial u} = \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u} \end{array}$$

By construction, the non-diagonal elements of its metric are zero:

$$\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} = -\frac{\partial y}{\partial v}\frac{\partial y}{\partial u} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} = 0$$

To “inflate” this orthogonal grid, the z coordinate can be introduced but made out to be a function of u or v alone:

$$\begin{array}{l} x = \Re\{f(u+iv)\} \\ y = \Im\{f(u+iv)\} \\ z = g(u) \end{array}$$

This two-dimensional coordinate system will also have zero non-diagonal elements:

$$\cancelto{0}{\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v}} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = 0 + g'(u)\cdot 0 = 0$$

this will also be true if u or v are scaled.

F.0.2. An Inflated Conformal Map. The map of $x + iy = \cosh(u + iv)$ found in Figure 4.2 can be re-scaled and inflated:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathcal{R}e\{\cosh(u+iv)\} \\ \mathcal{I}m\{\cosh(u+iv)\} \\ 0 \end{bmatrix} = \begin{bmatrix} \cosh u \cos v \\ \sinh u \sin v \\ 0 \end{bmatrix} \to \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boldsymbol{\gamma}(u, v) = \begin{bmatrix} \sqrt{1+\cos^2 u}\cos v \\ \cos u \sin v \\ \sin u \end{bmatrix}$$

with scaling function $u_{new} = \arccos(\sinh(u_{old}))$
This produces a solid that resembles (and will be referred to as) a pillow:

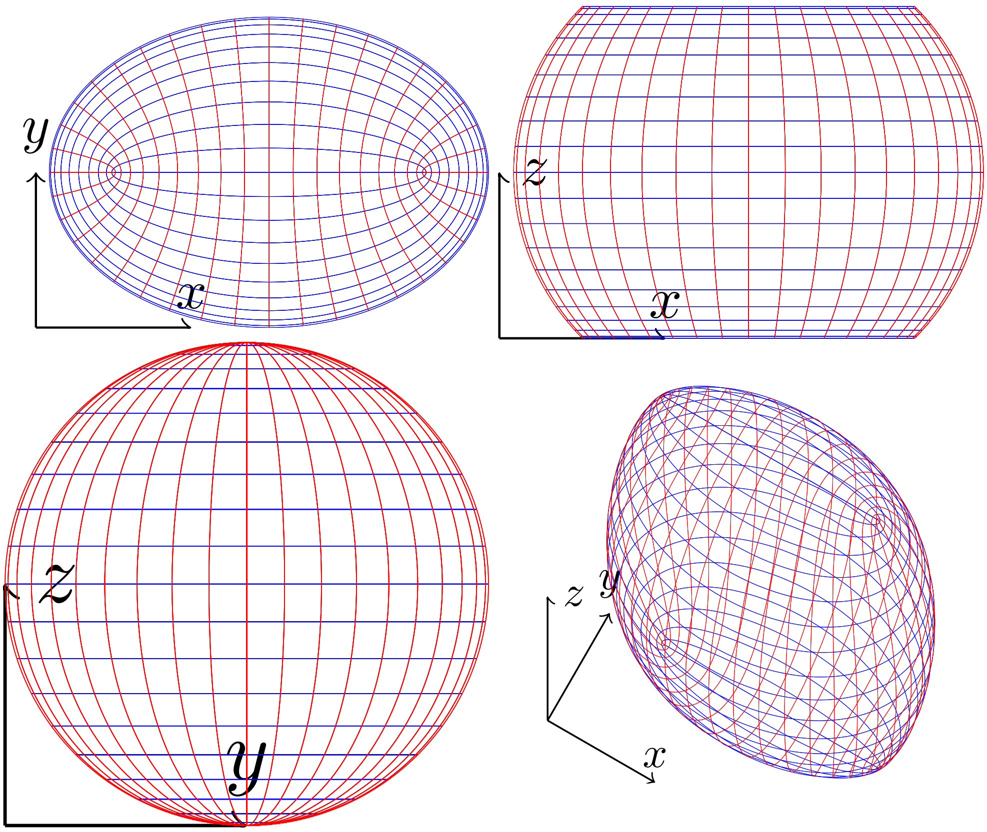


FIGURE F.1. Pillow produced by re-scaling and inflating $x + iy = \cosh(u + iv)$

It may be possible to produce near-orthogonal coordinate systems by extruding this pillow through the Gaussian surfaces it would produce if it was treated as if it was made of a charged perfectly conducting material.

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